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Improvable Students in School Choice

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Improvable Students in School Choice

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The Deferred Acceptance algorithm (DA) frequently produces Pareto inefficient allocations in school choice problems. While a number of efficient mechanisms that Pareto-dominate DA are available, a normative question remains unexplored: which students should benefit from efficiency enhancements? We address it by introducing the concept of *maximally improvable students*, who benefit in every improvement over DA that includes as many students as possible in set-inclusion terms. We prove that common mechanisms such as Efficiency-Adjusted DA (EADA) and Top Trading Cycles applied to DA (DA+TTC) can fall significantly short of this benchmark. These mechanisms may only improve two maximally-improvable students when up to $n - 1$ could benefit. Addressing this limitation, we develop the Maximum Improvement over DA mechanism (MIDA), which generates an efficient allocation that maximises the number of students improved over DA. We show that MIDA can generate fewer blocking pairs than EADA and DA+TTC, demonstrating that its distributional improvements need not come at the cost of high justified envy.

KEYWORDS. school choice, improvable students.

JEL CLASSIFICATION. C78, D47.

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1. INTRODUCTION

The student-proposing Deferred Acceptance (DA) algorithm is a cornerstone of market design, renowned for its theoretical elegance and practical applications. However, DA often produces student-school allocations that are Pareto-inefficient from the students’ perspective.¹ This inefficiency has motivated the development of efficient mechanisms that Pareto-dominate DA (which we call *fully-dominating*), improving outcomes for some students without disadvantaging others. Notable examples include Efficiency-Adjusted DA (EADA, [Kesten 2010](#)) and Top Trading Cycles using DA as endowments (DA+TTC, [Alcalde and Romero-Medina 2017](#)).²

Interestingly, in every school choice problem there are students who do not improve their allocation under any fully-dominating mechanism—these are known as unimprovable students ([Tang and Yu, 2014](#)). Identifying unimprovable students has been helpful for two important reasons. First, understanding which (and how many) students are unimprovable allows us to recognise the scope of efficiency adjustments performed by EADA, DA+TTC and any other fully-dominating mechanism in theory and practice. Second, it has allowed us to gain a deeper insight into fully-dominating mechanisms; this knowledge has translated into substantially simpler implementations of EADA.

This paper shifts focus to improvable students, which are those who improve their DA allocation under some fully-dominating mechanism. We do this using a graph-theoretic approach on DA’s *envy digraph*—a directed graph representation where a student points to another if they prefer the latter’s assignment—as it is known that every improvement over DA can be obtained with a trading cycle.

First, we focus on *universally improvable* students, which are those who must improve their allocation under any fully-dominating mechanism whenever DA is

¹In simulations, DA generates a Pareto-efficient allocation in fewer than 15% of cases whenever random preferences have some degree of heterogeneity, and never whenever the DA allocation is generated with real data ([Cantillon et al., 2024](#)).

²Many other mechanisms are (asymptotically) Pareto-efficient (including top trading cycles, random serial dictatorship, DA with circuit breaker, and others), but they do not Pareto-dominate DA.

inefficient. Even though DA is weakly Pareto-efficient (meaning that it never assigns all students to their least preferred choice), universally improvable students do not exist even in simple school choice problems (Proposition 1). While intuitive conditions guarantee their existence, they are strong and unlikely to hold in practice: one of them being that DA's envy digraph has exclusive trading cycles that do not share nodes with any other cycle in the directed envy graph (Proposition 2).

Given the non-existence of universally improvable students, we focus on a weaker condition: maximal improvability. A student is *maximally improvable* if they benefit from an efficiency adjustment in every fully-dominating mechanism that improves as many students as possible in set-inclusion terms. The concept of maximal improvability is weaker than universal improvability, yet it has a much more profound normative meaning: if an efficiency adjustment does not benefit some student, and yet an alternative improvement could include them without removing any other student, it seems reasonable and fair that the latter adjustment including strictly more students should be implemented instead. This normative principle matters from a policy perspective because it ensures that efficiency gains are distributed as widely as possible across the student population, promoting greater perceived fairness in the school choice process. Although we find that in some very complex school choice problems maximally improvable students fail to exist (Proposition 3), we can guarantee their existence whenever there is a key node in the envy digraph whose removal increases the number of strongly connected components of the resulting envy digraph (Proposition 4).

Having established a guiding normative principle for the design of fully-dominating mechanisms, we compare existing mechanisms such as EADA and DA+TTC against our desideratum: how many students among those who are maximally improvable actually benefit from their efficiency enhancement? In other words, which trading cycles are EADA and DA+TTC implementing? We find that both mechanisms can perform poorly on this metric: if there are n students, we can construct examples where $n - 1$ students are maximally improvable, and

yet EADA and DA+TTC forgo this large cycle involving almost all students, and perform a mere two-way exchange instead. Therefore, the improvement ratio of both mechanisms, which tells us how many efficiency swaps they miss, is $\frac{n-1}{2}$ (the largest possible ratio obtained by a fully-dominating mechanism) and grows linearly with the size of the school choice problem (Theorem 1).

The poor distribution of efficiency gains under these mechanisms motivates us to propose a new mechanism that distributes efficiency gains more evenly. This algorithm computes DA, and based on the corresponding envy digraph, chooses the collection of disjoint trading cycles that maximises the number of nodes involved. Alternatively, it chooses the fully-dominating mechanism that assigns as many students as possible to a school more preferred than their DA allocation. We call this mechanism *Maximum Improvement over Deferred Acceptance* (MIDA).

While the MIDA mechanism is imperfect (all mechanisms involve trade-offs), it sometimes generates fewer blocking pairs than EADA (Theorem 2). This is particularly notable as EADA is envy-minimal among efficient mechanisms (Doğan and Ehlers, 2021, Kwon and Shorrer, 2020) and is known to satisfy several weaker but meaningful notions of stability (Ehlers and Morrill, 2020, Troyan et al., 2020, Tang and Zhang, 2021, Reny, 2022). Even though it does not take priorities into account when choosing which trades to implement, its low justified envy emerges because it matches more students to more preferred schools, generating less envy overall—justified or otherwise. For school administrators, this offers a meaningful practical advantage: the ability to improve more students’ assignments while potentially creating fewer complaints about priority violations. It also generates fewer blocking pairs than DA+TTC, this finding is admittedly less surprising as DA+TTC does not have any weak stability guarantees and is known to potentially violate the priorities of almost all students (Example 8; Kesten, 2010).

