Gabih, Abdelali; Kondakji, Hakam; Wunderlich, Ralf

Power utility maximization with expert opinions at fixed arrival times in a market with hidden Gaussian drift

Annals of Operations Research

Provided in Cooperation with:
Springer Nature

Suggested Citation: Gabih, Abdelali; Kondakji, Hakam; Wunderlich, Ralf (2024) : Power utility maximization with expert opinions at fixed arrival times in a market with hidden Gaussian drift, Annals of Operations Research, ISSN 1572-9338, Springer US, New York, NY, Vol. 341, Iss. 2, pp. 897-936,
https://doi.org/10.1007/s10479-024-06172-5

Annals of Operations Research (2024) 341:897–936
https://doi.org/10.1007/s10479-024-06172-5

ORIGINAL RESEARCH

Power utility maximization with expert opinions at fixed arrival times in a market with hidden Gaussian drift

Abdelali Gabih[1] · Hakam Kondakji[2] · Ralf Wunderlich[3]

Received: 20 January 2023 / Accepted: 17 July 2024 / Published online: 11 August 2024

Abstract
In this paper we study optimal trading strategies in a financial market in which stock returns depend on a hidden Gaussian mean reverting drift process. Investors obtain information on that drift by observing stock returns. Moreover, expert opinions in the form of signals about the current state of the drift arriving at fixed and known dates are included in the analysis. Drift estimates are based on Kalman filter techniques. They are used to transform a power utility maximization problem under partial information into an optimization problem under full information where the state variable is the filter of the drift. The dynamic programming equation for this problem is studied and closed-form solutions for the value function and the optimal trading strategy of an investor are derived. They allow to quantify the monetary value of information delivered by the expert opinions. We illustrate our theoretical findings by results of extensive numerical experiments.

Keywords Power utility maximization · Partial information · Stochastic optimal control · Kalman-Bucy filter · Expert opinions · Black-Litterman model

Mathematics Subject Classification 91G10 · 93E20 · 93E11 · 60G35

1 Introduction

In dynamic portfolio selection problems optimal trading strategies depend crucially on the drift of the underlying asset return processes. That drift describes the expected asset returns, varies over time and is driven by certain stochastic factors such as dividend yields, the firm's

✉ Ralf Wunderlich
ralf.wunderlich@b-tu.de

Abdelali Gabih
a.gabih@uca.ma

Hakam Kondakji
kondakji@hsu-hh.de

[1] Equipe de Modélisation et Contrôle des Systèmes Stochastiques et Déterministes, Faculty of Sciences, Chouaib Doukkali University, 24000 El Jadida, Morocco

[2] Faculty of Economics und Social Sciences, Helmut Schmidt University, P.O. Box 700822, 22008 Hamburg, Germany

[3] Institute of Mathematics, Brandenburg University of Technology Cottbus-Senftenberg, P.O. Box 101344, 03013 Cottbus, Germany

return on equity, interest rates and macroeconomic indicators. This was already addressed in the seminal paper of Merton (1971, Sec. 9). We also refer to other early articles by Bielecki and Pliska (1999), Brennan et al. (1997) and Xia (2001). The dependence of the drift process on these factors is usually not perfectly known and some of the factors may be not directly observable. Therefore, it is reasonable to model the drift as an unobservable stochastic process for which only statistical estimates are available. Then solving the associated portfolio problems has to be based on such estimates of the drift process.

For the one-period Markowitz model the surprisingly large impact of statistical errors in the estimation of model parameters on mean-variance optimal portfolios is often reported in the literature, e.g. by Broadie (1993). Estimating the drift with a reasonable degree of precision based only on historical asset prices is known to be almost impossible. This is nicely shown in Rogers (2013, Chapter 4.2) for a model in which the drift is even constant. For a reliable estimate extremely long time series of data are required which are usually not available. Further, assuming a constant drift over longer periods of time is quite unrealistic. Drifts tend to fluctuate randomly over time and drift effects are often overshadowed by volatility.

For these reasons, portfolio managers and traders try to diversify their observation sources and also rely on external sources of information such as news, company reports, ratings and benchmark values. Further, they increasingly turn to data outside of the traditional sources that companies and financial markets provide. Examples are social media posts, internet search, satellite imagery, sentiment indices, pandemic data, product review trends and are often related to Big Data analytics. Finally, they use views of financial analysts or just their own intuitive views on the future asset performance.

In the literature these outside sources of information are coined *expert opinions* or more generally *alternative data*, see Chen and Wong (2022), Davis and Lleo (2022). In this paper we will use the first term. After an appropriate mathematical modeling as additional noisy observations they are included in the drift estimation and the construction of optimal portfolio strategies. That approach goes back to the celebrated Black-Litterman model which is an extension of the classical one-period Markowitz model, see Black and Litterman (1992). The idea is to improve return predictions using expert opinions by means of a Bayesian updating of the drift estimate.

Instead of a static one-period model we consider in this paper a continuous-time model for asset prices. Additional information in the form of expert opinions arrive repeatedly over time. Davis and Lleo (2013a) coined this approach "Black–Litterman in Continuous Time" (BLCT). More precisely, we study a hidden Gaussian model where asset returns are driven by an unobservable mean-reverting Gaussian process. Information on the drift is of mixed type. First investors observe stock prices or equivalently the return process. Moreover, investors may have access to expert opinions arriving at already known discrete time points in a form of unbiased drift estimates. Since the investors' ability to construct good trading strategies depends on the quality of the hidden drift estimation we study a filtering problem. There the aim is to find the conditional distribution of the drift given the available information drawn from the return observations and expert opinions.

For investors who observe only the return process that filter known as the classical Kalman–Bucy filter, see for example Liptser and Shiryaev (2001). Based on this one can derive the filter for investors who also observe expert opinions by a Bayesian update of the current drift estimate at each information date. This constitutes the above mentioned continuous-time version of the static Black–Litterman approach.

Utility maximization problems for partially informed investors have been intensively studied in the last years. For models with Gaussian drift we refer to Lakner (1998) and Brendle

(2006). Results for models in which the drift is described by a continuous-time hidden Markov chain are given in Rieder and Bäuerle (2005), Sass and Haussmann (2004) and more recently by Chen and Wong (2022). A good overview with further references and generalization can be found in Björk et al. (2010).

For the literature on BLCT in which expert opinions are included we refer to a series of papers (Gabih et al., 2014, 2020; Sass et al., 2017, 2021, 2022) of the present authors and of Sass and Westphal. They investigate utility maximization problems for investors with logarithmic preferences in market models with a hidden Gaussian drift process and discrete-time expert opinions. The case of continuous-time expert opinions and power utility maximization is treated in a series of papers by Davis and Lleo, see Davis and Lleo (2013a, 2020, 2022) and the references therein. Power utility maximization problems for expert opinions arriving randomly at the jump times of a Poisson process are treated in the recent work (Gabih & Wunderlich, 2023). Similar portfolio problems for drift processes described by continuous-time hidden Markov chains have been studied in Frey et al. (2012, 2014). Finally, we refer to our companion paper (Gabih et al., 2022) where we investigate the well posedness of power utility maximization problems which are addressed in the present paper.
Our contribution The new contribution to the literature in the present paper is the detailed analysis of the case of power utility maximization for market models with a hidden Gaussian drift and discrete-time expert opinions arriving at fixed and known time points. That case was not yet treated in the literature and is only considered in the PhD thesis of Kondakji (2019) on which this paper is based. Only recently, a few results of Kondakji (2019) were briefly mentioned in Sass et al. (2022) and applied in numerical experiments. Note that the approach in Gabih and Wunderlich (2023) for the case of randomly arriving expert opinions cannot be adopted to the case of fixed arrival times considered in this paper. In Gabih and Wunderlich (2023) the dynamic programming equation appears as a partial integro-differential equation which requires a numerical solution. Instead, in this paper we can derive closed-form solutions to the derived dynamic programming equation.

Our main results are presented in Theorems 6.1 and 6.3. Here we present a backward recursion for the value function and the optimal strategy of the partially informed investors observing also discrete-time expert opinions, and discuss verification issues. Another contribution are results of extensive numerical experiments which we present in Sect. 7. There we study in particular the asymptotic properties of filters, value functions and optimal strategies for high-frequency experts and the monetary value of expert opinions. These studies on high-frequency experts and their limiting behavior provide a link to the results for continuous-time expert opinions which are available from the works of Davis and Lleo, see Davis and Lleo (2013a, 2020, 2022).

The paper is organized as follows. In Sect. 2 we introduce the model for our financial market including the expert opinions and define information regimes for investors with different sources of information. Further, we formulate the portfolio optimization problem. Section 3 states for the different information regimes the filter equations for the corresponding conditional mean and conditional covariance process. Then it reviews properties of the filter, in particular the asymptotic filter behavior for high-frequency expert opinions. Section 4 is devoted to the solution of the power utility maximization problem. That problem is reformulated as an equivalent stochastic optimal control problem which can be solved by dynamic programming techniques. Solutions are presented for the fully informed investor. Section 5 presents the solution for partially informed investors combining return observations with diffusion type expert opinions and Sect. 6 studies the case of investors observing discrete-time experts. Section 7 illustrates the theoretical findings by numerical results.

Notation Throughout this paper, we use the notation I_d for the identity matrix in $\mathbb{R}^{d\times d}$, 0_d denotes the null vector in $\mathbb{R}^d$, $0_{d\times m}$ the null matrix in $\mathbb{R}^{d\times m}$. For a symmetric and positive-semidefinite matrix $A \in \mathbb{R}^{d\times d}$ we call a symmetric and positive-semidefinite matrix $B \in \mathbb{R}^{d\times d}$ the *square root* of A if $B^2 = A$. The square root is unique and will be denoted by $A^{1/2}$. For a vector X we denote by $\|X\|$ the Euclidean norm. Unless stated otherwise, whenever A is a matrix, $\|A\|$ denotes the spectral norm of A.

2 Financial market and optimization problem

2.1 Price dynamics

We model a financial market with one risk-free and multiple risky assets. The setting is based on Gabih et al. (2014, 2020) and Sass et al. (2017, 2022, 2021). For a fixed date $T > 0$ representing the investment horizon, we work on a filtered probability space $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{P})$, with filtration $\mathbb{G} = (\mathcal{G}_t)_{t\in[0,T]}$ satisfying the usual conditions. All processes are assumed to be $\mathbb{G}$-adapted.

We consider a market model for one risk-free asset and d risky securities. We follow an approach frequently used in the literature on optimal portfolio selection and consider discounted asset prices with the risk-free asset as numéraire. Then the risk-free asset has a constant price $S_t^0 = 1$. The excess log returns or risk premiums $R = (R^1, \dots, R^d)$ of the risky securities are described by stochastic processes defined by the SDE

$$dR_t = \mu_t\, dt + \sigma_R\, dW_t^R, \tag{1}$$

driven by a d_1-dimensional $\mathbb{G}$-adapted Brownian motion W^R with $d_1 \geq d$. In the remainder of this paper we will call R simply *returns*. $\mu = (\mu_t)_{t\in[0,T]}$ denotes the stochastic drift process which is described in detail below. The volatility matrix $\sigma_R \in \mathbb{R}^{d\times d_1}$ is assumed to be constant over time such that $\Sigma_R := \sigma_R\sigma_R^\top$ is positive definite. In this setting the discounted price process $S = (S^1, \dots, S^d)$ of the risky securities reads as

$$dS_t = diag(S_t)\, dR_t, \quad S_0 = s_0,$$

with some fixed initial value $s_0 = (s_0^1, \dots, s_0^d)$. Note that for the solution to the above SDE it holds

$$\begin{aligned}\log S_t^i - \log s_0^i &= \int_0^t \mu_s^i ds + \sum_{j=1}^{d_1}\left(\sigma_R^{ij} W_t^{R,j} - \frac{1}{2}(\sigma_R^{ij})^2 t\right)\\ &= R_t^i - \frac{1}{2}\sum_{j=1}^{d_1}(\sigma_R^{ij})^2 t, \quad i = 1, \dots, d.\end{aligned}$$

So we have the equality $\mathbb{G}^R = \mathbb{G}^{\log S} = \mathbb{G}^S$, where for a generic process X we denote by $\mathbb{G}^X$ the filtration generated by X. This is useful since it allows to work with R instead of S in the filtering part.

The dynamics of the drift process $\mu = (\mu_t)_{t\in[0,T]}$ in (1) are given by the stochastic differential equation (SDE)

$$d\mu_t = \kappa(\overline{\mu} - \mu_t)\, dt + \sigma_\mu\, dW_t^\mu, \tag{2}$$

where $\kappa \in \mathbb{R}^{d\times d}$, $\sigma_\mu \in \mathbb{R}^{d\times d_2}$ and $\overline{\mu} \in \mathbb{R}^d$ are constants such that all eigenvalues of κ have a positive real part (that is, $-\kappa$ is a stable matrix) and $\Sigma_\mu := \sigma_\mu \sigma_\mu^\top$ is positive definite. Further, W^μ is a d_2-dimensional Brownian motion such that $d_2 \geq d$. For the sake of simplification and shorter notation we assume that the Wiener processes W^R and W^μ driving the return and drift process, respectively, are independent. We refer to Brendle (2006), Colaneri et al. (2021) and Fouque et al. (2015) for the general case. Here, $\overline{\mu}$ is the mean-reversion level, κ the mean-reversion speed and σ_μ describes the volatility of μ. The initial value μ_0 is assumed to be a normally distributed random variable independent of W^μ and W^R with mean $\overline{m}_0 \in \mathbb{R}^d$ and covariance matrix $\overline{q}_0 \in \mathbb{R}^{d\times d}$ assumed to be symmetric and positive semi-definite. It is well-known that the solution to SDE (2) is known as Ornstein-Uhlenbeck process which is a Gaussian process given by

$$\mu_t = \overline{\mu} + e^{-\kappa t}\left[(\mu_0 - \overline{\mu}) + \int_0^t e^{\kappa s}\sigma_\mu dW_s^\mu\right], \quad t \geq 0. \tag{3}$$

2.2 Expert opinions

We assume that investors observe the return process R but they neither observe the factor process μ nor the Brownian motion W^R. They do however know the model parameters such as $\sigma_R, \kappa, \overline{\mu}, \sigma_\mu$ and the distribution $\mathcal{N}(\overline{m}_0, \overline{q}_0)$ of the initial value μ_0. Information about the drift μ can be drawn from observing the returns R. A special feature of our model is that investors may also have access to additional information about the drift in form of *expert opinions* such as news, company reports, ratings or their own intuitive views on the future asset performance. The expert opinions provide noisy signals about the current state of the drift arriving at known deterministic time points $0 = t_0 < t_1 < \ldots < t_{n-1} < T$. For the sake of convenience we also write $t_n = T$ although no expert opinion arrives at time t_n. The signals or "the expert views" at time t_k are modelled by $\mathbb{R}^d$-valued Gaussian random vectors $Z_k = (Z_k^1, \ldots, Z_k^d)^\top$ with

$$Z_k = \mu_{t_k} + \Gamma^{\frac{1}{2}}\varepsilon_k, \quad k = 0, \ldots, n-1, \tag{4}$$

where the matrix $\Gamma \in \mathbb{R}^{d\times d}$ is symmetric and positive definite. Further, $(\varepsilon_k)_{k=0,\ldots,n-1}$ is a sequence of independent standard normally distributed random vectors, i.e., $\varepsilon_k \sim \mathcal{N}(0, I_d)$. It is also independent of both the Brownian motions W^R, W^μ and the initial value μ_0 of the drift process. That means that, given μ_{t_k}, the expert opinion Z_k is $\mathcal{N}(\mu_{t_k}, \Gamma)$-distributed. So, Z_k can be considered as an unbiased estimate of the unknown state of the drift at time t_k.

Modeling expert opinions as normally distributed random variables corresponds well to a variety of additional information on average stock returns available in real-world markets. We refer to Davis and Lleo (2020) for more details about an appropriate preprocessing, debiasing and approximation of such extra information by Gaussian models. Let us briefly sketch the mathematical modeling of analyst views in terms of confidence intervals. Inspired by Davis and Lleo (2020) we consider the following example of a view at time $t = t_k$: "My research leads me to believe that the average stock return lies within a range of 6–10%, and I'm 90% confident about this". This view can be treated as a 90%-confidence interval for the unknown mean μ_{t_k} of a Gaussian distribution centered at 0.08, which is the observed Z_k. The corresponding variance Γ is chosen such that the boundaries of the interval are 0.06 and 0.10. We also want to emphasize that the Gaussian expert opinions allow to work with Kalman filtering techniques. For other distributions, in general no closed-form filters are available.

The matrix Γ is a measure of the expert's reliability. The diagonal entries Γ_{ii} are just the variances of the expert's estimates of the drift for the i-th asset at time t_k: the larger Γ_{ii} the less reliable is the expert's view about $\mu^i_{t_k}, i = 1, \dots, d$. The off-diagonal entries describe the correlation between the experts' views. An example for a diagonal matrix Γ, i.e., uncorrelated views Z^i_k about the i-th asset's drift, is obtained if the random vector Z_k contains the views Z^i_k of d independent analysts estimating the drift of a single asset only. That case is hard to justify in reality where one can observe so-called "groupthink" leading to positive correlations between the views of the analysts. However, correlations between the views are hard to calibrate. We refer to Davis and Lleo (2020) for more details.

The above model of discrete-time expert opinions can be modified such that expert opinions arrive not at fixed and known dates but at random times $(T_k)_{k\in\mathbb{N}}$. That approach together with results for filtering and maximization of log-utility was studied in detail in Sass et al. (2021). There the arrival times are modeled as the jump times of a Poisson process. The maximization of power utility is considered in Gabih and Wunderlich (2023).

One may also allow for relative expert views where experts give an estimate for the difference in the drift of two stocks instead of absolute views. This extension can be studied in Schöttle et al. (2010) where the authors show how to switch between these two models for expert opinions by means of a pick matrix.

In addition to expert opinions arriving at discrete time points we will also consider expert opinions arriving continuously over time as in Davis and Lleo (2013a, 2020) who called this approach "Black–Litterman in Continuous Time". This is motivated by the results of Sass et al. (2021, 2022) who show that asymptotically as the arrival frequency tends to infinity and the expert's variance Γ grows linearly in that frequency the information drawn from these expert opinions is essentially the same as the information one gets from observing yet another diffusion process. This diffusion process can then be interpreted as an expert who gives a continuous-time estimation about the current state of the drift. Another interpretation is that the diffusion process models returns of assets which are not traded in the portfolio but depend on the same stochastic factors and are observable by the investor. Let these continuous-time expert opinions be given by the diffusion process

$$dJ_t = \mu_t dt + \sigma_J dW^J_t, \tag{5}$$

where W^J_t is a d_3-dimensional Brownian motion independent of W^R_t and W^μ, and such that $d_3 \geq d$. The volatility matrix $\sigma_J \in \mathbb{R}^{d\times d_3}$ is assumed to be constant over time such that the matrix $\Sigma_J := \sigma_J \sigma_J^\top$ is positive definite. In Sects. 4.1 and 7.4 we show that based on this model and on the diffusion approximations provided in Sass et al. (2022) one can find efficient approximative solutions to utility maximization problems for partially informed investors observing high-frequency discrete-time expert opinions.

