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# Combining Savage and Laplace: a new approach to ambiguity

Ralf Diedrich<sup>1</sup>

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#### Abstract

This paper presents a new representation of preference orderings for the study of ambiguity-related decision-making. The central feature is a preference-based decomposition of subjective probabilities that provides information about inherent ambiguity. The probability decomposition is combined with a utility function reflecting the decision-maker's attitude toward ambiguity. The proposed theory generalizes Savage's SEU and allows for a straightforward measurement of ambiguity and ambiguity aversion while keeping concepts for measuring risk and risk attitudes unaffected. For the measurement of ambiguity, concepts of probability theory can be used since decision acts can be interpreted as two-dimensional probability distributions. The proposed measure of ambiguity aversion exploits the properties of the utility function in the same way as the Arrow/Pratt measure of risk aversion.

**Keywords** Ambiguity  $\cdot$  Knightian uncertainty  $\cdot$  Subjective probability  $\cdot$  Ambiguity aversion

JEL Classification D81

#### **1** Introduction

Savage's (1972) subjective expected utility (SEU) theory is one of the most important achievements in decision theory. It brilliantly combines a subjective interpretation of probability with von Neumann and Morgenstern's (1944) expected utility theory laying a solid foundation for manifold theoretical developments in information economics, principal-agent theory, and portfolio and capital market theory. The basic idea is that preferences for decision acts are based on probabilities for decision-relevant events. From which considerations, if any, these probabilities follow is beyond the scope of SEU. For the decision-maker, however, the extent to which

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probability estimates are supported by event-specific knowledge can make a difference. Following this idea, Knight (1921) argued that confidence in probabilities plays a role in decision-making. Ellsberg's (1961) famous thought experiments supported this view by suggesting that bets on events with well-specified probabilities are preferred over ambiguous bets where event-specific knowledge is not available. Numerous empirical studies have confirmed the relevance of ambiguity aversion for decision-making.

Ambiguity aversion implies a violation of Savage's Sure Thing Principle, which is the element of SEU that ensures the additivity of the probability measure. Consequently, the Sure Thing Principle was substituted by weaker assumptions leading to Choquet's expected utility (CEU) theory which applies nonadditive probabilities (capacities). For its axiomatization, Schmeidler (1989) used the Anscombe and Aumann (1963) approach assuming the existence of lotteries with exogenously determined probabilities. Gilboa (1987) applied Savage's setup and thus established CEU on a purely subjective basis. Schmeidler (1986) showed that CEU conforms to the minimization of expected utility across probability measures in the core of the capacity when nonadditive probabilities are convex, which can be interpreted as a special kind of ambiguity aversion. However, a separation of ambiguity perception and ambiguity attitude is hardly possible, since CEU differs from SEU only in the nonadditivity of probabilities. More recent approaches attempt to overcome this problem.

As a close relative to CEU, Gilboa and Schmeidler (1989) axiomatized maxmin expected utility (MEU) theory, which postulates the existence of a set of probability measures from which the most unfavorable one is chosen. A refinement is the  $\alpha$  -maxmin expected utility theory ( $\alpha$ -MEU), introduced by Ghirardato et al. (2004), which uses a weighted mean of expected utilities based on the most favorable and the most unfavorable subjective probability measure. The weight can be interpreted as a coefficient measuring the decision-maker's ambiguity aversion. An even better prerequisite for the separation of ambiguity perception and ambiguity attitude is provided by approaches that use second-order probabilities (Smooth Ambiguity Preferences) are used to account for ambiguity. Nau (2006), Ergin and Gul (2009), Seo (2009), and Denti and Pomatto (2022) presented alternative axiomatizations. An issue is that second-order probabilities and second-order utilities are difficult to determine empirically.<sup>1</sup>

This paper contributes to the literature by presenting a new representation of ambiguity-related preferences without nonadditive or second-order probabilities. The representation is computationally simple, entails SEU as a special case, explains all common types of Ellsberg phenomena, and offers a variety of possibilities for separating and measuring ambiguity perception and ambiguity attitude. Moreover, methods for the elicitation of the applied constructs are straightforward. The basic idea is that ambiguity results from a lack of knowledge that makes event-specific probability estimates impossible and requires the application

<sup>&</sup>lt;sup>1</sup> On this problem, see for example Epstein (2010).

of Laplace's (1812) principle of indifference. At the core of the representation is an additive decomposition of probabilities  $P(A) = Q(A, A) + Q(A, A^{C})$  indicating to what extent the indifference principle is used. If P(A) is based entirely on event-specific knowledge, event A is perfectly unambiguous and  $Q(A, A^C)$  vanishes. If, on the other hand, there is no event-specific knowledge and P(A) is only based on the indifference principle, event A is perfectly ambiguous and P(A) is distributed proportionally between Q(A, A) and  $Q(A, A^{C})$ . Complementary to the probability decomposition, the utility function of SEU is extended. u(x, x) is the utility of outcome x when the underlying event is unambiguous. u(x, z) refers to two alternatively possible outcomes x and z of one decision act and modifies the utility of these outcomes when there is ambiguity concerning the corresponding events. Combining the utility function with the probability decomposition results in the preference functional  $\sum_{i=1}^{n} \sum_{j=1}^{n} u(x_i, x_j) Q(A_i, A_j)$ . With two possible outcomes x and z associated with perfectly unambiguous events A and  $A^{C}$ , the preference value is  $u(x, x)Q(A, A) + u(z, z)Q(A^C, A^C) = u(x, x)P(A) + u(z, z)P(A^C)$ . In the case of an ambiguous event and an ambiguity-averse decision-maker, the preference value is  $u(x, x)Q(A, A) + u(x, z)Q(A, A^{C}) + u(z, z)Q(A^{C}, A^{C}) + u(z, x)Q(A^{C}, A) < u(x, x)P(A) + u(x, z)Q(A^{C}, A) < u(x, x)Q(A^{C}, A) < u(x, x)Q(A^{C}$  $u(z,z)P(A^{C}).$ 

The approach is based on Savage's framework, i.e., acts are modeled as mappings from states of the world to arbitrary outcomes. Hence, it differs conceptionally from approaches that use the Anscombe and Aumann (1963) framework or place restrictions on the outcome space. All probabilities are derived from preferences over acts. Thus, the approach also differs from those that use Savage's setup but rely on exogenously given probabilities. Klibanoff et al. (2005), for example, use Savage's setup but assume the existence of a subset of acts, called lotteries, that are measurable with respect to a partition of the state space over which probabilities are objectively given. Regarding the representation of preferences, the proposed approach uses different constructs (the probability decomposition and the extended utility function) and a different preference functional compared to CEU, MEU,  $\alpha$ -MEU, or Smooth Ambiguity Preferences. In general, it is a generalization of SEU but neither a generalization nor a specification of these approaches. Except for SEU-compliant preferences, it overlaps with CEU, for example, only under narrow conditions.<sup>2</sup> The distinction between perfectly unambiguous and perfectly ambiguous events finds some parallel in the approaches of Nau (2006) and Ergin and Gul (2009) who model the state space as the Cartesian product of two sets and use the resulting structure to distinguish different types of events. Here, in contrast, the state space remains unstructured. Differing from Sarin and Wakker (1992) and Epstein (1999), who use exogenously specified unambiguous events, all definitions needed for the characterization of perfectly unambiguous and perfectly ambiguous events refer to the preferences of the decision-maker. In this respect, parts of the approach are comparable to Epstein and Zhang (2001) and Zhang (2002), who also use preference-based definitions

<sup>&</sup>lt;sup>2</sup> Suppose CEU is based on the upper Choquet Integral and the capacity is superadditive. Then the proposed approach leads to the same preference order on the set of all acts with no more than three possible outcomes for  $u(x_i, x_j) = min\{v(x_i), v(x_j)\}$  and  $Q(A_i \cup A_j, A_i \cup A_j) = C(A_i \cup A_j)$ . Here  $v(x_i)$  is the utility according to CEU and  $C(A_i)$  is the capacity.

of unambiguous events. Preference-based definitions are introduced in this paper, however, for two classes of events, perfectly unambiguous and perfectly ambiguous events. Gul and Pesendorfer (2014) also use a utility function with two arguments, called interval utility, in their Expected Uncertain Utility Theory (EUU). In this theory, outcomes are modeled as prize intervals. The arguments of the utility function are the endpoints of these intervals. In contrast, the outcome space here is arbitrary and the utility function refers to two alternatively possible outcomes.

The proof of the central representation theorem is largely based on Savage's proof of SEU. After setting the model framework, a preference order is established and perfectly unambiguous events (S-events) are defined as events for which the Sure Thing Principle applies. Next, partitions of the state of the world are considered for which the Sure Thing Principle does not apply but to which probabilities can nevertheless be assigned using the indifference principle. On this basis, perfectly ambiguous events (L-events) are introduced. Additional axioms ensure that S-events and L-events do not overlap except for null and universal events. This makes it possible to separate perfectly ambiguous acts (L-acts) and perfectly unambiguous acts (S-acts), and to replace outcomes with L-acts in Savage's framework. Further parts of Savage's axiomatic system are adapted, and on this basis, a representation theorem for simple S-acts with L-acts as consequences is stated (Theorem 1). Subsequently, a link between perfectly ambiguous and perfectly unambiguous acts is established, enabling L-acts with an arbitrary number of possible outcomes to be traced back to a risky mixture of L-acts with two possible outcomes. This forms the basis for a representation theorem on L-acts that is based on the extended utility function (Theorem 2). By merging Theorem 1 and Theorem 2, the final theorem (Theorem 3) is developed. It establishes the intended representation based on the extended utility function and the probability decomposition for all simple acts.

Concerning the separation and measurement of ambiguity perception and ambiguity attitude, the presented approach opens up various new possibilities. First, based on the probability decomposition, a measure of event ambiguity is suggested. It is shown that this measure satisfies intuitively plausible requirements for measures of event ambiguity proposed in the literature. To derive a measure of act ambiguity, it is useful to interpret the probability decomposition as a two-dimensional probability distribution (SL-distribution) and to utilize concepts from probability theory. Following this idea, an ambiguity measure for decision acts based on the covariance of the corresponding SL-distributions is introduced. Finally, a measure of ambiguity aversion is presented that is linked to the characteristics of the utility function in a way that is similar to Arrow's (1971) and Pratt's (1964) measure of risk aversion. Altogether, the proposed approach offers possibilities for separating and measuring ambiguity perception and ambiguity attitude that go beyond those in CEU, MEU, and  $\alpha$ -MEU and are comparable to those in approaches based on secondorder concepts (Smooth Ambiguity Preferences). In contrast to these approaches, the proposed measures are based on "first-order concepts", in particular on the probability decomposition and the extended utility function. Hence, the problems associated with the use of second-order probabilities and second-order utilities should not occur. Finally, it is also noteworthy that the proposed approach is the only purely subjective approach without restrictions on the state space and the outcome space that separates ambiguity perception and ambiguity attitude and allows for different degrees of ambiguity aversion.

The paper is based on a specific perspective on decision-making under ambiguity. While ambiguity in more recent approaches is mostly associated with the existence of second-order probabilities, ambiguity is related here to different possibilities of assigning probabilities. The idea is that in the case of perfectly ambiguous events, the decision-maker has no event-specific knowledge upon which to determine probabilities. In the absence of other possibilities, he or she must apply the indifference principle. The resulting probability is less reliable, however. It appears uncertain due to the lack of event-specific knowledge. Ambiguity is thus defined as a lack of event-specific knowledge leading to uncertainty about probabilities that can only be determined by the indifference principle. Note that the indifference principle is referred to as a method for calculating probabilities when no event-specific knowledge is available. Central to the notion of ambiguity is the absence of event-specific knowledge that forces the application of the indifference principle, rather than the application of the indifference principle is<sup>3</sup>

The paper is organized as follows. In the next section, the probability decomposition, the utility function, and the representation of preferences are introduced. Subsequently, axioms are presented and representation theorems are derived. All relevant proofs are given in the appendix. The last part of the paper elaborates and illustrates ways to measure ambiguity and ambiguity attitudes. The paper concludes with remarks on special issues and future research.