On the incentives front, MIDA is manipulable but not obviously so in the sense of Troyan and Morrill (2020). Moreover, finding a successful manipulation requires solving several interdependent problems: computing the manipulated DA

outcome, constructing the resulting envy digraph, and identifying the new maximum set of disjoint cycles. We argue that MIDA's complexity arms it with a strategic complexity protection that is likely to make it non-manipulable in practice.

The main drawback of MIDA is that solving for the maximum collection of disjoint cycles is a computationally intensive problem, making MIDA more of a theoretical benchmark than a ready-to-implement mechanism. Yet, similar *cycle-packing* problems are routinely approximately solved by matching theorists in designing kidney exchange systems that have been successfully applied in the UK (Biro et al., 2009). This offers a ray of hope for the potential real-life implementation of MIDA-like mechanisms in the future.

In summary, our paper makes three main contributions. First, we establish a theoretical framework for analysing who benefits from Pareto improvements over DA, introducing concepts that complement the existing literature on unimprovable students and showing when their existence is guaranteed. Second, we identify a previously unrecognised limitation of existing mechanisms, quantifying exactly how far they fall short of achieving maximum possible improvements. Third, we propose and analyse MIDA, comparing its advantages and its weaknesses.

Outline. The remainder of this paper is organised as follows. Section 2 discusses the related literature. Section 3 introduces the model. Section 4 develops our theoretical analysis and presents the limitations of existing mechanisms that Pareto-dominate DA. Section 5 introduces the Maximal Improvement over DA mechanism, and discusses its practical advantages and limitations. Section 6 concludes.

2. RELATED LITERATURE

On DA's Pareto-inefficiency. The Pareto-inefficiency of DA (for students) has been recognised in the matching literature since its inception (Roth, 1982, Abdulkadiroğlu and Sönmez, 2003). Theoretical bounds on DA's inefficiency were established by Kesten (2010), with numerous empirical studies documenting the inefficiency in practice (Abdulkadiroğlu et al., 2005, Abdulkadiroğlu et al., 2009,

Che and Tercieux, 2019, Abdulkadiroğlu et al., 2020, Ortega and Klein, 2023). There is also a large literature establishing preference and priority domains under which DA's inefficiency disappears (e.g. Ergin, 2002, Cantillon et al., 2024).

Efficient mechanisms that dominate DA. The most prominent mechanism in this class is Efficiency-Adjusted Deferred Acceptance (EADA, Kesten, 2010). Over the past decade, EADA's properties and implementation have been extensively studied (Bando, 2014, Tang and Yu, 2014, Dur et al., 2019, Troyan et al., 2020, Troyan and Morrill, 2020, Ehlers and Morrill, 2020, Tang and Zhang, 2021, Doğan and Ehlers, 2021, Reny, 2022, Chen and Möller, 2023, Cerrone et al., 2024), demonstrating that it is possible to achieve an efficient improvement over DA while maintaining relatively low instability and manipulability.

Another well-known efficient mechanism that Pareto-dominates DA is DA+TTC, which applies the top trading cycles (TTC) procedure to the allocation obtained by DA. While frequently discussed informally, surprisingly little has been published on this mechanism, with only a few notable exceptions (Alcalde and Romero-Medina, 2017, Troyan et al., 2020). Although many other efficient mechanisms Pareto-dominate DA, they appear to be largely unexplored in the literature.

Other studies have examined conditions under which DA can be improved in a strategy-proof manner (Kesten and Kurino, 2019). Some have focused on refining DA while minimising the number of blocking pairs (Doğan and Ehlers, 2021, Kwon and Shorrer, 2020, Afacan et al., 2022), while others have explored improvements that are consistent when specific unimprovable students are removed (Doğan and Yenmez, 2020).

Maximality restrictions. Our notion of maximally improvable students requires that some students are improved under any maximal mechanism that Pareto-dominates DA. Maximality as a desideratum of assignments is very common in the computer science literature (see Biro et al. (2009), Krysta et al. (2014) and references therein), yet relatively few papers in economics have focused on it

(Noda, 2018, Andersson and Ehlers, 2020, Afacan et al., 2023, Afacan and Dur, 2023, Zhang, 2023).

3. MODEL

Following Abdulkadiroğlu and Sönmez (2003), a school choice problem P consists of a set of n students I and a set of schools S . Each student $i \in I$ has a strict preference relation \succ_i over the schools. Each school $s \in S$ has a quota of available seats q_s and a strict priority relation \triangleright_s over the students.

For a given school choice problem P , a *matching* μ is a mapping from I to S such that no school is matched to more students than its quota. We denote by μ_i the school to which student i is assigned.

The function $\text{rk}_i : S \rightarrow \{1, \dots, n\}$ specifies the rank of school s according to the preference relation \succ_i of student i :

$$\text{rk}_i(s) = |\{s' \in S : s' \succ_i s\}| + 1, \quad (1)$$

so that student i 's most preferred school gets a rank of 1.

A matching μ weakly Pareto-dominates matching ν if $\text{rk}_i(\mu_i) \leq \text{rk}_i(\nu_i)$ for all $i \in I$. A matching μ *Pareto-dominates* matching ν if μ weakly Pareto-dominates ν and there exists at least one student $j \in I$ such that $\text{rk}_j(\mu_j) < \text{rk}_j(\nu_j)$. A matching is Pareto-efficient if it is not Pareto-dominated by any other matching.

A *mechanism* is a function that maps a (probability distribution over) matching(s) to every school choice problem. We use $\text{DA}(P)$ to denote the unique student-optimal stable matching generated by the Deferred Acceptance algorithm in school choice problem P , and $\text{DA}_i(P)$ to denote the school to which student i is assigned under this matching.

In this paper, we focus on matchings that both Pareto-dominate DA and are efficient:

DEFINITION 1. A matching μ is *fully-dominating* if it weakly Pareto-dominates $\text{DA}(P)$ and is Pareto-efficient. We denote the set of fully-dominating matchings by $\mathcal{M}^{\text{DA}}(P)$.³

3.1 Notions of Improvability

Having established the basic framework for school choice problems, we now turn to the central concepts of our paper: different notions of improvability. We start with the concept of unimprovable students as proposed by [Tang and Yu \(2014\)](#).

DEFINITION 2. A student i is *unimprovable* if, for every fully-dominating matching $\mu \in \mathcal{M}^{\text{DA}}(P)$, we have $\mu_i = \text{DA}_i(P)$.