2.3 Investor filtration

We consider various types of investors with different levels of information. The information available to an investor is described by the *investor filtration* $\mathbb{F}^H = (\mathcal{F}^H_t)_{t\in[0,T]}$. Here, H denotes the information regime for which we consider the cases $H = R, Z, J, F$, and the investor with filtration $\mathbb{F}^H = (\mathcal{F}^H_t)_{t\in[0,T]}$ is called the H-investor. The R-investor observes only the return process R, the Z-investor combines return observations with the discrete-time expert opinions Z_k while the J-investor observes the return process together with the continuous-time expert J. Finally, the F-investor has full information and can observe the drift process μ directly and of course the return process. For stochastic drift this case is not

realistic, but we use it as a benchmark. It will serve as a limiting case for high-frequency expert opinions with fixed covariance matrix Γ, see Sect. 3.3, Thm. 3.9.

The σ-algebras $\mathcal{F}_t^H$ representing the H-investor's information at time $t \in [0,T]$ are defined at initial time $t=0$ by $\mathcal{F}_0^F = \sigma\{\mu_0\}$ for the fully informed investor and by $\mathcal{F}_0^H = \mathcal{F}_0^I \subset \mathcal{F}_0^F$ for $H = R, Z, J$, i.e., for the partially informed investors. Here, $\mathcal{F}_0^I$ denotes the σ-algebra representing prior information about the initial drift μ_0. More details on $\mathcal{F}_0^I$ are given below. For $t \in (0,T]$ we define

$$\begin{aligned}
\mathcal{F}_t^R &= \sigma(R_s,\ s \le t) \vee \mathcal{F}_0^I,\\
\mathcal{F}_t^Z &= \sigma(R_s,\ s \le t,\ (t_k, Z_k),\ t_k \le t) \vee \mathcal{F}_0^I,\\
\mathcal{F}_t^J &= \sigma(R_s, J_s,\ s \le t) \vee \mathcal{F}_0^I,\\
\mathcal{F}_t^F &= \sigma(R_s, \mu_s,\ s \le t).
\end{aligned}$$

We assume that the above σ-algebras $\mathcal{F}_t^H$ are augmented by the null sets of $\mathbb{P}$.

Note that all partially informed investors ($H = R, J, Z$) start at $t=0$ with the same initial information given by $\mathcal{F}_0^I$. The latter models prior knowledge about the drift process at time $t=0$, e.g., from observing returns or expert opinions in the past, before the trading period $[0,T]$. The expert opinion Z_0 arriving at time $t=0$ does not belong to this prior information and is therefore excluded from $\mathcal{F}_0^Z$ and only contained in $\mathcal{F}_t^Z$ for $t>0$. At first glance this may appear not consistent but it will facilitate below in Sect. 2.6 the formal definition of the monetary value of the expert opinions.

We assume that the conditional distribution of the initial value drift μ_0 given $\mathcal{F}_0^I$ is the normal distribution $\mathcal{N}(m_0, q_0)$ with mean $m_0 \in \mathbb{R}^d$ and covariance matrix $q_0 \in \mathbb{R}^{d\times d}$ assumed to be symmetric and positive semi-definite. In this setting typical examples are:

(a) The investor has no information about the initial value of the drift μ_0. However, he knows the model parameters, in particular the distribution $\mathcal{N}(\overline{m}_0, \overline{q}_0)$ of μ_0 with given parameters $\overline{m}_0$ and $\overline{q}_0$. This corresponds to $\mathcal{F}_0^I = \{\varnothing, \Omega\}$ and $m_0 = \overline{m}_0$, $q_0 = \overline{q}_0$.
(b) The investor can fully observe the initial value of the drift μ_0, which corresponds to $\mathcal{F}_0^I = \mathcal{F}_0^F = \sigma\{\mu_0\}$ and $m_0 = \mu_0(\omega)$ and $q_0 = 0$.
(c) Between the above limiting cases we consider an investor who has some prior but no complete information about μ_0 leading to $\{\varnothing, \Omega\} \subset \mathcal{F}_0^I \subset \mathcal{F}_0^F$.

2.4 Portfolio and optimization problem

We describe the self-financing trading of an investor by the initial capital $x_0 > 0$ and the $\mathbb{F}^H$-adapted trading strategy $\pi = (\pi_t)_{t\in[0,T]}$ where $\pi_t \in \mathbb{R}^d$. Here π_t^i represents the proportion of wealth invested in the i-th stock at time t. The assumption that π is $\mathbb{F}^H$-adapted models that investment decisions have to be based on information available to the H-investor which he obtains from observing assets prices ($H = R$) combined with expert opinions ($H = Z, J$) or with the drift process ($H = F$). Following the strategy π the investor generates a wealth process $(X_t^\pi)_{t\in[0,T]}$ whose dynamics reads as

$$\frac{dX_t^\pi}{X_t^\pi} = \pi_t^\top dR_t = \pi_t^\top \mu_t\, dt + \pi_t^\top \sigma_R\, dW_t^R, \quad X_0^\pi = x_0. \tag{6}$$

We denote by

$$\begin{aligned}
\mathcal{A}_0^H = \Big\{ &\pi = (\pi_t)_t : \pi_t \in \mathbb{R}^d, \pi \text{ is } \mathbb{F}^H\text{-adapted}, X_t^\pi > 0, \mathbb{E}\Big[\int_0^T \|\pi_t\|^2\, dt\Big] < \infty,\\
&\text{the process } \Lambda^H \text{ defined below in (24) satisfies } \mathbb{E}\big[\Lambda_T^H\big] = 1\Big\}
\end{aligned} \tag{7}$$

the class of *admissible trading strategies*. The last condition in the definition of $\mathcal{A}_0^H$ is needed to apply a change of measure technique for the solution of the optimization problem. More details are provided below in Sect. 4.2.

We assume that the investor wants to maximize the expected utility of terminal wealth for a given utility function $U : \mathbb{R}_+ \to \mathbb{R}$ modelling the risk aversion of the investor. In our approach we will use the function

$$\mathcal{U}_\theta(x) := \frac{x^\theta}{\theta}, \quad \theta \in (-\infty, 0) \cup (0, 1). \tag{8}$$

The limiting case for $\theta \to 0$ for the power utility function leads to the logarithmic utility $\mathcal{U}_0(x) := \ln x$, since we have $\mathcal{U}_\theta(x) - \frac{1}{\theta} = \frac{x^\theta - 1}{\theta} \xrightarrow[\theta \to 0]{} \log x$.

The optimization problem thus reads as

$$\mathcal{V}_0^H(x_0) := \sup_{\pi \in \mathcal{A}_0^H} \mathcal{D}_0^H(x_0; \pi) \quad \text{where} \quad \mathcal{D}_0^H(x_0; \pi) = \mathbb{E}\left[\mathcal{U}_\theta(X_T^\pi) \mid \mathcal{F}_0^H\right], \ \pi \in \mathcal{A}_0^H, \tag{9}$$

where we call $\mathcal{D}_0^H(x_0; \pi)$ *reward function* or *performance criterion* of the strategy π and $\mathcal{V}_0^H(x_0)$ *value function* to given initial capital x_0.

This is for $H \neq F$ a maximization problem under partial information since we have required that the strategy π is adapted to the investor filtration $\mathbb{F}^H$. However, the drift coefficient of the wealth equation (6) is not $\mathbb{F}^H$-adapted, it depends on the non-observable drift μ. Note that for $x_0 > 0$ the solution of the SDE (6) is strictly positive. This guarantees that X_T^π is in the domain of logarithmic and power utility.

2.5 Well posedness of the optimization problem

The analysis of utility maximization problem (9) requires conditions under which the problem is well-posed. Problem (9) is said to be *well-posed* for the H-investor, if there exists a constant $C_\mathcal{V}^H < \infty$ such that $\mathcal{V}_0^H(x_0) \leq C_\mathcal{V}^H$. Then, the maximum expected utility of terminal wealth cannot explode in finite time as it is the case for so-called nirvana strategies described e.g. in Kim and Omberg (1996) and Angoshtari (2014). Nirvana strategies generate in finite time a terminal wealth with a distribution leading to infinite expected utility although the realizations of terminal wealth may be finite.

In general, well posedness will depend not only on the initial capital x_0 but on the complete set of model parameters which are $T, \theta, d, \sigma_R, \sigma_\mu, \kappa, \overline{\mu}, x_0, \overline{m}_0, \overline{q}_0, m_0, q_0, \Gamma, n, \sigma_J$.

For power utility with parameter $\theta < 0$ we have $\mathcal{U}_\theta(x) < 0$ for all $x > 0$. Hence, in that case we can simply choose $C_\mathcal{V}^H = 0$ and the optimization problem is well-posed for all model parameters with negative θ. The logarithmic utility function ($\theta = 0$) is no longer bounded from above but we show below in Sect. 4.1 that the value function $\mathcal{V}_0^H(x_0)$ is bounded from above by some positive constant $C_\mathcal{V}^H$ for any selection of the remaining model parameters. More delicate is the case of power utility with positive parameter $\theta \in (0, 1)$ which is also not bounded from above. Here, well posedness is only guaranteed under certain restrictions on the choice of model parameters such as the investment horizon and parameters controlling the variance of the asset price and drift processes. For a market with a fully observed drift rate modeled by an Ornstein-Uhlenbeck process this phenomenon was already described in Kim and Omberg (1996). Further, it was also observed in Korn and Kraft (2004, Sec. 3) who coined it "I-unstability", in Angoshtari (2014, 2016), and Lee and Papanicolaou (2016) who studied power utility maximization problems and their well posedness for financial market

models with cointegrated asset price processes, and in Battauz et al. (2017) for markets with defaultbale assets.

For detailed investigation also for models with not directly observable drift and expert opinions we refer to our paper (Gabih et al., 2022) where we find sufficient conditions on the model parameters ensuring well posedness. They are given below in (32) and (37). Some results for markets with a single risky asset ($d = 1$) are also contained in Colaneri et al. (2021). One of the findings is that depending on the chosen parameters well posedness can be guaranteed only if the trading horizon T is smaller than some certain "explosion time". In the following we always assume that (9) constitutes a well-posed optimization problem.

2.6 Monetary value of information

In this subsection we want to express the value of information available to the H-investors in monetary terms. It is expected that the fully informed F-investor which can directly observe the drift has an advantage over the partially informed investors. In fact, an easy calculation as in Lee and Papanicolaou (2016, Subsec. 3.1) and (2022, Subsec. 3.3) shows that for $H = R, Z, J$ it holds $\mathcal{V}_0^H(x_0) \leq \mathbb{E}[\mathcal{V}_0^F(x_0) \mid \mathcal{F}_0^H]$. This inequality expresses the above advantage in mathematically. The difference between the right and left hand side of the inequality is termed in Lee and Papanicolaou (2016, Subsec. 3.1) loss of utility and constitutes a first measure for the value of information. However, utility functions and the derived value function to the utility maximization problems (9) do not carry a meaningful unit and therefore it is difficult to compare results for different utility functions. In order to derive quantities with a clear economic interpretation which allow to express the value of information in monetary terms we follow a utility indifference approach as in Brendle (2006), Lee and Papanicolaou (2016, Subsec. 3.1), (2014, Sec. 6).

First, we compare the fully informed F-investor with the other partially informed H-investors, $H = R, Z, J$. Recall that the fully informed F-investor can observe the drift. The R-investor can only observe stock returns while the Z- and J- investors have access to additional information and combine observations of stock return with expert opinions. Now we consider for $H = R, Z, J$ the initial capital $x_0^{H/F}$ which the F-investor needs to obtain the same maximized expected utility at time T as the partially informed H-investor who started at time 0 with wealth $x_0^H > 0$ which according to (9) is given by $\mathcal{V}_0^H(x_0)$. Following this utility indifference approach $x_0^{H/F}$ is obtained as solution of the following equation

$$\mathcal{V}_0^H(x_0^H) = \mathbb{E}\Big[\mathcal{V}_0^F\big(x_0^{H/F}\big) \mid \mathcal{F}_0^H\Big]. \tag{10}$$

The difference $x_0^H - x_0^{H/F} > 0$ can be interpreted as *loss of information* for the (non fully informed) H-investor measured in monetary units, while the ratio

$$\varepsilon^H := \frac{x_0^{H/F}}{x_0^H} \in (0, 1] \tag{11}$$

is a measure for the *efficiency* of the H-investor relative to the F-investor.

The above utility indifference approach can also be used to quantify the monetary value of the additional information delivered by the experts. We now compare the maximum expected utility of an R-investor who only observes returns with that utility of the H-investor for $H = Z, J$ who can combine return observations with information from the experts. Given that the R-investor is equipped with initial capital $x_0^R > 0$ we determine the initial capital

$x_0^{R/H}$ for the H-investor which leads to the same maximal expected utility, i.e $x_0^{R/H}$ is the solution of the equation

$$\mathcal{V}_0^R(x_0^R) = \mathbb{E}\Big[\mathcal{V}_0^H(x_0^{R/H}) \mid \mathcal{F}_0^R\Big].$$

Since we assume that at time $t = 0$ all partially informed investors have access to the same information about the drift, it holds $\mathcal{F}_0^R = \mathcal{F}_0^H = \mathcal{F}_0^I$ (see Sect. 2.3) and the above equation reads as

$$\mathcal{V}_0^R(x_0^R) = \mathcal{V}_0^H(x_0^{R/H}). \tag{12}$$

From the initial capital x_0^R the R-investor can put aside the amount $P_{Exp}^H := x_0^R - x_0^{R/H}$ to buy the information from the expert. The remaining capital $x_0^{R/H}$ can be invested in an $H-$optimal portfolio and leads to the same expected utility of terminal wealth as the R-optimal portfolio with initial capital x_0^R. Hence, P_{Exp}^H describes the *monetary value of the expert opinions* for the R-investor.

3 Partial information and filtering

The trading decisions of investors are based on their knowledge about the drift process μ. While the F-investor observes the drift directly, the H-investor for $H = R, Z, J$ has to estimate the drift. This leads us to a filtering problem with hidden signal process μ and observations given by the returns R and the sequence of expert opinions (t_k, Z_k). The *filter* for the drift μ_t is its projection on the $\mathcal{F}_t^H$-measurable random variables described by the conditional distribution of the drift given $\mathcal{F}_t^H$. The mean-square optimal estimator for the drift at time t, given the available information is the *conditional mean*

$$M_t^H := \mathbb{E}[\mu_t \mid \mathcal{F}_t^H].$$

The accuracy of that estimator can be described by the *conditional covariance matrix*

$$Q_t^H := \mathbb{E}[(\mu_t - M_t^H)(\mu_t - M_t^H)^\top \mid \mathcal{F}_t^H].$$

Since in our filtering problem the signal μ, the observations and the initial value of the filter are jointly Gaussian also the the conditional distribution of the drift is Gaussian and completely characterized by the conditional mean M_t^H and the conditional covariance Q_t^H.

3.1 Dynamics of the filter

We now give the dynamics of the filter processes M^H and Q^H for $H = R, J, Z$.

R-Investor The R-investor only observes returns and has no access to additional expert opinions, the information is given by $\mathbb{F}^R$. Then, we are in the classical case of the Kalman-Bucy filter, see e.g. Liptser and Shiryaev (2001), Theorem 10.3, leading to the following dynamics of M^R and Q^R.

Lemma 3.1 *The conditional distribution of the drift given the R-investor's observations is Gaussian. The conditional mean M^R follows the dynamics*

$$dM_t^R = \kappa(\overline{\mu} - M_t^R)\, dt + Q_t^R\, \Sigma_R^{-1/2}\, d\widetilde{W}_t^R. \tag{13}$$

The innovation process $\widetilde{W}^R = (\widetilde{W}^R_t)_{t\in[0,T]}$ *given by* $d\widetilde{W}^R_t = \Sigma_R^{-1/2}\left(dR_t - M^R_t dt\right)$, $\widetilde{W}^R_0 = 0$, *is a* d*-dimensional standard Brownian motion adapted to* $\mathbb{F}^R$.

The dynamics of the conditional covariance Q^R *is given by the Riccati differential equation*

$$dQ^R_t = (\Sigma_\mu - \kappa Q^R_t - Q^R_t\kappa^\top - Q^R_t\Sigma_R^{-1}Q^R_t)\,dt. \tag{14}$$

The initial values are $M^R_0 = m_0$ *and* $Q^R_0 = q_0$.

Note that the conditional covariance matrix Q^R_t satisfies an ordinary differential equation and is hence deterministic, whereas the conditional mean M^R_t is a stochastic process defined by an SDE driven by the innovation process $\widetilde{W}^R$.

J-Investor The J-investor observes a $2d$-dimensional diffusion process with components R and J. That observation process is driven by a (d_1+d_3)-dimensional Brownian motion with components W^R and W^J. Again, we can apply classical Kalman-Bucy filter theory as in Liptser and Shiryaev (2001) to deduce the dynamics of M^J and Q^J. We also refer to Lemma 2.2 in Sass et al. (2022).

Lemma 3.2 *The conditional distribution of the drift given the* J*-investor's observations is Gaussian. The conditional mean* M^J *follows the dynamics*

$$dM^J_t = \kappa(\overline{\mu} - M^J_t)\,dt + Q^J_t\,(\Sigma_R^{-1/2}, \Sigma_J^{-1/2})d\widetilde{W}^J_t.$$

The innovation process $\widetilde{W}^J = (\widetilde{W}^J_t)_{t\in[0,T]}$ *given by*

$$d\widetilde{W}^J_t = \begin{pmatrix}\Sigma_R^{-1/2}\left(dR_t - M^J_t dt\right)\\ \Sigma_J^{-1/2}\left(dJ_t - M^J_t dt\right)\end{pmatrix}, \quad \widetilde{W}^J_0 = 0,$$

is a $2d$*-dimensional standard Brownian motion adapted to* $\mathbb{F}^J$.

The dynamics of the conditional covariance Q^J *is given by the Riccati differential equation*

$$dQ^J_t = (\Sigma_\mu - \kappa Q^J_t - Q^J_t\kappa^\top - Q^J_t(\Sigma_R^{-1} + \Sigma_J^{-1})Q^J_t)\,dt.$$

The initial values are $M^J_0 = m_0$ *and* $Q^J_0 = q_0$.

Z-Investor The next lemma provides the filter for the Z-investor who combines continuous-time observations of stock returns and expert opinions received at discrete points in time. For a detailed proof we refer to Lemma 2.3 in Sass et al. (2017) and Lemma 2.3 in Sass et al. (2022).

