

Ewerhart, Christian

Working Paper

Games with continuous payoff functions and the problem of measurability

Working Paper, No. 467

Provided in Cooperation with:

Department of Economics, University of Zurich

Suggested Citation: Ewerhart, Christian (2025) : Games with continuous payoff functions and the problem of measurability, Working Paper, No. 467, University of Zurich, Department of Economics, Zurich,
<https://doi.org/10.5167/uzh-276544>

This Version is available at:

<https://hdl.handle.net/10419/316344>

Standard-Nutzungsbedingungen:

Die Dokumente auf EconStor dürfen zu eigenen wissenschaftlichen Zwecken und zum Privatgebrauch gespeichert und kopiert werden.

Sie dürfen die Dokumente nicht für öffentliche oder kommerzielle Zwecke vervielfältigen, öffentlich ausstellen, öffentlich zugänglich machen, vertreiben oder anderweitig nutzen.

Sofern die Verfasser die Dokumente unter Open-Content-Lizenzen (insbesondere CC-Lizenzen) zur Verfügung gestellt haben sollten, gelten abweichend von diesen Nutzungsbedingungen die in der dort genannten Lizenz gewährten Nutzungsrechte.

Terms of use:

Documents in EconStor may be saved and copied for your personal and scholarly purposes.

You are not to copy documents for public or commercial purposes, to exhibit the documents publicly, to make them publicly available on the internet, or to distribute or otherwise use the documents in public.

If the documents have been made available under an Open Content Licence (especially Creative Commons Licences), you may exercise further usage rights as specified in the indicated licence.



**University of
Zurich**^{UZH}

University of Zurich
Department of Economics

Working Paper Series
ISSN 1664-7041 (print)
ISSN 1664-705X (online)

Working Paper No. 467

Games with Continuous Payoff Functions and the Problem of Measurability

Christian Ewerhart

March 2025

Games with Continuous Payoff Functions and the Problem of Measurability

Christian Ewerhart*

March 25, 2025

Abstract. This paper examines the definition and continuity of expected payoffs in compact games with continuous payoff functions. There are three main results. First, we confirm that [Glicksberg's \(1952\)](#) original definition of expected payoffs as an iterated integral is mathematically sound under general conditions. Second, we show that the now more common definition as a single integral is both rigorous and equivalent to the original when strategy spaces are either Hausdorff or second countable. Third, we offer an alternative proof of the continuity of expected payoffs without imposing the Hausdorff separation axiom. Together, these results lead to a strengthening of Glicksberg's theorem on equilibrium existence in compact Hausdorff games with continuous payoff functions.

Keywords. Compact games, expected payoffs, weak* topology, measurability, continuity.

JEL classification. C72: Noncooperative Games

*) Department of Economics, University of Zurich; Schönberggasse 1, 8001 Zurich, Switzerland; e-mail: christian.ewerhart@econ.uzh.ch.

1 Introduction

In the theory of finite noncooperative games, pioneered by [von Neumann \(1928\)](#) and [Nash \(1950\)](#), the *mixed extension* requires players to choose probability distributions over pure strategies in order to maximize their expected payoffs.¹ This canonical construction stands in sharp contrast to the flexibility allowed in infinite games. A widely studied class of infinite games consists of those with compact Hausdorff strategy spaces and continuous payoff functions. Under such assumptions, [Glicksberg \(1952\)](#) defined a player’s expected payoff as an iterated integral, and asserted the continuity of the integral with respect to the weak* topology. Since the original work provided little detail on these issues, a small but significant body of literature, beginning with [Glycopantis and Muir \(2000\)](#), has established the *continuity of expected payoffs* under varying degrees of generality using different techniques. In contrast, there apparently does not exist any closer investigation of the issue of *measurability*. Yet, even for continuous payoff functions, it is not immediately clear that expected payoffs are always well-defined and independent of the chosen integral representation.

In this paper, we review the definition and continuity of expected payoffs in n -player games with compact pure strategy spaces and continuous payoff functions. The contribution is threefold. First, we verify that expected payoffs, if specified as iterated integrals, are well-defined for continuous functions under general conditions. Second, we show that the “modern” definition of a player’s expected payoff as a single integral is both admissible and equivalent to [Glicksberg’s \(1952\)](#) definition provided that either all strategy spaces are Hausdorff or all strategy spaces are second countable. Third and finally, we develop an alter-

¹Admitting such randomization is generally useful to obtain equilibria in games that lack convex strategy spaces or quasiconcave payoff functions ([Dasgupta and Maskin, 1986](#)).

native approach for deriving the continuity of expected payoffs. In contrast to prior work, however, the result is obtained without any reference to the Hausdorff separation axiom.

To understand why the measurability assumption on payoff functions is crucial in the analysis of mixed-strategy Nash equilibria, recall the definition of expected payoffs in terms of the Lebesgue integral, viz. as the supremum of the integral of all simple functions which are less or equal than the function to be integrated. Here, as usual, a function is called *simple* if it is a convex combination of indicator functions associated with measurable sets. If the expectation is formed over a product of pure strategy spaces, then the relevant class of measurable sets consists of “rectangles,” or more precisely of Cartesian products of Borel sets. Notably, for a bounded payoff function defined on a product of compact pure strategy spaces, the supremum mentioned above is always well-defined, regardless of the measurability properties of the payoff function. However, there is a reason why the Lebesgue integral is restricted to measurable functions. Measurability ensures that the integrand can be approximated uniformly by simple functions, i.e., the supremum norm of the absolute difference between the integrand and the approximating simple function can be made arbitrarily small. Therefore, for nonmeasurable payoff functions that cannot be approximated in that way, a gap remains between the payoff function and the pointwise supremum of the simple functions approximating the integrand from below. As a result, important properties of the integration operator, such as additivity of the formation of expectations, may be lost if the definition is extended to nonmeasurable functions. Thus, the problem of measurability cannot be easily resolved by modifying the definition of the Lebesgue integral over product measures.

Our central result (Proposition 2) says that the crucial measurability condi-

tions needed to represent expected payoffs in the usual way are satisfied if either all pure strategy spaces are Hausdorff or all strategy spaces are second countable.² A case in point is the class of compact games with metrizable strategy spaces which satisfy both criteria simultaneously. Interestingly, the argument leading to measurability of continuous payoff function with respect to the product of the respective Borel σ algebras differs across the two cases. For second countable spaces, the two relevant σ -algebras coincide, i.e., the product of the Borel σ -algebras and the Borel σ -algebra of the Cartesian product, which makes the proof of measurability relatively simple. For spaces that are not second countable, however, these two σ -algebras may differ.³ As a result, establishing the measurability with respect to the smaller σ -algebra requires additional arguments, including Urysohn's Lemma and the Stone-Weierstrass Approximation Theorem.

