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Modeling the term structure

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Modeling the Term Structure*

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Abstract

Based on an analysis of changes in the yields of German government bonds, we propose a simple model for the term structure of interest rates and show empirically that this model with two parameters (relating to the interest level and slope of the term structure) fits empirically well the data for a change horizon of one year or longer, especially in the low interest environment, and give examples for applications. In addition, we provide closed-form solutions for some interest bearing instruments and give a new interpretation for the convexity when this linear model for the term structure is used.

Keywords: Term Structure of Interest Rates, Duration and Convexity, Common Factors, Government Bonds

JEL classification: G21

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1 Introduction

The term structure of interest rates is an important concept for an overview of the interest environment in a given country. Understanding its shape and dynamics is of key interest for various market participants, not the least banks, which usually have a high dependence on interest income.

Interest rates vary over time and depend on the maturity. In other words, there is a term structure, i.e. a collection of interest rates that mature after different periods for each point in time, and one can also, for example, construct a time series for the overnight interest rate, a time series for the two-year interest rates and a times series for the ten-year interest rate.

The aim of this paper is (1) to show that for many applications (especially in the area of banking), two parameters (relating to the level and slope of the term structure) are sufficient to describe changes over one year or longer in the term structure in Germany, (2) to show that under this two-factor model, closed-form expressions can be derived for several key figures of interest bearing instruments and (3) to give a new interpretation to the convexity, namely as the sensitivity to slope changes in the term structure. In addition, we analyse changes in the term structure (yield curve for German government bonds) over different time horizons and discuss the common factors observed in these changes for different time spans.

Regarding the first point, there is a trade-off between the parsimony and the empirical fit of the model which also depends on the time span considered. With the help of information criteria, we analyse this trade-off and we empirically establish that this parsimonious model explains well yearly (or longer) term structure changes. Our model fills the gap between complex term structure models with many parameters which explain the interest rates for a wide range of maturities (see e.g. Litterman and Scheinkman (1991), Knez, Litterman, and Scheinkman (1994), Driessen, Melenberg, and Nijman (2003)), and stylized models used e.g. in economic textbooks or in stress testing, where there is often only one rate to characterize the whole term structure or where changes in the term structure are assumed to be parallel shifts (see e.g. Armerin, Jensen, and Björk (2007), Basel Committee on Banking Supervision (2016)).

Taking into account possible slope changes is especially relevant for banks, as they generate a substantial part of their earnings by term transformation, i.e. they grant long-term loans and finance these operations by short-term deposits, thereby they benefit from the usually higher interest rates for longer maturities. Our suggested model is the easiest way to include the slope of the term structure in the model for their interest business.

Using this model for the term structure, we give closed-form solutions for the present value, and the level and slope sensitivity of some interest bearing instruments, for instance for instruments that generate a constant or geometrically declining cash flow or for normal bonds. We are also able to derive closed-form solutions for a trading strategy that consist in investing in a revolving manner into par-yield bonds. Using additional assumptions, we can very much simplify the closed-form solutions without losing much precision.

Traditionally, the convexity is the summand relating to the second derivative of a Taylor-approximation of the present value of a financial instrument, where the first term is the far more important one and relates to the duration. In other words: without the duration, the convexity has no meaning. Applying the assumption of the simple

term structure model, we give a new interpretation to the convexity and show that the convexity is closely related to changes in the steepness of the term structure in our model.

The empirical part of our analysis is closely related to the works by Litterman and Scheinkman (1991) and Knez et al. (1994) who analyse short-term US money market returns and long-term US government bond returns and find that the movements of these returns can be well described by a limited set of factors. Litterman and Scheinkman (1991) describe and interpret a three-factor model, where the three factors correspond to term structure changes in the level, the slope and the curvature. The authors show that for selected time series, the three-factor model can explain most of the variation of returns, and they draw important conclusions for the setup of hedges from these insights. In a follow-up work, Knez et al. (1994) show that a four-factor model can be more appropriate in some cases, where they attribute the fourth (additional) factor to properties of private issuers.

Later, Driessen et al. (2003) extend the analysis to compare the influence of different factors for different jurisdictions. They analyse factors for bond returns of the USA, Germany and Japan and show that the first ("level") factor is a common factor across different countries. The second and third factor in their model correspond, however, to parallel shifts in selected countries, while the fourth and fifth factor describe changes in the slope of single countries. In a similar spirit, Juneja (2012) analyse bond returns from the USA, the UK and Germany and determine separate as well as common factors underlying these time series. Abbritti, Dell'Erba, Moreno, and Sola (2018) extend this idea and identify global factors underlying the change in yield curves across eight different economies.

While our general approach is similar to the mentioned articles, we reconsider the approach to analyse principal components of movements of the yield curve (in our case for German government bonds) for several reasons. First, the analyses cited above were performed on returns from certain historical time windows and mainly on US returns, relating thus to a different market environment compared to the European markets in the 21st century. We think it is worth reevaluating the common factors in more recent times and especially during the low-interest rate environment observed in the last decade. We therefore analyse different regimes of interest rates, e.g. compare the Euro era to the time before the introduction of the Euro, and also analyse the low-interest rate environment.

As a second point, several of the mentioned articles come to the conclusion that three or more parameters are an adequate choice for describing changes in the term structure. In our analysis, we provide an update and show that when aiming to model changes in the term structure as parsimonious as possible, even two parameters may be sufficient for many applications if one considers change periods of one year or more.

Our approach is to take historical time series of the term structure based on German government bond yields, where we deal with an estimated term structure. A similar approach has been used by Diebold, Rudebusch, and Aruoba (2006), but in contrast to this work, we do not relate the parameters of the term structure to macro factors and also do not try to predict the evolution of the yield curve based on the factors. Instead, we concentrate on the empirical fit of the linear model of the term structure and discuss the implications. We also do not deal with the explanation or forecasting of term premia (see, for instance Dai and Singleton (2002)). Instead, we take the historical interest rates for different maturities as given and try to model the relationship between the interest

rates of different maturities and points in time as parsimonious as possible, thereby always considering the model fit. There is a large literature about the precise modeling of the term structure, especially at daily, weekly or monthly frequency (see Adrian, Crump, and Moench (2013)). In our analysis, we compare different horizons and show that for a change horizon of one year or longer, our model provides a pragmatic solution to the trade-off of a parsimonious term structure model that still has a good fit to the empirical data.

In applications to the banking sector, we show how to use the approach to derive consistent and harmful stress scenarios, how to estimate the banks' earnings from term transformation and we estimate the "likelihood" of the increase in the interest level after the end of the low-interest environment.

The paper is structured as follows. In Section 2, we explain the setup for modeling the term structure and discuss the distinction between our proposed model and term structure models which incorporate future interest rate paths. In Section 3, we show how this model of the term structure can be used to value interest bearing instruments and we apply this term structure model to some specific cash flow distributions. In Section 4, the empirical data used in the study is described and, in Section 5, we give the results related to our parsimonious modeling of the term structure. In Section 6, we apply the model to banking. Section 7 concludes.

2 Modeling the term structure

2.1 General approach

We collect risk-free interest rates (zero-bond returns) $r_t(m)$ of different maturities $m = m_1, ..., m_N$ in the vector R_t, where the index t gives the point in time and m_1 to m_N the maturities of the interest rates:

$$R_t = \begin{pmatrix} r_t(m_1) \\ \vdots \\ r_t(m_N) \end{pmatrix} \tag{1}$$

This spans, for every point in time, a term structure of interest rates, i.e. a collection of interest rates that mature after different periods. In other words: The vector R_t has a cross-sectional dimension of the different maturities. To make this cross-sectional dimension more manageable, Nelson and Siegel (1987) model the term structure (for each point in time) as a function, depending only on a small number of parameters (here: four, namely $\alpha_{0,t}$, $\alpha_{1,t}$, $\alpha_{2,t}$ and $\lambda_t > 0$):

$$r_t^{NeSi}(m) = \alpha_{0,t} + \alpha_{1,t}\frac{1 - exp(-\lambda_t \cdot m)}{\lambda_t \cdot m} + \alpha_{2,t}\left(\frac{1 - exp(-\lambda_t \cdot m)}{\lambda_t \cdot m} - exp(-\lambda_t \cdot m)\right) \tag{2}$$

Svensson (1994) added a further term (similar to the last one, but with a different parameter λ_t from that in Equation (2)), yielding a model with six parameters.

Often it is analytically easier to deal with linear relationships, for instance, one can then estimate the coefficients with ordinary least squares (OLS) (see Equation (9)).

Diebold and Li (2006) turned Equation (2) into a linear relationship with three parameters by setting the parameter λ_t as time-constant, namely to $\bar{\lambda} = 12 \cdot 0.0609$ (see Equation (8)). Generally, a linear model for the term structure has the form:

$$r_t^{lin.Model}(m) = \alpha_{0,t} + \alpha_{1,t} \cdot f_1(m) + \ldots + \alpha_{n,t} \cdot f_n(m) \tag{3}$$

For econometric reasons, as the interest rates do not seem to be stationary (see, for instance, Diebold and Li (2006)), we often deal with changes in interest rates (where the $\triangle$-operator represents, in our case, monthly, quarterly, half-yearly, yearly, one-and half yearly or biennial changes). As the functions $f_i(m)$ do not depend on time-varying parameters (like λ_t in Equation (2)), we obtain:

$$\triangle r_t^{lin.Model}(m) = \beta_{0,t} + \beta_{1,t} \cdot f_1(m) + \ldots + \beta_{n,t} \cdot f_n(m) \tag{4}$$

with $\beta_{i,t} = \triangle \alpha_{i,t}$. Note, that a formulation in levels (with $\alpha_{i,t}$) contains nearly the same information as a specification in changes (with $\beta_{i,t}$). However, there is a slight differences: A term structure with $r_t(m) = \alpha_{0,t}$ implies a flat term structure, whereas $\triangle r_t(m) = \beta_{0,t}$ only implies a parallel shift, where the original term structure can have any shape.

In Section 3 of this paper, we mainly deal with a simple model for the term structure that includes the level and the steepness. We choose the following model to describe the term structure:

$$r_t^{HeMe}(m) = \alpha_{0,t} + \alpha_{1,t} \cdot m \tag{5}$$

In this model, the parameter $\alpha_{0,t}$ corresponds to the short-term interest level in time t and the parameter $\alpha_{1,t}$ gives the slope per unit of measurement, in our case years.

For comparison, we consider the following three linear models, which are all special cases of the model in Equation (3), namely a constant interest level (leading to a parallel shift), the model from above in Equation (5) and the model of Diebold and Li (2006) and compare their explanatory power:

$$\triangle r_t^{Parallel}(m) = \beta_{0,t} \tag{6}$$

.

$$\triangle r_t^{HeMe}(m) = \beta_{0,t} + \beta_{1,t} \cdot m \tag{7}$$

$$\triangle r_t^{DiLi}(m) = \beta_{0,t} + \beta_{1,t} \frac{1 - exp(-\bar{\lambda}m)}{\bar{\lambda}m} + \beta_{2,t} \left(\frac{1 - exp(-\bar{\lambda}m)}{\bar{\lambda}m} - exp(-\bar{\lambda}m) \right) \tag{8}$$

We regress the estimated changes $\triangle r_t^{Model}(m)$ of the three models on the true changes $\triangle r_t(m)$, having $dim(R_t) \cdot T_{Period}$ observations.[1]

[1] If we look at the level, not at changes in the term structure, the two-factor-model of Equation (7) outperforms the other models even more strongly. As it is more common in the field of banking to look at changes in the term structure (not so much at the level), we stick to changes in the term structure and look at different horizons.