Intuitively, an unimprovable student cannot benefit from any efficiency enhancement to DA, regardless of which particular improvement is chosen. Students who are *not* unimprovable are termed *improvable*. Note that, for any school choice problem, at least one unimprovable student always exists.

For a fully-dominating matching μ , we denote by $\mathcal{I}(\mu)$ the set of students who strictly improve upon their DA allocation:

$$\mathcal{I}(\mu) := \{i \in I : \text{rk}_i(\mu_i) < \text{rk}_i(\text{DA}_i(P))\} \quad (2)$$

Similarly (and abusing notation slightly), $\mathcal{I}(P)$ denotes the set of all students who can improve upon their DA allocation under some fully-dominating matching in school choice problem P :

$$\mathcal{I}(P) := \{i \in I : \exists \mu \in \mathcal{M}^{\text{DA}}(P) \text{ such that } i \in \mathcal{I}(\mu)\} \quad (3)$$

We now introduce our first novel concept: students who must benefit from any efficiency enhancement to DA.

³Fully-dominating matchings are similar to stable-dominating ones ([Alva and Manjunath, 2019](#)). Yet, our notion is stronger: a stable-dominating matching must merely Pareto-dominate some stable matching, but not necessarily the student-optimal one, and need not be Pareto-efficient.

DEFINITION 3. A student i is *universally improvable* if, in every fully-dominating matching $\mu \in \mathcal{M}^{\text{DA}}(P)$, we have $i \in \mathcal{I}(\mu)$.

Universally improvable students represent those who are guaranteed to benefit whenever DA's inefficiency is corrected, regardless of the specific improvement method chosen.

We also introduce a weaker notion that reflects the practical reality that efficiency improvements are typically designed to benefit as many students as possible:

DEFINITION 4. A fully-dominating matching μ is *maximal* if there is no other fully-dominating matching μ' such that $\mathcal{I}(\mu) \subsetneq \mathcal{I}(\mu')$. A student i is *maximally improvable* if, in every maximal fully-dominating matching μ , we have $i \in \mathcal{I}(\mu)$.

Maximally improvable students are those who benefit under any improvement that maximises the number of improved students in a set-inclusion sense. This concept is particularly relevant for practical market design, as it captures the students who should benefit if we aim to improve the placement of as many students as possible.

These definitions create a nested hierarchy: every universally improvable student is maximally improvable, and every maximally improvable student is improvable. This nested structure helps organise our subsequent analysis.

3.2 Envy Digraphs and Improvement Cycles

To analyse improvements over DA, we use the envy digraph proposed by [Ortega et al. \(2024\)](#), which illustrates who envies whom after the initial DA allocation (envy need not be justified).

DEFINITION 5. The *envy digraph* $\tilde{G}^{\text{DA}}(P)$ is a directed graph where:

- Nodes represent students in I .
- A directed edge (i, j) exists if student i envies student j 's assignment under DA, i.e. if $\text{rk}_i(\text{DA}_j(P)) < \text{rk}_i(\text{DA}_i(P))$.

We denote the edge (i, j) as $i \rightarrow j$. The in-degree of node i , denoted by $\deg^-(i)$, counts the number of edges pointing towards i . The out-degree $\deg^+(i)$ counts the number of edges pointing away from node i .

A *cycle* in the digraph is a sequence of nodes $(i_0, i_1, \dots, i_k, i_0)$ such that there is a directed edge between each pair of consecutive nodes and no edge is repeated. A *trading cycle* is a cycle in which every node appears exactly once, except for i_0 . Unless specifically stated otherwise, every cycle mentioned henceforth is a trading cycle. Abusing notation, we alternatively write cycles as $(i_0 \rightarrow i_1 \rightarrow \dots \rightarrow i_k \rightarrow i_0)$.

In practical applications, DA is rarely Pareto-efficient, as demonstrated by [Cantillon et al. \(2024\)](#) and others. Therefore, we focus on school choice problems where $\tilde{G}^{\text{DA}(P)}$ contains at least one cycle, indicating potential improvements.

To simplify our analysis, we work with an irreducible version of the envy digraph:

DEFINITION 6. The *pruned envy digraph* $G^{\text{DA}(P)}$ is obtained from $\tilde{G}^{\text{DA}(P)}$ by recursively removing:

- All nodes with in-degree or out-degree of zero (sources and sinks).
- All edges connected to these nodes.

The resulting graph has the property that every node has both in-degree and out-degree of at least 1. This pruning is justified because only unimprovable students are removed ([Tang and Yu, 2014](#)), $G^{\text{DA}(P)}$ therefore preserves all relevant information for studying improvable students.

3.3 Feedback Sets

To systematically identify and implement improvements over DA, we use the concept of a feedback set.

DEFINITION 7. A *feedback set* F is a collection of pairwise-disjoint cycles in $G^{\text{DA}(P)}$ such that, when all nodes in these cycles are removed (together with their adjacent edges), the resulting subgraph is acyclic.⁴

Feedback sets represent sets of disjoint trading cycles that can be simultaneously implemented to improve upon DA. We use $V(F)$ to denote the set of nodes (students) that are included in some cycle in F . If $i \in V(F)$, we say that F covers i .

DEFINITION 8. A feedback set F^* is *maximal* if there is no other feedback set F' such that $V(F^*) \subsetneq V(F')$. A feedback set F^* is *maximum* if, for any other feedback set F' , we have $|V(F')| \leq |V(F^*)|$.

Given a feedback set F , we define the matching μ^F as the allocation obtained when the trades in each cycle of F are implemented, starting from DA. Formally:

- For student $i \in V(F)$ belonging to cycle C , $\mu_i^F = \text{DA}_{\text{succ}(i)}(P)$, where $\text{succ}(i)$ denotes the successor of i in cycle $C \in F$.
- For student $i \notin V(F)$, $\mu_i^F = \text{DA}_i(P)$.

A *strongly connected component* (SCC) is a maximal subgraph where every node is reachable from every other node along directed paths following the edge direction. An SCC is *trivial* if it only includes one node. A *strong articulation point* is a node whose removal increases the number of strongly connected components in a directed graph.

3.4 Connection to Existing Results

Our analysis builds upon a well-known result which connects improvements over DA to cycles in envy digraphs:

⁴Feedback node sets are sometimes called feedback vertex sets (Karp, 2010). The concept of a “cycle cover” is related, but requires that every node must be part of a cycle. The problem of choosing the maximal feedback set is sometimes known as cycle packing—we will examine this connection in more detail in section 5 (Conlon et al., 2014, Biro et al., 2009).

