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REGULAR ARTICLE



On limiting behaviors of stepwise multiple testing procedures

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Abstract

Stepwise multiple testing procedures have attracted several statisticians for decades and are also quite popular with statistics users because of their technical simplicity. The Bonferroni procedure has been one of the earliest and most prominent testing rules for controlling the familywise error rate (FWER). A recent article established that the FWER for the Bonferroni method asymptotically (i.e., when the number of hypotheses becomes arbitrarily large) approaches zero under any positively equicorrelated multivariate normal framework. However, similar results for the limiting behaviors of FWER of general stepwise procedures are nonexistent. The present work addresses this gap in a unified manner by elucidating that, under the multivariate normal setups with some correlation structures, the probability of rejecting one or more null hypotheses approaches zero asymptotically for any step-down procedure. Consequently, the FWER and power of the step-down procedures also tend to be asymptotically zero. We also establish similar limiting zero results on FWER of other popular multiple testing rules, e.g., Hochberg's and Hommel's procedures. It turns out that, within our chosen asymptotic framework, the Benjamini-Hochberg method can hold the FWER at a strictly positive level asymptotically under the equicorrelated normality.

Keywords Familywise error rate · Multiple testing under dependence · Stepwise procedures · Benjamini–Hochberg method · Holm's method · Hommel's procedure

Mathematics Subject Classification 62J15 · 62F03

1 Introduction

Large-scale multiple testing problems arising in various scientific disciplines often study correlated variables simultaneously. For example, in microRNA expression data, several genes may cluster into groups through their transcription processes and pos-

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sess high correlations (Goeman and Solari 2014). The data observed from different locations and time periods in public health studies are generally spatially or serially correlated (Chandra and Bhattacharya 2019). FMRI studies and multistage clinical trials also involve variables with complex and unknown dependencies (Derado et al. 2010). Consequently, the study of the effect of correlation on dependent test statistics in simultaneous inference problems has attracted considerable attention recently.

Benjamini and Yekutieli (2001) proved that the Benjamini–Hochberg procedure (Benjamini and Hochberg 1995) controls the false discovery rate (FDR) at the desired level under positive regression dependency. Sarkar (2002) established some general results on FDR control under dependence. Storey and Tibshirani (2003) proposed methodologies for estimating the FDR for dependent test statistics. Simultaneous testing methods under dependence have also been studied by Sun and Cai (2007), Efron (2007), Liu et al. (2016) among others. Efron (2010b) mentions that the correlation penalty on the summary statistics depends on the root mean square (RMS) of the correlations. Efron (2010a) contains an excellent review of the relevant literature. Finner and Roters (2001a) discussed the behavior of expected type I errors of multiple level- α single-step test procedures based on exchangeable test statistics. They also studied (Finner and Roters 2001b) asymptotic (i.e., when the number of hypotheses tends to infinity) properties of the supremum of the expected type I error rate (EER) for some FDR-controlling stepwise procedures under independence. Fan et al. (2012) proposed a method of dealing with correlated test statistics with a known covariance structure. They capture the association between correlated statistics using the principal eigenvalues of the covariance matrix. Fan and Han (2016) extended this work when the underlying dependence structure is unknown. Qiu et al. (2005) demonstrated that many FDR controlling procedures lose power significantly under dependence. Huang and Hsu (2007) remark that stepwise decision rules based on modeling of the dependence structure are in general superior to their counterparts that do not consider the correlation.

There is relatively little literature on the performance of FWER controlling procedures under dependence. Das and Bhandari (2021) have established that the Bonferroni FWER is asymptotically a convex function in correlation ρ under the equicorrelated normal framework. Consequently, they show that the Bonferroni FWER is bounded above by $\alpha(1 - \rho)$, α being the target level. Dey and Bhandari (2023a) have improved this result by showing that the Bonferroni FWER asymptotically goes to zero for any strictly positive ρ . They have also extended this to arbitrarily correlated setups where the limiting infimum of the correlations is strictly positive. Dey (2022) has obtained upper bounds on the Bonferroni FWER in the equicorrelated and general setups with small and moderate dimensions. Finner and Roters (2002) derived explicit formulas for the distribution of the number of falsely rejected hypotheses in single-step, stepdown and step-up methods under the assumption of independent *p*-values. However, the role of correlation on the limiting behavior of the FWER for stepwise procedures is much less explored.

The present work addresses this problem by theoretically investigating the limiting FWER values of general step-down procedures under the correlated normal setup. These results provide new insights into the behavior of step-down decision procedures. By establishing the limiting performances of commonly used step-up methods, e.g.,

MTP	Limiting results on FWER	Dependence	
Bonferroni	Theorem 2.1 (Das and Bhandari 2021)	Equicorrelated	
	Theorem 2.2 (Dey and Bhandari 2023a)	Equicorrelated	
	Theorem 2.3 (Dey and Bhandari 2023a)	Non-negatively correlated	
Holm	Theorem 3.1 (Dey and Bhandari 2023a)	Equicorrelated	
	Theorem 3.2	Non-negatively correlated	
General step-down	Theorem 3.3	Non-negatively correlated	
Hochberg	Theorems 4.1, 4.2	Equicorrelated	
Benjamini-Hochberg	Theorem 4.3	Equicorrelated	
General step-up	Theorem 4.5	Equicorrelated	
Hommel	Theorems 5.1, 5.2, 5.3	Equicorrelated	

 Table 1
 Results on limiting behaviors of various FWER-controlling MTPs under correlated normal setup (the theorem numbers are as in this paper)

the Benjamini–Hochberg method and the Hochberg method, we have elucidated that the class of step-up procedures does not possess a similar *universal asymptotic zero* result as obtained in the case of step-down procedures. It is also noteworthy that most of our results are quite general since they accommodate any combination of true and false null hypotheses. We have also obtained the limiting powers of the stepwise procedures. Table 1 summarizes all the limiting FWER Results of various FWERcontrolling MTPs in different correlated normal setups.

This paper is structured as follows. We first formally introduce the framework with relevant notations and summarize some results on the limiting behavior of the Bonferroni procedure in the next section. Section 3 studies in detail the limiting behaviors of the FWER of step-down procedures in equicorrelated and some general normal setups. Section 4 is dedicated to similar results on Hochberg's and Benjamini–Hochberg procedures. Hommel's stepwise procedure is studied in Sect. 5. We present simulation studies in Sect. 6. We outline our contributions and discuss related problems briefly in Sect. 7.

2 Preliminaries

2.1 Testing framework

Here we discuss the simultaneous inference problem through a *Gaussian sequence model* framework (Das and Bhandari 2021; Dey and Bhandari 2023a; Dey 2022; Finner and Roters 2001a; Finner et al. 2007):

$$X_i \sim N(\mu_i, 1), \quad i \in \{1, ..., n\}$$

where X_i 's are dependent. Here the vector (X_1, \ldots, X_n) is assumed to follow the multivariate normal distribution. The variances are taken to be unity since the literature

on the asymptotic multiple testing theory often assumes that the variances are known [see, e.g., Abramovich et al. (2006); Bogdan et al. (2011); Das and Bhandari (2021); Dey and Bhandari (2023a); Donoho and Jin (2004)]. We are interested in the following multiple testing problem:

$$H_{0i}: \mu_i = 0$$
 vs $H_{1i}: \mu_i > 0, 1 \le i \le n.$

The intersection null hypothesis (also called the global null) $H_0 = \bigcap_{i=1}^n H_{0i}$ states that each μ_i is zero. Let \mathcal{A} denote the set of indices from $\{1, \ldots, n\}$ for which H_{0i} is true. So, under the global null, \mathcal{A} is $\{1, \ldots, n\}$. Throughout this work, $\Phi(\cdot)$ denotes the cumulative distribution function of N(0, 1) distribution and $\alpha \in (0, 1)$ denotes the target level of FWER control.

Let $R_n(T)$, $S_n(T)$ and $V_n(T)$ respectively denote the number of rejected hypotheses, the number of true rejections, and the number of type I errors of a multiple testing procedure (MTP henceforth) T and α be the desired level of FWER control. So, $R_n(T) = S_n(T) + V_n(T)$. The FWER of procedure T is given by

$$FWER_T(n,\alpha,\Sigma_n) = \mathbb{P}_{\Sigma_n}(V_n(T) \ge 1)$$
(1)

where Σ_n is the covariance matrix of (X_1, \ldots, X_n) . It is noteworthy that FWER is *not* the probability of making any type I errors when the global null hypothesis H_0 is true. A MTP is said to have *weak control* of the FWER if the FWER is less than or equal to the test level under the global null hypothesis. It has *strong control* of the FWER if the FWER is less than or equal to the level of the test under any configuration of true and false null hypotheses. In many of our results, we shall consider the probability in the r.h.s of (1) under the intersection null H_0 at first (and take that as the definition of FWER) for the sake of technical simplicity. Then we shall extend the results obtained in this case to any combination of true and false null hypotheses.