To determine the parameter vector β_t, we run (for each point in time t) the following OLS estimation for $\beta_t = (\beta_{0,t})$ (Equation (6)), $\beta_t = (\beta_{0,t}, \beta_{1,t})'$ (Equation (7)) and $\beta_t = (\beta_{0,t}, \beta_{1,t}, \beta_{2,t})'$ (Equation (8)):

$$\hat{\beta}_t = (X'X)^{-1}X' \triangle R_t. \tag{9}$$

Depending on the model, the matrix X (of dimension $dim(R_t) \times d$) consists of up to three columns d, where the first column is always composed of ones. In the case of Equation (7), the second column includes the maturities $m_1, ..., m_N$. In the case of Equation (8), the second and third columns include time-constant function values of the maturities m_1 to m_N. For instance for $m = 1.5$ (which is one and a half year or 18 months) we obtain 0.8377 and 0.1437. Equation (9) makes it possible to transform a stress scenario that includes changes in interest rates of various maturities into the number of parameters of the vector β.

2.2 Discussion of alternative approaches

The models above are meant to describe the term structure, but they do not deal with future paths of interest rates and their distribution as more complex stochastic models (see, for instance, Vasicek (1977), Cox, Ingersoll, and Ross (1985) and Heath, Jarrow, and Morton (1992)). That is, they do not aim to model the evolution of interest rates based on underlying mechanisms such as volatility and mean reversion. This sets limits to the applications of these model, e.g., instruments that are dependent on the interest rate volatility can not be priced.[2]

This could be considered as a shortcoming of the simple models above, as, to some extent, the models are modest in comparison with fully fledged (stochastic) interest models that provide information about the distribution of future interest rates. However, we see nevertheless merit in revisiting and analysing this type of models.

The knowledge about the distribution has little value if our knowledge about the portfolio composition (and thereby interest sensitivity) is limited. As to banks, the knowledge for outsiders is scarce because we have little information about their portfolio composition and the extent to which they pass on changes in the market rates to their bank rates. Often, the regulatory requirements stipulate only that banks report aggregate values close to the duration of their equity which do not reveal much on the underlying structure. Here, the simple models seem more suitable and sufficient to take into account the scarce information. Making use of the empirical distribution of the two parameters (level and slope) that govern the term structure in our model, it is possible to derive applications for banks (Section 6).

One has to bear in mind that even complex models are still models with a certain range of application. The application and interpretation of stochastic interest rate models requires profound mathematical knowledge. By contrast, our simple model provides closed-from solutions that can easily be interpreted and applied.

[2]For instance swaps can be priced with the deterministic models, but interest rate options like caps are out of scope.

3 Valuation of instruments

3.1 General

Generally speaking, the present value PV of a stream of risk-free cash flows CF is calculated as follows:

$$PV = \int_0^\infty CF(t) exp(-r(t)t) dt \tag{10}$$

In the following, we show how the simple linear model of the term structure of Equation (5) can be used to value interest bearing instruments. The simple term structure model of Equation (5) allows us to investigate changing slopes of the term structure, even if we use strict assumptions as to the shape of the term structure at the time of the shock. In other words, the shape of the term structure at the time of the shock only determines the weighting of future cash flows and some weighting schemes lead to closed-form expressions. For instance, if we assume a flat term structure at the time of the shock, we obtain manageable expressions without losing much precision (relative to the assumed true term structure model of Equation (5)). This holds, especially, if there are no cash flows in the far future, which is the case, for instance, for geometrically declining cash flows. By contrast, if we look at a normal bond (with no extreme values for the maturity and for the slope), the relative error is even below 5% for the relative level or slope shift (see Table 8).

In the following, we mainly deal with two assumptions: (i) we use the linear term structure model $r(t) = \alpha_0 + \alpha_1 \cdot t$ of Equation (5) (note that we now write t as dependent variable instead of m for the valuation of future cash-flows) and (ii) we calculate the present values and the derivatives at $\alpha_1 = 0$, i.e. at a flat term structure at the time of the shock (later in Table 2 we give also closed-form solutions for the term structure model (5) without the assumption of a flat term structure at the time of the shock):

$$PV \;|r(t) = \alpha_0 + \alpha_1 \cdot t, \alpha_1 = 0 \; = \int_0^\infty CF(t) \cdot exp(-\alpha_0 t) dt \tag{11}$$

$$\frac{\partial PV}{\partial \alpha_0} \;|r(t) = \alpha_0 + \alpha_1 \cdot t, \alpha_1 = 0 \; = -\int_0^\infty t \cdot CF(t) \cdot exp(-\alpha_0 t) dt \tag{12}$$

$$\frac{\partial PV}{\partial \alpha_1} \;|r(t) = \alpha_0 + \alpha_1 \cdot t, \alpha_1 = 0 \; = -\int_0^\infty t^2 \cdot CF(t) \cdot exp(-\alpha_0 t) dt \tag{13}$$

We define $f_\gamma(t) := CF(t) \cdot exp(-\alpha_0 t)/PV \;|r(t) = \alpha_0 + \alpha_1 \cdot t, \alpha_1 = 0$ as the density function of the variable γ, which is the standardized present value of cash flows (for a flat term structure). Under the two assumptions from above, we obtain (see Equation (11)):

$$\int_0^\infty f_\gamma(t) dt = 1 \tag{14}$$

Equations (12) and (13) become more handy:

Table 1: Relative slope sensitivity for different distributions

Cash flow distribution (present value)	Relative slope sensitivity C	Remark(s)
Bernoulli distribution, probabilities are 0.5	$2 \cdot D^2$	Special case of the beta distribution, $\beta \to 0$
Exponential distribution	$2 \cdot D^2$	
Uniform distribution	$4/3 \cdot D^2$	Special case of the beta distribution, $\beta = 1$
One-point distribution	D^2	corresponds to a zero bond

This table shows the relative slope sensitivity c for different cash flow distributions (where the present value of the cash flow γ is used) in dependence of the relative duration ($D = E(\gamma)$). Note that the term structure is assumed to be flat at the time of the shock, i.e. $\alpha_1 = 0$.

$$\begin{aligned} D = -\frac{\frac{\partial PV}{\partial \alpha_0}}{PV} \; |\alpha_1 = 0 &= \int_0^\infty t \cdot f_\gamma(t) dt \\ &= E(\gamma) \end{aligned} \tag{15}$$

$$\begin{aligned} C = -\frac{\frac{\partial PV}{\partial \alpha_1}}{PV} \; |\alpha_1 = 0 &= \int_0^\infty t^2 \cdot f_\gamma(t) dt \\ &= \big(E(\gamma)^2 + var(\gamma)\big) \end{aligned} \tag{16}$$

where $E(\cdot)$ and $var(\cdot)$ denote the expectation and the variance operator, respectively.

Please note that the expectation and the variance are calculated with respect to the present value of the cash flows γ, not with the nominal values CF; only if the interest level α_0 reaches zero, they are both equal.

From Equation (14), we see that the (with the present value) standardized cash flows (under the assumption of a flat term structure) $f_\gamma(t)$ corresponds to a density function for a random variable with only non-negative values, i.e. for the point in time when a certain standardized cash-flow is due. In Equation (16), we see that the relative slope sensitivity is closely related to the expectation and the variance. In Table 1, the relative slope sensitivity for different distributions are given. Apart from the case of the one-point-distribution, we see that the variance of the cash flows substantially contributes to the (relative) slope sensitivity, i.e. the present value effect of changes in the slope not only depends on the points in time when the cash flows on average take place, but also on their distribution.

In case of slope changes, the present value loss may be underestimated if one uses the

duration, as the following example shows. We start by the total differential of the present value (PV), using the term structure model of Equation (5), i.e. $\triangle r(t) = \triangle\alpha_0 + \triangle\alpha_1 \cdot t$ (at a flat term structure at the time of the shock), and we standardize this relationship with the present value and the Equations (15) and (16):

$$\frac{\triangle PV}{PV} = -D \cdot \triangle\alpha_0 - C \cdot \triangle\alpha_1 \tag{17}$$

Considering only slope changes, i.e. $\triangle\alpha_0 = 0$, and using the slope change to calculate the interest rate change at the relative duration D, i.e. $t = D$, so that $\triangle r = \triangle\alpha_1 \cdot D$, we obtain:

$$\begin{aligned} \frac{\triangle PV}{PV} &= -C \cdot \triangle\alpha_1 \\ &= -h \cdot D \cdot \triangle r \end{aligned} \tag{18}$$

with $h = C/D^2$. If the parameter h is equal to one, i.e. all cash-flows are concentrated in one point of time, Equation (18) is the well-known formula for the relative change in present value due to a parallel shift in the term structure of $\triangle r$, for instance, if the relative duration D is equal to 5 (years) and $\triangle r$ equals 300 basis points, then the relative loss in present value is 15%. According to Table 1, the parameter h is usually greater than one for slope changes, for instance equal to 4/3 for a standardized cash flow following the uniform distribution (third row). If one neglects this parameter although it is often larger than one in reality one underestimates the loss in present value if changes in the steepness are analyzed (here: by one third, for instance, if the calculated loss is 15% of the present value, the true loss is 20%).

3.2 Convexity

The change in present value can be represented as a sum of derivatives and powers of the change (here the change in the interest level α_0). The approximation with a small number of derivatives is the better the smaller the change.

$$\triangle PV = \frac{\partial PV}{\partial\alpha_0} \cdot \triangle\alpha_0 + 0.5 \cdot \frac{\partial^2 PV}{(\partial\alpha_0)^2} \cdot (\triangle\alpha_0)^2 + Rest \tag{19}$$

In finance, the first two summands are named: the first derivative is called the duration; the second derivative is the convexity.

In our model, there is a close connection to the concept of convexity $\frac{\partial^2 PV}{(\partial\alpha_0)^2}$:

Calculating the second derivative of Equation (10) and assuming $r(t) = \alpha_0$, we obtain:

$$\frac{\partial^2 PV}{(\partial\alpha_0)^2} \; | r(t) = \alpha_0 + \alpha_1 \cdot t, \alpha_1 = 0 \quad = \int_0^\infty t^2 \cdot CF(t) \cdot exp(-\alpha_0 t) dt \tag{20}$$

which is the same as the negative derivative with respect to α_1 (slope sensitivity as in Equation (13)). In other words, when assuming the linear model from above for the term structure, the convexity corresponds to the negative sensitivity to changes in the steepness of the term structure. In the context of the Taylor-approximation, it does not make sense to look at the convexity in isolation because the first and most important summand in

Equation (19), the duration times the change in the interest level, would be neglected. By contrast, in the case of the linear term structure model of Equation (5), the convexity is closely related to a change in the slope of the term structure, meaning that it is sensible to look at the second derivative in isolation, especially in banking where a substantial part of the net interest income comes from term transformation (see also Section 6.2).