To establish the existence of the iterated integral (Proposition 1) and the continuity of expected payoffs (Proposition 3), we use a specific characterization of joint continuity.⁴ We show that a function defined on a product of two topological spaces is jointly continuous if and only if the function is continuous in the first variable, with local uniformity over the second variable, and separately continuous in the second variable. Moreover, we note that compactness allows turning local uniformity into uniformity. Since the property of being continuous in the first argument, with uniformity in the second argument, is stable under the formation of expectations, these observations are precisely the tools that al-

²A topological space is *second countable* if there exists a countable collection of open sets such that every open set can be written as a union of sets from this collection.

³See Glycopantis and Muir (2000). The earliest reference to the problem of measurability in the game-theoretic literature seems to be Mertens (1986, p. 245), which at the same time seems to be the only earlier work allowing for randomization over non-Hausdorff strategy spaces.

⁴The term *joint continuity* will be used to emphasize that a function defined on a product space is continuous with respect to the product topology on its domain. E.g., in games with continuous payoff functions, payoff functions are assumed to be jointly continuous.

lows us to “lift” continuity properties from a game to its mixed extension. We introduce the corresponding characterization of joint continuity in a preliminary section before the main analysis.

Related literature. [Glicksberg’s \(1952\)](#) continuity assertion has been reviewed by a variety of authors. Specifically, in the general case, [Glycopantis and Muir \(2000\)](#) relied on the Stone-Weierstrass theorem, while [Zarichnyi \(2004\)](#) derived his results using the existence of so-called Milyutin maps. [Aliprantis et al. \(2006\)](#) used advanced methods from functional analysis to provide an alternative proof tailored to the important special case of metrizable strategy spaces. They also offered an excellent discussion of the literature. Further contributions include [Becker and Damianov \(2006\)](#), [Kozhan and Zarichnyi \(2008\)](#), and [Kim \(2014\)](#). Regarding the measurability issues addressed in the present paper, we are not aware of any work, neither within nor outside of that literature, that has investigated if, and if so why, the standard assumptions of Fubini’s Theorem hold under the general conditions of [Glicksberg’s \(1952\)](#) analysis.⁵

The rest of the paper is organized as follows. Section 2 concerns preliminaries. Section 3 reviews Glickberg’s definition of expected payoffs. The problem of measurability is discussed in Section 4. In Section 5, we establish the continuity of the expected payoff function. Section 6 concludes. There are two appendices: Appendix A provides the necessary mathematical background, while Appendix B contains formal proofs.

⁵In a contribution that apparently went unnoticed by the game theory community, [Glicksberg \(1962\)](#) proved a variant of our Proposition 1 for two factors, assuming that strategy spaces are locally compact Hausdorff and that the integrand is separately continuous. His proof uses methods from functional analysis. In the present paper, we provide a direct and elementary proof for $n \geq 2$ factors. Moreover, we do not impose the Hausdorff assumption.

2 Preliminaries

This section reviews the measurability of continuous functions (Subsection 2.1), introduces the weak* topology on the space of regular probability measures (Subsection 2.2), and provides a characterization of functions continuous on product spaces (Subsection 2.3). We assume familiarity with general topology and integration theory as reviewed in Appendix A.

2.1 The Measurability of Continuous Functions

Let X be a topological space, i.e., a set equipped with a topology \mathcal{T} consisting of open sets. The elements of $\mathcal{B}(X)$, defined as the smallest σ -algebra containing all open sets, are called *Borel sets*. The following lemma sets the stage for our analysis.

Lemma 1. *Any continuous mapping between two topological spaces is measurable with respect to the respective σ -algebras of Borel sets.*

Proof. Let $h : X \rightarrow X'$ be continuous, and let Σ' be the collection of subsets $M \subseteq X'$ such that $h^{-1}(M) \in \mathcal{B}(X)$. It is straightforward to check that Σ' is a σ -algebra. By continuity, the preimage of every open set in X' under h is open, hence an element of $\mathcal{B}(X)$. Since $\mathcal{B}(X')$ is the smallest σ -algebra containing the open sets, it follows that $\Sigma' \supseteq \mathcal{B}(X')$, proving the lemma. \square

By a *Borel probability measure* μ on X , we mean a countably additive nonnegative set function defined on $\mathcal{B}(X)$ such that $\mu(X) = 1$. For a function $f : X \rightarrow \mathbb{R}$ continuous on a compact space X , Lemma 1 implies integrability with respect to any Borel probability measure, which is a fact widely used in economic theory. Much of the subsequent analysis will be concerned with understanding how Lemma 1 extends to functions continuous on a product of topological spaces.

2.2 The Weak* Topology on the Space of Regular Probability Measures

A Borel probability measure μ on a topological space X is *regular* if for any $M \in \mathcal{B}(X)$ and any $\varepsilon > 0$, there is a closed set C and an open set U such that $C \subseteq M \subseteq U$ and $\mu(U \setminus C) < \varepsilon$.⁶ The regularity assumption ensures that a measure is well-behaved with respect to the underlying topology, which explains why it has been commonly imposed on Borel probability distributions that represent mixed strategies. We will need the following fact.

Lemma 2. *Suppose that X is Hausdorff and that $x \in X$. Then, the Dirac measure δ_x is regular.*

Proof. Let M be Borel. For $x \in M$, one notes that $C = \{x\}$ is closed by the Hausdorff assumption and that $U = X$ is open by the definition of the topology, while $\mu(U \setminus C) = \mu(X \setminus \{x\}) = 0$. For $x \notin M$, one notes similarly that $C = \emptyset$ is closed, and that $U = X \setminus \{x\}$ is open, while again $\mu(U \setminus C) = \mu(X \setminus \{x\}) = 0$. \square

We denote by $\mathcal{P}(X)$ the space of regular probability measures, and by $\mathcal{C}(X)$ the space of continuous functions on X . For X compact, we define the *weak** topology on $\mathcal{P}(X)$ as the coarsest topology such that the evaluation map $e_f : \mu \mapsto \int_X f(x) d\mu(x)$ is continuous for any $f \in \mathcal{C}(X)$.

Glicksberg (1952) introduced the topology on $\mathcal{P}(X)$ by reference to the weak* topology on the space of functionals on the Banach space $\mathcal{C}(X)$. While elegant, that approach presupposes the validity of the Riesz Representation Theorem, and therefore relies on the Hausdorff separation axiom. The following result ensures that our direct definition of the weak* topology, which does not depend on the Hausdorff assumption, is consistent with his definition.

⁶Dunford and Schwartz (1958, Def. III.5.11) reduces to our definition for nonnegative Borel set functions.

