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Reducing Obizhaeva–Wang-type trade execution problems to LQ stochastic control problems

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Abstract

We start with a stochastic control problem where the control process is of finite variation (possibly with jumps) and acts as integrator both in the state dynamics and in the target functional. Problems of such type arise in the stream of literature on optimal trade execution pioneered by Obizhaeva and Wang (J. Financ. Mark. 16:1–32, 2013) (models with finite resilience). We consider a general framework where the price impact and the resilience are stochastic processes. Both are allowed to have diffusive components. First we continuously extend the problem from processes of finite variation to progressively measurable processes. Then we reduce the extended problem to a linear–quadratic (LQ) stochastic control problem. Using the well-developed theory on LQ problems, we describe the solution to the obtained LQ one and translate it back to the solution for the (extended) initial trade execution problem. Finally, we illustrate our results by several examples. Among other things, the examples discuss the Obizhaeva–Wang model with random (terminal and moving) targets, the necessity to extend the initial trade execution problem to a reasonably large class of progressively measurable processes (even going beyond semimartingales), and the effects of diffusive components in the price impact process and/or the resilience process.

Keywords Optimal trade execution · Stochastic price impact · Stochastic resilience · Finite-variation stochastic control · Continuous extension of cost functional · Progressively measurable execution strategy · Linear–quadratic stochastic control · Backward stochastic differential equation

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1 Introduction

In the literature on optimal trade execution in illiquid financial markets, there arise stochastic control problems where the control is a process of finite variation (possibly with jumps) that acts as integrator both in the state dynamics and in the target functional. For brevity, we use the term *finite-variation (FV) stochastic control* for such problems. Notice that the class of FV stochastic control problems contains the class of singular stochastic control problems. In contrast, for control problems where the state is driven by a controlled stochastic differential equation (SDE) and the control acts as one of the arguments in that SDE and as one of the arguments in the integrand of the target functional, we use the term *standard stochastic control problems*.

In this article, we present a general solution approach to FV stochastic control problems that arise in the literature on optimal trade execution. We set up an FV stochastic control problem of the same type as in Obizhaeva and Wang [37] and its extensions like e.g. Alfonsi and Acevedo [4], Bank and Fruth [13], Fruth et al. [24, 25]. We then show how it can be transformed into a standard linear–quadratic (LQ) stochastic control problem which can be solved with the help of state-of-the-art techniques from stochastic optimal control theory.

In the introduction, we first describe the FV stochastic control problem and show-case its usage in finance, before presenting our solution approach, summarising our main contributions and embedding our paper into the literature.

1.1 Finite-variation (FV) stochastic control problem

As a starting point, we consider in this paper the following stochastic control problem. Fix $T > 0$ and let $(\Omega, \mathcal{F}_T, \mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]}, P)$ be a filtered probability space satisfying the usual conditions. Let ξ be an \mathcal{F}_T -measurable random variable and $\zeta = (\zeta_s)_{s \in [0, T]}$ a progressively measurable process both satisfying suitable integrability assumptions (see (2.3) below). Further, let $\lambda = (\lambda_s)_{s \in [0, T]}$ be a bounded progressively measurable process. Let $\gamma = (\gamma_s)_{s \in [0, T]}$ be a positive Itô process driven by some Brownian motion and $R = (R_s)_{s \in [0, T]}$ an Itô process driven by a (stochastically) correlated Brownian motion (see (2.1) and (2.2) below). Throughout the introduction, we fix $t \in [0, T]$ and $x, d \in \mathbb{R}$ and denote by $\mathcal{A}_t^{\text{fv}}(x, d)$ the set of all adapted, càdlàg, finite-variation (FV) processes $X = (X_s)_{s \in [t-, T]}$ satisfying $X_{t-} = x$, $X_T = \xi$ and appropriate integrability assumptions (see (2.5)–(2.7) below). Note that $X \in \mathcal{A}_t^{\text{fv}}(x, d)$ potentially has a jump at the initial time, i.e., X_t might differ from the initial position X_{t-} . To highlight that the full description of $X \in \mathcal{A}_t^{\text{fv}}(x, d)$ requires an assignment of the value X_{t-} and not only X_s , $s \in [t, T]$, we use the notation $(X_s)_{s \in [t-, T]}$ throughout the article. To each $X \in \mathcal{A}_t^{\text{fv}}(x, d)$, we associate a stochastic process $D^X = (D_s^X)_{s \in [t-, T]}$ satisfying

$$dD_s^X = -D_s^X dR_s + \gamma_s dX_s, \quad s \in [t, T], \quad D_{t-}^X = d. \quad (1.1)$$

We consider the FV stochastic control problem of minimising the cost functional

$$J_t^{\text{fv}}(x, d, X) = E_t \left[\int_{[t, T]} \left(D_{s-}^X + \frac{1}{2} \gamma_s \Delta X_s \right) dX_s + \int_t^T \lambda_s \gamma_s (X_s - \zeta_s)^2 ds \right] \quad (1.2)$$

over $X \in \mathcal{A}_t^{\text{fv}}(x, d)$, where $E_t[\cdot]$ is a shorthand notation for $E[\cdot | \mathcal{F}_t]$ and the integral $\int_{[t, T]} \cdots dX_s$ includes a potential jump of X at the initial time t .

1.2 Financial interpretation

Stochastic control problems with a cost functional of the form (1.2) or a special case thereof play a central role in the literature on optimal trade execution problems (see the literature discussion in Sect. 1.5 below). Consider an institutional investor who holds immediately prior to time $t \in [0, T]$ a position $x \in \mathbb{R}$ ($x > 0$ meaning a long position of x shares and $x < 0$ a short position of $-x$ shares) of a certain financial asset. The investor trades the asset during the period $[t, T]$ in such a way that at each time $s \in [t-, T]$, the position is given by the value X_s of the adapted, càdlàg, FV process $X = (X_s)_{s \in [t-, T]}$ (satisfying $X_{t-} = x$). More precisely, X_{s-} represents the position immediately prior to the trade at time s , while X_s is the position immediately after that trade.

The investor's goal is to reach the target position

$$X_T = \xi$$

during the course of the trading period $[t, T]$. Note that we allow ξ to be random to incorporate the possibility that the target position is not known at the beginning of trading, but only revealed at the terminal time T . Such situations may for example be faced by airline companies buying on forward markets the kerosene they need in T months. Their precise demand for kerosene at that future time depends on several factors, such as ticket sales and flight schedules, that are not known today, but are only gradually learned.

We assume that the market the investor trades in is illiquid, implying that the investor's trades impact the asset price. To model this effect, we assume (as is typically done in the literature on optimal trade execution) an additive impact on the price. This means that the realised price at which the investor trades at time $r \in [t, T]$ consists of an unaffected price S_r^0 plus a deviation D_r^X that is caused by the investor's trades during $[t, r]$.

We assume that the unaffected price process $S^0 = (S_r^0)_{r \in [0, T]}$ is a càdlàg martingale satisfying appropriate integrability conditions. Then integration by parts and the martingale property of S^0 ensure that expected trading costs due to S^0 are given by

$$E_t \left[\int_{[t, T]} S_r^0 dX_r \right] = E_t[\xi S_T^0] - x S_t^0.$$

Thus these costs do not depend on the investor's trading strategy X and are therefore neglected in the sequel (we refer to Ackermann et al. [1, Remark 2.2] for a more detailed discussion in the case $\xi = 0$).

The deviation process D^X associated to X is given by (1.1). Informally speaking, we see from (1.1) that a trade of size dX_s at time $s \in [t, T]$ impacts D^X by $\gamma_s dX_s$. So the factor γ_s determines how strongly the price reacts to trades, and the process γ is therefore called the *price impact process*. In particular, because γ is nonnegative, a buy trade $dX_s > 0$ leads to higher and a sell trade $dX_s < 0$ to lower prices.

The second component $-D_s^X dR_s$ in the dynamics (1.1) describes the behaviour of D^X when the investor is not trading. Typically, it is assumed that R is an increasing process such that in the absence of trades, D^X is reverting to 0 with relative rate dR_s . Therefore R is called the *resilience process*. We refer to Ackermann et al. [3] for a discussion of the effects of “negative” resilience, where R might also be decreasing. We highlight that in the present paper, we allow R to have a diffusive part.

In summary, we note that the deviation prior to a trade of the investor at a time $s \in [t, T]$ is given by D_{s-}^X , while it is $D_s^X = D_{s-}^X + \gamma_s \Delta X_s$ afterwards. We take the mean $D_{s-}^X + \frac{1}{2} \gamma_s \Delta X_s$ of these two values as the realised price per unit so that the investor’s overall trading costs due to D^X amount to $\int_{[t, T]} (D_{s-}^X + \frac{1}{2} \gamma_s \Delta X_s) dX_s$. This describes the first integral on the right-hand side of (1.2).

Assuming that λ is nonnegative, the second integral $\int_t^T \lambda_s \gamma_s (X_s - \zeta_s)^2 ds$ on the right-hand side of (1.2) can be understood as a risk term that penalises any deviation of the position X from the moving target ζ in a quadratic way. The parametrisation $\lambda_s \gamma_s$, $s \in [0, T]$, for the weight is chosen out of mathematical convenience since it makes some of the following assumptions and results shorter to state. Likewise, one can use $\tilde{\lambda}_s$, $s \in [0, T]$, as a weight and replace λ by $\tilde{\lambda}/\gamma$ in the subsequent assumptions and results. A possible and natural choice for the moving target ζ would be $\zeta_s = E_s[\xi]$, $s \in [0, T]$, so that the risk term ensures that any optimal strategy X does not deviate too much from the (expected) target position during the trading period.

1.3 Solution approach

The overarching goal of this paper is to show that the FV stochastic control problem (1.2) is equivalent to a standard LQ stochastic control problem (see Corollaries 3.3 and 3.4 below). The derivation of this result is based on the following insights.

The first observation is that in general, the functional (1.2) does not admit a minimiser in $\mathcal{A}_t^{\text{fv}}(x, d)$ (see Sect. 5.3 below for a specific example). In Ackermann et al. [1], the functional (1.2) was extended to a set of càdlàg semimartingales X , and it was shown that its minimum is attained in this set of semimartingales if and only if a certain process derived from the solution of an associated backward stochastic differential equation (BSDE) can be represented by a càdlàg semimartingale (see [1, Theorem 2.4]).

In the present work, we go one step further and extend the functional (1.2) to the set $\mathcal{A}_t^{\text{pm}}(x, d)$ of progressively measurable processes $X = (X_s)_{s \in [t, T]}$ satisfying appropriate integrability conditions (see (2.5) below) and the boundary conditions $X_{t-} = x$ and $X_T = \xi$. To do so, we first derive alternative representations of the first integral inside the expectation in (1.2) and the deviation in (1.1) that do not involve $X \in \mathcal{A}_t^{\text{fv}}(x, d)$ as an integrator (see Proposition 2.3). It follows that the resulting alternative representation of J_t^{fv} (see Proposition 2.4) is well defined not only on $\mathcal{A}_t^{\text{fv}}(x, d)$, but even on $\mathcal{A}_t^{\text{pm}}(x, d)$, and we denote this extended functional by J_t^{pm} .

(see Sect. 2.3). We next introduce a metric on $\mathcal{A}_t^{\text{pm}}(x, d)$ and prove that J_t^{pm} is the unique continuous extension of J_t^{fv} from $\mathcal{A}_t^{\text{fv}}(x, d)$ to $\mathcal{A}_t^{\text{pm}}(x, d)$ (see Theorem 2.8). In particular, the infima of J_t^{fv} over $\mathcal{A}_t^{\text{fv}}(x, d)$ and of J_t^{pm} over $\mathcal{A}_t^{\text{pm}}(x, d)$ coincide.

Next, for a given $X \in \mathcal{A}_t^{\text{pm}}(x, d)$, we identify the process

$$\overline{H}_s^X = \gamma_s^{-\frac{1}{2}} D_s^X - \gamma_s^{\frac{1}{2}} X_s, \quad s \in [t, T],$$

as a useful tool in our analysis. Despite X and D^X having discontinuous paths in general, the process \overline{H}^X , which we call the *scaled hidden deviation process*, is always continuous. Moreover, we show that \overline{H}^X can be expressed in feedback form as an Itô process with coefficients that are linear in $\gamma^{-\frac{1}{2}} D^X$ and \overline{H}^X (see Lemma 2.7). Subsequently, we reinterpret the process $\gamma^{-\frac{1}{2}} D^X$ as a control process u and \overline{H}^X as the associated state process. Since the cost functional J_t^{pm} is quadratic in \overline{H}^X and $u = \gamma^{-\frac{1}{2}} D^X$, we arrive at a standard LQ stochastic control problem (see (3.1) and (3.2)) whose minimal cost coincides with the infimum of J_t^{pm} over $\mathcal{A}_t^{\text{pm}}(x, d)$ (see Corollary 3.3). Importantly, there is also a one-to-one correspondence between square-integrable controls u for this standard problem and strategies $X \in \mathcal{A}_t^{\text{pm}}(x, d)$, which allows to recover the minimiser $X^* \in \mathcal{A}_t^{\text{pm}}(x, d)$ of J_t^{pm} from a minimiser u^* of the standard problem and vice versa (see Corollary 3.4).

We then solve the LQ stochastic control problem in (3.1) and (3.2) using techniques provided in the literature on stochastic optimal control theory. More precisely, we apply results from Kohlmann and Tang [34] (we indicate in Remark 4.1 how we could alternatively use results from Sun et al. [40]) to provide conditions that guarantee that an optimal control u^* exists (and is unique). This optimal control u^* in the LQ problem is characterised by two BSDEs; one is a quadratic BSDE of Riccati type, the other is linear, however, with unbounded coefficients (see Theorem 4.4). In Corollary 4.5, we translate everything back and obtain a unique optimal execution strategy in the class of progressively measurable processes in closed form (in terms of the solutions to the mentioned BSDEs).

1.4 Summary of our contributions

Our contributions are as follows:

(a) The Obizhaeva–Wang-type FV stochastic control problem (1.1), (1.2) is continuously extended to the set $\mathcal{A}_t^{\text{pm}}(x, d)$ of appropriate progressively measurable processes X .

(b) The problem (1.1), (1.2) is rather general. In particular, it includes the following features:

- Presence of random terminal and moving targets ξ and (ζ_s) .
- The price impact is a positive Itô process (γ_s) .
- The resilience is an Itô process (R_s) acting as an integrator in (1.1) (also see Remark 1.1 below).

(c) Via introducing the mentioned *scaled hidden deviation process* \overline{H}^X and reinterpreting the process $\gamma^{-\frac{1}{2}} D^X$ as a control in an (a priori different) stochastic control problem, the problem extended to $\mathcal{A}_t^{\text{pm}}(x, d)$ is reduced to an explicitly solvable LQ stochastic control problem. Thus a unique optimal execution strategy in $\mathcal{A}_t^{\text{pm}}(x, d)$ is obtained in closed form (in terms of the solutions to two BSDEs).

Remark 1.1 It is worth noting that in our current parametrisation, only processes R with dynamics $dR_s = \rho_s ds$ without a diffusive component were considered up to now in the literature on optimal trade execution in Obizhaeva–Wang-type models (and actually the term “resilience” was previously used for ρ). Moreover, in most papers, ρ is assumed to be positive, that is, only the case of an increasing R was studied previously.

1.5 Literature discussion

FV stochastic control problems arise in the group of literature on optimal trade execution in limit order books with finite resilience. The pioneering work by Obizhaeva and Wang [37] (first posted in 2005 on SSRN) models the price impact via a block-shaped limit order book, where the impact decays exponentially at a constant rate. This embeds into our model via a price impact process γ that is a positive constant and a resilience process R given by $R_s = \rho s$ with some positive constant $\rho > 0$. Alfonsi et al. [5] study constrained portfolio liquidation in the Obizhaeva–Wang model. Subsequent works within this group of literature either extend this framework in different directions or suggest alternative frameworks with similar features.

There is a subgroup of models which include more general limit order book shapes; see Alfonsi et al. [6], Alfonsi and Schied [7], Predoiu et al. [38]. Models in another subgroup extend the exponential decay of the price impact to general decay kernels; see Alfonsi et al. [8], Gatheral et al. [27]. Models with multiplicative price impact are analysed in Becherer et al. [17, 18]. We mention that in [18], the (multiplicative) deviation is of Ornstein–Uhlenbeck type and incorporates a diffusion term (but this is different from our diffusion term that results from a diffusive part in the resilience R). Superreplication and optimal investment in a block-shaped limit order book model with exponential resilience is discussed in Bank and Dolinsky [11, 12] and in Bank and Voß [16].

The present paper belongs to the subgroup of the literature studying time-dependent (possibly stochastic) price impact γ or resilience R in generalised Obizhaeva–Wang models. In this regard, we mention the works by Alfonsi and Acevedo [4], Bank and Fruth [13] and Fruth et al. [24], where deterministically varying price impact and resilience are considered. Fruth et al. [25] allow stochastically varying price impact (resilience is still deterministic) and study the arising optimisation problem over monotone strategies. Optimal strategies in a discrete-time model with stochastically varying resilience and constant price impact are derived in Siu et al. [39]. In Ackermann et al. [1, 3, 2], both price impact and resilience are stochastic.

We now describe the differences to our present paper in more detail. In [2], optimal execution is studied in discrete time via dynamic programming. In [1], the framework is closest to the one in this paper. Essentially, our current framework is the framework from [1] extended by a risk term with some moving target (ζ_s) , a possibly non-zero (random) terminal target ξ and a larger class of resilience processes (in [1], as in many previous papers, R is assumed to have the dynamics $dR_s = \rho_s ds$ and ρ is called resilience). In [3], the framework is similar to [1], while the aim is to study qualitative effects of “negative” resilience (in the sense that $\rho_s \leq 0$ with ρ as in the previous sentence).

Now, to compare the approach in the present paper with that in [1], we first recall that in [1], the FV stochastic control problem of the type (1.1), (1.2) is extended to allow càdlàg semimartingale trading strategies X , and the resulting optimal execution problem over semimartingales is studied. The approach in [1] is based on (1.1), (1.2) (extended with some additional terms), but this does not work beyond semimartingales as X acts as integrator there. In contrast, our continuous extension needs to employ essentially different ideas since we want to consider the set $\mathcal{A}_t^{\text{pm}}(x, d)$ of progressively measurable strategies (in particular, beyond semimartingales). This extension is indeed necessary to get an optimiser (see the discussion at the end of Sect. 5.3).

Especially with regard to our extension result, we now mention several papers where in different models with finite resilience, trading strategies are not restricted to be of finite variation. The first instance known to us is Lorenz and Schied [35], who discuss the dependence of optimal trade execution strategies on a drift in the unaffected price. In order to react to non-martingale trends, they allow càdlàg semimartingale trading strategies. Gârleanu and Pedersen [26, Sect. 1.3] allow strategies of infinite variation in an infinite-horizon portfolio optimisation problem under market frictions. Becherer et al. [19] prove a continuous extension result for gains of a large investor in the Skorokhod J_1 - and M_1 -topologies in the class of predictable strategies with càdlàg paths. As discussed in the previous paragraph in more detail, the strategies in [1] are càdlàg semimartingales. In Horst and Kivman [29], càdlàg semimartingale strategies emerge in the limiting case of a vanishing instantaneous impact parameter, where the initial modelling framework is inspired by Graewe and Horst [28] and Horst and Xia [31].

To complement the preceding discussion from another perspective, we mention Carmona and Webster [22] who examine high-frequency trading in limit order books in general (not necessarily related with optimal trade execution). It is very interesting that one of their conclusions is strong empirical evidence for the infinite-variation nature of trading strategies of high-frequency traders.

Finally, let us mention that in the context of trade execution problems, risk terms with zero moving target have been included e.g. in Ankirchner et al. [9], Ankirchner and Kruse [10] and Graewe and Horst [28]. Inequality terminal constraints have been considered in Dolinsky et al. [23], and risk terms with general terminal and moving targets appear in the models of e.g. Bank et al. [14], Bank and Voß [15], Horst and Naujokat [30] and Naujokat and Westray [36]. In particular, [10, 15, 23] consider random terminal targets ξ within trade execution models where position paths are required to be absolutely continuous functions of time. This restriction of the set of position paths entails technical difficulties that make these problems challenging to analyse. In particular, existence of admissible paths that satisfy the terminal constraint is far from obvious and can in general only be assured under further conditions on ξ . Since position paths in our model are allowed to jump at the terminal time, we do not face these challenges in our framework.

1.6 Structure of the present article

The paper is structured as follows. Section 2 is devoted to the continuous extension of our initial trade execution problem to the class of progressively measurable strate-

gies. Section 3 reduces the problem for the progressively measurable strategies to a standard LQ stochastic control problem. In Sect. 4, we present the solution to the obtained LQ problem and translate it back to the solution for the trade execution problem (extended to progressively measurable strategies). In Sect. 5, we illustrate our results with several examples. Finally, Sect. 6 contains the proofs together with some auxiliary results necessary for them.

2 From finite-variation to progressively measurable execution strategies

In this section, we first set up the FV stochastic control problem (see Sect. 2.1). In Sect. 2.2, we then derive alternative representations of the cost functional and the deviation process which do not require the strategies to be of finite variation. We use these results in Sect. 2.3 to extend the cost functional to progressively measurable strategies. In Sect. 2.5, we show that this is the unique continuous extension. Section 2.4 introduces the hidden deviation process as a key tool for the proofs of Sect. 2.5. All proofs of this section are deferred to Sect. 6.

2.1 The FV stochastic control problem

Fix $T > 0$ and $m \geq 2$, and let $(\Omega, \mathcal{F}_T, \mathbb{F} = (\mathcal{F}_s)_{s \in [0, T]}, P)$ be a filtered probability space satisfying the usual conditions and supporting an m -dimensional Brownian motion $W = (W^1, \dots, W^m)$ with respect to the filtration \mathbb{F} .