#### 2 Representation of Beliefs and Preferences

Let  $(\Omega, \mathscr{A}, P)$  be a probability space with state space  $\Omega$ ,  $\mathscr{A}$  the  $\sigma$ -algebra of decision-relevant events, and P a countably additive probability measure. Suppose  $\mathscr{A}$  is generated by two  $\sigma$ -algebras  $\mathscr{S}$  and  $\mathscr{L}$  which contain perfectly unambiguous and perfectly ambiguous events, respectively. Then  $Q : \mathscr{A} \times \mathscr{A} \to [0, 1]$  is an *SL*-decomposition of P if the following conditions hold:

- (1) Adaptation:  $Q(A, \Omega) = P(A)$  for all  $A \in \mathscr{A}$
- (2) Additivity:  $Q(\bigcup_{i=1}^{\infty} A_i, B) = \sum_{i=1}^{\infty} Q(A_i, B)$  for all pairwise disjoint  $A_i \in \mathcal{A}$  with  $i \in \mathbb{N}$  and all  $B \in \mathcal{A}$
- (3) Symmetry: Q(A, B) = Q(B, A) for all  $A, B \in \mathcal{A}$
- (4) Separation:  $Q(A, A^C) = 0$  for all  $A \in \mathscr{S}$
- (5) Dispersion:  $Q(A, A^C) = Q(A, \Omega) Q(A^C, \Omega)$  for all  $A \in \mathcal{L}$
- (6) Independence:  $Q(A \cap B, C \cap D) = Q(A, C)Q(B, D)$  for all  $A, C \in \mathscr{S}$  and  $B, D \in \mathscr{L}$

<sup>&</sup>lt;sup>3</sup> In the absence of any event-specific knowledge, a uniform assignment of probabilities, as resulting from indifference principle, can also be derived from the maximum entropy principle, Jaynes (2003), pp. 343. I would like to thank an anonymous reviewer who brought this to my attention.

It can be shown that an *SL*-decomposition always exists if  $P(A \cap B) = P(A)P(B)$  holds for all  $A \in \mathscr{S}$  and  $B \in \mathscr{L}$ .<sup>4</sup> If an *SL*-decomposition exists, it is uniquely determined.

Based on Q, the probability of an event can be decomposed into additively combined parts that show the extent to which it is associated with event classes  ${\mathscr S}$  and  $\mathscr{L}$ , respectively. For example, for an arbitrary event A with P(A) > 0, adaptation, additivity, and symmetry imply  $P(A) = Q(A, \Omega) = Q(A, A) + Q(A, A^{C})$ . Using additivity, Q(A, A) and  $Q(A, A^{C})$  can be further decomposed in any desired way, leading to finer and finer decompositions of P(A). To see how Q informs about the ambiguity of event A, suppose  $A \in \mathcal{S}$ . Separation yields P(A) = Q(A, A). Events with this property are called S-events because the existence of a subjective probability measure is implied by Savage's axioms. For the characterization of S-events as perfectly unambiguous, the Sure Thing Principle is crucial. It allows the evaluation of one part of an act independently of the other, which requires a strict separation of the corresponding events based on event-specific knowledge. Without such knowledge, changing outcomes in one part of the act can cast a different light on the outcomes of the other part.<sup>5</sup> As a consequence, the Sure Thing Principle implies that the decision-maker considers the corresponding S-events to be perfectly unambiguous. Since  $\mathscr{S}$  is an algebra,  $\Omega$  and  $\emptyset$  are S-events. If all events are S-events, i.e.,  $\mathscr{A} = \mathscr{S}$ , there is no ambiguity.

By separation,  $Q(A, A^C) = 0$  holds for all  $A \in \mathcal{S}$ . If, on the other hand,  $Q(A, A^C) > 0$ , *A* cannot be an S-event and must be at least partially ambiguous. Ambiguity comes into play with event class  $\mathcal{L}$ . The elements of  $\mathcal{L}$  are called *L*-events since, in the absence of any event-specific knowledge, Laplace's indifference principle can be used to determine probabilities. Due to the lack of event-specific knowledge, *L*-events are considered to be perfectly ambiguous. Since  $\mathcal{L}$  is an algebra,  $\Omega$  and  $\emptyset$  are *L*-events. If all events are *L*-events, i.e.,  $\mathcal{A} = \mathcal{L}$ , ambiguity is perfect. Note that adaptation, separation, and dispersion jointly exclude the possibility of an event *A* with 0 < P(A) < 1 that belongs to both,  $\mathcal{S}$  and  $\mathcal{L}$ . Moreover, adaptation and independence imply that *L*-events are not informative about *S*-events and vice versa, i. e.  $P(A \cap B) = P(A)P(B)$  for all  $A \in \mathcal{S}$  and  $B \in \mathcal{L}$ .<sup>6</sup>

Considering the decomposition  $P(A) = Q(A, A) + Q(A, A^C)$  with P(A) > 0 and  $Q(A, A^C) > 0$ , event ambiguity is perfect for  $A \in \mathcal{L}$ . In this case, adaptation and dispersion yield  $Q(A, A^C) = P(A)(1 - P(A))$ , indicating a uniform distribution of probability mass according to the indifference principle. If *A* is neither an *L*-event nor an *S*-event,  $Q(A, A^C) < P(A)(1 - P(A))$ . For example, if  $A = B \cap C$  with  $B \in \mathcal{S}$  and  $C \in \mathcal{L}$ , the properties of the *SL*-decomposition imply  $Q(A, A^C) = Q(A, B \setminus A)$ 

<sup>&</sup>lt;sup>4</sup> Define  $Q(A \cap B, C \cap D) = P(A \cap C)P(B)P(D)$  for  $A, C \in S$  and  $B, D \in S$  and use additivity and symmetry to extend this definition to countable unions of pairwise disjoint intersections of *S*-events and *L*-events.

<sup>&</sup>lt;sup>5</sup> Zhang (2002), p. 168.

<sup>&</sup>lt;sup>6</sup> Apply the independence condition with  $C = D = \Omega$ .

with  $Q(A, B \setminus A) < P(A)(1 - P(A))$ .<sup>7</sup> All in all, Q(A, A) reflects the unambiguous and  $Q(A, A^C)$  the ambiguous part of P(A). For an ambiguity-neutral decision-maker, the decomposition of P(A) into Q(A, A) and  $Q(A, A^C)$  is irrelevant. An ambiguity-averse decision-maker, however, evaluates outcomes differently depending on the degree to which the underlying events are subject to ambiguity.

The SL-decomposition Q is complemented by a utility function  $u: X \times X \to \mathbb{R}$ where X denotes the set of outcomes, u is symmetric, and  $u(x, x) \ge u(x, z) \ge u(z, z)$ holds for all  $x, z \in X$  with  $u(x, x) \ge u(z, z)$ . The arguments of u refer to alternatively possible outcomes of one decision act. The stand-alone utility of an outcome x is measured by u(x, x). With different outcomes x and z, u(x, z) captures additional preferential aspects measuring the decision-maker's ambiguity attitude. As the main result of this paper shows, preferences concerning simple acts can be represented by the preference functional  $\sum_{i=1}^{n} \sum_{j=1}^{n} u(x_i, x_j) Q(A_i, A_j)$  where  $x_i$  denotes the outcome associated with event  $A_i$ . According to the axioms presented in the next section, Qis uniquely determined, and u is unique up to positive linear transformations. The decision-maker follows SEU with utility function  $U(x_i) = u(x_i, x_i)$  for all  $x_i \in X$  if he or she perceives no ambiguity, i.e.,  $Q(A_i, A_i) = P(A_i)$  for all *i* and  $Q(A_i, A_i) = 0$  for all *i*, *j* with  $i \neq j$  and, consequently,  $\sum_{i=1}^{n} \sum_{j=1}^{n} u(x_i, x_j)Q(A_i, A_j) = \sum_{i=1}^{n} U(x_i)P(A_i)$ . An inspection of the preference functional shows that the same holds for  $u(x_i, x_i) = 0.5u(x_i, x_i) + 0.5u(x_i, x_i)^8$  which reflects a neutral attitude toward ambiguity. In all other cases, the preference functional indicates deviations from SEU according to the ambiguity of the decision-relevant events and the ambiguity attitude of the decision-maker.

As the representation of beliefs and preferences is intended to model ambiguity-related decision-making, it is illustrated by considering the classical Ellsberg experiments.

**Example 1** (Ellsberg, two urns) The decision-maker must choose between two urns, each containing 100 red or black balls. He or she knows from urn S that it contains 50 red and 50 black balls. The composition of balls in urn L is unknown. When a red ball is drawn, the decision-maker wins \$1. The draw of a red ball from urn S is regarded as an S-event, while the draw of a red ball from urn L is interpreted as an L-event. Thus,  $\mathscr{S} = \{\emptyset, \Omega, R^S, B^S\}$  and  $\mathscr{L} = \{\emptyset, \Omega, R^L, B^L\}^9$  The complete event algebra is  $\mathscr{A} = \{\emptyset, \Omega, R^S, B^S, R^L, B^L, R^S \cap R^L, R^S \cap B^L, B^S \cap R^L, R^S \cup R^L, R^S \cup B^L, B^S \cup R^L, R^S \cup B^L\}$ . Given the known distribution of red and black balls, the decision-maker's subjective probability of drawing a red ball from urn S is 0.5. The subjective probability for a draw of a red ball from urn L is also 0.5. It is based, however, on the indifference principle. Using these subjective probabilities, the properties of the *SL*-decomposition imply the entries in Table 1. For example,

<sup>&</sup>lt;sup>7</sup>  $Q(A, A^C) = Q(B \cap C, (B \cup C)^C) + Q(B \cap C, B \cap C^C) + Q(B \cap C, B^C \cap C) = Q(B \cap C, B \cap C^C) = P(B)$ P(C)(1 - P(C)) = P(A)(1 - P(C)) < P(A)(1 - P(A)) is implied by the properties of the *SL*-decomposition.

<sup>&</sup>lt;sup>8</sup> Insert  $u(x_i, x_j) = 0.5u(x_i, x_i) + 0.5u(x_j, x_j)$  and substitute  $\sum_{j=1}^{n} Q(A_i, A_j) = P(A_i)$  and  $\sum_{i=1}^{n} = Q(A_i, A_i) = P(A_i)$ , which follows from adaptation, additivity, and symmetry.

<sup>&</sup>lt;sup>9</sup> If *Rb* denotes a draw of a red ball from urn S and a draw of a black ball from urn L,  $R^S = Rr \cup Rb$ .

 $Q(R^S \cap R^L, R^S \cap B^L) = Q(R^S, R^S)Q(R^L, B^L) = P(R^S)P(R^L)P(B^L) = 0.125$ , where the first identity is implied by independence,  $Q(R^S, R^S) = P(R^S)$  follows from adaptation, additivity, and separation, and  $Q(R^L, B^L) = P(R^L)P(B^L)$  follows from adaptation and dispersion. For u(1, 1) = 1, u(0, 0) = 0, and u(1, 0) = u(0, 1) = 0.4, the preference values are 0.5 for urn S and 0.45 for urn L. The decision-maker prefers urn S because he or she is ambiguity-averse and the draw of a red ball from urn S is a less ambiguous event. For u(1, 0) = u(0, 1) = 0.5 the decision-maker is ambiguity-neutral and thus indifferent to urn S and urn L.

**Example 2** (Ellsberg, one urn) There is one urn containing 30 red balls and 60 black and yellow balls, the latter in unknown proportion. The decision-maker can bet on a draw of a red or a black ball. When a ball of the chosen color is drawn, he or she wins \$1. According to the known number of red balls, the subjective probability of drawing a red ball is 0.333. The probability of drawing a black ball is the same. However, since the number of black balls is not known, the indifference principle is needed for the determination of this probability. Table 2 displays the assumed *SL*-decomposition. For u(1, 1) = 1, u(0, 0) = 0 and u(1, 0) = u(0, 1) = 0.4, the preference values for R and B are  $0.\overline{33}$  and 0.30, respectively. The ambiguity-averse decision-maker prefers the bet on R since it is less ambiguous than the bet on B. For u(1, 0) = u(0, 1) = 0.5, the decision-maker is ambiguity-neutral and thus indifferent to a bet on R and a bet on B.

In this example, *B* cannot be interpreted as an *L*-event if *R* is assumed to be an *S*-event.<sup>10</sup> Hence, in order to calculate the *SL*-decomposition as in Example 1, suitable *S*-events and *L*-events must first be constructed. To this end, the situation is interpreted as follows: The considered urn contains 30 red and 60 non-red balls. If a red ball is drawn, event *R* occurs. Otherwise, the ball is replaced by a ball drawn from another urn with an unknown composition of 60 black and yellow balls. For this procedure, Table 2 can be calculated in the same way as Table 1. It should be noted, however, that the entries in Table 2 are basically not subject to calculations. The *SL*-decomposition is a subjective quantity derived from preferences. The above considerations only serve to clarify the concept of *S*-events and *L*-events. The decision-maker might interpret the events in question differently, or not worry about an interpretation at all.

