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Robust asymptotic insurance-finance arbitrage

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Abstract

This paper studies the valuation of insurance contracts linked to financial markets, for example through interest rates or in equity-linked insurance products. We build upon the concept of insurance-finance arbitrage as introduced by Artzner et al. (Math Financ, 2024), extending their work by incorporating model uncertainty. This is achieved by introducing statistical uncertainty in the underlying dynamics to be represented by a set of priors \mathcal{P} . Within this framework we propose the notion of *robust asymptotic insurance-finance arbitrage* (RIFA) and characterize the absence of such strategies in terms of the new concept of $Q\mathcal{P}$ -evaluations. This nonlinear two-step evaluation ensures absence of RIFA. Moreover, it dominates all two-step evaluations, as long as we agree on the set of priors \mathcal{P} . Our analysis highlights the role of $Q\mathcal{P}$ -evaluations in terms of showing that all two-step evaluations are free of RIFA. Furthermore, we introduce a doubly stochastic model to address uncertainty for surrender and survival, utilizing copulas to define conditional dependence. This setting illustrates how the $Q\mathcal{P}$ -evaluation can be applied for the pricing of hybrid insurance products, highlighting the flexibility and potential of the proposed approach.

Keywords Insurance-finance arbitrage under uncertainty · Robust QP -rule · Enlargement of filtration · Absence of robust insurance-finance arbitrage

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1 Introduction

This paper develops and characterizes the absence of insurance-finance arbitrage under model uncertainty. Our starting point is the observation that most insurance contracts are linked to financial markets, for example through interest rates or via direct links of the contractual benefits to stocks or indices. However, the modeling and the valuation of insurance contracts and products on financial markets are fundamentally different due to their distinct characteristics: insurance contracts are static and personalized products, whereas products on financial markets are standardized and traded frequently. Various approaches have been proposed in the literature as to how insurance and financial markets can be treated in a coherent manner, see e.g., [10, 11, 21, 23] and references therein. More recently, 2-step and 3-step approaches have been proposed as for example in [2, 7] and in [19].

In this paper we aim for a fundamental analysis of arbitrage in these markets and rely on the notion of insurance-finance arbitrage (IFA) introduced by [1]. In this approach the insurance company may issue contracts to a large number of clients on the one hand. On the other hand, it can simultaneously hedge its positions by trading on the financial market. In order to model the two information flows to which the insurance company has access, we work with two filtrations. The smaller filtration represents the publicly available information on the financial market, denoted by $\mathbb{F} = (\mathcal{F}_t)_{t \leq T}$, while the larger filtration $\mathbb{G} = (\mathcal{G}_t)_{t \leq T}$ additionally contains the insurer's information. Given a pricing measure Q on the financial market (Ω, \mathcal{F}) and a statistical measure P on (Ω, \mathcal{G}) a characterization of the absence of IFA in terms of the QP -rule is derived in [1].

Even if a large set of homogeneous data is available, statistical uncertainties in predicting the future evolution of insurance losses in the considered portfolio remain a problem that needs to be addressed. In this paper we therefore take this uncertainty into account.

To do so, we fix the nullsets \mathcal{N} on the financial market (Ω, \mathcal{F}) which together with the traded assets S determine the set of equivalent martingale measures \mathcal{Q} . Second, we consider a class of probabilistic models \mathcal{P} on (Ω, \mathcal{G}) , such that the measures in \mathcal{P} restricted to \mathcal{F} have exactly the nullsets \mathcal{N} and study the associated $Q\mathcal{P}$ -rule. Most notably, this framework allows us to model uncertainty on the insurance market under the assumption that we do not face any model risk on the financial market. Working with a class of potential models \mathcal{P} is in line with the growing literature on model risk and uncertainty, see e.g., [5, 6, 8, 9, 24, 28].

To the best of our knowledge, this is the first study of insurance-finance arbitrage under model uncertainty. More specifically, we prove a characterization of robust insurance-finance arbitrage (RIFA) by using the $Q\mathcal{P}$ -evaluation in Theorem 2.12.

Furthermore, we show that this result provides a theoretical foundation for a class of two-step evaluations introduced in [23], which is applied for the pricing of hybrid products depending on the financial market, as well as on other random sources. In particular, we prove that every two-step evaluation, which is the combination of a risk-free measure Q and a coherent \mathcal{F} -conditional risk measure, being continuous from below, equals a $Q\mathcal{P}$ -evaluation for a suitable subset \mathcal{P} on (Ω, \mathcal{G}) . Thus, every

two-step evaluation of this kind leads to a robust arbitrage-free price in the asymptotic insurance-finance setting.

We conclude the paper by suggesting possible applications in Sect. 4. First, we consider the case of two conditional independent random times such that the conditional distribution functions face a certain degree of uncertainty. Here, the random times represent the surrender time and the time of death of an insurance seeker. In this setting, we introduce a financial market via a Cox-Ross-Rubinstein model and consider finance-linked insurance benefits with surrender options. We compute the robust arbitrage-free price of these products numerically and find that the uncertainty in this example justifies that the robust arbitrage-free price is higher than the supremum of all arbitrage-free prices under each possible model. This underlines the importance of treating uncertainty in a systematic manner. Furthermore, we show that the well-known Cox model under uncertainty is contained in the outlined setting. Moreover, we generalize the setting by allowing dependence between the random times described by a copula.

The paper is structured as follows. In Sect. 2 we introduce the definition of a robust insurance-finance arbitrage and provide a characterization in our main result. After that, Sect. 3 studies the relation of the $Q\mathcal{P}$ -rule to two-step evaluations. Then, in Sect. 4 we study an insurance-finance market with two conditionally independent random times and numerically provide the robust arbitrage-free prices for certain hybrid products. In addition, we consider a copula framework for the two random times under uncertainty.

2 Robust asymptotic insurance-finance arbitrage

Let (Ω, \mathcal{G}) be a measurable space and denote by $\mathfrak{P}(\Omega, \mathcal{G})$ the set of all probability measures on this space. We consider a discrete time model with times $t = 0, \dots, T$ and introduce two different kinds of information flows described by the filtrations \mathbb{F} and \mathbb{G} on (Ω, \mathcal{G}) . The filtration $\mathbb{F} = (\mathcal{F}_t)_{t \leq T}$ represents *publicly available information* and contains all information available on the financial market. The filtration $\mathbb{G} = (\mathcal{G}_t)_{t \leq T}$ contains additional private information of the considered insurance company, which includes, for example, several datasets on its clients. In particular, $\mathbb{F} \subseteq \mathbb{G}$. Moreover, let $\mathcal{F}_0 = \mathcal{G}_0 = \{\emptyset, \Omega\}$ and \mathcal{F} be a σ -field such that $\mathcal{F}_T \subseteq \mathcal{F} \subseteq \mathcal{G}$.

Given $\mathcal{P} \subseteq \mathfrak{P}(\Omega, \mathcal{G})$, a set $A \subseteq \Omega$ is called \mathcal{P} -polar if $A \subseteq N$ for some $N \in \mathcal{G}$ satisfies $P(N) = 0$ for all $P \in \mathcal{P}$. Moreover, a property holds \mathcal{P} -quasi surely (\mathcal{P} -q.s.) if it holds outside a \mathcal{P} -polar set.

For a fixed σ -ideal of \mathcal{F} -nullsets \mathcal{N} , i.e., there exists a measure P_0 on (Ω, \mathcal{F}) such that \mathcal{N} are the nullsets of P_0 , we define the following sets of priors

$$\mathfrak{P}_{\mathcal{N}}(\Omega, \mathcal{F}) := \{P \in \mathfrak{P}(\Omega, \mathcal{F}) \mid \mathcal{N} \text{ are the nullsets of } P\} \quad (2.1)$$

and

$$\mathfrak{P}_{\mathcal{N}}(\Omega, \mathcal{G}) := \{P \in \mathfrak{P}(\Omega, \mathcal{G}) \mid \mathcal{N} \text{ are the nullsets of } P|_{\mathcal{F}}\}. \quad (2.2)$$

Hereafter, we fix a probability measure $P_0 \in \mathfrak{P}(\Omega, \mathcal{F})$, which determines the nullset \mathcal{N} on (Ω, \mathcal{F}) , and a subset of priors $\mathcal{P} \subseteq \mathfrak{P}_{\mathcal{N}}(\Omega, \mathcal{G})$, which specifies the uncertainty about the model.

In the following, we introduce the concept of insurance-finance arbitrage. For simplicity, we consider only a single insurer. We also assume that the insurer can contract with an arbitrarily high number of clients to reduce its risks, as specified in Assumption 2.1. To establish this, we consider a finite number of insurance seekers and study the limits of portfolio allocations - a technique inspired by large financial markets, see e.g., [15, 17, 18]. Such strategies may lead to *insurance arbitrages* and it is partly our aim to characterize when and how such arbitrages can be achieved and under which conditions they can be avoided.

At the same time, the insurance company also trades on the financial market, potentially leading to a *financial arbitrage*. Thus, the combination of these concepts results in an *insurance-finance arbitrage*.

2.1 The insurance contracts

Insurance contracts offer a variety of benefits at future times in exchange for a single premium or a premium stream. We work with discounted quantities and, without loss of generality, we consider a single premium paid at time 0 and an aggregated benefit received at future time T . More precisely, we denote by $p \in \mathbb{R}$ the premium to be paid at time 0 and a \mathcal{G} -measurable (discounted) benefit X^i to be received by the i th client at time T . This allows to cover a wide range of contracts, particularly contracts depending on financial markets, such as variable annuities.

We assume that all insurance seekers under consideration can be treated as homogeneous (under each model $P \in \mathcal{P}$) and each insurance seeker pays the same premium p in order to receive his or her personal benefit X^i . This idea is formalized in the following assumption, which is a generalization of the framework of actuarial mathematics to stochastic assets, as discussed in [1].

Assumption 2.1 For all $P \in \mathcal{P}$, the following holds:

- (i) $X^1, X^2, \dots \in L_+^2(\Omega, \mathcal{G}, P)$ are \mathcal{F} -conditionally independent.
- (ii) $E_P[X^i | \mathcal{F}] = E_P[X^1 | \mathcal{F}]$ for all $i \in \mathbb{N}$.
- (iii) $\text{Var}_P[X^i | \mathcal{F}] = \text{Var}_P[X^1 | \mathcal{F}]$ for all $i \in \mathbb{N}$.

Remark 2.2 (The implications of Assumption 2.1) Assumption 2.1 is a fairly weak assumption and includes a wide range of existing models:

- (i) Approaches where insurance benefits are i.i.d. and independent of the publicly available information.
- (ii) Any kind of variable annuities including \mathcal{F} -independent and i.i.d. survival (or surrender) times τ^i . This covers for example survival benefits of the form $X^i = \mathbb{1}_{\{\tau^i > T\}} e^{-rT} F(\tilde{S}_T)$ or payments at surrender times like $X^i = \mathbb{1}_{\{\tau^i = T\}} e^{-r\tau^i} F(\tilde{S}_{\tau^i})$, where \tilde{S} is the \mathbb{F} -adapted (undiscounted) stock price process and e^{-rt} is the deterministic discounting factor at time t . Similarly, stochastic interest rates or path-dependent payoffs can be included.

- (iii) Doubly-stochastic random times are also part of the framework, i.e., random times which are driven by a hazard rate λ which is \mathbb{F} -progressively measurable such that the process $\mathbb{1}_{\{\tau^i \leq t\}} - \int_{[0, t \wedge \tau^i]} \lambda_s ds$ is an \mathbb{F} -martingale. This allows to model important aspects like systemic risk or longevity risk, for example by incorporating factors which affect λ and hence all insurance seekers. It seems important to point out, that \mathbb{F} contains all publicly available information, and hence might be significantly larger than the filtration created by stock prices only. In particular, publicly available mortality tables would be included in \mathbb{F} . Intuitively, λ allows to cover risks which are observable by the public, such as a general crisis giving rise to systemic risk or increasing longevity reflected in publicly available life tables. The individual risk, referring to the risk beyond systematic risk that aligns with the individual characteristics of the policyholder, is subsequently captured by the model's additional stochastic component. We refer to Remark 4.3 which shows how to include increasing life expectations in the Gompertz model under Assumption 2.1.

However, our current framework does not include different cohorts. To incorporate for example different ages of the insurance seekers, a separate model for each age must be considered.

The insurance portfolio is obtained as a limit of *allocations* of contracts with a finite number of clients. An allocation at time 0, $\psi = (\psi^i)_{i \in \mathbb{N}}$, is c_{00} -valued, deterministic and non-negative, where c_{00} denotes the space of sequences with a finite number of non-zero elements. For $i \in \mathbb{N}$, $\psi^i \in \mathbb{R}_+$ denotes the size of the contract with the i th policyholder. The accumulated benefits and premiums associated with the allocation ψ are given by $\sum_{i \in \mathbb{N}} \psi^i X^i$ and $\sum_{i \in \mathbb{N}} \psi^i p$, while the associated profits and losses are denoted by

$$V_T(\psi) := \sum_{i \in \mathbb{N}} \psi^i (p - X^i).$$

An *insurance portfolio strategy* $\Psi := (\psi^n)_{n \in \mathbb{N}} \in c_{00}^{\mathbb{N}}$ is modeled as a sequence of allocations $\psi^n = (\psi^{n,i})_{i \in \mathbb{N}}$. Moreover, the profit and loss of an insurance portfolio strategy Ψ is given (if it exists) by

$$V_T(\Psi) := \lim_{n \rightarrow \infty} V_T(\psi^n) = \lim_{n \rightarrow \infty} \sum_{i \in \mathbb{N}} \psi^{n,i} (p - X^i).$$

We introduce the following *admissibility conditions* for an insurance portfolio strategy $\Psi = (\psi^n)_{n \in \mathbb{N}}$.

Assumption 2.3 (i) *Uniform boundedness*: There exists $C > 0$, such that

$$\|\psi^n\| := \sum_{i \in \mathbb{N}} \psi^{n,i} \leq C \text{ for all } n \geq 1.$$

(ii) *Convergence of the total mass:* There exists $\gamma \geq 0$ such that

$$\gamma = \lim_{n \rightarrow \infty} \|\psi^n\|.$$

(iii) *Convergence of the total wealth:* There exists a \mathbb{R} -valued random variable $V_T(\Psi)$, such that

$$V_T(\Psi) = \lim_{n \rightarrow \infty} V_T(\psi^n) \quad \mathcal{P}\text{-q.s.}$$

An insurance portfolio strategy $\Psi = (\psi^n)_{n \in \mathbb{N}}$ which satisfies Assumption 2.3 is called *\mathcal{P} -admissible*.

