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# Tropical convexity in location problems

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## Abstract

We investigate location problems where the optimal solution is found within the tropical convex hull of the given input points. Our initial focus is on geodesically star-convex sets, using the asymmetric tropical distance. We introduce the concept of tropically quasiconvex functions, which have sub-level sets with this shape, and are closely related to monotonic functions. Our findings demonstrate that location problems using tropically quasiconvex functions as distance measures will result in an optimal solution within the tropical convex hull of the input points. We also extend this result to cases where the input points are replaced with tropically convex sets. Finally, we explore the applications of our research in phylogenetics, highlighting the properties of consensus methods that arise from our class of location problems.

**Keywords** Tropical convexity · Tropically quasiconvex function · Monotonic norms · Tropical  $L^p$ -norm · Tropically convex consensus trees

**Mathematics Subject Classification** 14T90 · 26B25 · 52A30 · 90B85 · 92B10

## 1 Introduction

There has been a recent surge of interest in the study of location problems in tropical geometry, particularly in the application of tropical methods to data analysis. One of the earliest articles to explore this topic and recognize its significance for “tropical statistics” was written by Lin et al. (2017). In their work, they demonstrated that tropical convexity in tree spaces offers superior properties compared to the geometry of Billera, Holmes, and Vogtmann (BHV) from Billera et al. (2001). This discovery motivated them to suggest location estimators based on the symmetric tropical distance, which could potentially exploit tropical convexity. This approach could also provide a tropical approach to the consensus problem in phylogenetics (Bryant 2003).

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The relationship between the proposed location statistics and tropical convexity is not well understood. For instance, it has been observed that tropical Fermat–Weber points may fall outside the tropical convex hull of the given points (Lin et al. 2017, Example 26). However, it has since been discovered that Fermat–Weber points can also be found within the tropical convex hull (Page et al. 2020, Lemma 3.5). This lack of a clear connection makes it challenging to obtain solutions that can be interpreted in a phylogenetic context; see also Lin and Yoshida (2018).

Recently, we have demonstrated that examining the Fermat–Weber problem using an *asymmetric* distance function results in a more comprehensive explanation in terms of tropical convexity (Comănesci and Joswig 2024). This approach offers a distinct and well-defined solution to the consensus problem in phylogenetics based on tropical geometry. Furthermore, due to the connections to tropical convexity, we were able to achieve several desirable properties in consensus methods. It is worth noting that these favorable properties were solely attributed to tropical convexity, rather than the specific distance function used. This observation has motivated our exploration of alternative methods that exhibit similar properties.

In this paper, our focus is on location problems that can exploit tropical convexity. Specifically, we are interested in location estimators that fall within the tropical convex hull of the input points. These estimators are based on distances that reflect the tropical structure of the space and can be seen as a counterpart to similar studies on location problems and ordinary convexity.

Significant progress has been made in understanding the geometric properties of location problems and their relationship to ordinary convexity. The study of Chebyshev centers dates back to the 1960s, with the work of Garkavi (1964) and Klee (1960). Wendell and Hurter (1973) further investigated more general location problems in normed spaces, while Durier and Michelot (1985) focused on the geometric properties of Fermat–Weber problems with varying distances. Additionally, it has been proven that finding an optimal solution within the (ordinary) convex hull for any set of points is only possible in an inner product space with three or more dimensions. Durier (1994, 1997) obtained a general form of this result.

These results mentioned above show a strong relationship between ordinary convexity and a Euclidean structure. On the other hand, tropical convexity is linked to the lattice  $(\mathbb{R}^n, \leq)$ . Therefore, our focus should be on “monotonic” distances. To interpret geometrically monotonic functions in the quotient space  $\mathbb{R}^n/\mathbb{R}\mathbf{1}$ , we observe that all sub-level sets share a common characteristic: they are geodesically star-convex with respect to the asymmetric tropical distance. This can be seen by noting that geodesic segments are images of order segments in  $(\mathbb{R}^n, \leq)$ . The resulting sets, referred to as  $\Delta$ -star-convex, and functions, called  $\Delta$ -star-quasiconvex, are discussed in Sects. 2 and 3, respectively.

In Sect. 4, we focus on location problems where distances to the sites are measured by  $\Delta$ -star-quasiconvex functions. We demonstrate that this setting guarantees optimal locations in the tropical convex hull of the input. This highlights the differences between tropical and ordinary convexity, as the triangle inequality does not play a role in this setting. Additionally, this approach allows for a wide range of location problems, even when dissimilarities are not necessarily distances. While the triangle inequality is typically assumed in location science for geographic location (Laporte

et al. 2019), it is not always reasonable for more general data (Tversky 1977) and is not assumed in the construction of M-estimators (Huber 1981, Sect. 3.2).