#### 3 Axioms and Theorems

The decision situation is modeled as in Savage (1972).  $\Omega$  is the state space with states of nature *s*, *t*. Events are subsets of  $\Omega$  and are denoted by *A*, *B*, or *C*. The decision-relevant events form the  $\sigma$ -algebra  $\mathscr{A}$ .<sup>11</sup> *X* is the outcome space with outcomes

<sup>&</sup>lt;sup>10</sup> Remind that  $\mathscr{S}$  and  $\mathscr{L}$  are  $\sigma$ -algebras. Hence, if *B* and *Y* were *L*-events and *R* was an *S*-event,  $B \cup Y = R^C$  would have to be both an *S*-event and an *L*-event. Except for events *A* with P(A) = 0 or P(A) = 1 this is impossible due to the properties of the *SL*-decomposition.

<sup>&</sup>lt;sup>11</sup> In Savage's framework,  $\mathscr{A}$  is the power set of  $\Omega$ . Savage himself pointed out that his results remain valid if  $\mathscr{A}$  is a  $\sigma$ -algebra, Savage (1972), p. 42.

| Table 1         SL-decomposition in<br>Ellsberg's experiment with two<br>urns |                | $R^S \cap R^L$ | $R^S \cap B^L$ | $B^S \cap R^L$ | $B^S \cap B^L$ | Р    |
|---|----------------|----------------|----------------|----------------|----------------|------|
|   | $R^S \cap R^L$ | 0.125          | 0.125          | 0              | 0              | 0.25 |
|   | $R^S \cap B^L$ | 0.125          | 0.125          | 0              | 0              | 0.25 |
|   | $B^S \cap R^L$ | 0              | 0              | 0.125          | 0.125          | 0.25 |
|   | $B^S \cap B^L$ | 0              | 0              | 0.125          | 0.125          | 0.25 |
|   | Р              | 0.25           | 0.25           | 0.25           | 0.25           | 1    |

x, y, z. Acts f, g, h are modeled as  $\mathscr{A}$ -measurable functions. F is the set of all acts. f is a constant act if an outcome  $x \in X$  exists such that f(s) = x for all  $s \in \Omega$ . Hence, every  $x \in X$  can be identified with a constant act. An act is simple if it has finite support. Throughout this paper, only simple acts are considered. f A g B h is the act that leads to the outcome of act f(g) when event A(B) occurs and to the outcome of act h otherwise. It is presupposed that A and B are disjoint. (f A g) B h denotes  $f A \cap B g A^C \cap B h$ . When this notation is used, there are no restrictions on A or B.  $F_{i=1}^n f_i A_i$  abbreviates  $f_1 A_1 f_2 A_2 \dots f_{n-1} A_{n-1} f_n$ . Preferences are modeled by a binary preference relation  $\leq$  on F.

The first axiom states that  $\leq$  on *F* establishes a weak order:

Axiom 1 (Ordering)  $\leq$  on F is complete and transitive.

Preference relations  $\geq, \prec, \succ$  and  $\sim$  are defined in the usual way.  $A \in \mathscr{A}$  is a null event if  $fAh \leq gAh$  for all  $f, g, h \in F$ , as defined in Savage (1972). Axiom 1 implies  $\emptyset$  is a null event. The complement of a null event is a universal event, hence,  $\Omega$  is a universal event.  $\mathscr{N}$  is the set of null events,  $\mathscr{U}$  is the set of universal events. f is a certain act if an outcome  $x \in X$  and a universal event A exist such that f(s) = x for all  $s \in A$ .

For its particular importance, the following definition is specially labeled:

**Definition 1** (S-events)  $A \in \mathscr{A}$  is an S-event if  $hAf \leq hAg \Rightarrow h'Af \leq h'Ag$  for all  $f, g, h, h' \in F$ .

A partition of the state space into S-events is an S-partition.  $\mathscr{S}$  is the set of all S-events. An  $\mathscr{S}$ -measurable act is an S-act. The set of S-acts is denoted by  $F_S \subseteq F$ . The definition of null events implies universal events are S events. Hence, certain acts are S-acts.

The definition of *S*-events reflects that outcomes associated with *S*-events do not affect the evaluation of outcomes that may otherwise occur. It is an adapted version of Epstein and Zhang's (2001) definition of linearly unambiguous events<sup>12</sup> resp. Zhang's (2002) definition of unambiguous events<sup>13</sup> but differs from that definition

<sup>&</sup>lt;sup>12</sup> Epstein and Zhang (2001), p. 288.

<sup>&</sup>lt;sup>13</sup> Zhang (2002), p. 167.

| Table 2         SL-decomposition in           Ellsberg's experiment with one |   | R     | В     | Y     | Р     |
|--|---|-------|-------|-------|-------|
| urn  | R | 0.333 | 0     | 0     | 0.333 |
|  | В | 0     | 0.166 | 0.166 | 0.333 |
|  | Y | 0     | 0.166 | 0.166 | 0.333 |
|  | Р | 0.333 | 0.333 | 0.333 | 1     |

in two ways: First, it refers only to an event A, not to its complement  $A^C$ . Here, a restricted form of Savage's Sure Thing Principle ensures that  $A^C$  is an S-event whenever A is an S-event. Second, Epstein's and Zhang's definition refers to outcomes x and x' rather than acts h and h'. The restricted Sure Thing Principle, however, is formulated in terms of acts, and consequently the definition of S-events is based on acts as well. Before discussing further aspects of Definition 1, the addressed axiom is introduced:

**Axiom 2** (Restricted Sure Thing Principle)  $fAh \leq gAh \Rightarrow fAh' \leq gAh'$  for all  $f, g, h, h' \in F$  and all  $A \in \mathscr{S}$ .

The only difference between Axiom 2 and Savage's Sure Thing Principle concerns the event A which must be an S-event. Note, however, that S-events are not exogenously determined but introduced by definition and that f, g, h, and h' need not be S-acts as in Sarin and Wakker (1992).

Given an S-event A, Axiom 2 implies  $A^C$  is also an S-event. Moreover, Axiom 2 is sufficient to establish  $\mathscr{S}$  as an algebra. However, Savage's proof is based on a  $\sigma$ -algebra of events. To utilize this proof, it must be ensured that  $\mathscr{S}$  contains countable unions. In addition, the countable additivity of the subjective probability measure is required for the proofs of Theorem 2 and Theorem 3 below. Both requirements are fulfilled by the following axiom:

**Axiom 3** (Monotone Continuity) Let  $\{A_i\}$  be an event sequence with  $A_i \in \mathscr{S}$ and  $A_i \subseteq A_{i+1}$  for  $i \in \mathbb{N}$ . Then  $\bigcup_{i=1}^{\infty} A_i$  is an *S*-event and  $xA_i z \leq xBz$  for  $i \in \mathbb{N} \Rightarrow x \bigcup_{i=1}^{\infty} A_i z \leq xBz$  for all  $x, z \in X$  and  $B \in \mathscr{S}$ .

Note that  $\mathscr{S}$  is the set of *S*-events according to Definition 1 and not a primitive of the axiomatization. Therefore, Axiom 3 does not interfere with the setting specified in the prefix. Alternatively, the requirement that a countable union of *S*-events is an *S*-event could have been included in Axiom 2. It is stated here as part of Axiom 3 to separate behavioral and technical conditions. The second part of Axiom 3 establishes the monotone continuity of the probability relation and thus ensures the countable additivity of the probability measure on  $\mathscr{S}$ .<sup>14</sup> The use of technical

<sup>&</sup>lt;sup>14</sup> For the definition of monotone continuity with respect to a probability relation see Villegas (1964), p. 1789.

conditions that ensure countable additivity has become common in decision theory since Arrow's axiomatization of SEU.<sup>15</sup>

The first part of Axiom 3 guarantees that  $\mathscr{S}$  is not just an algebra but a  $\sigma$ -algebra, which is stated in Lemma 1 below.

#### **Lemma 1** $\mathscr{S}$ is a $\sigma$ -algebra.

The idea that  $\mathscr{S}$  is a  $\sigma$ -algebra is of particular importance for the interpretation of *S*-events. Zhang (1999), Zhang (2002) and Epstein and Zhang (2001) argue that unambiguous events cannot form an algebra as it is not plausible to assume that the intersection of two unambiguous events is unambiguous in general. They illustrate this reasoning by an Ellsberg urn with a total of 100 balls of different colors *R*, *B*, *Y*, and *W*, where R + B = B + Y = 50 is known. The authors term it intuitive that  $R \cup B$  and  $B \cup Y$  are unambiguous, whereas *B* is ambiguous, and, consequently, introduce the set of unambiguous events as Dynkin System. As every  $\sigma$ -algebra is a Dynkin System, the concept of *S*-events is more restrictive than Epstein and Zhang's concept of unambiguous events when the respective context is taken into account. For a clear distinction, *S*-events are referred to as perfectly unambiguous events.

The difference is due to different research objectives. One objective of Epstein and Zhang (2001) is to find a basis for probability sophisticated preferences.<sup>16</sup> To this end, the idea of unambiguous events must be as inclusive as possible to provide the broadest possible basis for decision theories that assume given probabilities. A more restrictive concept would face the problem that effects are attributed to ambiguity for which other explanations have been found, e.g., nonlinear preference functionals.<sup>17</sup> The present paper, in contrast, aims at an explanation for ambiguityrelated preferences by introducing a new representation formalism. Ambiguity is ascribed to a particular class of events, which are referred to as perfectly ambiguous. S-events, on the other hand, are completely unaffected by perfectly ambiguous events. Thus, they serve as an extreme limiting the range of possible ambiguity from below. For this purpose, Epstein and Zhang's concept of unambiguous events is too less restrictive. To see this, consider a second urn containing 100 balls of types R, B, Y, and W. The decision-maker knows that there are 25 balls of each type. According to the reasoning of Epstein and Zhang, the draw of  $R \cup B$  from this second urn is an unambiguous event. However, the decision-maker has more event-specific knowledge in this situation and may therefore perceive a draw from the second urn as less ambiguous than a draw from the first urn. A preference for betting on  $R \cup B$  from the second urn, however, can not be explained using Epstein and Zhang's concept of unambiguous events, since there is no way to distinguish between the two events in question.

**Example 3** (Epstein and Zhang) With respect to the situation outlined above (first urn), the *SL*-decomposition is shown in Table 3. Regardless of the utility

<sup>&</sup>lt;sup>15</sup> See Arrow (1972), p. 22.

<sup>&</sup>lt;sup>16</sup> Epstein and Zhang (2001), p. 268.

<sup>&</sup>lt;sup>17</sup> Epstein and Zhang (2001), p. 289.

| Table 3Epstein and Zhangexample (first urn) |       | $R_1$  | $B_1$  | $Y_1$  | $W_1$  | Р    |
|---|-------|--------|--------|--------|--------|------|
|   | $R_1$ | 0.1250 | 0.0625 | 0      | 0.0625 | 0.25 |
|   | $B_1$ | 0.0625 | 0.1250 | 0.0625 | 0      | 0.25 |
|   | $Y_1$ | 0      | 0.0625 | 0.1250 | 0.0625 | 0.25 |
|   | $W_1$ | 0.0625 | 0      | 0.0625 | 0.1250 | 0.25 |
|   | Р     | 0.25   | 0.25   | 0.25   | 0.25   | 1    |
|   |       |        |        |        |        |      |

function, the ambiguity-averse decision-maker is indifferent to bets on R, B, Y, or W. All these events are equally ambiguous. For u(1,1) = 1, u(0,0) = 0 and u(1,0) = u(0,1) = 0.4, the preference value for a bet on  $R_1 \cup B_1$  is 0.475, while the preference value for a bet on  $R_1 \cup Y_1$  is 0.45.  $R_1 \cup B_1$  is more unambiguous compared to  $R_1 \cup Y_1$  but not perfectly unambiguous. Hence, the decision-maker prefers to bet on  $R_1 \cup B_1$ . In contrast, the SL-decomposition for the second urn is given in Table 4. The events  $R_2 \cup B_2$  and  $R_2 \cup Y_2$  are perfectly unambiguous. The preference value for a bet on these events is 0.5 in both cases. The decision-maker prefers a bet on  $R_2 \cup B_2$  to a bet on  $R_1 \cup B_1$ .

The counterpart of S-events in terms of ambiguity are L-events, whose definition is based on L-partitions:

**Definition 2** (*L*-partitions) A partition  $\{C_i\}$  of  $\Omega$  with i = 1, ..., n and n > 1 is an *L*-partition if  $f C_i g C_i h \sim g C_i f C_i h$  for all  $C_i, C_i \in \{C_i\}$ , all  $f, g \in F_S$  and all  $h \in F$ .