The present work studies the limiting behaviors of $FWER_T$ for T belonging to a broad class of MTPs under two dependent setups:

- 1. The equicorrelated setup: The covariance matrix of (X_1, \ldots, X_n) is Γ_n . Γ_n has each off-diagonal entry equal to $\rho \ge 0$.
- 2. The non-negatively correlated setup: The covariance matrix of (X_1, \ldots, X_n) is Σ_n . Σ_n has non-negative off-diagonal entries. We denote the (i, j)'th entry of Σ_n as ρ_{ij} .

The equicorrelated setup (Cohen et al. 2009; Das and Bhandari 2021; Dey and Bhandari 2023a; Dey 2022; Finner and Roters 2001a; Finner et al. 2007; Roy and Bhandari 2024) is the intraclass covariance matrix model, characterizing the exchangeable situation. Although this is a special case of the second one, we are considering them separately since the proof of the result in the general case is based on the corresponding results in the equicorrelated case. The equicorrelated setup also encompasses the problem of comparing a control against several treatments. However, many scientific disciplines involve variables with more complex dependence structure (e.g., fMRI studies). These complex dependence scenarios need to be tackled with more general covariance matrices (Dey and Bhandari 2023a; Dey 2022). The second setup also includes the successive correlation covariance matrix, which covers change point problems (Cohen et al. 2009).

As with the notions of type I and type II error rates, the concept of power can be extended in various ways when moving from single to multiple hypothesis testing (Dudoit and Laan 2008). One such notion of power is AnyPwr (Dudoit and Laan 2008), which is the probability of rejecting at least one false null hypothesis. So, for a MTP *T*,

$$AnyPwr_T = \mathbb{P}(S_n(T) \ge 1).$$

Throughout this work, Γ_n denotes the $n \times n$ matrix with each diagonal entry equal to 1 and each off-diagonal entry equal to ρ . Also, Σ_n denotes the $n \times n$ correlation matrix with (i, j)'th entry equal to $\rho_{ij}, i \neq j$.

2.2 The Bonferroni procedure

The classic Bonferroni procedure is the best-known and one of the most frequently used MTP for controlling FWER. This single-step method sets the same cut-off for all the hypotheses. In one-sided settings, it rejects H_{0i} if $X_i > \Phi^{-1}(1 - \alpha/n) (= c_{Bon}, \text{say})$. So the Bonferroni FWER (for the covariance matrix Σ_n) is defined by

$$FWER_{Bon}(n, \alpha, \Sigma_n) = \mathbb{P}_{\Sigma_n} (X_i > c_{Bon} \text{ for some } i \in \mathcal{A}) = \mathbb{P}_{\Sigma_n} \left(\bigcup_{i \in \mathcal{A}} \{X_i > c_{Bon}\} \right).$$

We write $FWER_T(n, \alpha, \Gamma_n)$ as $FWER_T(n, \alpha, \rho)$ for simpler notation. Das and Bhandari (2021) obtain the following in the equicorrelated case:

Theorem 2.1 (Das and Bhandari 2021) Given any $\alpha \in (0, 1)$ and $\rho \in [0, 1]$, $FWER_{Bon}(n, \alpha, \rho)$ is asymptotically bounded by $\alpha(1 - \rho)$ under the global null hypothesis.

Dey and Bhandari (2023a) improve this result as follows:

Theorem 2.2 (Dey and Bhandari 2023a) *Given any* $\alpha \in (0, 1)$ *and* $\rho \in (0, 1]$ *, we have*

$$\lim_{n \to \infty} FWER_{Bon}(n, \alpha, \rho) = 0$$

under any configuration of true and false null hypotheses.

The proofs of Theorems 2.1 and 2.2 exploit an well known result on equicorrelated multivariate normal variables with equal marginal variances. Under the global null hypothesis, the sequence $\{X_r\}_{r\geq 1}$ is exchangeable in the equicorrelated normal setup. In other words,

$$(X_{i_1},\ldots,X_{i_k}) \sim N_k \left(\mathbf{0}_k,(1-\rho)I_k+\rho J_k\right)$$

where J_k is the $k \times k$ matrix of all ones. Thus, for each $i \ge 1$, $X_i = \theta + Z_i$ where θ has a normal distribution with mean 0, independent of $\{Z_n\}_{n\ge 1}$ and Z_i 's are i.i.d normal random variables. Cov $(X_i, X_j) = \rho$ gives $\operatorname{Var}(\theta) = \rho$. Hence, $\theta \sim N(0, \rho)$ and $Z_i \stackrel{iid}{\sim} N(0, 1 - \rho)$ for each $i \ge 1$.

The following result extends Theorem 2.2 to general correlated normal setups:

Theorem 2.3 (Dey and Bhandari 2023a) Let Σ_n be the correlation matrix of X_1, \ldots, X_n with (i, j)'th entry ρ_{ij} such that $\liminf \rho_{ij} = \delta > 0$. Then, for any $\alpha \in (0, 1)$, we have

$$\lim_{n \to \infty} FWER_{Bon}(n, \alpha, \Sigma_n) = 0$$

under any configuration of true and false null hypotheses.

Theorem 2.3, a much stronger result than Theorem 2.1, highlights the fundamental problem of using Bonferroni method in a simultaneous testing problem. Dey and Bhandari (2023a) establish Theorem 2.3 using a famous inequality due to Slepian:

Theorem 2.4 (Slepian 1962) Let **X** follow $N_k(\mathbf{0}, \Sigma)$, where Σ is a $k \times k$ correlation matrix. Let $\mathbf{a} = (a_1, \ldots, a_k)'$ be an arbitrary but fixed real vector. Consider the quadrant probability

$$g(k, \mathbf{a}, \Sigma) = \mathbb{P}_{\Sigma}\left[\bigcap_{i=1}^{k} \{X_i \leq a_i\}\right].$$

Let $R = (\rho_{ij})$ and $T = (\tau_{ij})$ be two positive semidefinite correlation matrices. If $\rho_{ij} \ge \tau_{ij}$ holds for all i, j, then $g(k, \mathbf{a}, R) \ge g(k, \mathbf{a}, T)$, i.e

$$\mathbb{P}_{R}\left[\bigcap_{i=1}^{k} \{X_{i} \leqslant a_{i}\}\right] \geqslant \mathbb{P}_{T}\left[\bigcap_{i=1}^{k} \{X_{i} \leqslant a_{i}\}\right]$$

holds for all $\mathbf{a} = (a_1, \dots, a_k)'$. Moreover, the inequality is strict if R, T are positive definite and if the strict inequality $\rho_{ij} > \tau_{ij}$ holds for some i, j.

Dey and Bhandari (2023a) showed the following regarding the asymptotic power of Bonferroni's method:

Theorem 2.5 (Dey and Bhandari 2023a) Consider the equicorrelated normal setup with equicorrelation $\rho \in (0, 1)$. Suppose $\sup \mu_i$ is finite. Then, for any $\alpha \in (0, 1)$, Any Pwr_{Bon} goes to zero as $n \to \infty$.

Throughout this work, P_i denotes the *p*-value corresponding to the *i*-th null hypothesis H_{0i} , $1 \le i \le n$. Also, let $P_{(1)} \le \ldots \le P_{(n)}$ be the ordered *p*-values. Let the null hypothesis corresponding to the p-value $P_{(i)}$ be denoted as $H_{(0i)}$, $1 \le i \le n$.

2.3 Step-down and step-up procedures

Single-step MTPs (e.g., Bonferroni's method, Sidak's method) compare the individual test statistics to the corresponding cut-offs simultaneously, and they stop after performing this simultaneous 'joint' comparison. Often stepwise methods possess greater power than the single-step procedures, while still controlling FWER (or, in general, the error rate under consideration) at the desired level.