We first fix some notation. For $t \in [0, T]$, conditional expectations with respect to \mathcal{F}_t are denoted by $E_t[\cdot]$. For $t \in [0, T]$ and a càdlàg process $X = (X_s)_{s \in [t-, T]}$, the jump at time $s \in [t, T]$ is denoted by $\Delta X_s = X_s - X_{s-}$. We follow the convention that for $t \in [0, T]$, $r \in [t, T]$ and a càdlàg semimartingale $L = (L_s)_{s \in [t-, T]}$, jumps of the càdlàg integrator L at time t contribute to integrals of the form $\int_{[t, r]} \cdots dL_s$. In contrast, we write $\int_{(t, r]} \cdots dL_s$ when we do not include jumps of L at time t in the integral. The notation $\int_t^T \cdots dL_s$ is sometimes used for continuous integrators L . For $n \in \mathbb{N}$ and $y \in \mathbb{R}^n$, let $|y| = (\sum_{j=1}^n y_j^2)^{\frac{1}{2}}$. For every $t \in [0, T]$, let $L^1(\Omega, \mathcal{F}_t, P)$ be the space of all (equivalence classes of) real-valued \mathcal{F}_t -measurable random variables Y such that $\|Y\|_{L^1} = E[|Y|] < \infty$. For every $t \in [0, T]$, let

$$\mathcal{L}_t^2 = \mathcal{L}^2(\Omega \times [t, T], \text{Prog}(\Omega \times [t, T]), dP \times ds|_{[t, T]})$$

denote the Hilbert space of all (equivalence classes of) real-valued progressively measurable processes $u = (u_s)_{s \in [t, T]}$ such that $\|u\|_{\mathcal{L}_t^2} = (E[\int_t^T u_s^2 ds])^{\frac{1}{2}} < \infty$.

The control problem we are about to set up requires as input the real-valued, \mathcal{F}_T -measurable random variable ξ and the real-valued, progressively measurable processes

$$\begin{aligned} \mu &= (\mu_s)_{s \in [0, T]}, & \sigma &= (\sigma_s)_{s \in [0, T]}, & \rho &= (\rho_s)_{s \in [0, T]}, & \eta &= (\eta_s)_{s \in [0, T]}, \\ \bar{F} &= (\bar{F}_s)_{s \in [0, T]}, & \zeta &= (\zeta_s)_{s \in [0, T]}, & \lambda &= (\lambda_s)_{s \in [0, T]}. \end{aligned}$$

We suppose that the processes μ , σ , ρ , η and λ are $(dP \times ds|_{[0,T]})$ -a.e. bounded. Moreover, we assume that \bar{r} is $[-1, 1]$ -valued. We define W^R by

$$dW_s^R = \bar{r}_s dW_s^1 + \sqrt{1 - |\bar{r}_s|^2} dW_s^2, \quad s \in [0, T], \quad W_0^R = 0$$

and refer to \bar{r} as the *correlation process*. The processes ρ and η give rise to the continuous semimartingale R with

$$dR_s = \rho_s ds + \eta_s dW_s^R, \quad s \in [0, T], \quad R_0 = 0, \quad (2.1)$$

which is called the *resilience process*. We use the processes μ and σ to define the positive continuous semimartingale γ by

$$d\gamma_s = \gamma_s(\mu_s ds + \sigma_s dW_s^1), \quad s \in [0, T], \quad (2.2)$$

with deterministic initial value $\gamma_0 > 0$. We refer to γ as the *price impact process*. Finally, we assume that ξ and ζ satisfy the integrability conditions

$$E[\gamma_T \xi^2] < \infty, \quad E\left[\int_0^T \gamma_s \zeta_s^2 ds\right] < \infty. \quad (2.3)$$

Remark 2.1 Note that the components W^3, \dots, W^m of the Brownian motion are not needed in the dynamics (2.1) and (2.2). We introduce these components already here because in Sect. 4, in order to apply the results from the literature on LQ stochastic control, we restrict the present setting by assuming that the filtration \mathbb{F} is generated by (W^1, \dots, W^m) . The components W^3, \dots, W^m therefore serve as further sources of randomness on which the model inputs may depend.

We next introduce the FV strategies we consider in the sequel. Given $t \in [0, T]$ and $d \in \mathbb{R}$, we associate to an adapted, càdlàg, FV process $X = (X_s)_{s \in [t-, T]}$ a process $D^X = (D_s^X)_{s \in [t-, T]}$ defined by

$$dD_s^X = -D_s^X dR_s + \gamma_s dX_s, \quad s \in [t, T], \quad D_{t-}^X = d. \quad (2.4)$$

If there is no risk of confusion, we sometimes simply write D instead of D^X in the sequel. For $t \in [0, T]$, $x, d \in \mathbb{R}$, we denote by $\mathcal{A}_t^{\text{fv}}(x, d)$ the set of all adapted, càdlàg, FV processes $X = (X_s)_{s \in [t-, T]}$ satisfying $X_{t-} = x$, $X_T = \xi$ and

$$E\left[\int_t^T \gamma_s^{-1} (D_s^X)^2 ds\right] < \infty, \quad (2.5)$$

$$E\left[\left(\int_t^T (D_s^X)^4 \gamma_s^{-2} \eta_s^2 ds\right)^{\frac{1}{2}}\right] < \infty, \quad (2.6)$$

$$E\left[\left(\int_t^T (D_s^X)^4 \gamma_s^{-2} \sigma_s^2 ds\right)^{\frac{1}{2}}\right] < \infty. \quad (2.7)$$

Any element $X \in \mathcal{A}_t^{\text{fv}}(x, d)$ is called an *FV execution strategy*. The process $D = D^X$ defined via (2.4) is called the associated *deviation process*.

For $t \in [0, T]$, $x, d \in \mathbb{R}$, $X \in \mathcal{A}_t^{\text{fv}}(x, d)$ and associated D^X , the cost functional J_t^{fv} is given by

$$J_t^{\text{fv}}(x, d, X) = E_t \left[\int_{[t, T]} D_s^X dX_s + \frac{1}{2} \int_{[t, T]} \gamma_s \Delta X_s dX_s + \int_t^T \lambda_s \gamma_s (X_s - \zeta_s)^2 ds \right] \quad (2.8)$$

(see the proofs of Proposition 2.4 and Lemma 2.7 for well-definedness). The FV stochastic control problem we consider consists of minimising the cost functional J_t^{fv} over $X \in \mathcal{A}_t^{\text{fv}}(x, d)$.

2.2 Alternative representations for the cost functional and the deviation process

For $t \in [0, T]$, we introduce an auxiliary process $v = (v_s)_{s \in [t, T]}$. It is defined to be the solution of

$$dv_s = v_s d(R_s + [R]_s), \quad s \in [t, T], \quad v_t = 1. \quad (2.9)$$

Observe that the inverse is given by

$$dv_s^{-1} = -v_s^{-1} dR_s, \quad s \in [t, T], \quad v_t^{-1} = 1. \quad (2.10)$$

Remark 2.2 Let $t \in [0, T]$ and $d \in \mathbb{R}$. With the definition of v in (2.9), for all adapted, càdlàg, FV processes $X = (X_s)_{s \in [t-, T]}$, the solution $D^X = (D_s^X)_{s \in [t-, T]}$ of the linear SDE (2.4) can be written as

$$D_s^X = v_s^{-1} \left(d + \int_{[t, s]} v_r \gamma_r dX_r \right), \quad s \in [t, T].$$

Proposition 2.3 Let $t \in [0, T]$ and $x, d \in \mathbb{R}$. Suppose X is an adapted, càdlàg, FV process with $X_{t-} = x$ and associated process D^X defined by (2.4). Then we have

$$\begin{aligned} & \int_{[t, T]} D_s^X dX_s + \frac{1}{2} \int_{[t, T]} \gamma_s \Delta X_s dX_s \\ &= \frac{1}{2} \left(\gamma_T^{-1} (D_T^X)^2 - \gamma_t^{-1} d^2 - \int_t^T (D_s^X)^2 v_s^2 d(v_s^{-2} \gamma_s^{-1}) \right) \end{aligned} \quad (2.11)$$

and

$$D_r^X = \gamma_r X_r + v_r^{-1} \left(d - \gamma_t x - \int_t^r X_s d(v_s \gamma_s) \right), \quad r \in [t, T]. \quad (2.12)$$

As a consequence of Proposition 2.3 and relying on (2.5)–(2.7), we can rewrite the cost functional J_t^{fv} as follows. To shorten notation, we introduce the stochastic process κ defined by

$$\kappa_s = \frac{1}{2} (2\rho_s + \mu_s - \sigma_s^2 - \eta_s^2 - 2\sigma_s \eta_s \bar{r}_s), \quad s \in [0, T]. \quad (2.13)$$

Proposition 2.4 *Let $t \in [0, T]$ and $x, d \in \mathbb{R}$. Suppose that $X \in \mathcal{A}_t^{\text{fv}}(x, d)$ with associated deviation process D^X defined by (2.4). Then $J_t^{\text{fv}}(x, d, X)$ in (2.8) admits the representation*

$$\begin{aligned} J_t^{\text{fv}}(x, d, X) &= \frac{1}{2} E_t \left[\gamma_T^{-1} (D_T^X)^2 + \int_t^T (D_s^X)^2 \gamma_s^{-1} 2\kappa_s ds + \int_t^T 2\lambda_s \gamma_s (X_s - \zeta_s)^2 ds \right] \\ &\quad - \frac{d^2}{2\gamma_t} \quad a.s. \end{aligned} \quad (2.14)$$

Remark 2.5 Analogues of Proposition 2.4 can be found in the literature in other related settings; see e.g. Fruth et al. [24, Lemmas 7.4 and 8.6] and Fruth et al. [25, proof of Lemma 5.3 in Appendix B]. A small technical point which might be worth noting is that we present a somewhat different proof below. The idea in [24, 25] is to derive an analogue of (2.11) by applying the substitution $dX_s = \gamma_s^{-1}(dD_s^X + D_s^X dR_s)$ and then to compute the expectation. Exactly the same idea would also work in our present setting, but it would result in more complicated calculations, and moreover, the right-hand side of (2.11) would then look rather different (but this would of course be an equivalent representation). The reason is that the process R , hence D^X , can have non-vanishing quadratic variation. Here we essentially express everything not through D^X , but rather through νD^X , which has finite variation by Remark 2.2 (as X has finite variation here). This allows reducing calculations and provides a somewhat more compact form of (2.11).

2.3 Progressively measurable execution strategies

We point out that the right-hand side of (2.14) is also well defined for progressively measurable processes X satisfying an appropriate integrability condition and with associated deviation D^X defined by (2.12) for which one assumes (2.5). This motivates the following extension of the setting from Sect. 2.1.

For $t \in [0, T]$, $x, d \in \mathbb{R}$ and a progressively measurable $X = (X_s)_{s \in [t-, T]}$ with $\int_t^T X_s^2 ds < \infty$ a.s. and $X_{t-} = x$, we define the process $D^X = (D_s^X)_{s \in [t-, T]}$ by

$$D_s^X = \gamma_s X_s + \nu_s^{-1} \left(d - \gamma_t x - \int_t^s X_r d(\nu_r \gamma_r) \right), \quad s \in [t, T], \quad D_{t-}^X = d \quad (2.15)$$

(recall ν from (2.9)). Notice that the condition $\int_t^T X_s^2 ds < \infty$ a.s. ensures that the stochastic integral in (2.15) is well defined. Again, we sometimes write D instead of D^X . Further, for $t \in [0, T]$, $x, d \in \mathbb{R}$, let $\mathcal{A}_t^{\text{pm}}(x, d)$ be the set of (equivalence classes of) progressively measurable processes $X = (X_s)_{s \in [t-, T]}$ with $X_{t-} = x$ and $X_T = \xi$ that satisfy $\int_t^T X_s^2 ds < \infty$ a.s. and such that condition (2.5) holds true for D^X defined by (2.15). To be precise, we stress that the equivalence classes for

$\mathcal{A}_t^{\text{pm}}(x, d)$ are understood with respect to the equivalence relation

$$\begin{aligned} X^{(1)} \sim X^{(2)} \quad \text{means} \quad & X^{(1)} = X^{(2)} \quad (dP \times ds)\text{-a.e. on } \Omega \times [t, T], \\ & X_{t-}^{(1)} = X_{t-}^{(2)} (=x) \text{ and } X_T^{(1)} = X_T^{(2)} (= \xi). \end{aligned} \quad (2.16)$$

Any element $X \in \mathcal{A}_t^{\text{pm}}(x, d)$ is called a *progressively measurable execution strategy*. Again the process $D = D^X$, now defined via (2.15), is called the associated *deviation process*. Clearly, we have $\mathcal{A}_t^{\text{fv}}(x, d) \subseteq \mathcal{A}_t^{\text{pm}}(x, d)$.

Given $t \in [0, T]$, $x, d \in \mathbb{R}$ and $X \in \mathcal{A}_t^{\text{pm}}(x, d)$ with associated D^X from (2.15), we define the cost functional J_t^{pm} by

$$\begin{aligned} J_t^{\text{pm}}(x, d, X) &= \frac{1}{2} E_t \left[\gamma_T^{-1} (D_T^X)^2 + \int_t^T (D_s^X)^2 \gamma_s^{-1} 2\kappa_s ds + \int_t^T 2\lambda_s \gamma_s (X_s - \zeta_s)^2 ds \right] \\ &\quad - \frac{d^2}{2\gamma_t}. \end{aligned} \quad (2.17)$$

Observe that we have the following corollary of Propositions 2.3 and 2.4.

Corollary 2.6 *Let $t \in [0, T]$, $x, d \in \mathbb{R}$ and $X \in \mathcal{A}_t^{\text{fv}}(x, d)$ with associated deviation process D^X from (2.4). Then X is in $\mathcal{A}_t^{\text{pm}}(x, d)$, D^X satisfies (2.15), and $J_t^{\text{fv}}(x, d, X) = J_t^{\text{pm}}(x, d, X)$.*

2.4 The hidden deviation process

For $t \in [0, T]$, $x, d \in \mathbb{R}$ and $X \in \mathcal{A}_t^{\text{pm}}(x, d)$ with associated deviation process D^X , we define H^X by $H_s^X = D_s^X - \gamma_s X_s$, $s \in [t, T]$. Observe that if the investor follows an FV execution strategy $X \in \mathcal{A}_t^{\text{fv}}(x, d)$ until time $s \in [t, T]$ and then decides to sell X_s units of the asset ($X_s < 0$ means buying) at time s , then by (2.4), the resulting deviation at time s equals $D_s^X - \gamma_s X_s$. The value of H_s^X hence represents the hypothetical deviation if the investor decides to close the position at time $s \in [t, T]$. We therefore call H^X the *hidden deviation process*.

Despite $X \in \mathcal{A}_t^{\text{pm}}(x, d)$ and D^X in general being discontinuous, the hidden deviation process H^X is always continuous. This can be seen from (2.15) and the fact that R (hence also ν) and γ are continuous. In the case of an FV execution strategy $X \in \mathcal{A}_t^{\text{fv}}(x, d)$, we have $dH_s^X = -D_s dR_s - X_s d\gamma_s$, $s \in [t, T]$. In particular, the infinitesimal change of the hidden deviation is driven by the changes of the resilience process and the price impact process.

For $t \in [0, T]$, $x, d \in \mathbb{R}$ and $X \in \mathcal{A}_t^{\text{pm}}(x, d)$, we furthermore introduce the *scaled hidden deviation process* \overline{H}^X defined by

$$\overline{H}_s^X = \gamma_s^{-\frac{1}{2}} H_s^X = \gamma_s^{-\frac{1}{2}} D_s^X - \gamma_s^{\frac{1}{2}} X_s, \quad s \in [t, T]. \quad (2.18)$$

From a mathematical viewpoint, the scaled hidden deviation plays an extremely important role in what follows. It is therefore instructive to see in what kind of units it

is measured. The unit of X is number (of shares), while both D^X and γ are measured in \$. Thus the scaled hidden deviation \bar{H}^X is measured in $\sqrt{\$}$.

Also for H^X and \bar{H}^X , we sometimes simply write H and \bar{H} , respectively. Note that due to (2.15), we have

$$\bar{H}_s^X = \gamma_s^{-\frac{1}{2}} \nu_s^{-1} \left(d - \gamma_t x - \int_t^s X_r d(\nu_r \gamma_r) \right), \quad s \in [t, T].$$

We next show that the scaled hidden deviation process \bar{H}^X satisfies a linear SDE and an L^2 -bound. Moreover, we derive a representation of J^{pm} in terms of \bar{H}^X .

Lemma 2.7 *Let $t \in [0, T]$, $x, d \in \mathbb{R}$ and $X \in \mathcal{A}_t^{\text{pm}}(x, d)$. Then we have*

$$\begin{aligned} d\bar{H}_s^X &= \left(\frac{1}{2} \left(\mu_s - \frac{1}{4} \sigma_s^2 \right) \bar{H}_s^X - \frac{1}{2} (2(\rho_s + \mu_s) - \sigma_s^2 - \sigma_s \eta_s \bar{r}_s) \gamma_s^{-\frac{1}{2}} D_s^X \right) ds \\ &\quad + \left(\frac{1}{2} \sigma_s \bar{H}_s^X - (\sigma_s + \eta_s \bar{r}_s) \gamma_s^{-\frac{1}{2}} D_s^X \right) dW_s^1 \\ &\quad - \eta_s \sqrt{1 - |\bar{r}_s|^2} \gamma_s^{-\frac{1}{2}} D_s^X dW_s^2, \quad s \in [t, T], \\ \bar{H}_t^X &= \frac{d}{\sqrt{\gamma_t}} - \sqrt{\gamma_t} x. \end{aligned} \quad (2.19)$$

Moreover, $E[\sup_{s \in [t, T]} (\bar{H}_s^X)^2] < \infty$ and

$$\begin{aligned} J_t^{\text{pm}}(x, d, X) &= \frac{1}{2} E_t \left[(\bar{H}_T^X + \sqrt{\gamma_T} \xi)^2 + \int_t^T 2(\kappa_s + \lambda_s) \gamma_s^{-1} (D_s^X)^2 ds \right] - \frac{d^2}{2\gamma_t} \\ &\quad + E_t \left[\int_t^T (\lambda_s (\bar{H}_s^X + \sqrt{\gamma_s} \zeta_s)^2 - 2\lambda_s (\bar{H}_s^X + \sqrt{\gamma_s} \zeta_s) \gamma_s^{-\frac{1}{2}} D_s^X) ds \right]. \end{aligned} \quad (2.20)$$

2.5 Continuous extension of the cost functional

Corollary 2.6 states that for FV execution strategies, the cost functionals J_t^{fv} and J_t^{pm} are the same. In this subsection, we show that J_t^{pm} can be considered as an extension of J_t^{fv} to progressively measurable strategies in the following way: We introduce a metric \mathbf{d} on $\mathcal{A}_t^{\text{pm}}(x, d)$ and show in Theorem 2.8 that (i) $J_t^{\text{pm}}(x, d, X)$ is continuous in the strategy $X \in \mathcal{A}_t^{\text{pm}}(x, d)$, (ii) $\mathcal{A}_t^{\text{fv}}(x, d)$ is dense in $\mathcal{A}_t^{\text{pm}}(x, d)$ and (iii) the metric space $(\mathcal{A}_t^{\text{pm}}(x, d), \mathbf{d})$ is complete.

The first and second parts of Theorem 2.8 mean that $J_t^{\text{pm}}(x, d, \cdot)$ under the metric \mathbf{d} is a unique continuous extension of $J_t^{\text{fv}}(x, d, \cdot)$ from $\mathcal{A}_t^{\text{fv}}(x, d)$ to $\mathcal{A}_t^{\text{pm}}(x, d)$. The third part of Theorem 2.8 means that under the metric \mathbf{d} , $\mathcal{A}_t^{\text{pm}}(x, d)$ is the largest space where such a continuous extension is uniquely determined by $J_t^{\text{fv}}(x, d, \cdot)$ on $\mathcal{A}_t^{\text{fv}}(x, d)$. This is because the completeness of $(\mathcal{A}_t^{\text{pm}}(x, d), \mathbf{d})$ is equivalent to the following statement: For any metric space $(\hat{\mathcal{A}}_t(x, d), \hat{\mathbf{d}})$ containing $\mathcal{A}_t^{\text{pm}}(x, d)$ and such that $\hat{\mathbf{d}}|_{\mathcal{A}_t^{\text{pm}}(x, d)} = \mathbf{d}$, the set $\mathcal{A}_t^{\text{pm}}(x, d)$ is closed in $\hat{\mathcal{A}}_t(x, d)$.

For $t \in [0, T]$, $x, d \in \mathbb{R}$ and $X, Y \in \mathcal{A}_t^{\text{pm}}(x, d)$ with associated deviation processes D^X, D^Y defined by (2.15), we define

$$\mathbf{d}(X, Y) = \left(E \left[\int_t^T (D_s^X - D_s^Y)^2 \gamma_s^{-1} ds \right] \right)^{\frac{1}{2}}. \quad (2.21)$$

Identifying any processes that are equal $(dP \times ds|_{[t, T]})$ -a.e., this is indeed a metric on $\mathcal{A}_t^{\text{pm}}(x, d)$; see Lemma 6.2.

Note that for fixed $t \in [0, T]$ and $x, d \in \mathbb{R}$, we may consider the cost functional in (2.17) as a function $J_t^{\text{pm}}(x, d, \cdot): (\mathcal{A}_t^{\text{pm}}(x, d), \mathbf{d}) \rightarrow (L^1(\Omega, \mathcal{F}_t, P), \|\cdot\|_{L^1})$. Indeed, using (2.5), Lemma 2.7, (2.3) and the boundedness of the input processes, we see that $J_t^{\text{pm}}(x, d, X) \in L^1(\Omega, \mathcal{F}_t, P)$ for all $X \in \mathcal{A}_t^{\text{pm}}(x, d)$.