LEMMA 1 (Tang and Yu (2014)). *Every Pareto improvement over DA corresponds to a set of trading cycles in $G^{\text{DA}(P)}$.*⁵

Consequently, every feedback set induces a matching that Pareto dominates DA. Moreover, we can derive the following relation between maximal feedback sets and fully-dominating matchings.

LEMMA 2. *For every fully-dominating matching μ , there exists a feedback set F such that $\mu = \mu^F$. Conversely, for every maximal feedback set F^* , there exists a fully-dominating matching μ such that $\mathcal{I}(\mu) = V(F^*)$.*

PROOF. That every fully-dominating matching can be obtained from some feedback set is a direct implication from Lemma 1. Note that not every allocation obtained by a feedback set is Pareto-efficient, not even when the feedback set is maximal, because further trading edges and cycles may appear after the cycles in the original feedback set are executed. However, we will show that any additional trading must be restricted to the nodes covered by F^* .

To prove the second statement, start with an arbitrary maximal feedback set F^* and implement the disjoint trading cycles to obtain μ^{F^*} . By construction, $\mathcal{I}(\mu^{F^*}) = V(F^*)$. If μ^{F^*} is Pareto-efficient we are done. So suppose it is not. Then μ^{F^*} is dominated by another matching ν . By Lemma 1, ν corresponds to a feedback set F_ν in $G^{\text{DA}(P)}$. Now, because ν Pareto dominates μ^{F^*} , we have that $V(F^*) \subseteq V(F_\nu)$. If the set inclusion is strict, then F^* is not maximal, and therefore $V(F^*) = V(F_\nu)$. Thus, ν is fully-dominating and $\mathcal{I}(\nu) = V(F^*)$. \square

This relationship between fully-dominating matchings and maximal feedback sets provides a framework to identify which students can benefit from improvements over DA, which we explore in the next section.

⁵Similar arguments have been previously made in the literature by Abdulkadiroğlu et al. (2009) and Erdil (2014). Ortega et al. (2024) extend the result to show that a student is improvable if they belong to a cycle in $G^{\text{DA}(P)}$.

4. RESULTS

4.1 Universally Improvable Students

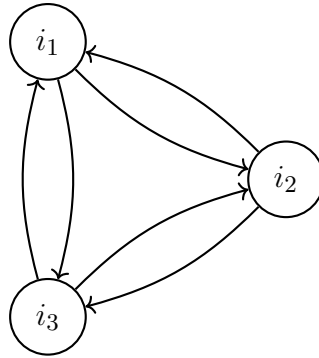
We begin by investigating which students are universally improvable—those who must benefit from any efficiency enhancement to DA. One might intuitively expect that whenever DA is inefficient, at least one student must be universally improvable (due to DA's weak Pareto-efficiency; [Roth 1982](#)).⁶ However, our first result shows this is not the case.

PROPOSITION 1. *There exist school choice problems in which DA is inefficient, yet no universally improvable students exist.*

PROOF. Consider the following school choice problem with four students and four schools, each with a quota of 1. DA's allocation appears in bold.

i_1	i_2	i_3	i_4	s_1	s_2	s_3	s_4
s_2	s_1	s_1	s_1	i_1	i_2	i_3	i_4
s_3	s_3	s_2	s_4	i_4	i_1	i_2	\cdot
s_1	s_2	s_3	\cdot	i_2	i_3	i_1	\cdot
\cdot	\cdot	\cdot	\cdot	i_3	\cdot	\cdot	\cdot

The pruned envy digraph $G^{\text{DA}(P)}$ appears below.



⁶Weak Pareto-efficiency is why the corresponding example cannot be simplified to only 3 students.

Note that, for each node, there exists a feedback set which excludes it. Therefore, no student is universally improvable in this example. □

In the example above, it might seem counter-intuitive not to perform the trade suggested by the maximal cycle, which strictly improves the DA placement of every improvable student. Yet, this is exactly what EADA and DA+TTC recommend: that only two students trade places: namely $i_1 \rightarrow i_2 \rightarrow i_1$.⁷ This observation motivates our subsequent analysis of maximally improvable students and the performance of existing mechanisms.

4.2 When do Universally Improvable Students Exist?

A universally improvable student exists if there exists a node covered by every feedback set. A sufficient condition for this to occur is the existence of *exclusive cycles*, which are those that do not intersect with any other cycle. We can find intuitive (albeit strict) conditions on preferences and priorities for the existence of exclusive cycles in DA's pruned envy digraph, as follows.

DEFINITION 9. A school choice problem admits an *exclusive cycle structure* if its set of students can be partitioned into two pairwise-disjoint subsets \mathcal{A}, \mathcal{B} , such that:

1. \mathcal{A} is the set of type-A students. Type-A students satisfy the following conditions:
 - No two type-A students share the same first, or second choice school.⁸ $S(\mathcal{A})$ denotes the set of top choice schools for type-A students.
 - For every student $i \in \mathcal{A}$, i 's second most preferred school s' is in $S(\mathcal{A})$ and i is among the $q_{s'}$ highest-priority students in it.

⁷In the first DA execution, s_4 is under-demanded and removed along with i_4 . In the second execution, s_3 is under-demanded and deleted it with i_3 (in this round s_2 is also under-demanded but now assigned to i_1 , who ranks it first).

⁸Formally, $|\{s \in S : \exists i \in \mathcal{A} : \text{rk}_i(s) = 1\}| = |\{s \in S : \exists i \in \mathcal{A} : \text{rk}_i(s) = 2\}| = |\mathcal{A}|$.

- $|\mathcal{A}| \geq 2$
- 2. \mathcal{B} is the set of type-B students. For every exclusive cycle, there exists at least q_s type-B students whose top choice, s , coincides with a type-A student's top choice. This school, s , prioritises all of the type-B students over the type-A student.

PROPOSITION 2. *If a school choice problem P admits an exclusive cycle structure, then every type-A student is universally improvable.*

PROOF. When executing DA, all type-A students apply to their most preferred schools. By definition of the exclusive cycle structure, one of them, say i_1 , is rejected by their top-school s_1 because enough type-B students with higher priority are assigned there.