Consider the set

$$\mathcal{S}_n = \left\{ \mathbf{t} = (t_1, \dots, t_n) \in \mathbb{R}^n : 0 \le t_1 \le \dots \le t_n \le 1 \right\}.$$

A *p*-value based step-down MTP uses a vector of cutoffs $\mathbf{u} = (u_1, \ldots, u_n) \in S_n$, and works as follows. The step-down method rejects a hypothesis $H_{(i)}$ if and only if $P_{(j)} \leq u_j$ for all $j \leq i$. In other words, the step-down MTP compares the most significant *p*-value $P_{(1)}$ with the smallest *u*-value u_1 at first and so on. One can also formally describe a step-down MTP as follows. Let $m_1 = \max \{i : P_{(j)} \leq u_j \text{ for} all \ j = 1, \ldots, i\}$. Then the step-down procedure based on critical values **u** rejects $H_{(01)}, \ldots, H_{(0m_1)}$.

Example 1 The Bonferroni method is a step-down procedure with $u_i = \alpha/n$, i = 1, ..., n.

Example 2 The Sidak method is a step-down MTP with $u_i = 1 - (1 - \alpha)^{1/n}$, i = 1, ..., n.

Example 3 The Holm (1979) method is a popular step-down MTP with $u_i = \alpha/(n - i + 1)$, i = 1, ..., n.

Example 4 Benjamini and Liu (1999a) introduced a step-down MTP with

$$u_i = \min\left(1, \frac{nq}{(n-i+1)^2}\right), \quad 1 \le i \le n \quad (0 < q < 1).$$

Example 5 Benjamini and Liu (1999b) studied another step-down MTP with

$$u_i = 1 - \left[1 - \min\left(1, \frac{nq}{n-i+1}\right)\right]^{1/(n-i+1)}, \quad 1 \le i \le n \quad (0 < q < 1).$$

Example 6 Benjamini and Liu (1999b) mentioned a Holm-type procedure with the critical values

$$u_i = 1 - (1 - q)^{1/(n - i + 1)}, \quad 1 \le i \le n \quad (0 < q < 1).$$

The step-up MTP also utilizes a set of critical values, say $\mathbf{u} = (u_1, \ldots, u_n) \in S_n$. But the step-up method is inherently different from the step-down method in the sense that it starts by comparing the least significant *p*-value $P_{(n)}$ with the largest *u*-value u_n and so on. Formally, the step-up method based on critical values \mathbf{u} rejects the hypotheses $H_{(01)}, \ldots, H_{(0m_2)}$, where $m_2 = \max \{i : P_{(i)} \le u_i\}$. If such a m_2 does not exist, then the procedure does not reject any null hypothesis.

Example 7 The Bonferroni correction is also a step-up MTP, where $u_i = \alpha/n$, i = 1, ..., n.

Example 8 The Sidak method is also a step-up procedure with $u_i = 1 - (1 - \alpha)^{1/n}$, i = 1, ..., n.

Example 9 Hochberg (1988) method is a popular step-up MTP with $u_i = \alpha/(n-i+1)$.

Example 10 The classic Benjamini and Hochberg (1995) method is a step-up procedure with $u_i = i\alpha/n$.

3 Limiting FWER of step-down procedures

Holm method (1979) is a step-down MTP which uses adjusted *p*-values and utilizes the Bonferroni inequality. It controls the FWER under any dependence of the test statistics. The following result is known on the limiting FWER of Holm's method under the equicorrelated normal framework:

Theorem 3.1 (Dey and Bhandari 2023a) Suppose $\mu^* = \sup \mu_i < \infty$. Then, under any configuration of true and false null hypotheses, we have

$$\lim_{n \to \infty} FWER_{Holm}(n, \alpha, \rho) = 0 \text{ for all } \alpha \in (0, 1) \text{ and } \rho \in (0, 1].$$

We extend this result to some non-negatively correlated normal setups:

Theorem 3.2 Let Σ_n be the correlation matrix of X_1, \ldots, X_n with (i, j)'th entry ρ_{ij} such that $\liminf \rho_{ij} = \delta > 0$. Suppose $\mu^* = \sup \mu_i < \infty$. Then, for any $\alpha \in (0, 1)$,

$$\lim_{n\to\infty}\mathbb{P}_{\Sigma_n}\bigg(R_n(Holm)\geq 1\bigg)=0$$

under any configuration of true and false null hypotheses. Consequently,

$$\lim_{n \to \infty} FWER_{Holm}(n, \alpha, \Sigma_n) = 0 \quad and \quad \lim_{n \to \infty} AnyPwr_{Holm}(n, \alpha, \Sigma_n) = 0.$$

We extend Theorem 3.2 to any step-down MTP below:

Theorem 3.3 Let Σ_n be the correlation matrix of X_1, \ldots, X_n with (i, j)'th entry ρ_{ij} such that $\liminf \rho_{ij} = \delta > 0$. Suppose $\sup \mu_i$ is finite and T is any step-down MTP controlling FWER at level $\alpha \in (0, 1)$. Then, for any $\alpha \in (0, 1)$,

$$\lim_{n\to\infty}\mathbb{P}_{\Sigma_n}\bigg(R_n(T)\geq 1\bigg)=0$$

under any configuration of true and false null hypotheses. Consequently,

 $\lim_{n\to\infty} FWER_T(n,\alpha,\Sigma_n) = 0 \quad and \quad \lim_{n\to\infty} AnyPwr_T(n,\alpha,\Sigma_n) = 0.$

Theorem 3.3 can be viewed as a *universal asymptotic zero* result since it encompasses all step-down FWER controlling procedures and also accommodates any configuration of true and false null hypotheses.

4 Limiting FWER of some step-up procedures

Let us consider a step-down procedure T_1 and a step-up procedure T_2 having the identical vector of cutoffs $\mathbf{u} = (u_1, \ldots, u_n) \in S_n$. We always have $m(T_1) \le m(T_2)$ where $m(T_1) = \max \{i : P_{(j)} \le u_j \text{ for all } j = 1, \ldots, i\}$ and $m(T_2) = \max \{i : P_{(i)} \le u_i\}$. This implies that the step-up MTP is at least as rejective as the step-down MTP (which uses the same cutoffs). This observation steers that we might not get a similar *uni-versal asymptotic zero* result for the class of step-up MTPs as obtained in the case of step-down procedures (Theorem 3.3). This is indeed the case as we shall show in the next two subsections the following:

- 1. Under the equicorrelated Gaussian sequence model, the FWER of Hochberg (1988) procedure asymptotically approaches zero as the number of tests becomes arbitrarily large.
- 2. Under the equicorrelated Gaussian sequence model and under H_0 , the Benjamini and Hochberg (1995) method with a pre-specified FDR level controls FDR at some strictly positive quantity which is a function of the chosen FDR level and the common correlation, even when the number of tests approaches infinity.

We have considered Hochberg's MTP in particular because it uses the same vector of cutoffs as Holm's MTP (note that Holm's MTP has the 'optimal' critical values in the class of step-down procedures).

Benjamini–Hochberg method, on the other hand, has been one of the most eminent MTPs proposed in the literature and also possesses some optimality properties both in frequentist and Bayesian paradigms of simultaneous inference. Bogdan et al. (2011) showed that the BH method is asymptotically Bayes optimal under sparsity in the normal scale mixture model under certain conditions on sparsity. In the classical paradigm, Guo and Rao (2008) proved that among all FDR-controlling methods belonging to a certain class, the BH method has the largest cut-offs for the *p*-values is the most rejective (the formal statement is given in Theorem 4.4).

4.1 Hochberg's procedure

Hochberg's (1988) MTP and Holm's sequentially rejective procedure use the same set of cutoffs; and hence, as mentioned earlier, Hochberg's method is sharper than Holm's MTP. Holm's MTP rejects a hypothesis only if its p-value and each of the smaller p-values are less than their corresponding cutoffs. Hochberg's method rejects

all hypotheses with smaller or equal *p*-values to that of any one found less than its cutoff.

The following result depicts the limiting behavior of the FWER of Hochberg's procedure under the correlated Gaussian sequence model:

Theorem 4.1 Consider the equicorrelated normal setup with correlation $\rho \in [0, 1)$. *Then,*

1. When $\rho = 0$ (i.e., the independent normal setup), we have

$$\lim_{n \to \infty} FWER_{Hochberg}(n, \alpha, 0) \in [1 - e^{-\alpha}, \alpha]$$

under the global null hypothesis.

2. When $\rho \in (0, 1)$, we have

$$\lim_{n \to \infty} FWER_{Hochberg}(n, \alpha, \rho) = 0$$

for any $\alpha \in (0, 1/2)$, under the global null hypothesis.

We now consider the free-combination condition (Holm 1979) under which any combination of the true and false hypotheses is possible.