Lemma 3.3 *The conditional distribution of the drift given the* Z*-investor's observations is Gaussian. The dynamics of the conditional mean and conditional covariance matrix are given as follows:*

(i) Between two information dates t_k *and* t_{k+1}, $k = 0,\dots,n-1$, *the conditional mean* M^Z_t *satisfies SDE (13), i.e.,*

$$dM^Z_t = \kappa(\overline{\mu} - M^Z_t)\,dt + Q^Z_t\,\Sigma_R^{-1/2}\,d\widetilde{W}^Z_t \quad \text{for } t\in[t_k, t_{k+1}).$$

The innovation process $\widetilde{W}^Z = (\widetilde{W}^Z_t)_{t\in[0,T]}$ *given by*

$$d\widetilde{W}^Z_t = \Sigma_R^{-1/2}\left(dR_t - M^Z_t dt\right), \quad \widetilde{W}^Z_0 = 0,$$

is a d-dimensional standard Brownian motion adapted to $\mathbb{F}^Z$.
The conditional covariance Q^Z satisfies the ordinary Riccati differential equation (14), i.e.,

$$dQ_t^Z = (\Sigma_\mu - \kappa Q_t^Z - Q_t^Z \kappa^\top - Q_t^Z \Sigma_R^{-1} Q_t^Z)\, dt.$$

The initial values are $M_{t_k}^Z$ and $Q_{t_k}^Z$, respectively, with $M_0^Z = m_0$ and $Q_0^Z = q_0$.

(ii) *At the information dates t_k, $k = 1, \ldots, n-1$, the conditional mean and covariance $M_{t_k}^Z$ and $Q_{t_k}^Z$ are obtained from the corresponding values at time t_{k-} (before the arrival of the view) using the update formulas*

$$M_{t_k}^Z = \rho_k M_{t_k-}^Z + (I_d - \rho_k) Z_k, \tag{15}$$

$$Q_{t_k}^Z = \rho_k Q_{t_k-}^Z, \tag{16}$$

with the update factor $\rho_k = \Gamma(Q_{t_k-}^Z + \Gamma)^{-1}$. At initial time $t = 0$ the above update formulas give M_{0+}^Z and Q_{0+}^Z based on the initial values $M_0^Z = m_0$ and $Q_0^Z = q_0$.

Note that the dynamics of M^Z and Q^Z between information dates are the same as for the R-investor, see Lemma 3.1. The values at an information date t_k are obtained from a Bayesian update. Further, we recall that for the R- and J-investor the conditional mean M^H is a diffusion process and the conditional covariance Q^H is a continuous and deterministic function. Contrary to that, for the Z-investor the conditional mean M^Z is a diffusion process between the information dates but shows jumps of random jump size at those dates. The conditional covariance Q^Z is piecewise continuous with deterministic jumps at the arrival dates t_k of the expert opinions.

3.2 Properties of the filter

In this and the next subsection we collect some properties of the filter processes. We start with a proposition stating in mathematical terms the intuitive property that additional information from the expert opinions improves drift estimates. Since the accuracy of the filter is measured by the conditional variance it is expected that this quantity for the Z- and J-investor who combine observations of returns and expert opinions is "smaller" than for the R-investor who observes returns only. Mathematically, this can be expressed by the partial ordering of symmetric matrices. For symmetric matrices $A, B \in \mathbb{R}^{d\times d}$ we write $A \preceq B$ if $B - A$ is positive semi-definite. Note that $A \preceq B$ implies that $\|A\| \le \|B\|$.

Proposition 3.4 (Sass et al. (2021), Lemma 2.4) *For $H = Z, J$ it holds $Q_t^H \preceq Q_t^R$ and there exists a constant $C_Q > 0$ such that $\|Q_t^H\| \le \|Q_t^R\| \le C_Q$ for all $t \in [0, T]$.*

At the arrival dates t_k of the expert opinions the expert's view Z_k lead to an update of the conditional mean M^Z given by (15). That update can be considered as a weighted mean of the filter estimate $M_{t_k-}^Z$ before the arrival and the expert opinion Z_k. The following proposition shows that the update improves the accuracy both of the estimate $M_{t_k-}^Z$ before the arrival as well as of the expert's estimate Z_k.

Proposition 3.5 (Sass et al. (2017), Proposition 2.2) *For $k = 0, \ldots, n-1$ it holds $Q_{t_k}^Z \preceq \Gamma$ and $Q_{t_k}^Z \preceq Q_{t_k-}^Z$.*

The following lemma provides the conditional distribution of the expert opinions Z_k given the available information of the Z-investor before the arrival of the expert's view.

Lemma 3.6 (Kondakji (2019), Lemma 3.1.6) *The conditional distribution of the expert opinions Z_k given $\mathcal{F}^Z_{t_k-}$ is the multivariate normal distribution $\mathcal{N}\left(M^Z_{t_k-},\ \Gamma + Q^Z_{t_k-}\right)$, $k = 0, \ldots, n-1$.*

According to this lemma we can choose a sequence of i.i.d. $\mathcal{F}^Z_{t_k-}$-measurable random vectors $U_k \sim \mathcal{N}(0, I_d)$, $k = 0, \ldots, n-1$ such that under $\mathbb{F}^Z$ it holds $Z_k - M^Z_{t_k-} = \left(\Gamma + Q^Z_{t_k-}\right)^{\frac{1}{2}} U_k$. From the update formula (15) we deduce that the increments of the filter process M^Z at the information dates t_k can be expressed as

$$M^Z_{t_k} - M^Z_{t_k-} = Q^Z_{t_k-}\left(\Gamma + Q^Z_{t_k-}\right)^{-\frac{1}{2}} U_k. \tag{17}$$

Further, the update formula (16) implies that the (deterministic) increments of the filter process Q^Z at the information dates t_k can be expressed as

$$\Delta Q^Z_{t_k} = Q^Z_{t_k} - Q^Z_{t_k-} = -Q^Z_{t_k-}\left(\Gamma + Q^Z_{t_k-}\right)^{-1} Q^Z_{t_k-}. \tag{18}$$

Remark 3.7 We mention some asymptotic properties of the conditional variances Q^H_t for $t \to \infty$. Sass et al. (2017, Theorem 4.1.) show that the conditional variances Q^R and Q^J for diffusion type observations stabilize for increasing t and tend to a finite limit. For the Z-investor receiving expert opinions at equidistant time points they show in Prop. 4.1 that the conditional variances $Q^Z_{t_k-}$ and $Q^Z_{t_k}$ before and after the arrival, respectively, stabilize and tend to (different) finite limits. We also refer to our numerical results presented in Sect. 7.2.

3.3 Asymptotic filter behavior for high-frequency expert opinions

In this subsection we provide results for the asymptotic behavior of the filters for a Z-investor when the number of expert opinions goes to infinity. This will be helpful for deriving approximations not only of the filters but also of solutions to the utility maximization problem (9) in case of high-frequency expert opinions. We will denote the arrival times of the expert's views by $t_k = t^{(n)}_k$ to emphasize the dependence on n. Then we have for all n that $0 = t^{(n)}_0 < t^{(n)}_1 < \ldots < t^{(n)}_{n-1} < T$. Again we set $t^{(n)}_n = T$. We also use an additional superscript n and write $(M^{Z,n})_{t\in[0,T]}$ and $(Q^{Z,n})_{t\in[0,T]}$ for the conditional mean and the conditional covariance matrix of the filter, respectively, in order to emphasize dependence of the filter processes on the number of expert opinions.

We distinguish two different asymptotic regimes. First the expert's variance Γ stays constant, second that variance grows linearly with the number n of expert opinions.

3.3.1 High-frequency expert opinions with fixed variance

For an increasing number of expert opinions with fixed variance Γ the investor receives more and more noisy signals about the current state of the drift μ of the same precision. Then, it can be expected that in the limit the filter process M^Z_t constitutes a perfectly accurate estimate which is equal to the actual drift μ_t, i.e., the investor has full information about the drift. This intuitive statistical consistency result has been rigorously proven in Sass et al. (2017) under the following

Assumption 3.8

1. The expert's covariance matrix Γ is (strictly) positive definite and does not depend on n.
2. For the mesh size $\delta_n = \max_{k=0,\dots,n-1} |t_{k+1}^{n)} - t_k^{(n)}|$ it holds $\lim_{n\to\infty} \delta_n = 0$.

Theorem 3.9 (Sass et al. (2017), Theorem 3.4.) *Under Assumption 3.8 it holds for all* $t \in (0, T]$ *for*

1. *the conditional covariance matrix* $\lim_{n\to\infty} \|Q_t^{Z,n}\| = 0$,
2. *the conditional mean* $\lim_{n\to\infty} \mathbb{E}\left[\left\|M_t^{Z,n} - \mu_t\right\|^2\right] = 0$.

3.3.2 High-frequency expert opinions with linearly growing variance

Now we consider another asymptotic regime arising in models in which a higher arrival frequency of expert opinions is only available at the cost of accuracy. We assume that the expert views arrive at equidistant time points and the variance Γ of the views Z_k grows linearly with n. This reflects that contrary to the above setting with constant variance now investors are not able to gain an arbitrary amount of information over a fixed time interval.

We recall the dynamics of the continuous expert opinions $J = (J_t)_{t\in[0,T]}$ given in (5) by the SDE $dJ_t = \mu_t\, dt + \sigma_J\, dW_t^J$, $J_0 = 0$, and make the following

Assumption 3.10

(1) The expert arrival dates are equidistant, i.e., $t_k = t_k^{(n)} = k\Delta_n$ for $k = 0, \dots, n-1$ with $\Delta_n = \frac{T}{n}$.
(2) The experts covariance matrix is given by $\Gamma = \Gamma^{(n)} = \frac{1}{\Delta_n}\sigma_J\sigma_J^\top$.
(3) The normally distributed random vectors (ε_k^n) in (4) are linked with the Brownian motion W^J from (5) via $\varepsilon_k^n = \frac{1}{\sqrt{\Delta_n}} \int_{[t_k^{(n)}, t_{k+1}^{(n)}]} dW_s^J$.

In view of the representation of expert opinions in (4) the third assumption implies that

$$Z_k^{(n)} = \mu_{t_k^{(n)}} + \frac{1}{\Delta_n}\sigma_J \int_{[t_k^{(n)}, t_{k+1}^{(n)}]} dW_s^J, \quad k = 0, \dots, n-1. \tag{19}$$

The following theorem shows that in the present setting the information obtained from observing the discrete-time expert opinions is asymptotically the same as that from observing the diffusion process J representing the continuous-time expert and defined in (5).

Theorem 3.11 (Sass et al. (2022), Theorems 3.2 and 3.3) *Let* $p \in [1, +\infty)$. *Under Assumption 3.10 it holds:*

(1) There exists a constant $K_Q > 0$ *such that* $\left\|Q_t^{Z,n} - Q_t^J\right\| \le K_Q \Delta_n$ *for all* $t \in [0, T]$.
In particular, it holds $\lim_{n\to\infty} \sup_{t\in[0,T]} \left\|Q_t^{Z,n} - Q_t^J\right\| = 0$.

(2) There exists a constant $K_{m,p} > 0$ *such that* $\mathbb{E}\left[\left\|M_t^{Z,n} - M_t^J\right\|^p\right] \le K_{m,p}\Delta_n$ *for all* $t \in [0, T]$.
In particular, it holds $\lim_{n\to\infty} \sup_{t\in[0,T]} \mathbb{E}\left[\left\|M_t^{Z,n} - M_t^J\right\|^p\right] = 0$.

4 Utility maximization

This section is devoted to the solution of the utility maximization problem (9). We briefly review in Sect. 4.1 the solution for logarithmic utility. For the more demanding case of power utility we reformulate problem (9) in Sect. 4.2 as an equivalent stochastic optimal control problem which can be solved by dynamic programming techniques. We present the solutions for the fully informed investor ($H = F$) in Sect. 4.3. Results for partially informed investors with diffusion type observations ($H = R, J$) and for the Z-investor observing discrete-time expert opinions will follow in Sects. 5 and 6.

4.1 Logarithmic utility

For an investor who wants to maximize expected logarithmic utility of terminal wealth optimization problem (9) reads as

$$\mathcal{V}_0^H(x_0) := \sup_{\pi \in \mathcal{A}_0^H} \mathbb{E}\left[\log(X_T^\pi) \mid \mathcal{F}_0^H\right]. \tag{20}$$

This optimization problem has been solved in Gabih et al. (2014) and generalized in Sass et al. (2017) and Kondakji (2019) in the context of the different information regimes addressed in this paper. In the sequel we state the obtained results.

Proposition 4.1 *The optimal strategy $(\pi_t^H)_{t\in[0,T]})$ for the optimization problem (20) is given in feedback form by $\pi_t^H = \Pi^H(t, M_t^H)$ where the optimal decision rule is given by*

$$\Pi^H(t,m) = \Sigma_R^{-1} m \quad \text{for } m \in \mathbb{R}^d,$$

and the optimal value is

$$\begin{aligned}\mathcal{V}_0^H(x_0) &= \log(x_0) + \frac{1}{2}\int_0^T \operatorname{tr}\left(\Sigma_R^{-1}\mathbb{E}[M_t^H (M_t^H)^\top]\right)dt \\ &= \log(x_0) + \frac{1}{2}\int_0^T \operatorname{tr}\left(\Sigma_R^{-1}\left(Var[\mu_t] + \mathbb{E}[\mu_t]\mathbb{E}[\mu_t^\top] - Q_t^H\right)\right)dt. \end{aligned} \tag{21}$$

We assumed in our model for the drift process μ in (2) that the matrix κ is positive definite. Using the closed-form solution of the SDE (2) given in (3) it can be deduced that the mean $\mathbb{E}[\mu_t]$ and covariance matrix $Var[\mu_t]$ are bounded. Further, it is known from Prop. 3.4 that also the conditional covariance matrix Q_t^H is bounded. Thus from representation (21) it can be derived that the value function $\mathcal{V}_0^H(x_0)$ is bounded. As already mentioned in Sect. 2.5 there exists some constant $C_\mathcal{V}^H > 0$ such that $\mathcal{V}_0^H(x_0) \le C_\mathcal{V}^H$.

Representation (21) also shows that the value function depends on the information regime H only via an integral functional of the conditional covariance $(Q_t^H)_{t\in[0,T]}$. This allows to carry over the convergence results for the conditional covariance matrices $Q^{Z,n}$ for $n \to \infty$ given in Theorems 3.11 and 3.9 to the value functions. We refer to Sass et al. (2017, Corollary 5.2.) for the convergence $\mathcal{V}_0^{Z,n}(x_0) \to \mathcal{V}_0^F(x_0)$ for the case of a fixed expert's variance Γ and to Sass et al. (2022, Corollary 5.2.) for the convergence $\mathcal{V}_0^{Z,n}(x_0) \to \mathcal{V}_0^J(x_0)$ for linearly growing variance.

4.2 Power utility

In this section we focus on the maximization of expected power utility as given in (8). That problem can be treated as a stochastic optimal control problem and solved using dynamic programming methods. We will apply a change of measure technique which was already used among others by Nagai and Peng (2002) and Davis and Lleo (2013a). This allows to study simplified control problems in which the state variables are reduced to the (slightly modified) filter processes of conditional mean whereas the wealth process can be removed from the state.

Performance criterion Recall equation (6) for the wealth process X^π saying that $dX_t^\pi / X_t^\pi = \pi_t^\top dR_t$. Rewriting SDE (1) for the return process R in terms of the innovations processes $\widetilde{W}^H$ given in Lemmas 3.1, 3.2 and 3.3 we obtain for $H = R, J, Z$ the $\mathbb{F}^H$-semimartingale decomposition of X^π (see also Lakner (1998), Sass and Haussmann (2004))

$$\frac{dX_t^\pi}{X_t^\pi} = \pi_t^\top M_t^H + \pi_t^\top \sigma_X^H \, d\widetilde{W}_t^H, \quad X_0^\pi = x_0,$$

where $\sigma_X^R = \sigma_X^Z = \Sigma_R^{1/2}$ and $\sigma_X^J = (\Sigma_R^{1/2}, \ 0_{d\times d})$. From the above wealth equation we obtain that for a given admissible strategy π the power utility of terminal wealth X_T^π is given by

$$\mathcal{U}_\theta(X_T^\pi) = \frac{1}{\theta}(X_T^\pi)^\theta = \frac{x_0^\theta}{\theta} \Lambda_T^H \exp\Big\{ \int_0^T b(M_s^H, \pi_s) ds \Big\}, \tag{22}$$

where for $m, \, p \in \mathbb{R}^d$

$$b(m, \, p) = \theta\Big(p^\top m - \frac{1-\theta}{2} \| p^\top \sigma_X \|^2 \Big) \quad \text{and} \tag{23}$$

$$\Lambda_T^H = \exp\Big\{ \theta \int_0^T \pi_s^\top \sigma_X \, d\widetilde{W}_s^H - \frac{1}{2}\theta^2 \int_0^T \| \pi_s^\top \sigma_X \|^2 \, ds \Big\}. \tag{24}$$

Since we require that admissible strategies π satisfy $\mathbb{E}[\Lambda_T^H] = 1$ we can define an equivalent probability measure $\overline{\mathbb{P}}^H$ by $\Lambda_T^H = \frac{d\overline{\mathbb{P}}^H}{d\mathbb{P}}$ for which Girsanov's theorem guarantees that the process $\overline{W}^H = (\overline{W}_t^H)_{t\in[0,T]}$ with

$$\overline{W}_t^H = \widetilde{W}_t^H - \theta \int_0^t \sigma_X^\top \pi_s \, ds, \quad t \in [0, T], \tag{25}$$

is a $(\mathbb{F}^H, \overline{\mathbb{P}}^H)$-standard Brownian motion. This change of measure allows to rewrite the performance criterion $\mathcal{D}_0^H(x_0; \pi) = \mathbb{E}\left[\mathcal{U}_\theta(X_T^\pi) \mid \mathcal{F}_0^H\right]$ of the utility maximization problem (9) for $\pi \in \mathcal{A}_0^H$ as

$$\begin{aligned} \mathcal{D}_0^H(x_0; \pi) &= \frac{x_0^\theta}{\theta} \mathbb{E}\Big[\Lambda_T^H \exp\Big\{ \int_0^T b(M_s^{H,m_0,q_0}, \pi_s) ds \Big\} \Big] \\ &= \frac{x_0^\theta}{\theta} \overline{\mathbb{E}}^H \Big[\exp\Big\{ \int_0^T b(M_s^{H,m_0,q_0}, \pi_s) ds \Big\} \Big]. \end{aligned} \tag{26}$$

Above we used the notation $\overline{\mathbb{E}}^H$ for the expectation under the measure $\overline{\mathbb{P}}^H$. Further, the notation M^{H,m_0,q_0} emphasizes the dependence of the filter processes M^H, Q^H on the initial values m_0, q_0 at time $t = 0$ and reflects the conditioning in $\mathbb{E}\left[\mathcal{U}_\theta(X_T^\pi) \mid \mathcal{F}_0^H\right]$ on the initial

information given by the σ-algebra $\mathcal{F}_0^H$. It turns out that the utility maximization problem (9) is equivalent to the maximization of

$$\overline{\mathcal{D}}_0^H(m_0, q_0; \pi) = \overline{\mathbb{E}}^H\Big[\exp\Big\{\int_0^T b(M_s^{H,m_0,q_0}, \pi_s)ds\Big\}\Big] \tag{27}$$

over all admissible strategies for $\theta \in (0, 1)$ while for $\theta < 0$ the above expectation has to be minimized. Note that it holds $\mathcal{D}_0^H(x_0; \pi) = \frac{x_0^\theta}{\theta}\overline{\mathcal{D}}_0^H(m_0, q_0; \pi)$. This allows us to remove the wealth process X from the state of the control problem which we formulate next.