2.2 The financial market

We introduce a financial market model in discrete time consisting of d risky assets $S = (S_t^1, \dots, S_t^d)_{t=0, \dots, T}$ on (Ω, \mathcal{F}) . For $i = 1, \dots, d$, the discounted price of the i th risky asset at time t is given by the \mathcal{F}_t -measurable random variable S_t^i and the bank account is given by $S^0 \equiv 1$. We assume that the insurance company trades with \mathbb{F} -trading strategies on the financial market, where a \mathbb{F} -trading strategy is a d -dimensional \mathbb{F} -predictable process $\xi = (\xi_t)_{t=1, \dots, T}$ with $\xi_t = (\xi_t^1, \dots, \xi_t^d)$. Note that each strategy ξ can be extended by $\xi^0 = (\xi_t^0)_{t=1, \dots, T}$ to a self-financing trading strategy $\bar{\xi} = (\xi_t^0, \dots, \xi_t^d)_{t=1, \dots, T}$, cf. [14, Remark 5.8]. The associated *gain process* at time $t = 1, \dots, T$ is then given by the discrete stochastic integral

$$(\xi \cdot S)_t := \sum_{s=1}^t \xi_s (S_s - S_{s-1}) = \sum_{s=1}^t \sum_{i=1}^d \xi_s^i (S_s^i - S_{s-1}^i).$$

The absence of arbitrage in this market with respect to one and thus any restricted measure $P|_{\mathcal{F}}$ for $P \in \mathcal{P} \subseteq \mathfrak{P}_{\mathcal{N}}(\Omega, \mathcal{G})$ can be characterized by the existence of an equivalent martingale measure, cf. [14, Theorem 5.16]. The set of all equivalent martingale measures is denoted by $\mathcal{M}_e(\mathbb{F})$ and defined by

$$\mathcal{M}_e(\mathbb{F}) := \{Q \in \mathfrak{P}_{\mathcal{N}}(\Omega, \mathcal{F}) \mid S \text{ is a } (Q, \mathbb{F})\text{-martingale}\}. \quad (2.3)$$

Remark 2.4 By taking into account a set of priors $\mathcal{P} \subseteq \mathfrak{P}_{\mathcal{N}}(\Omega, \mathcal{G})$ and according to the definition of $\mathfrak{P}_{\mathcal{N}}(\Omega, \mathcal{G})$ in (2.2), all nullsets on (Ω, \mathcal{F}) are already determined by \mathcal{N} . Thus, the uncertainty of our model refers only to the additional insurance part defined on (Ω, \mathcal{G}) .

2.3 Robust insurance-finance arbitrage

Finally, we introduce the *insurance-finance market* (S, \mathcal{X}, p) on (Ω, \mathcal{G}) consisting of the benefits $\mathcal{X} = (X^i)_{i \in \mathbb{N}}$, the premium p , and the discounted asset prices $S = (S_t)_{t=0, \dots, T}$.

In order to define a robust arbitrage in our setting, we use the concept of a robust arbitrage strategy introduced in [6]. Note, however, that we do not need to require convexity of \mathcal{P} for the characterization of the absence of a robust arbitrage strategy.

Definition 2.5 A \mathcal{P} -robust asymptotic insurance-finance arbitrage RIFA(\mathcal{P}) on the insurance-finance market (S, \mathcal{X}, p) is a pair (ξ, Ψ) consisting of an \mathbb{F} -predictable trading strategy ξ and a \mathcal{P} -admissible insurance portfolio strategy Ψ such that

$$(\xi \cdot S)_T + V_T(\Psi) \geq 0 \quad \mathcal{P}\text{-q.s.} \quad \text{and} \quad E_P[(\xi \cdot S)_T + V_T(\Psi)] > 0 \text{ for some } P \in \mathcal{P}. \quad (2.4)$$

If there exists no such pair (ξ, Ψ) satisfying (2.4), then there is no \mathcal{P} -robust asymptotic insurance-finance arbitrage, which we denote by NRIFA(\mathcal{P}).

Remark 2.6 If for all $P \in \mathcal{P}$ it holds NRIFA($\{P\}$) then NRIFA(\mathcal{P}) is also satisfied. However, the converse statement does not hold in general.

In the case of no model uncertainty, i.e., when $\mathcal{P} = \{P\}$ for a measure $P \in \mathfrak{P}_{\mathcal{N}}(\Omega, \mathcal{G})$, it is shown in Corollary 5.2 in [1] that if there exists $Q \in \mathcal{M}_e(\mathbb{F})$ such that

$$p \leq E_{Q \odot P}[X^1] := E_Q[E_P[X^1 | \mathcal{F}]], \quad (2.5)$$

then there is no asymptotic insurance-finance arbitrage. Thus, according to Remark 2.6, an insurance-finance market (S, \mathcal{X}, p) fulfills NRIFA(\mathcal{P}) if (2.5) holds for each $P \in \mathcal{P}$. However, it should be noted that these conditions are not necessary.

2.4 Uniform essential supremum

In order to characterize the absence of a \mathcal{P} -robust asymptotic insurance-finance arbitrage, we aim to identify assumptions that allow us to take into account a robust version of the conditional expectations $E_P[X | \mathcal{F}]$ for $P \in \mathcal{P}$ in (2.5). Given a set of priors $\mathcal{P} \subseteq \mathfrak{P}(\Omega, \mathcal{G})$, it is in general not possible to consider $\sup_{P \in \mathcal{P}} E_P[X | \mathcal{F}]$, as the conditional expectation is only defined P -a.s. and the priors in \mathcal{P} may have different nullsets. Furthermore, the supremum no longer needs to be measurable.

The natural approach to solve the measurability issue is to work with the essential supremum instead of the supremum. For a fixed probability measure $P \in \mathcal{P}(\Omega, \mathcal{G})$ and a set of random variables Φ on (Ω, \mathcal{G}) there is a random variable $Y =: \text{ess sup}^P \Phi$ such that

- (i) $Y \geq \varphi$ P -a.s. for all $\varphi \in \Phi$, and
- (ii) $Y \leq \psi$ P -a.s. for every random variable ψ satisfying $\psi \geq \varphi$ P -a.s. for all $\varphi \in \Phi$.

We refer to [14, Theorem A.37] for a proof of the existence of the essential supremum and to [3], in whose work a general construction of such a nonlinear conditional expectation is studied.

However, for a general set of priors $\mathcal{P} \subseteq \mathfrak{P}(\Omega, \mathcal{G})$ and a set of random variables Φ on (Ω, \mathcal{G}) there may be no uniform essential supremum, i.e., a random variable Y such that

$$Y = \operatorname{ess\,sup}^P \Phi \quad P\text{-a.s. for all } P \in \mathcal{P}. \quad (2.6)$$

The definition of the set $\mathfrak{P}_{\mathcal{N}}(\Omega, \mathcal{G})$ in (2.2) allows us to consider for $\mathcal{P} \subseteq \mathfrak{P}_{\mathcal{N}}(\Omega, \mathcal{G})$ the set of random variables $\Phi = (\varphi^P)_{P \in \mathcal{P}}$ such that

$$\varphi^P = E_P[X|\mathcal{F}] \quad P\text{-a.s. for all } P \in \mathcal{P}. \quad (2.7)$$

Indeed, we observe that by \mathcal{F} -measurability together with the fact that $P|_{\mathcal{F}} \sim P'|_{\mathcal{F}}$ for all $P, P' \in \mathcal{P}$, the conditional expectation $E_P[X|\mathcal{F}]$ is not only P -a.s. uniquely determined but also \mathcal{P} -q.s.. Moreover, there exists an uniform essential supremum which fulfills (2.6), as the following result demonstrates.

Lemma 2.7 *Let \mathcal{N} be the nullsets generated by the probability measure $P_0 \in \mathfrak{P}(\Omega, \mathcal{F})$, $\mathcal{P} \subseteq \mathfrak{P}_{\mathcal{N}}(\Omega, \mathcal{G})$ and Φ a set of \mathcal{F} -measurable random variables. Then*

$$\operatorname{ess\,sup}^{P_0} \Phi = \operatorname{ess\,sup}^P \Phi \quad P\text{-a.s. for all } P \in \mathcal{P}. \quad (2.8)$$

Proof We show that $\operatorname{ess\,sup}^{P_0} \Phi$ fulfills condition (i) and (ii) from the definition of the essential supremum for all $P \in \mathcal{P}$. For the first part, using the definition of $\operatorname{ess\,sup}^{P_0} \Phi$, we obtain that

$$\operatorname{ess\,sup}^{P_0} \Phi \geq \varphi \quad P_0\text{-a.s. for all } \varphi \in \Phi. \quad (2.9)$$

Since $\{\operatorname{ess\,sup}^{P_0} \Phi \geq \varphi\}$ is \mathcal{F} -measurable and $P|_{\mathcal{F}} \sim P_0$ for all $P \in \mathcal{P}$, (2.9) also holds P -a.s. for all $P \in \mathcal{P}$. Relying on the construction of the essential supremum, see, for example [14, Theorem A.37], there exists a countable subset $\Phi^* \subseteq \Phi$ such that

$$\operatorname{ess\,sup}^{P_0} \Phi(\omega) = \sup \Phi^*(\omega) \quad \text{for all } \omega \in \Omega.$$

Fix any $P \in \mathcal{P}$, then for each random variable ψ on (Ω, \mathcal{G}) such that

$$\psi \geq \varphi \quad P\text{-a.s. for all } \varphi \in \Phi,$$

we get $\psi \geq \sup \Phi^* = \operatorname{ess\,sup}^{P_0} \Phi$ P -a.s.. This shows the second part and the result is proven. \square

In the following, for the fixed measure P_0 , which generates the \mathcal{F} -nullsets \mathcal{N} and use the notation

$$\operatorname{ess\,sup} \Phi := \operatorname{ess\,sup}^{P_0} \Phi.$$

2.5 Characterization of robust insurance-finance arbitrage

In the next result we use the essential supremum to characterize NRIFA(\mathcal{P}). Furthermore, to prove this theorem, we use a measure extension $Q \odot P$ which was proposed in [25] and [12] (see also Proposition 4.1 in [1]). For the reader's convenience we state the existence result for such a measure.

Proposition 2.8 *Let (Ω, \mathcal{G}, P) be a probability space, $\mathcal{F} \subseteq \mathcal{G}$ a sub- σ -algebra, and $Q \sim P|_{\mathcal{F}}$ a probability measure on (Ω, \mathcal{F}) . Then, there exists a unique probability measure, denoted by $Q \odot P$, on (Ω, \mathcal{G}) such that $Q \odot P = Q$ on \mathcal{F} and $Q \odot P = P$ conditioned on \mathcal{F} .*

Remark 2.9 The measure $Q \odot P$ on (Ω, \mathcal{G}) from Proposition 2.8 can also be characterized by each of the following more explicit expressions:

- (i) For all $G \in \mathcal{G}$ we have $Q \odot P(G) = E_Q[E_P[\mathbb{1}_G | \mathcal{F}]]$.
- (ii) The density of $Q \odot P$ with respect to P is given by $dQ/dP|_{\mathcal{F}}$.

This also shows that the measures $Q \odot P$ and P are equivalent.

The arbitrage-free prices of a \mathcal{F}_T -measurable claim $H \geq 0$ in the financial market S can be described by the set of expectations under all $Q \in \mathcal{M}_e(\mathbb{F})$ such that $H \in L^1(Q)$, cf. [14, Theorem 5.29].

The subsequent lemma shows that, when a specific $P \in \mathcal{P}$ is chosen, it suffices to consider only those measures $Q \in \mathcal{M}_e(\mathbb{F})$ for which $H \in L^1(Q)$ and the density of Q with respect to P is bounded. The expectations of this subset of measures determines the set of arbitrage-free prices for H .

Lemma 2.10 *Let $H \geq 0$ be a discounted \mathcal{F}_T -measurable claim such that $H \in L^1(\mathcal{F}_T, Q)$ for some $Q \in \mathcal{M}_e(\mathbb{F})$ and let $P \in \mathfrak{P}_{\mathcal{N}}(\Omega, \mathcal{G})$. Then, there exists $Q^* \in \mathcal{M}_e(\mathbb{F})$ such that $H \in L^1(\mathcal{F}_T, Q^*)$, $E_Q[H] = E_{Q^*}[H]$ and $dQ^*/dP|_{\mathcal{F}} \in L^\infty(\mathcal{F}, P|_{\mathcal{F}})$.*

Proof We define the process $S^{d+1} = (S_t^{d+1})_{t=0, \dots, T}$ by $S_t^{d+1} = E_Q[H | \mathcal{F}_t]$. Then Q is a martingale measure for the extended market (S, S^{d+1}) . According to [14, Theorem 5.16], it follows that this extended market is free of arbitrage and that there is an equivalent measure $Q^* \sim P|_{\mathcal{F}}$ such that (S, S^{d+1}) is a (Q^*, \mathbb{F}) -martingale and $dQ^*/dP|_{\mathcal{F}} \in L^\infty(\mathcal{F}, P|_{\mathcal{F}})$. The martingale property for S implies $Q^* \in \mathcal{M}_e(\mathbb{F})$ and for S^{d+1} it implies $H = S_T^{d+1} \in L^1(Q^*)$ and $E_Q[H] = S_0^{d+1} = E_{Q^*}[H]$. \square

Lemma 2.11 *Let $A_1, \dots, A_N \in \mathcal{F}$ be a partition of Ω and $P_1, \dots, P_N \in \mathfrak{P}_{\mathcal{N}}(\Omega, \mathcal{G})$. Then the measure P on (Ω, \mathcal{G}) defined by*

$$P(G) := c^{-1} \sum_{i=1}^N P_i(G \cap A_i), \quad (2.10)$$

for $c = \sum_{i=1}^N P_i(A_i)$ is well-defined, a probability measure and $P \in \mathfrak{P}_{\mathcal{N}}(\Omega, \mathcal{G})$.

Proof We must show that $c > 0$. For the sake of contradiction we assume that $P_i(A_i) = 0$ for all $i = 1, \dots, N$. By the equivalence $P_i \in \mathfrak{P}_{\mathcal{N}}(\Omega, \mathcal{G})$ and $A_i \in \mathcal{F}$, it follows that $P_1(A_i) = 0$ for all $i = 1, \dots, N$ and thus $P_1(\Omega) = 0$. This is a contradiction and we obtain $c > 0$. Thus, P is well-defined and a probability measure on (Ω, \mathcal{G}) . In order to prove $P \in \mathfrak{P}_{\mathcal{N}}(\Omega, \mathcal{G})$ we show that for all $F \in \mathcal{F}$ it holds that $P_1(F) = 0$ if and only if $P(F) = 0$. First, we assume that $P_1(F) = 0$. Then, once more using the equivalence $P_i \in \mathfrak{P}_{\mathcal{N}}(\Omega, \mathcal{G})$ we obtain $P_i(F \cap A_i) \leq P_i(F) = 0$ for all $i = 1, \dots, N$ and it follows that $P(F) = 0$. On the other hand, if $P(F) = 0$, then it follows that $P_i(A_i \cap F) = 0$ and thus by the equivalence $P_1(A_i \cap F) = 0$ for all $i = 1, \dots, n$. This implies $P_1(F) = 0$. \square

The following result is the main result of this paper. It characterizes the existence of insurance-finance arbitrage, and thus establishes a fundamental no-arbitrage result in insurance-finance models.