Note that, by definition, all L-partitions are finite. This rules out the paradoxes associated with applying the indifference principle to an infinite number of events, e.g., the Bertrand paradox. The  $\sigma$ -algebra generated by L-partitions is denoted by  $\mathscr{L}$ . Elements of  $\mathscr{L}$  are L-events, and elements of an L-partition with two events are called  $R_L$ -events. Null events and universal events are L-events as a modification of an L-partition with respect to a null event leads to another L-partition. An L-act is an  $\mathscr{L}$ -measurable act, hence, certain acts are L-acts. The set of L-acts is  $F_L \subseteq F$ . For the interpretation of the approach, it is important to note that *L*-events and *L*-acts, like S-events and S-acts, are not exogenously given but derived from the preferences of the decision-maker.

| Table 4Modified Epstein andZhang example (second urn) |       | <i>R</i> <sub>2</sub> | <i>B</i> <sub>2</sub> | <i>Y</i> <sub>2</sub> | <i>W</i> <sub>2</sub> | Р    |
|---|-------|-----------------------|-----------------------|-----------------------|-----------------------|------|
|   | $R_2$ | 0.2500                | 0                     | 0                     | 0                     | 0.25 |
|   | $B_2$ | 0                     | 0.2500                | 0                     | 0                     | 0.25 |
|   | $Y_2$ | 0                     | 0                     | 0.2500                | 0                     | 0.25 |
|   | $W_2$ | 0                     | 0                     | 0                     | 0.2500                | 0.25 |
|   | Р     | 0.25                  | 0.25                  | 0.25                  | 0.25                  | 1    |

The definition of *L*-partitions is based on the idea that there is a class of decisionrelevant events that are not included in  $\mathscr{S}$  but to which probabilities can nevertheless be assigned. There is no event-specific knowledge concerning the elements of an L partition, however, the decision-maker is indifferent to permuting the consequences of these events. This is plausible only if he or she has no reason to believe that one event is more probable than another (the principle of insufficient reason). Consequently, the decision-maker considers the elements of an *L*-partition to be equally probable.

The next two axioms cover further properties of L-acts and L-events.

**Axiom 4** (Outcome Monotonicity for *L*-acts)  $x \leq z \Leftrightarrow x Ch \leq z Ch$  for all  $x, z \in X$ , all  $h \in F_L$ , and all  $C \in \mathscr{L} \setminus \mathscr{N}$ .

**Axiom 5** (Decomposability of *L*-events) There is an *L*-partition  $\{C_i\}$  with i = 1, ..., n such that  $xA \cup C_i z \sim x C_i z$  or  $xA \cup C_i z \sim z$  for all  $x, z \in X$  and all  $A \in \mathcal{L}$ .

Axiom 4 corresponds to Savage's P3, related to *L*-acts, and appears largely uncontroversial. An immediate consequence is:

#### **Lemma 2** *L*-partitions do not include null events.

The main function of Axiom 5 is guaranteeing the existence of *L*-partitions on which the construction of a subjective probability measure on  $\mathscr{L}$  can be based. To begin with, Axiom 5 is used to show that all *L*-events except null events and universal events can be decomposed into the elements of an *L*-partition.

**Lemma 3** For all  $A \in \mathscr{L} \setminus (\mathscr{N} \cup \mathscr{U})$  there is an L-partition  $\{C_i\}$  with i = 1, ..., n such that  $A \cap C_i = C_i$  or  $A \cap C_i = \emptyset$  for all i.

Taken together, Lemma 2 and Lemma 3 imply that *S*-events and *L*-events do not intersect except for null events and universal events:

Lemma 4  $\mathscr{S} \cap \mathscr{L} = \mathscr{N} \cup \mathscr{U}$ 

Consequently, S-acts and L-acts do not overlap except for certain acts. This opens the possibility to treat L-acts as consequences of S-acts and to adapt Savage's P3 to P6.

**Axiom 6** (*L*-act Monotonicity for *S*-acts)  $f \leq g \Leftrightarrow fAh \leq gAh$  for all  $f, g \in F_L$ , all  $h \in F$ , and all  $A \in \mathscr{S} \setminus \mathscr{N}$ .

**Axiom 7** (Weak Comparative Probability)  $f A g \leq f B g \Leftrightarrow f' A g' \leq f' B g'$  for all  $A, B \in \mathcal{S}$ , and all  $f, g, f', g' \in F_L$  such that  $f \succ g$  and  $f' \succ g'$ .

**Axiom 8** (Nontriviality) There exist  $x, z \in X$  such that  $x \succ z$ .

**Axiom 9** (Small Event Continuity) For all  $f, g \in F$  such that f > g and all  $h \in F_L$ , there exists an S-partition  $\{E_i\}$  with i = 1, ..., n such that  $f > hE_i g$  and  $hE_i f > g$  for all i = 1, ..., n.

Axioms 6 to 9 resemble Savage's P3 to P6, with outcomes replaced by *L*-acts. Note that the interpretation of consequences as acts is common in decision theory. The approach of Anscombe and Aumann (1963), for example, is characterized by the interpretation of consequences as lotteries. Even Savage (1972) considered the possibility of modeling so-called small world consequences as grand world acts, suggesting the idea of identifying consequences with acts.<sup>18</sup> Since constant acts are *L*-acts, Axiom 6 implies Savage's P3 if the considered event is an *S*-event. Together with the other axioms, Axioms 5 and 6 imply monotonicity with respect to all decision acts. Except for the replacement of outcomes by *L*-acts, Axioms 7 to 9 are well known and require no further comments. Designations come from Machina and Schmeidler (1992).

Given Axioms 1 to 9, Savage's representation theorem can be adapted:

**Theorem 1** (Subjective expected utility theory with *L*-acts as consequences) Suppose that Axioms 1 to 9 hold. Then there is a countably additive probability measure  $P_S : \mathscr{S} \to [0, 1]$  and a function  $U : F_L \to \mathbb{R}$  such that

$$F_{i=1}^n f_i A_i \leq F_{i=1}^n g_i A_i \Leftrightarrow \sum_{i=1}^n U(f_i) P_{\mathcal{S}}(A_i) \leq \sum_{i=1}^n U(g_i) P_{\mathcal{S}}(A_i)$$

holds for all  $f_i, g_i \in F_L$  and all  $A_i \in \mathscr{S}$  with i = 1, ..., n.  $P_S$  is uniquely determined, and U is unique up to positive linear transformations.

To prove Theorem 1, notice that Axioms 1, 2, 6, 7, and 9 imply P1 to P4 and P6 in Savage (1972), related to *L*-acts as consequences. The analogon of Savage's P5 concerning *L*-acts follows from Axiom 8 since constant acts are *L*-acts. Hence, the proof of Theorem 1 with respect to a finitely additive probability measure is given in Savage (1972). The countable additivity of  $P_S$  follows from Axiom 3, as proven in Villegas (1964).<sup>19</sup>

Theorem 1 may give the impression that it concerns the evaluation of a two-stage decision act: L-acts are determined in the first stage, and outcomes in the second. This reminds of the approach in Segal (1987), where ambiguous decision acts are modeled as two-stage lotteries and ambiguity is interpreted as uncertainty about

<sup>&</sup>lt;sup>18</sup> Savage (1972), pp. 82.

<sup>&</sup>lt;sup>19</sup> Villegas (1964), Theorem 2, p. 1794.

probability distributions. Crucial for Segal's approach is the assumption that the decision-maker does not apply the reduction of compound lotteries axiom.<sup>20</sup> Consequently, in his approach, two-stage decision acts are judged differently than single-stage acts, even when the probability distribution of outcomes is identical. In contrast, the approach presented here does not rely on this assumption. For illustrative purposes, it may be useful to imagine a stepwise resolution of uncertainty, but it is not part of the modeling of the decision situation. The representation of decision acts in Theorem 1 does not result from a temporal order of *S*-events and *L*-events but from their independence according to the definition of *L*-partitions and Lemma 4. Consequently, the temporal resolution of uncertainty does not matter for the validity of Theorem 1.

Theorem 1 can serve as a basis for decision theories that determine the utility of *L*-acts in different ways. Given Lemma 4, one possibility is to consider so-called horse lotteries with roulette lotteries in terms of *S*-acts as consequences, thus providing a subjective foundation for the modeling framework of Anscombe and Aumann (1963). The approach taken here involves relating *L*-acts and *S*-acts, thus establishing a link between risk and ambiguity. To this end, some additional concepts must be introduced: An *S*-partition  $\{D_i\}$  with i = 1, ..., n is uniform if  $f D_i g \sim f D_j g$  holds for all  $f, g \in F_L$  and all i, j. Hence, all events forming a uniform *S*-partition have the same probability. The elements of a uniform *S*-partition with two elements are called  $R_S$ -events. Two uniform *S*-partitions  $\{D_i\}$  with i = 1, ..., n and  $\{E_j\}$  with j = 1, ..., m are independent if the events  $D_i \cap E_j$  form a uniform *S*-partition. Using these concepts, Axiom 10 is stated as follows:

**Axiom 10** (*SL*-Isometry) Let *A* be an  $R_L$ -event, *B* an  $R_S$ -event,  $\{D_i\}$ ,  $\{E_i\}$  with i = 1, ..., n independent uniform *S*-partitions, and  $\{C_i\}$  with i = 1, ..., n an *L*-partition.  $\{E_i\}$  and  $\{B, B^C\}$  are independent. Then for all  $x_i, x_j \in X$ 

$$(F_{i=1}^n x_i D_i) A (F_{i=1}^n x_i E_i) \sim (F_{i=1}^n x_i C_i) B (F_{i=1}^n x_i E_i).$$

For the interpretation of Axiom 10, note that the indifference principle assigns the same probability to all elements of an *L*-partition. Hence,  $F_{i=1}^n x_i C_i$ ,  $F_{i=1}^n x_i D_i$ , and  $F_{i=1}^n x_i E_i$  produce identical outcomes with identical probabilities. Acts with this property will be called risk-equivalent. With this in mind, Axiom 10 states that the decision-maker is indifferent between an ambiguous mixture of two independent risk-equivalent *S*-acts and an unambiguous mixture of one of these *S*-acts and a riskequivalent *L*-act. Consequently, Axiom 10 makes it possible to trace the effects of ambiguity back to a systematic juxtaposition of two outcomes and the corresponding events.

Axiom 10 refers to uniform and independent *S*-partitions as devices for the mixture of *L*-acts and *S*-acts. The existence of these partitions is ensured by the following lemma, which essentially goes back to Savage:

<sup>&</sup>lt;sup>20</sup> Segal (1987), p. 177.

**Lemma 5** For all S-events A and all  $r \in [0, 1]$  there is an S-event  $B \subseteq A$  such that  $P_S(B) = rP_S(A)$ . It follows that

- there is an R<sub>s</sub>-event.
- there is a uniform S-partition  $\{D_i\}$  with i = 1, ..., n for all  $n \in \mathbb{N}$ .
- for all uniform S-partitions {D<sub>i</sub>} with i = 1,..., n there is an S-partition {E<sub>j</sub>} with j = 1,..., m that is independent from {D<sub>i</sub>}.

For the first statement see Savage (1972).<sup>21</sup> The other statements are immediate implications. Next, the existence of at least one  $R_L$ -event must be guaranteed:

Axiom 11 (Existence of  $R_L$ -events) There is an  $R_L$ -event.

Axiom 11 is a departure from SEU since, according to Lemma 4, an  $R_L$ -event is not an S-event. From a technical point of view, an  $R_L$ -event is needed for a constructive proof of the intended representation theorem using Axiom 10. Since the indifference principle implies equal probabilities for an  $R_L$ -event and its complement, Axiom 11 reminds of Ramsey's (1931) postulate of the existence of an ethically neutral event. Here, the required "neutrality" includes the idea that the event carries no decision-relevant information about S-events, which follows from the definition of L-partitions and Lemma 4.

With the introduction of Axiom 11, the prerequisites for the utilization of Axiom 10 have been completed. As a first consequence, Axiom 10 implies that preferences for *L*-acts depend only on the possible outcomes and the cardinality of the relevant *L*-partition:

**Lemma 6** Let  $\{C_i\}$  and  $\{D_i\}$  with i = 1, ..., n be arbitrary L-partitions. Then  $F_{i-1}^n x_i C_i \sim F_{i-1}^n x_i D_i$  holds for all  $x_i \in X$ .

Lemma 6 ensures that preferences for outcomes of *L*-acts can be separated from beliefs concerning *L*-events.

**Lemma 7** For all L-acts h = xAz with A an arbitrary  $R_L$ -event and outcomes x > z,  $\frac{3}{4}U(x) + \frac{1}{4}U(z) \ge U(h) \ge \frac{1}{4}U(x) + \frac{3}{4}U(z)$ .

Lemma 7 is an implication of Axiom 4 and Axiom 10 and shows that the utility of an L-act according to Theorem 1 is constrained by the utilities of possible outcomes, which excludes violations of the monotonicity principle.

In the next step, a utility function  $u : X \times X \to \mathbb{R}$  is introduced that will be used for the representation of preferences for *L*-acts. *u* is symmetric if u(x, z) = u(z, x) for all  $x, z \in X$ , and *u*-monotonic if  $u(x, x) \ge u(x, z) \ge u(z, z)$  holds for all  $x, z \in X$  with  $u(x, x) \ge u(z, z)$ .