Theorem 2.8 *Let $t \in [0, T]$ and $x, d \in \mathbb{R}$.*

(i) *Suppose that $X \in \mathcal{A}_t^{\text{pm}}(x, d)$. For every sequence $(X^n)_{n \in \mathbb{N}}$ in $\mathcal{A}_t^{\text{pm}}(x, d)$ with $\lim_{n \rightarrow \infty} \mathbf{d}(X^n, X) = 0$, we have $\lim_{n \rightarrow \infty} \|J_t^{\text{pm}}(x, d, X^n) - J_t^{\text{pm}}(x, d, X)\|_{L^1} = 0$.*

(ii) *For any $X \in \mathcal{A}_t^{\text{pm}}(x, d)$, there exists a sequence $(X^n)_{n \in \mathbb{N}}$ in $\mathcal{A}_t^{\text{fv}}(x, d)$ such that $\lim_{n \rightarrow \infty} \mathbf{d}(X^n, X) = 0$. In particular, we have*

$$\operatorname{ess\,inf}_{X \in \mathcal{A}_t^{\text{fv}}(x, d)} J_t^{\text{fv}}(x, d, X) = \operatorname{ess\,inf}_{X \in \mathcal{A}_t^{\text{pm}}(x, d)} J_t^{\text{pm}}(x, d, X). \quad (2.22)$$

(iii) *For any Cauchy sequence $(X^n)_{n \in \mathbb{N}}$ in $(\mathcal{A}_t^{\text{pm}}(x, d), \mathbf{d})$, there exists an element $X^0 \in \mathcal{A}_t^{\text{pm}}(x, d)$ such that $\lim_{n \rightarrow \infty} \mathbf{d}(X^n, X^0) = 0$.*

In Corollary 4.5 below, we provide sufficient conditions that ensure that the infimum on the right-hand side of (2.22) for $t = 0$ is indeed a minimum.

3 Reduction to a standard LQ stochastic control problem

In this section, we recast the problem of minimising J_t^{pm} over $X \in \mathcal{A}_t^{\text{pm}}(x, d)$ as a standard LQ stochastic control problem. All proofs of this section are given in Sect. 6.

3.1 The first reduction

Note that (2.20) shows that for $t \in [0, T]$, $x, d \in \mathbb{R}$ and $X \in \mathcal{A}_t^{\text{pm}}(x, d)$, the costs $J_t^{\text{pm}}(x, d, X)$ depend in a quadratic way on $(\bar{H}^X, \gamma^{-\frac{1}{2}} D^X)$. Moreover, (2.19) says that the dynamics of \bar{H}^X depend linearly on the pair $(\bar{H}^X, \gamma^{-\frac{1}{2}} D^X)$. These two observations suggest to view the minimisation problem for J_t^{pm} over progressively measurable execution strategies $X \in \mathcal{A}_t^{\text{pm}}(x, d)$ as an LQ stochastic control problem with state process \bar{H}^X and control $\gamma^{-\frac{1}{2}} D^X$. This motivates the following definitions.

For every $t \in [0, T]$, $x, d \in \mathbb{R}$ and $u \in \mathcal{L}_t^2$, we consider the state process \tilde{H}^u associated to u and defined by

$$\begin{aligned} d\tilde{H}_s^u &= \left(\frac{1}{2} \left(\mu_s - \frac{1}{4} \sigma_s^2 \right) \tilde{H}_s^u - \frac{1}{2} (2(\rho_s + \mu_s) - \sigma_s^2 - \sigma_s \eta_s \bar{r}_s) u_s \right) ds \\ &\quad + \left(\frac{1}{2} \sigma_s \tilde{H}_s^u - (\sigma_s + \eta_s \bar{r}_s) u_s \right) dW_s^1 - \eta_s \sqrt{1 - |\bar{r}_s|^2} u_s dW_s^2, \quad s \in [t, T], \\ \tilde{H}_t^u &= \frac{d}{\sqrt{\gamma_t}} - \sqrt{\gamma_t} x, \end{aligned} \quad (3.1)$$

and the cost functional J_t defined by

$$\begin{aligned} J_t &\left(\frac{d}{\sqrt{\gamma_t}} - \sqrt{\gamma_t} x, u \right) \\ &= \frac{1}{2} E_t \left[(\tilde{H}_T^u + \sqrt{\gamma_T} \xi)^2 + \int_t^T 2(\kappa_s + \lambda_s) u_s^2 ds \right] \\ &\quad + \frac{1}{2} E_t \left[\int_t^T (2\lambda_s (\tilde{H}_s^u + \sqrt{\gamma_s} \zeta_s)^2 - 4\lambda_s (\tilde{H}_s^u + \sqrt{\gamma_s} \zeta_s) u_s) ds \right]. \end{aligned} \quad (3.2)$$

Once again we sometimes simply write \tilde{H} instead of \tilde{H}^u . The LQ stochastic control problem is to minimise (3.2) over the set of admissible controls $u \in \mathcal{L}_t^2$.

For every progressively measurable execution strategy $X \in \mathcal{A}_t^{\text{pm}}(x, d)$, there exists a control $u \in \mathcal{L}_t^2$ such that the cost functional J_t^{pm} can be rewritten in terms of J_t (and $-\frac{d^2}{2\gamma_t}$). In fact, this is achieved by taking $u = \gamma^{-\frac{1}{2}} D^X$, as outlined in the motivation above. We state this as Lemma 3.1.

Lemma 3.1 *Let $t \in [0, T]$ and $x, d \in \mathbb{R}$. Suppose that $X \in \mathcal{A}_t^{\text{pm}}(x, d)$ with associated deviation D^X . Define u by $u_s = \gamma_s^{-\frac{1}{2}} D_s^X$, $s \in [t, T]$. Then $u \in \mathcal{L}_t^2$ and $J_t^{\text{pm}}(x, d, X) = J_t(\frac{d}{\sqrt{\gamma_t}} - \sqrt{\gamma_t} x, u) - \frac{d^2}{2\gamma_t}$ a.s.*

On the other hand, we may also start with $u \in \mathcal{L}_t^2$ and derive a progressively measurable execution strategy $X \in \mathcal{A}_t^{\text{pm}}(x, d)$ such that the expected costs match.

Lemma 3.2 *Let $t \in [0, T]$ and $x, d \in \mathbb{R}$. Suppose that $u \in \mathcal{L}_t^2$ and let \tilde{H}^u be the associated solution of (3.1). Define X by*

$$X_s = \gamma_s^{-\frac{1}{2}} (u_s - \tilde{H}_s^u), \quad s \in [t, T), \quad X_{t-} = x, \quad X_T = \xi.$$

Then $X \in \mathcal{A}_t^{\text{pm}}(x, d)$ and $J_t^{\text{pm}}(x, d, X) = J_t(\frac{d}{\sqrt{\gamma_t}} - \sqrt{\gamma_t} x, u) - \frac{d^2}{2\gamma_t}$ a.s.

Lemmas 3.1 and 3.2 together with Theorem 2.8 establish the following equivalence of the control problems pertaining to J_t^{fv} , J_t^{pm} and J_t .

Corollary 3.3 For $t \in [0, T]$ and $x, d \in \mathbb{R}$, we have

$$\begin{aligned} \operatorname{ess\,inf}_{X \in \mathcal{A}_t^{\text{fv}}(x, d)} J_t^{\text{fv}}(x, d, X) &= \operatorname{ess\,inf}_{X \in \mathcal{A}_t^{\text{pm}}(x, d)} J_t^{\text{pm}}(x, d, X) \\ &= \operatorname{ess\,inf}_{u \in \mathcal{L}_t^2} J_t \left(\frac{d}{\sqrt{\gamma_t}} - \sqrt{\gamma_t} x, u \right) - \frac{d^2}{2\gamma_t} \quad a.s. \end{aligned}$$

Furthermore, Lemmas 3.1, 3.2 and Corollary 3.3 provide a method to obtain an optimal progressively measurable execution strategy and potentially an optimal FV execution strategy from the standard optimal control problem, and vice versa.

Corollary 3.4 Let $t \in [0, T]$ and $x, d \in \mathbb{R}$.

(i) Suppose that $X^* \in \mathcal{A}_t^{\text{pm}}(x, d)$ minimises J_t^{pm} over $\mathcal{A}_t^{\text{pm}}(x, d)$ and let D^{X^*} be the associated deviation process. Then u^* defined by

$$u_s^* = \gamma_s^{-\frac{1}{2}} D_s^{X^*}, \quad s \in [t, T],$$

minimises J_t over \mathcal{L}_t^2 .

(ii) Suppose that $u^* \in \mathcal{L}_t^2$ minimises J_t over \mathcal{L}_t^2 and let \tilde{H}^{u^*} be the associated solution of (3.1) for u^* . Then X^* defined by

$$X_s^* = \gamma_s^{-\frac{1}{2}} (u_s^* - \tilde{H}_s^{u^*}), \quad s \in [t, T], \quad X_{t-}^* = x, \quad X_T^* = \xi,$$

minimises J_t^{pm} over $\mathcal{A}_t^{\text{pm}}(x, d)$.

Moreover, if $X^* \in \mathcal{A}_t^{\text{fv}}(x, d)$ (in the sense that there is an element of $\mathcal{A}_t^{\text{fv}}(x, d)$ in the equivalence class of X^* , see (2.16)), then X^* minimises J_t^{fv} over $\mathcal{A}_t^{\text{fv}}(x, d)$.

3.2 Formulation without cross-terms

Note that the last integral in the definition (3.2) of the cost functional J_t involves a product between the state process \tilde{H}^u and the control process u . A larger part of the literature on LQ optimal control considers cost functionals that do not contain such cross-terms. In particular, this applies to Kohlmann and Tang [34] whose results we apply in Sect. 4 below. For this reason, we provide in this subsection a reformulation of the control problem (3.1) and (3.2) that does not contain cross-terms. In order to carry out the transformation necessary for this, we need to impose in this subsection a further condition on our model inputs.

Assumption 3.5 We assume that there exists a constant $C \in [0, \infty)$ such that

$$|\lambda_{\cdot}| \leq C |\lambda_{\cdot} + \kappa_{\cdot}| \quad (dP \times ds)\text{-a.e. on } \Omega \times [0, T].$$

Under Assumption 3.5, we always set

$$\frac{\lambda_{\cdot}}{\lambda_{\cdot} + \kappa_{\cdot}} = 0 \quad \text{on the set } \{(\omega, s) : \lambda_s(\omega) + \kappa_s(\omega) = 0\}.$$

In this connection, we remark that the expression $\frac{\lambda}{\lambda+\kappa}$ appears in calculations below only in integrands, where the respective integrators L always have the form $dL_s = \cdots ds + \sum_i \cdots dW_s^i$. Notice that if progressively measurable processes Z and Z' are such that Z is integrable with respect to L and $Z = Z'$ ($dP \times ds$)-a.e., then also Z' is integrable with respect to L and the stochastic integrals $\int_0^\cdot Z_s dL_s$ and $\int_0^\cdot Z'_s dL_s$ are indistinguishable. Therefore the $(dP \times ds)$ -nullset in Assumption 3.5 never matters.

Now in order to get rid of the cross-term in (3.2), we transform for $t \in [0, T]$ any control process $u \in \mathcal{L}_t^2$ in an affine way to $\hat{u}_s = u_s - \frac{\lambda_s}{\lambda_s + \kappa_s}(\tilde{H}_s^u + \sqrt{\gamma_s} \zeta_s)$, $s \in [t, T]$. This leads to the new controlled state process $\hat{H}^{\hat{u}}$ defined by

$$\begin{aligned} d\hat{H}_s^{\hat{u}} &= \left(\frac{\mu_s}{2} - \frac{1}{8}\sigma_s^2 - \frac{\lambda_s}{\lambda_s + \kappa_s} \left(\rho_s + \mu_s - \frac{\sigma_s^2 + \sigma_s \eta_s \bar{r}_s}{2} \right) \right) \hat{H}_s^{\hat{u}} ds \\ &\quad - \left(\rho_s + \mu_s - \frac{\sigma_s^2 + \sigma_s \eta_s \bar{r}_s}{2} \right) \hat{u}_s ds \\ &\quad - \frac{\lambda_s}{\lambda_s + \kappa_s} \left(\rho_s + \mu_s - \frac{\sigma_s^2 + \sigma_s \eta_s \bar{r}_s}{2} \right) \sqrt{\gamma_s} \zeta_s ds \\ &\quad + \left(\frac{\sigma_s}{2} - \frac{\lambda_s}{\lambda_s + \kappa_s} (\sigma_s + \eta_s \bar{r}_s) \right) \hat{H}_s^{\hat{u}} dW_s^1 - (\sigma_s + \eta_s \bar{r}_s) \hat{u}_s dW_s^1 \\ &\quad - \frac{\lambda_s}{\lambda_s + \kappa_s} (\sigma_s + \eta_s \bar{r}_s) \sqrt{\gamma_s} \zeta_s dW_s^1 - \frac{\lambda_s}{\lambda_s + \kappa_s} \eta_s \sqrt{1 - |\bar{r}_s|^2} \hat{H}_s^{\hat{u}} dW_s^2 \\ &\quad - \eta_s \sqrt{1 - |\bar{r}_s|^2} \hat{u}_s dW_s^2 - \frac{\lambda_s}{\lambda_s + \kappa_s} \eta_s \sqrt{1 - |\bar{r}_s|^2} \sqrt{\gamma_s} \zeta_s dW_s^2, \quad s \in [t, T], \\ \hat{H}_t^{\hat{u}} &= \frac{d}{\sqrt{\gamma_t}} - \sqrt{\gamma_t} x. \end{aligned} \quad (3.3)$$

The meaning of (3.3) is that we only reparametrise the control ($u \mapsto \hat{u}$), but not the state variable ($\hat{H}^{\hat{u}} = \tilde{H}^u$); see Lemma 3.6 below for the formal statement.

For $t \in [0, T]$, $x, d \in \mathbb{R}$, $\hat{u} \in \mathcal{L}_t^2$ and associated $\hat{H}^{\hat{u}}$, we define the cost functional \hat{J}_t by

$$\begin{aligned} \hat{J}_t &\left(\frac{d}{\sqrt{\gamma_t}} - \sqrt{\gamma_t} x, \hat{u} \right) \\ &= E_t \left[\frac{1}{2} (\hat{H}_T^{\hat{u}} + \sqrt{\gamma_T} \xi)^2 \right. \\ &\quad \left. + \int_t^T \left(\frac{\lambda_s \kappa_s}{\lambda_s + \kappa_s} (\hat{H}_s^{\hat{u}} + \sqrt{\gamma_s} \zeta_s)^2 + (\lambda_s + \kappa_s) \hat{u}_s^2 \right) ds \right]. \end{aligned} \quad (3.4)$$

This cost functional does not exhibit cross-terms, but is equivalent to J_t of (3.2) in the sense of the following result.

Lemma 3.6 *Let Assumption 3.5 be in force. Let $t \in [0, T]$ and $x, d \in \mathbb{R}$.*

(i) *Consider $u \in \mathcal{L}_t^2$ with associated state process \tilde{H}^u defined by (3.1). Then the process \hat{u} defined by $\hat{u}_s = u_s - \frac{\lambda_s}{\lambda_s + \kappa_s}(\tilde{H}_s^u + \sqrt{\gamma_s} \zeta_s)$, $s \in [t, T]$, is in \mathcal{L}_t^2 and we have $\hat{H}^{\hat{u}} = \tilde{H}^u$ and $J_t(\frac{d}{\sqrt{\gamma_t}} - \sqrt{\gamma_t} x, u) = \hat{J}_t(\frac{d}{\sqrt{\gamma_t}} - \sqrt{\gamma_t} x, \hat{u})$.*

(ii) *Consider $\hat{u} \in \mathcal{L}_t^2$ with associated state process $\hat{H}^{\hat{u}}$ defined by (3.3). Then the process u defined by $u_s = \hat{u}_s + \frac{\lambda_s}{\lambda_s + \kappa_s}(\hat{H}_s^{\hat{u}} + \sqrt{\gamma_s} \zeta_s)$, $s \in [t, T]$, is in \mathcal{L}_t^2 and we have $\tilde{H}^u = \hat{H}^{\hat{u}}$ and $J_t(\frac{d}{\sqrt{\gamma_t}} - \sqrt{\gamma_t} x, u) = \hat{J}_t(\frac{d}{\sqrt{\gamma_t}} - \sqrt{\gamma_t} x, \hat{u})$.*

As a corollary, we obtain the following link between optimal controls for \hat{J}_t and J_t .

Corollary 3.7 *Let Assumption 3.5 be in force. Let $t \in [0, T]$ and $x, d \in \mathbb{R}$.*

(i) *Suppose that $u^* \in \mathcal{L}_t^2$ is an optimal control for J_t and let \tilde{H}^{u^*} be the solution of (3.1) for u^* . Then \hat{u}^* defined by*

$$\hat{u}_s^* = u_s^* - \frac{\lambda_s}{\lambda_s + \kappa_s}(\tilde{H}_s^{u^*} + \sqrt{\gamma_s} \zeta_s), \quad s \in [t, T],$$

is an optimal control in \mathcal{L}_t^2 for \hat{J}_t .

(ii) *Suppose that $\hat{u}^* \in \mathcal{L}_t^2$ is an optimal control for \hat{J}_t and let $\hat{H}^{\hat{u}^*}$ be the solution of (3.3) for \hat{u}^* . Then u^* defined by*

$$u_s^* = \hat{u}_s^* + \frac{\lambda_s}{\lambda_s + \kappa_s}(\hat{H}_s^{\hat{u}^*} + \sqrt{\gamma_s} \zeta_s), \quad s \in [t, T],$$

is an optimal control in \mathcal{L}_t^2 for J_t .

4 Solving the LQ control problem and the trade execution problem

We now solve the LQ control problem from Sect. 3 and consequently obtain a solution of the trade execution problem.

Remark 4.1 The solution approach of Kohlmann and Tang [34] which we are about to apply is built on the close connection between standard LQ stochastic control problems and Riccati-type BSDEs (BSRDEs). This connection is well known and dates back at least to Bismut (see e.g. Bismut [20, 21]). The central challenge in this approach is to establish the existence of a solution to the BSRDE. Kohlmann and Tang [34] prove such results in a general framework which in particular covers our problem formulation in Sect. 3.2 under appropriate assumptions.

There is a variety of further results in the literature on LQ stochastic control problems that provide existence results for BSRDEs under different sets of assumptions. A specific potential further possibility is for example to use the results of the recent article by Sun et al. [40] in our setting. The setup of [40] allows cross-terms in the cost functional and, more interestingly, the results in [40] hold under a uniform convexity assumption on the cost functional, which is a weaker requirement than the

usually imposed nonnegativity and positivity assumptions on the coefficients of the cost functional. However, in general, the terminal costs and the running costs in (3.2) (and also in (3.4)) contain terms such as $(\tilde{H}_T^\mu + \sqrt{\gamma_T} \xi)^2$ and $\lambda_s(\tilde{H}_s^\mu + \sqrt{\gamma_s} \zeta_s)^2$ which are inhomogeneous. Therefore the results of [40] are only directly applicable in the special case where $\xi = 0$ and at least one of λ and ζ vanishes. A possible route for future research could be to incorporate inhomogeneous control problems as presented in Sect. 3 into the framework of [40].

Throughout this section, let the following assumption be in force.

Assumption 4.2 In our general setting (see Sect. 2.1), we additionally assume that the filtration \mathbb{F} we work with is the augmented natural filtration of the Brownian motion (W^1, \dots, W^m) . Furthermore, we set the initial time to $t = 0$. We also assume that the stochastic processes λ and $\kappa = \frac{1}{2}(2\rho + \mu - \sigma^2 - \eta^2 - 2\sigma\eta\bar{r})$ are nonnegative ($dP \times ds|_{[0,T]}$ -a.e.

We emphasise at this point that the results presented in Sects. 2 and 3 are valid for more general filtrations and for processes λ and κ possibly taking negative values. This opens the way for applying Sects. 2 and 3 in other settings in future research.

Remark 4.3 Note that the assumption of nonnegativity of λ and κ is necessary to apply the results of Kohlmann and Tang [34]. Indeed, [34] requires that $\lambda + \kappa$ (the coefficient in front of \hat{u}^2 in (3.4)) and $\frac{\lambda\kappa}{\lambda+\kappa}$ (the coefficient in front of $(\hat{H}_s^\mu + \sqrt{\gamma_s} \zeta_s)^2$ in (3.4)) are nonnegative and bounded, which implies that λ and κ have to be nonnegative. Moreover, we note that nonnegativity of λ and κ ensures that Assumption 3.5 is satisfied. Further, we observe that $\lambda + \kappa$ and $\frac{\lambda\kappa}{\lambda+\kappa}$ are bounded, as required. Indeed, we clearly have $\frac{\lambda\kappa}{\lambda+\kappa} \leq \kappa$, and it remains to recall that μ, σ, ρ, η and λ are bounded and \bar{r} is $[-1, 1]$ -valued (see Sect. 2.1).

Note that the LQ control problem of Sect. 3.2, which consists of minimising the functional \hat{J}_0 in (3.4) with state dynamics given by (3.3), is of the form considered in [34, Eqs. (79)–(81)]. The solution can be described by two BSDEs [34, Eqs. (9) and (85)]. The first, [34, Eq. (9)], is a Riccati-type BSDE, which in our setting reads

$$dK_s = -g(s, K_s, L_s^1, L_s^2)ds + \sum_{j=1}^m L_s^j dW_s^j, \quad s \in [0, T], \quad K_T = \frac{1}{2}, \quad (4.1)$$

with the driver

$$\begin{aligned} g(s, K_s, L_s^1, L_s^2) &= \left(\mu_s + \frac{\lambda_s}{\lambda_s + \kappa_s} \left(\frac{\lambda_s}{\lambda_s + \kappa_s} (\sigma_s^2 + 2\sigma_s \eta_s \bar{r}_s + \eta_s^2) - 2(\rho_s + \mu_s) \right) \right) K_s \\ &\quad + \left(\sigma_s - \frac{\lambda_s}{\lambda_s + \kappa_s} 2(\sigma_s + \eta_s \bar{r}_s) \right) L_s^1 - \frac{\lambda_s}{\lambda_s + \kappa_s} 2\eta_s \sqrt{1 - |\bar{r}_s|^2} L_s^2 + \frac{\lambda_s \kappa_s}{\lambda_s + \kappa_s} \\ &\quad - \frac{\mathcal{N}_s^2}{\mathcal{D}_s} \end{aligned}$$

with

$$\begin{aligned} \mathcal{N}_s &:= \left(\rho_s + \mu_s - \frac{\lambda_s}{\lambda_s + \kappa_s} (\sigma_s^2 + 2\sigma_s \eta_s \bar{r}_s + \eta_s^2) \right) K_s \\ &\quad + (\sigma_s + \eta_s \bar{r}_s) L_s^1 + \eta_s \sqrt{1 - |\bar{r}_s|^2} L_s^2, \end{aligned} \quad (4.2)$$

$$\mathcal{D}_s := \lambda_s + \kappa_s + (\sigma_s^2 + 2\sigma_s \eta_s \bar{r}_s + \eta_s^2) K_s. \quad (4.3)$$

We call a pair (K, L) with $L = (L^1, L^2, \dots, L^m)$ a *solution* to the BSDE (4.1) if

- (i) K is an adapted, continuous, nonnegative and bounded process;
- (ii) $\mathcal{D} = \lambda + \kappa + (\sigma^2 + 2\sigma \eta \bar{r} + \eta^2) K > 0$ ($dP \times ds|_{[0, T]}$ -a.e.);
- (iii) $L^1, \dots, L^m \in \mathcal{L}_0^2$;
- (iv) the BSDE (4.1) is satisfied P -a.s.