Since each type-A student has higher priority at another's top-choice school (by definition), i_1 applies to their most preferred school in $S(\mathcal{A}) \setminus s_1$ (which is another type-A student's top choice) and is accepted, potentially displacing the type-A student i_2 for whom s_2 is top choice.⁹ This rejection pattern propagates cyclically through all type-A students until each is assigned to a school in $S(\mathcal{A})$ where they have a high priority, but not their own top choice. The ultimate type-A student in this rejection pattern will displace one of the original type-B students.

In the resulting DA matching, every type-A student i_j is assigned to some $s_k \in S(\mathcal{A}) \setminus s_j$ and thus envies the student assigned to s_j . Since all type-A students have distinct top choices, and only envy other type-A students, their envy relations form a cycle encompassing all type-A students. Furthermore, it is the only cycle in which any type-A student is involved. Any matching that Pareto-dominates DA must implement this entire cycle, for the trading among type-A students is disjoint from other envy relations and not implementing it would preserve Pareto inefficiency. Therefore, every type-A student is universally improvable. \square

⁹They could have been displaced by some other student, but i_1 's application confirms i_2 's rejection.

Although easy to describe, exclusive cycle structures are very demanding. As demonstrated in Proposition 1, universally improvable students may not exist even in simple school choice problems. This motivates our exploration of a weaker but more broadly applicable notion: maximally improvable students.

4.3 Maximally Improvable Students

The rarity of universally improvable students in practical settings calls for a more broadly applicable concept of improvability. We therefore introduce the notion of maximally improvable students—those who benefit in every maximal fully-dominating matching.

Maximal improvability is a weaker condition, but is a significantly more policy-relevant notion than universal improvability. It is unnecessary for a policymaker to focus on universally improvable students, as they will be improved in every fully-dominating mechanism. On the other hand, a maximally improvable student has a claim to experience an efficiency adjustment on their DA allocation if an efficiency adjustment could add the student into the trading cycles without excluding any other student.

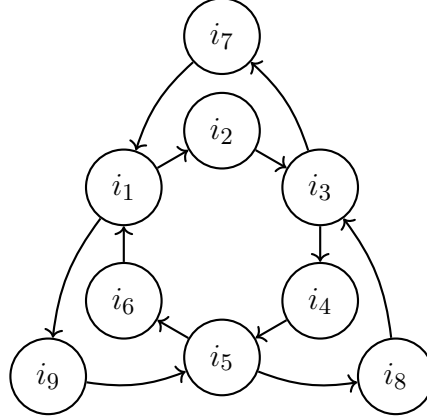
Although maximal improvability is a weaker concept than universal improvability, there are some (quite specific) school choice problems with no maximally improvable student.

PROPOSITION 3. *There exist school choice problems in which DA is inefficient, yet no maximally improvable students exist.*

PROOF. Consider the following school choice problem with 10 students and 10 schools with unit capacity. DA's allocation appears in bold.

i_1	i_2	i_3	i_4	i_5	i_6	i_7	i_8	i_9	i_{10}	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	s_{10}
s_2	s_3	s_4	s_5	s_6	s_1	s_1	s_3	s_5	s_3	i_1	i_2	i_3	i_4	i_5	i_6	i_7	i_8	i_9	i_{10}
s_9	s_2	s_7	s_4	s_8	s_6	s_7	s_8	s_9	s_{10}	i_6	i_1	i_{10}	i_3	i_4	i_5	i_3	i_5	i_1	.
s_1	.	s_3	.	s_5	i_7	.	i_2	.	i_9
.	i_8

The corresponding pruned envy digraph G is:



There are five cycles in the envy digraph above:

- $C_1: (i_1 \rightarrow i_2 \rightarrow i_3 \rightarrow i_7 \rightarrow i_1),$
 $C_2: (i_3 \rightarrow i_4 \rightarrow i_5 \rightarrow i_8 \rightarrow i_3),$
 $C_3: (i_5 \rightarrow i_6 \rightarrow i_1 \rightarrow i_9 \rightarrow i_5),$
 $C_4: (i_1 \rightarrow i_2 \rightarrow i_3 \rightarrow i_4 \rightarrow i_5 \rightarrow i_6 \rightarrow i_1),$
 $C_5: (i_1 \rightarrow i_9 \rightarrow i_5 \rightarrow i_8 \rightarrow i_3 \rightarrow i_7 \rightarrow i_1),$

The feedback set including any of the aforementioned cycles is maximal. $\{C_4\}$ and $\{C_5\}$ are maximum, thus their maximality is clear. Note that the feedback set including a single cycle among $\{C_1, C_2, C_3\}$ is also maximal; for example, C_1 is not contained in either $\{C_4\}$ because of i_7 nor in $\{C_5\}$ because of i_2 . For each node, we can construct a maximal feedback set that excludes them:

Node(s)	Maximal Feedback Set Excluding Them
i_7, i_8, i_9	$\{C_4\}$
i_2, i_4, i_6	$\{C_5\}$
i_1	$\{C_2\}$
i_3	$\{C_3\}$
i_5	$\{C_1\}$

Since for each student there exists a maximal improvement that does not include them, no student is maximally improvable in this example. \square

Whenever they exist, maximally improvable students provide a natural benchmark for evaluating mechanisms that aim to correct DA's inefficiency. A reasonably fair mechanism that considers the distributional impact of efficiency improvements should enhance the outcomes for most, if not all, of these maximally improvable students whenever they are present. This insight guides our subsequent analysis, where we evaluate the performance of existing mechanisms against this benchmark. But before that, we investigate the conditions in which maximally improvable students exist.

4.4 When do Maximally Improvable Students Exist?

Having established that maximally improvable students may not always exist, we now provide a sufficient condition on the types of envy digraphs that admit maximally improvable students.

PROPOSITION 4. *A student i is maximally improvable if they are covered by every maximal feedback set of $G^{\text{DA}(P)}$. Furthermore, i is covered by every maximal feedback set of $G^{\text{DA}(P)}$ if at least one of the following conditions holds:*

- (i) *i is a strong articulation point whose removal creates trivial SCCs jointly covering a set of nodes K , and there is a cycle C in $G^{\text{DA}(P)}$ such that $V(C) \subseteq K \cup \{i\}$.*
- (ii) *i appears in the intersection of multiple cycles and at least one of these cycles is included in every maximal feedback set.*
- (iii) *there exists a unique maximal feedback set.*

PROOF. First, we establish that maximally improvable nodes are covered by every maximal feedback set. By definition, a student i is maximally improvable if and only if i is improved in every maximal fully-dominating matching. Any maximal fully-dominating matching can be obtained by some feedback set by Lemma 2, and such a feedback set must be maximal, as otherwise there is an allocation that Pareto dominates DA that improves the placement of more students in set-inclusion terms. Since a student is improved only if they are covered by the feedback set, i is maximally improvable if it is covered by every maximal feedback set.