Theorem 4.2 Consider the multiple testing problem under the equicorrelated normal setup with equicorrelation $\rho \in (0, 1)$. Suppose $\sup \mu_i$ is finite. Then,

$$\lim_{n\to\infty}\mathbb{P}_{\Gamma_n}\bigg(R_n(Hochberg)\geq 1\bigg)=0$$

for any $\alpha \in (0, 1/2)$. Consequently, for any $\alpha \in (0, 1/2)$,

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$$\lim_{n \to \infty} FWER_{Hochberg}(n, \alpha, \Gamma_n) = \lim_{n \to \infty} AnyPwr_{Hochberg}(n, \alpha, \Gamma_n) = 0.$$

4.2 Benjamini–Hochberg procedure

The Benjamini–Hochberg method is the first FDR controlling procedure (Finner et al. 2007). Originally shown to be a valid FDR controlling method for independent *p*-values, it controls the FDR even if the test statistics exhibit some special dependence structure (e.g., the positive regression dependent setup). Formal treatments of these conditions and proofs can be found in Benjamini and Yekutieli (2001) and Sarkar (2002). Let i_{max} be the largest such *i* for which $p_{(i)} \leq i\alpha/n$. The BH procedure rejects $H_{0(i)}$ if $i \leq i_{\text{max}}$ and accepts $H_{0(i)}$ otherwise. Dey and Bhandari (2023a) evaluated the limiting FDR of Benjamini–Hochberg method. However, their proof had a technical gap. In their paper, it was incorrectly mentioned that $FWER_{BH}(n, \alpha, \rho) \leq \mathbb{P}_{\Gamma_n}(P_{(1)} \leq \alpha/n)$ under the global null. The correct identity is given by

$$FWER_{BH}(n,\alpha,\rho) = \mathbb{P}_{\Gamma_n}\left[\bigcup_{i=1}^n \left\{P_{(i)} < \frac{i\alpha}{n}\right\}\right]$$

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under the global null. We derive this identity in the proof of the following theorem. The revised statement on the limiting FDR of BH method along with its correct proof is given below:

Theorem 4.3 Consider the multiple testing problem under the equicorrelated normal setup with correlation ρ . Then, under the global null, for all $\alpha \in (0, 1)$ and for all $\rho \in (0, 1)$,

$$\lim_{n \to \infty} FDR_{BH}(n, \alpha, \rho) = 1 - \Phi\left[\inf_{t \in (0, 1)} \frac{\Phi^{-1}(1 - t\alpha) - \sqrt{1 - \rho} \cdot \Phi^{-1}(1 - t)}{\sqrt{\rho}}\right] > 0.$$

Also, $\lim_{n \to \infty} FDR_{BH}(n, \alpha, 0) = \lim_{n \to \infty} FDR_{BH}(n, \alpha, 1) = \alpha.$

Proof of Theorem 4.3 We have

$$FDR_{BH} = \mathbb{E}\left[\frac{V_n(BH)}{\max\left\{R_n(BH), 1\right\}}\right] = \mathbb{E}\left[\frac{V_n(BH)}{R_n(BH)} \mid V_n(BH) > 0\right] \mathbb{P}\left(V_n(BH) > 0\right).$$

Under the global null H_0 , all R_n rejected hypotheses are false rejections, hence $V_n(BH)/R_n(BH) = 1$ and FDR equals FWER. We shall work with FWER for the rest of this proof.

Suppose exactly n_0 null hypotheses are true. Then, it is a well-known fact (Benjamini and Hochberg 1995; Efron 2010a; Sarkar 2002) that, under the independent setup,

$$FDR_{BH}(n,\alpha,\rho) = \frac{n_0}{n}\alpha.$$

So, under the global null, $FDR_{BH}(n, \alpha, 0) = FWER_{BH}(n, \alpha, 0) = \alpha$. Now,

$$p_{(i)} \leq \frac{i\alpha}{n} \iff 1 - \Phi\left(X_{(n-i+1)}\right) \leq \frac{i\alpha}{n}$$
$$\iff 1 - \frac{i\alpha}{n} \leq \Phi\left(X_{(n-i+1)}\right)$$
$$\iff X_{(n-i+1)} \geq \Phi^{-1}\left(1 - \frac{i\alpha}{n}\right).$$

Consequently,

$$FWER_{BH}(n,\alpha,\rho) = \mathbb{P}_{\Gamma_n} \left[\bigcup_{i=1}^n \left\{ P_{(i)} < \frac{i\alpha}{n} \right\} \right]$$
$$= \mathbb{P}_{\Gamma_n} \left[\bigcup_{i=1}^n \left\{ X_{(n-i+1)} \ge \Phi^{-1} \left(1 - \frac{i\alpha}{n} \right) \right\} \right].$$

When $\rho = 1$, $X_i = X_j$ w.p 1. This implies

$$FWER_{BH}(n,\alpha,\rho) = \mathbb{P}\left[\bigcup_{i=1}^{n} \left\{ X \ge \Phi^{-1}\left(1-\frac{i\alpha}{n}\right) \right\}\right]$$
$$= \mathbb{P}\left[X \ge \Phi^{-1}\left(1-\alpha\right)\right] = \alpha \quad (X \sim N(0,1)).$$

Consider the case $0 < \rho < 1$ now. Then, $X_i = U_\rho + Z_i$ where $U_\rho \sim N(0, \rho)$ is independent of $Z_i \sim N(0, 1 - \rho)$. Here Z_i 's are i.i.d. So, under the global null,

$$\begin{aligned} FWER_{BH}(n,\alpha,\rho) = &\mathbb{P}_{\Gamma_n} \left[\bigcup_{i=1}^n \left\{ X_{(n-i+1)} \ge \Phi^{-1} \left(1 - \frac{i\alpha}{n} \right) \right\} \right] \\ = &\mathbb{P}_{\Gamma_n} \left[\bigcup_{i=1}^n \left\{ U_\rho + Z_{(n-i+1)} \ge \Phi^{-1} \left(1 - \frac{i\alpha}{n} \right) \right\} \right] \\ = &\mathbb{P}_{\Gamma_n} \left[\bigcup_{i=1}^n \left\{ U_\rho > \Phi^{-1} \left(1 - t_i \alpha \right) - Z_{(n-nt_i+1)} \right\} \right] \quad \text{where} \quad t_i = i/n. \\ = &\mathbb{P}_{\Gamma_n} \left[\bigcup_{i=1}^n \left\{ M > \frac{\Phi^{-1} \left(1 - t_i \alpha \right) - Z_{(n-nt_i+1)}}{\sqrt{\rho}} \right\} \right] \quad (M = U_\rho / \sqrt{\rho} \sim N(0, 1)) \\ = &\mathbb{P}_{\Gamma_n} \left[M > \min_{1 \le i \le n} \frac{\Phi^{-1} \left(1 - t_i \alpha \right) - Z_{(n-nt_i+1)}}{\sqrt{\rho}} \right] \\ = &\mathbb{P}_{\Gamma_n} \left[M > \min_{1 \le i \le n} \frac{\Phi^{-1} \left(1 - t_i \alpha \right) - Z_{(n-nt_i+1)}}{\sqrt{\rho}} \right] \end{aligned}$$

For $t \in (0, 1)$, $Z_{(n-nt+1)} = Z_{\left(n\left(1-t+\frac{1}{n}\right)\right)}$ converges in probability to (1-t)'th quantile of the distribution of Z_1 (i.e. $\sqrt{1-\rho} \cdot \Phi^{-1}(1-t)$) as $n \to \infty$. Now, let

$$s(t) = \frac{\Phi^{-1}(1 - t\alpha) - \sqrt{1 - \rho} \cdot \Phi^{-1}(1 - t)}{\sqrt{\rho}}.$$

Now, $g_n(t)$ converges to s(t) uniformly in any $t \in [a, b] \subset (0, 1)$ and sample point $w \in A$ with $\mathbb{P}(A) > 1 - \epsilon$. Let

$$t_0 := \arg \inf_{t \in (0,1)} s(t).$$

We shall show that

$$\min_{1\leq i\leq n}g_n(t_i)\to s(t_0).$$

Towards this, note that

$$\lim_{n \to \infty} \min_{a \le i/n \le b} g_n\left(\frac{i}{n}\right) \ge \lim_{n \to \infty} \min_{a \le t \le b} g_n(t) \ge g(t_0).$$

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Also, $g_n(t) \le g(t_0) + \delta$ for each $t \in (t_0 - \delta_1, t_0 + \delta_1)$. Thus, we have $\min_{1 \le i \le n} g_n(t_i) \rightarrow s(t_0)$. Hence,

$$\lim_{n \to \infty} FWER_{BH}(n, \alpha, \rho) = \mathbb{P}\left[M > \inf_{t \in (0, 1)} s(t)\right] = 1 - \Phi\left[\inf_{t \in (0, 1)} s(t)\right].$$

Now, $\inf_{t \in (0,1)} s(t) \le s(.5) < \infty$. So, $\Phi\left[\inf_{t \in (0,1)} s(t)\right] < 1$. Thus, $\lim_{n \to \infty} FWER_{BH}(n, \alpha, \rho) > 0$ for $\rho \in (0, 1)$.