<sup>&</sup>lt;sup>21</sup> Savage (1972), pp. 34.

**Lemma 8** There is a symmetric and u-monotonic function  $u : X \times X \to \mathbb{R}$  such that

$$U(F_{i=1}^{n}x_{i}C_{i}) = \frac{1}{n^{2}}\sum_{i=1}^{n}\sum_{j=1}^{n}u(x_{i},x_{j})$$

for all  $x_i, y_i \in X$  and all L-partitions  $C_i$  with i = 1, ..., n.

As the proof of Lemma 8 shows, *u* can be constructed starting from the utility function *U* in Theorem 1 by setting u(x, x) = U(x) for all  $x \in X$ . Since *u* has two arguments, however, it allows representing differing preferences for *S*-acts and *L*-acts. To establish a preference representation for all simple *L*-acts, a further implication of Axiom 4 is needed:

**Lemma 9** Each partition  $\{C_k\}$  of  $\Omega$  with  $C_k \in \mathcal{L}$  for all k has a finite number of nonnull events.

Lemma 9 is used, among other things, for the construction of a countably additive probability measure on  $\mathscr{L}$ . As a consequence of Lemma 8 and Lemma 9, the following result concerning preferences for simple *L*-acts can be derived:

**Theorem 2** (Representation of preferences for *L*-acts) Suppose Axioms 1 to 11 hold. Then there is a countably additive probability measure  $P_L : \mathcal{L} \to [0, 1]$  and a symmetric and u-monotonic function  $u : X \times X \to \mathbb{R}$  such that

$$F_{k=1}^{r} x_{k} C_{k} \leq F_{k=1}^{r} y_{k} C_{k} \Leftrightarrow$$

$$\sum_{k=1}^{r} \sum_{k'=1}^{r} u(x_{k}, x_{k'}) P_{L}(C_{k}) P_{L}(C_{k'}) \leq \sum_{k=1}^{r} \sum_{k'=1}^{r} u(y_{k}, y_{k'}) P_{L}(C_{k}) P_{L}(C_{k'})$$

for all  $x_k, y_k \in X$  and all  $C_k \in \mathcal{L}$  with k = 1, ..., r.  $P_L$  is uniquely determined, and u is unique up to positive linear transformations.

For u(x, z) = 0.5u(x, x) + 0.5u(z, z), the preference functional for *L*-acts resulting from Theorem 2 is identical to that for *S*-acts in Theorem 1. Hence, *L*-acts and *S*-acts are treated the same way, indicating that the decision-maker has a neutral attitude toward ambiguity. For u(x, z) < 0.5u(x, x) + 0.5u(z, z), the decision-maker is ambiguity-averse. Substitution of  $v(x_k) = \sum_{k'=1}^{r} u(x_k, x_{k'})P_L(C_{k'})$  and  $v(y_k) = \sum_{k'=1}^{r} u(y_k, y_{k'})P_L(C_{k'})$  shows that he or she still maximizes expected utility. The utility function, however, is modified in such a way that it effectively overweights less preferred outcomes. Moreover, the utility function is act-specific due to the inapplicability of the Sure Thing principle. The quadratic form of the preference functional in Theorem 2 was studied in detail in Chew et al. (1991) in the context of relaxations of the independence axiom. For  $u(x, z) = \min\{u(x, x), u(z, z)\}$ , it reflects a

special case of the rank-dependent utility theory of Quiggin (1982) and others.<sup>22</sup> The relationship is comparable to that between the approach developed here and CEU discussed in the introduction.

The final idea in this section is that all decision-relevant events can be interpreted as mixtures of S-events and L-events. To this end,  $\mathcal{K}$  is defined as the  $\sigma$ -algebra generated by  $\mathcal{S}$  and  $\mathcal{L}$ . The following axiom ensures that  $\mathcal{K}$  encompasses all decision-relevant events:

**Axiom 12** (Decision-relevant events) For all  $A \in \mathscr{A}$  there is an event  $B \subseteq A$  with  $B \in \mathscr{H}$  such that  $fA \setminus Bh \sim gA \setminus Bh$  for all  $f, g, h \in F$ .

 $A \setminus B$  in Axiom 12 is a null event. As null events are S-events,  $\mathscr{A}$  is the  $\sigma$ -algebra generated by  $\mathscr{S}$  and  $\mathscr{L}$ , i.e.,  $\mathscr{A} = \mathscr{K}$ .

By merging Theorem 1 and Theorem 2, the central result can now be stated as follows:

**Theorem 3** (*SL* expected utility theory) Suppose Axioms 1 to 12 hold. Then there are two  $\sigma$ -algebras  $\mathscr{S}$  and  $\mathscr{L}$  with  $\mathscr{S} \cap \mathscr{L} = \mathscr{N} \cup \mathscr{U}$  that jointly generate the  $\sigma$ -algebra  $\mathscr{A}$ , a countably additive probability measure  $P : \mathscr{A} \to [0, 1]$ , an *SL*-decomposition  $Q : \mathscr{A} \times \mathscr{A} \to [0, 1]$  of *P*, and a symmetric and u-monotonic function  $u : X \times X \to \mathbb{R}$ such that

$$F_{i=1}^{n} x_{i} A_{i} \leq F_{i=1}^{n} y_{i} A_{i} \Leftrightarrow$$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} u(x_{i}, x_{j}) Q(A_{i}, A_{j}) \leq \sum_{i=1}^{n} \sum_{j=1}^{n} u(y_{i}, y_{j}) Q(A_{i}, A_{j})$$

for all  $x_i, y_i \in X$  and all  $A_i \in \mathcal{A}$  with i = 1, ..., n. P and Q are uniquely determined, and u is unique up to positive linear transformations.

Theorem 3 uses the *SL*-decomposition Q defined in the last section and generalizes Theorem 1 and Theorem 2. If Q(A, A) = P(A) for all  $A \in \mathscr{A}$  or u(x, z) = 0.5u(x, x) + 0.5u(z, z) for all  $x, z \in X$ , the preference functional corresponds to SEU, i.e., Theorem 1. If Q(A, B) = P(A)P(B) for all  $A, B \in \mathscr{A}$ , the preference functional for *L*-acts according to Theorem 2 applies. Taken together, the diverging ambiguity of events and decision acts as well as different attitudes toward ambiguity can be represented, as discussed in the next section.

#### 4 Measurement of ambiguity and ambiguity aversion

Ambiguity can be measured with respect to events and to decision acts. The *SL*-decomposition and its interpretation suggest the following measure of event ambiguity:

<sup>&</sup>lt;sup>22</sup> Chew et al. (1991), p. 145.

**Definition 3** (Event ambiguity) The ambiguity of an event A is measured by  $\alpha(A) = Q(A, A^C)$ .

Definition 3 is intuitively plausible since  $Q(A, A^C)$  measures the part of the probability of event A that is not based on event-specific knowledge. In a more formal way,  $\alpha$  can be introduced by first defining a qualitative ambiguity relation and then showing that  $\alpha$  represents this relation. To this end, let  $x, z \in X$  be arbitrary outcomes with x > z. Further, for all events  $A \in \mathcal{A}$ , construct S-events  $A_S$  and  $A_S^C$  with  $xAz \sim xA_Sz$  and  $xA^Cz \sim xA_S^Cz$ , respectively, applying Theorem 3 and Lemma 5. An ambiguity-averse decision-maker prefers to bet on unambiguous events, hence  $A_S \cap A_S^C = \emptyset$  can be assumed without loss of generality.<sup>23</sup> A qualitative ambiguity relation  $\leq$  is then defined as follows: Event A is (weakly) less ambiguous than event B,  $A \leq B$ , if  $xA_S \cup A_S^C z \geq xB_S \cup B_S^C z$  holds for an ambiguity-averse decision-maker. Theorem 3 implies  $xA_S \cup A_S^C z \geq xB_S \cup B_S^C z$  iff  $Q(A, A^C) \geq Q(B, B^C)$  for u(x, z) < 0.5u(x, x) + 0.5u(z, z). Consequently,  $\alpha$  represents the ambiguity relation  $\leq$ . In view of Theorem 3, it is evident that  $\leq$  is independent of the defining outcomes and forms a weak order.

Another approach to introduce a measure of event ambiguity is to first establish properties that this measure should satisfy and then to develop an appropriate axiomatic system. Fishburn (1993) lists the following properties<sup>24</sup>:

- 1.  $A \leq B \Rightarrow \alpha(A) \leq \alpha(B)$  and  $A < B \Rightarrow \alpha(A) < \alpha(B)$
- 2.  $\alpha(\emptyset) = 0$  and  $\alpha(A) \ge 0$

3.  $\alpha(A) = \alpha(A^C)$ 

4.  $\alpha(A \cup B) + \alpha(A \cap B) \le \alpha(A) + \alpha(B)$ 

The proposed measure satisfies all these properties, which further underlines its plausibility. Property 1 requires  $\alpha$  to preserve the qualitative ambiguity relation, which follows from  $A \leq B \Leftrightarrow \alpha(A) \leq \alpha(B)$  derived above. Properties 2 to 4 can be readily deduced from the properties of the *SL*-decomposition listed in Section 2.<sup>25</sup> Properties 3 and 4, complementary equality and submodularity, respectively, are considered to be central to the distinction between event ambiguity and event probability. Complementary equality reflects the intuitively plausible idea that the ambiguity of an event is equal to that of its complement. Submodularity implies that the ambiguity of the union of two disjoint events is less than the sum of the stand-alone ambiguities.

To define a measure of act ambiguity, it is supposed that the SL-decomposition Q and the utility function u are given and that all outcomes are

 $<sup>\</sup>overline{^{23}}$  This follows from  $P(A_S) < P(A)$  and  $P(A_S^C) < P(A^C)$ .

<sup>&</sup>lt;sup>24</sup> Fishburn (1993), pp. 120.

<sup>&</sup>lt;sup>25</sup> Submodularity follows from  $Q(A \cup B, (A \cup B)^C) + Q(A \cap B, (A \cap B)^C) = P(A \cup B) - Q(A \cup B, A \cup B) + P(A \cap B) - Q(A \cap B, A \cap B) = P(A) + P(B) - Q(A, A) - Q(B, B) - Q(A, B) - Q(B, A) - Q(A \cap B, A \cap B) \le P(A) - Q(A, A) + P(B) - Q(B, B) = Q(A, A^C) + Q(B, B^C).$ 

measurable on a cardinal scale.  $q : \mathbb{R}^2 \to [0, 1]$  with  $q_{ij} = q(x_i, x_j) = Q(A_i, A_j)$ for  $A_i = \{s \in \Omega : f(s) = x_i\}$  and  $A_j = \{s \in \Omega : f(s) = x_j\}$  is referred to as *SL*distribution of decision act *f*. As a consequence of the properties of the *SL*decomposition, *q* can be interpreted as a symmetric two-dimensional probability distribution with marginal distribution *p* where  $p_i = p(x_i) = P(A_i)$  holds for  $A_i = \{s \in \Omega : f(s) = x_i\}$ . Based on *p*, the expected value  $\mu$  and the variance  $\sigma^2$ of outcomes can be calculated as usual. Two *SL*-distributions with identical marginal distributions are called risk-equivalent.

Based on the *SL*-distribution q, a convenient way to measure act ambiguity is as follows:

**Definition 4** (Act ambiguity) The ambiguity of an *SL*-distribution q with marginal distribution p is measured by  $\varphi = \sigma^2 - c$ , where  $\sigma^2$  denotes the variance of p and c the covariance of q.

 $0 \le c \le \sigma^2$  and, consequently,  $0 \le \varphi \le \sigma^2$  result from the characteristics of the *SL*-decomposition.  $\varphi = 0$  reflects a perfectly unambiguous outcome distribution where all relevant events are *S*-events.  $\varphi = \sigma^2$  results when act ambiguity is perfect, i.e., probabilities for relevant events are not based on event-specific knowledge but on the indifference principle. In all other cases, the decision act is more or less ambiguous since the outcomes depend on the occurrence of *S*-events and *L*-events.