As term structure changes in reality can be approximated by slope and intercept changes (no matter the change horizon), the convexity is likely a good approximation of a portfolio's sensitivity to slope changes. This interpretation is one of the main contributions of our paper.

3.3 Application to some instruments

Using the linear model with two parameters for the term structure of Equation (5), we can calculate, for some cash flow functions $CF(t)$ and instruments, the present value and the derivatives in an explicit manner, which we show in the Table 2 (not necessarily under the assumption of a flat term structure at the time of shock). We look at four different cash flow distributions, namely a constant cash flow for some time, a geometrically declining cash flow, a passive trading strategy and a normal bond.

The passive trading strategy consists in investing in default-free par-yield bonds in a revolving manner. Whenever such a bond matures, one reinvests into the then current par-yield bond of maturity M. The interest payments are taken out (see Memmel (2014) for a discussion of these payments). For instance, if one chooses the maturity M as 10 years, then every month (if the frequency is monthly[3]) $1/120 = 1/(10 \cdot 12)$ is invested in the then current par-yield bond. This strategy is feasible, no matter the dynamics of the term structure: whenever a bond matures, the repayment of the principal matches exactly the new investment in a bond that is quoted at par. With this strategy, we want to reproduce a bank's continuous business model, which consists in granting loans of a certain maturity in a revolving manner. The total investment of this strategy is one; however, the present value of this bond portfolio can deviate from one due to shocks in the term structure or a slope of the term structure different from zero; in the Equations (56) and (61), we show the change in present value of this bond portfolio due to level and slope shifts.

For all four cash flow distributions, we look at three different situations of the term structure at the time of the shock: First, the case of a normal term structure, i.e. we stipulate that the slope coefficient is positive (and not negative).[4] Second, the situation in which the slope is negligible and, third, the situation in which the level and the slope are both close to zero. The time span during the low-interest rate environment (2014-2022) in the Euro area or the interest environment in some countries (for instance, Japan or Switzerland) comes close to this situation.

[3]For practical reasons and not continuous as used in the calculations elsewhere in the paper.

[4]We assume a normal term structure mainly for technical reasons: in case of a normal term structure (i.e. a positive slope coefficient), the closed-form solution includes the wide-spread distribution function of a normally distributed random variable (often, its symbol is Φ). In case of an inverse term structure, there exists as well a closed-form solution, however it includes the less common imaginary error function.

[5]One obtains for the relative duration $d = M/2$ and for the relative convexity $c = m^2/3$, yielding $c = 4/3d^2$ which corresponds to Table 1, third row.

[6]One obtains for the relative duration $D = 1/\lambda$ and for the relative convexity $C = 2/\lambda^2$, yielding

Table 2: Closed-form solutions

Cash flow / instrument	Term structure at the time of the shock	Present value	Level change	Slope change
Constant cash flow (Appendix A.2)	Normal ($\alpha_1 > 0$)	Eq. (37)	Eq. (38)	Eq. (39)
	Flat ($\alpha_1 = 0$)	Eq. (40)	Eq. (41)	Eq. (42)
	Flat and zero[5] ($\alpha_0 = 0, \alpha_1 = 0$)	$k \cdot m$	$-k \cdot \frac{m^2}{2}$	$-k \cdot \frac{m^3}{3}$
Declining cash flow (Appendix A.3)	Normal ($\alpha_1 > 0$)	Eq. (43)	Eq. (44)	Eq. (45)
	Flat ($\alpha_1 = 0$)	$\frac{k}{\lambda+\alpha_0}$	Eq. (46)	Eq. (47)
	Flat and zero[6] ($\alpha_0 = 0, \alpha_1 = 0$)	$\frac{k}{\lambda}$	$-\frac{k}{\lambda^2}$	$-\frac{2 \cdot k}{\lambda^3}$
Passive trad. strategy (Appendix A.4)	Normal ($\alpha_1 > 0$)	Eq. (49)	Eq. (50)	Eq. (51)
	Flat ($\alpha_1 = 0$)	1	Eq. (56)	Eq. (61)
	Flat and zero ($\alpha_0 = 0, \alpha_1 = 0$)	1	$-\frac{m}{2}$	$-\frac{m^2}{3}$
Normal bond (Appendix A.5)	Normal ($\alpha_1 > 0$)	Eq. (63)	Eq. (64)	Eq. (65)
	Flat ($\alpha_1 = 0$)	Eq. (66)	Eq. (54)	Eq. (59)
	Flat and zero ($\alpha_0 = 0, \alpha_1 = 0$)	$1 + c \cdot m$	$-m \cdot \left(1 + c \cdot \frac{m}{2}\right)$	$-m^2 \left(1 + c \cdot \frac{m}{3}\right)$

This table shows closed-form solution for some cash-flow models and instruments, where k is the cash flow at the beginning, m is the maturity, $\lambda > 0$ is the decay rate and c is the coupon rate of the normal bond, which has a principal of one. Model for the term structure: $r(t) = \alpha_0 + \alpha_1 \cdot t$.

4 Data

For our analysis, we use zero-bond rates of different maturities, based on German government bonds and derived using the method according to Svensson (1994) with six parameters (see also Schich (1997) for the application to German data). Note that we are not dealing with single bonds, but with an already estimated term structure. The period covers nearly fifty years (1975-01 to 2023-12) and we use monthly data; in the paper, we have $dim(R_t) = 20$ maturities (maturities of up to 10 years in steps of 6 months) and $T_{Period} = 588$ monthly observations (49 years), yielding 11 760 observations.

In Table 3, we report summary statistics, where in each point in time, we perform a linear fit $r_t(m) = \alpha_{0,t} + \alpha_{1,t} \cdot m$ (either on the yield curve itself or on the respective changes over the different time horizons). While this could be considered as frontloading our results discussed below, this is primarily done to obtain comparable (and easily interpretable) information on the level and steepness.[7] The average steepness is 13.52 bp per year (first column, second row), meaning that on average, over the observed time span, for each additional year of maturity, the return increases by around 14 bp. As a concrete example, a bond with 10 years of maturity yields on average 1.35% p.a. more than the short-term (0 year) interest rate, as the ten years of maturity contribute 10 times 13.52 bp of additonal return. The statistics on the level of changes in the yield curve show (not surprisingly) that the longer the time horizon considered (going from "Change (1 month)" to "Change (24 months)" in Table 3), the larger changes (in absolute values) in the level are observed. When looking at both mean or median, the changes (in absolute values) in the level seem to increase broadly linearly with the time horizon. The negative values can certainly be attributed to the selected time span where interest rates overall decreased. Considering the steepness of changes, however, no systematic (linear) increase (in absolute values) is observed in mean or median values, while the standard deviation increases. The 99th percentile of yearly changes is about 390 bp (fifth column, seventh row), significantly more than the 200 bp of the Basel shock, which was informed by yearly changes. However, the interest rate changes tend to be larger for short maturities and when the interest level is higher, which was the case in the seventies and eighties of the last century.

5 Results

5.1 Changes of the term structure

We investigate changes of the term structure for the given maturities with the help of a principal component analysis (PCA). We use different time horizons to calculate changes in the term structure, i.e. changes of one month, three months, six months, 12 months, 18 months and 24 months. For illustration, the factor loadings of the three first components for yearly (12 months) changes are displayed in Figure 1. Here, it is important to note that the PCA is a completely statistical method, i.e. it is agnostic about possible structures (like level or steepness shifts) in the data. Nevertheless, the first component (i.e. the most

$C = 2D^2$ which corresponds to Table 1, last row.

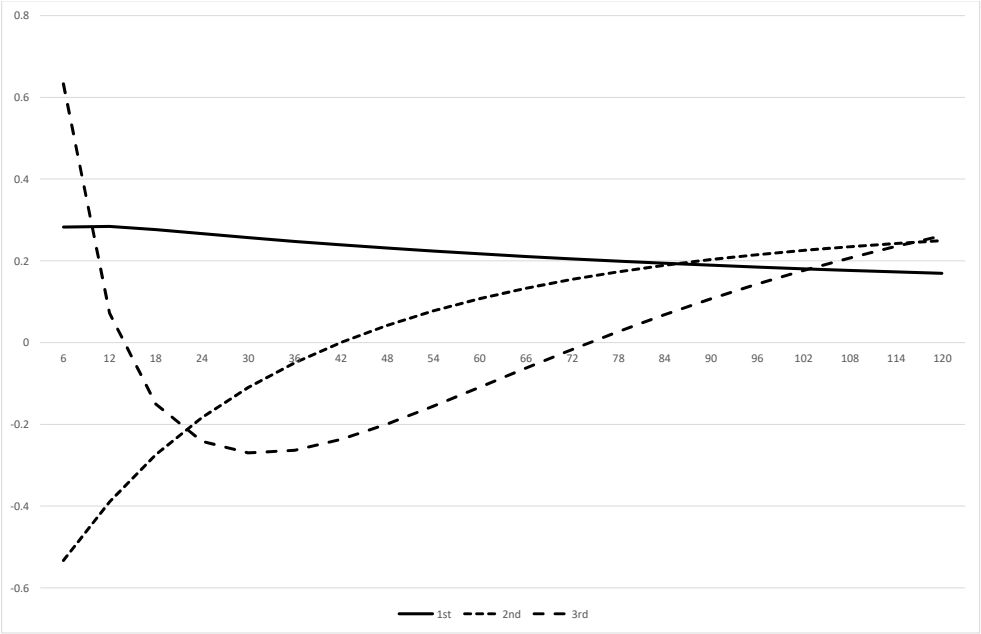
[7] In the following description, with "level" and "steepness" we refer to the results for level and steepness of the linear fit considering either the yield curve values (Term Structure row "Level") or calculated changes over the mentioned time horizon (e.g. "Term Structure Change (3 months)").

Table 3: Summary statistics

Term structure	Model parameter	Unit	Mean	SD	1st perc.	Median	99th perc.
Level	Level	Per cent	3.67	3.21	-0.99	3.69	11.65
	Steepness	bp per year	13.52	13.08	-21.39	14.80	39.08
Change (1 month)	Level	bp	-1.01	29.33	-91.49	-1.03	78.26
	Steepness	bp per year	-0.04	2.68	-6.57	-0.15	8.69
Change (3 months)	Level	bp	-3.02	62.38	-194.48	-2.09	172.11
	Steepness	bp per year	-0.13	5.10	-15.64	-0.34	15.04
Change (6 months)	Level	bp	-6.19	98.50	-308.89	-3.43	292.14
	Steepness	bp per year	-0.27	7.91	-23.36	-0.40	23.10
Change (12 months)	Level	bp	-13.20	150.56	-389.77	-11.37	391.51
	Steepness	bp per year	-0.41	11.78	-32.50	-0.31	30.78
Change (18 months)	Level	bp	-21.44	191.43	-522.50	-14.01	437.58
	Steepness	bp per year	-0.37	14.90	-35.75	-1.45	38.54
Change (24 months)	Level	bp	-30.07	222.29	-557.29	-34.36	579.11
	Steepness	bp per year	-0.36	17.06	-36.20	-2.68	44.26

This table shows summary statistics for the level of and changes in the term structure (Period: 1975-01 to 2023-12). "SD", "bp", "1st perc." and "99th perc." mean standard deviation, basis points, first percentile and 99th percentile. The summary statistics are based on the model for the term structure $r_t(m) = \alpha_{0,t} + \alpha_{1,t} \cdot m$ with $m = 0.5, 1, ..., 10$.