Theorem 2.12 *There is no \mathcal{P} -robust asymptotic insurance-finance arbitrage on (S, \mathcal{X}, p) if and only if one of the following statements is fulfilled.*

(i) *For all $P \in \mathcal{P}$ there is a measure $Q \in \mathcal{M}_e(\mathbb{F})$ such that*

$$p \leq E_Q \left[E_P[X^1 | \mathcal{F}] \right].$$

(ii) *There is a measure $Q \in \mathcal{M}_e(\mathbb{F})$ such that*

$$p < E_Q \left[\operatorname{ess\,sup}_{P \in \mathcal{P}} E_P[X^1 | \mathcal{F}] \right].$$

Proof We start with the *only if* direction and prove this using contraposition. Assume that neither (i) nor (ii) are fulfilled, i.e.,

$\overline{(i)}$ There is $P' \in \mathcal{P}$ such that for all $Q \in \mathcal{M}_e(\mathbb{F})$ it holds that

$$p > E_Q \left[E_{P'}[X^1 | \mathcal{F}] \right].$$

and

$\overline{(ii)}$ For all $Q \in \mathcal{M}_e(\mathbb{F})$ we have

$$p \geq E_Q \left[\operatorname{ess\,sup}_{P \in \mathcal{P}} E_P[X^1 | \mathcal{F}] \right].$$

We have to show that there is a \mathcal{P} -robust asymptotic insurance-finance arbitrage on (S, \mathcal{X}, p) . Fix some P' which satisfies $\overline{(i)}$ and define the set A by

$$A := \left\{ E_{P'}[X^1 | \mathcal{F}] < \operatorname{ess\,sup}_{P \in \mathcal{P}} E_P[X^1 | \mathcal{F}] \right\} \in \mathcal{F}.$$

First, we consider the case $P'(A) > 0$. Note that $Y := \operatorname{ess\,sup}_{P \in \mathcal{P}} E_P[X^1 | \mathcal{F}]$ can be interpreted as a contingent claim on (Ω, \mathcal{F}) . By (ii) we know that $p \geq \sup_{Q \in \mathcal{M}_e(\mathbb{F})} E_Q[Y]$. This means p is greater or equal than every arbitrage-free price for Y . Thus, as shown by [14, Theorem 5.29, Corollary 7.9], there exists a \mathbb{F} -predictable (super-hedging) strategy ξ such that

$$p + (\xi \cdot S)_T \geq \operatorname{ess\,sup}_{P \in \mathcal{P}} E_P[X^1 | \mathcal{F}] \quad P_0\text{-a.s.} \quad (2.11)$$

As both sides in (2.11) are \mathcal{F} -measurable and by the definition of \mathcal{P} , Equation (2.11) holds P -a.s. for all $P \in \mathcal{P}$. We define the admissible strategy $\Psi = (\psi^n)_{n \in \mathbb{N}}$ by

$$\psi^n := \frac{1}{n}(1, \dots, 1, 0, \dots) = \frac{1}{n}(\mathbb{1}_{\{k \leq n\}})_{k \in \mathbb{N}}. \quad (2.12)$$

According to Assumption 2.1 and [20, Theorem 3.5], we have that

$$V_T(\Psi) = p - E_P[X^1 | \mathcal{F}] \quad P\text{-a.s. for all } P \in \mathcal{P}.$$

The strategy (ξ, Ψ) is a robust asymptotic insurance-finance arbitrage since by (2.11) and the argumentation below it holds for all $P \in \mathcal{P}$ that

$$\begin{aligned} (\xi \cdot S)_T + V_T(\Psi) &= p + (\xi \cdot S)_T + V_T(\Psi) - p \\ &\geq \operatorname{ess\,sup}_{P \in \mathcal{P}} E_P[X^1 | \mathcal{F}] - E_P[X^1 | \mathcal{F}] \quad P\text{-a.s.} \\ &\geq 0 \quad P\text{-a.s.} \end{aligned}$$

Moreover, since $P'(A) > 0$, we obtain that

$$\operatorname{ess\,sup}_{P \in \mathcal{P}} E_P[X^1 | \mathcal{F}] - E_{P'}[X^1 | \mathcal{F}] \in L_+^0(\mathcal{F}, P') \setminus \{0\}. \quad (2.13)$$

Second, we consider the case $P'(A) = 0$. As $P'|_{\mathcal{F}} \sim Q$ for all $Q \in \mathcal{M}_e(\mathbb{F})$ and $A \in \mathcal{F}$, it also holds $Q(A) = 0$ and consequently $Q(A^c) = 1$, i.e.,

$$E_{P'}[X^1 | \mathcal{F}] = \operatorname{ess\,sup}_{P \in \mathcal{P}} E_P[X^1 | \mathcal{F}] \quad Q\text{-a.s.}$$

Then (i) yields that for all $Q \in \mathcal{M}_e(\mathbb{F})$

$$p > E_Q \left[E_{P'}[X^1 | \mathcal{F}] \right] = E_Q \left[\operatorname{ess\,sup}_{P \in \mathcal{P}} E_P[X^1 | \mathcal{F}] \right]. \quad (2.14)$$

Here, p is strictly greater than any arbitrage-free price for $\operatorname{ess\,sup}_{P \in \mathcal{P}} E_P[X^1 | \mathcal{F}]$ and thus also by [14, Theorem 5.29, Corollary 7.9] we can find a \mathbb{F} -predictable

strategy ξ such that

$$p + (\xi \cdot S)_T - \operatorname{ess\,sup}_{P \in \mathcal{P}} E_P[X^1 | \mathcal{F}] \in L_+^0(\mathcal{F}, P) \setminus \{0\} \text{ for all } P \in \mathcal{P}. \quad (2.15)$$

By once again choosing the insurance strategy Ψ given in (2.12), the pair (ξ, Ψ) is a \mathcal{P} -robust asymptotic insurance-finance arbitrage as it holds for all $P \in \mathcal{P}$ that

$$\begin{aligned} (\xi \cdot S)_T + V_T(\Psi) &= p + (\xi \cdot S)_T + V_T(\Psi) - p \\ &= p + (\xi \cdot S)_T - E_P[X^1 | \mathcal{F}] \quad P\text{-a.s.} \\ &\geq p + (\xi \cdot S)_T - \operatorname{ess\,sup}_{P \in \mathcal{P}} E_P[X^1 | \mathcal{F}] \in L_+^0(\mathcal{F}, P) \setminus \{0\}. \end{aligned}$$

This concludes the proof of the *only if* direction and we proceed with the *if* direction. If (i) holds, then for any $P \in \mathcal{P}$ we could assume that the martingale measure Q has bounded density with respect to $P|_{\mathcal{F}}$, cf. Lemma 2.10. According to Corollary 5.2 in [1], there exists no $\{P\}$ -robust asymptotic insurance-finance arbitrage for each fixed measure $P \in \mathcal{P}$ and thus NRIFA(\mathcal{P}) holds. For the reader's convenience we provide a detailed proof of this statement. Fix $P \in \mathcal{P}$. For the sake of contradiction assume that there is an admissible strategy (ξ, Ψ) such that

$$(\xi \cdot S)_T + V_T(\Psi) \in L_+^0(\mathcal{G}, P) \setminus \{0\}.$$

We define the process $Z = (Z_t)_{t=0, \dots, T}$ by

$$Z_0 = 0 \quad \text{and} \quad Z_t = E_{Q \odot P}[V_T(\Psi) | \mathcal{F}_t], \quad \text{for all } t = 1, \dots, T. \quad (2.16)$$

Using [1, Proposition B.1] and assumption (i) we obtain the following:

$$\begin{aligned} E_{Q \odot P}[Z_t] &= E_{Q \odot P}[V_T(\Psi)] \\ &= E_{Q \odot P}[\gamma(p - X^1)] \\ &= \gamma(p - E_{Q \odot P}[X^1]) \leq 0. \end{aligned}$$

This shows that Z is a local $(\mathbb{F}, Q \odot P)$ -supermartingale. By [14, Remark 9.5] the value process $(\xi \cdot S)$ is a local $(\mathbb{F}, Q \odot P)$ -martingale and thus $(\xi \cdot S) + Z$ is a local $(\mathbb{F}, Q \odot P)$ -supermartingale and fulfills

$$(\xi \cdot S)_T + Z_T = (\xi \cdot S)_T + V_T(\Psi) \geq 0.$$

By [14, Proposition 9.6] the process $(\xi \cdot S) + Z$ is a $(\mathbb{F}, Q \odot P)$ -supermartingale and thus

$$E_{Q \odot P}[(\xi \cdot S)_T + V_T(\Psi)] \leq E_{Q \odot P}[(\xi \cdot S)_0 + Z_0] = 0.$$

According to Remark 2.9 it holds $Q \odot P \sim P$ and we find a contradiction. This shows that (i) implies NRIFA(\mathcal{P}).

Let us now assume that (ii) holds. In a first step, we show that there exists a finite partition of Ω given by $A_1, \dots, A_N \in \mathcal{F}$ and measures $(P_i)_{i \leq N}$ with $P_i \in \mathcal{P}$ such that

$$p < \sum_{i=1}^N E_Q \left[\mathbb{1}_{A_i} E_{P_i} [X^1 | \mathcal{F}] \right].$$

Based on the definition of the essential supremum, cf. [14, Theorem A.37], there exists a countable subset $\mathcal{P}^* = (P^i)_{i \in \mathbb{N}} \subseteq \mathcal{P}$ such that

$$\operatorname{ess\,sup}_{P \in \mathcal{P}} E_P [X^1 | \mathcal{F}] = \sup_{P \in \mathcal{P}^*} E_P [X^1 | \mathcal{F}] = \lim_{N \rightarrow \infty} \sup_{P \in (P_i)_{i \leq N}} E_P [X^1 | \mathcal{F}] \quad P_0\text{-a.s.} \quad (2.17)$$

Moreover, as the essential supremum is \mathcal{F} -measurable, (2.17) also holds \mathcal{P} -q.s. and Q -a.s. for all $Q \in \mathcal{M}_e(\mathbb{F})$. Thus, using (ii) and monotone convergence, we get the following:

$$\begin{aligned} p &< E_Q \left[\operatorname{ess\,sup}_{P \in \mathcal{P}} E_P [X^1 | \mathcal{F}] \right] \\ &= E_Q \left[\lim_{N \rightarrow \infty} \sup_{P \in (P_i)_{i \leq N}} E_P [X^1 | \mathcal{F}] \right] \\ &= \lim_{N \rightarrow \infty} E_Q \left[\sup_{P \in (P_i)_{i \leq N}} E_P [X^1 | \mathcal{F}] \right]. \end{aligned}$$

Thus, there exists $N \in \mathbb{N}$ such that

$$p < E_Q \left[\sup_{P \in (P_i)_{i \leq N}} E_P [X^1 | \mathcal{F}] \right]. \quad (2.18)$$

For every $\ell \in 1, \dots, N$ we define the set B_ℓ as

$$B_\ell := \left\{ E_{P_\ell} [X^1 | \mathcal{F}] = \sup_{P \in (P_i)_{i \leq N}} E_P [X^1 | \mathcal{F}] \right\} \in \mathcal{F}$$

and $A_1 := B_1$ and $A_\ell := B_\ell \setminus \bigcup_{k < \ell} B_k$ for $\ell = 2, \dots, N$. Then A_1, \dots, A_N are disjoint and build a partition of Ω since for each $\omega \in \Omega$ there is $\ell \in \{1, \dots, N\}$ such that

$$E_{P_\ell} [X^1 | \mathcal{F}](\omega) = \sup_{P \in (P_i)_{i \leq N}} E_P [X^1 | \mathcal{F}](\omega)$$

and thus we obtain

$$\biguplus_{\ell=1}^N A_\ell = \bigcup_{\ell=1}^N B_\ell = \Omega.$$

Moreover, we have by (2.18) that

$$p < E_Q \left[\sum_{i=1}^N \mathbb{1}_{A_i} \sup_{P \in (P_i)_{i \leq N}} E_P[X^1 | \mathcal{F}] \right] = \sum_{i=1}^N E_Q \left[\mathbb{1}_{A_i} E_{P_i}[X^1 | \mathcal{F}] \right]. \quad (2.19)$$

Now, we define the measure $P' \in \mathfrak{P}_{\mathcal{N}}(\Omega, \mathcal{G})$ using Equation (2.10) and the partition A_1, \dots, A_N . Moreover, let $Q' \in \mathcal{M}_e(\mathbb{F})$ be an equivalent martingale measure such that

$$p < E_Q \left[\sum_{i=1}^N \mathbb{1}_{A_i} E_{P_i}[X^1 | \mathcal{F}] \right] = E_{Q'} \left[\sum_{i=1}^N \mathbb{1}_{A_i} E_{P_i}[X^1 | \mathcal{F}] \right] \quad (2.20)$$

and such that the density of Q' with respect to P' is bounded, cf. Lemma 2.10. Finally, we define the measure R on (Ω, \mathcal{G}) as

$$R(G) = E_{Q'} \left[\sum_{i=1}^N \mathbb{1}_{A_i} E_{P_i}[\mathbb{1}_G | \mathcal{F}] \right].$$

Since $Q' \odot P_i|_{\mathcal{F}} = Q'$, cf. Remark 2.9, and by the \mathcal{F} -measurability of A_1, \dots, A_N , the measure R is a probability measure on (Ω, \mathcal{G}) and satisfies $R|_{\mathcal{F}} = Q'$. Thus, we find that S is also a (\mathbb{F}, R) -martingale. Moreover, we obtain for all admissible insurance strategies Ψ with positive total mass $\gamma > 0$ that

$$\begin{aligned} E_R[V_T(\Psi)] &= E_R \left[\lim_{n \rightarrow \infty} \sum_{i \in \mathbb{N}} \psi^{i,n}(p - X^i) \right] \\ &= E_{Q'} \left[\sum_{i=1}^N \mathbb{1}_{A_i} E_{P_i} \left[\lim_{n \rightarrow \infty} \sum_{i \in \mathbb{N}} \psi^{i,n}(p - X^i) \middle| \mathcal{F} \right] \right] \\ &= E_{P'} \left[\left(\frac{dQ'}{dP'} \right) \sum_{i=1}^N \mathbb{1}_{A_i} E_{P_i} \left[\lim_{n \rightarrow \infty} \sum_{i \in \mathbb{N}} \psi^{i,n}(p - X^i) \middle| \mathcal{F} \right] \right] \\ &= \sum_{i=1}^N E_{P_i} \left[\left(\frac{dQ'}{dP'} \right) \mathbb{1}_{A_i} E_{P_i} \left[\lim_{n \rightarrow \infty} \sum_{i \in \mathbb{N}} \psi^{i,n}(p - X^i) \middle| \mathcal{F} \right] \right] \\ &= E_{Q'} \left[\sum_{i=1}^N \mathbb{1}_{A_i} \left(\gamma(p - E_{P_i}[X^1 | \mathcal{F}]) \right) \right] \\ &= \gamma \left(p - E_{Q'} \left[\sum_{i=1}^N \mathbb{1}_{A_i} E_{P_i}[X^1 | \mathcal{F}] \right] \right) \\ &< 0, \end{aligned} \quad (2.21)$$

where we use [1, Proposition B.1] for the fifth equality and (2.20) for the inequality. We now define the process $Z = (Z_t)_{t=0,\dots,T}$ by

$$Z_t = E_R[V_T(\Psi)|\mathcal{F}_t], \quad \text{for all } t = 0, \dots, T.$$

Z is a (\mathbb{F}, R) -martingale with $Z_0 < 0$ by (2.21). Let ξ be some \mathbb{F} -predictable strategy. Then, as in [14, Remark 9.5] the value process $(\xi \cdot S)$ is a local (\mathbb{F}, R) -martingale and thus $(\xi \cdot S) + Z$ is a local (\mathbb{F}, R) -martingale. For the sake of contradiction we assume that (ξ, Ψ) is a \mathcal{P} -robust asymptotic insurance-finance arbitrage such that Ψ has positive total mass $\gamma > 0$. Thus, it holds that

$$(\xi \cdot S)_T + Z_T \geq 0 \quad P\text{-a.s. for all } P \in \mathcal{P}. \quad (2.22)$$