Remark 1 Since we are considering the infimum of the function $s(\cdot)$ and since $s(0) = \infty = s(1)$, the previous result still holds good if one considers the closed interval [0, 1] in place of the open interval (0, 1).

Remark 2 Finner et al. (2007) studied the (limiting) empirical distribution function of the *p*-values and used those to study limiting behaviors of FDR. Their results are derived under general distributional setups and different values of ξ_n where ξ_n denotes the proportion of the true nulls. Our elementary proof, in contrast, uses standard analytic tools and provides a simple closed-form expression for the limiting FDR under the global null.

4.3 Other step-up procedures

We have discussed the limiting FWER values of two step-up procedures so far. In this subsection, we shall provide an upper bound on the limiting FWER of any step-up procedure satisfying some properties. Towards this, we discuss a special dependency property of test statistics introduced by Benjamini and Yekutieli (2001). They referred to this property as *positive regression dependency on each one from a subset* A, or PRDS on A. The notion of PRDS involves increasing sets.

Definition 1 A set $D \subset \mathbb{R}^k$ is called an *increasing set* if $\mathbf{a} \in D$ and $\mathbf{b} \ge \mathbf{a}$ imply that $\mathbf{b} \in D$.

Definition 2 *Property PRDS* Let $\mathbf{X} = (X_1, X_2, ..., X_n)$ be the vector of test statistics. We say that the PRDS property holds on \mathcal{A} if for any increasing set D, and for each $i \in \mathcal{A}$, $\mathbb{P} \{ \mathbf{X} \in D \mid X_i = x \}$ is nondecreasing in x.

Benjamini and Yekutieli (2001) established that the Benjamini–Hochberg method controls the FDR under the PRDS property. Let $\mathbf{u}_T = (u_1, \ldots, u_n) \in S_n$ denote the vector of critical values of the step-up MTP *T*. Guo and Rao (2008) showed the following.

Lemma 4.1 (Guo and Rao 2008) Let T be any step-up MTP having vector of critical values $\mathbf{u}_T \in S_n$. The following inequality holds under the PRDS property:

$$\sum_{k=1}^{n} \mathbb{P}\left(R_n(T) = k \mid P_i \le u_k\right) \le 1, \quad \text{for } i \in \mathcal{A}.$$
(2)

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Moreover, the above inequality becomes an equality under the independence of the test statistics.

They also constructed an example of the joint distribution of the p-values, under which the PRDS property fails to hold although the inequality (2) holds. Thus, it turns out that the inequality (2) is a strictly weaker property of the test statistics than the PRDS property. They further showed the following optimality property of the BH procedure:

Theorem 4.4 (Guo and Rao 2008) Let \mathcal{T} be the class of all step-up procedures with vector of cutoffs belonging to S_n and satisfying the inequality (2). Then, the Benjamini–Hochberg procedure is optimal in the class \mathcal{T} . That is, for any step-up procedure $T \in \mathcal{T}$ with vector of critical values $\mathbf{u}_T \in S_n$, if it can control the FDR at α , then $u_k \leq k\alpha/n$ for each $k \in \{1, ..., n\}$.

Theorems 4.3 and 4.4 result in the following:

Theorem 4.5 Let \mathcal{T} be the class of all step-up procedures with vector of cutoffs belonging to S_n and satisfying the inequality (2). Let $T \in \mathcal{T}$ be such that it controls the FDR at $\alpha \in (0, 1)$. Consider the equicorrelated normal setup with correlation ρ . Then, under the global null, for all $\alpha \in (0, 1)$ and for all $\rho \in (0, 1)$,

$$\lim_{n \to \infty} FDR_T(n, \alpha, \rho) \le 1 - \Phi\left[\inf_{t \in (0, 1)} \frac{\Phi^{-1} \left(1 - t\alpha\right) - \sqrt{1 - \rho} \cdot \Phi^{-1} (1 - t)}{\sqrt{\rho}}\right]$$

Proof of Theorem 4.5 We have, under the global null,

$$DR_{T}(n, \alpha, \rho) = FWER_{T}(n, \alpha, \rho)$$

= $\mathbb{P}_{\Gamma_{n}} \left[\bigcup_{i=1}^{n} \{ P_{(i)} < u_{i} \} \right]$
 $\leq \mathbb{P}_{\Gamma_{n}} \left[\bigcup_{i=1}^{n} \{ P_{(i)} < \frac{i\alpha}{n} \} \right]$ (using Theorem4.4)
= $FDR_{BH}(n, \alpha, \rho)$

Taking limit as $n \to \infty$ on both sides, we have

$$\lim_{n \to \infty} FDR_T(n, \alpha, \rho) \le \lim_{n \to \infty} FDR_{BH}(n, \alpha, \rho).$$

The rest follows from Theorem 4.3.

5 Hommel's procedure

F

We have focused on step-down and step-up procedures so far. However, many powerful MTPs proposed in the literature do not belong to the step-down or step-up categories. The Hommel (1988) procedure is such a *p*-value based MTP that controls the FWER.

The decisions for the individual hypotheses are performed in the following simple way:

Step 1. Compute $j = \max \{ i \in \{1, ..., n\} : P_{(n-i+k)} > k\alpha/i \text{ for } k = 1, ..., i \}.$

Step 2. If the maximum does not exist in Step 1, reject all the hypotheses. Otherwise, reject all H_i with $P_i \leq \alpha/j$.

Hommel's MTP is uniformly more powerful than the methods of Bonferroni, Holm, and Hochberg (Gou et al. 2014). The following two results depict the asymptotic behavior of the FWER of Hommel's procedure under the independent normal setup and under the positively equicorrelated normal setup, respectively.

Theorem 5.1 *Consider the multiple testing problem under the independent normal setup. Under the global null, we have*

$$\lim_{n \to \infty} FWER_{Hommel}(n, \alpha, 0) = 1 - e^{-\alpha}.$$

Theorem 5.2 Consider the multiple testing problem under the equicorrelated normal framework with correlation $\rho \in (0, 1)$. Then, for any $\alpha \in (0, 1)$,

$$\lim_{n \to \infty} FWER_{Hommel}(n, \alpha, \rho) = 0$$

with probability one under the global null hypothesis.

Proof of Theorem 5.1 For $1 \le i \le n$, we have $P_{(i)} = 1 - \Phi(X_{(n-i+1)})$. Putting i = n - j + k (here $1 \le j \le n$ and $1 \le k \le j$) gives $P_{(n-j+k)} = 1 - \Phi(X_{(j-k+1)})$, $k \le j$. Now,

$$P_{(n-j+k)} > \frac{k\alpha}{j} \iff 1 - \Phi\left(X_{(j-k+1)}\right) > \frac{k\alpha}{j}$$
$$\iff \Phi^{-1}\left(1 - \frac{k\alpha}{j}\right) > X_{(j-k+1)}$$
$$\iff \Phi^{-1}\left(1 - \frac{s\alpha}{t}\right) > X_{(n(t-s)+1)} \text{ where } s = k/n \text{ and } t = j/n.$$

For any $r \in (0, 1)$, $X_{(nr)}$ converges in probability to *r*'th quantile of the distribution of X_1 as $n \to \infty$. This implies, $X_{(n(t-s)+1)}$ converges in probability to $\Phi^{-1}(t-s)$ as $n \to \infty$. Thus, as $n \to \infty$,

$$P_{(n-j+k)} > \frac{k\alpha}{j} \iff \Phi^{-1}\left(1 - \frac{s\alpha}{t}\right) > \Phi^{-1}(t-s)$$
$$\iff 1 - \frac{s\alpha}{t} > t - s$$
$$\iff t - s\alpha > t(t-s)$$
$$\iff t(1-t) > s(\alpha - t).$$

We have $t \ge s$ and $1 > \alpha$. So, $t(1 - t) > s(\alpha - t)$ always holds. This means that the largest t for which $t(1 - t) > s(\alpha - t)$ holds for each $s \in (0, t]$ is 1. This in turn

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implies that, as $n \to \infty$, the largest integer $j \le n$ satisfying $P_{(n-j+k)} > \frac{k\alpha}{j}$ for all $k \in \{1, \dots, j\}$ is *n* with probability one. Thus, the Hommel's procedure is same as the Bonferroni's procedure as $n \to \infty$. Hence,

$$\lim_{n\to\infty} FWER_{Hommel}(n,\alpha,0) = 1 - e^{-\alpha}.$$

Proof of Theorem 5.2 For the equicorrelated normal framework with correlation $\rho \in$ (0, 1), for each $i \ge 1$, we have $X_i = U + Z_i$. Here $U \sim N(0, \rho)$ is independent of $\{Z_n\}_{n>1}$ and Z_i 's are i.i.d $N(0, 1 - \rho)$.