Figure 1: PCA: Factor loadings



This figure shows the factor loadings for the first three components of a principal component analysis (PCA) of yearly changes in interest rates of different maturities. German government bonds up to 120 months maturity in steps of 6 months. Monthly data; period: 1975-01 to 2023-12.

important one) is very similar to a parallel level shift (with longer maturities less affected) and the second component resembles a (concave) shift in the steepness. These results are in line with the findings in the literature (see Litterman and Scheinkman (1991), Knez et al. (1994), Bliss (1997) and Memmel (2014)).

The explained variance of changes in the term structure is shown in Table 4. Here, one can see that the explanatory power of the first factor increases with the time horizon over which changes are calculated. At the same time, the relevance of the second and third factor decreases with increasing change horizon. While for a change horizon of one month, the first factor explains "only" 83% and the second and third factors contribute 11% and 5%, respectively, the first factor can explain nearly 93% for a change horizon of 24 months, and the second and third factors only contribute 6% and 1%, respectively. This already indicates that the choice of an appropriate parsimonious approximation will depend on the time horizon.

We also regress the interest rate changes derived from the models (Equations (6), (7) and (8)) on the true interest rate changes and compare the coefficient of determination R^2 of the three different models. The results are shown in Table 5, where the following can be noted:

- The parallel shift of the term structure is often applied, for instance it is prescribed by bank regulators and used by supervisors in what is known as the Basel interest

Table 4: Explained Variance of Changes in the Term Structure

Change in the term structure	Contribution	Principal component		
		First	Second	Third
1 month	Additional	82.60%	11.01%	4.97%
	Cumulative	82.60%	93.61%	98.58%
3 months	Additional	88.56%	8.33%	2.37%
	Cumulative	88.56%	96.89%	99.26%
6 months	Additional	90.39%	7.85%	1.30%
	Cumulative	90.39%	98.24%	99.54%
12 months	Additional	91.46%	7.23%	1.01%
	Cumulative	91.46%	98.69%	99.70%
18 months	Additional	92.25%	6.75%	0.79%
	Cumulative	92.25%	98.99%	99.78%
24 months	Additional	92.92%	6.18%	0.71%
	Cumulative	92.92%	99.10%	99.81%

This table shows the fraction of the explained variance of the changes in the term structure over different time horizons, derived from a principal component analysis (PCA). German government bonds up to 120 months maturity in steps of 6 months. Monthly data; period: 1975-01 to 2023-12. For each of the change horizon, the additional contribution and the cumulative contribution are reported.

rate shock (see Basel Committee on Banking Supervision (2004)). One reason is the analytical simplification when using a parallel term structure shift. When looking at the first row of Table 5), there is also a profound empirical reason: A parallel shift explains up to 90% of the variation and is close to the theoretical maximum (see the first column of Table 4). The gap between the theoretical maximum and the parallel shift may be due to the fact that the interest rates of longer maturities are more sluggish and that the loadings for the first factor are not a parallel line, but tend to decrease (see Figure 1 for an illustration for yearly changes).

- The same holds true if we compare the R^2 of the two-factor model (in Table 5, second row) with the theoretical maximum (Table 4, cells relating to the second factor, cumulative values). Here, we assume that the model of linear changes (see Equation (7)) cannot be considered an exact fit for the real interest rate changes (see Figure 1, where the second component is not a straight line, but a concave curve).

- The three-factor model has the highest explained variance (see Table 5). However, the additional increase in the explained variance (relative to the two-factor model) seems relatively small, especially for longer change horizons. Note that we are only dealing with a section of the term structure (for insurance companies, longer maturities may be relevant); if we looked at the whole term structure (including longer maturities), the difference in explained variances relative to the two-factor model might be more relevant.

- When we consider shorter horizons of changes in the interest rates (Table 4; 1 month or, to some extent, 3 months), we see that the third principal component makes a

Table 5: Different models: coefficient of determination

Coefficient of determination	Change horizon					
	1 month	3 months	6 months	12 months	18 months	24 months
Parallel shift (see Eq. (6))	82.25	87.21	88.52	89.10	89.31	89.53
Two factors (see Eq.(7))	91.00	95.64	97.19	97.63	98.09	98.27
Three factors (see Eq. (8))	97.23	97.90	98.24	98.52	98.64	98.70

This table shows the coefficient of determination R^2 (in %) of three different interest rate models. German government bonds up to 120 months maturity in steps of 6 months. Monthly data; period: 1975-01 to 2023-12.

substantial contribution to explaining the variance. Looking at one-year or longer changes, this contribution is only about or less than one per cent.

5.2 Model selection with information criteria

There is a trade-off between finding the most parsimonious model (i. e. the model with the lowest number of parameters) and explaining the changes in interest rates of different maturities.We try to solve this trade-off by using two different information criteria: First, the information criterion AIC (Akaike information criterion) which takes into account the number of parameters and the model fit

$$AIC = -2 \cdot loglikelihood + 2 \cdot (\#parameters + 1) \tag{21}$$

and second, the Bayesian information criterion (BIC)

$$BIC = -2 \cdot loglikelihood + (\#parameters) \cdot log(\#observations) \tag{22}$$

For each month in our sample period 01/1975 to 12/2023 (which yields 588 observations), we regress the $N = 20$ interest rate changes for each of the three models on the corresponding true interest rate changes. We determine each month which of the three models performed best according to the two criteria. In Table 6, we report the shares relative to our sample period of 588 months, for the three models and for six change horizons for the AIC, and in Table 7, we report the equivalent information for the BIC. We see that the three factor model often performs best of the three models considered if the change horizon is relatively short; however we found only significant differences to the two-factor-model for the one month horizon, the shortest horizon considered in our study (see Appendix A.7). The turning point where the two factor model improves over the three factor model is around yearly changes, where we see that the performance is slightly better for the BIC. For longer horizons of 18 months or 24 months, the two factor model performs on average better than the three factor model according to both criteria. From this analysis, we conclude that the two factor model is not only easier to treat and analytically tractable, but for longer change horizons, the two-factor-model outperforms the three factor model.

Table 6: Information criterion AIC

Model for the term structure	Change horizon (months)					
	1 ***	3	6	12	18	24
Parallel shift (see Eq. (6))	0.5%	0.6%	0.2%	0.7%	0.5%	0.3%
Two factors (see Eq.(7))	39.5%	44.1%	46.1%	48.1%	54.3%	56.8%
Three factors (see Eq. (8))	60.0%	55.3%	53.7%	51.2%	45.2%	42.9%

This table shows how often the respective factor model for the term structure is the best one according to the information criterion AIC. Sample period: 1975-01 to 2023-12, 588 points in time. *** means a significance level of 1% for a test of the null-hypotheses that the 2- and 3-Factor-Model are equally good (see Appendix A.7).

Table 7: Information criterion BIC

Model for the term structure	Change horizon (months)					
	1 ***	3	6	12	18	24
Parallel shift (see Eq. (6))	3.4%	1.2%	1.0%	1.0%	1.2%	0.34%
Two factors (see Eq.(7))	39.1%	44.6%	47.6%	49.3%	54.8%	57.1%
Three factors (see Eq. (8))	57.5%	54.3%	51.4%	49.7%	44.1%	42.5%

This table shows how often the respective factor model for the term structure is the best one according to the information criterion BIC. Sample period: 1975-01 to 2023-12, 588 points in time. Deviations from 100% are due to rounding. *** means a significance level of 1% for a test of the null-hypotheses that the 2- and 3-Factor-Model are equally good (see Appendix A.7).

5.3 Analysis of subperiods

In this section, we analyse different periods to gather whether the results obtained in the two previous sections still hold for subsets of the time horizon.

First, we split our sample in the parts before and after the German reunification (1990-10). After the reunification, the two-factor-model performs well, but before the reunification, the three-factor-model outperforms the two-factor-model clearly. This is also interesting with respect to the findings e.g. in Litterman and Scheinkman (1991) or Knez et al. (1994) who find that three or even more factors are needed for an appropriate description. According to our analysis, while this is true for the past (when these analyses were performed), this does not hold anymore for the more recent term structure changes.

Second, we use the introduction of the euro (1999-01) as the starting point and set the end point to the end of the sample period (2023-12). For this "Euro era", the two-factor model outperforms the three-factor model according to the information criteria for all change horizons except 1 month (i.e. the respective proportion of the two-factor model in Tables 6 and 7 would be higher than that of the three-factor model). This indicates that especially for the nearer past, the two-factor model is even more relevant.

As complement, we next consider the "pre-Euro era" (1975-01 until 1998-12), in which we find that the three-factor model outperforms the two-factor model according to the information criteria for all change horizons.

To narrow this finding further down, as a third period, we look at the low-interest rate environment in the euro area (assumed starting point at 2014-06, end point 2022-06 [8]). Here, we find that the two-factor model even more clearly outperforms the three-factor model for all change horizons.

When considering again the complement (Euro era before the low-interest rate environment, i.e. 1999-01 until 2014-05), the results are rather mixed and similar to the ones shown for the entire time series in the Tables 6 and 7, i.e. the two- and three-factor models seem to be favorable for longer and shorter time horizons, respectively. We conclude from this that the two-factor model has its biggest merits in an environment with low interest levels.

6 Application to banks

6.1 Constructing stress scenarios

As said above, our modeling of the term structure neglects details about the distribution of future interest rates. However, we can establish an empirical distribution of the change in the future term structure, i.e. of the parameters β_0 and β_1 and thereby design stress scenarios. We use yearly changes because a yearly horizon seems relevant in the banking context. Due to the central limit theorem, there is good reason to believe that yearly changes (as the sum of e.g. daily changes) are normally distributed. In our sample, we have 49 yearly (1975-2023) non-overlapping observations. Below, we give two examples of how the knowledge of the distribution may be helpful to design stress scenarios:

1. Designing consistent stress scenarios for the whole term structure. Assume that the

[8] During this time span, the rate of European Central Bank's (ECB) deposit facility was negative.

term structure model in Equation (7) is valid and that the change in the interest rate $\triangle r\,(m_0)$ for any arbitrary maturity m_0 is given. Then the expected change in interest rate for any arbitrary maturity m_1 is:

$$E\left(\triangle r\,(m_1)\,|\triangle r\,(m_0)\right) = \left(1 + \frac{(m_1 - m_0)\,\sigma_{01} + (m_1 - m_0)\,m_0\sigma_1^2}{\sigma_0^2 + m_0^2 \cdot \sigma_1^2 + 2 \cdot m_0 \cdot \sigma_{01}}\right) \cdot \triangle r\,(m_0) \quad (23)$$

where σ_0^2 is the variance of the level change (in our study: 1.9422), σ_1^2 the variance of the slope change (in our study: 0.0117) and σ_{01} is the covariance between the two (in our study: -0.1023, see Appendix A.8 for a derivation of Equation (23)). This expression can thus be used to analyse expected changes in interest rates for different maturities in a way that leads to consistent changes that take into account the dependence structure of interest changes of different maturities. In our sample, the correlation between level and slope changes is negative ($\sigma_{01} < 0$) and more significant than the expression $m_0\sigma_1^2$. We would thus deduct that consistently with historical observations, the expected change should be smaller for longer maturities (which is also in line with the first principal component in Figure 1).