State process In view of the performance criterion (27) the state process of the associated control problem is the conditional mean M^H for which we need to express the dynamics under the measure $\overline{\mathbb{P}}^H$. Recall the $\mathbb{P}$-dynamics of M^H given in Lemma 3.1 through 3.3. Using (25) the dynamics under $\overline{\mathbb{P}}^H$ for $H = R, Z, J$ are obtained by expressing $\widetilde{W}^H$ in terms of $\overline{W}^H$ and leads to the SDE for $M^H = M^{H,m_0,q_0}$

$$dM_t^H = \alpha(M_t^H, Q_t^H; \pi_t)\,dt + \beta^H(Q_t^H)\,d\overline{W}_t^H, \quad M_0^H = m_0, \tag{28}$$

where for $m, p \in \mathbb{R}^d$ and $q \in \mathbb{R}^{d\times d}$

$$\alpha(m, q; p) = \kappa(\overline{\mu} - m) + \theta qp \quad \text{and} \quad \beta^H(q) = \begin{cases} q\Sigma_R^{-1/2}, & H = R, Z, \\ q(\Sigma_R^{-1/2}, \Sigma_J^{-1/2}), & H = J. \end{cases}$$

Note that for $H = Z$ the above SDE describes the dynamics only between two arrival dates t_{k-1} and t_k, $k = 1, \ldots, n$, whereas at the arrival dates t_k according to the updating formula (15) there are jumps of size $M_{t_k}^Z - M_{t_k-}^Z = (I_d - \rho_k)(Z_k - M_{t_k-}^Z)$. Further, note that the drift coefficient α in the SDE (28) for M^H now depends also on the conditional variance Q^H as well as on the strategy π. Since the conditional covariance Q^H is deterministic it is not affected by the change of measure.

The case of full information can formally be incorporated in our model with the settings $M^F = \mu$ and a state equation (28) with the coefficients $\alpha(m, q, p) = \kappa(\overline{\mu} - m)$, $\beta^F(q) = \sigma_\mu$ which are independent of q.

Markov Strategies To apply the dynamic programming approach to the optimization problem (27) the state process M^H needs to be a Markov process adapted to $\mathbb{F}^H$. To allow for this situation we restrict the set of admissible strategies to those of Markov type which are defined in terms of time and the state process M^H according to a given specified decision rule Π, i.e, $\pi_t = \Pi(t, M_t^H)$ for a some given measurable function $\Pi : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$. Below we will need some technical conditions on Π which we collect in the following

Assumption 4.2

(1) *Lipschitz condition*
There exists a constant $C_L > 0$ such that for all $m_1, m_2 \in \mathbb{R}^d$ and all $t \in [0, T]$ it holds

$$\|\Pi(t, m_1) - \Pi(t, m_2)\| \le C_L\|m_1 - m_2\|.$$

(2) *Linear growth condition*
There exists a constant $C_G > 0$ such that for all $m \in \mathbb{R}^d$ and all $t \in [0, T]$ it holds

$$\|\Pi(t, m)\| \le C_G(1 + \|m\|).$$

(3) *Integrability condition*
For the information regimes $H = R, J, Z$ the strategy processes π defined by $\pi_t = \Pi(t, M_t^H)$ on $[0, T]$ are such that the process Λ defined by (24) satisfies $\mathbb{E}[\Lambda_T^H] = 1$.

We denote by

$$\mathcal{A}^H := \Big\{ \Pi : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d : \ \Pi \text{ is a measurable function satisfying Assumption 4.2} \Big\}$$

the *set of admissible decision rules*.

Remark 4.3 The integrability condition guarantees that the Radon-Nikodym density process Λ^H given in (23) is a martingale, hence the equivalent measure $\overline{\mathbb{P}}^H$ is well-defined. A Markov strategy $\pi = (\pi_t)_{t\in[0,T]}$ with $\pi_t = \Pi(t, M_t^H)$ defined by an admissible decision rule Π is contained in the set of admissible strategies $\mathcal{A}_0^H$ given in (7) since by construction it is $\mathbb{F}^H$-adapted, the positivity of the wealth process X^π follows from (22). The integrability condition implies the square-integrability of π. Finally, the Lipschitz and linear growth condition ensure that SDE (28) for the dynamics for the controlled state process enjoys for all admissible strategies a unique strong solution.

Control problem We are now ready to formulate the stochastic optimal control problem with the state process M^H and a Markov control defined by the decision rule Π. The dynamics of the state process M^H are given in (28). We write $M_s^{H,\Pi,t,m}$ for the state process at time $s \in [t,T]$ controlled by the decision rule Π and starting at time $t \in [0,T]$ with initial value $M_0^H = m \in \mathbb{R}^d$. Note that $M_s^{H,\Pi,t,m}$ depends on the conditional covariance Q^H which is deterministic and can be computed offline. Therefore Q^H needs not to be included as state process of the control problem. Further, we remove the initial value $q_0 = Q_t^H$ from the notation but keep in mind the dependence of $M_s^{H,\Pi,t,m}$ on Q^H.

To solve the control problem (27) we will apply the dynamic programming approach which requires the introduction of the following reward and value functions. For all $t \in [0,T]$ and $\Pi \in \mathcal{A}^H$ the associated reward function or performance criterion of the control problem (27) reads as

$$D^H(t,m;\Pi) := \overline{\mathbb{E}}^H\Big[\exp\Big\{ \int_t^T b(M_s^{H,\Pi,t,m}, \Pi(s, M_s^{H,\Pi,t,m}))ds \Big\}\Big], \quad \text{for } \Pi \in \mathcal{A}^H, \tag{29}$$

while the associated value function reads

$$V^H(t,m) := \begin{cases} \sup\limits_{\Pi\in\mathcal{A}^H} D^H(t,m;\Pi), & \theta \in (0,1), \\ \inf\limits_{\Pi\in\mathcal{A}^H} D^H(t,m;\Pi), & \theta \in (-\infty,0), \end{cases} \tag{30}$$

and it holds $V^H(T,m) = D^H(T,m;\Pi) = 1$. In the sequel we will concentrate on the case $0 < \theta < 1$, the necessary changes for $\theta < 0$ will be indicated where appropriate.

In view of relation (26) and the above transformations the value function of the original utility maximization problem (9) can be obtained from the value function of control problem (30) by

$$\mathcal{V}_0^H(x_0) = \frac{x_0^\theta}{\theta} V^H(0,m_0). \tag{31}$$

4.3 Full information

The utility maximization problem for the case of full information $H = F$ is already investigated in Kim and Omberg (1996), Brendle (2006) and Davis and Lleo (2013b, Chapter 2). In our analysis it will serve as a reference case for the comparison with results for partial information. Recall that for $H = F$ the state process is set to be the drift, i.e., $M^F = \mu$ whereas for the

partially informed investors the state M^H is the conditional mean process under the measure $\overline{\mathbb{P}}^H$. In the state equation (28) for $H = F$ the coefficients read as $\alpha(m, q, p) = \kappa(\overline{\mu} - m)$, $\beta^F(q) = \sigma_\mu$. Below we only present the results for the associated control problem (30) which will serve as a reference when we investigate the other information regimes of partial information. For details we refer to Kondakji (2019, Sec. 4.1, 5.1).

Well posedness We assume that the model parameters $\sigma_R, \sigma_\mu, \kappa, T, \theta$ are such that the following sufficient condition for the well posedness of the control problem under full information is satisfied. It is derived in our paper (Gabih et al., 2022, Corollary 3.4) and requires that the terminal value problem for the Riccati equation

$$\frac{dA_\gamma(t)}{dt} = -2A_\gamma(t)\Sigma_\mu A_\gamma(t) + \kappa^\top A_\gamma(t) + A_\gamma(t)\kappa - \gamma\Sigma_R^{-1}, \quad A_\gamma(T) = 0_{d\times d} \tag{32}$$

has for $\gamma = \gamma_0 = \frac{\theta}{2(1-\theta)^2}$ a bounded solution on [0,T]. That condition implies restrictions to the choice of model parameters for $\theta \in (0, 1)$, that is for investors which are less risk-averse than log-utility maximizing investor. For $\theta < 0$ it is always fulfilled. This follows from Theorem 5.2 in Fleming and Rishel (1975). For $\theta < 0$ we have $\gamma_0 < 0$ and $\gamma_0\Sigma_R^{-1}$ is negative semindefinite. Then the above theorem says that the solution A_{γ_0} to the Riccati ODE (34) does not explode on $(-\infty, T)$ and is thus always bounded on $[0, T]$.

Theorem 4.4 *For the control problem (30) under full information ($H = F$) the optimal decision rule is for $t \in [0, T)$, $m \in \mathbb{R}^d$ given by*

$$\Pi^F = \Pi^F(t, m) = \frac{1}{(1-\theta)}\Sigma_R^{-1}m. \tag{33}$$

The value function for $t \in [0, T]$, $m \in \mathbb{R}^d$ is given by

$$V^F(t, m) = \exp\left\{m^\top A^F(t)m + m^\top B^F(t) + C^F(t)\right\},$$

where A^F, B^F and C^F are functions on $[0, T]$ taking values in $\mathbb{R}^{d\times d}$, $\mathbb{R}^d$ and $\mathbb{R}$, respectively, satisfying a terminal value problem for the following system of ODEs

$$\begin{aligned}
\frac{dA^F(t)}{dt} &= -2A^F(t)\Sigma_\mu A^F(t) + \kappa^\top A^F(t) + A^F(t)\kappa - \frac{1}{2}\frac{\theta}{1-\theta}\Sigma_R^{-1}, && A^F(T) = 0_{d\times d},\\
\frac{dB^F(t)}{dt} &= -2A^F(t)\kappa\overline{\mu} + \left[\kappa^\top - 2A^F(t)\Sigma_\mu\right]B^F(t), && B^F(T) = 0_d,\\
\frac{dC^F(t)}{dt} &= -\frac{1}{2}(B^F(t))^\top\Sigma_\mu B^F(t) - (B^F(t))^\top\kappa\overline{\mu} - \mathrm{tr}\{\Sigma_\mu A^F(t)\}, && C^F(T) = 0.
\end{aligned} \tag{34}$$

A proof and a detailed verification analysis can be found in Davis and Lleo (2013b, Chapter 2).

Boundedness of A^F, B^F, C^F The differential equations for A^F, B^F, C^F are coupled. The ODE for A^F is an autonomous matrix Riccati ODE which can be solved first. Given the solution to A^F, one can solve the linear ODE for B^F, and then one can find C^F by integrating the r.h.s. of the last ODE. Therefore, a bounded solution A^F implies boundedness of B^F and C^F on $[0, T]$. Since the Riccati equation for A^F is a special case of (32) it is sufficient to require the condition

$$\text{the solution } A_\gamma \text{ to (32) is for } \gamma = \gamma^F = \frac{\theta}{2(1-\theta)} \text{ bounded on [0,T],} \tag{35}$$

to be satisfied. For $\theta < 0$ we have $\gamma^F < 0$ and as above for A_{γ_0}, from Fleming and Rishel (1975, Theorem 5.2) it follows that the solution $A^F = A_{\gamma^F}$ to (32) does not explode on $(-\infty, T)$ and is thus always bounded on $[0, T]$.

For $\theta \in (0, 1)$ and the scalar case, that is $d = 1$, condition (35) follows immediately from the well posedness condition in (32). To see this, we introduce for $G \in \mathbb{R}$ the notation $h_\gamma(G)$ for the r.h.s. of (32) such that this ODE reads $\frac{d}{dt}A_{\gamma_0} = h_{\gamma_0}(A_{\gamma_0})$ and the Riccati equation for $A^F = A_{\gamma^F}$ as $\frac{d}{dt}A_{\gamma^F} = h_{\gamma^F}(A_{\gamma^F})$. Since for $\theta \in (0, 1)$ it holds $\gamma_0 > \gamma^F > 0$ and Σ_R^{-1} is positive it holds for the r.h.s. of the above ODEs $h_{\gamma_0}(G) < h_{\gamma^F}(G)$. Since the terminal conditions $A_{\gamma_0}(T) = A_{\gamma^F}(T) = 0$ are the same for both equations it can be deduced that the solutions satisfy $A_{\gamma_0}(t) \geq A_{\gamma^F}(t) = A^F(t)$ for $t \in [0, T)$. From Roduner (1994, Theorem 1.2) it follows that for $\gamma > 0$ the solutions of the Riccati equations are nonnegative on $[0, T]$. Thus, we have $A_{\gamma_0}(t) \geq A^F(t) \geq 0$ on $[0, T)$ and the boundedness of A_{γ_0} implies that A^F is bounded.

Monetary value of information In order to quantify the monetary value of information we have introduced in Sect. 2.6 the quantity $x_0^{H/F}$ for which we have to evaluate an expectation given in (10). The latter can be given in terms of the functions A^F, B^F, C^F appearing in the above theorem. Using (31) we find for $H = R, J, Z$

$$\mathbb{E}\Big[\mathcal{V}_0^F\big(x_0^{H/F}\big) \mid \mathcal{F}_0^H\Big] = \mathbb{E}\Big[\frac{\big(x_0^{H/F}\big)^\theta}{\theta} V^F(0, \mu_0) \mid \mathcal{F}_0^H\Big] = \frac{\big(x_0^{H/F}\big)^\theta}{\theta}\mathbb{E}\Big[V^F(0, \mu_0) \mid M_0^H, Q_0^H\Big].$$

For $M_0^H = m$ and $Q_0^H = q$ the conditional distribution of μ_0 given $\mathcal{F}_0^H$ is the normal distribution $\mathcal{N}(m, q)$, i.e. $\mu_0 = m + q^{\frac{1}{2}}\varepsilon$ with $\varepsilon \sim \mathcal{N}(0, I_d)$. Hence we have

$$\mathbb{E}\Big[V^F(0, \mu_0) \mid M_0^H = m, Q_0^H = q\Big] = \mathbb{E}\Big[V^F\big(0, m + q^{1/2}\varepsilon\big)\Big].$$

This approach can be extended to the case of arbitrary time points $t \in [0, T]$ with $M_t^H = m$ and $Q_t^H = q$ by means of the following function

$$\overline{V}^F(t, m, q) := \mathbb{E}\Big[V^F\big(t, m + q^{\frac{1}{2}}\varepsilon\big)\Big], \qquad \forall t \in [0, T]. \tag{36}$$

The following Lemma gives an explicit form for computing $\overline{V}^F(t, m, q)$.

Lemma 4.5 (Kondakji (2019), Lemma 5.1.3) *Under the assumption that the eigenvalues of the matrix $I_d - 2A^F(t)q$ are positive, it holds*

$$\overline{V}^F(t, m, q) = \exp\Big\{m^\top \overline{A}^F(t, q)\, m + m^\top \overline{B}^F(t, q)m + \overline{C}^F(t, q)\Big\},$$

for $t \in [0, T]$, $m \in \mathbb{R}^d$, $q \in \mathbb{R}^{d\times d}$, where

$$\begin{aligned}
\overline{A}^F(t, q) &= (I_d - 2A^F(t)q)^{-1}A^F(t),\\
\overline{B}^F(t, q) &= (I_d - 2A^F(t)q)^{-1}B^F(t),\\
\overline{C}^F(t, q) &= C^F(t) + \frac{1}{2}\big(B^F(t)\big)^\top (I_d - 2A^F(t)q)^{-1}q\,B^F(t) - \frac{1}{2}\log\det(\mathrm{I_d} - 2\mathrm{A^F(t)q})
\end{aligned}$$

with A^F, B^F and C^F given in Theorem 4.4.

5 Partially informed investors observing diffusion processes

In this section we start to solve the control problem (30) for partially informed investors. We consider investors observing the diffusion processes R and/or J, i.e., the information regimes $H = R, J$. The case of the information regime $H = Z$ with discrete-time expert opinions follows in Sect. 6.

Well posedness In Gabih et al. (2022, Corollary 3.7) we show that for $\theta < 0$ the control problem (30) for $H = R, J, Z$ is always well posed. However, for $\theta \in (0, 1)$ in addition to the condition (32) ensuring well posedness of the control problem for the fully informed investor one now also has to impose the condition

$$\text{the eigenvalues of } I_d - 2(1-\theta)(t)Q_t^H \text{ are positive on } [0,T] \text{ for } \gamma_0 = \frac{\theta}{2(1-\theta)^2}. \quad (37)$$

Condition (37) says that the conditional covariance Q^H must not be "too big" such that the eigenvalues of $I_d - 2(1-\theta)(t)Q_t^H$ are positive. Recall, Q^H is deterministic and can be computed offline, and from Proposition 3.4 we know that Q^H is bounded.

For solving control problem (30) we apply dynamic programming techniques. Starting point is the dynamic programming principle given in the following lemma. For the proof we refer to Frey et al. (2014, Prop. 6.2) and Pham (1998, Prop. 3.1).

Lemma 5.1 (Dynamic Programming Principle) *For every $t \in [0,T]$, $m \in \mathbb{R}^d$ and for every stopping time τ with values in $[t,T]$ it holds*

$$V^H(t,m) = \sup_{\Pi \in \mathcal{A}^H} \overline{\mathbb{E}}^H\left[\exp\left\{ \int_t^\tau b(M_s^{H,\Pi,t,m}, \Pi(s, M_s^{H,\Pi,t,m}))ds \right\} V^H(\tau, M_\tau^{H,\Pi,t,m}) \right]. \quad (38)$$

From the dynamic programming principle the dynamic programming equation (DPE) for the value function presented in Theorem 5.2 can be deduced. That equation constitutes a necessary optimality condition and allows to derive the optimal decision rule. We recall that we focus on the solution for $\theta \in (0,1)$, the case $\theta < 0$ follows analogously by changing sup into inf in (38). For convenience we introduce the shorthand notation

$$\overline{\Sigma}_H^{-1} = \begin{cases} \Sigma_R^{-1}, & H = R, Z \\ \Sigma_R^{-1} + \Sigma_J^{-1}, & H = J. \end{cases} \quad (39)$$

Theorem 5.2 (Dynamic programming equation)

(1) In the case of diffusion type observations. i.e., $H = R, J$, the value function V^H satisfies for $t \in [0,T)$ and $m \in \mathbb{R}^d$ the PDE

$$\begin{aligned} 0 = {} & \frac{\partial}{\partial t} V^H(t,m) + D_m^\top V^H(t,m)\left(\kappa(\overline{\mu} - m) + \frac{\theta}{(1-\theta)} Q_t^H \Sigma_R^{-1} m\right) \\ & + \frac{1}{2} \operatorname{tr}\left\{ D_{mm} V^H(t,m)\left(Q_t^H \overline{\Sigma}_H^{-1} Q_t^H\right)\right\} + \frac{\theta}{2(1-\theta)} m^\top \Sigma_R^{-1} m V^H(t,m) \\ & + \frac{\theta}{2(1-\theta)} \frac{1}{V^H(t,m)} D_m^\top V^H(t,m)\left(Q_t^H \Sigma_R^{-1} Q_t^H\right) D_m V^H(t,m), \end{aligned} \quad (40)$$

with the terminal condition $V^H(T,m) = 1$ and $D_m V^H$, $D_{mm} V^H$ denoting the gradient and Hessian matrix of V^H, respectively.