Given that $R|_{\mathcal{F}} \sim P'|_{\mathcal{F}} \ll \frac{1}{N} \sum_{i=1}^n P_i|_{\mathcal{F}}$, equation (2.22) is also true R -a.s., i.e.,

$$(\xi \cdot S)_T + Z_T \geq 0 \quad R\text{-a.s.} \quad (2.23)$$

Thus, according to [14, Proposition 9.6] the process $(\xi \cdot S) + Z$ is a (\mathbb{F}, R) -supermartingale and we obtain

$$E_R[(\xi \cdot S)_T + Z_T] \leq (\xi \cdot S)_0 + Z_0 < 0.$$

This contradicts (2.23) and there cannot exist a \mathcal{P} -robust asymptotic insurance-finance arbitrage (ξ, Ψ) such that Ψ has positive total mass $\gamma > 0$. However, given that the pure financial market is arbitrage-free with respect to \mathbb{F} -predictable trading strategies ξ , there could also not be an arbitrage (ξ, Ψ) such that Ψ has total mass $\gamma = 0$. Overall, this leads to a contradiction and thus the result is proven. \square

Remark 2.13 Let us compare Theorem 2.12 in the case of $\mathcal{P} = \{P\}$ with Theorem 1 in [26]. According to Assumptions 2.1 and 2.3, each condition (i) and (ii) in Theorem 2.12 implies the absence of arbitrage in the sense of Definition 2.5 and there is no need for an additional boundedness assumption on the density of the corresponding martingale measures. By contrast, in [26] the boundedness assumption on the density of the equivalent martingale measure is essential and cannot be substituted by means of Lemma 2.10, cf. Example 2 in [26].

Motivated by Theorem 2.12 and the previous discussion (see e.g., Equation (2.5)), we define the following robust version of the QP -rule.

Definition 2.14 Let $\mathcal{P} \subseteq \mathfrak{P}_{\mathcal{N}}(\Omega, \mathcal{G})$ and $Q \in \mathcal{M}_e(\mathbb{F})$. Then, for $X \geq 0$ \mathcal{P} -q.s., we define the Q - \mathcal{P} -evaluation of X by

$$E_{Q \odot \mathcal{P}}[X] := E_Q \left[\operatorname{ess\,sup}_{P \in \mathcal{P}} E_P[X|\mathcal{F}] \right]. \quad (2.24)$$

Note that $Q \odot \mathcal{P}$ does not define a probability measure as it is the case for $Q \odot P$.

Remark 2.15 (On the choice of Q) If the financial market is complete, $\mathcal{M}_e(\mathbb{F}) = \{Q\}$ and there is only one possible choice for Q . If the market is incomplete, the choice of Q becomes more difficult. When sufficiently many traded derivatives are available, a market-consistent Q can be obtained by calibrating the model to those derivatives.

In the insurance context, the most challenging calibration problems occur when long maturities (like 10 to 30 years) are considered. Here, there are few or even no tradeable instruments and a good calibration requires much more effort, in particular to exclude model risk.

Remark 2.16 If the set $\{E_P[X|\mathcal{F}] | P \in \mathcal{P}\}$ is directed upward, i.e., for all $P, P' \in \mathcal{P}$ there exists $\tilde{P} \in \mathcal{P}$ such that

$$\max\{E_P[X|\mathcal{F}], E_{P'}[X|\mathcal{F}]\} \leq E_{\tilde{P}}[X|\mathcal{F}] \quad P_0\text{-a.s.}, \quad (2.25)$$

then, as demonstrated by [14, Theorem A.37], there exists a sequence of measures $(P_n)_{n \in \mathbb{N}} \subset \mathcal{P}$ such that

$$E_{P_n}[X|\mathcal{F}] \nearrow \operatorname{ess\,sup}_{P \in \mathcal{P}} E_P[X|\mathcal{F}] \quad P_0\text{-a.s. for } n \rightarrow \infty.$$

Using monotone convergence we find that

$$\begin{aligned} E_Q \left[\operatorname{ess\,sup}_{P \in \mathcal{P}} E_P[X|\mathcal{F}] \right] &= \lim_{n \rightarrow \infty} E_Q \left[E_{P_n}[X|\mathcal{F}] \right] \\ &\leq \sup_{P \in \mathcal{P}} E_Q [E_P[X|\mathcal{F}]] \\ &\leq E_Q \left[\operatorname{ess\,sup}_{P \in \mathcal{P}} E_P[X|\mathcal{F}] \right] \end{aligned}$$

and that

$$E_{Q \odot \mathcal{P}}[X] = \sup_{P \in \mathcal{P}} E_Q [E_P[X|\mathcal{F}]].$$

Consequently, for a set of priors \mathcal{P} that is directed upwards in the sense of Equation (2.25), there is no \mathcal{P} -robust asymptotic insurance-finance arbitrage if and only if there is no asymptotic insurance-finance arbitrage with respect to P for all $P \in \mathcal{P}$.

Remark 2.17 We now briefly consider the case of \mathbb{G} -trading strategies on the financial market introduced in Sect. 2.2, i.e., ξ is a d -dimensional \mathbb{G} -predictable process. This reflects the fact that the insurer has access to information on the financial market as well as on the insurance market. In order to define a no-arbitrage condition for the financial market, we introduce some more notation. Let $\mathcal{P} \subseteq \mathfrak{P}_{\mathcal{N}}(\Omega, \mathcal{G})$. We say that a measure $Q \in \mathfrak{P}(\Omega, \mathcal{G})$ is *dominated* by \mathcal{P} if there exists $P \in \mathcal{P}$ such that $Q \ll P$, and in this case we write $Q \lll \mathcal{P}$. Next, we define the set

$$\mathcal{M}(\mathbb{G}) := \{Q \in \mathfrak{P}(\Omega, \mathcal{G}) \mid Q \lll \mathcal{P} \text{ and } S \text{ is a } (Q, \mathbb{G})\text{-martingale}\}. \quad (2.26)$$

In this case we assume that for all $P \in \mathcal{P}$ there exists $Q \in \mathcal{M}(\mathbb{G})$ such that $P \ll Q$. Then, according to [6, Theorem 4.5], there is no \mathcal{P} -robust arbitrage, denoted by $\text{NA}(\mathcal{P}, \mathbb{G})$, on the financial market, which means that for all \mathbb{G} -trading strategies

$$(\xi \cdot S)_T \geq 0 \quad \mathcal{P}\text{-q.s.} \quad \text{implies} \quad (\xi \cdot S)_T = 0 \quad \mathcal{P}\text{-q.s.} \quad (2.27)$$

Given that every \mathbb{F} -trading strategy is also a \mathbb{G} -trading strategy, it is clear that $\text{NA}(\mathcal{P}, \mathbb{G})$ implies $\text{NA}(\mathcal{P}, \mathbb{F})$, where \mathbb{G} and \mathbb{F} refers here to the \mathbb{G} -trading strategies and \mathbb{F} -trading strategies, respectively. It thus follows that $\text{NRIFA}(\mathcal{P}, \mathbb{G})$ implies $\text{NRIFA}(\mathcal{P}, \mathbb{F})$ and thus (i) or (ii) in Theorem 2.12 is satisfied. Here, $\text{NRIFA}(\mathcal{P}, \mathbb{G})$ is defined as in Definition 2.5, but with a \mathbb{G} -trading strategy ξ . However, the corresponding *if* direction of Theorem 2.12 is more delicate and could therefore serve as a topic for future research.

3 Robust two-step evaluation

In this section we show that Theorem 2.12 provides a theoretical foundation for the so-called *two-step evaluation*, cf. [11, 23]. This kind of evaluation is used for the pricing of hybrid products depending on the financial market, as well as on other random sources, e.g., individual risks depending on the policy holder of an insurance contract. The idea of a two-step evaluation is to combine actuarial techniques with concepts from financial mathematics. In the following, we recap the idea and the concept. However, note that in contrast to the existing literature we do not fix any probability measure on (Ω, \mathcal{G}) , but only a prior P_0 on the measurable space (Ω, \mathcal{F}) .

Let X be a \mathcal{G} -measurable random variable, representing the discounted payoff of an insurance product. In a first step we consider the \mathcal{F} -conditional risk of X , i.e. $\rho_{\mathcal{F}}(X)$, where $\rho_{\mathcal{F}}$ is a suitable \mathcal{F} -conditional risk measure defined on the space of bounded random variables on (Ω, \mathcal{G}) , which is denoted by $\mathcal{L}_b(\Omega, \mathcal{G})$. This corresponds to an actuarial evaluation resulting in a \mathcal{F} -measurable random variable $\rho_{\mathcal{F}}(X)$ defined on the financial market $(\Omega, \mathcal{F}, P_0)$. In the second step we price $\rho_{\mathcal{F}}(X)$ on the financial market under an equivalent risk-neutral measure $Q \in \mathcal{M}_e(\mathbb{F})$. Combining these, we arrive at the following two-step evaluation:

$$\pi : \mathcal{L}_b(\Omega, \mathcal{G}) \rightarrow \mathbb{R}, \quad \pi(X) = E_Q[\rho_{\mathcal{F}}(-X)]. \quad (3.1)$$

As already mentioned in [1], the QP -evaluation in (2.5) is a two-step evaluation with the \mathcal{F} -conditional risk measure $\rho_{\mathcal{F}}(X) = E_P[X|\mathcal{F}]$ as well as the new concept of the robust QP -evaluation from Definition 2.14. In the following we recap some well-known facts for conditional risk measures in order to highlight that for specific coherent \mathcal{F} -conditional risk measures $\rho_{\mathcal{F}}$ the two-step evaluation in (3.1) can be rewritten by a QP -evaluation for a suitable choice of \mathcal{P} .

3.1 Robust representation of conditional risk measures

For the reader's convenience we recall the definition of conditional risk measures (see e.g., [14]).

Definition 3.1 A map $\rho_{\mathcal{F}} : \mathcal{L}_b(\Omega, \mathcal{G}) \rightarrow L^\infty(\Omega, \mathcal{F}, P_0)$ is called a *convex \mathcal{F} -conditional risk measure*, if for all $X, Y \in \mathcal{L}_b(\Omega, \mathcal{G})$ the following holds P_0 -a.s.:

- (i) *Conditional cash invariance*: $\rho_{\mathcal{F}}(X + \bar{X}) = \rho_{\mathcal{F}}(X) - \bar{X}$ for any $\bar{X} \in \mathcal{L}_b(\Omega, \mathcal{F})$.
- (ii) *Monotonicity*: $X \leq Y$ implies $\rho_{\mathcal{F}}(X) \geq \rho_{\mathcal{F}}(Y)$.
- (iii) *Normalization*: $\rho_{\mathcal{F}}(0) = 0$.
- (iv) *Conditional convexity*: $\rho_{\mathcal{F}}(\lambda X + (1 - \lambda)Y) \leq \lambda \rho_{\mathcal{F}}(X) + (1 - \lambda) \rho_{\mathcal{F}}(Y)$ for $\lambda \in \mathcal{L}_b(\Omega, \mathcal{F})$ with $0 \leq \lambda \leq 1$.

A convex \mathcal{F} -conditional risk measure $\rho_{\mathcal{F}}$ is called *coherent* if it also satisfies the following condition:

- (v) *Conditional positive homogeneity*: $\rho_{\mathcal{F}}(\lambda X) = \lambda \rho_{\mathcal{F}}(X)$ for $\lambda \in \mathcal{L}_b(\Omega, \mathcal{F})$ with $\lambda \geq 0$.

We say that $\rho_{\mathcal{F}}$ is *continuous from below* if

- (vi) *Continuity from below*: $X_n \nearrow X$ pointwise on Ω implies $\rho_{\mathcal{F}}(X_n) \searrow \rho_{\mathcal{F}}(X)$.

Moreover, we say that $\rho : \mathcal{L}_b(\Omega, \mathcal{G}) \rightarrow \mathbb{R}$ is a (*coherent*) *convex risk measure* if $\mathcal{F} = \{\Omega, \emptyset\}$.

Definition 3.2 Let S be a financial market on $(\Omega, \mathcal{F}, \mathbb{F}, P_0)$ and denote by $\mathcal{M}_e(\mathbb{F})$ the set of all martingale measures which are equivalent to P_0 . A map $\pi : \mathcal{L}_b(\Omega, \mathcal{G}) \rightarrow \mathbb{R}$ is called *two-step evaluation* if there is a \mathcal{F} -conditional risk measure $\rho_{\mathcal{F}}$ and an equivalent martingale measure $Q \in \mathcal{M}_e(\mathbb{F})$ such that $\pi(X) = E_Q[\rho_{\mathcal{F}}(-X)]$ for all $X \in \mathcal{L}_b(\Omega, \mathcal{G})$.

We now show that Theorem 2.12 provides an economic foundation for the pricing of finance-linked insurance products via two-step evaluations. Indeed, by using results from [14], we formulate sufficient condition for a conditional risk measure $\rho_{\mathcal{F}}$ in a two-step evaluation $\pi(X) = E_Q[\rho_{\mathcal{F}}(-X)]$ such that π is a Q - \mathcal{P} -evaluation. In this way, the two-step evaluation π characterizes the \mathcal{P} -robust asymptotic insurance-finance arbitrage-free price as presented in Theorem 2.12.