2. Finding the stress scenario that hurts most, i.e. the combination of β_0 and $\beta_1$that yields the maximal loss in present value for a given instrument and a significance level p. We assume that a bank's interest business can be represented by the passive trading strategy of Section 3.3 where the derivatives of the present value with respect to the parameters α_0 and α_1 according to Table 2 are set to γ_0 and γ_1.We further assume that the loss in present value of the equity is the relevant metric.[9] This yields to optimized changes β_0^* and β_1^*:

$$\beta_0^* = \frac{\gamma_0 \cdot \sigma_0^2 + \gamma_1 \cdot \sigma_{01}}{2 \cdot \lambda} \quad (24)$$

$$\beta_1^* = \frac{\gamma_0 \cdot \sigma_{01} + \gamma_1 \cdot \sigma_1^2}{2 \cdot \lambda} \quad (25)$$

with

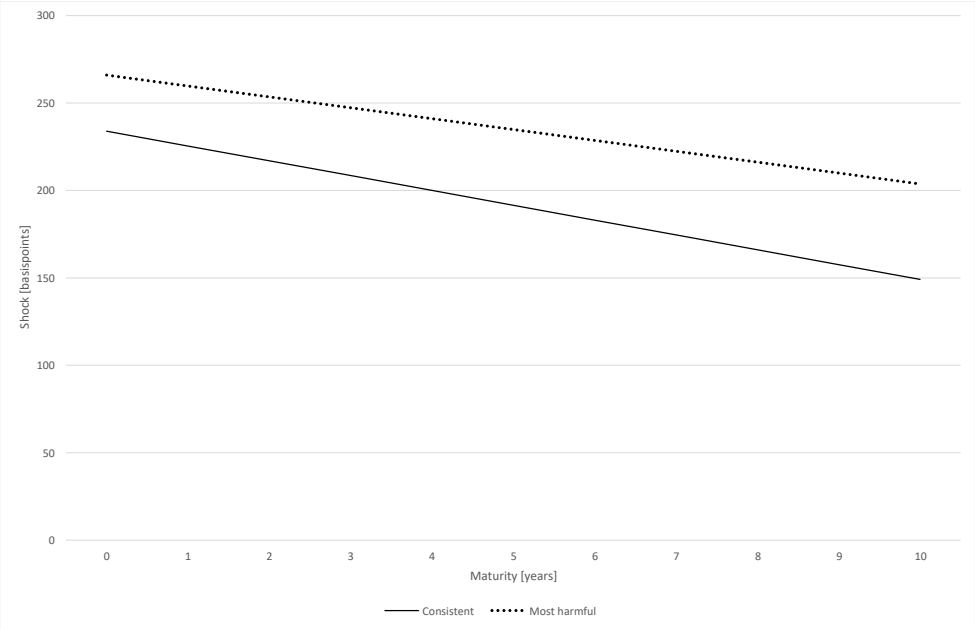
$$\lambda = -\sqrt{-\frac{\gamma_0^2 \cdot \sigma_0^2 + 2 \cdot \gamma_0 \cdot \gamma_1 \cdot \sigma_{01} + \gamma_1^2 \cdot \sigma_1^2}{8 \cdot ln(p)}} \quad (26)$$

where again σ_0^2 is the variance of the level change, σ_1^2 the variance of the slope change and σ_{01} is the covariance between the two (see Appendix A.9 for a derivation of the Equations (24) to (26), parameter values in our study given above).

In Figure 2, the two interest rate shocks are shown for a certain parameter constellation (Example 1: 4-year interest rate change as 200 bp given and Example 2: passive trading strategy with a maturity of 10 years, 10% significance level and a normal term structure at the time of the shock (average values in Table 3 for the parameters $\alpha_0 = 3.67\%$ and $\alpha_1 = 13.52$ bp/year). One can thus deduct that a high shock at small maturities and

[9]Even if we can describe the banks by the passive trading strategy, according to accounting rules the banks do not have to disclose all present value losses. Therefore, the present value of the equity is only then the relevant metric for the bank management if it steers the bank by the present value of the bank's equity (and not by accounting figures, for instance the return of the equity on the balance sheet).

Figure 2: Interest rate shocks



This figure shows the two interest rate scenarios derived in Section 6.1. The 'consistent shock' relates to a shock for the 4-year interest rate of 200 basis points. For the other maturities, the expected shock (given the shock for the 4-year interest rate) is calculated according to Equation (23). The 'most harmful shock' relates to the passive trading strategy of Section 3.3 for $M = 10$ years, a probability of $p = 10\%$ and the average values in Table 3 for the parameter α_0 and α_1, which are needed to compute the sensitivities according the Equations (50) and (51).

lower shock at higher maturities is the most harmful scenario for the assumed bank. This could also be used to design hedging strategies.

6.2 Earnings from term transformation

We assume that a bank's interest business can be modeled by the passive trading strategy of Section 3.3. This is, for instance, the case if this bank grants loans of a certain maturity in a revolving manner. For this strategy, we know the average return and its risk, measured by its duration. In addition, we often have, for a bank, supervisory risk figures for its term transformation risk, for instance the loss in present value as a consequence of a 200-bp-upward shock of the term structure.

We scale the (excess) return of the passive trading strategy such that bank i's risk from term transformation corresponds to the risk of this scaled strategy. In other words, we determine bank i's interest rate risk exposure and those of the passive trading strategy's and assume that this is same ratio as the bank's earnings from term transformation to the excess return of the passive trading strategy. The following formula results (see Appendix

(A.10)):

$$share_{TT,i} = \frac{1}{NIM_i} \cdot ER_i \cdot D_{mod,i} \cdot rr \tag{27}$$

where NIM_i is the net interest margin, i.e. net interest income over total assets, ER_i is the equity ratio (equity over total assets), $D_{mod,i}$ is the bank i's modified duration of its equity (often close to supervisory figures)[10] and rr is the passive trading strategy's remuneration of risk. For the German banking system, we can make the following guess $Share_{TT,DE} = 12.5\%$.[11]

6.3 Increase in the interest level

In summer 2022, the low-interest environment ended in the euro-area; the end was followed by a sharp increase in the interest level. In an out-of-sample study, we ask how extreme (i.e. unlikely) this increase was. To answer this question, we calculate the return parameters until summer 2022 and apply them to the (realized) yearly changes of the level β_0 and the slope β_1 from summer 2022 to summer 2023.

We obtain the following expression for the probability that such an (or more extreme) event occurs (see Appendix A.11).

$$p = exp\left(-\frac{\frac{1}{2}\beta_0^2\sigma_1^2 + \frac{1}{2}\beta_1^2\sigma_0^2 - \beta_0\beta_1\sigma_{01}}{\sigma_0^2\sigma_1^2 - \sigma_{01}^2}\right) \tag{28}$$

where σ_0^2 is the variance of the level change, σ_1^2 the variance of the slope change and σ_{01} is the covariance between the two. Note that the estimates for the return parameters differ from the ones of Section 6.1 because the estimation period ended in summer 2022. For the whole sample (i.e. starting from January 1975), we estimate a probability of $p = 10\%$, i.e. 10% of the (realized) yearly changes in the term structure were equal to or more extreme than the development after the end of the low-interest rate environment. If we look only at the period after the introduction of the euro, i.e. the parameters σ_0^2, σ_1^2 and σ_{01} are estimated in the period from January 1999 to June 2022, this probability is close to zero ($p = 0.4\%$).

7 Conclusion

In our empirical study for Germany, we show that changes in the term structure over different change horizons can be described by a small number of factors and that defining an appropriate parsimonious model depends on the change horizon and on the historical period considered. We show that yearly (or longer) changes in the term structure can be

[10] For instance, in Germany, banks have to report the present value losses as a consequence of a 200-bp-shift of the term structure. As the reporting includes all the interest bearing products (i.e. on the asset- and liability-side), it corresponds to the equity's present value loss. As the shift-size is known (200 bp), one can compute something close to the equity duration.

[11] The following values (averages of the last five years) are used: $NIM_{DE} = 0.92\%$, $ER_{DE} = 7.88\%$, and $D_{mod,DE} = 4.65$. The remuneration of risk for the passive trading strategy is 0.0031 (see Appendix (A.10)).

reasonably well described by a two-factor model, where one factor relates to the interest level and one to the slope of the term structure. This finding is especially relevant in the context of banking, where the planning horizon is often one year or longer. Furthermore, we discuss these results for different eras during the past 50 years and show that the two-factor model is especially relevant for the recent low-interest rate environment. We apply our findings to banks and show how to construct stress scenarios that are consistent in themselves and that are tailored to the banks' business model. We furthermore analyse the banks' earnings from term transformation and estimate how extreme the increase in the interest level was after the low-interest rate environment.

In addition, we show a new interpretation for the convexity and give closed-form solutions for some interest bearing instruments. The expressions in case of a term structure close to zero are often very handy and make it possible to have a first guess about the effects as a consequence of shocks to the term structure. The simple model proposed may thus give practitioners tools at hand to estimate broadly the effects of changes in the interest rate environment on banks'portfolios.

A Appendix

A.1 Useful integrals

For $\delta > 0$ and $m > 0$, we obtain:

$$\int_0^m exp(-\delta t)dt = \frac{1 - exp(-\delta m)}{\delta} \tag{29}$$

$$\int_0^m t \cdot exp(-\delta t)dt = \frac{1}{\delta^2}\left(1 - (1 + \delta m)\, exp(-\delta m)\right) \tag{30}$$

$$\int_0^m t^2 \cdot exp(-\delta t)dt = \frac{1}{\delta^3}\left(2 - \left(2 + 2\delta m + \delta^2 m^2\right) exp(-\delta m)\right) \tag{31}$$

$$\int_0^m exp\left(-at - bt^2\right) dt = \left(\Phi\left(\sqrt{2b} \cdot m + \frac{a}{\sqrt{2b}}\right) - \Phi\left(\frac{a}{\sqrt{2b}}\right)\right) \cdot \sqrt{\frac{\pi}{b}} \cdot exp\left(\frac{a^2}{4b}\right) \tag{32}$$

where $b > 0$ and $\Phi(\cdot)$ is the cumulative density function of the standard normal distribution. The equality of both sides of the equation can be seen by using the following relationship: $\int_f^e \frac{1}{\sqrt{2\pi}} exp\left(-1/2z^2\right) dz = \Phi(e) - \Phi(f)$ and we replace $-at - bt^2$ by $-1/2z^2 + a^2/(4b)$ in the left-hand side of Equation (32), and we apply the substitution method, using $z(t) = \sqrt{2b} \cdot t + \frac{a}{\sqrt{2b}}$, so that $e = z(m) = \sqrt{2b} \cdot m + \frac{a}{\sqrt{2b}}$, $f = z(0) = \frac{a}{\sqrt{2b}}$ and $dt = dz \cdot \frac{1}{\sqrt{2b}}$.