**Example 4** (Measurement of Act ambiguity) The monetary outcome x of an investment depends on a perfectly unambiguous factor  $z_s$  and a perfectly ambiguous factor  $z_L$ ,  $x = z_S + z_L$ . Possible realizations of  $z_S$  are 0, 100, 200, or 300 with underlying S-events B<sub>1</sub>, B<sub>2</sub>, B<sub>3</sub>, and B<sub>4</sub>, and probabilities 0.2, 0.3, 0.4, and 0.1, respectively.  $z_L$  results in -100, 0, or 100 with underlying L-events  $C_1$ ,  $C_2$ , and  $C_3$ , and probabilities 0.2, 0.6, and 0.2, respectively. Overall, the expected outcome is  $\mu = \mu_S + \mu_L = 140 + 0 = 140$ . From the properties of S-events and L-events,  $z_{\rm S}$  and  $z_{\rm L}$  are necessarily uncorrelated, hence the outcome variance is  $\sigma^2 = \sigma_s^2 + \sigma_L^2 = 8,400 + 4,000 = 12,400$ . To determine the relevant *SL*-distribution, note that  $z_S$  is perfectly unambiguous, i.e.,  $q(z_S^i, z_S^{i'}) = p(z_S^i)$  for i = i' and  $q(z_S^i, z_S^{i'}) = 0$ otherwise, while  $z_L$  is perfectly ambiguous, hence  $q(z_I^j, z_I^{j'}) = p(z_I^j)p(z_I^{j'})$  for all j, j'. The resulting SL-distribution is shown in Table 5. For example, the properties of SL-decompositions imply  $q(200, 300) = Q((B_1 \cap C_3) \cup (B_2 \cap C_2) \cup (B_3 \cap C_1),$  $(B_2 \cap C_3) \cup (B_3 \cap C_2) \cup (B_4 \cap C_1)) = Q(B_2 \cap C_2, B_2 \cap C_3) + Q(B_3 \cap C_1, B_3 \cap C_2) = Q(B_2 \cap C_3) + Q(B_3 \cap C_1) + Q(B_3 \cap C_2) = Q(B_3 \cap C_3) + Q(B_3 \cap C_3) +$  $P(B_2)P(C_1)P(C_3) + P(B_3)P(C_1)P(C_2) = 0.06$ . The covariance of q is c = 8,400. Consequently, according to Definition 4,  $\varphi = 4,000$  is calculated for the ambiguity of the investment. It results solely from the perfectly ambiguous factor  $z_L$  and equals its ambiguity  $\varphi_L$ .

| Table 5         SL-distribution of           investment outcome |      | -100  | 0     | 100   | 200   | 300   | 400   | р    |
|---|------|-------|-------|-------|-------|-------|-------|------|
|   | -100 | 0.008 | 0.024 | 0.008 | 0     | 0     | 0     | 0.04 |
|   | 0    | 0.024 | 0.084 | 0.060 | 0.012 | 0     | 0     | 0.18 |
|   | 100  | 0.008 | 0.060 | 0.132 | 0.084 | 0.016 | 0     | 0.30 |
|   | 200  | 0     | 0.012 | 0.084 | 0.160 | 0.060 | 0.004 | 0.32 |
|   | 300  | 0     | 0     | 0.016 | 0.060 | 0.052 | 0.012 | 0.14 |
|   | 400  | 0     | 0     | 0     | 0.004 | 0.012 | 0.004 | 0.02 |
|   | р    | 0.04  | 0.18  | 0.30  | 0.32  | 0.14  | 0.02  | 1    |
|   |      |       |       |       |       |       |       |      |

As a starting point for the consideration of ambiguity aversion, note that risk aversion can be defined and measured with regard to the utility function U(x) = u(x, x) in the usual way. To define a measure of ambiguity aversion remind that an ambiguity-neutral decision-maker acts as if all decision acts were *S*-acts. This is the case if and only if  $\sum_{i=1}^{n} \sum_{j=1}^{n} u(x_i, x_j)q_{ij} = \sum_{i}^{n} u(x_i, x_i)p_i$  holds for all *SL* -distributions *q* with marginal distributions *p*. Accordingly, a decision-maker is (strictly) ambiguity-averse if  $\sum_{i=1}^{n} \sum_{j=1}^{n} u(x_i, x_j)q_{ij} < \sum_{i}^{n} u(x_i, x_i)p_i$  for all  $f \in F \setminus F_S$ . Thus, an ambiguity-averse decision-maker strictly prefers an *S*-act against all other decision acts that have the same probability distribution of outcomes but are not *S*-acts. Monotony and differentiability presupposed, a necessary and sufficient condition for this to hold is  $u_{11}(x, z) > 0$  for all  $x, z \in X$ . Consequently, a measure of ambiguity aversion can be defined as follows:

**Definition 5** (Ambiguity aversion) The local ambiguity aversion of a decision-maker is measured by  $A(x, z) = \frac{u_{11}(x, z)}{u_{10}(x, z) + u_{01}(x, z)}$ 

If A(x, z) has a negative sign, the decision-maker is ambiguity-loving. The measure A(x, z) remains, like the Arrow/Pratt measure, unaffected by positive linear transformations of the utility function. Note that ambiguity aversion is in general independent of risk aversion, i.e., a risk-averse decision-maker can be ambiguity-neutral and a risk-neutral decision-maker can be ambiguity-averse. The measurement of ambiguity aversion is illustrated by example 5.

#### **Example 5** (Measurement of Ambiguity aversion)

Suppose  $u(x, z) = 0.5(x + z) - 0.5(a + b)(x^2 + z^2) + bxz$ . The domain of *u* is restricted to outcomes in the interval [0, 1]. To ensure that *u* exhibits weak risk aversion and weak ambiguity aversion,  $a, b \ge 0$  and a + b < 0.5 is presupposed. The Arrow/Pratt measure is  $R(x) = \frac{2a}{1-2ax}$ . For a > 0, *u* indicates increasing absolute risk aversion as typical for the quadratic utility function in SEU. For b > 0, the measure of ambiguity aversion  $A(x, z) = \frac{b}{1-a(x+z)}$  reflects absolute ambiguity aversion increasing in both arguments. In analogy to the  $\mu$ - $\sigma^2$ -principle, the following result for all simple *SL*-distributions *q* with support [0, 1] can be derived:

$$\sum_{i=1}^{n} \sum_{j=1}^{n} u(x_i, x_j) q_{ij} = \mu - a(\mu^2 + \sigma^2) - b\varphi$$

The last term on the right-hand side refers to the measure of act ambiguity according to Definition 4. Given  $\mu$  and  $\sigma^2$ , preference increases with decreasing ambiguity when the decision-maker is ambiguity-averse, i.e., b > 0. For b = 0, the decisionmaker is ambiguity-neutral and follows SEU with  $U(x) = x - ax^2$ . If risk aversion is held constant, the ambiguity-neutral decision-maker achieves the same utility level as the ambiguity-averse decision-maker with a risk-equivalent but perfectly unambiguous act.

#### 5 Concluding remarks

A central feature of the proposed theory is the possibility to separate ambiguity and ambiguity attitudes in a relatively simple way. Together with the additivity of the *SL*-decomposition, this allows analyzing the consequences of ambiguity and ambiguity aversion in model-theoretic and empirical studies in quite the same way as the consequences of risk and risk aversion. In doing so, all results concerning risk remain valid as the theory presented is an extension of SEU. From a technical point of view, it is particularly advantageous that the *SL*-distributions used to model decision acts can be interpreted as two-dimensional probability distributions. This makes it possible to apply concepts from probability theory when modeling and measuring ambiguity. In this respect, the proposed theory is similar to approaches that use second-order probabilities to model ambiguity.

As for the axiomatic basis of the proposed theory, the number of axioms could be an issue. The presented axiomatic system includes twelve axioms, while Savage's and Gilboa's axiomatizations of SEU and CEU, respectively, each consist of six axioms. Reducing the number of axioms by merging axioms (e.g., Axioms 4 and 6) is possible, but only to a limited extent and at the cost of more complicated and less plausible axioms. This is also a consequence of the model framework which is subject to comparatively few restrictions. In general, the number of axioms decreases with increasing structural specifications. Axiomatic systems based on Savage's approach, for example, usually include more axioms than those based on Anscombe and Aumann's framework. With this in mind and given the higher complexity resulting from the distinction between *S*-events and *L*-events, it seems questionable whether the axiomatic basis of the proposed theory can be reduced to a significantly smaller number of axioms without imposing additional structural specifications.

Considering the axioms in detail, Axiom 10 and Axiom 12 are of particular interest. Axiom 10 connects preferences for *S*-acts and *L*-acts thus establishing a link between risk and ambiguity. The central question here is whether Axiom 10 leads to reasonable and empirically tenable hypotheses about ambiguity-related decision-making. Axiom 12 raises some interesting epistemic questions, for example: Are there possibilities of assigning probabilities to events beyond knowledge-based separation and indifference-based dispersion? Or, more closely to the topic of this paper: Are there sources of ambiguity that With regard to the latter point, it could also be argued that the presented approach basically considers only perfectly unambiguous and perfectly ambiguous events and that there are ambiguous events that cannot be traced back to such events. The idea is that if necessary, the  $\sigma$ -algebra of decision-relevant events can always be extended in such a way that suitable *L*-events and *S*-events can be identified. If this is the case, the argument leads to the debate as to whether axioms in decision theory may only refer to actually existing events or whether fictitious events may also be used for axiomatization. Savage's P6, for example, was criticized for its use of infinitely divisible events. Today, most decision theorists seem to agree that referring to fictitious events is acceptable.

From the constructive proof of the theorems in the last section, it is easy to derive a procedure for eliciting the SL decomposition Q and the utility function u of an individual decision-maker. An issue here is that in this context events are needed that are S-events and/or L-events from the decision-maker's perspective. The situation is comparable to elicitation techniques for SEU that use certain events. As with certain events, different decision-makers may interpret different events as S-events or L-events. For a purely subjective theory, this is not a conceptual problem. It requires, however, special care in the conceptualization of elicitation procedures.

Further research questions are manifold. Of particular interest is an ongoing investigation of the relationship between ambiguity and information. In the context of the presented theory, it seems plausible to assume that subjective probabilities should be updated according to Bayes' rule. However, it is by no means self-evident how the *SL*-decomposition must be adjusted. This may depend on the extent to which the information concerns *S*-events or *L*-events. An investigation of this issue could contribute to a better understanding of the relationship between ambiguity and information and also raise new questions concerning ambiguity-related behavior.

#### Appendix

**Proof of Lemma 1**  $\emptyset$  and *S* are *S*-events according to Definition 1 and Axiom 1. Axiom 2 implies  $A^C$  is an *S*-event if *A* is an *S*-event. Suppose  $hA \cup Bf = hA \setminus BhB \setminus AhA \cap Bf \leq hA \setminus BhB \setminus AhA \cap Bg = hA \cup Bg$  for any  $A, B \in \mathscr{S}$  and  $f, g, h \in F$ . Since  $A^C$  and  $B^C$  are *S*-events, Axiom 2 implies  $h'A \setminus BhB \setminus Ah'A \cap Bf \leq h'A \setminus BhB \setminus Ah'A \cap Bg$  and  $h'A \cup Bf \leq h'A \cup Bg$  for all  $h' \in F$ . Consequently,  $A \cup B \in \mathscr{S}$ .  $\mathscr{S}$  is an algebra. According to Axiom 3,  $\mathscr{S}$  is a  $\sigma$ -algebra.

**Proof of Lemma 2** Let  $\{C_i\}$  for i = 1, ..., n be any *L*-partition. Suppose  $C_j$  is a null event. For f = x and g = h = z with x > z Definition 2 yields  $x C_j z C_i z \sim z C_j x C_i z$  for all i. Axiom 1 and the definition of null events result in  $z \sim x C_i z$ . Axiom 4 implies  $C_i$  must also be a null event. Hence, all elements of the considered

*L*-partition are null events, which contradicts Axiom 8. Consequently, no element of an *L*-partition is a null event.  $\Box$ 

**Proof of Lemma 3** According to Axiom 5, for all  $A \in \mathscr{L}(\mathscr{N} \cup \mathscr{U})$  there is an *L*-partition  $\{D_i\}$  with i = 1, ..., n such that  $xA \cap D_i z \sim xD_i z$  or  $xA \cap D_i z \sim z$  for all  $x, z \in X$ . Assume without loss of generality  $xA \cap D_i z \sim xD_i z$  for i = 1, ..., j,  $A \cap D_i \in \mathscr{N} \setminus \emptyset$  for i = j + 1, ..., k, and  $A \cap D_i = \emptyset$  for i = k + 1, ..., n. Then define  $C_1 = (A \cap D_1) \cup \bigcup_{i=j+1}^k (A \cap D_i), C_i = (A \cap D_i)$  for  $i = 2, ..., j, C_i = D_i \setminus (A \cap D_i)$  for i = j + 1, ..., k,  $C_i = D_i$  for i = k + 1, ..., n - 1, and  $C_n = D_n \cup \bigcup_{i=1}^j D_i \setminus (A \cap D_i)$ .  $D_i \setminus (A \cap D_i)$  for i = 1, ..., j are null events according to Axiom 4. Hence,  $\{C_i\}$  with i = 1, ..., n is an *L*-partition according to Definition 2, the definition of null events and Axiom 1. Moreover,  $A \cap C_i = C_i$  or  $A \cap C_i = \emptyset$  for all i.