The requirements of nonnegativity and boundedness of K can be explained here by the fact that under mild conditions, such a solution exists (see Theorem 4.4 below). Condition (ii) ensures that there is no problem with division in the driver of (4.1) where the quantity \mathcal{D} appears in the denominator. Moreover, it is worth noting that for $K \geq 0$, we always have $\mathcal{D} \geq 0$ as $\sigma^2 + 2\sigma \eta \bar{r} + \eta^2 = (\sigma + \eta \bar{r})^2 + \eta^2(1 - |\bar{r}|^2)$ and $\lambda + \kappa \geq 0$. From this, we also see that the quantity \mathcal{D} can vanish only in “very degenerate” situations. Hence condition (ii) is quite natural.

To shorten notation, we introduce for a solution (K, L) of the BSDE (4.1) the process $\theta = (\theta_s)_{s \in [0, T]}$ by

$$\theta_s = \frac{\mathcal{N}_s}{\mathcal{D}_s} \quad (4.4)$$

with $\mathcal{N}_s, \mathcal{D}_s$ from (4.2) and (4.3). Next, we consider the second BSDE [34, (85)], which is linear and reads in our setting

$$\begin{aligned} d\psi_s &= -f(s, \psi_s, \phi_s^1, \phi_s^2) ds + \sum_{j=1}^m \phi_s^j dW_s^j, \quad s \in [0, T], \\ \psi_T &= -\frac{1}{2} \sqrt{\gamma_T} \xi, \end{aligned} \quad (4.5)$$

with the driver

$$\begin{aligned}
 & f(s, \psi_s, \phi_s^1, \phi_s^2) \\
 &= \left(\frac{\mu_s}{2} - \frac{\sigma_s^2}{8} - \left(\rho_s + \mu_s - \frac{\sigma_s^2 + \sigma_s \eta_s \bar{r}_s}{2} \right) \left(\frac{\lambda_s}{\lambda_s + \kappa_s} + \theta_s \right) \right) \psi_s \\
 &+ \left(\frac{\sigma_s}{2} - (\sigma_s + \eta_s \bar{r}_s) \left(\frac{\lambda_s}{\lambda_s + \kappa_s} + \theta_s \right) \right) \left(\phi_s^1 + \frac{\lambda_s}{\lambda_s + \kappa_s} (\sigma_s + \eta_s \bar{r}_s) \sqrt{\gamma_s} \zeta_s K_s \right) \\
 &- \eta_s \sqrt{1 - |\bar{r}_s|^2} \left(\frac{\lambda_s}{\lambda_s + \kappa_s} + \theta_s \right) \left(\phi_s^2 + \frac{\lambda_s}{\lambda_s + \kappa_s} \eta_s \sqrt{1 - |\bar{r}_s|^2} \sqrt{\gamma_s} \zeta_s K_s \right) \\
 &+ \frac{\lambda_s}{\lambda_s + \kappa_s} \sqrt{\gamma_s} \zeta_s \left(\left(\rho_s + \mu_s - \frac{\sigma_s^2 + \sigma_s \eta_s \bar{r}_s}{2} \right) K_s \right. \\
 &\quad \left. + (\sigma_s + \eta_s \bar{r}_s) L_s^1 + \eta_s \sqrt{1 - |\bar{r}_s|^2} L_s^2 \right) - \frac{\lambda_s \kappa_s}{\lambda_s + \kappa_s} \sqrt{\gamma_s} \zeta_s.
 \end{aligned}$$

A pair (ψ, ϕ) with $\phi = (\phi^1, \phi^2, \dots, \phi^m)$ is called a *solution* to the BSDE (4.5) if

- (i) ψ is an adapted continuous process with $E[\sup_{s \in [0, T]} \psi_s^2] < \infty$;
- (ii) ϕ is progressively measurable with $\int_0^T |\phi_s|^2 ds < \infty$ P -a.s.;
- (iii) the BSDE (4.5) is satisfied P -a.s.

For a solution (K, L) of the BSDE (4.1) and a corresponding solution (ψ, ϕ) of the BSDE (4.5), we define the process θ^0 for $s \in [0, T]$ by

$$\begin{aligned}
 \theta_s^0 = & \left(\left(\rho_s + \mu_s - \frac{\sigma_s^2 + \sigma_s \eta_s \bar{r}_s}{2} \right) \psi_s + \frac{\lambda_s}{\lambda_s + \kappa_s} \sqrt{\gamma_s} \zeta_s (\sigma_s^2 + 2\sigma_s \eta_s \bar{r}_s + \eta_s^2) K_s \right. \\
 & \left. + (\sigma_s + \eta_s \bar{r}_s) \phi_s^1 + \eta_s \sqrt{1 - |\bar{r}_s|^2} \phi_s^2 \right) / \mathcal{D}_s,
 \end{aligned} \tag{4.6}$$

where \mathcal{D}_s is from (4.3). We further introduce for $x, d \in \mathbb{R}$ and $s \in [0, T]$ the SDE

$$d\hat{H}_s^* = \hat{H}_s^* d\mathcal{Y}_s + d\mathcal{Z}_s, \quad \hat{H}_0^* = \frac{d}{\sqrt{\gamma_0}} - \sqrt{\gamma_0} x, \tag{4.7}$$

where for $s \in [0, T]$,

$$\begin{aligned} d\mathcal{Y}_s &= \left(\frac{\mu_s}{2} - \frac{\sigma_s^2}{8} - \left(\rho_s + \mu_s - \frac{\sigma_s^2 + \sigma_s \eta_s \bar{r}_s}{2} \right) \left(\frac{\lambda_s}{\lambda_s + \kappa_s} + \theta_s \right) \right) ds \\ &\quad + \left(\frac{\sigma_s}{2} - (\sigma_s + \eta_s \bar{r}_s) \left(\frac{\lambda_s}{\lambda_s + \kappa_s} + \theta_s \right) \right) dW_s^1 \\ &\quad - \eta_s \sqrt{1 - |\bar{r}_s|^2} \left(\frac{\lambda_s}{\lambda_s + \kappa_s} + \theta_s \right) dW_s^2, \\ d\mathcal{Z}_s &= \left(\rho_s + \mu_s - \frac{\sigma_s^2 + \sigma_s \eta_s \bar{r}_s}{2} \right) \left(\theta_s^0 - \sqrt{\gamma_s} \zeta_s \frac{\lambda_s}{\lambda_s + \kappa_s} \right) ds \\ &\quad + (\sigma_s + \eta_s \bar{r}_s) \left(\theta_s^0 - \sqrt{\gamma_s} \zeta_s \frac{\lambda_s}{\lambda_s + \kappa_s} \right) dW_s^1 \\ &\quad + \eta_s \sqrt{1 - |\bar{r}_s|^2} \left(\theta_s^0 - \sqrt{\gamma_s} \zeta_s \frac{\lambda_s}{\lambda_s + \kappa_s} \right) dW_s^2. \end{aligned}$$

We shall show that the solution \hat{H}^* of (4.7) is the optimal state process in the stochastic control problem to minimise \hat{J}_0 of (3.4). Notice that \hat{H}^* can be easily expressed via \mathcal{Y} and \mathcal{Z} in closed form.

In the next result, we summarise consequences from Kohlmann and Tang [34] in our setting to obtain a minimiser of \hat{J}_0 in (3.4) and a representation of the minimal costs.

Theorem 4.4 *Under Assumption 4.2, assume there exists $\varepsilon \in (0, \infty)$ with $\lambda + \kappa \geq \varepsilon$ ($dP \times ds|_{[0, T]}$)-a.e. or $\sigma^2 + 2\sigma\eta\bar{r} + \eta^2 \geq \varepsilon$ ($dP \times ds|_{[0, T]}$)-a.e. Then:*

- (i) *There exists a unique solution (K, L) of the BSDE (4.1). If $\sigma^2 + 2\sigma\eta\bar{r} + \eta^2 \geq \varepsilon$ ($dP \times ds|_{[0, T]}$)-a.e., there exists $c \in (0, \infty)$ with $P[K_s \geq c \text{ for all } s \in [0, T]] = 1$.*
- (ii) *There exists a unique solution (ψ, ϕ) of the BSDE (4.5).*
- (iii) *Let $x, d \in \mathbb{R}$, and let \hat{H}^* be the solution of the SDE (4.7). Then \hat{u}^* defined by*

$$\hat{u}_s^* = \theta_s \hat{H}_s^* - \theta_s^0, \quad s \in [0, T], \quad (4.8)$$

is the unique optimal control in \mathcal{L}_0^2 for \hat{J}_0 , and \hat{H}^ is the corresponding state process (i.e., $\hat{H}^* = \hat{H}^{\hat{u}^*}$).*

- (iv) *Let $x, d \in \mathbb{R}$. The costs associated to the optimal control (4.8) are given by*

$$\begin{aligned} \inf_{\hat{u} \in \mathcal{L}_0^2} \hat{J}_0 \left(\frac{d}{\sqrt{\gamma_0}} - \sqrt{\gamma_0} x, \hat{u} \right) &= \hat{J}_0 \left(\frac{d}{\sqrt{\gamma_0}} - \sqrt{\gamma_0} x, \hat{u}^* \right) \\ &= K_0 \left(\frac{d}{\sqrt{\gamma_0}} - \sqrt{\gamma_0} x \right)^2 - 2\psi_0 \left(\frac{d}{\sqrt{\gamma_0}} - \sqrt{\gamma_0} x \right) + C_0, \end{aligned}$$

where

$$\begin{aligned}
 C_0 = & \frac{1}{2} E_0[\gamma_T \xi^2] - E_0 \left[\int_0^T (\theta_s^0)^2 (\lambda_s + \kappa_s + (\sigma_s^2 + 2\sigma_s \eta_s \bar{r}_s + \eta_s^2) K_s) ds \right] \\
 & + E_0 \left[\int_0^T K_s \frac{\lambda_s^2}{(\lambda_s + \kappa_s)^2} \gamma_s \zeta_s^2 (\sigma_s^2 + 2\sigma_s \eta_s \bar{r}_s + \eta_s^2) ds \right] \\
 & + E_0 \left[\int_0^T 2 \frac{\lambda_s}{\lambda_s + \kappa_s} \sqrt{\gamma_s} \zeta_s \psi_s \left(\rho_s + \mu_s - \frac{\sigma_s^2 + \sigma_s \eta_s \bar{r}_s}{2} \right) ds \right] \\
 & + E_0 \left[\int_0^T 2 \frac{\lambda_s}{\lambda_s + \kappa_s} \sqrt{\gamma_s} \zeta_s (\phi_s^1 (\sigma_s + \eta_s \bar{r}_s) + \phi_s^2 \eta_s \sqrt{1 - |\bar{r}_s|^2}) ds \right] \\
 & + E_0 \left[\int_0^T \frac{\lambda_s \kappa_s}{\lambda_s + \kappa_s} \gamma_s \zeta_s^2 ds \right]. \tag{4.9}
 \end{aligned}$$

Proof Observe that the problem in Sect. 3.2 fits into the framework in Kohlmann and Tang [34, Sect. 5]. In particular, the coefficients in the SDE (3.3) for $\widehat{H}^{\hat{u}}$ and the cost functional \hat{J}_0 (see (3.4)) are bounded and the inhomogeneities are in \mathcal{L}_0^2 . Moreover, $\frac{1}{2}, \frac{\lambda\kappa}{\lambda+\kappa}$ and $\lambda + \kappa$ are nonnegative, and the filtration in this section by assumption is generated by the Brownian motion (W^1, \dots, W^m) .

(i) If $\lambda + \kappa \geq \varepsilon$, this is an immediate consequence of [34, Theorem 2.1]. In the case $\sigma^2 + 2\sigma\eta\bar{r} + \eta^2 \geq \varepsilon$, this is an application of [34, Theorem 2.2].

(ii) This is due to [34, Theorem 5.1].

(iii) The first part of [34, Theorem 5.2] yields the existence of a unique optimal control \hat{u}^* which is given in feedback form by the formula $\hat{u}^* = \theta \widehat{H}^{\hat{u}^*} - \theta^0$. We obtain (4.7) by plugging this into (3.3).

(iv) The second part of [34, Theorem 5.2] provides us with the optimal costs. \square

By applying Corollaries 3.7 and 3.4, we obtain a solution to the trade execution problem of Sect. 2.

Corollary 4.5 *Impose Assumption 4.2 and assume there exists $\varepsilon \in (0, \infty)$ with $\lambda + \kappa \geq \varepsilon$ ($dP \times ds|_{[0,T]}$)-a.e. or $\sigma^2 + 2\sigma\eta\bar{r} + \eta^2 \geq \varepsilon$ ($dP \times ds|_{[0,T]}$)-a.e. Let (K, L) be the unique solution of the BSDE (4.1), (ψ, ϕ) the unique solution of the BSDE (4.5), and recall the definitions (4.4) of θ and (4.6) of θ^0 . Let $x, d \in \mathbb{R}$. Then X^* defined by*

$$\begin{aligned}
 X_{0-}^* &= x, & X_T^* &= \xi, \\
 X_s^* &= \gamma_s^{-\frac{1}{2}} \left(\left(\theta_s + \frac{\lambda_s}{\lambda_s + \kappa_s} - 1 \right) \widehat{H}_s^* + \gamma_s^{\frac{1}{2}} \zeta_s \frac{\lambda_s}{\lambda_s + \kappa_s} - \theta_s^0 \right), & s &\in [0, T),
 \end{aligned}$$

with \widehat{H}^* from (4.7) is the unique (up to $(dP \times ds|_{[0,T]})$ -nullsets) optimal execution strategy in $\mathcal{A}_0^{\text{pm}}(x, d)$ for J_0^{pm} . The associated costs are given by

$$\begin{aligned} & \inf_{X \in \mathcal{A}_0^{\text{pm}}(x, d)} J_0^{\text{pm}}(x, d, X) \\ &= J_0^{\text{pm}}(x, d, X^*) \\ &= K_0 \left(\frac{d}{\sqrt{\gamma_0}} - \sqrt{\gamma_0} x \right)^2 - 2\psi_0 \left(\frac{d}{\sqrt{\gamma_0}} - \sqrt{\gamma_0} x \right) + C_0 - \frac{d^2}{2\gamma_0} \end{aligned}$$

with C_0 from (4.9).

Remark 4.6 (i) Note that the BSDE (4.1) contains neither ξ nor ζ . In particular, the solution component K and the process θ from (4.4) do not depend on the choice of ξ or ζ (although they depend on the choice of λ). In contrast, the BSDE (4.5) involves both ξ and ζ . If $\xi = 0$ and at least one of λ and ζ is equivalent to 0, then (ψ, ϕ) from (4.5), θ^0 from (4.6) and C_0 from (4.9) vanish.

(ii) Under the assumptions of Corollary 4.5, we have $K_0 \leq \frac{1}{2}$. This is a direct consequence of Corollary 4.5 and (i) above. Indeed, choose $\xi = 0$ and $\zeta = 0$ (by (i), this does not affect K). Then Corollary 4.5 and (i) show for the optimal strategy X^* from Corollary 4.5 that $J_0^{\text{pm}}(1, 0, X^*) = K_0\gamma_0$. The suboptimal FV execution strategy $X_{0-} = 1$, $X_s = 0$, $s \in [0, T]$, in $\mathcal{A}_0^{\text{fv}}(1, 0)$ incurs the costs $J_0^{\text{pm}}(1, 0, X) = \frac{\gamma_0}{2}$, and hence $K_0 \leq \frac{1}{2}$.

(iii) Our present setting essentially includes that in Ackermann et al. [1], where $\xi = 0$, $\lambda = 0$ and $\eta = 0$ (and therefore the processes ζ and \bar{r} are not needed; cf. (2.1) and (2.8)). The word “essentially” relates to different integrability conditions and to the fact that in [1], the formulation is for a continuous local martingale and a general filtration instead of Brownian motion with a Brownian filtration. For $\xi = 0$, $\lambda = 0$ and $\eta = 0$, the FV control problem associated with (2.4)–(2.8) is extended in [1] to a problem where the control X is a càdlàg semimartingale that acts as an integrator in the extended state dynamics of the form (2.4) and with a target functional of the form (2.8). Here, the word “extended” relates to the fact that (2.4) and (2.8) need to be extended with certain additional terms when allowing general semimartingale strategies; see [1]. In [1], the existence of an optimal semimartingale strategy as well as its form (when it exists) are characterised in terms of a certain process β , which is in turn defined via a solution (Y, Z, M^\perp) to a certain quadratic BSDE (see [1, Eq. (3.2)]). It is worth noting that if $\xi = 0$, $\lambda = 0$ and $\eta = 0$, all formulas in the present section simplify greatly, and in particular, the BSDE (4.1) above is equivalent to the BSDE (3.2) in [1]. The relation is $Y = K$, $Z = L^1$, $dM_s^\perp = \sum_{j=2}^m L_s^j dW_s^j$, where for the sake of a fair comparison, we consider the subsetting in [1] where the filtration is generated by the m -dimensional Brownian motion (W^1, \dots, W^m) and the continuous local martingale M is W^1 . Further, for $\xi = 0$, $\lambda = 0$ and $\eta = 0$, our process θ from (4.4) reduces to the above-mentioned process $\tilde{\beta}$ (see [1, Eq. (3.5)]), while (ψ, ϕ) from (4.5), θ^0 from (4.6) and C_0 from (4.9) vanish.

(iv) It is also instructive to compare Corollary 4.5, where we obtain that the control problem extended to $\mathcal{A}_0^{\text{pm}}(x, d)$ always admits a minimiser, with [1, Theorem 3.4],

where it turns out that an optimal semimartingale strategy can fail to exist. See the discussion at the end of Sect. 5.3 for a specific example.

4.1 On the continuity of optimal position paths

In the setting of Obizhaeva and Wang [37], optimal position paths X^* exhibit jumps (so-called block trades) at times 0 and T , but are continuous on the interior $(0, T)$ (see also Sect. 5.1 below). An interesting question is whether the continuity on $(0, T)$ prevails in the generalised setting considered in the present paper. This is not reasonable to expect when we have a risk term with a “sufficiently irregular” process ζ , and indeed, we see in Example 5.1 below that the continuity of X^* on $(0, T)$ can fail (this is discussed in Remark 5.2).

More interestingly, such a continuity can already fail even without the risk term (i.e., $\lambda = 0$) and with terminal target $\xi = 0$. Indeed, consider the setting with $\sigma = 0$, $\lambda = 0$, $\xi = 0$ and non-diffusive resilience process R given by $R_s = \rho s$ (with ρ a deterministic constant). Then it follows from [1, Example 6.2] that continuity of the price impact process γ is not sufficient for continuity of the optimal position paths X^* on $(0, T)$: It is shown that if the paths of γ are absolutely continuous, then a jump of the weak derivative of γ on $(0, T)$ already causes X^* to jump on $(0, T)$.

Moreover, it is possible that the random terminal target position ξ causes the optimal position path X^* to jump in $(0, T)$ with all other input processes being continuous. We present an example for this phenomenon in Sect. 5.2.

A way to obtain sufficient conditions for the continuity of X^* on $(0, T)$ consists of combining Corollary 4.5 with path regularity results for BSDEs. Indeed, if the coefficient processes $\rho, \mu, \sigma, \eta, \bar{r}, \lambda, \zeta$ are continuous and if one can ensure that the solution components L^1, L^2 and ϕ^1, ϕ^2 (which correspond to the martingale representation part of the solution) of the BSDE (4.1) resp. (4.5) have continuous sample paths, then Corollary 4.5 ensures that X^* also has continuous sample paths on $(0, T)$. Results that guarantee continuity of BSDE solutions in a Markovian framework, including the quadratic case, can for example be found in Imkeller and Dos Reis [32].

5 Examples

In this section, we apply the preceding results in specific case studies.

5.1 The Obizhaeva–Wang model with random targets

The models developed by Obizhaeva and Wang [37] can be considered as special cases of the setup in Sect. 2. Indeed, we obtain the problem of [37, Sect. 6] by setting $\mu \equiv 0$, $\sigma \equiv 0$, $\eta \equiv 0$, $\bar{r} \equiv 0$, $\lambda \equiv 0$ and choosing $\rho \in (0, \infty)$ and $\xi \in \mathbb{R}$ as deterministic constants.

Example 5.1 In this example, we apply our results (in particular, Corollary 4.5) and provide closed-form solutions (see (5.5) below) for optimal progressively measurable execution strategies in versions of these problems which allow general random terminal targets ξ and general running targets ζ .