From the website WolframAlpha (2023), we obtain:

$$\begin{aligned}\int_0^m t\cdot exp\left(-at-bt^2\right)dt &= \frac{1-exp\left(-a\cdot m-b\cdot m^2\right)}{2b}\\ &- \frac{1}{2\cdot b^{3/2}}\cdot\sqrt{\pi}\cdot a\cdot exp\left(\frac{a^2}{4b}\right)\\ &\cdot \left(\Phi\left(\sqrt{2b}\cdot m+\frac{a}{\sqrt{2b}}\right)-\Phi\left(\frac{a}{\sqrt{2b}}\right)\right)\end{aligned} \tag{33}$$

$$\begin{aligned}\int_0^m t^2 exp\left(-at-bt^2\right)dt &= \frac{exp\left(-m\cdot(a+b\cdot m)\right)}{4\cdot b^{5/2}}\cdot\left[\sqrt{\pi}\left(a^2+2b\right)exp\left(\frac{(a+2bm)^2}{4b}\right)\right.\\ &\cdot \left(\Phi\left(\sqrt{2b}\cdot m+\frac{a}{\sqrt{2b}}\right)-\Phi\left(\frac{a}{\sqrt{2b}}\right)\right)\\ &- \left.\sqrt{b}\left(a\cdot\left(exp\left(m\cdot(a+b\cdot m)\right)-1\right)+2\cdot b\cdot m\right)\right]\end{aligned} \tag{34}$$

$$\int_0^\infty t^2 exp\left(-at-bt^2\right)dt = \frac{\sqrt{\pi}\left(a^2+2b\right)exp\left(\frac{a^2}{4b}\right)\Phi\left(-\frac{a}{\sqrt{2b}}\right)-a\sqrt{b}}{4\cdot b^{5/2}} \tag{35}$$

and

$$\begin{aligned}\int_0^m t^3 exp\left(-at-bt^2\right)dt &= \frac{exp\left(-m\cdot(a+b\cdot m)\right)}{8\cdot b^{7/2}}\\ &\cdot \left[\sqrt{b}\left(\left(a^2+4b\right)\cdot exp\left(m(a+bm)\right)-a^2+2abm-4b^2m^2-4b\right)\right.\\ &- \sqrt{\pi}\cdot a\left(a^2+6b\right)exp\left(\frac{(a+2bm)^2}{4b}\right)\\ &\cdot \left.\left(\Phi\left(\sqrt{2b}\cdot m+\frac{a}{\sqrt{2b}}\right)-\Phi\left(\frac{a}{\sqrt{2b}}\right)\right)\right]\end{aligned} \tag{36}$$

where $b>0$ and $\Phi(\cdot)$ is the cumulative density function of the standard normal distribution.

A.2 Constant cash flows

We consider the case of constant cash flows $CF(t)=k$ in the future for $0\le t\le m$ (and zero after m). With the help of Equation (32), we can determine the present value of the payment stream (for a normal term structure) as

$$\begin{aligned}PV\ |r(t)=\alpha_0+\alpha_1\cdot t,\alpha_1>0 &= k\cdot\left(\Phi\left(\sqrt{2\alpha_1}\cdot m+\frac{\alpha_0}{\sqrt{2\alpha_1}}\right)-\Phi\left(\frac{\alpha_0}{\sqrt{2\alpha_1}}\right)\right)\\ &\cdot\sqrt{\frac{\pi}{\alpha_1}}\cdot exp\left(\frac{\alpha_0^2}{4\alpha_1}\right).\end{aligned} \tag{37}$$

where $\Phi(\cdot)$ is the cumulative density function of the standard normal distribution.

The derivative of the present value in Equation (37) is

$$\frac{\partial PV}{\partial \alpha_0} = PV \cdot \left(\frac{\frac{1}{\sqrt{2\cdot\alpha_1}} \left(\varphi\left(\sqrt{2\alpha_1} \cdot m + \frac{\alpha_0}{\sqrt{2\alpha_1}} \right) - \varphi\left(\frac{\alpha_0}{\sqrt{2\alpha_1}} \right) \right)}{\Phi\left(\sqrt{2\alpha_1} \cdot m + \frac{\alpha_0}{\sqrt{2\alpha_1}} \right) - \Phi\left(\frac{\alpha_0}{\sqrt{2\alpha_1}} \right)} + \frac{\alpha_0}{2 \cdot \alpha_1} \right). \tag{38}$$

where $\varphi(\cdot)$ is the density function of the standard normal distribution. Using Equation (33) for the slope sensitivity of the present value in Equation (37), we obtain:

$$\begin{aligned} \frac{\partial PV}{\partial \alpha_1} &= -k \cdot \frac{exp\left(-m(\alpha_0 + \alpha_1 m)\right)}{4 \cdot \alpha_1^{5/2}} \cdot \left[\sqrt{\pi} \left(\alpha_0^2 + 2\alpha_1 \right) exp\left(\frac{(\alpha_0 + 2\alpha_1 m)^2}{4\alpha_1} \right) \right. \\ &\cdot \left(\Phi\left(\sqrt{2\alpha_1} \cdot m + \frac{\alpha_0}{\sqrt{2\alpha_1}} \right) - \Phi\left(\frac{\alpha_0}{\sqrt{2\alpha_1}} \right) \right) \\ &- \left. \sqrt{\alpha_1} \left(\alpha_0 \cdot \left(exp\left(m \cdot (\alpha_0 + \alpha_1 m) \right) - 1 \right) + 2\alpha_1 m \right) \right] \end{aligned} \tag{39}$$

For a flat term structure at the time of the shock ($\alpha_1 = 0$), we obtain (See Equation (11) in connection with Equation (29))

$$PV \; |r(t) = \alpha_0 + \alpha_1 \cdot t, \alpha_1 = 0 \; = k \cdot (1 - exp(-\alpha_0 m))/\alpha_0. \tag{40}$$

For the derivative with respect to the interest level we get (using the Equations (12) and (30)):

$$\frac{\partial PV}{\partial \alpha_0} \; |r(t) = \alpha_0 + \alpha_1 \cdot t, \alpha_1 = 0 \; = -\frac{k}{\alpha_0^2} \left(1 - (1 + \alpha_0 m) \, exp(-\alpha_0 m) \right) \tag{41}$$

where m is the maturity.
Using the Equations (13) and (31), we obtain for the slope sensitivity:

$$\frac{\partial PV}{\partial \alpha_1} \; |r(t) = \alpha_0 + \alpha_1 \cdot t, \alpha_1 = 0 \; = -\frac{k}{\alpha_0^3} \left(2 - \left(2 + 2\alpha_0 m + \alpha_0^2 m^2 \right) exp(-\alpha_0 m) \right) \tag{42}$$

Using the rule of l'Hôpital, we obtain the expressions of Table (2) (third row) for the Equations (40) to (42).

A.3 Geometrically declining cash flows

We look at the case of a geometrically declinig distribution, i.e. $CF(t) = k \cdot exp(-\lambda \cdot t)$, where λ is the decay parameter. The present value can be calculated with the help of Equation (32) (for $m \to \infty$, $a = \alpha_0 + \lambda$ and $\Phi(-x) = 1 - \Phi(x)$):

$$\begin{aligned} PV \; |r(t) = \alpha_0 + \alpha_1 \cdot t, \alpha_1 > 0 \; &= k \cdot \Phi\left(-\frac{\alpha_0 + \lambda}{\sqrt{2\alpha_1}} \right) \\ &\cdot \sqrt{\frac{\pi}{\alpha_1}} \cdot exp\left(\frac{(\alpha_0 + \lambda)^2}{4\alpha_1} \right) \end{aligned} \tag{43}$$

where $\Phi(\cdot)$ is the cumulative density function of the standard normal distribution. The derivative of the present value in Equation (43) is

$$\frac{\partial PV}{\partial \alpha_0} = -PV \cdot \left(\frac{\frac{-1}{\sqrt{2 \cdot \alpha_1}} \varphi\left(\frac{\alpha_0+\lambda}{\sqrt{2\alpha_1}}\right)}{\Phi\left(-\frac{\alpha_0+\lambda}{\sqrt{2\alpha_1}}\right)} + \frac{\alpha_0+\lambda}{2 \cdot \alpha_1} \right). \tag{44}$$

where $\varphi(\cdot)$ is the density function of the standard normal distribution. Using Equation (35) for the slope sensitivity of the present value in Equation (43), we obtain:

$$\frac{\partial PV}{\partial \alpha_1} = \frac{\sqrt{\pi}\left((\alpha_0+\lambda)^2 + 2\alpha_1\right) exp\left(\frac{(\alpha_0+\lambda)^2}{4\alpha_1}\right) \Phi\left(-\frac{\alpha_0+\lambda}{\sqrt{2\alpha_1}}\right) - (\alpha_0+\lambda)\sqrt{\alpha_1}}{4 \cdot \alpha_1^{5/2}} \tag{45}$$

Under the assumption of a flat term structure at the time of the shock, the present value of this strategy is $k/(\lambda+\alpha_0)$ (see Equation (29) for $m \to \infty$ and $a = \alpha_0 + \lambda$) and the negative duration is (see Equation (30) for $m \to \infty$ and $a = \alpha_0 + \lambda$):

$$\frac{\partial PV}{\partial \alpha_0} \; | r(t) = \alpha_0 + \alpha_1 \cdot t, \alpha_1 = 0 \; = -\frac{k}{(\lambda+\alpha_0)^2} \tag{46}$$

and the slope sensitivity is (see Equation (31) for $m \to \infty$ and $a = \alpha_0 + \lambda$):

$$\frac{\partial PV}{\partial \alpha_1} \; | r(t) = \alpha_0 + \alpha_1 \cdot t, \alpha_1 = 0 \; = -\frac{2k}{(\lambda+\alpha_0)^3} \tag{47}$$

A.4 Passive trading strategy

The cash-flow CF in time t is (when the trading strategy in $t_0 = 0$ is interrupted; for $0 < t \leq m$, after $t = m$, the cash-flow is zero):

$$CF(t) = \frac{1}{m} + \frac{m-t}{m} \cdot c_{Pari} \tag{48}$$

where $c_{pari} = \left(1 - exp\left(-\alpha_0 m - \alpha_1 m^2\right)\right) / \int_0^m exp\left(-\alpha_0 t - \alpha_1 t^2\right) dt$ is the coupon of a par-yield bond (The integral can be solved using Equation (32) to obtain a closed-form solution). The present value is (using the term structure model of Equation (5) and Equations (32) and (33)):

$$\begin{aligned}
PV &= \left(\frac{1}{m} + c_{Pari}\right) \cdot \int_0^m exp\left(-\alpha_0 t - \alpha_1 t^2\right) dt - \frac{c_{Pari}}{m} \int_0^m t \cdot exp\left(-\alpha_0 t - \alpha_1 t^2\right) dt \\
&= \left(\frac{1}{M} + c_{Pari}\right) \cdot \left(\Phi\left(\sqrt{2\alpha_1} \cdot M + \frac{\alpha_0}{\sqrt{2\alpha_1}}\right) - \Phi\left(\frac{\alpha_0}{\sqrt{2\alpha_1}}\right)\right) \cdot \sqrt{\frac{\pi}{\alpha_1}} \cdot exp\left(\frac{\alpha_0^2}{4\alpha_1}\right) \\
&- \frac{c_{Pari}}{m} \cdot \frac{exp\left(-m \cdot (\alpha_0 + \alpha_1 \cdot m)\right)}{4 \cdot b^{3/2}} \cdot \sqrt{\pi} \cdot a \cdot exp\left(\frac{(\alpha_0 + 2\alpha_1 m)^2}{4\alpha_1}\right) \\
&\cdot \left(\Phi\left(\sqrt{2\alpha_1} \cdot m + \frac{\alpha_0}{\sqrt{2\alpha_1}}\right) - \Phi\left(\frac{\alpha_0}{\sqrt{2\alpha_1}}\right)\right) \\
&- \frac{c_{Pari}}{m} \cdot \frac{\left(exp\left(m \cdot (\alpha_0 + \alpha_1 \cdot m)\right) - 1\right)}{2\alpha_1}
\end{aligned} \tag{49}$$