(2) The candidate optimal decision rule is for $t \in [0, T)$ and $m \in \mathbb{R}^d$ given by

$$\Pi^H = \Pi^H(t,m) = \frac{1}{1-\theta}\Sigma_R^{-1}\left(m + \frac{1}{V^H(t,m)}Q_t^H D_m V^H(t,m)\right). \tag{41}$$

Proof Let $t, \tau \in [0, T)$ with $\tau > t$ for some fixed time point t. Then the dynamic programming principle (38) and the continuity of $M_s^{H,\Pi,t,m}$ on $[0, T)$ imply

$$\begin{aligned} V^H(t,m) &= \lim_{\tau \searrow t} V^H(\tau, m) \\ &= \lim_{\tau \searrow t} \sup_{\Pi \in \mathcal{A}^H} \overline{\mathbb{E}}^H\Big[\exp\Big\{\int_t^\tau b(M_s^{H,\Pi,t,m}, \Pi(s, M_s^{H,\Pi,t,m}))ds\Big\} V^H(\tau, M_\tau^{H,\Pi,t,m})\Big]. \end{aligned} \tag{42}$$

For the state process $M = M^{H,\Pi,t,m}$ given in (28) the associated generator $\mathcal{L} = \mathcal{L}^p$ applied to a function $g \in C^2(\mathbb{R}^d)$ for fixed $p = \Pi(t,m)$ reads

$$\mathcal{L}g(m) = D_m^\top g(m)\alpha(m,q,p) + \frac{1}{2}\operatorname{tr}\Big\{D_{mm}g(m)\beta^H(q)\big(\beta^H(q)\big)^\top\Big\}.$$

From (42) we obtain using Dynkin's formula for $V^H(\tau, M_\tau^{H,\Pi,t,m})$ and standard arguments of the dynamic programming approach the following PDE

$$\begin{aligned} 0 =& \frac{\partial}{\partial t}V_k(t,m) + D_m^\top V^H(t,m)\kappa(\overline{\mu} - m) + \frac{1}{2}\operatorname{tr}\Big\{D_{mm}V^H(t,m)Q_t^H\overline{\Sigma}_H^{-1}Q_t^H\Big\} \\ &+ \sup_{p\in\mathbb{R}^d}\Big\{D_m^\top V^H(t,m)\theta Q_t^H p + \theta\Big(p^\top m - \frac{1-\theta}{2}p^\top\Sigma_R p\Big)V^H(t,m)\Big\}. \end{aligned} \tag{43}$$

The maximizer for the supremum appearing in (43) yields the optimal decision rule $\Pi^H = \Pi^H(t,m)$ which is given in (41). Plugging the expression for the maximizer Π^Z back into the DPE (43) we obtain the PDE (40). □

The above dynamic programming equation can be solved using an ansatz as in (44) below and leads to closed-form expressions for the value function V^H and the optimal decision rule Π^H in terms of solutions of some ODEs. The results are given in the next theorem and are known for $H = R$ already from Brendle Brendle (2006). For the case $H = J$ and details of the proof we refer Kondakji (2019, Sec. 5.2).

Theorem 5.3 (Solution of DPE and optimal decision rule)

(1) In the case of diffusion type observations, that is $H = R, J$, the solution to the dynamic programming equation equation (40) is given for $t \in [0, T]$, $m \in \mathbb{R}^d$ by

$$V^H(t,m) = \exp\Big\{m^\top A^H(t)m + m^\top B^H(t) + C^H(t)\Big\}. \tag{44}$$

The functions $A^H(t)$, $B^H(t)$ and $C^H(t)$ staisfy the system of ODEs

$$\begin{aligned} \frac{dA^H(t)}{dt} =& -2A^H(t)Q_t^H\Big(\frac{\theta}{1-\theta}\Sigma_R^{-1} + \overline{\Sigma}_H^{-1}\Big)Q_t^H A^H(t) \\ &+ \Big(\kappa^\top - \frac{\theta}{1-\theta}\Sigma_R^{-1}Q_t^H\Big)A^H(t) \\ &+ A^H(t)\Big(\kappa - \frac{\theta}{1-\theta}Q_t^H\Sigma_R^{-1}\Big) - \frac{\theta}{2(1-\theta)}\Sigma_R^{-1}, \end{aligned}$$

$$\frac{dB^H(t)}{dt} = \left[\kappa^\top - 2A^H(t)Q_t^H\Big(\frac{\theta}{1-\theta}\Sigma_R^{-1} + \overline{\Sigma}_H^{-1}\Big)Q_t^H - \frac{\theta}{1-\theta}\Sigma_R^{-1}Q_t^H\right]B^H(t) - 2A^H(t)\kappa\overline{\mu},$$

$$\frac{dC^H(t)}{dt} = -(B^H(t))^\top\Big[\kappa\overline{\mu} + \frac{1}{2}Q_t^H\Big(\frac{\theta}{1-\theta}\Sigma_R^{-1} + \overline{\Sigma}_H^{-1}\Big)Q_t^H B^H(t)\Big] \operatorname{tr}\left\{Q_t^H\Sigma_R^{-1}Q_t^H A^H(t)\right\},$$

with terminal values $A^H(T) = 0_{d\times d}$, $B^H = 0_d$, $C^H = 0$.

(2) *The candidate optimal decision rule is for* $t \in [0,T)$, $m \in \mathbb{R}^d$, *given by*

$$\Pi^H(t,m) = \frac{1}{1-\theta}\Sigma_R^{-1}\left(m + Q_t^H(2A^H(t)m + B^H(t))\right). \tag{45}$$

The next proposition shows that the functions A^H, B^H, C^H solving the system of ODEs in Theorem 5.3 can be expressed by the solutions A^F, B^F, C^F to the ODEs for the full information problem given in Theorem 4.4 via the functions $\overline{A}^F, \overline{B}^F, \overline{C}^F$ given in Lemma 4.5. The proof is given in Kondakji, (2019, Lemma 5.2.1). The result will facilitate the proof of boundedness of A^H, B^H, C^H.

Proposition 5.4 *If the eigenvalues of* $I_d - 2A^F(t)Q_t^H$ *are positive for all* $t \in [0,T]$ *then it holds for the functions* A^H, B^H *and* C^H *on* $[0,T]$

$$A^H(t) = \overline{A}^F\left(t, Q_t^H\right),\ B^H(t) = \overline{B}^F\left(t, Q_t^H\right),\ C^H(t) = \overline{C}^F\left(t, Q_t^H\right) - \theta\Delta_X^H(t), \tag{46}$$

where the functions $\overline{A}^F$, $\overline{B}^F$ *and* $\overline{C}^F$ *are given in Lemma 4.5 and*

$$\Delta_X^H(t) := \frac{1}{2}\log\frac{det(I_d - 2\xi^H(t)Q_t^H)}{det(I_d - 2A^F(t)Q_t^H)} + \underline{K}^H(t) - \overline{K}^H(t), \quad t \in [0,T]. \tag{47}$$

The function $\xi^H(t)$ *satisfies on* $[0,T]$ *the Riccati equation*

$$\frac{d\xi^H(t)}{dt} = -2\xi^H(t)\Sigma_\mu\xi^H(t) + \kappa^\top\xi^H(t) + \xi^H(t)\kappa + \frac{1}{2}\overline{\Sigma}_H,\ \xi^H(T) = 0,$$

where $\overline{\Sigma}_H$ *is given in (39). The functions* $\underline{K}^H(t)$, $\overline{K}^H(t)$ *are given by*

$$\underline{K}^H(t) = \int_t^T \operatorname{tr}\{\Sigma_\mu\left(A^F(u) - \xi^H(u)\right)\}du,$$

$$\overline{K}^J(t) = \frac{1}{2}\int_t^T \operatorname{tr}\{Q_u^J\Sigma_J^{-1}(I_d - 2Q_u^J A^F(u))\}du \quad \text{and} \quad \overline{K}^R(t) = 0.$$

Boundedness of $\mathbf{A^H}, \mathbf{B^H}, \mathbf{C^H}$ In view of Proposition 5.4 this property holds under condition (35) saying that A^F is bounded, and if the eigenvalues of $I_d - 2(1-\theta)A^F(t)Q_t^H$ are positive on $[0,T]$. Note that if A^F is bounded then B^F and C^F are also bounded. Thus, the expressions $\overline{A}^F\left(t, Q_t^H\right), \overline{B}^F\left(t, Q_t^H\right), \overline{B}^F\left(t, Q_t^H\right)$ in (46) with $\overline{A}^F, \overline{B}^F$ given in Lemma 4.5 are bounded on $[0,T]$. This proves the boundedness of A^H and B^H. Finally, the boundedness of C^H follows from integrating the r.h.s. of the ODE for C^H given in Theorem 5.3, which is bounded.

Recall, for $\theta < 0$ we have that A^F is always bounded and negative semindefinite. Analogously to the approach in our paper (Gabih et al., 2022, Section 3) one can show that the eigenvalues of $K(t) = I_d - 2(1-\theta)A^F(t)Q_t^H$ are bounded below by 1 and thus positive.

However, for $\theta \in (0, 1)$ the eigenvalues of $K(t)$ are positive only if the conditional covariance Q_t^H is on $[0, T]$ not "too large". Since if λ is an eigenvalue of $A^F(t)Q_t^H$ then $1 - 2(1-\theta)\lambda$ is an eigenvalue of $K(t)$ one has to require that $\lambda_{\max}(A^F(t)Q_t^H) < \frac{1}{2(1-\theta)}$ for all $t \in [0, T]$. Here, $\lambda_{\max}(G)$ denotes the largest eigenvalue of a generic matrix G.

Verification We derived the above candidate solution to the control problem (30) using the classical stochastic control approach. To ensure that the solution to the DPE is indeed the value function V, the candidate optimal decision rule Π^H indeed satisfies $V^H(t, m) = D^H(t, m; \Pi^H)$, and that Π^H defines an optimal strategy process via $\pi_t^H = \Pi^H(t, M_t^H)$ that is admissible, one needs to prove a suitable verification theorem. Such a verification theorem is given in Hata and Sheu (2018, Theorem 4.1) for a slightly general setting. That paper considers a combined consumption and investment problem in which the investor is also allowed to consume portfolio wealth and aims to maximize the expectation of the sum of the power utility of terminal wealth and aggregated running power utility of consumption. Further, Hata and Sheu (2018) allows for correlation between return and drift process, non-negative interest rate for the risk-free asset, and for discounting the utility. The findings can be directly adopted to the control problem (30) for $H = R, J$ if the utility for consumption formally is set to zero, leading to zero optimal consumption, and removing the consumption from the strategy process.

In Hata and Sheu (2018) the authors take advantage of the consideration of a logarithmic transformation of the performance criterion and study the associated DPE for $\log V^H$. One of the key assumptions for the above mentioned verifications results is the boundedness of the functions A^H, B^H, C^H on $[0, T]$. Further, the linearity of the optimal decision rule Π^H w.r.t. the state variable m, see (45), is exploited to prove, that the optimal strategy process π^H generated by Π^H, is admissible. In particular, using a result of Bensoussan (1992, Lemma 4.1.1) which is also given in Nagai (2015, Lemma 5.1), it can be deduced that the associated density process Λ^H defining the change of measure and given in (24) satisfies $\mathbb{E}[\Lambda_T^H] = 1$.

Remark 5.5 The optimal decision rules given in (41) and (45) can be rewritten as

$$\begin{aligned}\Pi^H(t, m) &= \Pi^F(t, m) + \frac{1}{(1-\theta)V^H(t, m)}\Sigma_R^{-1} Q_t^H D_y V^H(t, m) \\ &= \Pi^F(t, m) + \frac{1}{1-\theta}\Sigma_R^{-1}\, Q_t^H(2A^H(t)m + B^H(t)).\end{aligned}$$

Thus, Π^H can be decomposed into two parts. The first part Π^F is the optimal decision rule of the fully informed investor given in (33). To obtain the value of the strategy process π_t at time t the fully informed investor plugs in for m the current value of the drift μ_t, whereas the partially informed H-investor plugs in the filter estimate M_t^H. In the literature Π^F is also known as *myopic decision rule*. The second part is the "correction term" $\Pi^H - \Pi^F$ which is known as *drift risk* of the partially informed H-investor since it accounts for the investor's uncertainty about the current value of the non-observable drift, see Rieder and Bäuerle (2005, Remark 1) and Frey et al. (2012, Remark 5.2). The decomposition shows that, in contrast to the case of log utility (see Sass et al. (2017)), the so-called *certainty equivalence principle* does not apply to power utility. It states that the optimal strategy under partial information is obtained by replacing the unknown drift μ_t by the filter estimate M_t^H in the formula for the optimal strategy under full information.

Remark 5.6 It is well-known that log-utility can be embedded in the family of power utilities using the relation $\mathcal{U}_\theta(x) - 1/\theta = (x^\theta - 1)/\theta \to \log x$ for $\theta \to 0$. Replacing $\mathcal{U}_\theta(x)$ by $\mathcal{U}_\theta(x) - 1/\theta$ in the utility maximization problem (9) will only lead to an additive shift of the

value function while the optimal strategy remains unchanged. Hence, it can be expected that the optimal decision rule given (45) converges for $\theta \to 0$ to the optimal decision rule for log-utility $\Pi^H_{\log}(t,m) = \Sigma_R^{-1} m$ given in Proposition 4.1. And in fact, this is the case, since for $\theta \to 0$ the solutions to the ODEs for A^H, B^H converge to zero. First, the r.h.s. of the Riccati ODE for A^H is a quadratic expression of A^H with the absolute term $-\frac{\theta}{2(1-\theta)}\Sigma_R^{-1}$. The latter vanishes for $\theta = 0$. Since the terminal value at time $t = T$ is zero, the solution A^H is zero on $[0, T]$. Thus, the linear ODE for B^H is homogeneous with zero terminal value, and the solution B^H is zero on $[0, T]$. Finally, the correction term or drift risk $\Pi^H - \Pi^F$ containing $2A^H(t)m + B^H(t)$ vanishes for $\theta \to 0$.

By comparing the optimal decision rules for the R, J and F-investor for identical values m of the conditional mean and q of the conditional covariance, some interesting properties result, which are formulated in the following lemma. For the proof we refer to Kondakji (2019, Lemma 5.2.4 and 5.2.5).

Lemma 5.7

1. *If* $M_t^R = M_t^J = m$ *and* $Q_t^R = Q_t^J = q$ *then it holds* $\Pi^R(t,m) = \Pi^J(t,m)$.
2. *If* $M_t^H = \mu_t = m$ *and* $Q_t^H = 0$ *then it holds* $\Pi^F(t,m) = \Pi^H(t,m)$ *for* $H = R, J$.

Monetary value of information Theorem 5.3 allows to derive explicit expressions for the initial investment $x_0^{R/H}$ and $x_0^{H/F}$ introduced in Sect. 2.6 and needed to evaluate the efficiency of the R- and J-investor and the monetary value of expert opinions.

Lemma 5.8 (Kondakji (2019), Lemma 5.3.1) *For the initial capital* $x_0^{H/F}$ *and the efficiency* ε^H *defined in (10) and (11), respectively, it holds for* $H = R, J$

$$x_0^{H/F} = x_0^H \exp\{-\Delta_X^H(0)\} \quad \text{and} \quad \varepsilon^H = \exp\{-\Delta_X^H(0)\}.$$

For the initial capital $x_0^{R/J}$ *defined in (12) and the monetary value of expert opinions* P_{Exp}^J *it holds*

$$P_{Exp}^J = x_0^R - x_0^{R/J} \quad \text{with} \quad x_0^{R/J} = \exp\{-\Delta_X^J(0) + \Delta_X^R(0)\},$$

where $\Delta_X^J(0)$ *and* $\Delta_X^R(0)$ *are given in (47).*

6 Partially informed investors observing discrete-time expert opinions

After solving the control problem (30) for partially informed investors observing the diffusion processes this section presents the solution for investors observing returns and discrete-time expert opinions, i.e., the information regime $H = Z$. As in Sect. 5 for $H = R, J$ we impose the pair of well posedness conditions (32) and (37). We again apply the dynamic programming principle to derive the DPE for the value function and introduce the notation

$$V_k(t,m) = \begin{cases} V^Z(t,m), & \text{for } t \in [t_{k-1}, t_k), \\ v_k(y) = V^Z(t_k-, m) = \lim\limits_{t \nearrow t_k} V^Z(t,m), & \text{for } t = t_k, \end{cases}$$

for $k = 1, \ldots, n$, i.e., $V_k : [t_{k-1}, t_k] \to \mathbb{R}$ denotes the value function on the k-th interval between two subsequent information dates and $v_k(y)$ its left-hand limit at t_k. Note that for $t_n = T$ we have $v_n(m) = V(T,m) = 1$.

Theorem 6.1 (Dynamic programming equation)

(1) Let the value function be defined piecewise for $t \in [t_{k-1}, t_k)$, $k = 1, \dots n$, and $m \in \mathbb{R}^d$ by $V^Z(t,m) = V_k(t,m)$. Then the functions V_k satisfy on (t_0, t_1) and $[t_{k-1}, t_k)$, $k = 2, \dots, n$, the PDE (40) with the terminal conditions

$$V_k(t_k, m) = v_k(m) = \overline{\mathbb{E}}^Z\Big[V_{k+1}\big(t_k, M_{t_k}^{\Pi, t_k-, m}\big)\Big], \quad k = 1, \dots, n-1. \tag{48}$$

For $t = t_n = T$ it holds $V_n(T,m) = 1$ and for $t = t_0 = 0$

$$V_1(0,m) = v_0(m) = \overline{\mathbb{E}}^Z\Big[V_1\big(0, M_{0+}^{\Pi,0,m}\big)\Big], \quad m \in \mathbb{R}^d. \tag{49}$$

(2) The candidate optimal decision rule is for $t \in [t_{k-1}, t_k)$, $k = 1, \dots n$, given by

$$\Pi^Z = \Pi^Z(t,m) = \Pi^F(t,m) + \frac{1}{1-\theta}\Sigma_R^{-1} Q_t^Z \frac{D_m V_k(t,m)}{V_k(t,m)}, \tag{50}$$

where Π^F is given in (33).