**Proof of Lemma 4** According to Lemma 3, for all  $A \in \mathscr{L}(\mathcal{N} \cup \mathscr{U})$  there is an *L*-partition  $\{C_i\}$  with i = 1, ..., n such that  $A = \bigcup_{i=1}^{j} C_i$  with  $1 \le j < n$ . Suppose,  $\bigcup_{i=1}^{j} C_i$  with  $1 \le j < n$  is an *S*-event. According to Lemma 2,  $A^C$  is also an *S*-event. For *S*-acts f = xAz, g = h = z with x > z, Definition 2 yields  $xC_j z = fC_j gC_{j+1}h \sim gC_j fC_{j+1}h = z$ . Since  $C_j$  is not a null event according to Lemma 2, there is a contradiction to Axiom 4. Consequently, *A* cannot be an *S*-event.  $\mathscr{L}$  contains no *S*-events except null events and universal events.

**Proof of Lemma 6** Considering first the case n = 2,  $R_S$ -events D, E, and B are constructed using Lemma 5.  $\{D, D^c\}$  and  $\{E, E^c\}$  as well as  $\{B, B^c\}$  and  $\{E, E^c\}$  are presumed to be independent. Let C and A be arbitrary  $R_L$ -events. For L-acts  $g = x_1 A x_2$ ,  $g' = x_2 A x_1$  and  $f = x_1 C x_2$  with arbitrary  $x_1, x_2 \in X$ , Axiom 10 implies:

$$(x_1 D x_2) A (x_1 E x_2) = x_1 D \cap E x_2 g D \cap E^c g' D^c \cap E x_2 \sim f B (x_1 E x_2)$$

Theorem 1 results in:

$$\frac{1}{4}\left(U(x_1) + U(g) + U(g') + U(x_2)\right) = \frac{1}{2}U(f) + \frac{1}{4}\left(U(x_1) + U(x_2)\right)$$

U(g) = U(g') follows from Definition 2 and Theorem 1. Hence, U(g) = U(f) and therefore  $x_1 A x_2 \sim x_1 C x_2$ .

Concerning the case n > 2, Axiom 10 and Axiom 1 imply:

$$(F_{i=1}^{n} x_{i} D_{i}) B(F_{i=1}^{n} x_{i} E_{i}) \sim (F_{i=1}^{n} x_{i} C_{i}) B(F_{i=1}^{n} x_{i} E_{i})$$

for arbitrary  $x_i \in X$ , where  $\{B, B^c\}$  and  $\{E_i\}$  are independent uniform *S*-partitions. With  $g = F_{i=1}^n x_i D_i$  and  $f = F_{i=1}^n x_i C_i$ , U(g) = U(f) and thus  $F_{i=1}^n x_i D_i \sim F_{i=1}^n x_i C_i$  follows from Theorem 1.

**Proof of Lemma 7** Lemma 5 is used to construct an  $R_s$ -event B, and independent uniform S-partitions  $\{D_i^n\}$  and  $\{E_i^n\}$  with i = 1, ..., n for  $n = 2, 3, ..., \{B_i^n\}$  and  $\{E_i^n\}$  are presumed to be independent. Then, let A be an arbitrary  $R_L$ -event, and  $\{C_i^n\}$  with i = 1, ..., n L-partitions for n = 2, 3, ... Consider h = xAz, and h' = zAx with

x > z. Lemma 6 and Theorem 1 imply U(h) = U(h'). According to Axiom 10,  $(xD_1^n z)A(xE_1^n z) \sim (xC_1^n z)B(xE_1^n, z)$  holds for all *n*. With  $f^n = xC_1^n z$ , Theorem 1 implies:

$$\frac{1}{n^2} U(x) + \frac{2(n-1)}{n^2} U(h) + \frac{(n-1)^2}{n^2} U(z) = \frac{1}{2} U(f^n) + \frac{1}{2n} U(x) + \frac{n-1}{2n} U(z).$$

Axiom 4 and Lemma 2 yield  $x C_1^n z > z$  and, hence,  $U(f^n) > U(z)$  according to Theorem 1. Without loss of generality, U(z) > 0 can be assumed. Thus,

$$\frac{1}{2}U(z) + \frac{1}{2n}U(x) + \frac{n-1}{2n}U(z) < \frac{1}{n^2}U(x) + \frac{2(n-1)}{n^2}U(h) + \frac{(n-1)^2}{n^2}U(z)$$

and consequently

$$\frac{1}{4}\frac{n-2}{n-1}U(x) + \frac{1}{4}\frac{3n-2}{n-1}U(z) < U(h).$$

Calculation of the limit shows  $\frac{1}{4}U(x) + \frac{3}{4}U(z) \le U(h)$ . Analogously,  $\frac{3}{4}U(x) + \frac{1}{4}U(z) \ge U(h)$  follows from considering  $g^n = z C_1^n x$ .

**Proof of Lemma 8** Applying Lemma 5, an  $R_S$ -event *B* and uniform *S*-partitions  $\{D_i\}$  and  $\{E_i\}$  with i = 1, ..., n are constructed in such a way that  $\{B, B^C\}$  and  $\{E_i\}$  as well as  $\{D_i\}$  and  $\{E_i\}$  are independent. According to Axiom 11, there is an  $R_L$ -event *A* that is used for the construction of *L*-acts  $h_{ii'} = x_i A x_{i'}$ . Axiom 10 implies:

$$F_{i=1}^{n} (F_{i'=1}^{n} (x_{i} A x_{i'}) D_{i}) E_{i'} = (F_{i=1}^{n} x_{i} D_{i}) A (F_{i=1}^{n} x_{i} E_{i}) \sim f B (F_{i=1}^{n} x_{i} E_{i})$$

Theorem 1 results in:

$$\frac{1}{n^2} \sum_{i=1}^n \sum_{i'=1}^n U(h_{ii'}) = \frac{1}{2} U(f) + \frac{1}{2n} \sum_{i=1}^n U(x_i)$$

 $x_i = x_{i'}$  implies  $U(h_{ii'}) = U(x_i) = U(x_{i'})$ .  $x_i \sim x_{i'}$  results in  $U(h_{ii'}) = U(x_i) = U(x_{i'})$ according to Axiom 4 and Axiom 1. Without loss of generality,  $x_i > x_{i'}$  is presumed. Define  $u : X \times X \to \mathbb{R}$  by  $u(x_i, x_i) = U(x_i)$ ,  $u(x_{i'}, x_{i'}) = U(x_{i'})$  and  $u(x_i, x_{i'}) = 2U(h_{ii'}) - \frac{1}{2}U(x_i) - \frac{1}{2}U(x_{i'})$ . u is symmetric since  $U(h_{ii'}) = U(h_{i'i})$ . According to Lemma 7,

$$\frac{3}{4} U(x_i) + \frac{1}{4} U(x_{i'}) \ge U(h_{ii'}) \ge \frac{1}{4} U(x_i) + \frac{3}{4} U(x_{i'})$$

Thus,  $u(x_i, x_i) \ge u(x_i, x_{i'}) \ge u(x_{i'}, x_{i'})$ , i.e., *u* is *u*-monotonic. Solving  $u(x_i, x'_i)$  for  $U(h_{ii'})$  and substituting yields

$$\frac{1}{n^2} \sum_{i=1}^n \sum_{i'=1}^n \left( \frac{1}{4} u(x_i, x_i) + \frac{1}{4} u(x_{i'}, x_{i'}) + \frac{1}{2} u(x_i, x_{i'}) \right) = \frac{1}{2} U(f) + \frac{1}{2n} \sum_{i=1}^n u(x_i, x_i).$$
nce
$$U(f) = \frac{1}{2} \sum_{i=1}^n \sum_{i=1}^n u(x_i, x_i)$$

Hence,  $U(f) = \frac{1}{n^2} \sum_{i=1}^n \sum_{i'=1}^n u(x_i, x_{i'}).$ 

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**Proof of Lemma 9** Let  $\{D_k\}$  with  $D_k \in \mathscr{L}$  be an arbitrary partition of  $\Omega$ . According to Axiom 5, there is an *L*-partition  $\{C_i\}$  with i = 1, ..., n such that  $xD_k \cap C_i z \sim xC_i z$ or  $xD_k \cap C_i z \sim z$  with x > z for all  $D_k$ . Axiom 4 implies  $D_k$  must be a null event if  $xD_k \cap C_i z \sim z$  holds for all i. Thus, for each  $D_k \in \mathscr{L} \setminus \mathscr{N}$ , there must be at least one  $C_i$  such that  $xD_k \cap C_i z \sim xC_i z$ . Axiom 4 implies  $C_i \setminus D_k \in \mathscr{N}$  in these cases. Hence, for this  $C_i, D_{k^*} \cap C_i = D_{k^*} \cap (C_i \setminus D_{k^*}) \in \mathscr{N}$  and therefore  $xD_{k^*} \cap C_i z \sim z$ for all  $k^* \neq k$ . Consequently,  $xD_k \cap C_i z \sim xC_i z$  with given  $C_i$  holds for at most one  $D_k$ . As  $\{C_i\}$  is finite, there is only a finite number of  $D_k \in \mathscr{L} \setminus \mathscr{N}$ .

**Proof of Theorem 2** According to Axiom 5, there is an L-partition  $\{D_i\}$  with i = 1, ..., n such that  $x C \cap D_i z \sim x D_i z$  or  $x C \cap D_i z \sim z$  with  $x \succ z$  for all  $C \in \mathscr{L}$ . Assume without loss of generality  $x C \cap D_i z \sim x D_i z$  for i = 1, ..., r and  $x C \cap D_i z \sim z$  otherwise. On this basis, define  $P_L : \mathscr{L} \to [0,1]$  by  $P_L(C) = \frac{r}{r}$ for  $C \in \mathscr{L} \setminus \mathscr{N}$  and  $P_L(C) = P_S(C) = 0$  for  $C \in \mathscr{N}$ . Obviously, P is finitely additive. For the proof of countable additivity, let  $\{C_k\}$  with  $k \in \mathbb{N}$  be an arbitrary collection of disjoint L-events.  $\bigcup_{k=1}^{\infty} C_k$  is an L-event since  $\mathscr{L}$  is a  $\sigma$ algebra. According to Lemma 9,  $\{C_k\}$  contains only a finite number of nonnull events. Assume without loss of generality  $C_k$  is nonnull for k = 1, ..., r. Note that null events are S-events and  $P_S$  is countably additive. Hence,  $P_L(\bigcup_{k=1}^{\infty} C_k) = P_L(\bigcup_{k=1}^{r} C_k) + P_S(\bigcup_{k=r+1}^{\infty} C_k) = \sum_{k=1}^{r} P_L(C_k) + \sum_{k=r+1}^{\infty} P_S(C_k) = \sum_{k=1}^{\infty} P_L(C_k) .$ To show that  $P_L$  is independent from  $\{D_i\}$  assume that there is another L-partition  $\{E_i\}$  fulfilling the requirements of Axiom 5. Then, according to Axiom 5,  $xD_i \cap E_i z \sim xE_i z \sim xD_i z$  or  $xD_i \cap E_j z \sim z$  for all  $E_j$  and  $D_i$ . Hence,  $\{E_j\}$  and  $\{D_i\}$  differ only by null events. Thus,  $P_L$  is independent from  $\{D_i\}$  and uniquely determined.

Now consider  $F_{k=1}^r x_k C_k$  and  $F_{k=1}^r y_k C_k$ . According to Axiom 4 and the definition of null events,  $C_k \in \mathscr{L} \setminus \mathscr{N}$  can be supposed without loss of generality. Axiom 5 implies that there is an *L*-partition  $\{E_i\}$  with i = 1, ..., n such that  $x C_k \cap E_i z \sim x E_i z$ or  $x C_k \cap E_i z \sim z$  with  $x \succ z$  for all  $C_k$ . Assume without loss of generality  $x C_k \cap E_i z \sim x E_i z$  for  $i = j_{k-1} + 1, ..., j_k$  with  $j_0 = 0$  and  $j_r = n$ , and  $x C_k \cap E_i z \sim z$ in all other cases. Hence,  $P_L(C_k) = \frac{j_k - j_{k-1}}{n}$ . Consider  $D_k = \bigcup_{i=j_{k-1}+1}^{j_k} E_i$  for k = 1, ..., r.  $\{D_k\}$  is a partition of  $\Omega$  with  $P_L(D_k) = P_L(C_k)$ . Furthermore,  $D_k$  and  $C_k$ differ only by null events. Hence, Axiom 1 and the definition of null events imply  $F_{k=1}^r x_k C_k \sim F_{k=1}^r x_k D_k$  resp.  $F_{k=1}^r y_k C_k \sim F_{k=1}^r y_k D_k$ . Furthermore,  $F_{k=1}^r x_k D_k \sim$   $F_{i=1}^n x_i E_i$  and  $F_{k=1}^r y_k D_k \sim F_{i=1}^n y_i E_i$  with  $x_i = x_k$  resp.  $y_i = y_k$  for  $i = j_{k-1} + 1, \dots, j_k$ . Lemma 8 and Axiom 1 imply

 $F_{k=1}^r x_k C_k \leq F_{k=1}^r y_k C_k \Leftrightarrow$ 

 $\sum_{k=1}^{r} \sum_{k'=1}^{r} u(x_k, x_{k'}) P_L(C_k) P_L(C_{k'}) \ge \sum_{k=1}^{r} \sum_{k'=1}^{r} u(y_k, y_{k'}) P_L(C_k) P_L(C_{k'}).$ 

linear this also holds positive Obviously, for transformations of *u*. Suppose u' is another symmetric and *u*-monotonic utility function that can be used to represent the preferences of the decision-maker. Considering S-acts, Theorem 1 implies u'(x, x) is a positive linear transformation of u(x, x). Moreover, considering an arbitrary L-act f = x C z,  $U'(f) = u'(x,x)P_L(C)^2 + 2u'(x,z)P_L(C)(1 - P_L(C)) + u'(z,z)(1 - P_L(C))^2$  must be a positive linear transformation of  $U(f) = u(x, x)P_L(C)^2 + 2u(x, z)P_L(C)(1 - P_L(C))$  $+u(z,z)(1-P_{I}(C))^{2}$ . Thus, u'(x,z) is a positive linear transformation of u(x,z). 