Lemma 3.3 Let $\rho_{\mathcal{F}} : \mathcal{L}_b(\Omega, \mathcal{G}) \rightarrow L^\infty(\Omega, \mathcal{F}, P_0)$ be a convex \mathcal{F} -conditional risk measure which is continuous from below. Then $\rho_{\mathcal{F}}$ is represented by

$$\rho_{\mathcal{F}}(X) = \operatorname{ess\,sup}_{P \in \mathfrak{P}^{P_0}(\Omega, \mathcal{G})} \left(E_P[-X | \mathcal{F}] - \alpha_{\mathcal{F}}^{\min}(P) \right), \quad (3.2)$$

where the acceptance set $\mathcal{A}_{\mathcal{F}}$, the penalty function $\alpha_{\mathcal{F}}^{\min}$ and the set of priors $\mathfrak{P}^{P_0}(\Omega, \mathcal{G})$ are defined by

$$\mathcal{A}_{\mathcal{F}} := \{X \in \mathcal{L}_b(\Omega, \mathcal{G}) \mid \rho_{\mathcal{F}}(X) \leq 0\},$$

$$\alpha_{\mathcal{F}}^{\min}(P) := \operatorname{ess\,sup}_{X \in \mathcal{A}_{\mathcal{F}}} E_P[-X|\mathcal{F}],$$

and

$$\mathfrak{P}^{P_0}(\Omega, \mathcal{G}) := \{P \in \mathfrak{P}(\Omega, \mathcal{G}) \mid P|_{\mathcal{F}} = P_0\} \subseteq \mathfrak{P}_{\mathcal{N}}(\Omega, \mathcal{G}). \quad (3.3)$$

If, in addition, $\rho_{\mathcal{F}}$ is a coherent \mathcal{F} -conditional risk measure, then there is a subset $\mathcal{P} \subseteq \mathfrak{P}^{P_0}(\Omega, \mathcal{G})$ such that

$$\rho_{\mathcal{F}}(X) = \operatorname{ess\,sup}_{P \in \mathcal{P}} E_P[-X|\mathcal{F}]. \quad (3.4)$$

Proof In the unconditional case the first statement follows from Theorem 4.16 and Theorem 4.22 as proven in [14]. Using the same idea as in the proof of Theorem 11.2 in [14] yields the conditional statement. For the second statement we refer to Corollary 4.19 in [14] for the unconditional case. Using similar arguments to those in Corollary 11.6 in [14], the conditional statement follows. \square

Thus, Lemma 3.3 shows that every two-step evaluation $\pi(X) = E_Q[\rho_{\mathcal{F}}(-X)]$ given by an equivalent martingale measure $Q \in \mathcal{M}_e(\mathbb{F})$ and a coherent \mathcal{F} -conditional risk measure $\rho_{\mathcal{F}}$ which is continuous from below can be written as Q - \mathcal{P} -evaluation $\pi(X) = E_Q[\operatorname{ess\,sup}_{P \in \mathcal{P}} E_P[X|\mathcal{F}]]$ for a suitable subset $\mathcal{P} \subseteq \mathfrak{P}^{P_0}(\Omega, \mathcal{G})$.

Remark 3.4 If we a priori fix the nullsets on (Ω, \mathcal{G}) by a probability measure P on (Ω, \mathcal{G}) such that $P|_{\mathcal{F}} = P_0$, then we can define the conditional risk measure $\rho_{\mathcal{F}}$ on $L^\infty(\Omega, \mathcal{G}, P)$, instead of working with $\mathcal{L}_b(\Omega, \mathcal{G})$. In this case $\rho_{\mathcal{F}}$ only needs to be continuous from above, as opposed to satisfying the stronger assumption of continuity from below, in order to have a representation as in Lemma 3.3, cf. Theorem 4.33 and Theorem 11.2 in [14]. The drawback of this approach is that we consider uncertainty in a narrow sense because we fix all relevant nullsets on (Ω, \mathcal{G}) using a single probability measure P . Nevertheless, this also leads to a robust pricing problem in the spirit of Sect. 2, which will be discussed in more detail in Remark 3.8.

Next, we provide some examples for the set $\mathcal{P} \subseteq \mathfrak{P}^{P_0}(\Omega, \mathcal{G})$ in Lemma 3.3 and the associated \mathcal{F} -conditional risk measure in Equation (3.4).

Example 3.5 Let P be a probability measure on (Ω, \mathcal{G}) such that $P|_{\mathcal{F}} = P_0$. We consider the set of priors $\mathcal{P}_\lambda \subseteq \mathfrak{P}^{P_0}(\Omega, \mathcal{G})$ given by

$$\mathcal{P}_\lambda := \left\{ \tilde{P} \in \mathfrak{P}^{P_0}(\Omega, \mathcal{G}) \mid \tilde{P} \ll P \text{ with } d\tilde{P}/dP \leq \lambda^{-1} \text{ } P\text{-a.s.} \right\} \quad \text{for } \lambda \in (0, 1). \quad (3.5)$$

In this case, the associated risk measure $\rho_{\mathcal{F}}(X)$ is the *conditional average value at risk*, denoted by $\operatorname{AVR}_\lambda(X|\mathcal{F})$, see also Definition 11.8 in [14]. Note that the set \mathcal{P}_λ is dominated by the probability measure P .

Example 3.6 Let P be a probability measure on (Ω, \mathcal{G}) such that $P|_{\mathcal{F}} = P_0$. We consider the set of priors $\mathcal{P}_c \subseteq \mathfrak{P}^{P_0}(\Omega, \mathcal{G})$ for $c > 0$ given by

$$\mathcal{P}_c := \left\{ \tilde{P} \in \mathfrak{P}^{P_0}(\Omega, \mathcal{G}) \mid H(\tilde{P}|P) \leq c \right\}, \quad (3.6)$$

where $H(\tilde{P}|P)$ denotes the *relative entropy of \tilde{P} with respect to P* and is defined by

$$H(\tilde{P}|P) := \begin{cases} E_{\tilde{P}} \left[\log \frac{d\tilde{P}}{dP} \right], & \text{if } \tilde{P} \ll P \\ +\infty, & \text{otherwise.} \end{cases}$$

Here, the associated risk measure $\rho_{\mathcal{F}}(X)$ is the *coherent entropic risk measure*, introduced in [13]. As in Example 3.5, the set \mathcal{P}_c is dominated by the probability measure P .

In the following proposition we show that pricing with two-step evaluations leads to arbitrage-free premiums in the sense of Sect. 2. This remarkable result is a consequence of Lemma 3.3 and Theorem 2.12.

Proposition 3.7 *Let S be a financial market on (Ω, \mathcal{F}) and let π be a two-step evaluation with \mathcal{F} -conditional convex risk measure of the form (3.4) for the set $\mathcal{P} = \{P \in \mathfrak{P}_{\mathcal{N}}(\Omega, \mathcal{G}) \mid \alpha_{\mathcal{F}}^{\min}(P) < \infty\}$. Assume that $\mathcal{X} = (X^i)_{i \in \mathbb{N}}$ is a sequence of insurance benefits fulfilling Assumption 2.1 and assume that $p < \pi(X^1)$. Then there is NRIFA(\mathcal{P}) with respect to the insurance-finance market (S, \mathcal{X}, p) .*

Proof The result is a direct consequence of Theorem 2.12, since the $Q\mathcal{P}$ -evaluation is an upper bound for the two-step evaluation π and the chosen premium p is even smaller by assumption. \square

3.2 Construction of conditional iid copies

Our next goal is to apply Theorem 2.12 in the context of a robust two-step evaluation. More specifically, given a random variable \tilde{X} describing an insurance benefit, we determine a robust arbitrage-free premium p for \tilde{X} by using Theorem 2.12 and by taking into account some actuarial constraints, which are reflected by the set of priors $\mathcal{P} \subseteq \mathfrak{P}_{\mathcal{N}}(\Omega, \mathcal{G})$ (see e.g., the sets \mathcal{P}_{λ} and \mathcal{P}_c in Example 3.5 and 3.6, respectively).

To do so, two factors must to be considered. First, the assumptions of Theorem 2.12 must be satisfied. To this end, we construct a sequence of benefits $(X^j)_{j \in \mathbb{N}}$ which are copies of \tilde{X} and which satisfy Assumption 2.1. Second, the set of priors \mathcal{P} must be shifted to the product space where we model the benefits $(X^j)_{j \in \mathbb{N}}$. We observe that these steps contain some subtleties which we discuss in more detail in Remark 3.8 after formally introducing the setting.

Let $(\Omega^F, \mathcal{F}^F)$ and $(\Omega^I, \mathcal{F}^I)$ be two measurable spaces on which we model purely financial and purely insurance events, respectively. On the product space $(\Omega^F \times \Omega^I, \mathcal{F}^F \otimes \mathcal{F}^I)$ we introduce the stochastic process $\tilde{S} = (\tilde{S}_t)_{t=0, \dots, T}$ and the random variable \tilde{X} describing the financial market and a single insurance benefit, respectively. Moreover, let P_0 be a measure on $(\Omega^F \times \Omega^I, \mathcal{F}^F \otimes \{\emptyset, \Omega^I\})$ which

determines the nullsets in $\mathcal{F}^F \otimes \{\emptyset, \Omega\}$ and \mathcal{P} be a set of probability measures on $(\Omega^F \times \Omega^I, \mathcal{F}^F \otimes \mathcal{F}^I)$ such that

$$P|_{\mathcal{F}^F \otimes \{\emptyset, \Omega^I\}} \sim P_0 \text{ for all } P \in \mathcal{P}.$$

We now shift \mathcal{P} to a set of priors $\mu_{\mathcal{P}}$ on $(\Omega^F \times (\Omega^I)^{\mathbb{N}}, \mathcal{F}^F \otimes (\mathcal{F}^I)^{\otimes \mathbb{N}})$. Furthermore, on this space we copy the financial market \tilde{S} to S and construct insurance benefits $(X^j)_{j \in \mathbb{N}}$ which are *iid* conditionally on S such that for all $P \in \mathcal{P}$ the law of (S, X^j) for $j \in \mathbb{N}$ under μ^P coincides with the law of (\tilde{S}, \tilde{X}) under P .

Remark 3.8 We also emphasize that if the set \mathcal{P} is dominated by a measure $P \in \mathcal{P}$, as is the case in Example 3.5 and 3.6, this will no longer hold for the shifted set $\mu_{\mathcal{P}}$. The reason for this is that absolute continuity of measures is not stable under countable products. Therefore, the seemingly not robust problem in the dominated case on $(\Omega^F \times \Omega^I, \mathcal{F}^F \otimes \mathcal{F}^I, \mathcal{P})$ is indeed a robust pricing problem on $(\Omega^F \times (\Omega^I)^{\mathbb{N}}, \mathcal{F}^F \otimes (\mathcal{F}^I)^{\otimes \mathbb{N}}, \mu_{\mathcal{P}})$.

To be precise, we define

$$\begin{aligned} \tilde{\Omega} &:= \Omega^F \times \Omega^I, & \Omega &:= \Omega^F \times (\Omega^I)^{\mathbb{N}}, \\ \tilde{\mathcal{F}} &:= \mathcal{F}^F \otimes \{\Omega^I, \emptyset\}, & \mathcal{F} &:= \mathcal{F}^F \otimes \{\Omega^I, \emptyset\}^{\otimes \mathbb{N}}, \\ \tilde{\mathcal{G}} &:= \mathcal{F}^F \otimes \mathcal{F}^I, & \mathcal{G} &:= \mathcal{F}^F \otimes (\mathcal{F}^I)^{\otimes \mathbb{N}}. \end{aligned}$$

We denote by $\tilde{\omega} = (\tilde{\omega}^F, \tilde{\omega}^I)$ an element in $\tilde{\Omega}$ and by $\omega = (\omega^F, (\omega_j^I)_{j \in \mathbb{N}})$ an element in Ω . Furthermore, we introduce the following projections on $\tilde{\Omega}$:

$$\begin{aligned} \tilde{\pi}_{\Omega^F}: \tilde{\Omega} &\rightarrow \Omega^F, & \tilde{\pi}_{\Omega^F}(\tilde{\omega}) &= \tilde{\omega}^F, \\ \tilde{\pi}_{\Omega_j^I}: \tilde{\Omega} &\rightarrow \Omega_j^I, & \tilde{\pi}_{\Omega_j^I}(\tilde{\omega}) &= \tilde{\omega}_j^I, \end{aligned}$$

as well as the following the projections on Ω :

$$\begin{aligned} \pi_{\Omega^F}: \Omega &\rightarrow \Omega^F, & \pi_{\Omega^F}(\omega) &= \omega^F, \\ \pi_{\Omega_j^I}: \Omega &\rightarrow \Omega_j^I, & \pi_{\Omega_j^I}(\omega) &= \omega_j^I. \end{aligned}$$

Given a measure P on $(\tilde{\Omega}, \tilde{\mathcal{G}})$, the aim is to define a probability measure μ_P on (Ω, \mathcal{G}) which fulfills the following properties:

$$\text{The law of } (\pi_{\Omega^F}, \pi_{\Omega_j^I}) \text{ under } \mu_P \text{ equals } P \text{ for all } j \in \mathbb{N}, \quad (3.7)$$

and

$$(\pi_{\Omega_j^I})_{j \in \mathbb{N}} \text{ are } \mathcal{F}\text{-conditionally independent under } \mu_P. \quad (3.8)$$

If P is a product measure given by $P = P_F \otimes P_I$ for measures P_F on $(\Omega^F, \mathcal{F}^F)$ and P_I on $(\Omega^I, \mathcal{F}^I)$, then the measure μ_P can be defined by $\mu_P = P_F \otimes (P_I)^{\otimes \mathbb{N}}$. Otherwise, we construct μ_P via disintegration as follows. For some measure P on $(\tilde{\Omega}, \tilde{\mathcal{G}})$ the measure μ_P is defined as

$$\mu_P(A \times B) := \int \mathbb{1}_{\tilde{\pi}_{\Omega^F}^{-1}(A)}(P^{\tilde{\pi}_{\Omega^I}|\tilde{\mathcal{F}}})^{\otimes \mathbb{N}}(B) dP \quad \text{for } A \in \mathcal{F}^F \text{ and } B \in (\mathcal{F}^I)^{\otimes \mathbb{N}}, \quad (3.9)$$

where $P^{\tilde{\pi}_{\Omega^I}|\tilde{\mathcal{F}}}$ denotes the regular version of the conditional probability of $\tilde{\pi}_{\Omega^I}$ given $\tilde{\mathcal{F}}$ (see e.g., [16, Chapter 8]). Note that we implicitly assume its existence. This is no restriction, however, because Ω^F , representing a financial market with $d + 1$ assets and T time steps, can always be assumed to have the form $\Omega^F = \mathbb{R}^{(d+1) \times (T+1)}$ and, thus, is a Borel space. Moreover, using a monotone class argument, it follows that

$$\begin{aligned} (P^{\tilde{\pi}_{\Omega^I}|\tilde{\mathcal{F}}})^{\otimes \mathbb{N}}: (\mathcal{F}^I)^{\otimes \mathbb{N}} \times \tilde{\Omega} &\rightarrow [0, 1] \\ (B, \tilde{\omega}) &\mapsto (P^{\tilde{\pi}_{\Omega^I}|\tilde{\mathcal{F}}}(\cdot, \tilde{\omega}))^{\otimes \mathbb{N}}(B) \end{aligned}$$

is a probability kernel from $(\tilde{\Omega}, \tilde{\mathcal{G}})$ to (Ω, \mathcal{G}) and, thus, the measure μ_P is well defined. If P is a measure on $(\tilde{\Omega}, \tilde{\mathcal{F}})$, then (3.9) defines a measure on (Ω, \mathcal{F}) . In this case we have $B = \Omega^{\mathbb{N}}$. Note that by using (3.9) we get the following:

$$\mu_P(A \times (\Omega^I)^{\mathbb{N}}) = P(A \times \Omega^I) \quad \text{for all } A \in \mathcal{F}^F$$

and thus $P|_{\tilde{\mathcal{F}}} \sim P_0$ implies $\mu_P|_{\mathcal{F}} \sim \mu_{P_0}$ for all $P \in \mathcal{P}$.