We establish Theorem 5.2 in the following steps:

1. Showing that as $n \to \infty$,

$$P_{(n-j+k)} > \frac{k\alpha}{j}$$
 for all $k = 1, \dots, j \iff U < \min_{0 < s < t} f(s)$

where $f(s) = \Phi^{-1} (1 - s\alpha/t) - \sqrt{1 - \rho} \cdot \Phi^{-1}(t - s)$.

2. Showing that

$$\Phi\left(\frac{-U - \Phi^{-1}(\alpha)}{\sqrt{1 - p}}\right) > t \text{ implies } U < \min_{0 < s < t} f(s).$$

3. Showing that, for each positive integer *m*,

$$FWER_{Hommel}(n, \alpha, \rho) \leq \mathbb{P}\left[P_{(1)} \leqslant \frac{1}{t_0} \cdot \frac{\alpha}{n}\right] + \mathbb{P}(U \geq m)$$

where $t_0 = \max_t \{t \in (0, 1) : \min_{0 \le s \le t} f(s) > U\}.$

We explicate the steps now.

Similar to the previous proof, we have

$$\begin{split} P_{(n-j+k)} &> \frac{k\alpha}{j} \iff 1 - \Phi\left(X_{(j-k+1)}\right) > \frac{k\alpha}{j} \\ \iff \Phi^{-1}\left(1 - \frac{k\alpha}{j}\right) > X_{(j-k+1)} \\ \iff \Phi^{-1}\left(1 - \frac{k\alpha}{j}\right) > U + Z_{(j-k+1)} \\ \iff \Phi^{-1}\left(1 - \frac{s\alpha}{t}\right) > U + Z_{(n(t-s)+1)} \quad \text{where } s = k/n \text{ and } t = j/n. \end{split}$$

For any $r \in (0, 1)$, $Z_{(nr)}$ converges in probability to r'th quantile of the distribution of Z_1 as $n \to \infty$. This implies, $Z_{(n(t-s)+1)}$ converges in probability to $\sqrt{1-\rho} \cdot \Phi^{-1}(t-s)$ as $n \to \infty$. Thus, as $n \to \infty$,

$$P_{(n-j+k)} > \frac{k\alpha}{j} \iff U < \Phi^{-1}\left(1 - \frac{s\alpha}{t}\right) - \sqrt{1 - \rho} \cdot \Phi^{-1}(t-s).$$

Proceeding in the same way as in the proof of Theorem 4.3, one may show that, as $n \to \infty$,

$$P_{(n-j+k)} > \frac{k\alpha}{j} \text{ for all } k = 1, \dots, j \iff U < \min_{0 < s < t} f(s)$$
(3)

completing the proof of step 1.

Now, t > t - s as s > 0. This implies $\Phi^{-1}(t) > \Phi^{-1}(t - s)$. Consequently, for each s > 0, f(s) > g(s) where $g(s) = \Phi^{-1} \left(1 - \frac{s\alpha}{t}\right) - \Phi^{-1}(t)$. Thus,

$$g(s) > U \implies f(s) > U.$$

Now,

$$\begin{split} g(s) > U &\iff \Phi^{-1} \left(1 - \frac{s\alpha}{t} \right) - \sqrt{1 - \rho} \cdot \Phi^{-1}(t) > U \\ &\iff \Phi^{-1} \left(1 - \frac{s\alpha}{t} \right) > U + \sqrt{1 - \rho} \cdot \Phi^{-1}(t) \\ &\iff 1 - \frac{s\alpha}{t} > \Phi \left(U + \sqrt{1 - \rho} \cdot \Phi^{-1}(t) \right) \\ &\iff \frac{\Phi \left(-U - \sqrt{1 - \rho} \cdot \Phi^{-1}(t) \right)}{\alpha} > \frac{s}{t}. \end{split}$$

Therefore, if $\frac{\Phi(-U-\sqrt{1-\rho}\cdot\Phi^{-1}(t))}{\alpha} > 1$ then $\forall s \in (0, t), g(s) > U$. Hence, $\frac{\Phi(-U-\sqrt{1-\rho}\cdot\Phi^{-1}(t))}{\alpha} > 1$ implies f(s) > U for all $s \in (0, t)$. Now,

$$\frac{\Phi\left(-U - \sqrt{1 - \rho} \cdot \Phi^{-1}(t)\right)}{\alpha} > 1 \iff -U - \sqrt{1 - \rho} \cdot \Phi^{-1}(t) > \Phi^{-1}(\alpha)$$
$$\iff \Phi\left(\frac{-U - \Phi^{-1}(\alpha)}{\sqrt{1 - \rho}}\right) > t.$$

Therefore, we have established the following:

$$\Phi\left(\frac{-U - \Phi^{-1}(\alpha)}{\sqrt{1 - p}}\right) > t \text{ implies } U < \min_{0 < s < t} f(s), \tag{4}$$

completing step 2. Thus,

$$t_0 := \max_t \left\{ t \in (0,1) : \min_{0 < s < t} f(s) \ge U \right\} \ge \max_t \left\{ t \in (0,1) : \Phi\left(\frac{-U - \Phi^{-1}(\alpha)}{\sqrt{1 - \rho}}\right) > t \right\}.$$

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Now, U < r implies $t_0 \ge \varepsilon_r$ where

$$\varepsilon_r = \Phi\left(\frac{-r - \Phi^{-1}(\alpha)}{\sqrt{1-p}}\right).$$

So, for every $m \in \mathbb{N}$, there exists $\varepsilon_m > 0$ such that $t_0 > \varepsilon_m$ if U < m. In other words, there is ε_m such that $t_0 > \varepsilon_m > 0$ with probability at least $\mathbb{P}(U < m)$. This implies, t_0 is bounded away from zero with probability one. Now, let

$$j_0 = \max_{1 \leq j \leq n} \left\{ P_{(n-j+k)} > \frac{k\alpha}{j} \text{ for all } k = 1, \dots, j \right\}.$$

Evidently, $j_0 \ge nt_0$. Consequently, under the global null,

$$FWER_{Hommel}(n, \alpha, \rho) = \mathbb{P}\left[\bigcup_{i=1}^{n} \left\{P_{i} \leq \frac{\alpha}{j_{0}}\right\}\right]$$
$$\leq \mathbb{P}\left[\bigcup_{i=1}^{n} \left\{P_{i} \leq \frac{\alpha}{nt_{0}}\right\}\right] + \mathbb{P}(U \geq m)$$
$$= \mathbb{P}\left[P_{(1)} \leq \frac{1}{t_{0}} \cdot \frac{\alpha}{n}\right] + \mathbb{P}(U \geq m).$$

This completes the proof of Step 3. Now, $\mathbb{P}(U \ge m) \le \epsilon$ for all $\epsilon > 0$ as $m \to \infty$. We claim now that

$$\mathbb{P}\left[P_{(1)} \leqslant \frac{1}{t_0} \cdot \frac{\alpha}{n}\right] \longrightarrow 0 \quad \text{as} \quad n \to \infty.$$

Its proof is precisely the same as the proof of Theorem 2 of Dey and Bhandari (2023a) and we therefore omit it. The rest is obvious. \Box

Remark 3 Suppose a > 0. The proof of Theorem 2 of Dey and Bhandari (2023a) also culminates in the following:

$$\mathbb{P}_{\Gamma_n}\left[P_{(1)}\leqslant a\cdot\frac{\alpha}{n}\right]\longrightarrow 0 \quad \text{as} \ n\to\infty$$

for each $\rho \in (0, 1)$. Then, invoking Slepian's inequality, we have the following: Let Σ_n be the correlation matrix of X_1, \ldots, X_n having (i, j)'th entry ρ_{ij} with lim inf $\rho_{ij} = \delta > 0$. Suppose $\mu^* = \sup \mu_i < \infty$. Then, for any $\alpha \in (0, 1)$,

$$\mathbb{P}_{\Sigma_n}\left[P_{(1)}\leqslant a\cdot \frac{\alpha}{n}\right]\longrightarrow 0 \text{ as } n\to\infty.$$

Note that this is a much stronger result than Theorem 3.2.