We now prove that a student i appears in every maximal feedback set if conditions (i), (ii), or (iii) hold.

(i) Suppose i is a strong articulation point that creates k trivial SCCs covering a set of nodes K , and there is a cycle C such that $V(C) \subseteq K \cup \{i\}$. If a maximal feedback set F^* did not cover i , then $F^* \cup \{C\}$ would be a larger feasible feedback set. This is because, if F^* does not cover i , then it cannot cover any node in K either, as nodes in the appearing trivial SCCs are not connected to the rest of the graph unless i is involved.

(ii) Suppose $i \in \{C_1 \cap \dots \cap C_k\}$ and for any maximal feedback set F^* , there exists $C_j \in \{C_1, \dots, C_k\}$ such that $C_j \in F^*$. Then $i \in V(F^*)$ for any maximal feedback set, and is thus maximally improvable by the first part of this Proposition.

(iii) If there exists a unique maximal feedback set, then every node in it is maximally improvable. \square

A practical implication of these conditions is that students who serve as "bridges" in the envy graph are likely to be maximally improvable. This provides a computationally efficient way to identify at least some maximally improvable students, as finding strong articulation points can be done in linear time (Italiano et al., 2012).

Maximally improvable students represent those who have a normative claim to benefit from efficiency adjustments whenever possible. Having established conditions for their existence, a practical question emerges: do existing fully-dominating mechanisms like EADA and DA+TTC actually improve these students? To address this question, we now analyse the performance of these mechanisms against the benchmark of maximum possible improvements. We emphasise that finding all strong articulation points of a digraph can be done in linear time, and therefore verifying condition 1 can be done in polynomial time.

4.5 Limitations of Existing Mechanisms

Do existing fully-dominating mechanisms like EADA and DA+TTC successfully identify and improve maximally-improvable students whenever they exist? And

how big are the improvements generated by these known mechanisms compared to the largest improvement possible? To answer this, we need to understand how close these mechanisms come to achieving the maximum possible improvements.

To quantify this gap, we define the improvement ratio of a mechanism as follows: Let $\mathcal{I}(M, P)$ denote the set of students improved by mechanism M in problem P . We define the maximum improvement $\mathcal{I}(\mu^*, P)$ as:

$$\mathcal{I}(\mu^*, P) := \max_{\mu \in \mathcal{M}^{\text{DA}}(P)} |\mathcal{I}(\mu, P)| \quad (4)$$

Let \mathcal{P}_n denote the class of all school choice problems with n students. For any mechanism M that Pareto-dominates DA, we define its *improvement ratio* as:

$$\rho(M; n) := \max_{P \in \mathcal{P}_n} \frac{|\mathcal{I}(\mu^*, P)|}{|\mathcal{I}(M, P)|} \quad (5)$$

For example, an improvement ratio of 2 means that the maximal mechanism may generate twice as many improvements as mechanism M . Our next Theorem shows that this ratio can grow linearly with the size of the problem.

THEOREM 1 (Distributional Limitations of Existing Mechanisms). *For every positive integer $n \geq 4$,*

$$\rho(\text{EADA}; n) = \rho(\text{DA+TTC}; n) = \frac{n-1}{2} \quad (6)$$

PROOF. *Upper Bound.* First we prove that the improvement ratio cannot exceed $\frac{n-1}{2}$. For the numerator: The maximum number of improvable students is $n-1$, as there is always at least one unimprovable student (Tang and Yu, 2014). For the denominator: The minimum number of students improved by EADA and DA+TTC is 2, because a trading cycle requires at least two students.¹⁰ Therefore, the ratio cannot exceed $\frac{n-1}{2}$.

¹⁰We have assumed that DA is inefficient; otherwise, the minimum would be 0 and the ratio undefined.

Tightness. The example in Proposition 1 illustrates a school choice problem achieving the $\frac{n-1}{2}$ threshold when $n = 4$. We now construct an school choice problem with n students and schools (all with unit capacity) where this improvement ratio is achieved for an arbitrary value of $n > 4$.

Let school priorities be such that, for every school s_k except s_1 :

$$s_k : i_k \triangleright_{s_k} i_{k-1} \quad (7)$$

Students' preferences are such that, for every student $i_k \in I \setminus \{i_1, i_{n-2}, i_{n-1}, i_n\}$:

$$i_k : s_1 \succ_{i_k} s_{k+1} \succ_{i_k} s_k \quad (8)$$

The remaining priorities and preferences are:

$$s_1 : i_1 \triangleright_{s_1} i_n \triangleright_{s_1} i_2 \quad (9)$$

$$i_1 : s_2 \succ_{i_1} s_{n-1} \succ_{i_1} s_1 \quad (10)$$

$$i_{n-2} : s_1 \succ_{i_{n-2}} s_2 \succ_{i_{n-2}} s_{n-2} \quad (11)$$

$$i_{n-1} : s_1 \succ_{i_{n-1}} s_2 \succ_{i_{n-1}} s_{n-1} \quad (12)$$

$$i_n : s_1 \succ_{i_n} s_n \quad (13)$$

A table depicting these preferences and priorities appears below:

i_1	\dots	i_j	\dots	i_{n-2}	i_{n-1}	i_n	s_1	\dots	s_k	\dots	s_n
s_2	\dots	s_1	\dots	s_1	s_1	s_1	i_1	\dots	i_k	\dots	i_n
s_{n-1}	\dots	s_{j+1}	\dots	s_2	s_2	s_n	i_n	\dots	i_{k-1}	\dots	\cdot
s_1	\dots	s_j	\dots	s_{n-2}	s_{n-1}	\cdot	i_2	\dots	\cdot	\dots	\cdot

The table below demonstrates this construction for $n = 7$. The DA allocation appears in bold. The allocation that maximises the number of improvements appears in circles, whereas the trade recommended by both EADA and DA+TTC appears in squares.

i_1	i_2	i_3	i_4	i_5	i_6	i_7	s_1	s_2	s_3	s_4	s_5	s_6	s_7
s_2	s_1	s_1	s_1	s_1	s_1	s_1	i_1	i_2	i_3	i_4	i_5	i_6	i_7
s_6	s_3	s_4	s_5	s_2	s_2	s_7	i_7	i_1	i_2	i_3	i_4	i_5	i_6
s_1	s_2	s_3	s_4	s_5	s_6	\cdot	i_2	\cdot	\cdot	\cdot	\cdot	\cdot	\cdot

We now demonstrate that the unique stable matching assigns each student i_k to the school with the corresponding index s_k .