Proof Let $t, \tau \in (0, t_1)$ or $t, \tau \in [t_{k-1}, t_k)$, $k = 2, \dots n$, with $\tau > t$ for a fixed time point t. Analogously to the proof of Theorem 5.2 the dynamic programming principle (38) and the continuity of $M_s^{H,\Pi,t,m}$ on $(0, t_1)$ and $[t_{k-1}, t_k)$, $k = 2, \dots n$, imply that V_k satisfies the PDE (40) and that the optimal decision rule is given as in (50).

It remains to prove the terminal conditions in (48) and relation (49) for the initial time $t = 0$. We fix an information date t_k, $k = 1, \dots, n-1$ and apply again the dynamic programming principle (38) where we set $\tau = t_k$ and consider the following limit

$$\begin{aligned} v_k(m) &= V_k(t_k-, m) = V^Z(t_k-, m) \\ &= \lim_{t \nearrow t_k} \sup_{\Pi \in \mathcal{A}^Z} \overline{\mathbb{E}}^Z\Big[\exp\Big\{\int_t^{t_k} b(M_s^{Z,\Pi,t,m}, \Pi(s, M_s^{Z,\Pi,t,m}))ds\Big\} V^Z(t_k, M_{t_k}^{Z,\Pi,t,m})\Big] \\ &= \sup_{\Pi \in \mathcal{A}^Z} \overline{\mathbb{E}}^Z\Big[V_{k+1}\big(t_k, M_{t_k}^{Z,\Pi,t_k-,m}\big)\Big]. \end{aligned}$$

The above expectation depends only on the distribution of the jump size $M_{t_k} - M_{t_k-}$ of the state process which is independent of the decision rule Π. Thus, we can omit the supremum in the last equation and get $v_k(m) = \overline{\mathbb{E}}^Z\Big[V_{k+1}\big(t_k, M_{t_k}^{Z,\Pi,t_k-,m}\big)\Big]$.

For time $t = 0$ we have $\mathcal{F}_0^Z = \mathcal{F}_0^I$ which represents the prior information on the initial drift μ_0 but does not yet contain the first expert opinion Z_0. As above, the dynamic programming principle yields for $t = 0$ and $\tau \in (0, t_1)$

$$\begin{aligned} V^Z(0,m) &= V_1(0,m) \\ &= \lim_{\tau \searrow 0} \sup_{\Pi \in \mathcal{A}^Z} \overline{\mathbb{E}}^Z\Big[\exp\Big\{\int_0^{\tau} b(M_s^{Z,\Pi,0,m}, \Pi(s, M_s^{Z,\Pi,0,m}))ds\Big\} V^Z(\tau, M_\tau^{Z,\Pi,0,m})\Big] \\ &= \sup_{\Pi \in \mathcal{A}^Z} \overline{\mathbb{E}}^Z\Big[V_1\big(0+, M_{0+}^{Z,\Pi,0,m}\big)\Big] = v_0(m). \end{aligned}$$

□

Remark 6.2 The above theorem shows that at the information dates $t_k, k = 1, \dots, n-1$, the value function exhibits jumps of size $V^Z(t_k, m) - V^Z(t_k-, m) = V_{k+1}(t_k, m) - v_k(m)$. Note that we excluded the information of the first expert opinion Z_0 from the initial σ-algebra $\mathcal{F}_0^Z$. Therefore V exhibits at time $t = 0$ a jump of size $V^Z(0+, m) - V^Z(0, m) = V_1(0,m) - v_0(m)$.

For $H = Z$ the DPE appears as a system of coupled terminal value problems for the PDE (40) for V_k which are tied together by the terminal conditions (48). The latter appear as pasting conditions for the value function described by V_k and V_{k+1} on two subsequent intervals divided by the information date t_k. Therefore that system can be solved recursively for $k = n, \ldots, 1$ starting with $V_n(t_n, m) = V_n(T, m) = 1$. From Sect. 5 it is already known that the DPEs for the control problems for the information regimes $H = R, J$ can be solved explicitly using an exponential ansatz leading to the results given in Theorem 5.3. We apply this idea to our problem and make for $t \in [t_{k-1}, t_k),\ k = 1, \ldots n$, the ansatz

$$V_k(t, m) = \exp\{m^\top A_k(t)m + B_k^\top(t)m + C_k(t)\} \tag{51}$$

where A_k is a function on $[t_{k-1}, t_k]$ taking values in the set of real symmetric $d \times d$ matrices, whereas B_k and C_k are some functions on $[t_{k-1}, t_k]$ with values in $\mathbb{R}^d$ and $\mathbb{R}$, respectively, which have to be determined.

Theorem 6.3 (Solution of dynamic programming equation and optimal decision rule)

(1) The solution to the dynamic programming equation given in (48) and (49) is for $[t_{k-1}, t_k),\ k = 1, \ldots n$ *and* $m \in \mathbb{R}^d$ *given by*

$$V_k(t, m) = \exp\{m^\top A_k(t)m + B_k^\top(t)m + C_k(t)\},$$

where the functions A_k, B_k *and* C_k *satisfy on* $t \in (t_0, t_1)$ *and* $t \in [t_{k-1}, t_k),\ k = 2, \ldots n$, *the system of ODEs given in Theorem 5.3 for* $H = R$ *with terminal values for* $t = t_k$

$$A_k(t_k) = \Psi_k\, A_{k+1}(t_k), \tag{52}$$

$$B_k(t_k) = \Psi_k\, B_{k+1}(t_k), \tag{53}$$

$$C_k(t_k) = C_{k+1}(t_k) - \frac{1}{2}\, B_{k+1}^\top(t_k)\, \Psi_k\, \Delta Q_{t_k}^Z B_{k+1}(t_k) + \frac{1}{2} \log\det \Psi_k, \tag{54}$$

where $\Delta Q_{t_k}^Z$ *is the increment of the conditional variance at* t_k *and given in (18). Further,*

$$\Psi_k := \left(I_d + 2\, A_{k+1}(t_k) \Delta Q_{t_k}^Z\right)^{-1} \quad \text{for } k = 1, \ldots, n-1. \tag{55}$$

For $k = n$ *it holds* $A_n(t_n) = 0_{d\times d}$, $B_n(t_n) = 0_{d\times 1}$, $C_n(t_n) = 0$.
For $t_0 = 0$ *the values of* A_1, B_1, C_1 *are obtained from the formulas (52), (53), (54) replacing* $A_{k+1}(t_k)$, $B_{k+1}(t_k)$, $C_{k+1}(t_k)$ *by* $A_1(0+)$, $B_1(0+)$, $C_1(0+)$, *respectively.*

(2) The candidate optimal decision rule is for $t \in [t_{k-1}, t_k),\ k = 1, \ldots n, m \in \mathbb{R}^d$, *given by*

$$\Pi^Z = \Pi^Z(t, m) = \Pi^F(t, m) + \frac{1}{1-\theta} \Sigma_R^{-1} Q_t^Z \left(2\, A_k(t)m + B_k(t)\right),$$

where Π^F *is given in (33).*

Proof Plugging ansatz (51) for V_k into PDE (40) with $\overline{\Sigma}^Z = \Sigma_R$ and equate coefficients in front of m yields the ODEs given in Theorem 5.3 for $H = R$. The terminal value $V(T, m) = V_n(t_n, m) = 1$ implies the given terminal values for A_n, B_n, C_n at the terminal time $t_n = T$. The other terminal values follow from the evaluation of the expectation on the right side of (48). Using the shorthand notation $\widetilde{A} = A_{k+1}(t_k)$, $\widetilde{B} = B_{k+1}(t_k)$, $\widetilde{C} = C_{k+1}(t_k)$ and ansatz (51) for $V_{k+1}(t_k)$ we obtain

$$\begin{aligned} v_k(m) &= \overline{\mathbb{E}}^Z\left[V_{k+1}\left(t_k, M_{t_k}^{\Pi, t_k-, m}\right)\right] \\ &= \overline{\mathbb{E}}^Z\left[\exp\left\{ \left(M_{t_k}^{\Pi, t_k-, m}\right)^\top \widetilde{A}\, M_{t_k}^{\Pi, t_k-, m} + \widetilde{B}^\top M_{t_k}^{\Pi, t_k-, m} + \widetilde{C} \right\}\right]. \end{aligned}$$

Completing the square with respect to $M_{t_k}^{\Pi,t_k-,m}$ yields

$$v_k(m) = \overline{\mathbb{E}}^Z\Big[\exp\Big\{\big(M_{t_k}^{\Pi,t_k-,m} + \frac{1}{2}\widetilde{A}^{-1}\widetilde{B}\big)^\top \widetilde{A}\,\big(M_{t_k}^{\Pi,t_k-,m} + \frac{1}{2}\widetilde{A}^{-1}\widetilde{B}\big)\Big\} \cdot \exp\Big\{-\frac{1}{4}\widetilde{B}^\top\widetilde{A}^{-1}\widetilde{B} + \widetilde{C}\Big\}\Big].$$

The above expectation can be evaluated using (Gabih et al., 2022, Lemma 3.5) saying that for a d-dimensional Gaussian random vector $Y \sim \mathcal{N}(\mu_Y, \Sigma_Y)$, $b \in \mathbb{R}^d$ and a symmetric and invertible matrix $U \in \mathbb{R}^{d\times d}$ such that all eigenvalues of $I_d - 2U\Sigma_Y$ are positive, it holds

$$\overline{\mathbb{E}}^Z\big[\exp\{(Y+b)^\top U(Y+b)\}\big] = \big(\det(\mathrm{I_d} - 2\,\mathrm{U}\Sigma_\mathrm{Y})\big)^{-1/2} \times \exp\big\{(\mu_Y+b)^\top\,(I_d - 2\,U\Sigma_Y)^{-1}U\,(\mu_Y+b)\big\}.$$

We set $U = \widetilde{A}$ and $b = \frac{1}{2}\widetilde{A}^{-1}\widetilde{B}$ and $Y = M_{t_k}^{\Pi,t_k-,m} \sim \mathcal{N}(\mu_Y, \Sigma_Y)$. For mean and covariance of Y update formula (17) and relation (18) imply $\mu_Y = m$ and $\Sigma_Y = Q^Z_{t_k-}(\Gamma + Q^Z_{t_k-})^{-1}Q^Z_{t_k-} = -\Delta Q^Z_{t_k}$. We obtain

$$v_k(m) = \exp\Big\{\big(m + \frac{1}{2}\widetilde{A}^{-1}\widetilde{B}\big)^\top\,\Psi_k\widetilde{A}\big(m + \frac{1}{2}\widetilde{A}^{-1}\widetilde{B}\big) - \frac{1}{4}\widetilde{B}^\top\widetilde{A}^{-1}\widetilde{B} + \widetilde{C} + \frac{1}{2}\log\det\Psi_k\Big\}.$$

and rearranging terms yields

$$v_k(m) = \exp\Big\{m^\top\Psi_k\widetilde{A}\,m\; + \widetilde{B}^\top\Psi_k\,m\; + \widetilde{C} - \frac{1}{2}\widetilde{B}^\top\Psi_k\,\Delta Q^Z_{t_k}\widetilde{B} + \frac{1}{2}\log\det\Psi_\mathrm{k}\Big\},$$

On the other hand we have from ansatz (51) $v_k(m) = V_k(t_k, m) = \exp\big\{m^\top A_k(t_k)m + B_k^\top(t_k)m + C_k(t_k)\big\}$.

By comparing the coefficients in front of m in the last two expressions for v_k and substituting back the expressions for $\widetilde{A}$, $\widetilde{B}$, $\widetilde{C}$ we obtain the desired result.

The proof for $t = t_0 = 0$ is analogous. Finally, the expression for the optimal decision rule follows if ansatz (51) for the value function V^Z is substituted into (50). □

Remark 6.4 Analysing the update formulas (52) through (54) for expert opinions which become less and less reliable in the sense that $\|\Gamma^{-1}\| \to 0$ it can be seen that $\Delta Q^Z_{t_k}$ tend to 0_d and the update-factors Ψ_k given in (55) tend to I_d. As a consequence the functions A_k, B_k and C_k define a smooth value function V^Z on $[0, T]$ which equals the value function V^R of the R-investor. This is as expected since in the considered limiting case for the Z-investors expert opinions do not provide any additional information about the drift.

In Remark 5.6 we have seen that for $H = R, J$ and $\theta \to 0$ the optimal decision rule $\Pi^H(t, m)$ converges to $\Pi^H_{\log}(t, m) = \Sigma_R^{-1}m$ which is optimal for log-utility, see Proposition 4.1. The same holds for $H = Z$ since the functions A^k, B^k and therefore the correction term $\Pi^Z - \Pi^F$ vanish for $\theta = 0$. This can be seen from the vanishing solutions of the ODEs for A_k, B_k between the information dates. This implies that the updates (52), (53) at the informations dates also vanish.

Boundedness of $\mathbf{A_k}, \mathbf{B_k}, \mathbf{C_k}$ The proof of this property is based on the relations $A_k(t) = \overline{A}^F\big(t, Q^Z_t\big)$ and $B_k(t) = \overline{B}^F\big(t, Q^Z_t\big)$ on $[t_{k-1}, t_k]$, $k = 1, \ldots, n$. They can be verified by some calculations showing that between the information dates the left and right hand sides satisfy the same ODEs as in the corresponding proof for A^H, B^H for $H = R, J$ in Kondakji (2019, Lemma 5.2.1). There are also the same terminal conditions at the information dates which are given by the update formulas (52) and (52). In view of the definition of $\overline{A}^F$, $\overline{B}^F$

in Lemma 4.5, the boundedness of A_k, B_k follows from condition (35) saying that A^F is bounded. Then, also B^F and C^F are bounded.

Further, the eigenvalues of $I_d - 2(1-\theta)A^Z(t)Q_t^Z$ have to be positive on $[0, T]$. As already explained for $H = R, J$ this always holds true for $\theta < 0$ while for $\theta \in (0, 1)$ one has to require $\lambda_{\max}(A^F(t)Q_t^Z) < \frac{1}{2(1-\theta)}$. From Prop. 3.4 it is known that $Q_t^Z \preceq Q_t^R$ from which one can deduce $\lambda_{\max}(A^F(t)Q_t^Z) \leq \lambda_{\max}(A^F(t)Q_t^R) < \frac{1}{2(1-\theta)}$. Thus, it is enough to check the condition for $H = R$.

Finally, The boundedness of C_k follows from integrating the r.h.s. of the ODE for C_k which is bounded and the updates given in (52) which are also bounded.

Verification The key idea for the verification is that between the information dates the functions V_k satisfy the same DPE as the value function V^R of the R-investor given in (40). This allows to rely on the verification results for $H = R$ and to iterate them backward in time. Starting point is a control problem on $[t_{n-1}, T]$ with the modified initial time $t = t_{n-1}$ instead of $t = 0$ and initial value $m_0 = M^Z_{t_{n-1}}$ and $q_0 = Q^Z_{t_{tn-1}}$. This is a control problem for the R-investor on $[t_{n-1}, T]$ for which $V_n(t, m)$ and $\Pi^Z(t, m)$ for all (t, m) are verfied to be the value function and the optimal decision rule, respectively. Here, V_n satisfies the terminal condition $V(T, m) = 1$, as for $H = R$.

Next, we consider the control problem on $[t_{n-2}, t_{n-1}]$ with initial time $t = t_{n-2}$. At the terminal time t_{n-1} the terminal condition is obtained from the dynamic programming principle (38) leading to the expression for $V_{n-1}(t_{n-1}, m)$ given in (48). Again we can apply the verification results for $H = R$ on this time interval. Note, that these results also work for nonzero terminal conditions. They only require the boundedness of the solutions of the ODEs for A_k, B_k, C_k which already checked above. Continuing this iteration completes the verification.

Monetary value of information For the initial investments $x_0^{R/Z}$ and $x_0^{Z/F}$ introduced in Sect. 2.6 and needed to evaluate the efficiency of the Z-investor and the monetary value of expert opinions one can derive the following expressions.

Lemma 6.5 (Kondakji (2019), Lemma 6.3.1) *For the initial capital $x_0^{Z/F}$ and the efficiency ε^Z defined in (10) and (11), respectively, it holds*

$$x_0^{Z/F} = x_0^Z \exp\{-\Delta_X^Z(0)\} \quad \text{and} \quad \varepsilon^Z = \exp\{-\Delta_X^Z(0)\}.$$

For the initial capital $x_0^{R/Z}$ defined in (12) and the monetary value of expert opinions P^Z_{Exp} it holds

$$P^Z_{Exp} = x_0^R - x_0^{R/Z} \text{ with } x_0^{R/Z} = \exp\{-\Delta_X^Z(0) + \Delta_X^R(0)\},$$

where $\Delta_X^Z(0)$ and $\Delta_X^R(0)$ are given in (47) where for $H = Z$ the settings $Q^H = Q^Z$, $\overline{\Sigma}^Z = \Sigma_R$ and $\overline{K}^Z(t) = 0$ apply.

7 Numerical results

In this section we illustrate the theoretical findings of the previous sections with results of some numerical experiments.

Table 1 Model parameters for the numerical experiments

Drift mean reversion level	$\overline{\mu}$	0.05	Investment horizon	T	1 year
Mean reversion speed	κ	3	Power utility parameter	θ	-0.3
Volatility	σ_μ	1	Volatility of stock	σ_R	0.25
Mean of μ_0	$\overline{m}_0$	$\overline{\mu} = 0.05$	Volatility of cont.-time experts	σ_J	0.2
Variance of μ_0	$\overline{q}_0$	$\frac{\sigma_\mu^2}{2\kappa} = 0.1\overline{6}$	Variance of discr.-time experts	Γ	0.4
Filter initial value M_0^H	m_0	$\overline{m}_0 = 0$	Number of expert opinions	n	10
Initial value Q_0^H	q_0	$\overline{q}_0 = 0.1\overline{6}$			

7.1 Model parameters

Our experiments are based on a stock market model where the unobservable drift $(\mu_t)_{t\in[0,T]}$ follows an Ornstein-Uhlenbeck process as given in (2) whereas the volatility is known and constant. For simplicity, we assume that there is only one risky asset in the market, i.e. $d = 1$.

If not stated otherwise our numerical experiments are based on model parameters as given in Table 1. The parameter of the utility function is chosen to be negative, $\theta = -0.3$. This corresponds to an investor which is more risk averse than a log-utility maximizing investor trading the Kelly portfolio. This is the relevant case for most investors. Note that for a negative θ the optimization problem is always well-posed, see our paper (Gabih et al., 2022).