Lemma 3. *Suppose that X is Hausdorff. Then, the weak* topology defined above is identical to the subspace topology induced on $\mathcal{P}(X)$ by the weak* topology on the Banach space of bounded linear functionals on $\mathcal{C}(X)$.*

Proof. See Appendix [B.1](#). □

2.3 Continuous Functions on Product Spaces

The following characterization of the continuity of a function defined on a product of two topological spaces is the key to our Propositions [1](#) and [3](#) below. While the proof contains few surprises, we have been unable to find a suitable reference.⁷

Lemma 4. *Let X, Y be arbitrary topological spaces and let $f : X \times Y \rightarrow \mathbb{R}$. Then, $f(x, y)$ is jointly continuous in (x, y) if the following two conditions are satisfied:*

- (i) $f(x, y)$ is continuous in x , uniformly over $y \in Y$;
- (ii) $f(x, y)$ is continuous in y , for any $x \in X$.

Conversely, if $f(x, y)$ is jointly continuous in (x, y) and, in addition, Y is compact, then conditions (i) and (ii) hold.

Proof. See Appendix [B.2](#). □

The proof follows a similar idea as the Heine-Cantor Lemma which states that a continuous function on a compact space is uniformly continuous. However, our result concerns uniformity with respect to a different variable. In the case of metric spaces, the first part of the lemma is readily verified using the epsilon-delta criterion for continuity at a point in X . However, we need the general

⁷E.g., [Munkres \(2014, Ch. 2, p. 110\)](#) stated that there is “no useful criterion” for the continuity of a mapping that is defined on the product of topological spaces.

case. The second part of the lemma assumes that Y is compact. Compactness is needed for this direction because the joint continuity in (x, y) merely implies that the continuity in x is *locally* uniform in y . Thus, compactness ensures that we can deduce uniformity on all of Y .

3 Glicksberg's Definition of Expected Payoffs

Let $G = (S_i, u_i)_{i=1}^n$ be an n -person game, where S_i is player i 's pure strategy space and $u_i : S_1 \times \dots \times S_n \rightarrow \mathbb{R}$ is player i 's payoff function. We will assume throughout that each S_i is a compact topological space, and that each u_i is continuous, for any $i \in \{1, \dots, n\}$. In the *mixed extension* of G , each player $i \in \{1, \dots, n\}$ chooses some regular probability measure $\mu_i \in \mathcal{P}(S_i)$. Given the strategy profile

$$\mu = (\mu_1, \dots, \mu_n) \in \mathcal{P}(S_1) \times \dots \times \mathcal{P}(S_n),$$

we initially follow [Glicksberg \(1952\)](#) in defining player i 's expected payoff as the *iterated* integral

$$E_\mu[u_i] = \int_{S_1} \left\{ \dots \left\{ \int_{S_n} u_i(s_1, \dots, s_n) d\mu_n(s_n) \right\} \dots \right\} d\mu_1(s_1).$$

It should be noted that we have not used the concept of a product measure so far. Neither have we invoked any variant of the Fubini-Tonelli theorem. The following result clarifies the nature of Glicksberg's definition.

Proposition 1. *Under the assumptions made above, i.e., the compactness of strategy spaces and the continuity of u_i , the following is true:*

- (i) *The iterated integral $E_\mu[u_i]$ is well-defined;*
- (ii) *the value of $E_\mu[u_i]$ does not depend on the order of integration.*

Proof. See Appendix [B.3](#). □

Thus, even in the absence of the Hausdorff assumption, Glicksberg's original definition is both mathematically sound and economically meaningful. The iterated integral is well-defined because, given the continuity of u_i , the inner integrals are jointly continuous in the remaining strategy profile, as shown using Lemma 4. Since continuity implies univariate measurability (Lemma 1), the existence of iterated integrals follows by induction, establishing claim (i).⁸

The second claim says that any renaming of the players leaves expected payoffs invariant. Although not needed to establish Glicksberg's existence theorem for mixed-strategy Nash equilibria, this property is obviously desirable for any definition of expected payoffs. The proof relies again on Lemma 4, but drawing also from the proof of Glicksberg (1962, Thm. 3.1). Specifically, in the case of two factors, the order of integration does not matter if one of the probability measures is a convex combination of Dirac measures. Moreover, as we show, any Borel probability measure can be approximated arbitrarily well in the weak* topology by a convex combination of Dirac measures.

4 The Problem of Measurability

The following result provides conditions under which the modern definition of a player's expected payoff, expressed as a single integral over the product of pure strategy spaces, is both rigorous and equivalent to Glicksberg's (1952) definition.

Proposition 2. *Suppose that u_i is continuous. Suppose also that all S_i are compact and nonempty, and that at least one of the following conditions is satisfied:*

- (i) *all S_i are Hausdorff;*
- (ii) *all S_i are second countable.*

⁸It may well be conjectured that this argument was known to Glicksberg at the time, and considered too elementary to be reported.

Then u_i is measurable with respect to $\mathcal{B}(S_1) \otimes \dots \otimes \mathcal{B}(S_n)$, and the integral

$$\tilde{E}_\mu[u_i] = \int_{S_1 \times \dots \times S_n} u_i(s_1, \dots, s_n) d(\mu_1 \times \dots \times \mu_n)(s_1, \dots, s_n)$$

is well-defined and equal to the iterated integral representation, i.e., $\tilde{E}_\mu[u_i] = E_\mu[u_i]$

Proof. See Appendix B.4. □

Thus, if all strategy spaces are Hausdorff, or all strategy spaces are second countable, or both (the metrizable case), [Glicksberg's \(1952\)](#) original definition of expected payoffs in terms of an iterated integral is equivalent to the definition, common in the literature, in terms of a single integral over a product measure. When strategy spaces are neither Hausdorff nor second countable, however, the two definitions may differ. In fact, the integral in the statement of [Proposition 2](#) might not be well-defined. This is because a function continuous with respect to the product topology on $S_1 \times \dots \times S_n$ is, by [Lemma 1](#), measurable with respect to $\mathcal{B}(S_1 \times \dots \times S_n)$, but not necessarily with respect to $\mathcal{B}(S_1) \otimes \dots \otimes \mathcal{B}(S_n)$. Thus, a problem arises whenever

$$\mathcal{B}(S_1) \otimes \dots \otimes \mathcal{B}(S_n) \subsetneq \mathcal{B}(S_1 \times \dots \times S_n).$$

In this case, measurability with respect to $\mathcal{B}(S_1) \otimes \dots \otimes \mathcal{B}(S_n)$ is a strictly stronger condition than measurability with respect to $\mathcal{B}(S_1 \times \dots \times S_n)$. However, the stronger condition is crucial for the validity of Fubini's Theorem ([Folland, 1999](#), p. 68 and Ex. 47). The proof of [Proposition 2](#) shows that this complication does not arise for Hausdorff spaces or second countable spaces. As discussed in the Introduction, the two cases differ in their logic. Specifically, if all strategy spaces are second countable spaces, then the two σ -algebras just mentioned are identical. In contrast, if all strategy spaces are Hausdorff but not second countable, then

the two σ -algebras may differ, but we can still prove that jointly continuous payoff functions are measurable with respect to the smaller σ -algebra, which is all we need to apply Fubini's Theorem.