For the sensitivities with respect to the level and the slope, we obtain (closed-form solutions can be obtained using the Equations (33), (34) and (36)):

$$\frac{\partial PV}{\partial \alpha_0} = -\left(\frac{1}{m} + c_{Pari}\right) \cdot \int_0^m t \cdot exp\left(-\alpha_0 t - \alpha_1 t^2\right) dt + \frac{c_{Pari}}{m} \int_0^m t^2 exp\left(-\alpha_0 t - \alpha_1 t^2\right) dt \tag{50}$$

$$\frac{\partial PV}{\partial \alpha_1} = -\left(\frac{1}{m} + c_{Pari}\right) \cdot \int_0^m t^2 exp\left(-\alpha_0 t - \alpha_1 t^2\right) dt + \frac{c_{Pari}}{m} \int_0^m t^3 exp\left(-\alpha_0 t - \alpha_1 t^2\right) dt \tag{51}$$

Now, we turn to the case of a flat term structure at the time of shock. The present value of a default-free normal bond given the linear term structure model $r(t) = \alpha_0 + \alpha_1 \cdot t$ of Equation (5) is:

$$PV = \int_0^m c \cdot exp\left(-\alpha_0 t - \alpha_1 t^2\right) dt + exp\left(-\alpha_0 m - \alpha_1 m^2\right) \tag{52}$$

with c as the coupon and m as the maturity. The derivation with respect to the level α_0 at $\alpha_1 = 0$ is:

$$\frac{\partial PV}{\partial \alpha_0} \; | r(t) = \alpha_0 + \alpha_1 \cdot t, \alpha_1 = 0 \; = -c \int_0^m t \cdot exp\left(-\alpha_0 t\right) dt - M exp\left(-\alpha_0 m\right) \tag{53}$$

Applying (30), we get

$$\frac{\partial PV}{\partial \alpha_1} \; | r(t) = \alpha_0 + \alpha_1 \cdot t, \alpha_1 = 0 \; = -\frac{c}{\alpha_0^2} + \frac{c}{\alpha_0^2} \cdot exp\left(-\alpha_0 m\right) + \frac{c \cdot m}{\alpha_0} \cdot exp\left(-\alpha_0 m\right) - m \cdot exp\left(-\alpha_0 m\right) \tag{54}$$

For par-yield bonds, we have $\alpha_0 = c$ and we obtain:

$$dPV0 := \frac{\partial PV}{\partial \alpha_0} \; | r(t) = \alpha_0 + \alpha_1 \cdot t, \alpha_1 = 0 \; = -\frac{1}{\alpha_0} \cdot \left(1 - exp\left(-\alpha_0 m\right)\right) \tag{55}$$

For the passive trading strategy, we obtain:

$$\overline{dPV0} := \frac{1}{m} \int_0^M dPV0(t) dt = -\frac{1}{\alpha_0^2 m} \cdot \left(\alpha_0 m - 1 + exp\left(-\alpha_0 m\right)\right) \tag{56}$$

which corresponds to the expression in Memmel (2011).

By applying the rule of l'Hôpital twice, we obtain for a small interest level α_0 (which is shown in Table (2)):

$$\lim_{\alpha_0 \to 0} \overline{dPV0} = -\frac{1}{2} m \tag{57}$$

Similarly to Equation (53), we can calculate the derivation with respect to the steep-

ness α_1 at $\alpha_1 = 0$ and get:

$$\frac{\partial PV}{\partial \alpha_1} \; | r(t) = \alpha_0 + \alpha_1 \cdot t, \alpha_1 = 0 \quad = -c \int_0^{´\mathrm{m}} t^2 \cdot exp\left(-\alpha_0 t\right) dt - m^2 exp\left(-\alpha_0 m\right) \tag{58}$$

Applying (31), we obtain

$$\begin{aligned} \frac{\partial PV}{\partial \alpha_1} \; | r(t) = \alpha_0 + \alpha_1 \cdot t, \alpha_1 = 0 \quad &= -\frac{2c}{\alpha_0^3} + \frac{2c}{\alpha_0^3} \cdot exp\left(-\alpha_0 m\right) + \frac{2\alpha_0 c \cdot m}{\alpha_0^3} \cdot exp\left(-\alpha_0 m\right) \\ &+ \frac{\alpha_0^2 c M^2}{\alpha_0^3} \cdot exp\left(-\alpha_0 m\right) - m^2 exp\left(-\alpha_0 m\right) \end{aligned} \tag{59}$$

For par-yield bonds, we have $\alpha_0 = c$, i.e. the flat interest level α_0 corresponds to the coupon c and we obtain:

$$dPV1 := \frac{\partial PV}{\partial \alpha_1} \; | r(t) = \alpha_0 + \alpha_1 \cdot t, \alpha_1 = 0 \quad = -2 \frac{1 - (1 + \alpha_0 m) \cdot exp\left(-\alpha_0 m\right)}{\alpha_0^2} \tag{60}$$

For the passive trading strategy, we obtain:

$$\overline{dPV1} := \frac{1}{m} \int_0^m dPV1(t) dt = -\frac{2}{\alpha_0^3 m} \left(\alpha_0 m - 2 + (2 + \alpha_0 m)\, exp(-\alpha_0 m)\right) \tag{61}$$

By applying the rule of l'Hôpital several times, we obtain for a small interest level α_0 (which is shown in Table 2):

$$\lim_{\alpha_0 \to 0} \overline{dPV1} = -\frac{1}{3} m^2 \tag{62}$$

A.5 Normal bond

The linear modeling of the term structure from above makes it possible to derive a closed-form solution for the present value of a (bullet) bond given a term structure with strictly positive slope ($\alpha_1 > 0$) (see Equation (52) in connection with (32)):

$$\begin{aligned} PV \; | r(t) = \alpha_0 + \alpha_1 \cdot t, \alpha_1 > 0 \quad &= c \cdot \left(\Phi\left(\sqrt{2\alpha_1} \cdot m + \frac{\alpha_0}{\sqrt{2\alpha_1}} \right) - \Phi\left(\frac{\alpha_0}{\sqrt{2\alpha_1}} \right) \right) \\ &\cdot \sqrt{\frac{\pi}{\alpha_1}} \cdot exp\left(\frac{\alpha_0^2}{4\alpha_1} \right) + exp\left(-\alpha_0 m - \alpha_1 m^2\right) \end{aligned} \tag{63}$$

with $\Phi(\cdot)$ as the cumulative density function of the standard normal distribution, c as the coupon and M as the maturity (the principal is 1). Using Equation (38) and Equation (39), we can determine the derivative of Equation (63) with respect to the level (α_0) and the slope (α_1):

$$\begin{aligned}\frac{\partial PV}{\partial \alpha_0} &= -c \cdot \left(\Phi\left(\sqrt{2\alpha_1}\cdot m + \frac{\alpha_0}{\sqrt{2\alpha_1}}\right) - \Phi\left(\frac{\alpha_0}{\sqrt{2\alpha_1}}\right)\right) \cdot \sqrt{\frac{\pi}{\alpha_1}} \cdot exp\left(\frac{\alpha_0^2}{4\alpha_1}\right) \\ &\cdot \left(\frac{\frac{1}{\sqrt{2\cdot\alpha_1}}\left(\varphi\left(\sqrt{2\alpha_1}\cdot m + \frac{\alpha_0}{\sqrt{2\alpha_1}}\right) - \varphi\left(\frac{\alpha_0}{\sqrt{2\alpha_1}}\right)\right)}{\Phi\left(\sqrt{2\alpha_1}\cdot m + \frac{\alpha_0}{\sqrt{2\alpha_1}}\right) - \Phi\left(\frac{\alpha_0}{\sqrt{2\alpha_1}}\right)} + \frac{\alpha_0}{2\cdot\alpha_1}\right) \\ &- m \cdot exp\left(-\alpha_0 m - \alpha_1 m^2\right) \end{aligned} \tag{64}$$

$$\begin{aligned}\frac{\partial PV}{\partial \alpha_1} &= -c \cdot \frac{exp\left(-m(\alpha_0+\alpha_1 m)\right)}{4\cdot\alpha_1^{5/2}} \cdot \left[\sqrt{\pi}\left(\alpha_0^2 + 2\alpha_1\right) exp\left(\frac{(\alpha_0+2\alpha_1 m)^2}{4\alpha_1}\right)\right. \\ &\cdot \left(\Phi\left(\sqrt{2\alpha_1}\cdot m + \frac{\alpha_0}{\sqrt{2\alpha_1}}\right) - \Phi\left(\frac{\alpha_0}{\sqrt{2\alpha_1}}\right)\right) \\ &- \sqrt{\alpha_1}\left(\alpha_0 \cdot \left(exp\left(m\cdot(\alpha_0+\alpha_1 m)\right) - 1\right) + 2\alpha_1 m\right)] \\ &- m^2 exp\left(-\alpha_0 m - \alpha_1 m^2\right) \end{aligned} \tag{65}$$

If we assume a flat term structure at the time of the shock (see Equation (40), we obtain:

$$PV \;|r(t) = \alpha_0 + \alpha_1 \cdot t, \alpha_1 = 0 \;= c \cdot (1 - exp(-\alpha_0 m))/\alpha_0 + exp\left(-\alpha_0 m\right) \tag{66}$$

The level and slope sensitivities are given in the Equations (54) and (59). Using the rule of l'Hôpital for the part of the coupon payments, one obtains the expressions in the last row of Table 2.

A.6 Example: Normal bond

We assume that the term structure model of Equation (5) is the true one, we apply this model (see Equations (63), (64) and (65)) to a bond at the average values for the level and the slope (see Table 3) that is line with the underlying of the Euro-BUND-future contract, i.e. $m = 10$ years and coupon $c = 6\%$. The values and the relative errors when using the approximations of Table 2 are displayed in Table 8.