Execution of DA. In DA's round 1, student i_1 applies to s_2 , whereas all other students apply to s_1 . School s_1 temporarily accepts i_n (its highest-priority applicant) and rejects all others. In round 2, rejected students i_2 through i_{n-1} make their next applications. Students i_2 through i_{n-3} apply to their second-choice schools s_3 through s_{n-2} , while i_{n-2} and i_{n-1} both apply to s_2 . School s_2 temporarily accepts i_1 (who applied in round 1) and rejects i_{n-2} and i_{n-1} . In subsequent rounds, a cascade of rejections occurs as students apply to their next preferred schools, eventually resulting in each student i_k being matched with school s_k .

The uniqueness of the set of stable matchings can be verified by observing that the school-proposing DA algorithm yields the same outcome, with each school immediately being matched to its highest-priority student.

Analysis of EADA. Under EADA, we begin by identifying under-demanded schools. First, s_n is under-demanded and is removed together with i_n . In the next iteration, s_3, s_4, \dots, s_{n-1} are under-demanded and removed jointly with i_3, i_4, \dots, i_{n-1} , respectively. At this stage, only schools s_1 and s_2 remain with students i_1 and i_2 . These students form a trading cycle where i_1 improves to s_2 and i_2 improves to s_1 . Thus, EADA improves exactly 2 students.

Analysis of DA+TTC. In DA+TTC, we begin with the DA allocation where each student i_k is assigned to school s_k . In the TTC algorithm, only students i_1 and i_2 form a trading cycle, as i_1 points to i_2 and i_2 points to i_1 . After executing this trade,

students i_3 through i_n remain at their DA allocations which are preferred to any remaining allocation. Therefore, DA+TTC also improves exactly 2 students.

Maximum Improvement. There exists, however, a fully-dominating matching μ^* obtained by implementing the cycles $(i_1 \rightarrow i_{n-1} \rightarrow i_1)$ and $(i_2 \rightarrow i_3 \rightarrow \dots \rightarrow i_{n-2} \rightarrow i_2)$, which improves $n - 1$ students. Under this matching, student i_n receives the same allocation as under DA, while all other students receive a more preferred school.

Therefore, the improvement ratio equals $\frac{|I(\mu^*, P)|}{|I(M, P)|} = \frac{n-1}{2}$ for any $n \geq 4$, establishing the tightness of our bound. \square

This result reveals a substantial limitation of existing mechanisms. In large school choice problems, EADA and DA+TTC might improve only a small fraction of the students who could benefit from Pareto improvements over DA. Moreover, this finding has important implications for the treatment of maximally improvable students. Since both EADA and DA+TTC can fall short of the maximum possible improvements, they may fail to improve all maximally improvable students. This observation suggests the need for alternative mechanisms that better address the potential efficiency gains available in school choice problems.

In the next section, we introduce the Maximal Improvement DA mechanism (MIDA), which systematically outperforms both EADA and DA+TTC by prioritising larger improvements.

5. MAXIMUM IMPROVEMENT OVER DA MECHANISM (MIDA)

Our theoretical analysis has revealed that existing mechanisms can fall significantly short of maximising the number of students who benefit from efficiency enhancements to DA. The Maximum Improvement over Deferred Acceptance (MIDA) mechanism directly addresses this limitation by identifying and implementing improvements that benefit the maximum possible number of students. Formally, MIDA is defined as follows.

DEFINITION 10. Given a school choice problem P , MIDA operates as follows:

Algorithm 1 Maximum Improvement over Deferred Acceptance (MIDA)

- 1: Compute the student-optimal stable matching $\text{DA}(P)$
 - 2: Set $\mu^0 = \text{DA}(P)$
 - 3: Set $t = 0$
 - 4: **repeat**
 - 5: Generate the pruned envy digraph G^{μ^t}
 - 6: **if** G^{μ^t} contains no cycles **then**
 - 7: **break**
 - 8: **end if**
 - 9: Find a maximum feedback set F^t in G^{μ^t}
 - 10: Implement the trades in F^t to produce matching μ^{t+1}
 - 11: $t = t + 1$
 - 12: **until** convergence
 - 13: Return the final matching μ^t
-

We now establish the main theoretical properties of MIDA, including its improvement over existing mechanisms:

THEOREM 2 (Properties of MIDA). *MIDA has the following properties:*

1. *It is Pareto-efficient.*
2. *It Pareto-dominates DA.*
3. *It maximises the number of students who benefit from Pareto improvements over DA.*
4. *It improves every maximally improvable student (when they exist).*
5. *It can generate strictly fewer blocking pairs than EADA and DA+TTC.*
6. *It is non-obviously manipulable.*

PROOF. The first three properties follow directly from MIDA's construction: it implements only cycles from the envy digraph (ensuring Pareto-dominance over

DA), the absence of further trading cycles ensures Pareto-efficiency, and its definition ensures that it improves as many students as any other fully-dominating mechanism.

For property 4, note that any maximum feedback set is also maximal in the set-inclusion sense.

To prove property 5, that MIDA can generate fewer blocking pairs than EADA, consider the following school choice problem with 5 students and 5 schools with unit capacity. DA's allocation appears in bold.

i_1	i_2	i_3	i_4	i_5	s_1	s_2	s_3	s_4	s_5
s_2	s_2	s_2	s_4	s_5	i_2	i_5	i_1	i_3	i_4
s_3	s_5	s_5	s_2	s_2	\cdot	i_4	i_3	i_4	i_2
\cdot	s_1	s_3	s_5	\cdot	\cdot	i_1	\cdot	\cdot	i_3
\cdot	\cdot	s_4	\cdot	\cdot	\cdot	i_2	\cdot	\cdot	i_5
\cdot	\cdot	\cdot	\cdot	\cdot	\cdot	i_3	\cdot	\cdot	\cdot

MIDA chooses the maximal cycle in this case ($i_1 \rightarrow i_5 \rightarrow i_4 \rightarrow i_3 \rightarrow i_1$). There are two blocking pairs: (i_2, s_5) and (i_3, s_5) . However, EADA implements the shorter cycle ($i_4 \rightarrow i_3 \rightarrow i_5 \rightarrow i_4$), generating three blocking pairs, namely (i_1, s_2) , (i_2, s_2) , (i_2, s_5) .