5 Continuity of Expected Payoffs

The following result establishes the continuity of expected payoffs with respect to the product of the respective weak* topologies.

Proposition 3. *Suppose that each S_i is compact. Suppose also that u_i continuous on X . Then, the mapping that assigns to each $\mu = (\mu_1, \dots, \mu_n) \in \mathcal{P}(S_1) \times \dots \times \mathcal{P}(S_n)$ the expected payoff $E_\mu[u_i]$ is continuous in the product of the weak* topologies.*

Proof. Suppose first that $n = 2$. Then, as seen in the proof of Proposition 1, $E_{\mu_1, \mu_2}[u_i] = E_{\mu_1}[E_{\mu_2}[u_i]]$ is continuous in μ_2 , uniformly over μ_1 . Moreover, since $E_{\mu_2}[u_i]$ is separately continuous in s_1 , for any given μ_2 , the definition of the weak* topology implies that $E_{\mu_1, \mu_2}[u_i]$ is separately continuous in μ_1 , for any μ_2 . Hence, using Lemma 4, $E_{\mu_1, \mu_2}[u_i]$ is jointly continuous in (μ_1, μ_2) . This proves the claim in the case $n = 2$. The case where $n \geq 3$ is obtained as follows. First, one shows that

$$E_{\mu_n}[u_i] = \int_{S_n} u_i(s_1, \dots, s_n) d\mu_n(s_n)$$

is continuous in (s_1, \dots, s_{n-1}) , uniformly over μ_n , while $E_{\mu_n}[u_i]$ is continuous in μ_n , for any fixed (s_1, \dots, s_{n-1}) . Therefore, $E_{\mu_n}[u_i]$ is jointly continuous in $(s_1, \dots, s_{n-1}, \mu_n)$. Using the argument developed above, this implies that the iterated expectation $E_{\mu_{n-1}}[E_{\mu_n}[u_i]]$ is jointly continuous in $(s_1, \dots, s_{n-2}, \mu_{n-1}, \mu_n)$, etc. Straightforward induction yields then the claim for general $n \geq 3$. \square

From a modern perspective, the equilibrium notion used by [Glicksberg \(1952\)](#) is nonstandard insofar that player i 's expected payoff was defined as the iterated integral $E_\mu(u_i)$ (see the text before Proposition 1). By a *mixed-strategy Nash equilibrium* for an n -player game G , we mean here a profile $\mu = (\mu_1, \dots, \mu_n)$ of regular probability distributions such that no player i has an incentive to deviate, if her expected payoff is defined as the single integral $\tilde{E}_\mu(u_i)$ over the product measure (see Proposition 2). The following result is, therefore, actually a strengthening of Glicksberg's original result.

Corollary 1. *Suppose that strategy spaces are nonempty, compact, and Hausdorff, and that payoff functions are continuous. Then, there exists a mixed-strategy Nash equilibrium even if expected payoffs are defined as single integrals.*

Proof.⁹ In the Hausdorff case, compactness of $\mathcal{P}(S_i)$ follows via the Riesz Representation Theorem from Alaoglu's Theorem ([Reny, 1999](#), fn. 20).¹⁰ Moreover, $\mathcal{P}(S_i)$ is nonempty because pure strategy spaces are nonempty and Dirac measures are regular in the Hausdorff case (Lemma 2). Next, by Propositions 1 through 3, each player i 's expected payoff function $\tilde{E}_\mu(u_i)$ is well-defined and jointly continuous in the product of the respective weak* topologies. Finally, consider the linear space V_i generated by $\mathcal{P}(S_i)$, equipped with the corresponding weak* topology. By Lemma B.1 in the Appendix, $\mathcal{P}(S_i)$ is a subspace of V_i . Now, V_i locally convex because the standard base of the weak* topology in V_i consists of convex sets. Further, V_i is weak* Hausdorff.¹¹ Therefore, the mixed

⁹We include an outline of the proof in particular to provide details on why $\mathcal{P}(S_i)$ is a nonempty subspace of a locally convex Hausdorff space. These are issues that have likewise received little discussion in [Glicksberg \(1952\)](#).

¹⁰For discussion, see [Glycopantis and Muir \(2004\)](#).

¹¹To see why, let $v = \lambda\mu + \rho\nu$ and $\hat{v} = \hat{\lambda}\hat{\mu} + \hat{\rho}\hat{\nu}$ be distinct elements of V_i , with $\lambda, \rho, \hat{\lambda}, \hat{\rho} \in \mathbb{R}$ and $\mu, \nu, \hat{\mu}, \hat{\nu} \in \mathcal{P}(S_i)$. Then, there is a Borel set $M \subseteq S_i$ such that, in straightforward extension of the notation, $v(M) > \hat{v}(M)$. By regularity, we find a closed set $C \subseteq M$ and an open set $U \supseteq M$ such that $\mu(U \setminus C) < \varepsilon$, $\nu(U \setminus C) < \varepsilon$, $\hat{\mu}(U \setminus C) < \varepsilon$, and $\hat{\nu}(U \setminus C) < \varepsilon$. Given

extension satisfies the assumptions of [Glicksberg's \(1952\)](#) fixed point theorem, proving the claim. \square

Thus, with respect to [Glicksberg's \(1952\)](#) existence theorem for mixed-strategy Nash equilibria, our modest contribution is the observation that his definition of expected payoffs is rigorous under the general assumptions of his analysis (because he assumed that all pure strategy spaces are Hausdorff), and that the way in which he chose to define expected payoffs is equivalent to the modern definition.

6 Conclusion

The main takeaway from the present analysis is that expected payoffs are well-defined under [Glicksberg's \(1952\)](#) general conditions. This holds even regardless of the details of their representation as an integral. However, as explained, there are measurability assumptions implicit in the way in which expected payoffs are commonly represented. Specifically, Fubini's theorem, which allows to switch between iterated expectations and total expectations, assumes that the integrand is measurable in an abstract measure-theoretic setting. In the game-theoretic context where σ -algebras are derived from topologies, this concept of measurability with respect to the product of Borel σ -algebras is easily confused with the more permissive notion of measurability with respect to the Borel σ -algebra of the product topology. As demonstrated, however, only the latter concept is generally implied by the continuity of the underlying payoff function, while the former requires additional assumptions (such as Hausdorffness or second countability) as well as a different proof.

that S_i is Hausdorff compact, Urysohn's Lemma implies the existence of $f \in \mathcal{C}(S_i)$ such that $e_f(v) + (\lambda + \rho)\varepsilon \geq v(M) > \hat{v}(M) \geq e_f(\hat{v}) - (\hat{\lambda} + \hat{\rho})\varepsilon$. For $\varepsilon > 0$ sufficiently small, this implies $e_f(v) > e_f(\hat{v})$, which proves the claim.