The mean and variance of the initial value μ_0 of the drift are the parameters of the stationary distribution of the drift process which is known to be Gaussian with mean $\overline{\mu}$ and variance $\sigma_\mu^2/(2\kappa)$. Hence, the drift process μ is stationary on $[0, \infty)$ and for the chosen parameters there is a 90% probability for values in the interval $(-0.75, 0.85)$ centered at $\overline{\mu} = 0.05$, and a probability around 2/3 for values in $(0.35, 0.45)$. The filter processes M^H and Q^H are also initialized with the stationary mean and variance modeling partially informed investors which only know the model parameters but have no additional prior knowledge about the drift. The volatility σ_J of the continuous-time expert opinions J is chosen as 0.2 and slightly smaller than the volatility $\sigma_R = 0.25$ of the return process R. Hence the observations of J are more informative than those of the returns R.

7.2 Filter

In this subsection we want to illustrate our theoretical findings on filtering based on different information regimes by results on a simulation study. Figure 1 shows in the top panel simulated paths of the two diffusion type observation processes which are the return process R and the continuous-time expert opinion process J associated to the drift process μ. The drift process which is not observable by the investors is plotted in the middle panel. The top panel also presents for comparison the cumulated drift process representing the drift component in the dynamics of both R and J, see (1) and (5). Expert's views Z_k which are noisy observations of the drift process μ at the information dates and forming the additional information of the Z-investor are shown as red crosses in the middle panel. From the observed quantities the filter of the R-, J- and Z-investor are computed in terms of the pair (M^H, Q^H). For $H = R, Z, J$ the conditional expectations M^H are plotted against time in the middle panel while the conditional variances Q^H are shown in the bottom panel.

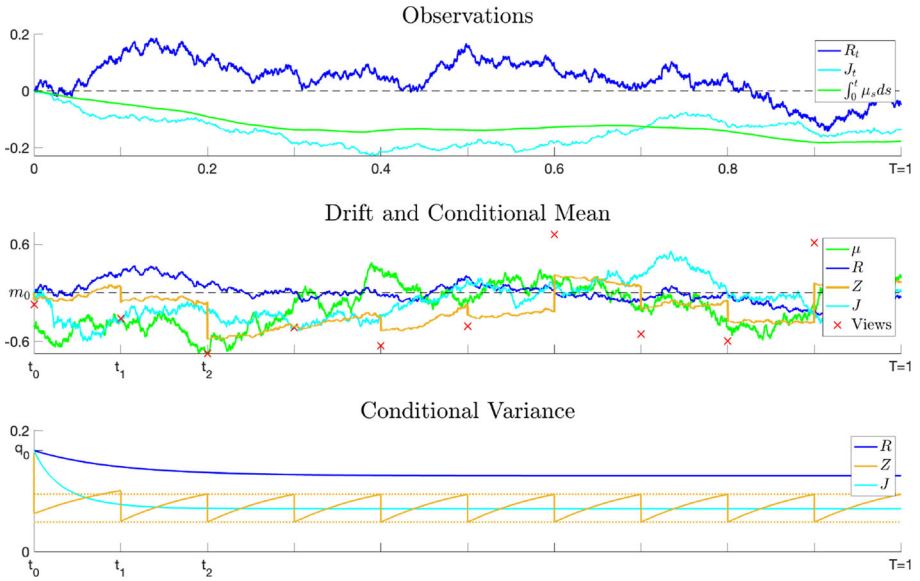

Fig. 1 Observation and filter processes. Top: Diffusion-type observation processes $\mathbb{R}$ and J. Middle: Drift μ, expert views Z_k and conditional mean M^H for $H = R, Z, J$. Bottom: Conditional variance Q^H

Recall that Q^R and Q^J as well as Q^Z for any $n \in \mathbb{N}$ are deterministic. In the bottom panel one sees that for any fixed $t \in [0, T]$, the value of Q_t^J as well as the value of Q_t^Z is less or equal than the value of Q_t^R. This shows that additional information by the expert opinions improves the accuracy of the filter estimate. It confirms the underlying theoretical result on the partial ordering of the conditional covariance matrices as stated in Proposition 3.4. The updates in the filter of the Z-investor at the information dates of the expert's views decrease the conditional variance and lead to a jump of the conditional mean M^Z. These are typically jumps towards the hidden drift μ, of course this depends on the actual value of the expert's view. Note that the drift estimate M^R of the R-investor is quite poor and mostly fluctuates just around the mean-reversion level $\overline{\mu}$. However, the expert opinions visibly improve the drift estimate.

For increasing t the conditional variances Q_t^R and Q_t^J approach a finite value. An associated convergence result for $t \to \infty$ has been proven in Proposition 4.6 of Gabih et al. (2014) for markets with $d = 1$ stock and generalized in Theorem 4.1 of Sass et al. (2017) for markets with an arbitrary number of stocks. For the Z-investor we observe an almost periodic behavior of the conditional variance $(Q_t^Z)_{t\geq 0}$. The asymptotic behavior for $t \to \infty$ and the derivation of upper and lower bounds have been studied in detail in Gabih et al. (2014, Prop. 4.6) for $d = 1$ and in Sass et al. (2017, Sec. 4.2) for the general case. These bounds are shown as dashed lines in the bottom panel.

Next we perform some experiments illustrating the theoretical results from Sect. 3.2 on the asymptotic filter behavior for increasing number n of expert opinions. We distinguish two cases. First, the expert's variance Γ stays constant leading to convergence to full information, i.e., mean square convergence of $M^{Z,n}$ to μ and $Q^{Z,n} \to 0$ on $(0, T]$, see Theorem 3.9. Second, that variance grows linearly with n leading to convergence of the filter processes of the Z-invester to those of the J-investor, see Theorem 3.11. For that experiment the expert's

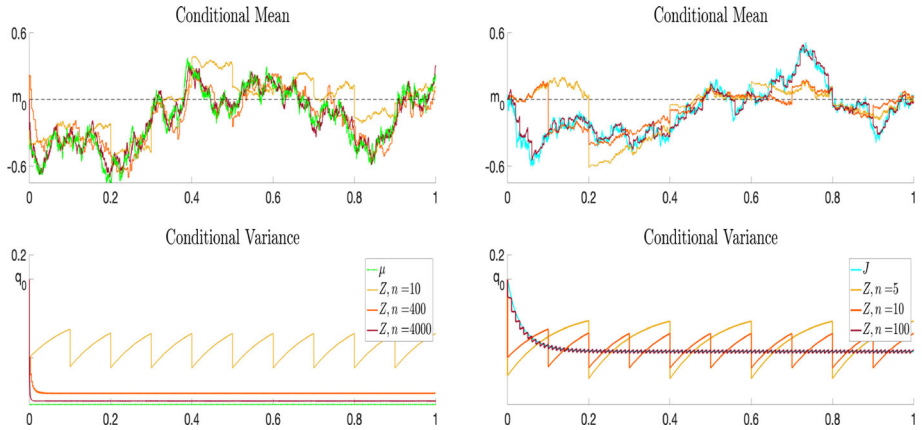

Fig. 2 Asymptotic behavior of the filter for $n \to \infty$ Left: Constant expert's variance $\Gamma = 0.4$, convergence to full information Right: Linearly growing expert's variance $\Gamma^{(n)} = \frac{n}{T}\sigma_J^2$, diffusion approximation

views are generated as in (19), i.e., the Gaussian random variables ε_k in (4) are linked with the Brownian motion W^J from (5) driving the continuous-time expert opinion process J.

Figure 2 shows on the left side for the experiment with constant variance Γ the conditional mean $M^{Z,n}$ and the drift μ (top) and the conditional variance $Q^{Z,n}$ (bottom) for $n = 10, 400, 4000$. It can be nicely seen that for increasing n the conditional variance tends to zero while the conditional mean approaches the drift process for any $t \in (0, T]$. In the limit for $n \to \infty$ the Z-investor has full information about the drift process. The panels on the right side shows the results for the experiment with linearly growing variance for which we consider the cases $n = 5, 10, 100$. It can be seen that the both filter processes $M^{Z,n}$ and $Q^{Z,n}$ approach the corresponding processes of the J-investor for any $t \in [0, T]$ in accordance with Theorem 3.11. Contrary to the first experiment we observe that this convergence is much faster.

Note that for the chosen parameters from Table 1 we have for $n = 10$ expert opinions that $\Gamma = \Gamma^{(n)} = \frac{n}{T}\sigma_J^2 = 0.4$, i.e., the same expert's variances for both experiments. This yields for $n = 10$ identical conditional variances as it can be seen in the two bottom panels. However, the paths of the conditional mean are different since the expert's views Z_k in the experiment with linearly growing expert's variance are linked to the Brownian motion W^J, see (19), whereas in the left panels they are not.

7.3 Value function

In this subsection we present for the case of power utility solutions to the control problem (30). In particular, we analyze the value functions $V^H(t, m)$ and the associated optimal decision rules $\Pi^H(t, m)$ for the different information regimes H. They are given in Theorems 4.4, 5.3 and 6.3 for $H = F$, R, J and Z, respectively. We recall relation (31) saying that the solution of the original problem of maximizing expected power utility can be obtained from V^H by the relation $\mathcal{V}_0^H(x_0) = \frac{x_0^\theta}{\theta} V^H(0, m_0)$. For $H = F$ the relation also holds true if the initial value m_0 of the conditional mean is replaced by the initial value μ_0 of the drift.

Figure 3 shows in the upper part the value functions $V^H(t, m)$ for $H = R, Z, J, F$ plotted against time t (left) for a fixed value for the drift estimate $m = m^* = 0.05$ and plotted against

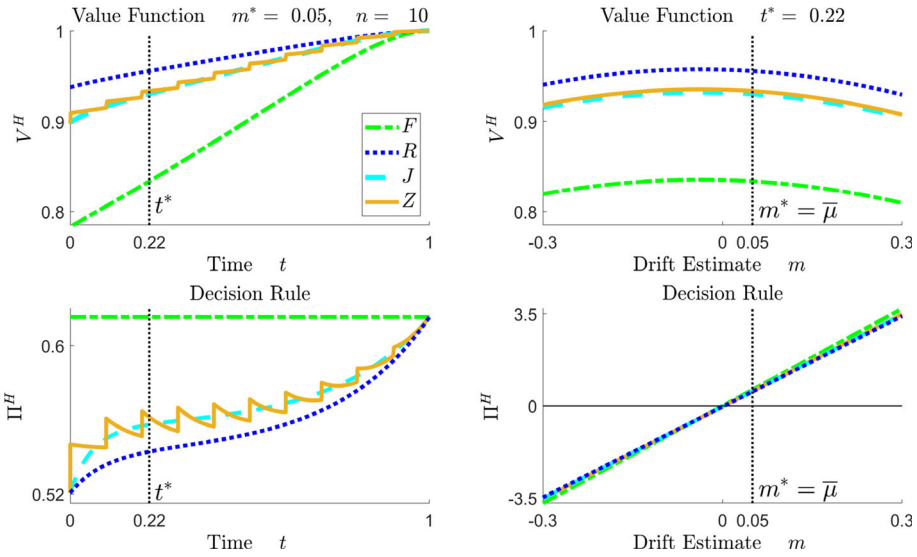

Fig. 3 Value functions and optimal decision rules. Top: Value functions $V^H(t,m)$ for $H = F, R, J, Z$ ($n = 10$) depending on t/m (left/right). Bottom: Optimal decision rule $\Pi^H(t,m)$ depending on t/m (left/right)

m (right) for fixed time $t = t^* = 0.22$. The lower panels show the corresponding decision rules $\Pi^H(t,m)$ for the partially informed investors.

For the value functions we can observe that they increase with time t and reach the value 1 at terminal time T which follows from the definition of the performance criterion in (29). The value function of the Z-investor exhibits jumps at the information dates. The upper right panel illustrates that the value functions are exponentials of a quadratic function of the drift estimate m. For almost all (t,m) the value function of the fully informed investor $V^F(t,m)$ is smaller than those those of the partially observed investors. We recall relation (31) saying that $\mathcal{V}_0^H(x_0) = \theta^{-1} x_0^\theta V^H(0,m_0)$ and that we work with a negative parameter of the utility function ($\theta = -0.3$). Hence, order relations for the maximized expected utilities $\mathbb{E}\left[\mathcal{U}_\theta(X_T^\pi) \mid \mathcal{F}_0^H\right]$ represented by $\mathcal{V}_0^H$ are reversed to relations for value functions V^H. Further, the value functions of the Z- and J-investor with access to additional information from the expert opinions are smaller than the value function of R-investor observing returns only. We note, that these relations do not hold in general, except for $t = 0$.

The lower plots show the optimal decision rules $\Pi^H(t,m)$ which are given in (33), (45) and (50). They are all of the form, see Remark 5.5 and (41),

$$\Pi^H(t,m) = \Pi^F(t,m) + \frac{1}{1-\theta}\Sigma_R^{-1} Q_t^H\left(2\,A^H(t)m + B^H(t)\right) \text{ with } \Pi^F(m) = \frac{1}{1-\theta}\Sigma_R^{-1} m.$$

Here, Π^F constitutes the myopic decision rule whereas the correction term $\Pi^H - \Pi^F$ describes the drift risk of the partially informed H-investor. All decision rules are linear in the drift estimate m as it can be seen from the lower right panel. For drift estimates m much larger (smaller) than the mean reversion level of the drift $\overline{\mu}$ the investors holds a long (short) position in the stock which are smaller (in absolute terms) for the fully informed investor than for partially informed ones. The figures further show that the drift risk decreases over time (in absolute terms) and vanishes at terminal time T. For $H = Z, J$ it is smaller than for $H = R$ indicating that more information about the hidden drift leads to decision rules closer

to the myopic decision rule. This effect is also supported by the observation that the decision rule of the Z-investor exhibits jumps at the information dates towards the myopic decision rule. The arrival of an additional information improves the filter estimate of the hidden drift and decreases the correction term.

We refer to Kondakji (2019, Sec. 8.3) for results for a positive parameter θ, i.e., a relative risk aversion $1-\theta$ larger than for log-utility. Contrary to the present case with $\theta = -0.3$ in which the control problem (30) is a minimization problem we face a maximization problem for $\theta > 0$. There are similar results but the monotonicity properties w.r.t. time t and the ordering of the value function and optimal decision rules for the different information regimes H are reversed.

7.4 High-frequency experts

In this subsection we want to study the asymptotic behavior of the value functions and optimal decision rules of the Z-investor for growing number n of expert opinions. For the case of log-utility it is known that the convergence results for the filters as given in Theorems 3.9 and 3.11 carry over directly to the convergence of value functions. The proof is straightforward and relies on representations of the value function as in (21) in terms of an integral functional of the (deterministic) conditional variance Q^Z.

However, in case of power utility that approach can no longer be adopted since the performance criterion in (29) and consequently the value functions $V^H(t,m)$ are now given in terms of expectations of the exponential of quite involved integral functionals of the filter processes M^H under the equivalent measure $\overline{\mathbb{P}}^H$ introduced in Sect. 4.2. Hence, the value functions $V^H(t,m)$ depend on the complete filter distribution, not only on its second-order moments. Further, for power utility the optimal strategies do not depend only on the current drift estimate but contain correction terms depending on the distribution of future drift estimates. A formal and rigorous proof of convergence of value functions is ongoing work and deferred to a forthcoming publication. It is based on the L^p-convergence of conditional mean processes for arbitrary $p \geq 2$ as it can be deduced from Theorems 3.9 and 3.11.

Our numerical results presented below provide a strong support of the convergence of value functions also for power utility. As in Sect. 3.2 we consider two different asymptotic regimes which are obtained if the expert's variance Γ is either fixed or grows linear in n. In order to emphasize the dependence of the value function and optimal decision rule of the Z-investor on n we use the notation $V^{Z,n}$ and $\Pi^{Z,n}$.

Figure 4 presents results of experiments for linearly growing variance $\Gamma^{(n)} = \frac{n}{T}\sigma_J^2$ for which we have convergence of the filter processes $M^{Z,n}$, $Q^{Z,n}$ to the diffusion limit given by filter processes of the J-investor observing a continuous-time expert opinion process. The top panels show the value function $V^{Z,n}(t,m)$ while the bottom panels present the optimal decision rule $\Pi^{Z,n}(t,m)$ of the Z-investor observing $n = 5, 10, 100$ expert opinions. For comparison we also show the results for the R- and J-investor. For increasing n both $V^{Z,n}$ and $\Pi^{Z,n}$ quickly approach the corresponding quantities of the J-investor. This shows that for the chosen parameters quite accurate diffusion approximations of solutions to the control problem for the Z-investor are available already for moderate numbers n of expert opinions. Since the latter require less computational effort this is very helpful for deriving approximations not only for the value functions but also for related quantities such as efficiencies and prices of expert opinions introduced in Sect. 2.6 and considered in the next subsection.