Let $x, d \in \mathbb{R}$. Suppose that $\mu \equiv 0$, $\sigma \equiv 0$, $\eta \equiv 0$ and $\bar{r} \equiv 0$. Furthermore, assume that $\rho \in (0, \infty)$ and $\lambda \in [0, \infty)$ are deterministic constants. We take some ξ and ζ as specified in Sect. 2.1 (in particular, see (2.3)). Note that the conditions of Theorem 4.4 and Corollary 4.5 hold true and that $\gamma_s = \gamma_0$ for all $s \in [0, T]$. In the current setting, the BSDE (4.1) reads

$$dK_s = \left(\frac{\rho^2}{\rho + \lambda} K_s^2 + \frac{2\lambda\rho}{\rho + \lambda} K_s - \frac{\lambda\rho}{\rho + \lambda} \right) ds + \sum_{j=1}^m L_s^j dW_s^j, \quad s \in [0, T],$$

$$K_T = \frac{1}{2}. \quad (5.1)$$

By Theorem 4.4, there exists a unique solution (K, L) . Since the driver and the terminal condition in (5.1) are deterministic, we obtain $L \equiv 0$, and hence (5.1) is in fact a scalar Riccati ODE with constant coefficients. Such an equation can be solved explicitly, and in our situation, we obtain in the case $\lambda > 0$ that

$$K_s = \frac{1}{2} \frac{\lambda \tanh\left(\frac{\sqrt{\lambda}\rho(T-s)}{\sqrt{\lambda+\rho}}\right) + \sqrt{\lambda(\rho+\lambda)}}{\left(\frac{\rho}{2} + \lambda\right) \tanh\left(\frac{\sqrt{\lambda}\rho(T-s)}{\sqrt{\lambda+\rho}}\right) + \sqrt{\lambda(\rho+\lambda)}}, \quad s \in [0, T],$$

and in the case $\lambda = 0$ that

$$K_s = \frac{1}{2 + (T-s)\rho}, \quad s \in [0, T]. \quad (5.2)$$

The process θ from (4.4) is here given by $\theta_s = \frac{\rho}{\lambda+\rho} K_s$, $s \in [0, T]$. The BSDE (4.5) becomes

$$d\psi_s = \left(\frac{\rho\lambda}{\lambda+\rho} + \rho\theta_s \right) \psi_s ds + \frac{\rho\lambda}{\lambda+\rho} \sqrt{\gamma_0} \zeta_s (1 - K_s) ds$$

$$+ \sum_{j=1}^m \phi_s^j dW_s^j, \quad s \in [0, T],$$

$$\psi_T = -\frac{1}{2} \sqrt{\gamma_0} \xi. \quad (5.3)$$

Again by Theorem 4.4, there exists a unique solution (ψ, ϕ) . The solution component ψ is given by

$$\psi_s = \frac{1}{\Gamma_s} \sqrt{\gamma_0} \left(-\frac{1}{2} \Gamma_T E_s[\xi] - \frac{\rho\lambda}{\lambda+\rho} E_s \left[\int_s^T \Gamma_r (1 - K_r) \zeta_r dr \right] \right), \quad s \in [0, T],$$

where we have introduced

$$\Gamma_s = \exp \left(-\rho \int_0^s \left(\frac{\lambda}{\lambda+\rho} + \theta_r \right) dr \right)$$

$$= \exp \left(-\frac{\rho}{\lambda+\rho} \left(\lambda s + \rho \int_0^s K_r dr \right) \right), \quad s \in [0, T]. \quad (5.4)$$

The process θ^0 in (4.6) is given by $\theta_s^0 = \frac{\rho}{\lambda + \rho} \psi_s$, $s \in [0, T]$. Further, the SDE (4.7) reads

$$\begin{aligned} d\hat{H}_s^* &= -\rho \left(\frac{\lambda}{\lambda + \rho} + \theta_s \right) \hat{H}_s^* ds + \rho \left(\theta_s^0 - \sqrt{\gamma_0} \zeta_s \frac{\lambda}{\lambda + \rho} \right) ds, \quad s \in [0, T], \\ \hat{H}_0^* &= \frac{d}{\sqrt{\gamma_0}} - \sqrt{\gamma_0} x, \end{aligned}$$

and has the solution

$$\hat{H}_s^* = \Gamma_s \left(\frac{d}{\sqrt{\gamma_0}} - \sqrt{\gamma_0} x + \rho \int_0^s \Gamma_r^{-1} \left(\theta_r^0 - \sqrt{\gamma_0} \zeta_r \frac{\lambda}{\lambda + \rho} \right) dr \right), \quad s \in [0, T],$$

with Γ given by (5.4). It then follows from Corollary 4.5 that the stochastic process X^* defined by $X_{0-}^* = x$, $X_T^* = \xi$ and

$$\begin{aligned} X_s^* &= \gamma_0^{-\frac{1}{2}} \left(\left(\theta_s - \frac{\rho}{\lambda + \rho} \right) \hat{H}_s^* - \theta_s^0 \right) + \zeta_s \frac{\lambda}{\lambda + \rho} \\ &= \frac{\rho}{\lambda + \rho} (1 - K_s) \Gamma_s \left(x - \frac{d}{\gamma_0} + \frac{\rho}{\lambda + \rho} \int_0^s \Gamma_r^{-1} \left(\lambda \zeta_r - \frac{\rho}{\sqrt{\gamma_0}} \psi_r \right) dr \right) \\ &\quad + \frac{\rho}{\lambda + \rho} \left(\frac{\lambda}{\rho} \zeta_s - \frac{1}{\sqrt{\gamma_0}} \psi_s \right), \quad s \in [0, T], \end{aligned} \quad (5.5)$$

is the (up to $(dP \times ds)_{|[0, T]}$ -nullsets unique) execution strategy in $\mathcal{A}_0^{\text{pm}}(x, d)$ that minimises J_0^{pm} .

Remark 5.2 From Example 5.1, we see that discontinuities of the target process ζ can cause jumps of the optimal position path X^* in $(0, T)$. Indeed, as θ , θ^0 and \hat{H}^* are continuous, it follows from (5.5) that in the case $\lambda > 0$, paths of the optimal strategy X^* inherit discontinuities from ζ on $(0, T)$ (in particular, X^* jumps on $(0, T)$ whenever ζ does).

In the next example, we study the case $\lambda \equiv 0$ in more detail.

Example 5.3 In the setting of Example 5.1, suppose that $\lambda \equiv 0$. If the terminal target $\xi \in \mathbb{R}$ is a deterministic constant, it follows from Obizhaeva and Wang [37, Proposition 3] that the optimal FV execution strategy is given by

$$X_s^* = \left(x - \xi - \frac{d}{\gamma_0} \right) \frac{1 + (T - s)\rho}{2 + T\rho} + \xi, \quad s \in [0, T]. \quad (5.6)$$

So the optimal strategy consists of potential block trades (jumps of X^*) at the time points 0 and T and a continuous linear trading program on $[0, T)$. In the following, we analyse how this structure changes when we allow a random terminal target ξ .

First recall that the solution of the BSDE (5.1) is given in this case by (5.2). It follows that Γ from (5.4) simplifies to $\Gamma_s = \frac{2 + (T - s)\rho}{2 + T\rho}$, $s \in [0, T]$. For the solution

component ψ of the BSDE (5.3), we thus obtain

$$\psi_s = -\frac{\sqrt{\gamma_0}}{2 + (T-s)\rho} E_s[\xi], \quad s \in [0, T].$$

The optimal strategy from (5.5) on $[0, T)$ becomes

$$\begin{aligned} X_s^* &= (1 - K_s)\Gamma_s \left(x - \frac{d}{\gamma_0} - \rho \int_0^s \Gamma_r^{-1} \frac{1}{\sqrt{\gamma_0}} \psi_r dr \right) - \frac{1}{\sqrt{\gamma_0}} \psi_s \\ &= \left(x - \frac{d}{\gamma_0} \right) \frac{1 + (T-s)\rho}{2 + T\rho} + \rho(1 + (T-s)\rho) \int_0^s \frac{E_r[\xi]}{(2 + (T-r)\rho)^2} dr \\ &\quad + \frac{E_s[\xi]}{2 + (T-s)\rho}, \quad s \in [0, T). \end{aligned} \quad (5.7)$$

Integration by parts implies that (note that $(E_r[\xi])_{r \in [0, T]}$ is a continuous martingale)

$$\begin{aligned} \int_0^s \frac{E_r[\xi]}{(2 + (T-r)\rho)^2} dr &= \int_0^s E_r[\xi] d \frac{1}{\rho(2 + (T-r)\rho)} \\ &= \frac{E_s[\xi]}{\rho(2 + (T-s)\rho)} - \frac{E_0[\xi]}{\rho(2 + T\rho)} \\ &\quad - \int_0^s \frac{1}{\rho(2 + (T-r)\rho)} dE_r[\xi], \quad s \in [0, T). \end{aligned}$$

Substituting this into (5.7) yields for $s \in [0, T)$ that

$$\begin{aligned} X_s^* &= \left(x - E_0[\xi] - \frac{d}{\gamma_0} \right) \frac{1 + (T-s)\rho}{2 + T\rho} + E_s[\xi] - \int_0^s \frac{1 + (T-s)\rho}{2 + (T-r)\rho} dE_r[\xi] \\ &= \left(x - E_0[\xi] - \frac{d}{\gamma_0} \right) \frac{1 + (T-s)\rho}{2 + T\rho} + E_0[\xi] \\ &\quad + \int_0^s \left(1 - \frac{1 + (T-s)\rho}{2 + (T-r)\rho} \right) dE_r[\xi]. \end{aligned}$$

We finally obtain the alternative representation

$$\begin{aligned} X_s^* &= \left(x - E_0[\xi] - \frac{d}{\gamma_0} \right) \frac{1 + (T-s)\rho}{2 + T\rho} \\ &\quad + E_0[\xi] + \int_0^s \frac{1 + (s-r)\rho}{2 + (T-r)\rho} dE_r[\xi], \quad s \in [0, T), \end{aligned}$$

for (5.7). We see that this optimal strategy $X^* \in \mathcal{A}_0^{\text{pm}}(x, d)$ consists of two additive parts. The first part corresponds exactly to the optimal deterministic strategy in (5.6), where the deterministic terminal target is replaced by the expected terminal target $E_0[\xi]$. The second part represents fluctuations around this deterministic strategy which incorporate updates about the random terminal target ξ . Note that this stochastic integral vanishes in expectation although it is *not* a martingale (indeed, the time s is not only the upper bound of integration, but also appears in the integrand).

5.2 A discontinuous optimal position path for continuous inputs

We now show that the optimal strategy can have jumps inside $(0, T)$ even if all input processes, including ζ , are continuous. Let $x, d \in \mathbb{R}$. Take $\lambda \equiv 0, \zeta \equiv 0, \eta \equiv 0, \bar{r} \equiv 0, \mu \equiv 0$ and assume that $\sigma \in (0, \infty)$ and $\rho \in (\frac{\sigma^2}{2}, \infty)$ are deterministic constants. Moreover, we later consider an appropriate random terminal target ξ satisfying the assumptions of Sect. 2.1 to produce a jump of the optimal strategy.

Note that the conditions of Theorem 4.4 and Corollary 4.5 hold true. In particular, there exists a unique solution (K, L) of the BSDE (4.1), and it is given by (compare also with [1, Sect. 5.2]) $L \equiv 0$ and

$$K_s = \frac{\rho - \frac{\sigma^2}{2}}{\sigma^2} \bigg/ \mathcal{W} \left(\frac{\rho - \frac{\sigma^2}{2}}{\sigma^2} e^{c_0 - \frac{\rho^2}{\sigma^2} s} \right), \quad s \in [0, T],$$

where \mathcal{W} denotes the Lambert W -function and $c_0 = \ln 2 + \frac{1}{\sigma^2}(2\rho - \sigma^2 + \rho^2 T)$. The process θ from (4.4) becomes

$$\theta_s = \frac{\rho K_s}{\rho - \frac{\sigma^2}{2} + \sigma^2 K_s}, \quad s \in [0, T],$$

and both θ and K are deterministic, increasing, continuous, $(0, 1/2]$ -valued functions.

For some $t_0 \in (0, T)$, let

$$\begin{aligned} \xi = & -2\gamma_0^{-\frac{1}{2}} \exp \left(-\frac{\sigma}{2} W_T^1 + \frac{3}{8} \sigma^2 T + \left(\rho - \frac{\sigma^2}{2} \right) \int_0^T \theta_s ds \right) \\ & \times \left(\sigma \int_{t_0}^T \Gamma_s \theta_s ds + \int_{t_0}^T \Gamma_s dW_s^1 \right), \end{aligned}$$

where

$$\Gamma_t = \exp \left(-\frac{\sigma^2}{8} t - \left(\rho - \frac{\sigma^2}{2} \right) \int_0^t \theta_s ds \right), \quad t \in [0, T].$$

Note that ξ is \mathcal{F}_T -measurable and $E[\gamma_T \xi^2] < \infty$. The terminal target ξ here is defined in such a way that the unique solution (ψ, ϕ) of the BSDE (4.5) (cf. Theorem 4.4) is given by $\phi^1 = 1_{[t_0, T]}$, $\phi^j \equiv 0$, $j \in \{2, \dots, m\}$, and

$$\psi_t = \begin{cases} 0, & 0 \leq t < t_0, \\ \Gamma_t^{-1} \left(\sigma \int_{t_0}^t \Gamma_s \theta_s ds + \int_{t_0}^t \Gamma_s dW_s^1 \right), & t_0 \leq t \leq T. \end{cases}$$

It follows for the process θ^0 in (4.6) that

$$\theta_t^0 = \begin{cases} 0, & 0 \leq t < t_0, \\ \frac{(\rho - \frac{\sigma^2}{2}) \psi_t + \sigma}{\rho - \frac{\sigma^2}{2} + \sigma^2 K_t}, & t_0 \leq t \leq T. \end{cases}$$

We thus have

$$\Delta\theta_{t_0}^0 = \frac{\sigma}{\rho - \frac{\sigma^2}{2} + \sigma^2 K_{t_0}} > 0.$$

From Corollary 4.5, we obtain the existence of a unique optimal strategy X^* and that $X_s^* = \gamma_s^{-\frac{1}{2}}((\theta_s - 1)\widehat{H}_s^* - \theta_s^0)$, $s \in [0, T]$. Since γ, θ and \widehat{H}^* (see (4.7)) are continuous and $\Delta\theta_{t_0}^0 > 0$, we obtain that $\Delta X_{t_0}^* = -\gamma_{t_0}^{-\frac{1}{2}}\Delta\theta_{t_0}^0 < 0$. Hence the optimal strategy has a jump at $t_0 \in (0, T)$.

5.3 An example where J_0^{fv} does not admit a minimiser

Let $x, d \in \mathbb{R}$ with $x \neq \frac{d}{\gamma_0}$. Suppose that $\sigma \equiv 0, \eta \equiv 0, \lambda \equiv 0, \bar{r} \equiv 0, \zeta \equiv 0, \xi = 0$. Choose μ to be a bounded deterministic càdlàg function such that there exists a constant $\delta \in (0, T)$ with μ having infinite variation on $[0, T - \delta]$, and take $\rho \in \mathbb{R} \setminus \{0\}$ such that there exists $\varepsilon \in (0, \infty)$ with $2\rho + \mu \geq \varepsilon$. Note that this corresponds to the setting in Ackermann et al. [1, Example 6.4]. Moreover, observe that the conditions of Corollary 4.5 are satisfied.

In the current setting, the BSDE (4.1) becomes

$$dK_s = \left(-\mu_s K_s + \frac{2(\rho + \mu_s)^2 K_s^2}{2\rho + \mu_s} \right) ds + \sum_{j=1}^m L_s^j dW_s^j, \quad s \in [0, T],$$

$$K_T = \frac{1}{2}.$$

Its solution is given by $(K, 0)$, where (see also Y in [1, Sect. 6])

$$K_s = e^{\int_s^T \mu_r dr} \left/ \left(\int_s^T \frac{2(\rho + \mu_r)^2}{2\rho + \mu_r} e^{\int_r^T \mu_\ell d\ell} dr + 2 \right) \right., \quad s \in [0, T],$$

is a deterministic continuous function of finite variation. We have that

$$\theta_s = \frac{2(\rho + \mu_s)}{2\rho + \mu_s} K_s, \quad s \in [0, T],$$

which is the same as $\widetilde{\beta}$ in [1, Example 6.4]. The solution of the BSDE (4.5) is given by $(\psi, \phi) = (0, 0)$, and it holds that $\theta^0 \equiv 0$. Furthermore, (4.7) reads

$$d\widehat{H}_s^* = \left(\frac{\mu_s}{2} - (\rho + \mu_s)\theta_s \right) \widehat{H}_s^* ds, \quad s \in [0, T], \quad \widehat{H}_0^* = \frac{d}{\sqrt{\gamma_0}} - \sqrt{\gamma_0} x,$$

and is solved by the continuous deterministic FV function

$$\widehat{H}_s^* = \left(\frac{d}{\sqrt{\gamma_0}} - \sqrt{\gamma_0} x \right) \exp \left(\int_0^s \left(\frac{\mu_r}{2} - (\rho + \mu_r)\theta_r \right) dr \right), \quad s \in [0, T],$$

which is nonvanishing due to our assumption $x \neq \frac{d}{\gamma_0}$.

Note that at this point, it is easy to explain why we exclude the case $x = \frac{d}{\gamma_0}$ in this example. Namely, if $x = \frac{d}{\gamma_0}$, we get $\widehat{H}^* \equiv 0$ and then the optimal strategy is to close the position immediately, i.e., $X_{0-}^* = x$, $X_s^* = 0$, $s \in [0, T]$, which is always an FV strategy.

To continue, by Corollary 4.5, there exists a (up to $(dP \times ds|_{[0,T]})$ -nullsets) unique minimiser X^* of J_0^{pm} in $\mathcal{A}_0^{\text{pm}}(x, d)$, and it is given by

$$X_{0-}^* = x, \quad X_T^* = 0, \quad X_s^* = \gamma_s^{-\frac{1}{2}}(\theta_s - 1)\widehat{H}_s^*, \quad s \in [0, T].$$

Assume by way of contradiction that there exists a minimiser X^0 of J_0^{fv} in the set $\mathcal{A}_0^{\text{fv}}(x, d)$ of FV execution strategies. We know from Corollary 3.3 that X^0 is then also a minimiser of J_0^{pm} in the set $\mathcal{A}_0^{\text{pm}}(x, d)$ of progressively measurable execution strategies. It follows that $X^0 = X^*$ ($dP \times ds|_{[0,T]}$ -a.e.). Since \widehat{H}^* is nowhere 0, we obtain that

$$1 + \frac{\gamma^{\frac{1}{2}} X^0}{\widehat{H}^*} = \theta \quad (dP \times ds|_{[0,T]})\text{-a.e.} \quad (5.8)$$

Observe that the left-hand side is a process of finite variation. On the other hand, our assumption on μ easily yields that θ has infinite variation. This contradiction proves that in the setting of this example, J_0^{fv} does not admit a minimiser in $\mathcal{A}_0^{\text{fv}}(x, d)$.

We can say even more: In this example, there exists no optimal semimartingale strategy, where under a *semimartingale strategy*, we formally understand a semimartingale that is an element of $\mathcal{A}_0^{\text{pm}}(x, d)$. Indeed, if we had a semimartingale X^0 as a minimiser, we should still get (5.8) (with a semimartingale X^0). The left-hand side would then be a semimartingale. On the other hand, it is shown in [1, Example 6.4] that there does not exist any semimartingale β with $\beta = \theta$ ($dP \times ds|_{[0,T]}$ -a.e.). Thus the cost functional does not have a minimiser in the set of semimartingales, but we are now able to find a minimiser in the set of progressively measurable execution strategies.

5.4 An example with a diffusive resilience

As already mentioned in the introduction, the literature on optimal trade execution in Obizhaeva–Wang-type models typically assumes that R is an increasing process. In Ackermann et al. [1] and Ackermann et al. [3], R is allowed to have finite variation. Now we consider an example with a truly diffusive R .

Let $x, d \in \mathbb{R}$ with $x \neq \frac{d}{\gamma_0}$. Let $\xi = 0$, $\lambda \equiv 0$, $\zeta \equiv 0$ and $\mu \equiv 0$. Moreover, suppose that $\bar{r} \in [-1, 1]$ and $\eta, \rho, \sigma \in \mathbb{R}$ are deterministic constants such that

$$\kappa = \frac{1}{2}(2\rho - \sigma^2 - \eta^2 - 2\sigma\eta\bar{r}) > 0 \quad \text{and} \quad \sigma^2 + \eta^2 + 2\sigma\eta\bar{r} > 0$$

(in particular, we thus need $\rho > 0$). Note that the assumptions of Corollary 4.5 are satisfied. We moreover remark that the subsetting with $\eta \equiv 0$ corresponds to the setting in [1, Sect. 5.2]. Therefore the difference to [1, Sect. 5.2] is that we now consider a more general resilience.