We have further a few examples of location problems from the literature that fall in our setting. In particular, location problems involving the symmetric and asymmetric tropical distances. However, the former case might contain cases where some optima are outside the tropical convex hull of the input. So what is the precise distinction between the symmetric and the asymmetric tropical distances that causes the above behaviour? We show that *strict*  $\Delta$ -star-convexity is the answer. This motivates that study of regularized versions discussed in Sect. 4.2.

In Sect. 5, we briefly demonstrate that our results can be extended to the case where the sites are tropically convex sets. Section 6 focuses on the main application of our work to phylogenetics: the tropical approach to consensus methods. Our general framework encompasses a wide range of tropically convex consensus methods, as defined in Comănesci and Joswig (2024, Sect. 5). Additionally, we expand upon the list of desirable properties for these consensus methods outlined in the aforementioned work. Finally, we conclude with Sect. 7, which highlights our findings and suggests potential avenues for future research.

## 2 Order and convexity

The purpose of this section is to fix the notation and to introduce the types of convex sets that will be used later. They will be related to the natural pointwise order on  $\mathbb{R}^n$ .

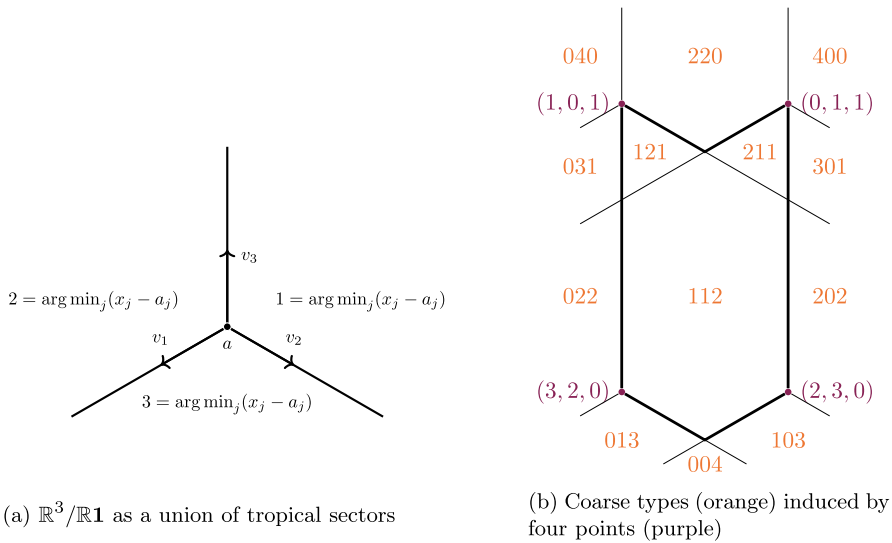
Firstly, we focus on tropically convex sets. To do this, we need to define the *max-tropical semiring*  $\mathbb{T} = (\mathbb{R} \cup \{-\infty\}, \vee, +)$  where  $x \vee y = \max(x, y)$ . This is a special case of an *idempotent algebra* (Kolokoltsov and Maslov 1997, Sect. 1.1), and more details can be found in Joswig (2021). The operations of the semiring can be extended naturally to the “vector space”  $\mathbb{T}^n$ , which can be seen as a semimodule from an algebraic perspective. However, we will approach semimodules from a geometric standpoint, which will lead us to the definition of tropical cones. We will only consider subsets of  $\mathbb{R}^n$ , as we will not be looking at points with infinite entries.

**Definition 1** A *max-tropical cone*  $K \subset \mathbb{R}^n$  is a set closed under max-tropical linear combinations:  $(x + \lambda \mathbf{1}) \vee (y + \mu \mathbf{1}) \in K$  for all  $x, y \in K$  and  $\lambda, \mu \in \mathbb{R}$ . The image of a max-tropical cone in  $\mathbb{R}^n / \mathbb{R}\mathbf{1}$  is called a *max-tropically convex set*.

**Remark 2** Tropical cones are semilattices with respect to the partial order  $\leq$ , as the element  $x \vee y$  is the smallest upper bound of  $x$  and  $y$  in  $K$  for every  $x, y \in K$ . In other words, a max-tropical cone is a *join semilattice* (Davey and Priestley 2002, p. 170).