**Proof of Theorem 3** According to Axiom 12, for all  $A \in \mathscr{A}$  there is an event  $B \subseteq A$  with  $B \in \mathscr{H}$ , such that  $A \setminus B$  is a null event. As null events are *S*-events, *A* is the  $\sigma$ -algebra generated by  $\mathscr{S}$  and  $\mathscr{L}$ , i. e.  $\mathscr{A} = \mathscr{K}$ .  $\mathscr{S} \cap \mathscr{L} = \mathscr{N} \cup \mathscr{U}$  follows from Lemma 4. Consequently, for any  $A \in \mathscr{A}$  there are at most countable partitions  $\{B_j\}$  and  $\{C_k\}$  of *S*-events and *L*-events, respectively, such that  $A = \bigcup_{\{j,k:D_{jk} \subseteq A\}} D_{jk}$  with  $D_{jk} = B_j \cap C_k$ . Each partition of *L*-events has only a finite number of nonnull events according to Lemma 9. Hence, there is only a finite number of nonnull intersections  $D_{jk}$ . As null events are *S*-events, all these  $D_{jk}$  are *S*-events. Thus, it can be assumed without loss of generality that  $\{B_j\}$  and  $\{C_k\}$  are finite.

Define  $P : \mathscr{A} \to [0, 1]$  by  $P(A) = \sum_{\{j,k:D_{jk} \subseteq A\}} P_S(B_j) P_L(C_k)$  with respect to appropriate partitions  $\{B_j\}$  and  $\{C_k\}$ . Obviously, P is a finitely additive probability measure with  $P(A) = P_S(A)$  for  $A \in \mathscr{S}$  and  $P(A) = P_L(A)$  for  $A \in \mathscr{S}$ . Countable additivity is implied by the countable additivity of  $P_S$  and the fact that all null events are both *L*-events and *S*-events. *P* is uniquely determined, which follows from the uniqueness and the additivity of  $P_L$  and  $P_S$ . Furthermore, define  $Q : \mathscr{A} \times \mathscr{A} \to [0, 1]$  by

$$Q(A_1, A_2) = \sum_{\{j,k: D_{jk} \subseteq A_1\}} \left( P_S(B_j) P_L(C_k) \sum_{\{j,k': D_{jk'} \subseteq A_2\}} P_L(C_{k'}) \right)$$

Q is an *SL*-decomposition of *P*. This is verified as follows:  $\mathscr{A}$  is the algebra generated by  $\mathscr{S}$  and  $\mathscr{L}$ . Furthermore, Q fulfills all requirements listed in Sect. 2:

$$\begin{aligned} \text{Adaptation} : & \mathcal{Q}(A_1, \Omega) = \sum_{(j,k:D_{jk} \in A_1)} \left( P_S(B_j) P_L(C_k) \sum_{(j,k':D_{jk'} \in \Omega)} P_L(C_{k'}) \right) \\ &= \sum_{(j,k:D_{jk} \in A_1)} P_S(B_j) P_L(C_k) = P(A_1) \text{ for } A_1 \in \mathscr{A} \end{aligned}$$

$$\begin{aligned} \text{Additivity} : & \mathcal{Q}(A_1 \cup A_2, A_3) = \sum_{(j,k:D_{jk} \in A_1 \cup A_2)} \left( P_S(B_j) P_L(C_k) \sum_{(j,k':D_{jk'} \in A_3)} P_L(C_{k'}) \right) \\ &= \sum_{(j,k:D_{jk} \in A_1)} \left( P_S(B_j) P_L(C_k) \sum_{(j,k':D_{jk'} \in A_3)} P_L(C_{k'}) \right) + \sum_{(j,k:D_{jk'} \in A_3)} P_L(C_{k'}) \sum_{(j,k':D_{jk'} \in A_3)} P_L(C_{k'}) \right) \\ &= Q(A_1, A_3) + Q(A_2, A_3) \text{ for } A_1, A_2, A_3 \in \mathscr{A} \text{ with } A_1 \cap A_2 = \emptyset \end{aligned}$$

$$\begin{aligned} \text{Symmetry} : & Q(A_1, A_2) = \sum_{(j,k:D_{jk'} \in A_1)} \left( P_S(B_j) P_L(C_k) \sum_{(j,k':D_{jk'} \in A_2)} P_L(C_{k'}) \right) \\ &= \sum_{(j,k:D_{jk'} \in A_2)} \left( P_S(B_j) P_L(C_k) \sum_{(j,k':D_{jk'} \in A_1)} P_L(C_{k'}) \right) = Q(A_2, A_1) \text{ for } A_1, A_2 \in \mathscr{A} \end{aligned}$$

$$\begin{aligned} \text{Separation} : & Q(A_1, A_1^C) = \sum_{(j,k:D_{jk'} \in A_1)} \left( P_S(B_j) P_L(C_k) \sum_{(j,k':D_{jk'} \in A_1^C)} P_L(C_{k'}) \right) \\ &= 0 \text{ for } A_1 \in \mathscr{A} \end{aligned}$$

$$\begin{aligned} \text{Dispersion} : & Q(A_1, A_1^C) = \sum_{(j,k:D_{jk'} \in A_1)} \left( P_S(B_j) P_L(C_k) \sum_{(j,k':D_{jk'} \in A_1^C)} P_L(C_{k'}) \right) \\ &= P(A_1^C) \sum_{(j,k:D_{jk'} \in A_1} P_S(B_j) P_L(C_k) = P(A_1^C) P(A_1) = Q(A_1, \Omega) Q(A_1^C, \Omega) \text{ for } A_1 \in \mathscr{A} \end{aligned}$$

$$\begin{aligned} \text{Independence} : & Q(A_1, A_2, A_3 \cap A_4) = \sum_{(j,k:D_{jk'} \in A_1^C)} P_L(C_{k'}) \\ &= \sum_{(j,k:D_{jk'} \in A_1 \cap A_2, A_3 \cap A_4) = \sum_{(j,k:D_{jk'} \in A_1^C)} P_S(B_j) P_L(C_k) \sum_{(j,k':D_{jk'} \in A_1^C)} P_S(B_j) P_L(C_k) \\ &= \left( \sum_{(j,k:D_{jk'} \in A_1 \cap A_2)} P_S(B_j) P_L(C_k) \right) \left( \sum_{(j,k':D_{jk'} \in A_1^C)} P_S(B_j) P_L(C_k) \right) \\ &= \sum_{(j,k:D_{jk'} \in A_1 \cap A_2} P_S(B_j) P_L(C_k) \sum_{(j,k':D_{jk'} \in A_2)} P_S(B_j) P_L(C_k) \right) P(A_4) \\ &= \sum_{(j,k:D_{jk'} \in A_1 \cap A_2} P_S(B_j) P_L(C_k) \left( \sum_{(j,k':D_{jk'} \in A_2)} P_S(B_j) P_L(C_k) \right) P(A_4) \\ &= \sum_{(j,k:D_{jk'} \in A_1 \cap A_2} P_S(B_j) P_L(C_k) \sum_{(j,k':D_{jk'} \in A_2)} P_S(B_j) P_L(C_k) \right) P(A_4) \\ &= \sum_{(j,k:D_{jk'} \in A_1} \left( P_S(B_j) P_L(C_k) \sum_{(j,k':D_{jk'} \in A_2)} P_S(B_j) P_L(C_k) \right) P(A_4) \\ &= \sum_{(j,k:D_{jk'} \in A_1} \left( P_S(B_j) P_L(C_k) \sum_{(j,k':D_$$

For verification, note that for all j,  $\sum_{\{k:D_{jk} \subseteq A\}} P_L(C_k) = P(A)$  for all  $A \in \mathcal{L}$ . Countable additivity is relevant only for *S*-events and follows from the countable additivity of  $P_S$ . *Q* is uniquely determined, which follows from the uniqueness of  $P_S$  and  $P_L$ .

For the next part of the proof, the following substitutions are used:

$$F_{i=1}^{n} x_{i} A_{i} = F_{j=1}^{m} F_{k=1}^{r} x_{jk} D_{jk} = F_{j=1}^{m} (F_{k=1}^{r} x_{jk} C_{k}) B_{j}$$

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with  $x_{jk} = x_i$  for  $D_{jk} \subseteq A_i$ .  $F_{i=1}^n y_i A_i = F_{j=1}^m (F_{k=1}^r y_{jk} C_k) B_j$  results analogously. According to Theorem 1, there is a function  $U : X \to \mathbb{R}$  such that

$$F_{i=1}^n x_i A_i \leq F_{i=1}^n y_i A_i \Leftrightarrow \sum_{j=1}^m U(f_j) P(B_j) \leq \sum_{j=1}^m U(g_j) P(B_j).$$

with  $f_j = F_{k=1}^r x_{jk} C_k$  and  $g_j = F_{k=1}^r y_{jk} C_k$ . *U* is uniquely determined except for positive linear transformations. According to Theorem 2, there is a symmetric and *u*-monotonic function  $u : X \times X \to \mathbb{R}$  such that

$$\sum_{j=1}^{m} U(f_j) P(B_j) = \sum_{j=1}^{m} \left( \sum_{k=1}^{r} \sum_{k'=1}^{r} u(x_{jk}, x_{jk'}) P(C_k) P(C_{k'}) \right) P(B_j).$$

Additivity of Q implies  $P(C_k) P(C_{k'}) P(B_j) = Q(D_{jk}, D_{jk'})$  for all k, k'. Hence,

$$\sum_{j=1}^{m} U(f_j) P(B_j) = \sum_{j=1}^{m} \sum_{k=1}^{r} \sum_{k'=1}^{r} u(x_{jk}, x_{jk'}) Q(D_{jk}, D_{jk'}).$$

The separation property of Q implies  $Q(D_{jk}, D_{j'k'}) = 0$  for  $j \neq j'$ . Consequently,

$$\sum_{i=1}^{m} U(f_j) P(B_j) = \sum_{j=1}^{m} \sum_{j'=1}^{m} \sum_{k=1}^{r} \sum_{k'=1}^{r} u(x_{jk}, x_{j'k'}) Q(D_{jk}, D_{j'k'}).$$

Finally,  $x_{jk}$  is replaced by  $x_i$ . Moreover, all  $D_{jk}$  belonging to  $A_i$  are combined, and  $Q(A_i, A_{i'}) = \sum_{\{j,k:D_{jk} \subseteq A_i\}} \sum_{\{j',k':D_{j'k'} \subseteq A_{i'}\}} Q(D_{jk}, D_{j'k'})$  is substituted. Thus,

$$\sum_{j=1}^{m} U(f_j) P(B_j) = \sum_{i=1}^{n} \sum_{i'=1}^{n} u(x_i, x_{i'}) Q(A_i, A_{i'}).$$

In the same way,  $\sum_{j=1}^{m} U(g_j)P(B_j) = \sum_{i=1}^{n} \sum_{i'=1}^{n} u(y_i, y_{i'})Q(A_i, A_{i'})$  is derived. Hence,

$$\sum_{i=1}^{n} \sum_{i'=1}^{n} u(x_i, x_{i'}) Q(A_i, A_{i'}) \le \sum_{i=1}^{n} \sum_{i'=1}^{n} u(y_i, y_{i'}) Q(A_i, A_{i'}).$$

*u* is unique up to positive linear transformations according to Theorem 1 and Theorem 2.  $\Box$ 

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