The Riccati BSDE (4.1) becomes

$$\begin{aligned} dK_s &= \frac{(\rho K_s + (\sigma + \eta\bar{r})L_s^1 + \eta\sqrt{1 - |\bar{r}|^2}L_s^2)^2}{(\sigma^2 + \eta^2 + 2\sigma\eta\bar{r})K_s + \kappa} ds - \sigma L_s^1 ds \\ &\quad + \sum_{j=1}^m L_s^j dW_s^j, \quad s \in [0, T], \\ K_T &= \frac{1}{2}. \end{aligned}$$

This has the solution $(K, L) = (K, 0)$ with, for $s \in [0, T]$,

$$K_s = \frac{\kappa}{\sigma^2 + \eta^2 + 2\sigma\eta\bar{r}} \Big/ \mathcal{W}\left(\frac{\kappa}{\sigma^2 + \eta^2 + 2\sigma\eta\bar{r}} \exp\left(c - \frac{\rho^2 s}{\sigma^2 + \eta^2 + 2\sigma\eta\bar{r}}\right)\right),$$

where $c = \ln 2 + \frac{2\kappa + \rho^2 T}{\sigma^2 + \eta^2 + 2\sigma\eta\bar{r}}$ (compare also with [1, Sect. 5.2]) and \mathcal{W} denotes the Lambert W -function. We further have

$$\theta_s = \frac{\rho K_s}{(\sigma^2 + \eta^2 + 2\sigma\eta\bar{r})K_s + \kappa}, \quad s \in [0, T].$$

Observe that $(\psi, \phi) = (0, 0)$ is the solution of (4.5) in the present setting and that we get $\theta^0 \equiv 0$ in (4.6). Moreover, the SDE (4.7) reads

$$\begin{aligned} d\hat{H}_s^* &= \left(-\frac{\sigma^2}{8} - \left(\rho - \frac{\sigma^2 + \sigma\eta\bar{r}}{2}\right)\theta_s\right)\hat{H}_s^* ds + \left(\frac{\sigma}{2} - (\sigma + \eta\bar{r})\theta_s\right)\hat{H}_s^* dW_s^1 \\ &\quad - \eta\sqrt{1 - |\bar{r}|^2}\theta_s\hat{H}_s^* dW_s^2, \quad s \in [0, T], \end{aligned}$$

with $\hat{H}_0^* = \frac{d}{\sqrt{\gamma_0}} - \sqrt{\gamma_0}x$. Hence we obtain for all $s \in [0, T]$ that

$$\begin{aligned} \hat{H}_s^* &= \left(\frac{d}{\sqrt{\gamma_0}} - \sqrt{\gamma_0}x\right) \\ &\quad \times \exp\left(\frac{\sigma}{2}W_s^1 - (\sigma + \eta\bar{r})\int_0^s \theta_r dW_r^1 - \eta\sqrt{1 - |\bar{r}|^2}\int_0^s \theta_r dW_r^2\right) \\ &\quad \times \exp\left(-\frac{\sigma^2 s}{4} - (\rho - \sigma^2 - \sigma\eta\bar{r})\int_0^s \theta_r dr - \frac{\sigma^2 + \eta^2 + 2\sigma\eta\bar{r}}{2}\int_0^s \theta_r^2 dr\right). \end{aligned}$$

It follows from Corollary 4.5 that the optimal execution strategy is given by

$$\begin{aligned} X_s^* &= \left(x - \frac{d}{\gamma_0}\right)(1 - \theta_s) \exp\left(-(\sigma + \eta\bar{r})\int_0^s \theta_r dW_r^1 - \eta\sqrt{1 - |\bar{r}|^2}\int_0^s \theta_r dW_r^2\right) \\ &\quad \times \exp\left(-(\rho - \sigma^2 - \sigma\eta\bar{r})\int_0^s \theta_r dr - \frac{\sigma^2 + \eta^2 + 2\sigma\eta\bar{r}}{2}\int_0^s \theta_r^2 dr\right) \end{aligned}$$

for $s \in [0, T)$.

We can show here that K and θ both are continuous, deterministic, increasing, $(0, 1/2]$ -valued functions of finite variation. Since $\theta < 1$, the optimal strategy on $[0, T)$ always has the same sign as $x - \frac{d}{\gamma_0}$. Moreover, the optimal strategy is stochastic and has infinite variation, as in [1, Sect. 5.2]. But in contrast to [1, Sect. 5.2] where the price impact always has infinite variation, we can here set $\sigma \equiv 0$ for a choice of $\eta^2 \in (0, 2\rho)$. In this case, the price impact $\gamma \equiv \gamma_0$ is a deterministic constant, but the optimal strategy still has infinite variation (due to the infinite variation in the resilience R).

Observe furthermore that by making use of η and \bar{r} , we can choose the parameters in the current setting in such a way that $\kappa > 0$ and $\sigma^2 + \eta^2 + 2\sigma\eta\bar{r} > 0$ are satisfied, but condition (3.1) in [1], i.e., $2\rho - \sigma^2 > 0$, is violated.

With regard to Sect. 5.3, we remark that in both sections, there is no optimal strategy in $\mathcal{A}_0^{\text{fv}}(x, d)$. But in contrast to Sect. 5.3, in the current section, there exists an optimal semimartingale strategy.

5.5 Cancellation of infinite variation

We now present an example where the infinite variation in the price impact process γ is “cancelled” by the infinite variation in the resilience process R and we obtain an optimal strategy X^* of finite variation.

Let $x, d \in \mathbb{R}$. Let $\xi = 0, \lambda \equiv 0, \zeta \equiv 0$ and $\mu \equiv 0$. Suppose that $\bar{r} = -1$ and $\rho > 0$ are deterministic constants, and assume that η and σ are progressively measurable, $(dP \times ds|_{[0, T]})$ -a.e. bounded processes such that $\eta = \sigma$ $(dP \times ds|_{[0, T]})$ -a.e. It then holds $(dP \times ds|_{[0, T]})$ -a.e. that $\sigma^2 + \eta^2 + 2\sigma\eta\bar{r} = 0$ and $\kappa = \rho > 0$. In particular, the assumptions of Corollary 4.5 are satisfied.

The BSDE

$$dK_s = \rho K_s^2 ds - \sigma_s L_s^1 ds + \sum_{j=1}^m L_s^j dW_s^j, \quad s \in [0, T], \quad K_T = \frac{1}{2},$$

which is the BSDE (4.1) in the present setting, has the solution $(K, L) = (K, 0)$ with

$$K_s = \frac{1}{2 + (T - s)\rho}, \quad s \in [0, T]$$

(cf. Sect. 5.1). We obtain $\theta \equiv K$, that $(\psi, \phi) = (0, 0)$ is the solution of (4.5), and that $\theta^0 \equiv 0$. It follows that (4.7) has the solution, for $s \in [0, T]$,

$$\hat{H}_s^* = \left(\frac{d}{\sqrt{\gamma_0}} - \sqrt{\gamma_0} x \right) \exp \left(-\frac{1}{4} \int_0^s \sigma_r^2 dr - \rho \int_0^s K_r dr + \frac{1}{2} \int_0^s \sigma_r dW_r^1 \right).$$

For the optimal execution strategy from Corollary 4.5, we then compute that

$$X_s^* = \left(x - \frac{d}{\gamma_0} \right) \frac{1 + (T - s)\rho}{2 + T\rho}, \quad s \in [0, T].$$

The optimal strategy in the current setting with general stochastic $\sigma = \eta$ and negative correlation $\bar{r} = -1$ is thus the same as in the Obizhaeva–Wang setting $\sigma = 0 = \eta$

(cf. Obizhaeva and Wang [37, Proposition 3]; see also [1, Sect. 4.2]). In particular, the optimal strategy is deterministic and of finite variation, although the price impact γ and the resilience R are both stochastic and of infinite variation (at least if $\sigma = \eta$ is nonvanishing).

We finally remark that this setting does not reduce to the Obizhaeva–Wang setting $\sigma = 0 = \eta$. Indeed, while the optimal strategies for $\sigma = 0 = \eta$ and for general stochastic $\sigma = \eta$ with correlation $\bar{r} = -1$ coincide, this is not true for the associated deviation processes. In general, we have

$$D_s^{X^*} = -\gamma_0 \left(x - \frac{d}{\gamma_0} \right) \frac{1}{2 + T\rho} \exp \left(\int_0^s \eta_r dW_r^1 - \frac{1}{2} \int_0^s \eta_r^2 dr \right), \quad s \in [0, T],$$

which for a nonvanishing η and $x \neq \frac{d}{\gamma_0}$ has infinite variation, but is constant in the Obizhaeva–Wang setting (take $\eta = 0$).

6 Proofs

In this section, we provide the proofs for the results in Sects. 2 and 3. We furthermore state and prove some auxiliary results used in the proofs of the main results.

For reference in several proofs, note that the order book height, i.e., the reciprocal of the price impact, has dynamics

$$d\gamma_s^{-1} = \gamma_s^{-1} \left(-(\mu_s - \sigma_s^2)ds - \sigma_s dW_s^1 \right), \quad s \in [0, T]. \quad (6.1)$$

We moreover observe that Itô's lemma gives

$$\begin{aligned} d\gamma_s^{\frac{1}{2}} &= \gamma_s^{\frac{1}{2}} \left(\frac{1}{2}\mu_s - \frac{1}{8}\sigma_s^2 \right) ds + \frac{1}{2}\gamma_s^{\frac{1}{2}}\sigma_s dW_s^1, \quad s \in [0, T], \\ d\gamma_s^{-\frac{1}{2}} &= \gamma_s^{-\frac{1}{2}} \left(-\frac{1}{2}\mu_s + \frac{3}{8}\sigma_s^2 \right) ds - \frac{1}{2}\gamma_s^{-\frac{1}{2}}\sigma_s dW_s^1, \quad s \in [0, T]. \end{aligned} \quad (6.2)$$

Proof of Proposition 2.3 Integration by parts implies for all $s \in [t, T]$ that

$$\begin{aligned} d(v_s D_s) &= v_s dD_s + D_s dv_s + d[v, D]_s \\ &= -v_s D_s dR_s + v_s \gamma_s dX_s + v_s D_s dR_s + v_s D_s d[R]_s + d[v, D]_s \\ &= v_s \gamma_s dX_s + v_s D_s d[R]_s + d[v, D]_s. \end{aligned}$$

Since

$$d[v, D]_s = v_s d[R, D]_s = -v_s D_s d[R]_s, \quad s \in [t, T],$$

it follows that the process $\tilde{D}_s = v_s D_s$, $s \in [t, T]$, $\tilde{D}_{t-} = d$, satisfies

$$d\tilde{D}_s = d(v_s D_s) = v_s \gamma_s dX_s, \quad s \in [t, T]. \quad (6.3)$$

In particular, \tilde{D} is of finite variation.

The facts that $\Delta D_s = \gamma_s \Delta X_s$ and $d\tilde{D}_s = v_s \gamma_s dX_s$, $s \in [t, T]$, imply that

$$\begin{aligned} \int_{[t,T]} (2D_{s-} + \gamma_s \Delta X_s) dX_s &= \int_{[t,T]} (2D_{s-} + \Delta D_s) dX_s \\ &= \int_{[t,T]} (2D_{s-} + \Delta D_s) \gamma_s^{-1} v_s^{-1} d\tilde{D}_s \\ &= \int_{[t,T]} v_s^{-2} \gamma_s^{-1} (2\tilde{D}_{s-} + \Delta \tilde{D}_s) d\tilde{D}_s \\ &= \int_{[t,T]} \varphi_s d(\tilde{D}_s^2), \end{aligned} \quad (6.4)$$

where we set $\varphi_s = v_s^{-2} \gamma_s^{-1}$, $s \in [t, T]$, and in the last equality use that

$$d(\tilde{D}_s^2) = (2\tilde{D}_{s-} + \Delta \tilde{D}_s) d\tilde{D}_s, \quad s \in [t, T],$$

as \tilde{D} has finite variation. Summing up, (6.4) yields

$$\begin{aligned} &\int_{[t,T]} D_{s-} dX_s + \frac{1}{2} \int_{[t,T]} \gamma_s \Delta X_s dX_s \\ &= \frac{1}{2} \left(\tilde{D}_T^2 \varphi_T - \tilde{D}_{t-}^2 \varphi_t - \int_t^T \tilde{D}_s^2 d\varphi_s \right) \\ &= \frac{1}{2} \left(\gamma_T^{-1} D_T^2 - \gamma_t^{-1} d^2 - \int_t^T D_s^2 v_s^2 d(v_s^{-2} \gamma_s^{-1}) \right). \end{aligned}$$

In order to show (2.12), we first obtain from (6.3) and integration by parts that

$$v_r D_r - d = v_r \gamma_r X_r - \gamma_t x - \int_{[t,r]} X_s d(v_s \gamma_s) - \int_{[t,r]} d[v\gamma, X]_s, \quad r \in [t, T].$$

This implies that

$$D_r = \gamma_r X_r + v_r^{-1} \left(d - \gamma_t x - \int_t^r X_s d(v_s \gamma_s) \right), \quad r \in [t, T]. \quad \square$$

Proof of Proposition 2.4 We first consider the integrator $v^{-2} \gamma^{-1}$ on the right-hand side of (2.11). Integration by parts and (2.10) yield for all $s \in [t, T]$ that

$$\begin{aligned} d(v_s^{-2} \gamma_s^{-1}) &= v_s^{-1} d(\gamma_s^{-1} v_s^{-1}) + \gamma_s^{-1} v_s^{-1} d v_s^{-1} + d[v^{-1}, \gamma^{-1} v^{-1}]_s \\ &= 2v_s^{-1} \gamma_s^{-1} d v_s^{-1} + v_s^{-2} d \gamma_s^{-1} + v_s^{-1} d[\gamma^{-1}, v^{-1}]_s + d[v^{-1}, \gamma^{-1} v^{-1}]_s \\ &= -2v_s^{-2} \gamma_s^{-1} dR_s + v_s^{-2} d \gamma_s^{-1} - v_s^{-2} d[\gamma^{-1}, R]_s + d[v^{-1}, \gamma^{-1} v^{-1}]_s. \end{aligned}$$

Note that for all $s \in [t, T]$, we have

$$\begin{aligned} d[v^{-1}, \gamma^{-1}v^{-1}]_s &= -v_s^{-1}d[R, \gamma^{-1}v^{-1}]_s \\ &= -v_s^{-1}d\left[R, \int_t^\cdot \gamma^{-1}dv^{-1} + \int_t^\cdot v^{-1}d\gamma^{-1}\right]_s \\ &= -v_s^{-1}\gamma_s^{-1}d[R, v^{-1}]_s - v_s^{-2}d[R, \gamma^{-1}]_s \\ &= v_s^{-2}\gamma_s^{-1}d[R]_s - v_s^{-2}d[R, \gamma^{-1}]_s. \end{aligned}$$

It hence follows for all $s \in [t, T]$ that

$$d(v_s^{-2}\gamma_s^{-1}) = -2v_s^{-2}\gamma_s^{-1}dR_s + v_s^{-2}d\gamma_s^{-1} - 2v_s^{-2}d[\gamma^{-1}, R]_s + v_s^{-2}\gamma_s^{-1}d[R]_s.$$

Plugging this into (2.11) from Proposition 2.3, we obtain that

$$\begin{aligned} &\int_{[t, T]} D_s dX_s + \frac{1}{2} \int_{[t, T]} \gamma_s \Delta X_s dX_s \\ &= \frac{1}{2} \left(\gamma_T^{-1} D_T^2 - \gamma_t^{-1} d^2 \right. \\ &\quad \left. - \int_t^T D_s^2 (d\gamma_s^{-1} + \gamma_s^{-1} d[R]_s - 2\gamma_s^{-1} dR_s - 2d[\gamma^{-1}, R]_s) \right). \end{aligned} \quad (6.5)$$

We further have by (2.1) and (6.1) that for all $s \in [t, T]$,

$$\begin{aligned} &d\gamma_s^{-1} + \gamma_s^{-1}d[R]_s - 2\gamma_s^{-1}dR_s - 2d[\gamma^{-1}, R]_s \\ &= -\gamma_s^{-1}(\mu_s - \sigma_s^2)ds - \gamma_s^{-1}\sigma_s dW_s^1 + \gamma_s^{-1}\eta_s^2 ds - 2\gamma_s^{-1}\rho_s ds - 2\gamma_s^{-1}\eta_s dW_s^R \\ &\quad + 2\gamma_s^{-1}\sigma_s \eta_s \bar{r}_s ds \\ &= -\gamma_s^{-1}(2\rho_s + \mu_s - \sigma_s^2 - \eta_s^2 - 2\sigma_s \eta_s \bar{r}_s)ds - \gamma_s^{-1}\sigma_s dW_s^1 - 2\gamma_s^{-1}\eta_s dW_s^R. \end{aligned} \quad (6.6)$$

It follows from (2.5) and the boundedness of the input processes that

$$E\left[\left|\int_t^T D_s^2 \gamma_s^{-1}(2\rho_s + \mu_s - \sigma_s^2 - \eta_s^2 - 2\sigma_s \eta_s \bar{r}_s)ds\right|\right] < \infty.$$

The Burkholder–Davis–Gundy inequality together with (2.7) shows that

$$E\left[\sup_{r \in [t, T]} \left|\int_t^r D_s^2 \gamma_s^{-1} \sigma_s dW_s^1\right|\right] \leq cE\left[\left(\int_t^T D_s^4 \gamma_s^{-2} \sigma_s^2 ds\right)^{\frac{1}{2}}\right] < \infty$$

for some constant $c \in (0, \infty)$. We therefore get $E_t[\int_t^T D_s^2 \gamma_s^{-1} \sigma_s dW_s^1] = 0$. Similarly, (2.6) implies that $E_t[\int_t^T 2D_s^2 \gamma_s^{-1} \eta_s dW_s^R] = 0$. It thus follows from (6.5), (6.6)

and (2.13) that

$$\begin{aligned} E_t \left[\int_{[t, T]} D_s dX_s + \frac{1}{2} \int_{[t, T]} \gamma_s \Delta X_s dX_s \right] \\ = \frac{1}{2} E_t \left[\gamma_T^{-1} D_T^2 + \int_t^T D_s^2 \gamma_s^{-1} 2\kappa_s ds \right] - \frac{d^2}{2\gamma_t}. \end{aligned}$$

By the definition (2.8) of J_t^{fv} , this proves (2.14). \square

The dynamics that we compute in the following lemma are used in the proofs of Lemmas 2.7 and 6.5.

Lemma 6.1 *Let $t \in [0, T]$, $x, d \in \mathbb{R}$. Assume that X is a progressively measurable process such that $\int_t^T X_s^2 ds < \infty$ a.s. For $\alpha_s = \gamma_s^{-\frac{1}{2}} v_s^{-1}$, $s \in [t, T]$, and*

$$\beta_s = d - \gamma_t x - \int_t^s X_r d(v_r \gamma_r), \quad s \in [t, T],$$

we then have for all $s \in [t, T]$ that

$$\begin{aligned} d(\alpha_s \beta_s) &= -\gamma_s^{\frac{1}{2}} X_s \left((\mu_s + \rho_s + \eta_s^2 + \sigma_s \eta_s \bar{r}_s) ds + (\sigma_s + \eta_s \bar{r}_s) dW_s^1 + \eta_s \sqrt{1 - |\bar{r}_s|^2} dW_s^2 \right) \\ &\quad + \alpha_s \beta_s \left(-\rho_s - \frac{1}{2} \mu_s + \frac{3}{8} \sigma_s^2 + \frac{1}{2} \sigma_s \eta_s \bar{r}_s \right) ds \\ &\quad + \alpha_s \beta_s \left(\left(-\eta_s \bar{r}_s - \frac{1}{2} \sigma_s \right) dW_s^1 - \eta_s \sqrt{1 - |\bar{r}_s|^2} dW_s^2 \right) \\ &\quad + \gamma_s^{\frac{1}{2}} X_s \left(\frac{3}{2} \eta_s \sigma_s \bar{r}_s + \frac{1}{2} \sigma_s^2 + \eta_s^2 \right) ds. \end{aligned} \quad (6.7)$$

Proof Integration by parts implies that for $s \in [t, T]$,

$$d(\alpha_s \beta_s) = -\alpha_s X_s d(v_s \gamma_s) + \beta_s d(\gamma_s^{-\frac{1}{2}} v_s^{-1}) - X_s d[\gamma^{-\frac{1}{2}} v^{-1}, v \gamma]_s. \quad (6.8)$$

Furthermore, integration by parts, (2.9), (2.1) and (2.2) yield

$$\begin{aligned}
 d(\nu_s \gamma_s) &= \nu_s d\gamma_s + \gamma_s \nu_s dR_s + \gamma_s \nu_s d[R]_s + \nu_s d[R, \gamma]_s \\
 &= \nu_s \gamma_s \mu_s ds + \nu_s \gamma_s \sigma_s dW_s^1 + \nu_s \gamma_s \rho_s ds \\
 &\quad + \nu_s \gamma_s \eta_s \bar{r}_s dW_s^1 + \nu_s \gamma_s \eta_s \sqrt{1 - |\bar{r}_s|^2} dW_s^2 + \nu_s \gamma_s \eta_s^2 ds + \nu_s \gamma_s \sigma_s \eta_s \bar{r}_s ds \\
 &= \nu_s \gamma_s ((\mu_s + \rho_s + \eta_s^2 + \sigma_s \eta_s \bar{r}_s) ds \\
 &\quad + (\sigma_s + \eta_s \bar{r}_s) dW_s^1 + \eta_s \sqrt{1 - |\bar{r}_s|^2} dW_s^2). \tag{6.9}
 \end{aligned}$$

Also by integration by parts and (2.10), (2.1) and (6.2), we get

$$\begin{aligned}
 d(\gamma_s^{-\frac{1}{2}} \nu_s^{-1}) &= -\gamma_s^{-\frac{1}{2}} \nu_s^{-1} dR_s + \nu_s^{-1} d\gamma_s^{-\frac{1}{2}} - \nu_s^{-1} d[R, \gamma^{-\frac{1}{2}}]_s \\
 &= -\gamma_s^{-\frac{1}{2}} \nu_s^{-1} \rho_s ds - \gamma_s^{-\frac{1}{2}} \nu_s^{-1} \eta_s \bar{r}_s dW_s^1 - \gamma_s^{-\frac{1}{2}} \nu_s^{-1} \eta_s \sqrt{1 - |\bar{r}_s|^2} dW_s^2 \\
 &\quad + \gamma_s^{-\frac{1}{2}} \nu_s^{-1} \left(-\frac{1}{2} \mu_s + \frac{3}{8} \sigma_s^2 \right) ds - \frac{1}{2} \gamma_s^{-\frac{1}{2}} \nu_s^{-1} \sigma_s dW_s^1 \\
 &\quad + \frac{1}{2} \gamma_s^{-\frac{1}{2}} \nu_s^{-1} \sigma_s \eta_s \bar{r}_s ds \\
 &= \alpha_s \left(-\rho_s - \frac{1}{2} \mu_s + \frac{3}{8} \sigma_s^2 + \frac{1}{2} \sigma_s \eta_s \bar{r}_s \right) ds \\
 &\quad + \alpha_s \left(\left(-\eta_s \bar{r}_s - \frac{1}{2} \sigma_s \right) dW_s^1 - \eta_s \sqrt{1 - |\bar{r}_s|^2} dW_s^2 \right). \tag{6.10}
 \end{aligned}$$