If one replaces X_i by $X_i + \mu_i$ and U by $U + \mu_i$ in the proof of Theorem 5.2, one would obtain the following result:

Theorem 5.3 Consider the equicorrelated normal setup with equicorrelation $\rho \in (0, 1)$. Suppose sup μ_i is finite. Then, for any $\alpha \in (0, 1)$,

$$\lim_{n\to\infty}\mathbb{P}_{\Gamma_n}\bigg(R_n(Hommel)\geq 1\bigg)=0$$

with probability one under any configuration of true and false null hypotheses. Consequently, $FWER_{Hommel}(n, \alpha, \rho)$ and $AnyPwr_{Hommel}(n, \alpha, \Gamma_n)$ tend to zero with probability one as $n \to \infty$.

6 Simulations and discussion

In this section we fix the desired level α at .05. We estimate the FWERs of Holm's, Hochberg's, Hommel's, and Benjamini–Hochberg method for the equicorrelated normal setup under the global null. For fixed values of (n, ρ) , we adopt a simulation scheme similar to the ones in Dey (2022) and Dey and Bhandari (2023b):

- 1. We generate 100,000 *n*-variate equicorrelated multivariate normal observations (each of the *n* components has zero mean, unit variance and each pair of components has common correlation ρ).
- 2. For each of the 100,000 replications, we obtain the *n p*-values from the *n* components using the relation $p_i = 1 \Phi(X_i)$.
- 3. For each of the four MTPs and for each of the replications, we check whether there is at least one rejection. To do this for Hommel procedure, we have used the *mtp* function from the R package *elitism*. For the other three procedures, we have used the *p.adjust* function from the R package *stats*.
- 4. The FWER for a particular MTP is estimated as the number of times that MTP makes at least one rejection (since we are considering global null), divided by 100,000.

We present the simulation results for the estimated FWER of Holm, Hochberg, Hommel, and BH methods for different combinations of (n, ρ) in Fig. 1. The estimated FWER values are mentioned in Table 2.

We observe from Fig. 1 that, for each ρ , the FWER values of Holm, Hochberg and Hommel decrease with increasing *n*. The convergence to zero is much faster for $\rho = .9$ compared to $\rho = .1$ for all these three procedures. Also, the simulation study explicates that for small ρ and moderately large *n*, the Holm and Hochberg methods have very similar performances.

For the BH FWER, we have also plotted the true limiting expression from Theorem 4.3 along with the simulation results to get an idea of the quality of asymptotic approximation. In Fig. 1, the true limiting BH FWERs for $\rho = .1, .5, .9$ are represented by the square sign, the plus sign, and the diamond sign respectively. For n = 100,000 and $\rho = .5, .9$, the BH FWER at $\alpha = .05$ are estimated as .01276 and .02022. These values are close to the corresponding limiting values .0103 and .0194 respectively. However, for $\rho = .1$, the BH FWER value at n = 100,000 is still far off from the



Fig. 1 Estimates of FWER($n, \alpha = .05, \rho$) of four procedures for $\rho = .1, .5, .9$ and n = 100, 1000, 10,000, 1,000, 00 under the global null. The solid, dashed and dotted lines correspond to the FWER values for $\rho = .1, .5$ and .9 respectively

limiting value .0076. As mentioned in the preceding paragraph, this slow convergence to the limit for small values of ρ is true for the other three procedures also.

One also observes from Fig. 1 and Table 2 that, for large *n*, the BH FDR tends to decrease at first and then increases as ρ increases, as also mentioned by Das and Bhandari (2020).

7 Concluding remarks

In recent years, substantial efforts have been made to understand the properties of multiple testing procedures under dependence. The work by Dey and Bhandari (2023a) sheds light on the extent of the conservativeness of the Bonferroni method under dependent setups. However, there is little literature on the effect of correlation on general step-down or step-up procedures. This paper addresses this gap in a unified manner by investigating the limiting behaviors of several testing rules under the correlated Gaussian sequence model. We have proved asymptotic zero results for some popular MTPs controlling FWER at a pre-specified level. Specifically, we have shown that the limiting FWER approaches zero for any step-down rule provided the infimum of the correlations is strictly positive.

Correlation	Procedure	Number of hypotheses (<i>n</i>)			
(ρ)		100	1000	10,000	100,000
	Holm	.04645	.04531	.04352	.04285
.1	Hochberg	.04645	.04531	.04352	.04285
	Hommel	.04653	.04533	.04353	.04285
	BH	.04942	.04855	.04749	.04721
	Holm	.02588	.01711	.01181	.00782
.5	Hochberg	.02588	.01711	.01181	.00782
	Hommel	.02612	.01725	.01193	.00801
	BH	.03177	.02257	.01660	.01276
	Holm	.00482	.00134	.00037	.00008
.9	Hochberg	.00639	.00276	.00130	.00069
	Hommel	.01149	.00701	.00446	.00310
	BH	.02168	.02035	.02024	.02022

Table 2 Estimates of $FWER(n, \alpha = .05, \rho)$ of Holm, Hochberg, Hommel and BH Method under the global null

Huang and Hsu (2007) show that both Holm's and Hochberg's methods are special cases of partition testing. Holm's MTP tests each partition hypothesis using the maximum order statistic, setting a cutoff utilizing the Bonferroni inequality. Hochberg's procedure, on the contrary, tests each partition hypothesis using each of the order statistics, using a set of cutoffs utilizing Simes' inequality. It is natural to expect partition testing utilizing the joint distribution of the test statistics is sharper than partition testing based on probabilistic inequalities. Our results elucidate that, at least under the correlated Gaussian sequence model setup with many hypotheses, Holm's MTP and Hochberg's MTP do not have significantly different performances in that they both have asymptotic zero FWER and asymptotic zero power.

The Benjamini–Hochberg procedure has been one of the most studied MTP and has several desirable optimality properties (Bogdan et al. 2011; Guo and Rao 2008). It is astonishing to note that, among all the procedures studied in this paper, the BH method is the only one which can hold the FWER at a strictly positive level asymptotically under the equicorrelated normal setup. An interesting problem would be to study the limiting power of the Benjamini–Hochberg method.

Hommel's method is more rejective than Hochberg's MTP (and consequently, Holm's and Bonferroni's methods) (Gou et al. 2014). Yet, within our chosen asymptotic framework, this has asymptotic zero FWER and asymptotic zero power.

Finally, there are possible scopes of interesting extensions in several directions. One extension is to consider more general distributional setups. Another is to study the (limiting) behaviors of Hochberg, Hommel, and Benjamini–Hochberg procedures under general dependent normality. The primary tool in establishing universal asymptotic zero results for the step-down MTPs is Slepian's inequality which compares the quadrant probabilities of two normal random vectors. However, for the step-up procedures, the FWERs become functions of several order statistics. Hence we can not directly

apply Slepian's inequality in these scenarios. Indeed, Finner et al. (2007) remark that it is challenging to deal with false discoveries in models with complicated dependence structures, e.g., in a multivariate Gaussian model with a general covariance matrix. It is also interesting to theoretically investigate whether similar asymptotic results hold for other classes of MTPs, e.g., the class of consonant procedures (Westfall et al. 1999).

Appendix

This section contains the proofs of various results stated in the paper.