The preceding remark highlights the connection between tropical cones and the order  $\leq$  on  $\mathbb{R}^n$ . However, our focus will be on tropical convexity in  $\mathbb{R}^n / \mathbb{R}\mathbf{1}$ , which was introduced by Develin and Sturmfels (2004). This concept exhibits more similarities to traditional convexity, as elaborated in Joswig (2021, Chapter 5). So our all concepts will deal with the quotient vector space  $\mathbb{R}^n / \mathbb{R}\mathbf{1}$ .

**Definition 3** The *max-tropical convex hull* of a set  $A \subset \mathbb{R}^n / \mathbb{R}\mathbf{1}$  is the smallest max-tropically convex set containing  $A$  and we denote it by  $\text{tconv}^{\max}(A)$ .



**Fig. 1** We identify  $\mathbb{R}^3/\mathbb{R}\mathbf{1}$  as a plane such that the basis  $\{e_1, e_2, e_3\}$  preserves its symmetry. Its image in  $\mathbb{R}^2$  is indicated by the vectors  $v_1, v_2, v_3$  in (a). The corresponding rays bound the min-tropical sectors. Figure b contains the tropical convex hull (thick boundary) of four points (purple). Their coarse types (orange) are described in Example 5

For our practical purposes, we will use a characterization of tropical convex hulls in terms of min-tropical sectors. We use the notation

$$S_i^{\min} := \{x \in \mathbb{R}^n/\mathbb{R}\mathbf{1} : x_i \leq x_j \ \forall j \in [n]\} = \{x \in \mathbb{R}^n/\mathbb{R}\mathbf{1} : i \in \arg \min_j x_j\}.$$

A min-tropical sector with apex  $a \in \mathbb{R}^n/\mathbb{R}\mathbf{1}$  is the set of the form

$$S_i^{\min}(a) := a + S_i^{\min} = \{x \in \mathbb{R}^n/\mathbb{R}\mathbf{1} : i \in \arg \min_j (x_j - a_j)\}$$

for some  $i \in [n]$ .

**Proposition 4** ((Joswig, 2021, Proposition 5.37)) *A point  $x \in \mathbb{R}^n/\mathbb{R}\mathbf{1}$  belongs to  $\text{tconv}^{\max}(A)$  if and only if for each  $i \in [n]$  there exists  $a^{(i)} \in A$  such that  $x \in S_i^{\min}(a^{(i)})$ .*

**Example 5** In Fig. 1, we illustrate how tropical objects are visualized in  $\mathbb{R}^3/\mathbb{R}\mathbf{1}$ . Each point is represented by three coordinates corresponding to the elementary basis vectors  $e_1, e_2$ , and  $e_3$ . These coordinates are then mapped to the quotient vector space  $\mathbb{R}^3/\mathbb{R}\mathbf{1}$ , where the additional rule  $e_1 + e_2 + e_3 = \mathbf{0}$  applies. By taking these considerations into account, we can represent the space  $\mathbb{R}^3/\mathbb{R}\mathbf{1}$  as a two-dimensional plane by considering a linear transformation defined on the basis vectors, where  $e_i \mapsto v_i$  and  $v_1, v_2, v_3 \in \mathbb{R}^2$  with the constraint that  $v_1 + v_2 + v_3 = \mathbf{0}$ .

Most tropical geometry papers focus on the “tropical basis”  $v_1 = (-1, -1)$ ,  $v_2 = (1, 0)$ , and  $v_3 = (0, 1)$ . However, this identification breaks the symmetry, so we prefer

to take  $v_1, v_2, v_3$  as three points on the unit circle at equal distance from each other. They are depicted in Fig. 1a and form the Mercedes–Benz frame (Waldron 2018, pp. 1, 3).

With these considerations, Fig. 1a displays the min-tropical sectors with apex  $a$ , covering the entire space  $\mathbb{R}^3/\mathbb{R}\mathbf{1}$ . For instance, the sector  $S_1^{\min}(a)$  is represented by the cone with extremal rays  $v_2$  and  $v_3$  and it is indicated in the picture by the label  $1 = \arg \min_j (x_j - a_j)$  which characterizes its interior.