It would be of some interest to extend the main results of the present analysis to the discontinuous case (Reny, 1999; Monteiro and Page, 2007; Prokopovych and Yannelis, 2014). Among the conditions sufficient for better reply-security in the mixed extension, it seems desirable, for instance, to explore the conditions under which uniform payoff security in the underlying game, potentially combined with the upper semicontinuity of the total payoff function, implies the measurability of all payoff functions with respect to the product σ -algebra. Such questions, however, lie beyond the limited scope of the present analysis.

Appendix A: Mathematical Tools

This section provides basic definitions and results regarding compact topological spaces (Subsection A.1) and integration over general measure spaces (Subsection A.2).

A.1 Compact Topological Spaces

A *topological space* is a pair (X, \mathcal{T}) , consisting of a set X , and a collection \mathcal{T} of subsets of X , such that: (i) $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$; (ii) the union of any collection of sets in \mathcal{T} is also in \mathcal{T} ; and (iii) the intersection of any finite number of sets in \mathcal{T} is also in \mathcal{T} . In this case, \mathcal{T} is called a *topology* on X , and the elements of \mathcal{T} are called *open sets*. A set $C \subseteq X$ is *closed* if its complement $X \setminus C$ is in \mathcal{T} . Given two topologies \mathcal{T} and \mathcal{T}' on the same set X , if $\mathcal{T} \subseteq \mathcal{T}'$, then \mathcal{T} is said to be *coarser* than \mathcal{T}' . Given a topological space (X, \mathcal{T}) and a subset $Y \subseteq X$, the *subspace topology* on Y is defined as $\mathcal{T}|_Y = \{Y \cap U : U \in \mathcal{T}\}$. The space $(Y, \mathcal{T}|_Y)$ is then called a *subspace* of (X, \mathcal{T}) . Given topological spaces (X, \mathcal{T}) and (X', \mathcal{T}') , a mapping $h : X \rightarrow X'$ is *continuous* if, for any $U \in \mathcal{T}'$, the *preimage* $h^{-1}(U) = \{x \in X : h(x) \in U\}$ is in \mathcal{T} . A topological space

(X, \mathcal{T}) is *compact* if every covering of X by open sets has a finite subcover. That is, for any collection $\{U_\alpha\}_{\alpha \in \mathcal{A}} \subseteq \mathcal{T}$ such that $X = \bigcup_{\alpha \in \mathcal{A}} U_\alpha$, there exists a finite subcollection $\{U_{\alpha_1}, U_{\alpha_2}, \dots, U_{\alpha_K}\} \subseteq \{U_\alpha\}_{\alpha \in \mathcal{A}}$ such that $X = \bigcup_{k=1}^K U_{\alpha_k}$. A topological space (X, \mathcal{T}) is *Hausdorff* if, for any two distinct points $x, y \in X$, there exist disjoint open sets $U, V \in \mathcal{T}$ such that $x \in U$ and $y \in V$. Given topological spaces $(X_1, \mathcal{T}_1), \dots, (X_n, \mathcal{T}_n)$, the *product topology* on the Cartesian product $X = X_1 \times \dots \times X_n$ is the coarsest topology \mathcal{T} such that the canonical projection $\pi_i : X \rightarrow X_i$ is continuous for all $i \in \{1, \dots, n\}$. Then, by *Tychonoff's Theorem*, (X, \mathcal{T}) is compact if each (X_i, \mathcal{T}_i) is compact. Given a topological space (X, \mathcal{T}) , a *basis* of its topology \mathcal{T} is a collection of open sets such that any open set can be written as a union of such sets. A topological space (X, \mathcal{T}) is *second countable* if it has a countable basis.¹² To lighten the notation, the explicit reference to \mathcal{T} is typically dropped. For further background on general topology, see Kelley (1975).

A.2 Integration over General Measure Spaces

A σ -algebra Σ on a set X is a collection of subsets of X that satisfies the following properties: (i) $X \in \Sigma$; (ii) if $M \in \Sigma$, then $X \setminus M \in \Sigma$; and (iii) if $\{M_k\}_{k=1}^\infty$ is a countable collection of sets in Σ , then $\bigcup_{k=1}^\infty M_k \in \Sigma$. The elements of Σ are called *measurable* sets. A *measurable space* is a pair (X, Σ) , where X is a nonempty set and Σ is a σ -algebra on X . Given measurable spaces (X, Σ) and (X', Σ') , a mapping $h : X \rightarrow X'$ is *measurable* if, for any $M \in \Sigma'$, the preimage $h^{-1}(M)$ is in Σ .

Let (X, Σ) be a measurable space. A *set function* is a mapping $\mu : \Sigma \rightarrow \mathbb{R}$. We call a set function *nonnegative* if it does not attain negative values. A set

¹²For compact Hausdorff spaces, second countability is equivalent to metrizability by Urysohn's Metrizability Theorem.

function is *countably additive* if, for any countable collection $\{M_k\}_{k=1}^{\infty}$ of pairwise disjoint sets in Σ , we have $\mu(\bigcup_{k=1}^{\infty} M_k) = \sum_{k=1}^{\infty} \mu(M_k)$. A *probability measure* is a countably additive, nonnegative set function μ satisfying $\mu(X) = 1$. For any $x \in X$, the *Dirac measure* δ_x , defined by $\delta_x(M) = 1$ if $x \in M$ and by $\delta_x(M) = 0$ if $x \notin M$, is a probability measure.

A function f on X is called *simple* if there exists a representation $f(x) = \sum_{k=1}^K \gamma_k \mathbf{1}_{M_k}(x)$ for constants $\gamma_1, \dots, \gamma_K \in \mathbb{R}$ and some partition $M_1 \cup \dots \cup M_K = X$ into measurable subsets $M_1, \dots, M_K \in \Sigma$. For any probability measure μ , the integral of a simple function g is $\int_X g(x) d\mu(x) = \sum_{k=1}^K \gamma_k \mu(M_k)$. Any measurable bounded function $f : X \rightarrow \mathbb{R}$ may be approximated uniformly from below by a sequence of simple functions $\{f_m\}_{m=1}^{\infty}$.¹³ The *Lebesgue integral* of f is given as

$$\int_X f(x) d\mu(x) = \sup_{m \rightarrow \infty} \int_X f_m(x) d\mu(x).$$

Given measurable spaces $(X_1, \Sigma_1), \dots, (X_n, \Sigma_n)$, the *product σ -algebra* $\Sigma_1 \otimes \dots \otimes \Sigma_n$ is defined as a smallest σ -algebra on $X_1 \times \dots \times X_n$ that contains all product sets $M_1 \times \dots \times M_n$, where $M_i \in \Sigma_i$ for $i \in \{1, \dots, n\}$. Given probability measures μ_1, \dots, μ_n on $(X_1, \Sigma_1), \dots, (X_n, \Sigma_n)$, respectively, the *product measure*, denoted by $\mu_1 \times \dots \times \mu_n$, is defined as the unique probability measure on $\Sigma_1 \otimes \dots \otimes \Sigma_n$ such that

$$(\mu_1 \times \dots \times \mu_n)(M_1 \times \dots \times M_n) = \mu_1(M_1) \cdot \dots \cdot \mu_n(M_n)$$

for all $M_1 \in \Sigma_1, \dots, M_n \in \Sigma_n$. For further background on integration over general measure spaces, see [Dunford and Schwartz \(1958, Ch. III\)](#) or [Royden and Fitzpatrick \(2010, Sec. 18\)](#).