Let $\tilde{\mathbb{F}} = (\tilde{\mathcal{F}}_t)_{t \leq T}$ with $\tilde{\mathcal{F}}_t := \mathcal{F}_t^F \vee \{\Omega^I, \emptyset\}$, be a filtration on $(\tilde{\Omega}, \tilde{\mathcal{F}})$. Hereafter, we assume that $\tilde{S} = (\tilde{S}_t)_{t \leq T}$ is a $\tilde{\mathbb{F}}$ -adapted stochastic process on $(\tilde{\Omega}, \tilde{\mathcal{F}})$ describing the prices in a financial market. Moreover, let $\mathcal{M}_e(\tilde{\mathbb{F}})$ be the set of all martingale measures on $(\tilde{\Omega}, \tilde{\mathcal{F}})$ which are equivalent some fixed measure P_0 on $(\tilde{\Omega}, \tilde{\mathcal{F}})$. The insurance and financial filtration on $(\tilde{\Omega}, \tilde{\mathcal{G}})$ is denoted by $\tilde{\mathbb{G}} = (\tilde{\mathcal{G}}_t)_{t \leq T}$ and the insurance benefit, a random variable on $(\tilde{\Omega}, \tilde{\mathcal{G}})$, is denoted by \tilde{X} . In order to shift all quantities to (Ω, \mathcal{G}) we define

$$\begin{aligned} \mathbb{F} &= (\mathcal{F}_t)_{t \leq T} \quad \text{with} \quad \mathcal{F}_t := \mathcal{F}_t^F \vee \{\Omega^I, \emptyset\}^{\otimes \mathbb{N}}, \\ \mathbb{G} &= (\mathcal{G}_t)_{t \leq T} \quad \text{with} \quad \mathcal{G}_t := \sigma((\pi_{\Omega^F}, \pi_{\Omega_j^I}): \Omega \rightarrow (\tilde{\Omega}, \tilde{\mathcal{G}}_t) | j \in \mathbb{N}), \end{aligned}$$

and

$$\begin{aligned} S_t &:= \tilde{S}_t \circ (\pi_{\Omega^F}, \pi_{\Omega_1^I}) = \tilde{S}_t \circ (\pi_{\Omega^F}, \pi_{\Omega_j^I}) \quad \text{for all } j \in \mathbb{N}, t = 0, \dots, T, \\ X^j &:= \tilde{X} \circ (\pi_{\Omega^F}, \pi_{\Omega_j^I}) \quad \text{for all } j \in \mathbb{N}. \end{aligned}$$

Note that \tilde{S}_t is assumed to be measurable with respect to $\tilde{\mathcal{F}}_t \subseteq \tilde{\mathcal{F}}$ and, thus, it does not depend on the second coordinate $\tilde{\omega}^I$.

We show that the measure μ_P defined in (3.9) satisfies the desired properties in (3.7) and (3.8) and is the unique measure with this property. For the reader's convenience we provide the proofs of these results in detail in the Appendix.

Proposition 3.9 *The measure μ_P , as defined by (3.9), is the unique measure on (Ω, \mathcal{G}) which fulfills (3.7) and (3.8).*

Next, we characterize the set of all equivalent martingales measures on (Ω, \mathcal{F}) .

Proposition 3.10 *The set $\mathcal{M}_e(\mathbb{F})$ of all measures on (Ω, \mathcal{F}) such that S is a \mathbb{F} -martingale and which are equivalent to μ_{P_0} is given by*

$$\mathcal{M}_e(\mathbb{F}) = \{\mu_Q | Q \in \mathcal{M}_e(\tilde{\mathbb{F}})\}.$$

Proposition 3.11 *For any $P \in \mathcal{P}$ and $Q \in \mathcal{M}_e(\tilde{\mathbb{F}})$ we have the following:*

$$\begin{aligned} E_Q[E_P[\tilde{X}|\tilde{\mathcal{F}}]] &= E_{\mu_Q}[E_{\mu_P}[X^1|\mathcal{F}]], \\ E_Q[\operatorname{ess\,sup}_{P \in \mathcal{P}} E_P[\tilde{X}|\tilde{\mathcal{F}}]] &= E_{\mu_Q}\left[\operatorname{ess\,sup}_{P \in \mathcal{P}} E_{\mu_P}[X^1|\mathcal{F}]\right]. \end{aligned}$$

We emphasize that Theorem 2.12 and Proposition 3.11 build a foundation of two-step evaluations from a new perspective. We shift the insurance benefit to an insurance-finance market such that the assumptions for Theorem 2.12 are fulfilled and characterize the robust insurance-finance arbitrage-free prices therein. Then, Proposition 3.11 shows that the prices in the shifted insurance-finance market coincide with the two-step evaluation of the initial benefit.

Summarizing, the $Q\mathcal{P}$ -evaluation provides a valuation methodology which excludes a robust insurance-finance arbitrage in the above sense. While the argument involves the conditional strong law of large numbers, and thus infinitely many insurance contracts, the $Q\mathcal{P}$ -evaluation also provides a highly reasonable valuation rule when only finitely many contractors are available. However, with only finitely many contracts, the no-arbitrage concept becomes less powerful: this phenomenon arises, because the theoretical limit, the conditional expectation, is not fully reached in this case. Consequently, this would permit higher prices without necessarily creating an insurance-finance arbitrage since always a small risk remains.

4 Modeling of insurance-finance markets

In this section we provide models for insurance-finance markets and calculate the robust insurance-finance arbitrage-free premium by means of the $Q\mathcal{P}$ -evaluation, cf. Theorem 2.12 and Definition 2.14.

As in Sect. 2.2, let S^0 be the bank account and denote by $S^1 = (S_t^1)_{t=0,\dots,T}$ the non-discounted price process of a risky asset on (Ω, \mathcal{F}) . We fix the \mathcal{F} -nullsets \mathcal{N}

generated by a probability measure $P_0 \in \mathfrak{P}(\Omega, \mathcal{F})$. We assume that the filtration \mathbb{F} is generated by S^0 and S^1 . Next, we introduce the \mathbb{N}_0 -valued random variables τ^1 and τ^2 representing the time of death and the time of surrender of a policy holder, respectively. Let the σ -algebra \mathcal{G} be given by $\mathcal{G} = \mathcal{F} \vee \sigma(\tau^1) \vee \sigma(\tau^2)$ and the filtration \mathbb{G} given by $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$, where \mathbb{H} is the filtration generated by the processes $(\mathbb{1}_{\{\tau^1 \leq t\}})_{t=0, \dots, T}$ and $(\mathbb{1}_{\{\tau^2 \leq t\}})_{t=0, \dots, T}$. Note that τ^1 and τ^2 are \mathbb{G} -stopping times but, in general, they are not \mathbb{F} -stopping times.

Given a parameter set Θ , we introduce the law of the stopping times (τ^1, τ^2) under the parameterized set of priors $\mathcal{P}_\Theta = (P^\theta)_{\theta \in \Theta} \subseteq \mathfrak{P}_{\mathcal{N}}(\Omega, \mathcal{G})$. In particular, we assume that for each $P^\theta \in \mathcal{P}_\Theta$ the conditional laws of τ^1 and τ^2 are given by

$$P^\theta(\tau^1 \leq t | \mathcal{F}) := F_1^{\mathcal{F}}(\theta, t) \quad \text{and} \quad P^\theta(\tau^2 \leq t | \mathcal{F}) := F_2^{\mathcal{F}}(\theta, t) \quad \text{for } t \in \mathbb{N}_0, \quad (4.1)$$

where for fixed $\theta \in \Theta$ the mappings $F_1^{\mathcal{F}}(\theta, \cdot)$ and $F_2^{\mathcal{F}}(\theta, \cdot)$ are \mathcal{F} -conditional distribution functions.

4.1 Modeling under conditional independence

In this subsection we assume that under every $P^\theta \in \mathcal{P}_\Theta$ the random variables τ^1 and τ^2 are \mathcal{F} -conditionally independent, i.e.,

$$P^\theta(\tau^1 \leq s, \tau^2 \leq t | \mathcal{F}) := F_1^{\mathcal{F}}(\theta, s) F_2^{\mathcal{F}}(\theta, t) \quad \text{for } s, t \in \mathbb{N}_0. \quad (4.2)$$

In order to determine the law of (S, τ^1, τ^2) under P^θ it now remains to introduce the restricted measures $P^\theta|_{\mathcal{F}} \sim P_0$ and use disintegration. However, we could assume that $P^\theta|_{\mathcal{F}} = P_0$ for all $\theta \in \Theta$ since the $Q\mathcal{P}$ -evaluation is invariant under the specific choice of the measures $\{P^\theta|_{\mathcal{F}} | \theta \in \Theta\}$ as the set of equivalent martingale measures $\mathcal{M}_e(\mathbb{F})$ only depends on the nullsets \mathcal{N} generated by P_0 .

We introduce the discounted survival benefit X_{survival} and the discounted surrender benefit $X_{\text{surrender}}$ by

$$X_{\text{survival}} := \mathbb{1}_{\{\tau^1 > T, \tau^2 > T\}} Y^1 (S_T^0)^{-1}, \quad (4.3)$$

$$X_{\text{surrender}} := \sum_{t=1}^{T-1} \mathbb{1}_{\{\tau^1 > t, \tau^2 = t\}} Y_t^2 (S_t^0)^{-1}, \quad (4.4)$$

where Y^1 is a \mathcal{F} -measurable random variable and $Y^2 := (Y_t^2)_{t=0, \dots, T}$ is a \mathbb{F} -adapted process. The insurance benefit X is then given by

$$X := X_{\text{survival}} + X_{\text{surrender}}. \quad (4.5)$$

Policy holder with such a policy receive the payment Y^1 at maturity T if they survives until T and do not surrender before time T . If they surrender at time $t < T$ they receive the payment Y_t^2 .

Example 4.1 We define the process $V = (V_t)_{t=0,\dots,T}$ by

$$V_t := K(t) + (S_t - K(t))^+ \quad \text{with} \quad K(t) := (1 + r_G)^t K \quad (4.6)$$

for $K \in \mathbb{R}_+$ and $r_G > -1$. Here, r_G denotes the interest rate associated with a guarantee K . Furthermore, we assume that in case a policy holder surrenders the contract at time t , he will receive the payment $(1-l)V_t$ for $l \in [0, 1]$, where l denotes the penalty in form of a proportional deduction of the actual value. Thus, we set $Y^1 := V_T$ and $Y_t^2 := (1-l)V_t$ for $t = 0, \dots, T-1$.

Example 4.2 We consider the parameter set $\Theta := B \times C := [\underline{b}, \bar{b}] \times [\underline{c}, \bar{c}] \subseteq \mathbb{R}_{>0}^2$ and assume that for every $\theta = (b, c) \in \Theta$ the conditional distribution function of the time of death in (4.1) is given by the well-known Gompertz model, i.e.,

$$F_1^{\mathcal{F}}(\theta, t) = P^\theta(\tau^1 \leq t | \mathcal{F}) := 1 - \exp\left(-\sum_{s=0}^{t-1} b e^{cs}\right). \quad (4.7)$$

In this example we assume that a policy holder is not allowed to surrender, which can be realized by setting $\tau_2 := +\infty$.

We set $S_t^0 := (1+r)^t$ for some risk-free rate $r > 0$ and assume that S^1 follows a Cox-Ross-Rubinstein model (CRR model) (see Section 5.5 in [14]). This means that the stock price at time $t = 1, \dots, T$ is given by the higher value $S_t^1 = S_{t-1}^1(1+u)$ for $t = 1, \dots, T$ with probability $0 < p < 1$ and by the lower value $S_t^1 = S_{t-1}^1(1+v)$ for $t = 1, \dots, T$ with probability $1-p$, such that $-1 < v < u$ and $v < r < u$. We denote by $R_t := S_t^1/S_{t-1}^1$ the returns at time t for $t = 1, \dots, T$. Then the unique equivalent martingale measure in this market can be characterized by the measure Q on (Ω, \mathcal{F}) such that R_1, \dots, R_T are independent and

$$Q(R_t = 1+u) = \frac{r-v}{u-v} \quad \text{and} \quad Q(R_t = 1+v) = \frac{u-r}{u-v} \quad \text{for all } t = 1, \dots, T. \quad (4.8)$$

We apply the $Q_{\mathcal{P}\Theta}$ -evaluation to $X = X_{\text{survival}}$ given by (4.3) and Example 4.1 and get

$$\begin{aligned} E_{Q \odot \mathcal{P}\Theta}[X] &= E_Q \left[\operatorname{ess\,sup}_{\theta \in \Theta} \left[\mathbb{1}_{\{\tau^1 > T\}} V_T (1+r)^{-T} | \mathcal{F} \right] \right] \\ &= E_Q \left[V_T (1+r)^{-T} \operatorname{ess\,sup}_{\theta \in \Theta} (1 - F_1^{\mathcal{F}}(\theta, T)) \right] \\ &= E_Q \left[V_T (1+r)^{-T} \operatorname{ess\,sup}_{\theta \in \Theta} \exp\left(-\sum_{s=0}^{T-1} b e^{cs}\right) \right] \\ &= (1+r)^{-T} \exp\left(-\sum_{s=0}^{T-1} \underline{b} e^{cs}\right) \left(K(T) + \mathbb{E}_Q \left[(S_T^1 - K(T))^+ \right] \right), \end{aligned}$$

where the arbitrage-free price for $(S_T^1 - K(T))^+$ is given by

$$\begin{aligned} E_Q[(S_T - K(T))^+] \\ = \sum_{k=0}^T \binom{T}{k} \left(\frac{r-v}{u-v} \right)^k \left(\frac{u-r}{u-v} \right)^{T-k} (S_0^1(1+u)^k(1+v)^{T-k} - K(T))^+. \end{aligned}$$

In this example the map $(b, c) \mapsto E_{P^\theta} [\mathbb{1}_{\{\tau^1 > T\}} V_T(1+r)^{-T} \mathcal{F}]$ is strictly decreasing in b and c . Thus, based on Remark 2.16, it holds that

$$E_{Q \odot \mathcal{P}_\Theta} [\mathbb{1}_{\{\tau^1 > T\}} V_T(1+r)^{-T}] = \sup_{\theta \in \Theta} E_{Q \odot P^\theta} [\mathbb{1}_{\{\tau^1 > T\}} V_T(1+r)^{-T}]. \quad (4.9)$$

In other words, the \mathcal{P}_Θ -robust price equals the worst-case price of all possible models.