Proof of Theorem 3.2 We have,

$$\mathbb{P}_{\Sigma_n}(R_n(Holm) \ge 1) = \mathbb{P}_{\Sigma_n}(P_{(1)} \le \alpha/n) = \mathbb{P}_{\Sigma_n}(X_{(n)} \ge c_{Bon}).$$

Theorem 2.4 gives $\mathbb{P}_{\Sigma_n}(X_{(n)} \ge c_{Bon}) \le \mathbb{P}_{\Gamma_n}(X_{(n)} \ge c_{Bon})$. Without any loss of generality, we may assume $X_i \sim N(\mu_i, 1)$ ($\mu_i > 0$) for $1 \le i \le n_1$ and for $n_1 < i \le n$, $X_i \sim N(0, 1)$. Thus,

$$\begin{split} & \mathbb{P}_{\Sigma_{n}}(R_{n}(Holm) \geq 1) \\ \leq & \mathbb{P}_{\Gamma_{n}}(X_{(n)} \geq c_{Bon}) \\ = & 1 - \mathbb{P}_{\Gamma_{n}}(X_{(n)} \leq c_{Bon}) \\ = & 1 - \mathbb{P}_{\Gamma_{n}}\left[\bigcap_{i=1}^{n_{1}} \{\theta + Z_{i} + \mu_{i} \leq c_{Bon}\} \bigcap_{i=n_{1}+1}^{n} \{\theta + Z_{i} \leq c_{Bon}\}\right] \\ = & 1 - \mathbb{E}_{\theta}\left[\left\{\prod_{i=1}^{n_{1}} \Phi\left(\frac{c_{Bon} - \theta - \mu_{i}}{\sqrt{1 - \rho}}\right)\right\} \cdot \Phi^{n-n_{1}}\left(\frac{c_{Bon} - \theta}{\sqrt{1 - \rho}}\right)\right] \\ \leq & 1 - \mathbb{E}_{\theta}\left[\Phi^{n}\left(\frac{c_{Bon} - \theta - \mu^{\star}}{\sqrt{1 - \rho}}\right)\right]. \end{split}$$

The last quantity above tends to zero asymptotically since $\mu^* < \infty$ (its proof is exactly similar to that of Theorem 2 of Dey and Bhandari (2023a). The rest follows by noting that $R_n(Holm) \ge \max\{V_n(Holm), S_n(Holm)\}$.

We establish Theorem 3.3 using the following result:

Theorem 7.1 (Gordon and Salzman 2008) Let *T* be a step-down MTP based on the set of cut-offs $\mathbf{u} \in S^n$. If $FWER_T \le \alpha < 1$, then $u_i \le \alpha/(n-i+1), i = 1, ..., n$.

Proof of Theorem 3.3 We have,

$$FWER_T(n, \alpha, \Sigma_n) = \mathbb{P}_{\Sigma_n}(V_n(T) \ge 1) \le \mathbb{P}_{\Sigma_n}(R_n(T) \ge 1)$$
$$= \mathbb{P}_{\Sigma_n}(P_{(1)} \le u_1)$$
$$\le \mathbb{P}_{\Sigma_n}(P_{(1)} \le \alpha/n).$$

The last step above follows since we have $u_1 \le \alpha/n$ from Theorem 7.1. The rest is obvious from Theorem 3.2.

Proof of Theorem 4.1 We have, under the global null,

$$FWER_{Hochberg}(n, \alpha, 0) = \mathbb{P}_{I_n} \left[\bigcup_{i=1}^n \left\{ P_{(i)} \leqslant \frac{\alpha}{n-i+1} \right\} \right]$$
$$\geq \mathbb{P}_{I_n} \left[P_{(1)} \leqslant \frac{\alpha}{n} \right]$$
$$\xrightarrow[n \to \infty]{} 1 - e^{-\alpha}.$$

Also, Hochberg's procedure controls FWER at level α (Hochberg 1988). So,

$$1 - e^{-\alpha} \leq \lim_{n \to \infty} FWER_{Hochberg}(n, \alpha, 0) \leq \alpha.$$

Also, $\lim_{\alpha \to 0} \frac{1-e^{-\alpha}}{\alpha} = 1$. Thus, we have, as $\alpha \to 0$, $\lim_{n \to \infty} \frac{FWER_{Hochberg}(n,\alpha,0)}{\alpha} = 1$. This completes the proof of the first part.

Consider the covariance matrix Γ_n now, with $\rho \in (0, 1)$. Under the global null, we have

$$X_i = U + Z_i$$

where $U \sim N(0, \rho), Z_i \stackrel{i.i.d}{\sim} N(0, 1 - \rho)$. Here U and Z_i are independent. Now,

$$P_{(i)} \leqslant \frac{\alpha}{n-i+1}$$

$$\iff 1 - \frac{\alpha}{n-i+1} \le \Phi\left(X_{(n-i+1)}\right)$$

$$\iff \Phi^{-1}\left(1 - \frac{\alpha}{n-i+1}\right) \leqslant U + Z_{(n-i+1)}$$

$$\iff \Phi^{-1}\left(1 - \frac{\alpha}{n-i+1}\right) \leqslant U + \sqrt{1-\rho} \cdot \Phi^{-1}\left(1 - \frac{i}{n}\right)$$

(for all sufficiently large values of *n*).

Therefore, for all sufficiently large values of *n*, we have

$$P_{(i)} \leq \frac{\alpha}{n-i+1}$$

$$\iff -U - \sqrt{1-\rho} \cdot \Phi^{-1} \left(1 - \frac{i}{n}\right) \leq -\Phi^{-1} \left(1 - \frac{\alpha}{n-i+1}\right)$$

$$\iff -U + \sqrt{1-\rho} \cdot \Phi^{-1} \left(\frac{i}{n}\right) \leq \Phi^{-1} \left(\frac{\alpha}{n-i+1}\right)$$

$$\iff \frac{-U}{\Phi^{-1} \left(\frac{\alpha}{n-i+1}\right)} + \frac{\sqrt{1-\rho} \cdot \Phi^{-1} \left(\frac{i}{n}\right)}{\Phi^{-1} \left(\frac{\alpha}{n-i+1}\right)} \geq 1 \quad (\text{since } \alpha \in (0, 1/2))$$

$$\iff \lim_{n \to \infty} \frac{\Phi^{-1} \left(\frac{i}{n}\right)}{\Phi^{-1} \left(\frac{\alpha}{n-i+1}\right)} \geq \frac{1}{\sqrt{1-\rho}}.$$

Thus, we have i/n < 1/2, because otherwise the limiting ratio of $\Phi^{-1}\left(\frac{i}{n}\right)$ and $\Phi^{-1}\left(\frac{\alpha}{n-i+1}\right)$ can not be positive. So, we have

$$\frac{i}{n} < \frac{\alpha}{n-i+1} < 1/2.$$

This implies $i(n - i + 1) < \alpha \cdot n$. But this is not valid for any value of i in $\{1, \ldots, n\}$. Consequently, the limiting FWER is zero.

Proof of Theorem 4.2 Let U and Z_i be as in the preceding proof. We have,

$$\mathbb{P}_{\Gamma_n}\left(R_n(Hochberg) \ge 1\right)$$

= $\mathbb{P}_{\Gamma_n}\left[\bigcup_{i=1}^n \left\{P_{(i)} \leqslant \frac{\alpha}{n-i+1}\right\}\right]$
= $\mathbb{P}_{\Gamma_n}\left[\bigcup_{i=1}^n \left\{\Phi^{-1}\left(1-\frac{\alpha}{n-i+1}\right) \le U+Z_{(n-i+1)}\right\}\right]$
= $\mathbb{P}_{\Gamma_n}\left[\bigcup_{i\in\mathcal{A}} \left\{c_{\alpha,n-i+1} \le U+Z_{(n-i+1)}\right\}\bigcup_{i\in\mathcal{A}^c} \left\{c_{\alpha,n-i+1} \le U+Z_{(n-i+1)}\right\}\right].$

In the last step above, $c_{\alpha,n-i+1}$ denotes $\Phi^{-1}\left(1-\frac{\alpha}{n-i+1}\right)$. Here, $Z_i = \mu_i + V_i$ where V_i 's are i.i.d $N(0, 1-\rho)$ variables, μ_i is zero for $i \in A$

Here, $Z_i = \mu_i + V_i$ where V_i 's are i.i.d $N(0, 1 - \rho)$ variables, μ_i is zero for $i \in A$ and is strictly positive otherwise. So, Z_i always lies in $[V_i, V_i + \sup \mu_i]$. This implies, for all $i \in \{1, ..., n\}$,

$$Z_{(n-i+1)} \le V_{(n-i+1)} + \sup \mu_i.$$

Hence, we have

$$\mathbb{P}_{\Gamma_n}\left(R_n(Hochberg) \ge 1\right)$$
$$\leq \mathbb{P}_{\Gamma_n}\left[\bigcup_{i=1}^n \left\{\Phi^{-1}\left(1 - \frac{\alpha}{n-i+1}\right) \le U + V_{(n-i+1)} + \sup \mu_i\right\}\right]$$

Proceeding exactly in the same way as in the earlier proof, one obtains that the upper bound tends to zero asymptotically. Consequently, the probability of rejecting any null hypothesis approaches zero.

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