¹³For this, it suffices to partition the real line into half-open intervals of length $\varepsilon > 0$, and to define the M_k 's as the respective preimages, of which only finitely many are nonempty.

Appendix B: Proofs

This section contains technical proofs omitted from the body of the paper.

B.1 Proof of Lemma 3

The proof makes use of the following lemma.

Lemma B.1 (Grothendieck). *Let X be equipped with the coarsest topology \mathcal{T} such that any mapping $g_\alpha : X \rightarrow X_\alpha$ ($\alpha \in \mathcal{A}$) is continuous, and let $h : Y \rightarrow X$ be an arbitrary mapping. Then, the coarsest topology on Y such that h is continuous is the coarsest topology such that any mapping $(g_\alpha \circ h) : Y \rightarrow X_\alpha$ ($\alpha \in \mathcal{A}$) is continuous. In particular, if h is an injection, then this property characterizes the subspace topology $\mathcal{T}|_Y$ on Y .*

Proof. See [Grothendieck \(1973, pp. 1-2\)](#). □

After these preparations, we will now prove Lemma 3. By the Riesz Representation Theorem, the linear space generated by the regular probability measures on $\mathcal{B}(X)$ is isometrically isomorphic to the Banach space of bounded linear functions on $\mathcal{C}(X)$. The claim therefore follows from Lemma B.1 above. □

B.2 Proof of Lemma 4

(If) Let $(x_0, y_0) \in X \times Y$, and take some $\varepsilon > 0$. Then, by uniform continuity, there exists an open neighborhood $U_x \subseteq X$ of x_0 such that

$$|f(x, y) - f(x_0, y)| < \frac{\varepsilon}{2},$$

for any $x \in U_x$ and any $y \in Y$. Next, by continuity with respect to y , there exists an open neighborhood $U_y \subseteq Y$ of y_0 such that

$$|f(x_0, y) - f(x_0, y_0)| < \frac{\varepsilon}{2},$$

for any $y \in U_y$. By the triangle inequality,

$$\begin{aligned} |f(x, y) - f(x_0, y_0)| &< |f(x, y) - f(x_0, y)| + |f(x_0, y) - f(x_0, y_0)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon, \end{aligned}$$

for any $(x, y) \in U_x \times U_y$. Clearly, $U_x \times U_y \subseteq X \times Y$ is an open neighborhood of (x_0, y_0) . Hence, given that $\varepsilon > 0$ and $(x_0, y_0) \in X \times Y$ were arbitrary, the mapping f is continuous.

(Only if) Conversely, suppose that $f(x, y)$ is jointly continuous in (x, y) , and let $\varepsilon > 0$. Then, for any $(x_0, y_0) \in X \times Y$, we find an open neighborhood $V \subseteq X \times Y$ such that

$$|f(x, y) - f(x_0, y_0)| < \frac{\varepsilon}{2},$$

for any $(x, y) \in V$. By the definition of the product topology, there exist open neighborhoods $U_x \subseteq X$ of x_0 and $U_y \subseteq Y$ of y_0 , respectively, such that $U_x \times U_y \subseteq V$. In particular, the inequality above holds for any $x \in U_x$ and any $y \in U_y$. Since $x_0 \in U_x$, this yields

$$|f(x_0, y) - f(x_0, y_0)| < \frac{\varepsilon}{2},$$

for any $y \in U_y$. Using the triangle inequality another time, we see that

$$\begin{aligned} |f(x, y) - f(x_0, y)| &< |f(x, y) - f(x_0, y)| + |f(x_0, y) - f(x_0, y_0)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon, \end{aligned}$$

for any $x \in U_x$ and any $y \in U_y$. Thus, f is continuous at $x = x_0$, locally uniformly over $y \in Y$. We keep now $x_0 \in X$ fixed. Then, depending on $y_0 \in U$, the set $U_y \equiv U_y(y_0)$ is an open neighborhood of y_0 , and similarly, $U_x \equiv U_x(y_0)$

is an open neighborhood of x_0 . Given that Y is compact, we find finitely many $y_1, \dots, y_K \in Y$ such that

$$Y = \bigcup_{k=1}^K U_{y_k}(y_k).$$

Hence, for any $y \in Y$, there is some $k \in \{1, \dots, K\}$ such that $y \in U_{y_k}(y_k)$. Then, for any $x \in \tilde{U}_x$ with

$$\tilde{U}_x = \bigcap_{k=1}^K U_x(y_k),$$

we have

$$|f(x, y) - f(x_0, y)| < \varepsilon.$$

Thus, given that ε was likewise arbitrary, $f(x, y)$ is continuous at $x = x_0$, uniformly over $y \in Y$. As $x_0 \in X$ was arbitrary, this shows that $f(x, y)$ is continuous in y , uniformly over all y . As the continuity of $f(x, y)$ with respect to y is immediate, the claim follows. \square

B.3 Proof of Proposition 1

(i) Given that u_i is continuous and bounded, the integral $\int_{S_n} u_i(s_1, \dots, s_n) d\mu_n(s_n)$ is well-defined. Moreover, since S_n is compact, Lemma 4 implies that $u_i(s_1, \dots, s_n)$ is jointly continuous in (s_1, \dots, s_{n-1}) , uniformly over all s_n . Therefore, the inner integral $\int_{S_n} u_i(s_1, \dots, s_n) d\mu_n(s_n)$ is jointly continuous in (s_1, \dots, s_{n-1}) . The claim now follows by induction over n .

(ii) Let $\hat{\mathcal{P}}(X)$ denote the space of Borel probability measures on a topological space X , equipped with the weak* topology. By Lemma B.1, $\mathcal{P}(X)$ is a subspace of $\hat{\mathcal{P}}(X)$, and any Dirac measure δ_x , with $x \in X$, is an element of $\hat{\mathcal{P}}(X)$. We will need the following auxiliary result.