Remark 4.3 This example can be generalized by considering random \mathcal{F} -measurable risk factors b and c taking values in $[\underline{b}, \bar{b}]$ and $[\underline{c}, \bar{c}]$, respectively. Note that this is in line with Assumption 2.1.

Example 4.4 We extend Example 4.2 and consider an insurance benefit which includes a surrender option. The parameter set Θ is now given by $\Theta := A \times B \times C \times D := [\underline{a}, \bar{a}] \times [\underline{b}, \bar{b}] \times [\underline{c}, \bar{c}] \times [\underline{d}, \bar{d}] \subseteq \mathbb{R} \times \mathbb{R}_{>0}^3$ and for every $\theta = (a, b, c, d) \in \Theta$ the conditional distribution functions of τ^1 and τ^2 in (4.1) are given by (4.7) and

$$F_2^{\mathcal{F}}(\theta, t) = P^\theta(\tau^2 \leq t | \mathcal{F}) := 1 - \exp \left(-\frac{1}{d} \sum_{s=0}^{t-1} (a - S_s^1)^2 \right). \quad (4.10)$$

The conditional probability to surrender before or at time t in (4.10) tends to one if $(a - S_s)^2$ increases. The intuition behind this, which is related to the definition of the insurance benefit V in (4.6), is as follows: if the value of the asset decreases, the value of the benefit V will also decline. Therefore, the insurance seeker faces the risk of ending up with $K(t)$ and thus the probability of needing to surrender increases. Conversely, if the value of the asset increases, the value of V increases as well, giving the insurance seeker an incentive to surrender. Obviously, in both cases these considerations further depend on the penalty parameter l . In summary, it is more likely that the insurance seeker surrenders in case the value of the asset deviates too much from the level a .

In applications, we can choose, for example, the parameter set Θ as the confidence intervals around the empirically observed values of (a, b, c, d) . The $Q \odot \mathcal{P}_\Theta$ -evaluation of the benefit X given by (4.5) and Example 4.1 is then given by

$$\begin{aligned} E_{Q \odot \mathcal{P}_\Theta}[X] \\ = E_{Q \odot \mathcal{P}_\Theta} \left[\mathbb{1}_{\{\tau^1 > T, \tau^2 > T\}} V_T(1+r)^{-T} + \sum_{t=1}^{T-1} \mathbb{1}_{\{\tau^2 = t\}} \mathbb{1}_{\{\tau^1 > t\}} (1-l) V_t(1+r)^{-t} \right] \\ = E_Q \left[\text{ess sup}_{\theta \in \Theta} E_{P^\theta} \left[\mathbb{1}_{\{\tau^1 > T, \tau^2 > T\}} V_T(1+r)^{-T} \right. \right. \end{aligned}$$

$$\begin{aligned}
& + \sum_{t=1}^{T-1} \mathbb{1}_{\{\tau^2=t\}} \mathbb{1}_{\{\tau^1>t\}} (1-l)V_t(1+r)^{-t} \Big| \mathcal{F} \Big] \Big] \\
& = E_Q \left[\operatorname{ess\,sup}_{\theta \in \Theta} \left((1 - F_1^{\mathcal{F}}(\theta, T))(1 - F_2^{\mathcal{F}}(\theta, T))V_T(1+r)^{-T} \right. \right. \\
& \quad \left. \left. + \sum_{t=1}^{T-1} (F_2^{\mathcal{F}}(\theta, t) - F_2^{\mathcal{F}}(\theta, t-1))(1 - F_1^{\mathcal{F}}(\theta, t))(1-l)V_t(1+r)^{-t} \right) \right] \\
& = E_Q \left[\operatorname{ess\,sup}_{\theta \in \Theta} \left(\exp \left(- \sum_{s=0}^{T-1} be^{cs} \right) \exp \left(- \frac{1}{d} \sum_{s=0}^{T-1} (a - S_s^1)^2 \right) \frac{V_T}{(1+r)^T} \right. \right. \\
& \quad \left. \left. + (1-l) \sum_{t=1}^{T-1} \left(\exp \left(- \frac{1}{d} \sum_{s=0}^{t-2} (a - S_s^1)^2 \right) \right. \right. \right. \\
& \quad \left. \left. \left. - \exp \left(- \frac{1}{d} \sum_{s=0}^{t-1} (a - S_s^1)^2 \right) \right) \exp \left(- \sum_{s=0}^{t-1} be^{cs} \right) \frac{V_t}{(1+r)^t} \right) \right] \\
& =: E_Q \left[\operatorname{ess\,sup}_{\theta \in \Theta} G(\omega, a, b, c, d) \right], \tag{4.11}
\end{aligned}$$

For fixed $\omega \in \Omega$ it is easy to see that the function $G_\omega := G(\omega, \cdot) : \Theta \rightarrow \mathbb{R}$ is decreasing in b and c . Additionally, a short calculation shows that G_ω is increasing in d . Thus (4.11) falls to

$$E_{Q \odot \mathcal{P}_\Theta}[X] = E_Q \left[\operatorname{ess\,sup}_{(a,d) \in A \times D} G(\omega, a, \underline{b}, \underline{c}, \bar{d}) \right]. \tag{4.12}$$

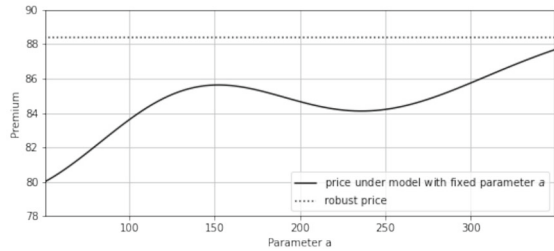
Moreover, we want to compare the robust price in (4.12) with the supremum over all classical $Q P^\theta$ -evaluations for $\theta \in \Theta$, which is given by

$$\sup_{\theta \in \Theta} E_{Q \odot P^\theta}[X] = \sup_{\theta \in \Theta} E_Q [E_{P^\theta}[X | \mathcal{F}]] = \sup_{a \in A} E_Q [G(\omega, a, \underline{b}, \underline{c}, \bar{d})]. \tag{4.13}$$

For the numerical evaluation we consider the time horizon $T = 8$ and use the following parameters. In the CRR model we set $S_0^1 = 100$, $r = 0.05$, $u = 0.1$, and $v = -0.1$. Furthermore, we fix the strike $K = 100$, the penalty parameter $l = 0.1$, and the guaranteed interest rate $r_G = 0.01$. The parameter set Θ is given by $\Theta := [50, 340] \times [0.02, 0.03] \times [0.01, 0.05] \times [10^4, 10^5]$. To calculate the value of the robust $Q \mathcal{P}_\Theta$ -evaluation in (4.12), we maximize for each path $\omega \in \Omega$ the function G_ω over Θ by using the Nelder–Mead method. As already noted above, due to the monotonicity properties of G_ω with respect to b, c and d , we only optimize over $a \in A$. In particular, we get the following numerical results for $E_{Q \odot \mathcal{P}_\Theta}[X]$ and $\sup_{\theta \in \Theta} E_{Q \odot P^\theta}[X]$:

$$E_{Q \odot \mathcal{P}_\Theta}[X] = 88.38 \quad \text{and} \quad \sup_{\theta \in \Theta} E_{Q \odot P^\theta}[X] = 87.61, \tag{4.14}$$

Fig. 1 Asymptotic insurance-finance arbitrage-free premium for $a \in [50, 350]$ and fixed parameters \underline{b} , \underline{c} and \underline{d}



with the difference $\Delta := E_{Q \odot \mathcal{P}_\Theta}[X] - \sup_{\theta \in \Theta} E_{Q \odot P^\theta}[X]$ given by $\Delta = 0.67$. These results indicate that the robust price, which reflects the model risk and which guarantees that the finance-insurance market is arbitrage-free, is strictly greater than the worst case price of all possible models. We illustrate the different premiums in Figure 1.

4.2 Modeling conditional dependence with copulas

In Sect. 4.1 we assumed that the random times τ^1 and τ^2 are \mathcal{F} -conditionally independent, see Eq. (4.2). We now drop this assumption and allow some dependence between τ^1 and τ^2 which is modelled by a family of copulas $(C_\theta)_{\theta \in \Theta}$. We assume that under each $P^\theta \in \mathcal{P}_\Theta$ the law of (τ^1, τ^2) is described by its marginal distribution functions $F_1^{\mathcal{F}}(\theta, \cdot)$ and $F_2^{\mathcal{F}}(\theta, \cdot)$ and the copula $C^\theta : [0, 1]^2 \rightarrow [0, 1]$ such that

$$P^\theta(\tau^1 \leq s, \tau^2 \leq t | \mathcal{F}) = C^\theta(F_1^{\mathcal{F}}(\theta, s), F_2^{\mathcal{F}}(\theta, t)) \quad \text{for } s, t \in \mathbb{N}_0. \quad (4.15)$$

We introduce the following notation for the marginal conditional survival probabilities:

$$\bar{F}_1^{\mathcal{F}}(\theta, t) := 1 - F_1^{\mathcal{F}}(\theta, t) \quad \text{and} \quad \bar{F}_2^{\mathcal{F}}(\theta, t) := 1 - F_2^{\mathcal{F}}(\theta, t) \quad \text{for } t \in \mathbb{N}_0.$$

The conditional survival function of (τ^1, τ^2) is then given by

$$\begin{aligned} P^\theta(\tau^1 > s, \tau^2 > t | \mathcal{F}) \\ = \bar{F}_1^{\mathcal{F}}(\theta, s) + \bar{F}_2^{\mathcal{F}}(\theta, t) - 1 + C^\theta(1 - \bar{F}_1^{\mathcal{F}}(\theta, s), 1 - \bar{F}_2^{\mathcal{F}}(\theta, t)), \end{aligned} \quad (4.16)$$

which can be described by $\hat{C}^\theta : \Theta \times [0, 1]^2 \rightarrow [0, 1]$ defined by

$$\hat{C}^\theta(u, v) := u + v - 1 + C^\theta(1 - u, 1 - v),$$

such that (4.16) can be rewritten in

$$P^\theta(\tau^1 > s, \tau^2 > t | \mathcal{F}) = \hat{C}^\theta(\bar{F}_1^{\mathcal{F}}(\theta, s), \bar{F}_2^{\mathcal{F}}(\theta, t)).$$

Note that \hat{C}^θ is also a copula, cf. [22, Section 2.6].

Example 4.5 A suitable class of copulas C^θ that depends on only one parameter is the class of Archimedean copulas (see [22, Section 4.2] or [27] for further details and related literature). Most of these copulas can be represented by an explicit formula, which leads to a high tractability for applications. The copula $C^\theta(u^1, u^2) = C(u^1, u^2) := u^1 u^2$ reflects the setting of Sect. 4.1, cf. Equation (4.2).

Let us consider the insurance benefit X given in (4.5) and evaluate this benefit with the $Q_{\mathcal{P}_\Theta}$ -rule. For $Q \in \mathcal{M}_e(\mathbb{F})$ we get

$$\begin{aligned}
 & E_{Q \odot \mathcal{P}_\Theta} [X] \\
 &= E_{Q \odot \mathcal{P}_\Theta} \left[\mathbb{1}_{\{\tau^1 > T, \tau^2 > T\}} Y^1 (1+r)^{-T} + \sum_{t=1}^{T-1} \mathbb{1}_{\{\tau^1 > t, \tau^2 = t\}} Y_t^2 (1+r)^{-t} \right] \\
 &= E_Q \left[\operatorname{ess\,sup}_{\theta \in \Theta} E_{P^\theta} \left[\mathbb{1}_{\{\tau^1 > T, \tau^2 > T\}} Y^1 (1+r)^{-T} \right. \right. \\
 &\quad \left. \left. + \sum_{t=1}^{T-1} \mathbb{1}_{\{\tau^1 > t, \tau^2 = t\}} Y_t^2 (1+r)^{-t} \middle| \mathcal{F} \right] \right] \\
 &= E_Q \left[\operatorname{ess\,sup}_{\theta \in \Theta} \left(Y^1 (1+r)^{-T} P^\theta[\tau^1 > T, \tau^2 > T | \mathcal{F}] \right. \right. \\
 &\quad \left. \left. + \sum_{t=1}^{T-1} Y_t^2 (1+r)^{-t} P^\theta[\tau^1 > t, \tau^2 = t | \mathcal{F}] \right) \right] \\
 &= E_Q \left[\operatorname{ess\,sup}_{\theta \in \Theta} \left(Y^1 (1+r)^{-T} \hat{C}^\theta(\bar{F}_1^\mathcal{F}(\theta, T), \bar{F}_2^\mathcal{F}(\theta, T)) \right. \right. \\
 &\quad \left. \left. + \sum_{t=1}^{T-1} Y_t^2 (1+r)^{-t} \left(\hat{C}^\theta(\bar{F}_1^\mathcal{F}(\theta, t), \bar{F}_2^\mathcal{F}(\theta, t-1)) \right. \right. \right. \\
 &\quad \left. \left. \left. - \hat{C}^\theta(\bar{F}_1^\mathcal{F}(\theta, t), \bar{F}_2^\mathcal{F}(\theta, t)) \right) \right) \right], \tag{4.17}
 \end{aligned}$$

where we use in (4.17) that

$$\begin{aligned}
 P^\theta(\tau^1 > t, \tau^2 = t | \mathcal{F}) &= P^\theta(\tau^1 > t, \tau^2 > t-1 | \mathcal{F}) - P^\theta(\tau^1 > t, \tau^2 > t | \mathcal{F}) \\
 &= \hat{C}^\theta(\bar{F}_1^\mathcal{F}(\theta, t), \bar{F}_2^\mathcal{F}(\theta, t-1)) - \hat{C}^\theta(\bar{F}_1^\mathcal{F}(\theta, t), \bar{F}_2^\mathcal{F}(\theta, t)).
 \end{aligned}$$

Example 4.6 We briefly show that the Cox model fits in the framework introduced above. To do this, let $\Lambda^i = (\Lambda_t^i)_{t=0, \dots, T}$ with $\Lambda_0^i = 0$ for $i = 1, 2$ be two increasing processes on $(\Omega, \mathcal{F}, P_0)$ and E_1 and E_2 two random variables on (Ω, \mathcal{G}) . For the parameter set

$$\Theta := \Gamma^1 \times \Gamma^2 \times \Xi := [\underline{\gamma}^1, \overline{\gamma}^1] \times [\underline{\gamma}^2, \overline{\gamma}^2] \times [\underline{\xi}, \overline{\xi}] \subseteq \mathbb{R}_{>0}^2 \times \mathbb{R},$$

we choose $\mathcal{P}_\Theta \subseteq \mathfrak{P}_{\mathcal{N}}(\Omega, \mathcal{G})$ such that for any $P^\theta \in \mathcal{P}_\Theta$ with $\theta = (\gamma^1, \gamma^2, \xi) \in \Theta$ it holds that

$$E^1 \sim^{P^\theta} \exp(\gamma^1) \quad \text{and} \quad E^2 \sim^{P^\theta} \exp(\gamma^2).$$

Moreover, E^1 and E^2 are assumed to be conditionally independent of \mathcal{F} under each $P_\theta \in \mathcal{P}_\Theta$. We define the default times τ^1 and τ^2 using

$$\tau^i := \inf\{t \in \mathbb{N}_0 \mid \Lambda_t^i \geq E^i\}, \quad i = 1, 2,$$

where we use the convention $\inf \emptyset := \infty$. In this case $F_1^{\mathcal{F}}(\theta, t)$ and $F_2^{\mathcal{F}}(\theta, t)$ are given by

$$F_1^{\mathcal{F}}(\theta, t) = 1 - e^{-\gamma^1 \Lambda_t^1} \quad \text{and} \quad F_2^{\mathcal{F}}(\theta, t) = 1 - e^{-\gamma^2 \Lambda_t^2} \quad \text{for } t \in \mathbb{N}_0.$$

Remark 4.7 A Cox model with a single default time under uncertainty is studied in [4]. In contrast to Example 4.6, the authors consider uncertainty on the financial market while the law of the random variable E is fixed.