Figure 5 shows results of the experiment with fixed variance $\Gamma = 0.4$ for which we have convergence to full information, i.e., mean-square convergence of $M^{Z,n}$ to μ and $Q^{Z,n} \to 0$

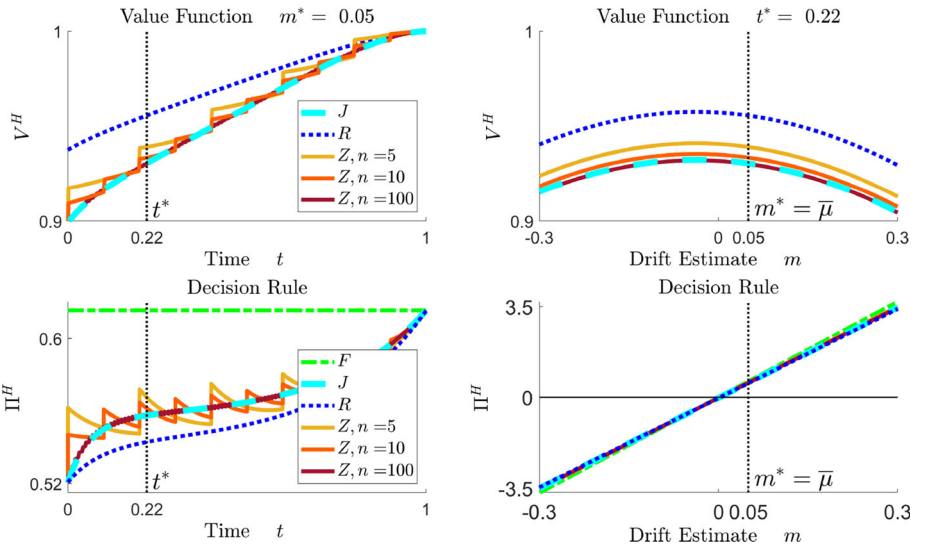

Fig. 4 Asymptotic behavior of value functions and optimal decisions rules for growing n and $\Gamma^{(n)} = \frac{n}{T}\sigma_J^2$. Top: Value functions $V^H(t, m)$ depending on t/m (left/right) for $H = R, J, Z$ Bottom: Optimal decision rules $\Pi^H(t, m)$ for $H = F, R, J, Z$ depending on t/m (left/right)

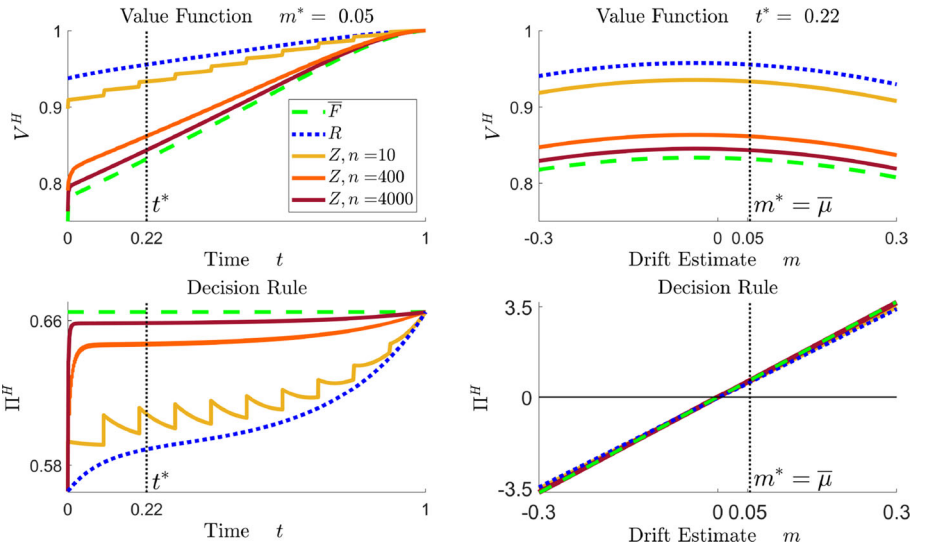

Fig. 5 Asymptotic behavior of value functions and optimal decision rules for growing n and $\Gamma = 0.4$ (fixed). Top: Value function $\overline{V}^F(t, m, Q_t^{Z,4000})$ and $V^H(t, m)$ depending on t/m (left/right) for $H = R, Z$. Bottom: Optimal decision rules $\Pi^H(t, m)$ for $H = F, R, Z$ depending on t/m (left/rights)

on $(0, T]$. As in Fig. 4 we plot $V^{Z,n}$ and $\Pi^{Z,n}$ against time t and drift estimate m, but now for $n = 10, 400, 4000$. We expect that $|V^{Z,n}(t, m) - \overline{V}^F(t, m, Q_t^{Z,n})|$ converges to zero where $\overline{V}^F$ is the conditional expectation of the value function $V^F(t, \mu_t)$ of the fully informed investor given $\mathcal{F}_t^{Z,n}$. That function is introduced in (36) and Lemma 4.5 provides a closed-form expression. The upper panels show for comparison $\overline{V}^F(t, m, Q_t^{Z,4000})$ and also the value function of the R-investor while the bottom panels also show the decision rules of the R- and F-investor. The latter is independent of time t and defines the myopic decision rule. We observe that for increasing n the value function and the optimal decision rule of the Z-investor approach $\overline{V}^F$ and the myopic decision rule, respectively. However, compared to the case of linearly growing expert's variance (see Fig. 4) the convergence is much slower. This was already observed in Sect. 7.2 for the convergence of filter processes.

We note again that for the chosen parameters we have for $n = 10$ expert opinions that $\Gamma = \Gamma^{(n)} = \frac{n}{T}\sigma_J^2 = 0.4$. This yields that for $n = 10$ the value function and decision rule for the experiment with linear growing expert's variance $\Gamma^{(n)}$ coincide with those for the experiment with constant variance Γ.

7.5 Monetary value of information

We conclude this section with some results of experiments illustrating the concepts of efficiency and price of expert opinions introduced in Sect. 2.6 for the description of the monetary value of information.

Efficency

Recall that we followed an utility indifference approach and considered the initial capital $x_0^{H/F}$ which the fully informed F–investor needs to obtain the same maximized expected utility at time T as the partially informed H-investor who started at time 0 with wealth $x_0^H > 0$. That wealth is given in Eq. (10) as the solution of the equation $\mathcal{V}_0^H(x_0^H) = \mathbb{E}\big[\mathcal{V}_0^F\big(x_0^{H/F}\big) \mid \mathcal{F}_0^H\big]$ for $H = R, Z, J$. The difference $x_0^H - x_0^{H/F} > 0$ describes the loss of information for the partially informed H-investor relative to the F-investor measured in monetary units. The ratio $\varepsilon^H = x_0^{H/F}/x_0^H \in (0, 1]$ introduced in (11) is a measure for the efficiency of the H-investor. We refer to Lemma 5.8 and 6.5 where we give explicit expressions for the above quantities for $H = R, J$ and $H = Z$, respectively.

In Fig. 6 we compare the efficiencies of the Z-investor for increasing n and parameter of the utility function $\theta = \pm 0.3$. In the left panel the expert's variance is kept constant and equal to $\Gamma = 0.4$. Then, the Z-investor asymptotically for $n \to \infty$ has full information about the hidden drift. The figure shows that the Z-investor's efficiency increases with n starting with the efficiency of the R-investor (blue) and approaching 1 which is the efficiency of the fully informed investor (green). Note that the investment horizon is $T = 1$ year such that arrival of the expert opinions once per year, month, week, day, hour or minute corresponds to $n = 1, 12, 52, 365, 8.760$ or 525.600, respectively. Comparing the efficiencies for different parameters θ it can be seen that an investor with the positive parameter $\theta = 0.3$, i.e., less risk averse than the log-utility investor ($\theta = 0$), achieves smaller efficiencies than an investor with the negative parameter $\theta = -0.3$. Note that the latter is more risk averse than the log-utility investor. Additional experiments have shown that the efficiency increases with increasing risk aversion $1 - \theta$.

In the right panel in Fig. 6 we show results of experiments in which the expert's variance Γ grows linearly with n. In that setting we expect convergence to the diffusion limit represented

by the J-investor. Here, the Z-investor's efficiency again increases with n starting with the efficiency of the R-investor (blue) but now approaches the efficiency ε^J of the J-investor (light blue) which is less than 1. As already observed for the value functions in Sect. 7.4 that convergence is much faster than the convergence to full information for fixed Γ. The diffusion limit ε^J provides quite accurate approximations for $\varepsilon^{Z,n}$ already for $n \approx 50$, i.e., weekly expert's views.

Price of the experts

In Sect. 2.6 we also used the utility indifference approach to derive a measure for the monetary value of the additional information delivered by the experts. The idea was to equate the maximum expected utilities of an R-investor who only observes returns of the H-investor for $H = Z, J$. The latter combines return observations with information from the experts. Given the R-investor is equipped with initial capital $x_0^R > 0$ one computes the initial capital $x_0^{R/H} \leq x_0^R$ for the H-investor which leads to the same maximum expected utility, we refer to Eq. (12). Then the R-investor could put aside from its initial capital x_0^R the amount $P_{Exp}^H := x_0^R - x_0^{R/H} \geq 0$ to buy the information from the expert. The remaining capital $x_0^{R/H}$ is invested in an H-optimal portfolio and providing the same expected utility of terminal wealth as the R-optimal portfolio starting with initial capital x_0^R. We refer to Lemma 5.8 and 6.5 where we give explicit expressions for P_{Exp}^H for $H = J$ and $H = Z$, respectively.

Figure 7 shows the above decomposition of the initial capital of the R-investor for $x_0^R = 1$. In the left panel we fix the expert's variance $\Gamma = 0.4$ and plot $P_{Exp}^{Z,n}$ against n. As expected

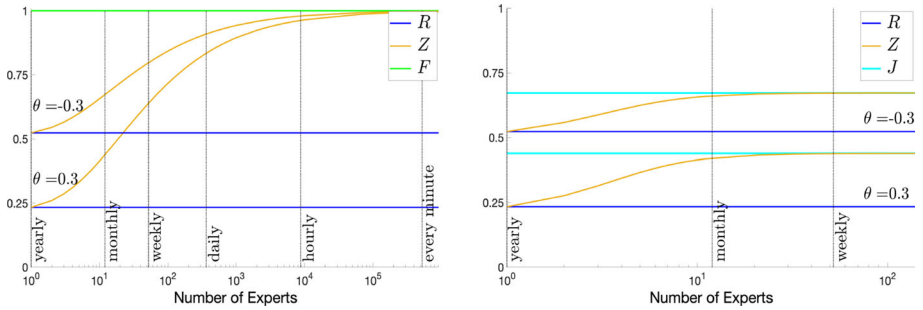

Fig. 6 Efficiency of the Z-investor for increasing n and power utility function with $\theta = \pm 0.3$: Left: Expert's variance $\Gamma = 0.4$ fixed; Right: Expert's variance $\Gamma^{(n)} = \frac{n}{T}\sigma_J^2$ linearly growing

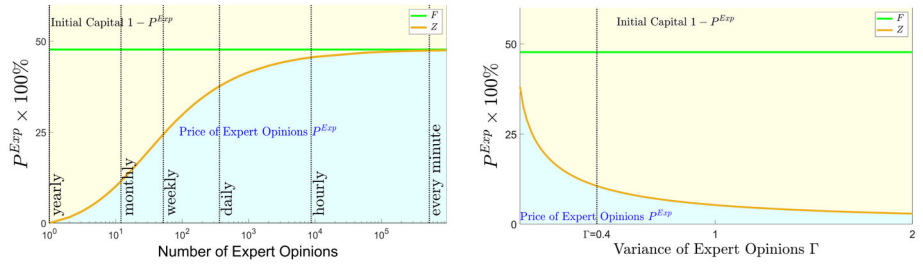

Fig. 7 Price of the experts for power utility with $\theta = -0.3$: Left: Increasing number n of expert opinions and expert's variance $\Gamma = 0.4$ fixed; Right: Increasing expert's variance Γ and $n = 10$ fixed

that price increases with the number of expert opinions but for $n \to \infty$, i.e., in the full information limit, the price reaches a saturation level which is given by $x_0^R - x_0^{R/F} = 1 - \varepsilon^R$.

The right panel shows results for fixed $n = 10$ but growing variance Γ. Then the expert's views provide less and less information about the hidden drift leading to a decreasing price P^Z_{Exp} approaching zero for $\Gamma \to \infty$, i.e., for fully non-informative expert's views. On the other hand, in the limiting case for $\Gamma \to 0$ at each of the $n = 10$ information dates the Z-investor has full information about the drift process. Note that full information is not available for all $t \in (0, T]$ but only at finitely many information dates and thus P^Z_{Exp} is for $\Gamma \to 0$ moving towards but not reaching the full information limit $1 - \varepsilon^R$.

Acknowledgements The authors thank Dorothee Westphal and Jörn Sass (TU Kaiserslautern) and Benjamin Auer (Friedrich Schiller University Jena) for valuable discussions that improved this paper. We also would like to thank two anonymous referees for helpful comments and suggestions.

Funding Open Access funding enabled and organized by Projekt DEAL.

Declarations

Conflict of interest The authors have no Conflict of interest to declare that are relevant to the content of this article. No funding was received to assist with the preparation of this manuscript.

References

Angoshtari, B. (2014). Stochastic modeling and methods for portfolio managment in cointegrated markets. PhD thesis, University of Oxford, https://ora.ox.ac.uk/objects/uuid:1ae9236c-4bf0-4d9b-a694-f08e1b8713c0.

Angoshtari, B. (2016). Portfolio on the market-neutrality of optimal pairs-trading strategies. arXiv:1608.08268 [q-fin.PM].

Battauz, A., De Donno, M., & Sbuelz, A. (2017). Reaching nirvana with a defaultable asset? *Decisions in Economics and Finance, 40*, 31–52.

Bensoussan, A. (1992). *Stochastic control of partially observable systems*. Cambridge: Cambridge University Press.

Bielecki, T. .R., & Pliska, S. .R. (1999). Risk-sensitive dynamic asset management. *Applied Mathematics and Optimization, 39*, 337–360.

Björk, T., Davis, M. H. A., & Landén, C. (2010). Optimal investment with partial information. *Mathematical Methods of Operations Research, 71*, 371–399.

Black, F., & Litterman, R. (1992). Global portfolio optimization. *Financial Analysts Journal, 48*(5), 28–43.

Brendle, S. (2006). Portfolio selection under incomplete information. *Stochastic Processes and their Applications, 116*(5), 701–723.

Brennan, M. J., Schwartz, E. S., & Lagnado, R. (1997). Strategic asset allocation. *Journal of Economic Dynamics and Control, 21*(8), 1377–1403.

Broadie, M. (1993). Computing efficient frontiers using estimated parameters. *Annals of Operations Research, 45*, 21–58.

Chen, K., & Wong, Y. (2022). Duality in optimal consumption–investment problems with alternative data. arXiv:2210.08422 [q-fin.MF].

Colaneri, K., Herzel, S., & Nicolosi, M. (2021). The value of knowing the market price of risk. *Annals of Operations Research, 299*, 101–131.
Davis, M., & Lleo, S. (2013). Black-Litterman in continous time: The case for filtering. *Quantitative Finance Letters, 1*, 30–35.
Davis, M., & Lleo, S. (2013). *Risk-sensitive investment management*. Singapore: World Scientific.
Davis, M., & Lleo, S. (2020). Debiased expert opinions in continuous-time asset allocation. *Journal of Banking & Finance, 113*, 105759.
Davis, M., & Lleo, S. (2022). Jump-diffusion risk-sensitive benchmarked asset management with traditional and alternative data. *Annals of Operations Research*. https://doi.org/10.1007/s10479-022-05130-3
Fleming, W., & Rishel, R. (1975). *Deterministic and stochastic optimal control*. New York: Springer.
Fouque, J. P., Papanicolaou, A., & Sircar, R. (2015). Filtering and portfolio optimization with stochastic unobserved drift in asset returns. *SSRN Electronic Journal, 13*(4), 935–953.
Frey, R., Gabih, A., & Wunderlich, R. (2012). Portfolio optimization under partial information with expert opinions. *International Journal of Theoretical and Applied Finance, 15*, 1250009.
Frey, R., Gabih, A., & Wunderlich, R. (2014). Portfolio optimization under partial information with expert opinions: Dynamic programming approach. *Communications on Stochastic Analysis, 8*(1), 49–71.
Gabih, A., Kondakji, H., Sass, J., & Wunderlich, R. (2014). Expert opinions and logarithmic utility maximization in a market with Gaussian drift. *Communications on Stochastic Analysis, 8*(1), 27–47.
Gabih, A., Kondakji, H., & Wunderlich, R. (2020). Asymptotic filter behavior for high-frequency expert opinions in a market with Gaussian drift. *Stochastic Models, 36*(4), 519–547.
Gabih, A., Kondakji, H., & Wunderlich, R. (2022). Well posedness of utility maximization problems under partial information in a market with Gaussian drift. submitted, arXiv:2205.08614 [q-fin.PM].
Gabih, A., & Wunderlich, R. (2023). Portfolio optimization in a market with hidden Gaussian drift and randomly arriving expert opinions: Modeling and theoretical results. arXiv:2308.02049 [q-fin.PM].
Hata, H., & Sheu, S. J. (2018). An optimal consumption and investment problem with partial information. *Advances in Applied Probability, 50*, 131–153.
Kim, T. S., & Omberg, E. (1996). Dynamic non myopic portfolio behavior. *The Review of Financial Studies, 9*(1), 141–16.
Kondakji, H. (2019). Optimal portfolios for partially informed investors in a financial market with gaussian drift and expert opinions (in German). PhD Thesis BTU Cottbus-Senftenberg. Available at https://opus4.kobv.de/opus4-btu/frontdoor/deliver/index/docId/4736/file/Kondakji_Hakam.pdf.
Korn, R., & Kraft, H. (2004). On the stability of continuous-time portfolio problems with stochastic opportunity set. *Mathematical Finance, 14*(3), 403–414.
Lakner, P. (1998). Optimal trading strategy for an investor: The case of partial information. *Stochastic Processes and their Applications, 76*, 77–97.
Lee, S., & Papanicolaou, A. (2016). Pairs trading of two assets with uncertainty in co-integration's level of mean reversion. *International Journal of Theoretical and Applied Finance, 19*(08), 1650054.
Liptser, R. S., & Shiryaev, A. N. (2001). *Statistics of random processes: General theory* (2nd ed.). New York: Springer.
Merton, R. (1971). Optimal consumption and portfolio rules in a continuous time model. *Journal of Economic Theory, 3*, 373–413.
Nagai, H. (2015). H-J-B equations of optimal consumption-investment and verification theorems. *Applied Mathematics & Optimization, 71*, 279–311.
Nagai, H., & Peng, S. (2002). Risk-sensitive dynamic portfolio optimization with partial information on infinite time horizon. *The Annals of Applied Probability, 12*, 173–195.
Pham, H. (1998). Optimal stopping of controlled jump diffusion processes: A viscosity solution approach. *Journal of Mathematical Systems, estimations, and Control, 8*(1), 1–27.
Putschögl, W., & Sass, J. (2008). Optimal consumption and investment under partial information. *Decisions in Economics and Finance, 31*, 131–170.
Rieder, U., & Bäuerle, N. (2005). Portfolio optimization with unobservable Markov-modulated drift process. *Journal of Applied Probability, 43*, 362–378.
Roduner, C. (1994). Die Riccati-Gleichung, IMRT-Report Nr. 26, ETH Zurich.
Rogers, L. C. G. (2013). *Optimal investment*. Briefs in quantitative financeBerlin-Heidelberg: Springer.
Sass, J., & Haussmann, U. G. (2004). Optimizing the terminal wealth under partial information: The drift process as a continuous time Markov chain. *Finance and Stochastics, 8*, 553–577.
Sass, J., Westphal, D., & Wunderlich, R. (2017). Expert opinions and logarithmic utility maximization for multivariate stock returns with Gaussian drift. *International Journal of Theoretical and Applied Finance, 20*(4), 1–41.
Sass, J., Westphal, D., & Wunderlich, R. (2021). Diffusion approximations for randomly arriving expert opinions in a financial market with Gaussian drift. *Journal of Applied Probability, 58*(1), 197–216.

Sass, J., Westphal, D., & Wunderlich, R. (2022). Diffusion approximations for periodically arriving expert opinions in a financial market with Gaussian drift. *Stochastic Models*. https://doi.org/10.1080/15326349.2022.2100423

Schöttle, K., Werner, R., & Zagst, R. (2010). Comparison and robustification of Bayes and Black-Litterman models. *Mathematical Methods of Operations Research, 71*, 453–475.

Xia, Y. (2001). Learning about predictability: The effects of parameter uncertainty on dynamic asset allocation. *Journal of Finance, 56*(1), 205–246.