It follows from (6.9) and (6.10) that for all $s \in [t, T]$,

$$\begin{aligned}
 d[\gamma^{-\frac{1}{2}} \nu^{-1}, \nu \gamma]_s &= \gamma_s^{\frac{1}{2}} \left(-\eta_s \bar{r}_s - \frac{1}{2} \sigma_s \right) (\sigma_s + \eta_s \bar{r}_s) ds - \gamma_s^{\frac{1}{2}} \eta_s^2 (1 - |\bar{r}_s|^2) ds \\
 &= -\gamma_s^{\frac{1}{2}} \left(\frac{3}{2} \eta_s \sigma_s \bar{r}_s + \frac{1}{2} \sigma_s^2 + \eta_s^2 \right) ds. \tag{6.11}
 \end{aligned}$$

We then plug (6.9)–(6.11) into (6.8), which yields (6.7). \square

Proof of Lemma 2.7 We set $\alpha_s = \gamma_s^{-\frac{1}{2}} \nu_s^{-1}$, $s \in [t, T]$, and

$$\beta_s = d - \gamma_t x - \int_t^s X_r d(\nu_r \gamma_r), \quad s \in [t, T].$$

It then holds that $\bar{H}_s = \alpha_s \beta_s$, $s \in [t, T]$. We use Lemma 6.1 and substitute

$$-\gamma^{\frac{1}{2}} X = \bar{H} - \gamma^{-\frac{1}{2}} D$$

in (6.7) to obtain for all $s \in [t, T]$ that

$$\begin{aligned}
 d\overline{H}_s &= (\overline{H}_s - \gamma_s^{-\frac{1}{2}} D_s) \left(\mu_s + \rho_s - \frac{1}{2} \sigma_s \eta_s \overline{r}_s - \frac{1}{2} \sigma_s^2 \right) ds \\
 &\quad + (\overline{H}_s - \gamma_s^{-\frac{1}{2}} D_s) (\sigma_s + \eta_s \overline{r}_s) dW_s^1 + \eta_s \sqrt{1 - |\overline{r}_s|^2} dW_s^2 \\
 &\quad + \overline{H}_s \left(-\rho_s - \frac{1}{2} \mu_s + \frac{3}{8} \sigma_s^2 + \frac{1}{2} \sigma_s \eta_s \overline{r}_s \right) ds \\
 &\quad + \overline{H}_s \left(\left(-\eta_s \overline{r}_s - \frac{1}{2} \sigma_s \right) dW_s^1 - \eta_s \sqrt{1 - |\overline{r}_s|^2} dW_s^2 \right) \\
 &= -\gamma_s^{-\frac{1}{2}} D_s \left(\left(\mu_s + \rho_s - \frac{1}{2} \sigma_s \eta_s \overline{r}_s - \frac{1}{2} \sigma_s^2 \right) ds \right. \\
 &\quad \left. + (\sigma_s + \eta_s \overline{r}_s) dW_s^1 + \eta_s \sqrt{1 - |\overline{r}_s|^2} dW_s^2 \right) \\
 &\quad + \overline{H}_s \left(\left(\frac{1}{2} \mu_s - \frac{1}{8} \sigma_s^2 \right) ds + \frac{1}{2} \sigma_s dW_s^1 \right).
 \end{aligned}$$

This proves (2.19). In particular, \overline{H} satisfies an SDE that is linear in \overline{H} and $\gamma^{-\frac{1}{2}} D$. Due to the boundedness of $\rho, \mu, \sigma, \eta, \overline{r}$, the coefficients of the SDE are bounded. Moreover, $E[\int_t^T (\gamma_s^{-\frac{1}{2}} D_s)^2 ds] < \infty$ by (2.5), and $\overline{H}_t = \gamma_t^{-\frac{1}{2}} d - \gamma_t^{\frac{1}{2}} x$ (cf. (2.18)) is square-integrable. We thus obtain $E[\sup_{s \in [t, T]} \overline{H}_s^2] < \infty$ (see for instance Zhang [41, Theorems 3.2.2 and 3.3.1]).

We next prove that the cost functional (2.17) admits the representation (2.20). To this end, note that (2.18) implies for all $s \in [t, T]$ that

$$\begin{aligned}
 \gamma_s (X_s - \zeta_s)^2 &= (\gamma_s^{-\frac{1}{2}} D_s - \overline{H}_s - \gamma_s^{\frac{1}{2}} \zeta_s)^2 \\
 &= \gamma_s^{-1} D_s^2 - 2\gamma_s^{-\frac{1}{2}} D_s (\overline{H}_s + \gamma_s^{\frac{1}{2}} \zeta_s) + (\overline{H}_s + \gamma_s^{\frac{1}{2}} \zeta_s)^2.
 \end{aligned}$$

Due to the assumption (2.3) on ζ and the fact that $E[\sup_{s \in [t, T]} \overline{H}_s^2] < \infty$, we obtain $E_t[\int_t^T (\overline{H}_s + \gamma_s^{\frac{1}{2}} \zeta_s)^2 ds] < \infty$ a.s. Combining this with (2.5) and the Cauchy–Schwarz inequality yields $E_t[\int_t^T |\gamma_s^{-\frac{1}{2}} D_s (\overline{H}_s + \gamma_s^{\frac{1}{2}} \zeta_s)| ds] < \infty$ a.s. Since λ is bounded, we conclude that

$$\begin{aligned}
 E_t \left[\int_t^T \lambda_s \gamma_s (X_s - \zeta_s)^2 ds \right] &= E_t \left[\int_t^T \lambda_s \gamma_s^{-1} D_s^2 ds \right] \\
 &\quad + E_t \left[\int_t^T \lambda_s (\overline{H}_s + \gamma_s^{\frac{1}{2}} \zeta_s)^2 ds \right] \\
 &\quad - 2E_t \left[\int_t^T \lambda_s \gamma_s^{-\frac{1}{2}} D_s (\overline{H}_s + \gamma_s^{\frac{1}{2}} \zeta_s) ds \right], \quad (6.12)
 \end{aligned}$$

where all conditional expectations are well defined and finite. Moreover, (2.18) implies that $\gamma_T^{-\frac{1}{2}} D_T = \bar{H}_T + \gamma_T^{\frac{1}{2}} X_T$ and thus $\gamma_T^{-1} D_T^2 = (\bar{H}_T + \sqrt{\gamma_T} \xi)^2$. Inserting this and (6.12) into (2.17), we obtain (2.20). \square

Lemma 6.2 *Let $t \in [0, T]$ and $x, d \in \mathbb{R}$. Then (2.21) defines a metric on $\mathcal{A}_t^{\text{pm}}(x, d)$ (identifying any processes that are equal $(dP \times ds|_{[t, T]})$ -a.e.).*

Proof Note first that $\mathbf{d}(X, Y) \geq 0$ for all $X, Y \in \mathcal{A}_t^{\text{pm}}(x, d)$ and that $\mathbf{d}(X, Y)$ is finite due to (2.5). Symmetry of \mathbf{d} is obvious. The triangle inequality follows from the Cauchy–Schwarz inequality. Now take $X, Y \in \mathcal{A}_t^{\text{pm}}(x, d)$ with associated deviation processes D^X, D^Y . If $X = Y$ $(dP \times ds|_{[t, T]})$ -a.e., it follows that also $\gamma^{-\frac{1}{2}} D^X = \gamma^{-\frac{1}{2}} D^Y$ $(dP \times ds|_{[t, T]})$ -a.e. and thus

$$\mathbf{d}(X, Y) = \left(E \left[\int_t^T (\gamma_s^{-\frac{1}{2}} D_s^X - \gamma_s^{-\frac{1}{2}} D_s^Y)^2 ds \right] \right)^{\frac{1}{2}} = 0.$$

For the other direction, suppose that $\mathbf{d}(X, Y) = 0$. This implies that

$$\gamma^{-\frac{1}{2}} D^X - \gamma^{-\frac{1}{2}} D^Y = 0 \quad (dP \times ds|_{[t, T]})\text{-a.e.}$$

By the definition of D^X and D^Y , it further follows from multiplying by $\nu\gamma^{\frac{1}{2}}$ that

$$\nu_s \gamma_s (X_s - Y_s) = \int_t^s (X_r - Y_r) d(\nu_r \gamma_r) \quad (dP \times ds|_{[t, T]})\text{-a.e.} \quad (6.13)$$

Observe that $\nu\gamma > 0$. Define $L = (L_s)_{s \in [t, T]}$ by $L_t = 0$, $dL_s = \nu_s^{-1} \gamma_s^{-1} d(\nu_s \gamma_s)$ and notice that L has the form $dL_s = \cdots ds + \cdots dW_s^1 + \cdots dW_s^2$ by (6.9). Denote by K the right-hand side of (6.13) and observe that K is a continuous “version” of the process $\nu\gamma(X - Y)$, where the word “version” is understood in the sense that the processes are equal $(dP \times ds|_{[t, T]})$ -a.e. Using the above form of dL_s , we obtain from (6.13) that K satisfies the stochastic integral equation

$$K_s = \int_t^s K_r dL_r, \quad s \in [t, T].$$

This has the unique solution $K \equiv 0$. As $X - Y$ is a “version” of $\nu^{-1} \gamma^{-1} K$ (in the above sense), we conclude that $X = Y$ $(dP \times ds|_{[t, T]})$ -a.e. \square

We now prepare the proof of Theorem 2.8. The next result on the scaled hidden deviation is helpful there to show convergence of the cost functional.

Lemma 6.3 *Let $t \in [0, T]$, $x, d \in \mathbb{R}$ and $X \in \mathcal{A}_t^{\text{pm}}(x, d)$ with associated deviation process D and scaled hidden deviation process \bar{H} . Suppose in addition that $(X^n)_{n \in \mathbb{N}}$ is a sequence in $\mathcal{A}_t^{\text{pm}}(x, d)$ such that $\lim_{n \rightarrow \infty} E[\int_t^T (D_s^n - D_s)^2 \gamma_s^{-1} ds] = 0$ for the associated deviation processes D^n , $n \in \mathbb{N}$. For the associated scaled hidden deviation processes \bar{H}^n , $n \in \mathbb{N}$, we then have $\lim_{n \rightarrow \infty} E[\sup_{s \in [t, T]} (\bar{H}_s^n - \bar{H}_s)^2] = 0$.*

Proof Define $\delta \bar{H}^n = \bar{H}^n - \bar{H}$, $n \in \mathbb{N}$, and for $n \in \mathbb{N}$, $s \in [t, T]$, $z \in \mathbb{R}$, let

$$\begin{aligned} b_s^n(z) &= -\frac{1}{2}(2(\rho_s + \mu_s) - \sigma_s^2 - \sigma_s \eta_s \bar{r}_s)(\gamma_s^{-\frac{1}{2}} D_s^n - \gamma_s^{-\frac{1}{2}} D_s) + \frac{1}{2}\left(\mu_s - \frac{1}{4}\sigma_s^2\right)z, \\ a_s^n(z) &= \left(-(\sigma_s + \eta_s \bar{r}_s)(\gamma_s^{-\frac{1}{2}} D_s^n - \gamma_s^{-\frac{1}{2}} D_s) + \frac{1}{2}\sigma_s z, \right. \\ &\quad \left. - \eta_s \sqrt{1 - |\bar{r}_s|^2}(\gamma_s^{-\frac{1}{2}} D_s^n - \gamma_s^{-\frac{1}{2}} D_s)\right). \end{aligned}$$

In view of (2.19), it then holds for all $n \in \mathbb{N}$ that

$$d(\delta \bar{H}_s^n) = b_s^n(\delta \bar{H}_s^n)ds + a_s^n(\delta \bar{H}_s^n)d\begin{pmatrix} W_s^1 \\ W_s^2 \end{pmatrix}, \quad s \in [t, T], \quad \delta \bar{H}_t^n = 0.$$

Linearity of b^n , a^n , $n \in \mathbb{N}$, and boundedness of μ , ρ , σ , η , \bar{r} imply that there exists a constant $c_1 \in (0, \infty)$ such that for all $n \in \mathbb{N}$ and all $z_1, z_2 \in \mathbb{R}$, it holds $(dP \times ds|_{[t, T]})$ -a.e. that

$$\begin{aligned} |b^n(z_1) - b^n(z_2)| + |a^n(z_1) - a^n(z_2)| &\leq \frac{1}{2}\left|\mu - \frac{1}{4}\sigma^2\right||z_1 - z_2| + \frac{1}{2}|\sigma||z_1 - z_2| \\ &\leq c_1|z_1 - z_2|. \end{aligned}$$

By boundedness of μ , ρ , σ , η , \bar{r} and Jensen's inequality, there exists $c_2 \in (0, \infty)$ such that for all $n \in \mathbb{N}$, we have

$$E\left[\left(\int_t^T |b_s^n(0)|ds\right)^2\right] + E\left[\int_t^T |a_s^n(0)|^2 ds\right] \leq c_2 E\left[\int_t^T (D_s^n - D_s)^2 \gamma_s^{-1} ds\right].$$

For instance Zhang [41, Theorem 3.2.2] (see also [41, Theorem 3.4.2]) now implies that there exists a constant $c_3 \in (0, \infty)$ such that for all $n \in \mathbb{N}$, we have

$$\begin{aligned} E\left[\sup_{s \in [t, T]} |\bar{H}_s^n - \bar{H}_s|^2\right] &\leq c_3 E\left[\left(\int_t^T |b_s^n(0)|ds\right)^2 + \int_t^T |a_s^n(0)|^2 ds\right] \\ &\leq c_2 c_3 E\left[\int_t^T (D_s^n - D_s)^2 \gamma_s^{-1} ds\right], \end{aligned}$$

and the right-hand side tends to 0 by assumption. \square

In order to establish existence of an appropriate approximating sequence in Theorem 2.8, we rely on Lemma 6.4 below. For its statement and the proof of the second part of Theorem 2.8, we introduce the process $Z = (Z_s)_{s \in [0, T]}$ defined by

$$Z_s = \exp\left(-\int_0^s \left(\frac{1}{2}\sigma_u + \eta_u \bar{r}_u\right)dW_u^1 - \int_0^s \eta_u \sqrt{1 - |\bar{r}_u|^2} dW_u^2\right). \quad (6.14)$$

Observe that by Itô's lemma, Z solves the SDE

$$\begin{aligned} dZ_s &= \frac{Z_s}{2} \left(\left(\frac{1}{2} \sigma_s + \eta_s \bar{r}_s \right)^2 + \eta_s^2 (1 - |\bar{r}_s|^2) \right) ds \\ &\quad - Z_s \left(\frac{1}{2} \sigma_s + \eta_s \bar{r}_s, \eta_s \sqrt{1 - |\bar{r}_s|^2} \right) d \begin{pmatrix} W_s^1 \\ W_s^2 \end{pmatrix}, \quad s \in [0, T], \\ Z_0 &= 1. \end{aligned} \quad (6.15)$$

Lemma 6.4 *Let $t \in [0, T]$ and $u \in \mathcal{L}_t^2$. Then there exists a sequence $(v^n)_{n \in \mathbb{N}}$ of bounded càdlàg FV processes such that*

$$\lim_{n \rightarrow \infty} E \left[\int_t^T \left(\frac{u_s}{Z_s} - v_s^n \right)^2 Z_s^2 ds \right] = 0.$$

In particular, for the sequence $(u^n)_{n \in \mathbb{N}}$ defined by $u^n = v^n Z$, $n \in \mathbb{N}$, each u^n is a càdlàg semimartingale with $E[\sup_{s \in [t, T]} |u_s^n|^p] < \infty$ for all $p \geq 2$ (in particular, $u^n \in \mathcal{L}_t^2$), and

$$\lim_{n \rightarrow \infty} E \left[\int_t^T (u_s - u_s^n)^2 ds \right] = 0.$$

Proof Define A by $A_s = \int_0^s Z_r^2 dr$, $s \in [0, T]$, and v by $v_s = \frac{u_s}{Z_s}$, $s \in [t, T]$. We verify the assumptions of Karatzas and Shreve [33, Lemma 3.2.7]. The process A is continuous, adapted and nondecreasing. Note that boundedness of σ , η and \bar{r} implies that the coefficients of (6.15) are bounded. It follows for any $p \geq 2$ that $E[\sup_{s \in [0, T]} |Z_s|^p] < \infty$ (see e.g. [41, Theorem 3.4.3]) and hence

$$E[A_T] = E \left[\int_0^T Z_r^2 dr \right] < \infty.$$

Since $u \in \mathcal{L}_t^2$, we have that v is progressively measurable and satisfies

$$E \left[\int_t^T v_s^2 dA_s \right] = E \left[\int_t^T u_s^2 ds \right] < \infty.$$

Thus [33, Lemma 3.2.7] applies and yields the existence of a sequence $(\hat{v}^n)_{n \in \mathbb{N}}$ of (càdlàg) simple processes such that

$$\lim_{n \rightarrow \infty} E \left[\int_t^T (v_s - \hat{v}_s^n)^2 dA_s \right] = 0.$$

Define $v_s^n(\omega) = \lim_{r \downarrow s} \hat{v}_r^n(\omega)$, $s \in [t, T)$, $\omega \in \Omega$, $n \in \mathbb{N}$, and $v_T^n = 0$, $n \in \mathbb{N}$. Then $(v^n)_{n \in \mathbb{N}}$ is a sequence of bounded càdlàg FV processes such that

$$\lim_{n \rightarrow \infty} E \left[\int_t^T (v_s - v_s^n)^2 dA_s \right] = 0.$$

Note that for each $n \in \mathbb{N}$, u^n defined by $u_s^n = v_s^n Z_s$, $s \in [t, T]$, is càdlàg. Since v^n is bounded for each $n \in \mathbb{N}$ and $E[\sup_{s \in [0, T]} |Z_s|^p] < \infty$ for all $p \geq 2$, we have that $E[\sup_{s \in [t, T]} |u_s^n|^p]$ is finite for each $n \in \mathbb{N}$, $p \geq 2$. It furthermore holds that

$$E\left[\int_t^T (u_s - u_s^n)^2 ds\right] = E\left[\int_t^T (v_s - v_s^n)^2 dA_s\right] \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad \square$$

For the part in Theorem 2.8 on completeness of $(\mathcal{A}_t^{\text{pm}}(x, d), \mathbf{d})$, we show how to construct an execution strategy $X^0 \in \mathcal{A}_t^{\text{pm}}(x, d)$ based on a square-integrable process u^0 and a process H^0 that satisfies the SDE (2.19) (with u^0 instead of $\gamma^{-\frac{1}{2}}D$). This result is also crucial for Lemma 3.2.

Lemma 6.5 *Let $t \in [0, T]$ and $x, d \in \mathbb{R}$. Suppose that $u^0 = (u_s^0)_{s \in [t, T]} \in \mathcal{L}_t^2$ and let H^0 be given by $H_t^0 = \frac{d}{\sqrt{\gamma_t}} - \sqrt{\gamma_t}x$ and, for $s \in [t, T]$,*

$$\begin{aligned} dH_s^0 = & \left(\frac{1}{2} \left(\mu_s - \frac{1}{4} \sigma_s^2 \right) H_s^0 - \frac{1}{2} (2(\rho_s + \mu_s) - \sigma_s^2 - \sigma_s \eta_s \bar{r}_s) u_s^0 \right) ds \\ & + \left(\frac{1}{2} \sigma_s H_s^0 - (\sigma_s + \eta_s \bar{r}_s) u_s^0 \right) dW_s^1 - \eta_s \sqrt{1 - |\bar{r}_s|^2} u_s^0 dW_s^2. \end{aligned} \quad (6.16)$$

Define X^0 by

$$X_s^0 = \gamma_s^{-\frac{1}{2}} (u_s^0 - H_s^0), \quad s \in [t, T), \quad X_{t-}^0 = x, \quad X_T^0 = \xi.$$

Then $X^0 \in \mathcal{A}_t^{\text{pm}}(x, d)$, and the associated deviation process is $D^0 = \gamma X^0 + \gamma^{\frac{1}{2}} H^0$.