Lemma B.2. *The set of convex combinations of Dirac measures on a topological space X is weak* dense in $\hat{\mathcal{P}}(X)$.*

Proof. The proof is adapted from [Bogachev \(2007, Ex. 8.1.6, p. 176\)](#). A base of the weak* topology on $\widehat{\mathcal{P}}(X)$ is the collection of all sets

$$U_{f_1, \dots, f_K, \varepsilon}(\mu) = \left\{ \nu \in \widehat{\mathcal{P}}(X) : \left| \int_X f_k(x) d\mu(x) - \int_X f_k(x) d\nu(x) \right| < \varepsilon, k = 1, \dots, K \right\},$$

where $\nu \in \widehat{\mathcal{P}}(X)$, f_1, \dots, f_K are continuous functions on X , and $\varepsilon > 0$. It suffices to show that any such set contains a convex combination of Dirac measures. For this, choose, for any k , a simple function g_k uniformly approximating f_k such that $|f_k(x) - g_k(x)| < \frac{\varepsilon}{2}$ for any $x \in X$. Select one element from each measurable set M in the join of the partitions associated with g_1, \dots, g_K , and attach the weight $\mu(M)$ to the corresponding Dirac measure. This delivers a probability measure ν^* that is a convex combination of Dirac measures. Moreover, by construction, $\int_X g_k(x) d\mu(x) = \int_X g_k(x) d\nu^*(x)$, so that ν^* is contained in $U_{f_1, \dots, f_K, \varepsilon}(\mu)$. \square

We are now in the position to prove Proposition 1(ii). We first address the case $n = 2$. Keep μ_1 fixed. Then, $E_{\mu_1}[E_{\mu_2}[u_i]] = E_{\mu_2}[E_{\mu_1}[u_i]]$ is true if μ_2 is a convex combination of Dirac measures. By Lemma B.2 above, any $\mu_2 \in P(S_2)$ is the weak* limit of a generalized sequence of such convex combinations of Dirac measures. It therefore suffices to show that the iterated integrals, considered as functionals of μ_2 alone, are weak* continuous on $\widehat{\mathcal{P}}(S_2)$. By assumption, u_i is jointly continuous in (s_1, s_2) . Moreover, S_2 is compact. Hence, using Lemma 4, $u_i(s_1, s_2)$ is continuous in s_1 , uniformly over s_2 . Taking expectations,

$$\int_{S_2} u_i(s_1, s_2) d\mu_2(s_2)$$

is continuous in s_1 , uniformly over μ_2 . Clearly, $u_i(s_1, s_2)$ is continuous in s_2 , for any s_1 . By the definition of the weak-* topology, $E_{\mu_2}[u_i]$ is continuous in μ_2 , for any fixed s_1 . Using Lemma 4 another time, $E_{\mu_2}[u_i]$ is jointly continuous in

(s_1, μ_2) . But S_1 is compact. Hence, using Lemma 4 still another time, $E_{\mu_2}[u_i]$ is continuous in μ_2 , uniformly over s_1 . Taking expectation, $E_{\mu_1}[E_{\mu_2}[u_i]]$ is continuous in μ_2 , even uniformly over μ_1 . On the other hand, $E_{\mu_2, \mu_1}[u_i] = E_{\mu_2}[E_{\mu_1}[u_i]]$ is continuous in μ_2 , as follows from the definition of the weak* topology and the continuity of $E_{\mu_1}[u_i]$ with respect to s_2 . This proves that the iterated integrals are continuous in μ_2 , for any fixed μ_1 , and therefore proves the claim for $n = 2$. The case where n is general follows now by induction, where one uses the facts that inner integrals are jointly continuous in the remaining pure strategy profile, as shown in part (i), and any permutation of the set of players may be written as a concatenation of transpositions that swap two neighboring players. \square

B.4 Proof of Proposition 2

(i) The claim follows from the following lemma.

Lemma B.3. *Let X_1, \dots, X_n be compact Hausdorff spaces and $f : X_1 \times \dots \times X_n \rightarrow \mathbb{R}$ be jointly continuous. Then, f is measurable with respect to the product of the respective Borel σ -algebras, $\mathcal{B}(X_1) \otimes \dots \otimes \mathcal{B}(X_n)$.*

Proof. We follow the steps of the proof outlined in Lang (1993, pp. 273-274) for the case $n = 2$, making suitable adaptations to account for the case of general n . Consider the set $\mathfrak{A} \subseteq \mathcal{C}(X_1 \times \dots \times X_n)$ consisting of all finite sums of functions of the form

$$(x_1, \dots, x_n) \mapsto \varphi_1(x_1) \cdot \dots \cdot \varphi_n(x_n),$$

where $\varphi_i \in \mathcal{C}(X_i)$ for each $i \in \{1, \dots, n\}$. We verify that \mathfrak{A} satisfies the conditions of the Stone-Weierstrass theorem. First, it is straightforward to check that \mathfrak{A} is an *algebra*, i.e., a linear space of functions closed under pointwise multiplication. Second, for distinct points $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$, there exists i

with $x_i \neq y_i$. As X_i is Hausdorff and compact, it is normal. Hence, by Urysohn's Lemma, we find $\varphi_i \in \mathcal{C}(X_i)$ separating x_i and y_i . Then, letting $\varphi_j \equiv 1$ for all $j \neq i$, the above product function is an element of \mathfrak{A} that separates x and y . Third, the constant function 1 is obviously in \mathfrak{A} . Since $X_1 \times \dots \times X_n$ is compact, the Stone-Weierstrass Theorem implies that \mathfrak{A} is uniformly dense in $\mathcal{C}(X_1 \times \dots \times X_n)$. Now, we claim that each $g \in \mathfrak{A}$ is measurable with respect to $\mathcal{B}(X_1) \otimes \dots \otimes \mathcal{B}(X_n)$. Indeed, being induced by a section as in [Folland \(1999, Prop. 2.34\)](#), the canonical projection $\mathcal{B}(X_1) \otimes \dots \otimes \mathcal{B}(X_n) \rightarrow \mathcal{B}(X_i)$ is measurable, and the same is true for finite sums and products of measurable functions, which proves the claim. As f is the uniform limit of such g , it is likewise measurable. \square

(ii) The proof makes use of the following fact.

Lemma B.4. *Let (X, \mathcal{T}) be the product of topological spaces $(X_1, \mathcal{T}_1), \dots, (X_n, \mathcal{T}_n)$. Then, $\mathcal{B}(X) \supseteq \mathcal{B}(X_1) \otimes \dots \otimes \mathcal{B}(X_n)$. Moreover, the set inclusion is an equality if (X_i, \mathcal{T}_i) is second countable, for $i \in \{1, \dots, n\}$.*