It's interesting that MIDA does not take priorities into account when choosing the trading cycles, and yet it can generate fewer blocking pairs. This occurs because a greater number of students are improving their allocation, thus becoming less envious overall.

Although Proposition 3 in [Doğan and Ehlers \(2021\)](#) shows an example where a fully-dominating mechanism generates fewer blocking pairs than EADA, in that case MIDA and EADA coincide. MIDA generating fewer blocking pairs than EADA is notable because EADA is justified envy minimal in a set-inclusion sense ([Kwon and Shorrer, 2020](#), [Afacan et al., 2022](#)).¹¹

¹¹When additionally imposing strategy-proofness, TTC is envy minimal in one-to-one matching problems in a set inclusion sense ([Abdulkadiroğlu et al., 2020](#)). When priorities are weak, finding

To show that MIDA can generate fewer blocking pairs than DA+TTC, consider the following Example with five students and five schools with unit capacity. DA's allocation appears in bold.

i_1	i_2	i_3	i_4	i_5	s_1	s_2	s_3	s_4	s_5
s_2	s_1	s_1	s_1	s_1	i_1	i_2	i_3	i_4	i_5
s_4	s_3	s_2	s_2	s_5	i_5	i_4	i_4	i_5	\cdot
s_1	s_2	s_3	s_4	\cdot	i_4	i_3	\cdot	\cdot	\cdot
\cdot	\cdot	\cdot	\cdot	\cdot	i_2	i_1	\cdot	\cdot	\cdot

In DA+TTC, students i_1 and i_2 trade their DA allocations, generating four blocking pairs: (i_4, s_1) , (i_5, s_1) , (i_3, s_2) , (i_4, s_2) . In contrast, MIDA executes the following trades: $(i_1 \rightarrow i_4 \rightarrow i_1)$, $(i_2 \rightarrow i_3 \rightarrow i_2)$, generating only one blocking pair: (i_5, s_1) .

Finally, property 6, that MIDA is not-obviously manipulable, follows because it Pareto-dominates DA and from Theorem 2 in [Troyan and Morrill \(2020\)](#).¹² \square

While MIDA offers significant advantages in terms of efficiency and distributional benefits, it also has several limitations:

1. *Computational Complexity*: The equivalent problem of finding the minimal feedback node set of $G^{\text{DA}(P)}$ is known to be NP-hard ([Bafna et al., 1999](#)), making exact computation challenging for large school choice problems. However, approximation algorithms with bounded approximation ratios exist and have been successfully implemented in similar contexts such as kidney

a matching that is Pareto-efficient and envy-minimal is an NP-hard problem ([Abdulkadiroğlu and Grigoryan, 2020](#)).

¹²Non-obvious-manipulability requires that there is no manipulation that is better than truthfulness in either the best- or worst-case scenario. The relevance of this concept is evidenced by the growing number of papers that have used it to establish possibility results for mechanisms that are both efficient and NOM across a wide range of economic settings ([Aziz and Lam, 2021](#), [Ortega and Segal-Halevi, 2022](#), [Psomas and Verma, 2022](#), [Archbold et al., 2023](#), [Shinozaki, 2023](#), [Arribillaga and Bonifacio, 2024, 2025](#), [Arribillaga and Risma, 2025](#), [Troyan, 2024](#), [Sirguiado, 2025](#)).

exchange matching (Biro et al., 2009), offering practical solutions for real-world applications.

2. *Instability*: MIDA is not stable, which is unavoidable since no mechanism can be both stable and efficient. There might be smaller improvements that generate fewer blocking pairs (see Example 3 in Doğan and Ehlers (2021)), though as we have shown, MIDA can sometimes generate fewer blocking pairs than mechanisms specifically designed to minimise justified envy.
3. *Manipulability*: While MIDA is not strategy-proof, it satisfies the weaker but meaningful notion of non-obvious-manipulability. Furthermore, its computational complexity provides a natural protection against manipulation. Finding an optimal manipulation strategy requires solving multiple inter-related NP-hard problems, creating a form of "strategic complexity protection" that makes truthful preference revelation the most practical approach for students.¹³ Whether MIDA is manipulated in practice is an interesting question to be answered empirically.
4. *Multiplicity*: As shown in Proposition 1, MIDA can be set-valued. A selection criterion among multiple MIDA outcomes would need to be defined for practical implementation, such as choosing the improvement that generates fewer blocking pairs or higher rank-efficiency.

These limitations highlight that MIDA represents a useful theoretical benchmark for what is achievable by school choice efficiency enhancements, but its practical application is considerably complicated by computational complexity barriers, potential solution multiplicity, and implementation challenges. While approximation algorithms exist, the overall complexity of deploying MIDA in real-world school assignment systems remains a significant obstacle.

Finally, we note that MIDA prioritises the number of students who benefit over the magnitude of their improvement. That is, moving from an 8th-choice school to the 7th is treated the same as moving to the 1st choice. To address this issue, a

¹³EADA is significantly more complex than DA, and despite this, it is less manipulated in practice, suggesting that when facing complex algorithms, subjects could resort to truthful reporting (Cerrone et al., 2024).

weighted variant of MIDA can be implemented. Analogous to the standard envy digraph, we construct a weighted envy digraph, where student i points to student j if i prefers $DA_j(P)$ to $DA_i(P)$, and the weight of the edge $i \rightarrow j$ is given by $w_{i,j} = rk_i(DA_i(P)) - rk_i(DA_j(P))$. This encodes the magnitude of possible improvements into rank terms. The weighted-MIDA mechanism then selects a feedback set that maximises the total weight of edges in the executed cycles.

It is straightforward to verify that this mechanism also Pareto-dominates DA and maintains efficiency, albeit with similar computational hardness. We believe that alternative weight functions in this framework could enable the incorporation of distributional justice objectives, such as affirmative action or diversity goals—a promising direction for future research.

6. CONCLUSION

In this paper, we have argued that a maximally improvable student should receive a more preferred allocation than their DA assignment whenever any efficiency-enhancing mechanism is applied. We have shown that existing efficient and Pareto-dominating mechanisms fall short of satisfying this principle, and we propose a new mechanism called MIDA designed to improve the distributional welfare gains achieved through efficiency enhancements. Somewhat surprisingly, MIDA may generate fewer blocking pairs than EADA and DA+TTC.

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