A.7 Tests statistics

Let p_i with $i = 1, 2, 3$ be the probabilities that the optimal model $X_t = (X_{t,1}, X_{t,2}, X_{t,3})$ in $t = 1, ..., T$ consists of i factors, where $X_{t,i}$ can take either the value 0 or 1 (for the best performing one). For instance, if in time $t = 10$ the two factor model is the optimal one, i.e. performs best, then $X_{10,2} = 1$ and $X_{10,1} = X_{10,3} = 0$. As the sum of X_t is multinomially distributed,[12] we have:

$$cov\left(X_{t,i}, X_{t,j}\right) = \begin{cases} p_i \cdot (1 - p_j) & \forall i = j \\ -p_i p_j & \forall i \neq j \end{cases} \tag{67}$$

[12] see, for instance, Weisstein (2024)

Table 8: Example: Normal bond

Term structure		Present	Level shift		Slope shift	
	Measure	value	Abs.	Rel.	Abs.	Rel.
Normal ($\alpha_1 > 0$)	Value	1.088	8.265	7.599	74.605	68.595
Flat ($\alpha_1 = 0$)	Value	1.195	9.286	7.770	84.508	70.716
	Rel. error	9.9%	12.4%	2.3%	13.3%	3.1%
Flat and zero	Value	1.600	13.000	8.125	120.000	75.000
($\alpha_0 = 0, \alpha_1 = 0$)	Rel. error	47.1%	57.3%	6.9%	60.8%	9.3%

This table shows the present value and the (negative) derivatives with respect to level and slope for a normal bond (maturity $M = 10$ years, coupon $c = 6\%$, principal = 1, which is line with the underlying bond of the Euro BUND future contract) at the mean values of Table 3 (term structure model of Equation (5) with level: 3.67% and slope: 13.52 bp/year). "Abs." means the (negative) deviation; "Rel." means the (negative) deviation relative to the present value. "Rel. error" means the relative error, where the true term structure model is the linear one of Equation (5).

Accordingly and taking account of the central limit theorem, we have

$$\sqrt{T}\left((\hat{p}_i - \hat{p}_j) - (p_i - p_j)\right) \longrightarrow N\left(0; p_i \cdot (1 - p_i) + 2p_i p_j + p_j \cdot (1 - p_j)\right) \tag{68}$$

with $\hat{p}_i = 1/T \cdot \sum_{t=1}^{T} X_{ti}$.

If the null-hypothesis is $p_i = p_j$, we obtain:

$$\sqrt{T}\left((\hat{p}_i - \hat{p}_j)\right) \longrightarrow N\left(0; 2p_i\right) \tag{69}$$

For the empirical implementation, we replaced the variance $2p_i$ in the distribution (69) by the sum $\hat{p}_i + \hat{p}_j$.

Note that T means the number of non-overlapping observations in the time dimension. For instance, if we look at yearly changes, the number of non-overlapping observations is not 588 monthly observations (from January 1975 to December 2023), but 49 yearly observations (from 1975 to 2023).

A.8 Derivation of the conditional expectation

We assume that the term structure model in Equation (7) is valid and that the changes in the level (β_0) and in the slope (β_1) of the term structure are normally distributed. We write down the unconditional distribution (where we assume that the expectations of the unconditional level and slope change are zero, see Table 3 for an empirical justification):

$$\begin{pmatrix} \beta_0 \\ \beta_1 \\ \triangle r\left(m_0\right) \\ \triangle r\left(m_1\right) \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}; \begin{pmatrix} \sigma_0^2 & \sigma_{01} & \sigma_{0\triangle r0} & \sigma_{0\triangle r1} \\ \sigma_{01} & \sigma_1^2 & \sigma_{1\triangle r0} & \sigma_{1\triangle r1} \\ \sigma_{0\triangle r0} & \sigma_{1\triangle r0} & \sigma_{\triangle r0}^2 & \sigma_{\triangle r0\triangle r1} \\ \sigma_{0\triangle r1} & \sigma_{1\triangle r1} & \sigma_{\triangle r0\triangle r1} & \sigma_{\triangle r1}^2 \end{pmatrix}\right) \tag{70}$$

with $\triangle r\left(m_i\right) = \beta_0 + m_i \cdot \beta_1$, $\sigma_{\triangle ri}^2 = \sigma_0^2 + m_i^2 \cdot \sigma_1^2 + 2 \cdot m_i \cdot \sigma_{01}$, $\sigma_{0\triangle ri} = \sigma_0^2 + m_i \cdot \sigma_{01}$, $\sigma_{1\triangle r0} = \sigma_{01} + m_i \cdot \sigma_1^2$ and $\sigma_{\triangle r0\triangle r1} = \sigma_0^2 + (m_0 + m_1)\,\sigma_{01} + m_0 m_1 \sigma_1^2$. All interest rate changes are normally distributed if the linear term structure model (7) applies and the level and

slope changes β_0 and β_1 are jointly normally distributed. In this case, we can make use of an approach which was inspired by an approach of Black and Litterman (1992). Using the result for the expectation of condition normal distributions[13] $E\left(\triangle r\left(m_1\right)|\triangle r\left(m_0\right)\right) = \sigma_{\triangle r0\triangle r1}/\sigma^2_{\triangle r0} \cdot \triangle r\left(m_0\right)$, we can write:

$$E\left(\triangle r\left(m_1\right)|\triangle r\left(m_0\right)\right) = \frac{\sigma_0^2 + \left(m_0 + m_1\right)\sigma_{01} + m_0 m_1 \sigma_1^2}{\sigma_0^2 + m_0^2 \cdot \sigma_1^2 + 2 \cdot m_0 \cdot \sigma_{01}} \cdot \triangle r\left(m_0\right) \tag{71}$$

which corresponds - after some transformations - to Equation (23).

A.9 Change parameters

The change in present value is the weighted sum of the changes ($\beta = (\beta_0, \beta_1)'$) in the term structure where the weights are the derivatives of the present value according to Table 2, i.e. $\partial PV/\partial\alpha_0$ and $\partial PV/\partial\alpha_1$. To see this, we calculate the total differential

$$\begin{aligned} \triangle PV &= \frac{\partial PV}{\partial \alpha_0} \cdot \triangle\alpha_0 + \frac{\partial PV}{\partial \alpha_1} \cdot \triangle\alpha_1 \\ &= \gamma'\beta \end{aligned} \tag{72}$$

with $\gamma = (\partial PV/\partial\alpha_0, \partial PV/\partial\alpha_1)'$ and $\beta = (\triangle\alpha_0, \triangle\alpha_1)'$. We assume that the changes in the term structure are normally distributed

$$\beta \sim N\left(0, \Sigma\right) \tag{73}$$

then the following expression is chi-squared distributed with the rank of the covariance matrix Σ as the degrees of freedom (See, for instance, Greene (2012)):

$$\beta'\Sigma^{-1}\beta \sim \chi^2(rg(\Sigma)) \tag{74}$$

For $rg(\Sigma) = 2$ (as in our case), we can establish the critical value (cr) as a closed-form expression, namely as $cr = -ln\left(p\right) \cdot 2$ where p is the significance level[14] (see Leemis and McQueston (2008) for the relationship of a chi-squared distribution with two degrees of freedom and the exponential distribution with the parameter $\lambda = \frac{1}{2}$).

We look for extreme losses and, at the same time, the critical value (cr) must not be exceeded. The Lagrangian function is:

$$L = \gamma'\beta + \lambda \cdot \left(\beta'\Sigma^{-1}\beta - cr\right) \tag{75}$$

Calculating the derivative with respect to β, we obtain

$$\frac{\partial L}{\partial \beta} = \gamma + 2 \cdot \lambda\Sigma^{-1}\beta \tag{76}$$

[13] See, for example, Greene (2012).

[14] In this case, the expression cr gives the critical value that a chi-squared distributed random variable with two degrees of freedom exceeds this critical value with probability p. This probability is just the complement to 100% of distribution function (cdf); therefore the quantile function (in the case of two degrees of freedom) is $q = -2 \cdot ln\left(1 - p\right)$.

Setting Equation (76) to zero and solving it for β, we get:

$$\beta^* = \frac{1}{2 \cdot \lambda} \Sigma \gamma \tag{77}$$

which corresponds to the Equations (24) and (25) for the case of a dimension of $rg(\Sigma) = 2$.

Setting Equation (77) into the constraint $cr = \beta' \Sigma^{-1} \beta$, we obtain:

$$\lambda^2 = -\frac{1}{4 \cdot cr} \gamma' \Sigma \gamma \tag{78}$$

Equation (78) has two solutions, one leading to the maximum and one to the minimum. The solution leading to the minimum in the case of a dimension of $rg(\Sigma) = 2$ is given in Equation (26).

With the help of Equation (72) in which we set in the optimized changes of Equation (77) (and the λ of Equation (26)), we can determine the Value-at Risk (VaR) of the change in present value ($\triangle PV$). Note that this VaR differs from the usual VaR because there are two (instead of one) random variables and that to achieve unambiguousness we have to make the additional assumption of maximizing losses (see the Equations (75ff)). Note as well that the sensitivities γ_0 and γ_1 depend on the maturity m (see Table 2).

A.10 Earnings from term transformation

According to Equation (5), the return $ret(m)$ in excess of the risk-free short-term rate is

$$\begin{aligned} ret(m) &= r(m) - r(0) \\ &= \alpha_0 + \alpha_1 \cdot m - \alpha_0 \end{aligned} \tag{79}$$

From Table 3, we obtain the average value for the steepness of the term structureα_1 as 13.52 bp per year and a duration $D(TS(m))) = 4.35$ (Equation (50) at the average values of Table 3 and $m = 10$ years, which yields a remuneration of risk of $rr = (0.001352 \cdot 10)/4.35 = 0.0031$. To derive a bank's (implicit) income from term transformation, we assume that the bank's maturity mismatch results from investing in the passive trading strategy, which yields a scaling factor sc_i of:

$$sc_i = \frac{D_{mod,i} \cdot Eq_i}{D(TS(m))} \tag{80}$$

where $D_{mod,i}$ is the modified equity duration of bank i and Eq_i is its equity in euro.

$$\begin{aligned} E\left(NII_{TT,i}\right) &= sc_i \cdot ret(m) \\ &= \frac{D_{mod,i} \cdot Eq_i}{D(TS(m))} \cdot ret(m) \\ &= TA_i \cdot \frac{Eq_i}{TA_i} \cdot D_{mod,i} \cdot \frac{ret(m)}{D(TS(m))} \end{aligned} \tag{81}$$

where TA_i denotes bank i's total assets. If we divide Equation (81) by bank i's net interest income NII_i, we obtain Equation (27).

A.11 Extreme event

We assume that the distribution (74) is valid and the special case just underneath applies (i.e. $rg(\Sigma) = 2$, see also Footnote 14). Then:

$$\begin{aligned} Pr\left(\beta'\Sigma^{-1}\beta \leq -2 \cdot ln(1-p)\right) &= p \\ Pr\left(1 - exp\left(-\frac{\beta'\Sigma^{-1}\beta}{2}\right) \leq p\right) &= p \end{aligned} \tag{82}$$

Therefore, the expression $1 - exp\left(-\frac{\beta'\Sigma^{-1}\beta}{2}\right)$ for $rg(\Sigma) = 2$ is standard uniformly distributed ($U(0,1)$), i.e.

$$1 - exp\left(-\frac{\frac{1}{2}\beta_0^2\sigma_1^2 + \frac{1}{2}\beta_1^2\sigma_0^2 - \beta_0\beta_1\sigma_{01}}{\sigma_0^2\sigma_1^2 - \sigma_{01}^2}\right) \sim U(0,1) \tag{83}$$

where the matrix expressions $\Sigma = \begin{pmatrix} \sigma_0^2 & \sigma_{01} \\ \sigma_{01} & \sigma_1^2 \end{pmatrix}$ and $\beta = (\beta_0, \beta_1)'$ are replaced by the corresponding scalars. This is given in Equation (28), where the relationship is used that $1 - X \sim U(0,1)$ if $X \sim U(0,1)$.[15]

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[15] See Nicolas (2013).

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