Proof. The case $n = 2$ corresponds to [Dudley \(2018, Prop. 4.1.7\)](#). For the reader's convenience, we include the proof. For any set $A \subseteq X_1$, let \mathcal{U}_A be the set of all $B \subseteq X_2$ such that $A \times B \in \mathcal{B}(X)$. If A is open, then $A \times X_2 \in \mathcal{B}(X)$. Now $B \mapsto A \times B$ preserves set operations, specifically: for any $B \subseteq X_2$,

$$A \times (X_2 \setminus B) = (A \times X_2) \setminus (A \times B),$$

and for any $B_n \subseteq X_2$,

$$\bigcup_n (A \times B_n) = A \times \bigcup_n B_n.$$

It follows that \mathcal{U}_A is a σ -algebra of subsets of X_2 . It includes \mathcal{T}_2 and hence $\mathcal{B}(X_2)$. Then, for $B \in \mathcal{B}(X_2)$, let $\mathcal{T}_1(B)$ be the set of all $A \subseteq X_1$ such that $A \times B \in \mathcal{B}(X)$. Then $X_1 \in \mathcal{T}_1(B)$, and $\mathcal{T}_1(B)$ is a σ -algebra. It includes \mathcal{T}_1 , and

hence $\mathcal{B}(X_1)$. Thus, the product σ -algebra of the Borel σ -algebras is included in the Borel σ -algebra $\mathcal{B}(X)$ of the product. Conversely, suppose that (X_1, \mathcal{T}_1) and (X_2, \mathcal{T}_2) are second countable. The product topology has a base \mathcal{W} consisting of all sets $A \times B$ where A belongs to a countable base of \mathcal{T}_1 and B to a countable base of \mathcal{T}_2 . Then the σ -algebra generated by \mathcal{W} is the Borel σ -algebra of the product topology. It is clearly included in the product σ -algebra. This completes the proof in the case $n = 2$. To deal with the case of general n , we proceed by induction. Suppose that the inclusion has been verified for $(n - 1)$ factors. Then,

$$\begin{aligned}
\mathcal{B}(X_1 \times \dots \times X_n) &= \mathcal{B}((X_1 \times \dots \times X_{n-1}) \times X_n) \\
&\supseteq \mathcal{B}(X_1 \times \dots \times X_{n-1}) \otimes \mathcal{B}(X_n) \\
&\supseteq (\mathcal{B}(X_1) \otimes \dots \otimes \mathcal{B}(X_{n-1})) \otimes \mathcal{B}(X_n) \\
&= \mathcal{B}(X_1) \otimes \dots \otimes \mathcal{B}(X_n),
\end{aligned}$$

because the formation of products of σ -algebras is associative ([Dunford and Schwartz, 1958](#), p. 195) as well as monotonic with respect to set-inclusion. Hence, the inclusion holds also for n factors. Suppose, finally, that the inclusion is an equality for up to $(n - 1)$ second countable factors. Recalling that a product of two second countable spaces is second countable ([Kelley, 1975](#), Ch. 3, Problem M), the inclusion is an equality also for n factors. This completes the proof. \square

We turn to the proof of the second claim in Proposition [2](#). As u_i is jointly continuous, it is measurable with respect to $\mathcal{B}(S_1 \times \dots \times S_n)$. By Lemma [B.4](#) above, u_i is measurable with respect to $\mathcal{B}(S_1) \otimes \dots \otimes \mathcal{B}(S_n)$. As u_i is bounded, it therefore satisfies the assumptions of Fubini's Theorem ([Dunford and Schwartz, 1958](#), Thm. III.11.9). The claim follows.

References

- Aliprantis, C. D., Glycopantis, D., and Puzzello, D. (2006). The joint continuity of the expected payoff functions. *Journal of Mathematical Economics*, 42(2):121–130. [5](#)
- Becker, J. G. and Damianov, D. S. (2006). On the existence of symmetric mixed strategy equilibria. *Economics Letters*, 90(1):84–87. [5](#)
- Bogachev, V. I. (2007). *Measure Theory, Volume 2*. Springer, Berlin, Heidelberg. [21](#)
- Dasgupta, P. and Maskin, E. (1986). The existence of equilibrium in discontinuous economic games, I: Theory. *The Review of Economic Studies*, 53(1):1–26. [2](#)
- Dudley, R. M. (2018). *Real Analysis and Probability*. Chapman and Hall/CRC, New York. [23](#)
- Dunford, N. and Schwartz, J. T. (1958). *Linear Operators, Part I: General Theory*. John Wiley & Sons, Hoboken, NJ. [7](#), [17](#), [24](#)
- Folland, G. B. (1999). *Real Analysis: Modern Techniques and their Applications*, volume 40. John Wiley & Sons, Hoboken, NJ. [11](#), [23](#)
- Glicksberg, I. L. (1952). A further generalization of the Kakutani fixed point theorem, with application to Nash equilibrium points. *Proceedings of the American Mathematical Society*, 3(1):170–174. [1](#), [2](#), [5](#), [7](#), [9](#), [10](#), [11](#), [13](#), [14](#)
- Glicksberg, I. L. (1962). Weak compactness and separate continuity. *Pacific Journal of Mathematics*, 12:205–214. [5](#), [10](#)

- Glycopantis, D. and Muir, A. (2000). Continuity of the payoff functions. *Economic Theory*, 16:239–244. [2](#), [4](#), [5](#)
- Glycopantis, D. and Muir, A. (2004). The compactness of $Pr(K)$. *Advances in Mathematical Economics*, 6:39–53. [13](#)
- Grothendieck, A. (1973). *Topological vector spaces*. Gordon and Breach, New York. [18](#)
- Kelley, J. L. (1975). *General Topology*. Courier Dover, Mineola, NY. [16](#), [24](#)
- Kim, W. K. (2014). Existence of a mixed equilibrium for a compact generalized strategic game. *Journal of Mathematical Analysis and Applications*, 420(2):942–953. [5](#)
- Kozhan, R. and Zarichnyi, M. (2008). Nash equilibria for games in capacities. *Economic Theory*, 35:321–331. [5](#)
- Lang, S. (1993). *Real and Functional Analysis*, volume 142 of *Graduate Texts in Mathematics*. Springer, 3rd edition. [22](#)
- Mertens, J.-F. (1986). The minmax theorem for u.s.c.-l.s.c. payoff functions. *International Journal of Game Theory*, 15(4):237. [4](#)
- Monteiro, P. K. and Page, F. H. J. (2007). Uniform payoff security and Nash equilibrium in compact games. *Journal of Economic Theory*, 134(1):566–575. [15](#)
- Munkres, J. R. (2014). *Topology*. Prentice Hall, Upper Saddle River, NJ, 2nd edition. [8](#)
- Nash, J. F. (1950). Equilibrium points in n -person games. *Proceedings of the National Academy of Sciences*, 36(1):48–49. [2](#)

- Prokopovych, P. and Yannelis, N. C. (2014). On the existence of mixed strategy Nash equilibria. *Journal of Mathematical Economics*, 52:87–97. [15](#)
- Reny, P. J. (1999). On the existence of pure and mixed strategy Nash equilibria in discontinuous games. *Econometrica*, 67(5):1029–1056. [13](#), [15](#)
- Royden, H. and Fitzpatrick, P. M. (2010). *Real Analysis*. China Machine Press, Beijing. [17](#)
- von Neumann, J. (1928). Zur Theorie der Gesellschaftsspiele. *Mathematische Annalen*, 100(1):295–320. (in German). [2](#)
- Zarichnyi, M. (2004). Continuity of the payoff function revisited. *Economics Bulletin*, 3(14):1–4. [5](#)