Conclusion

In this paper we introduce the notion of robust asymptotic insurance-finance arbitrage and characterize the absence of such an arbitrage. Our main theorem provides an economic foundation for the pricing of finance-linked insurance products via $Q\mathcal{P}$ -evaluations. Moreover, we show that valuation principles from a specific class of two-step evaluations can be written as $Q\mathcal{P}$ -evaluation. We also apply our results to model an insurance-finance market, before concluding with some numerical observations.

Appendix A. Proofs of Sect. 3

Proposition A.1 *The law of $(\pi_{\Omega^F}, \pi_{\Omega_j^I})$ under μ_P equals P for all $j \in \mathbb{N}$. Moreover, given any $j \in \mathbb{N}$, the law of (S, X^j) under μ_P equals the law of (\tilde{S}, \tilde{X}) under P .*

Proof Let $A \in \mathcal{F}^F$ and $C \in \mathcal{F}^I$. Then it holds that

$$\begin{aligned} \mu_P^{(\pi_{\Omega^F}, \pi_{\Omega_j^I})}(A \times C) &= \int \mathbb{1}_{\{\tilde{\pi}_{\Omega^F} \in A\}} (P^{\tilde{\pi}_{\Omega^I} | \tilde{\mathcal{F}}})^{\otimes \mathbb{N}} (\Omega^I \times \cdots \times \Omega^I \times C \times \Omega^I \times \cdots) dP \\ &= \int \mathbb{1}_{\{\tilde{\pi}_{\Omega^F} \in A\}} P^{\tilde{\pi}_{\Omega^I} | \tilde{\mathcal{F}}}(C) dP \\ &= P(A \times C) \quad \text{for all } j \in \mathbb{N}, \end{aligned}$$

where we use the definition of the conditional probability and the tower property in the third equality. The second statement follows by

$$\mu_P^{(S, X^j)} = \left(\mu_P^{(\pi_{\Omega^F}, \pi_{\Omega_j^I})} \right)^{(\tilde{S}, \tilde{X})} = P^{(\tilde{S}, \tilde{X})} \quad \text{for all } j \in \mathbb{N}.$$

□

Proposition A.2 Let $j \in \mathbb{N}$ and ν_P be a measure on (Ω, \mathcal{G}) which fulfills (3.7), i.e.,

$$\nu_P^{(\pi_{\Omega^F}, \pi_{\Omega_j^I})} = P \quad \text{for all } j \in \mathbb{N}.$$

Then it holds for all $A \in \mathcal{F}^I$ that

$$\begin{aligned} E_{\nu_P}[\mathbb{1}_{\{\pi_{\Omega_j^I} \in A\}} | \mathcal{F}] \\ = E_P[\mathbb{1}_{\{\tilde{\pi}_{\Omega_\ell^I} \in A\}} | \tilde{\mathcal{F}}] \circ (\pi_{\Omega^F}, \pi_{\Omega_\ell^I}) \quad \nu_P\text{-a.s. for all } A \in \mathcal{F}^I \text{ and } \ell \in \mathbb{N}. \end{aligned} \quad (\text{A.1})$$

Proof Let $B \in \mathcal{F}$ and $A \in \mathcal{F}^I$. Then we have

$$\begin{aligned} \int \mathbb{1}_B \mathbb{1}_{\{\pi_{\Omega_j^I} \in A\}} d\nu_P &= \int (\mathbb{1}_{\pi_{\Omega^F}(B)} \circ \pi_{\Omega^F})(\mathbb{1}_A \circ \pi_{\Omega_j^I}) d\nu_P \\ &= \int (\mathbb{1}_{\pi_{\Omega^F}(B)} \circ \tilde{\pi}_{\Omega^F})(\mathbb{1}_A \circ \tilde{\pi}_{\Omega_\ell^I}) dP \\ &= \int (\mathbb{1}_{\pi_{\Omega^F}(B)} \circ \tilde{\pi}_{\Omega^F}) E_P[(\mathbb{1}_A \circ \tilde{\pi}_{\Omega_\ell^I}) | \tilde{\mathcal{F}}] dP \\ &= \int \mathbb{1}_B E_P[(\mathbb{1}_A \circ \tilde{\pi}_{\Omega_\ell^I}) | \tilde{\mathcal{F}}] \circ ((\pi_{\Omega^F}, \pi_{\Omega_\ell^I})) d\nu_P, \end{aligned}$$

where we use (3.7) in the second equality and $\tilde{\pi}_{\Omega^F}^{-1}(\pi_{\Omega^F}(F)) \in \tilde{\mathcal{F}}$ in the third equality. □

Proposition A.3 The projections $(\pi_{\Omega_j^I})_{j \in \mathbb{N}}$ are \mathcal{F} -conditionally iid under μ_P and consequently also the insurance benefits $(X^j)_{j \in \mathbb{N}}$ are \mathcal{F} -conditionally iid under μ_P .

Proof We must show that for every finite subset $J \subset \mathbb{N}$ and $A_j \in \mathcal{F}^I$ for all $j \in J$ that

$$E_{\mu_P} \left[\prod_{j \in J} \mathbb{1}_{\{\pi_{\Omega_j^I} \in A_j\}} \middle| \mathcal{F} \right] = \prod_{j \in J} E_{\mu_P} \left[\mathbb{1}_{\{\pi_{\Omega_j^I} \in A_j\}} \middle| \mathcal{F} \right].$$

This follows because for every $B \in \mathcal{F}$, it holds that

$$\begin{aligned}
 E_{\mu_P} \left[\mathbb{1}_B \prod_{j \in J} \mathbb{1}_{\{\pi_{\Omega_j^I} \in A_j\}} \right] &= E_P \left[\mathbb{1}_{\tilde{\pi}_{\Omega^F}^{-1}(\pi_{\Omega^F}(B))} \prod_{j \in J} P^{\tilde{\pi}_{\Omega^I} | \tilde{\mathcal{F}}} (A_j) \right] \\
 &= E_P \left[\mathbb{1}_{\tilde{\pi}_{\Omega^F}^{-1}(\pi_{\Omega^F}(B))} \prod_{j \in J} E_P \left[\mathbb{1}_{\{\tilde{\pi}_{\Omega^I} \in A_j\}} | \tilde{\mathcal{F}} \right] \right] \\
 &= E_{\mu_P} \left[\mathbb{1}_B \prod_{j \in J} E_P \left[\mathbb{1}_{\{\tilde{\pi}_{\Omega^I} \in A_j\}} | \tilde{\mathcal{F}} \right] \right] \circ (\pi_{\Omega^F}, \pi_{\Omega_1^I}) \\
 &= \int \mathbb{1}_B \prod_{j \in J} E_{\mu_P} [\mathbb{1}_{\{\pi_{\Omega_j^I} \in A_j\}} | \mathcal{F}] d\mu_P,
 \end{aligned}$$

where we use the definition of μ_P given in (3.9) in the first equality and Proposition A.2 in the fourth equality. \square

Proposition A.4 *The measure μ_P defined by (3.9) is the unique measure on (Ω, \mathcal{G}) which fulfills (3.7) and (3.8).*

Proof Using Proposition A.1 and Proposition A.3 the measure μ_P fulfills (3.7) and (3.8). Moreover, let ν_P be a measure on (Ω, \mathcal{G}) which also satisfies (3.7) and (3.8) (where μ_P is replaced by ν_P). Let $J \subset \mathbb{N}$ be a finite subset and $A \in \mathcal{F}^F$ and $C_j \in \mathcal{F}^I$ for all $j \in J$. Then for D given by

$$D = \pi_{\Omega^F}^{-1}(A) \cap \bigcap_{j \in J} \pi_{\Omega_j^I}^{-1}(C_j) \quad (\text{A.2})$$

we get the following

$$\begin{aligned}
 \nu_P(D) &= E_{\nu_P} \left[\mathbb{1}_{\pi_{\Omega^F}^{-1}(A)} E_{\nu_P} \left[\prod_{j \in J} \mathbb{1}_{\pi_{\Omega_j^I} \in C_j} | \mathcal{F} \right] \right] \\
 &= E_{\nu_P} \left[\mathbb{1}_{\pi_{\Omega^F}^{-1}(A)} \prod_{j \in J} E_{\nu_P} \left[\mathbb{1}_{\pi_{\Omega_j^I} \in C_j} | \mathcal{F} \right] \right] \\
 &= E_{\nu_P} \left[\mathbb{1}_{\pi_{\Omega^F}^{-1}(A)} \prod_{j \in J} E_P \left[\mathbb{1}_{\tilde{\pi}_{\Omega^I} \in C_j} | \tilde{\mathcal{F}} \right] \right] \circ (\pi_{\Omega^F}, \pi_{\Omega_1^I}) \\
 &= E_P \left[\mathbb{1}_{\tilde{\pi}_{\Omega^F}^{-1}(A)} \prod_{j \in J} E_P \left[\mathbb{1}_{\tilde{\pi}_{\Omega^I} \in C_j} | \tilde{\mathcal{F}} \right] \right],
 \end{aligned}$$

where we use (3.8) in the third equality and Proposition A.2 in the fourth equality. The same calculations can be done for μ_P . The sets in (A.2) form a \cap -stable generator of \mathcal{G} and thus we obtain $\mu_P = \nu_P$. This demonstrates the uniqueness. \square

Proposition A.5 *The set $\mathcal{M}_e(\mathbb{F})$ of all measures on (Ω, \mathcal{F}) such that S is a \mathbb{F} -martingale and which are equivalent to μ_{P_0} is given by*

$$\mathcal{M}_e(\mathbb{F}) = \{\mu_Q | Q \in \mathcal{M}_e(\tilde{\mathbb{F}})\}.$$

Proof We show that for all $t, s \in \{0, \dots, T\}$ it holds that

$$E_{\mu_Q}[S_t | \mathcal{F}_s] = E_Q[\tilde{S}_t | \tilde{\mathcal{F}}_s] \circ (\pi_{\Omega^F}, \pi_{\Omega_t^I}). \quad (\text{A.3})$$

Let $B_s \in \mathcal{F}_s$, then we have

$$\begin{aligned} E_{\mu_Q}[\mathbb{1}_{B_s} S_t] &= E_{\mu_Q} \left[\mathbb{1}_{B_s} \tilde{S}_t \circ (\pi_{\Omega^F}, \pi_{\Omega_t^I}) \right] \\ &= E_Q \left[\mathbb{1}_{\tilde{\pi}_{\Omega^F}^{-1}(\pi_{\Omega^F}(B_s))} \tilde{S}_t \right] \\ &= E_Q \left[\mathbb{1}_{\tilde{\pi}_{\Omega^F}^{-1}(\pi_{\Omega^F}(B_s))} E_Q[\tilde{S}_t | \tilde{\mathcal{F}}_s] \right] \\ &= E_{\mu_Q} \left[\mathbb{1}_{B_s} E_Q[\tilde{S}_t | \tilde{\mathcal{F}}_s] \circ (\pi_{\Omega^F}, \pi_{\Omega_t^I}) \right] \end{aligned} \quad (\text{A.4})$$

Thus, it follows (A.3) and so we can conclude that $Q \in \mathcal{M}_e(\tilde{\mathbb{F}})$ implies $\mu_Q \in \mathcal{M}_e(\mathbb{F})$. The equivalence of μ_{P_0} and μ_Q follows from the equivalence of P_0 and Q . For the other inclusion, let $R \in \mathcal{M}_e(\mathbb{F})$. We now show that there exists $Q \in \mathcal{M}_e(\tilde{\mathbb{F}})$ such that $R = \mu_Q$. Define Q by

$$Q := R^{\pi_{\Omega^F}}(\tilde{\pi}_{\Omega^F}(\cdot)).$$

Then, by changing the roles of π_{Ω^F} and $\tilde{\pi}_{\Omega^F}$ the result follows using (A.4). \square

Proposition A.6 *For any $P \in \mathcal{P}$ and $Q \in \mathcal{M}_e(\tilde{\mathbb{F}})$ we have the following:*

$$\begin{aligned} E_Q[E_P[\tilde{X} | \tilde{\mathcal{F}}]] &= E_{\mu_Q}[E_{\mu_P}[X^1 | \mathcal{F}]], \\ E_Q \left[\operatorname{ess\,sup}_{P \in \mathcal{P}} E_P[\tilde{X} | \tilde{\mathcal{F}}] \right] &= E_{\mu_Q} \left[\operatorname{ess\,sup}_{P \in \mathcal{P}} E_{\mu_P}[X^1 | \mathcal{F}] \right]. \end{aligned}$$

Proof By using the same arguments as in the proof of Proposition 3.10, we find that

$$E_{\mu_P}[X^1 | \mathcal{F}] = E_P[\tilde{X} | \tilde{\mathcal{F}}] \circ (\pi_{\Omega^F}, \pi_{\Omega_t^I}).$$

This implies that

$$E_{\mu_Q}[E_{\mu_P}[X^1 | \mathcal{F}]] = E_{\mu_Q}[E_P[\tilde{X} | \tilde{\mathcal{F}}] \circ (\pi_{\Omega^F}, \pi_{\Omega_t^I})] = E_Q[E_P[\tilde{X} | \tilde{\mathcal{F}}]].$$

The second statement follows analogously. \square

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