Proof First, X^0 is progressively measurable and has initial value $X_{t-}^0 = x$ and terminal value $X_T^0 = \xi$. Furthermore, it holds that

$$\int_t^T (X_s^0)^2 ds \leq 2 \int_t^T \gamma_s^{-1} (u_s^0)^2 ds + 2 \int_t^T \gamma_s^{-1} (H_s^0)^2 ds < \infty \quad \text{a.s.}$$

since γ and H^0 have a.s. continuous paths and $E[\int_t^T (u_s^0)^2 ds] < \infty$. We are therefore able to define D^0 by (2.15). Moreover, set $\alpha_s = \gamma_s^{-\frac{1}{2}} v_s^{-1}$, $s \in [t, T]$, and

$$\beta_s = d - \gamma_t x - \int_t^s X_r^0 d(v_r \gamma_r), \quad s \in [t, T].$$

It follows from Lemma 6.1 and $-\gamma_s^{\frac{1}{2}} X_s^0 = H_s^0 - u_s^0$, $s \in [t, T]$, that for $s \in [t, T]$,

$$\begin{aligned} d(\alpha_s \beta_s) &= (H_s^0 - u_s^0) \left(\left(\mu_s + \rho_s - \frac{1}{2} \sigma_s \eta_s \bar{r}_s - \frac{1}{2} \sigma_s^2 \right) ds \right. \\ &\quad \left. + (\sigma_s + \eta_s \bar{r}_s) dW_s^1 + \eta_s \sqrt{1 - |\bar{r}_s|^2} dW_s^2 \right) \\ &\quad + \alpha_s \beta_s \left(-\rho_s - \frac{1}{2} \mu_s + \frac{3}{8} \sigma_s^2 + \frac{1}{2} \sigma_s \eta_s \bar{r}_s \right) ds \\ &\quad + \alpha_s \beta_s \left(\left(-\eta_s \bar{r}_s - \frac{1}{2} \sigma_s \right) dW_s^1 - \eta_s \sqrt{1 - |\bar{r}_s|^2} dW_s^2 \right). \end{aligned}$$

We combine this with

$$\begin{aligned} dH_s^0 &= -u_s^0 \left(\left(\mu_s + \rho_s - \frac{1}{2} \sigma_s \eta_s \bar{r}_s - \frac{1}{2} \sigma_s^2 \right) ds \right. \\ &\quad \left. + (\sigma_s + \eta_s \bar{r}_s) dW_s^1 + \eta_s \sqrt{1 - |\bar{r}_s|^2} dW_s^2 \right) \\ &\quad + H_s^0 \left(\left(\frac{1}{2} \mu_s - \frac{1}{8} \sigma_s^2 \right) ds + \frac{1}{2} \sigma_s dW_s^1 \right), \quad s \in [t, T], \end{aligned}$$

to obtain for all $s \in [t, T]$ that

$$\begin{aligned} d(\alpha_s \beta_s - H_s^0) &= (\alpha_s \beta_s - H_s^0) \left(-\rho_s - \frac{1}{2} \mu_s + \frac{3}{8} \sigma_s^2 + \frac{1}{2} \sigma_s \eta_s \bar{r}_s \right) ds \\ &\quad + (\alpha_s \beta_s - H_s^0) \left(\left(-\eta_s \bar{r}_s - \frac{1}{2} \sigma_s \right) dW_s^1 - \eta_s \sqrt{1 - |\bar{r}_s|^2} dW_s^2 \right). \end{aligned} \quad (6.17)$$

Note that $\alpha_t \beta_t = \gamma_t^{-\frac{1}{2}} d - \gamma_t^{\frac{1}{2}} x = H_t^0$. We thus conclude that 0 is the unique solution of (6.17), and hence

$$H_s^0 = \alpha_s \beta_s = \gamma_s^{-\frac{1}{2}} v_s^{-1} \left(d - \gamma_t x - \int_t^s X_r^0 d(v_r \gamma_r) \right), \quad s \in [t, T].$$

This gives $D^0 = \gamma X^0 + \gamma^{\frac{1}{2}} H^0$, i.e., $D_s^0 = \gamma_s^{\frac{1}{2}} u_s^0$, $s \in [t, T]$ and $D_T^0 = \gamma_T \xi + \gamma_T^{\frac{1}{2}} H_T^0$. Then $E[\int_t^T (u_s^0)^2 ds] < \infty$ immediately yields (2.5), proving that $X^0 \in \mathcal{A}_t^{\text{pm}}(x, d)$. \square

We finally are able to prove Theorem 2.8.

Proof of Theorem 2.8 (i) Denote by D, D^n , $n \in \mathbb{N}$, the deviation processes associated to X, X^n , $n \in \mathbb{N}$, and let \bar{H} and \bar{H}^n , $n \in \mathbb{N}$, be the corresponding scaled hidden

deviation processes. By Lemma 2.7, we have for all $n \in \mathbb{N}$ that

$$\begin{aligned} & |J_t^{\text{pm}}(x, d, X^n) - J_t^{\text{pm}}(x, d, X)| \\ &= \left| \frac{1}{2} E_t \left[\int_t^T \gamma_s^{-1} ((D_s^n)^2 - D_s^2) 2(\kappa_s + \lambda_s) ds \right] \right. \\ &\quad \left. - 2E_t \left[\int_t^T \lambda_s \gamma_s^{-\frac{1}{2}} (D_s^n (\overline{H}_s^n + \gamma_s^{\frac{1}{2}} \xi_s) - D_s (\overline{H}_s + \gamma_s^{\frac{1}{2}} \xi_s)) ds \right] \right. \\ &\quad \left. + E_t \left[\int_t^T \lambda_s ((\overline{H}_s^n + \gamma_s^{\frac{1}{2}} \xi_s)^2 - (\overline{H}_s + \gamma_s^{\frac{1}{2}} \xi_s)^2) ds \right] \right. \\ &\quad \left. + \frac{1}{2} E_t [(\overline{H}_T^n + \gamma_T^{\frac{1}{2}} \xi)^2 - (\overline{H}_T + \gamma_T^{\frac{1}{2}} \xi)^2] \right|. \end{aligned}$$

Boundedness of $\lambda, \bar{r}, \rho, \mu, \eta$ and σ implies (recall also (2.13)) that there exists a constant $c \in (0, \infty)$ such that for all $n \in \mathbb{N}$,

$$\begin{aligned} & E[|J_t^{\text{pm}}(x, d, X^n) - J_t^{\text{pm}}(x, d, X)|] \\ &\leq E[|(\overline{H}_T^n + \gamma_T^{\frac{1}{2}} \xi)^2 - (\overline{H}_T + \gamma_T^{\frac{1}{2}} \xi)^2|] + cE \left[\int_t^T |\gamma_s^{-1} ((D_s^n)^2 - D_s^2)| ds \right] \\ &\quad + cE \left[\int_t^T |\gamma_s^{-\frac{1}{2}} (D_s^n (\overline{H}_s^n + \gamma_s^{\frac{1}{2}} \xi_s) - D_s (\overline{H}_s + \gamma_s^{\frac{1}{2}} \xi_s))| ds \right] \\ &\quad + cE \left[\int_t^T |(\overline{H}_s^n + \gamma_s^{\frac{1}{2}} \xi_s)^2 - (\overline{H}_s + \gamma_s^{\frac{1}{2}} \xi_s)^2| ds \right]. \end{aligned} \quad (6.18)$$

We treat the terminal costs first. For all $n \in \mathbb{N}$, we have

$$\begin{aligned} & E[|(\overline{H}_T^n + \gamma_T^{\frac{1}{2}} \xi)^2 - (\overline{H}_T + \gamma_T^{\frac{1}{2}} \xi)^2|] \\ &= E[|(\overline{H}_T^n)^2 + 2\overline{H}_T^n \gamma_T^{\frac{1}{2}} \xi - \overline{H}_T^2 - 2\overline{H}_T \gamma_T^{\frac{1}{2}} \xi|] \\ &\leq E[|(\overline{H}_T^n)^2 - \overline{H}_T^2|] + 2E[|(\overline{H}_T^n - \overline{H}_T) \gamma_T^{\frac{1}{2}} \xi|] \\ &\leq E[|(\overline{H}_T^n)^2 - \overline{H}_T^2|] + 2(E[(\overline{H}_T^n - \overline{H}_T)^2])^{\frac{1}{2}} (E[\gamma_T \xi^2])^{\frac{1}{2}}. \end{aligned}$$

From

$$\lim_{n \rightarrow \infty} E \left[\int_t^T (D_s^n - D_s)^2 \gamma_s^{-1} ds \right] = 0 \quad (6.19)$$

(cf. (2.21)) and Lemma 6.3, we have that

$$\lim_{n \rightarrow \infty} E \left[\sup_{s \in [t, T]} |\overline{H}_s^n - \overline{H}_s|^2 \right] = 0. \quad (6.20)$$

Since furthermore $E[\gamma_T \xi^2] < \infty$, we obtain that

$$\lim_{n \rightarrow \infty} E[|(\overline{H}_T^n + \gamma_T^{\frac{1}{2}} \xi)^2 - (\overline{H}_T + \gamma_T^{\frac{1}{2}} \xi)^2|] = 0.$$

The second term in (6.18) converges to 0 using (6.19). For the third term in (6.18), we have for all $n \in \mathbb{N}$ that

$$\begin{aligned} & E \left[\int_t^T |\gamma_s^{-\frac{1}{2}} (D_s^n (\overline{H}_s^n + \gamma_s^{\frac{1}{2}} \zeta_s) - D_s (\overline{H}_s + \gamma_s^{\frac{1}{2}} \zeta_s))| ds \right] \\ & \leq E \left[\int_t^T (|\overline{H}_s + \gamma_s^{\frac{1}{2}} \zeta_s| |D_s^n - D_s| \gamma_s^{-\frac{1}{2}} + \gamma_s^{-\frac{1}{2}} |D_s^n| |\overline{H}_s^n - \overline{H}_s|) ds \right] \\ & \leq \left(E \left[\int_t^T (\overline{H}_s + \gamma_s^{\frac{1}{2}} \zeta_s)^2 ds \right] \right)^{\frac{1}{2}} \left(E \left[\int_t^T (D_s^n - D_s)^2 \gamma_s^{-1} ds \right] \right)^{\frac{1}{2}} \\ & \quad + \left(E \left[\int_t^T \gamma_s^{-1} (D_s^n)^2 ds \right] \right)^{\frac{1}{2}} T^{\frac{1}{2}} \left(E \left[\sup_{s \in [t, T]} |\overline{H}_s^n - \overline{H}_s|^2 \right] \right)^{\frac{1}{2}}. \end{aligned} \quad (6.21)$$

Lemma 2.7 and (2.3) yield $E[\int_t^T (\overline{H}_s + \gamma_s^{\frac{1}{2}} \zeta_s)^2 ds] < \infty$. Moreover, due to (6.19), $E[\int_t^T \gamma_s^{-1} (D_s^n)^2 ds]$ is bounded uniformly in $n \in \mathbb{N}$. So it follows from (6.19)–(6.21) that the third term in (6.18) converges to 0 as $n \rightarrow \infty$. The last term in (6.18) converges to 0 using (2.3) and (6.20). This proves (i).

(ii) Suppose that $X \in \mathcal{A}_t^{\text{pm}}(x, d)$. Let u be defined by

$$u_s = \gamma_s^{-\frac{1}{2}} D_s, \quad s \in [t, T],$$

where D denotes the deviation process associated to X . Then u is progressively measurable, and (2.5) gives that $E[\int_t^T u_s^2 ds] < \infty$. By Lemma 6.4, there exists a sequence $(v^n)_{n \in \mathbb{N}}$ of bounded càdlàg FV processes such that

$$\lim_{n \rightarrow \infty} E \left[\int_t^T \left(\frac{u_s}{Z_s} - v_s^n \right)^2 Z_s^2 ds \right] = 0,$$

where Z is defined in (6.14). Set $u^n = v^n Z$, $n \in \mathbb{N}$. This is a sequence of càdlàg semimartingales in \mathcal{L}_t^2 which satisfies

$$\lim_{n \rightarrow \infty} \|u - u^n\|_{\mathcal{L}_t^2} = 0. \quad (6.22)$$

Moreover, for all $n \in \mathbb{N}$ and $p \geq 2$, we have $E[\sup_{s \in [t, T]} |u_s^n|^p] < \infty$. For each u^n , let H^n be the solution of (6.16). We then define a sequence of càdlàg semimartingales X^n , $n \in \mathbb{N}$, by

$$X_s^n = \gamma_s^{-\frac{1}{2}} (u_s^n - H_s^n), \quad s \in [t, T], \quad X_{t-}^n = x, \quad X_T^n = \xi.$$

By Lemma 6.5, $X^n \in \mathcal{A}_t^{\text{pm}}(x, d)$ for all $n \in \mathbb{N}$ and for the associated deviation process $D^n = \gamma X^n + \gamma^{\frac{1}{2}} H^n$. It follows for all $n \in \mathbb{N}$ that

$$D_s^n = \gamma_s^{\frac{1}{2}} u_s^n, \quad s \in [t, T].$$

Therefore, we have for all $n \in \mathbb{N}$ that

$$\mathbf{d}(X^n, X) = \left(E \left[\int_t^T (D_s^n - D_s)^2 \gamma_s^{-1} ds \right] \right)^{\frac{1}{2}} = \left(E \left[\int_t^T (u_s^n - u_s)^2 ds \right] \right)^{\frac{1}{2}}.$$

Due to (6.22), we thus have $\lim_{n \rightarrow \infty} \mathbf{d}(X^n, X) = 0$. We next show for all $n \in \mathbb{N}$ that X^n has finite variation. To this end, we observe that for all $n \in \mathbb{N}$ and $s \in [t, T)$, integration by parts gives

$$dX_s^n = \gamma_s^{-\frac{1}{2}} d(u_s^n - H_s^n) + (u_s^n - H_s^n) d\gamma_s^{-\frac{1}{2}} + d[\gamma^{-\frac{1}{2}}, u^n - H^n]_s. \quad (6.23)$$

Again using integration by parts and (6.15) yields for all $n \in \mathbb{N}$ and $s \in [t, T]$ that

$$\begin{aligned} du_s^n &= v_s^n dZ_s + Z_s dv_s^n + d[v^n, Z]_s \\ &= \frac{1}{2} u_s^n \left(\left(\frac{1}{2} \sigma_s + \eta_s \bar{r}_s \right)^2 + \eta_s^2 (1 - |\bar{r}_s|^2) \right) ds - u_s^n \left(\frac{1}{2} \sigma_s + \eta_s \bar{r}_s \right) dW_s^1 \\ &\quad - u_s^n \eta_s \sqrt{1 - |\bar{r}_s|^2} dW_s^2 + Z_s dv_s^n. \end{aligned}$$

This and (6.16) imply for all $n \in \mathbb{N}$ and $s \in [t, T]$ that

$$\begin{aligned} \gamma_s^{-\frac{1}{2}} d(u_s^n - H_s^n) &= \gamma_s^{-\frac{1}{2}} \left(\rho_s + \mu_s + \frac{1}{2} \eta_s^2 - \frac{3}{8} \sigma_s^2 \right) u_s^n ds \\ &\quad - \gamma_s^{-\frac{1}{2}} \left(\frac{1}{2} \mu_s - \frac{1}{8} \sigma_s^2 \right) H_s^n ds \\ &\quad + \gamma_s^{-\frac{1}{2}} \frac{1}{2} \sigma_s (u_s^n - H_s^n) dW_s^1 + \gamma_s^{-\frac{1}{2}} Z_s dv_s^n. \end{aligned} \quad (6.24)$$

Moreover, it follows from (6.2) for all $n \in \mathbb{N}$ and $s \in [t, T]$ that

$$\begin{aligned} (u_s^n - H_s^n) d\gamma_s^{-\frac{1}{2}} &= (u_s^n - H_s^n) \gamma_s^{-\frac{1}{2}} \left(-\frac{1}{2} \mu_s + \frac{3}{8} \sigma_s^2 \right) ds \\ &\quad - (u_s^n - H_s^n) \gamma_s^{-\frac{1}{2}} \frac{1}{2} \sigma_s dW_s^1. \end{aligned} \quad (6.25)$$

We combine (6.23)–(6.25) to obtain for all $n \in \mathbb{N}$ and $s \in (t, T)$ that

$$\begin{aligned} dX_s^n &= \gamma_s^{-\frac{1}{2}} u_s^n \left(\rho_s + \frac{1}{2} \mu_s + \frac{1}{2} \eta_s^2 \right) ds - \gamma_s^{-\frac{1}{2}} H_s^n \frac{1}{4} \sigma_s^2 ds + \gamma_s^{-\frac{1}{2}} Z_s dv_s^n \\ &\quad + d[\gamma^{-\frac{1}{2}}, u^n - H^n]_s. \end{aligned}$$

Since v^n has finite variation for all $n \in \mathbb{N}$, this representation shows that also X^n has finite variation for all $n \in \mathbb{N}$. Note that for all $n \in \mathbb{N}$, by Proposition 2.3, the process (2.4) associated to the càdlàg FV process X^n is nothing but D^n . Since η is bounded, there exists $c \in (0, \infty)$ such that for all $n \in \mathbb{N}$, we have

$$E\left[\left(\int_t^T (D_s^n)^4 \gamma_s^{-2} \eta_s^2 ds\right)^{\frac{1}{2}}\right] = E\left[\left(\int_t^T (u_s^n)^4 \eta_s^2 ds\right)^{\frac{1}{2}}\right] \leq c E\left[\sup_{s \in [t, T]} (u_s^n)^2\right] < \infty.$$

This implies (2.6). Similarly, by boundedness of σ , we obtain (2.7). We thus conclude that $X^n \in \mathcal{A}_t^{\text{fv}}(x, d)$ for all $n \in \mathbb{N}$.

(iii) Let $(X^n)_{n \in \mathbb{N}}$ be a Cauchy sequence in $(\mathcal{A}_t^{\text{pm}}(x, d), \mathbf{d})$. For $n \in \mathbb{N}$, we denote by D^n the deviation process associated to X^n . Then $(\gamma^{-\frac{1}{2}} D^n)_{n \in \mathbb{N}}$ is a Cauchy sequence in the Hilbert space $(\mathcal{L}_t^2, |\cdot|_{\mathcal{L}_t^2})$, and so there exists $u^0 \in \mathcal{L}_t^2$ such that

$$\lim_{n \rightarrow \infty} \|\gamma^{-\frac{1}{2}} D^n - u^0\|_{\mathcal{L}_t^2} = 0.$$

Define X^0 by $X_{t-}^0 = x$, $X_T^0 = \xi$, $X_s^0 = \gamma_s^{-\frac{1}{2}}(u_s^0 - H_s^0)$, $s \in [t, T]$, where H^0 is given by (6.16). By Lemma 6.5, it holds that $X^0 \in \mathcal{A}_t^{\text{pm}}(x, d)$. We furthermore obtain from Lemma 6.5 that the associated deviation process is $D^0 = \gamma X^0 + \gamma^{\frac{1}{2}} H^0$. By the definition of X^0 , this yields $\gamma_s^{-\frac{1}{2}} D_s^0 = u_s^0$, $s \in [t, T]$. It follows that

$$\mathbf{d}(X^n, X^0) = \left(E\left[\int_t^T (\gamma_s^{-\frac{1}{2}} D_s^n - \gamma_s^{-\frac{1}{2}} D_s^0)^2 ds\right]\right)^{\frac{1}{2}} = \|\gamma^{-\frac{1}{2}} D^n - u^0\|_{\mathcal{L}_t^2}$$

and hence $\lim_{n \rightarrow \infty} \mathbf{d}(X^n, X^0) = 0$. \square

Proof of Lemma 3.1 By its definition, u is progressively measurable and due to (2.5) satisfies $E[\int_t^T u_s^2 ds] < \infty$; hence $u \in \mathcal{L}_t^2$. Let

$$\overline{H}_s = \gamma_s^{-\frac{1}{2}} D_s - \gamma_s^{\frac{1}{2}} X_s, \quad s \in [t, T],$$

be the scaled hidden deviation (2.18) associated to X . We can substitute $u = \gamma^{-\frac{1}{2}} D$ in the cost functional (2.20) and also in the dynamics (2.19) of \overline{H} . Observe that \overline{H} follows the same dynamics as the state process \tilde{H} associated to u (see (3.1)), and that the initial values $\overline{H}_t = \frac{d}{\sqrt{\gamma_t}} - \sqrt{\gamma_t} x = \tilde{H}_t$ are equal. Therefore \overline{H} and \tilde{H} coincide, which completes the proof. \square

Proof of Lemma 3.2 It follows from Lemma 6.5 that $X \in \mathcal{A}_t^{\text{pm}}(x, d)$. Moreover, by Lemma 6.5, the associated deviation satisfies $D = \gamma X + \gamma^{\frac{1}{2}} \tilde{H}$, that is, it holds that

$$D_s = \gamma_s^{\frac{1}{2}} u_s, \quad s \in [t, T],$$

and \tilde{H} is the scaled hidden deviation of X . Thus $J_t^{\text{pm}}(x, d, X)$ is given by (2.20). In the definition (3.2) of J_t , we may replace u under the integrals with respect to

Lebesgue measure by $\gamma^{-\frac{1}{2}}D$. This shows that

$$J_t^{\text{pm}}(x, d, X) = J_t\left(\frac{d}{\sqrt{\gamma_t}} - \sqrt{\gamma_t}x, u\right) - \frac{d^2}{2\gamma_t}. \quad \square$$

Proof of Lemma 3.6 (i) First, \hat{u} is progressively measurable. Furthermore, using

$$E\left[\int_t^T u_s^2 ds\right] < \infty, \quad E\left[\sup_{s \in [t, T]} \tilde{H}_s^2\right] < \infty, \quad E\left[\int_0^T \gamma_s \zeta_s^2 ds\right] < \infty$$

and Assumption 3.5 implies that $E[\int_t^T \hat{u}_s^2 ds] < \infty$. Hence $\hat{u} \in \mathcal{L}_t^2$. Substituting

$$u_s = \hat{u}_s + \frac{\lambda_s}{\lambda_s + \kappa_s}(\tilde{H}_s + \sqrt{\gamma_s} \zeta_s), \quad s \in [t, T],$$

in (3.1) leads to (3.3). For the cost functional, observe that

$$\begin{aligned} & \frac{1}{2}(2\lambda_s(\tilde{H}_s + \sqrt{\gamma_s} \zeta_s)^2 - 4\lambda_s(\tilde{H}_s + \sqrt{\gamma_s} \zeta_s)u_s) + (\kappa_s + \lambda_s)u_s^2 \\ &= \lambda_s(\tilde{H}_s + \sqrt{\gamma_s} \zeta_s)^2 - (\lambda_s + \kappa_s)\frac{\lambda_s^2}{(\lambda_s + \kappa_s)^2}(\tilde{H}_s + \sqrt{\gamma_s} \zeta_s)^2 \\ & \quad + (\lambda_s + \kappa_s)\left(u_s - \frac{\lambda_s}{\lambda_s + \kappa_s}(\tilde{H}_s + \sqrt{\gamma_s} \zeta_s)\right)^2 \\ &= \frac{\lambda_s \kappa_s}{\lambda_s + \kappa_s}(\tilde{H}_s + \sqrt{\gamma_s} \zeta_s)^2 + (\lambda_s + \kappa_s)\hat{u}_s^2, \quad s \in [t, T]. \end{aligned} \quad (6.26)$$

(ii) Note that (3.3) is an SDE that is linear in \hat{H} , \hat{u} and $\sqrt{\gamma}\zeta$. Furthermore, the boundedness of ρ , μ , σ , η , \bar{r} and Assumption 3.5 imply that the coefficients of the SDE are bounded. Since moreover $E[\int_t^T ((\hat{u}_s)^2 + \gamma_s \zeta_s^2) ds] < \infty$ and \hat{H}_t is square-integrable, we get that $E[\sup_{s \in [t, T]} \hat{H}_s^2] < \infty$ (see e.g. [41, Theorems 3.2.2 and 3.3.1]). We can thus argue similarly to (i) that $u \in \mathcal{L}_t^2$. A substitution of \hat{u} in (3.3) yields (3.1). A reverse version of the calculation in (6.26) proves equality of the cost functionals. \square

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