In Fig. 1b, we consider a set of four points  $V = \{(0, 1, 1), (1, 0, 1), (3, 2, 0), (2, 3, 0)\}$  and their max-tropical convex hull, which is represented by the thick line on the boundary. The other lines correspond to the boundaries of min-tropical sectors based on the four points. These lines subdivide  $\mathbb{R}^3/\mathbb{R}\mathbf{1}$  into polyhedral parts,<sup>1</sup> each with a three-digit code representing its *coarse type* (Joswig 2021, Sect. 4.4).

The coarse types and the tropical convex hull are related through Proposition 4. For example, the cell with label “112” can be expressed as the intersection

$$S_3^{\min}(0, 1, 1) \cap S_3^{\min}(1, 0, 1) \cap S_1^{\min}(3, 2, 0) \cap S_2^{\min}(2, 3, 0).$$

It is worth noting that the subscripts 1 and 2 occur once, while the subscript 3 occurs twice, as indicated by the label “112.” In other words, the label indicates the number of indices 1, 2, and 3 in the inclusion of min-tropical sectors based at the points of  $V$ . Furthermore, according to Proposition 4, this cell is contained in  $\text{tconv}^{\max}(V)$ .

On the other hand, cells with a 0 in their coarse type are not included in  $\text{tconv}^{\max}(V)$ . For example, if the coarse type is “202,” it means that there is no point  $v \in V$  whose  $S_2^{\min}(v)$  contains the cell. In other words, there is no candidate for  $a^{(2)}$  as described in Proposition 4.

We now move to another type of convexity in  $\mathbb{R}^n/\mathbb{R}\mathbf{1}$  related to the lattice  $(\mathbb{R}^n, \leq)$  which is related to its *order segments*, namely the sets of the form  $[a, b]_{\leq} := \{x \in \mathbb{R}^n : a \leq x \leq b\}$  defined for  $a, b \in \mathbb{R}^n$  with  $a \leq b$ .

Firstly, we will describe the relationship between the order and the metric. We will begin by discussing the *asymmetric tropical distance*  $d_{\Delta}$ , which is defined as

$$d_{\Delta}(a, b) := \sum_i (b_i - a_i) - n \min_j (b_j - a_j)$$

for  $a, b \in \mathbb{R}^n/\mathbb{R}\mathbf{1}$ . This distance was first introduced by Amini and Manjunath (2010), but its name comes from Comănesci and Joswig (2024). The main characteristic of this distance is that its unit ball is a regular simplex, as indicated by the subscript  $\Delta$ .

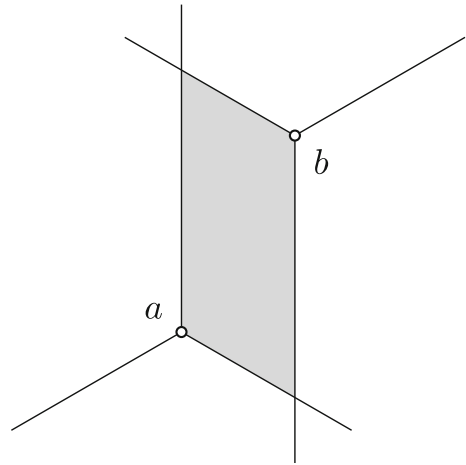
**Definition 6** For two points  $a, b \in \mathbb{R}^n/\mathbb{R}\mathbf{1}$  we define the (oriented) *geodesic segment* between  $a$  and  $b$  under  $d_{\Delta}$  as  $[a, b]_{\Delta} := \{x \in \mathbb{R}^n : d_{\Delta}(a, x) + d_{\Delta}(x, b) = d_{\Delta}(a, b)\}$ .

If we also take into account the *max-tropical sectors*, which are sets of the form

$$S_i^{\max}(a) := \{x \in \mathbb{R}^n/\mathbb{R}\mathbf{1} : i \in \arg \max_j (x_j - a_j)\},$$

<sup>1</sup> The union of boundaries of tropical sectors with a given apex forms a tropical hyperplane. In our two-dimensional case, a tropical line is a union of three rays, as in Fig. 1a. Thus, the subdivision from Fig. 1b comes from a tropical hyperplane arrangement.

**Fig. 2** The geodesic segment  $[a, b]_{\Delta}$  is marked with grey



we can obtain a clear description of  $[a, b]_{\Delta}$ , as shown in Fig. 2.

**Lemma 7** For every  $a, b \in \mathbb{R}^n / \mathbb{R}\mathbf{1}$  we have  $[a, b]_{\Delta} = S_i^{\min}(a) \cap S_i^{\max}(b)$  where  $i$  is any index from  $\arg \min_j (b_j - a_j)$ .

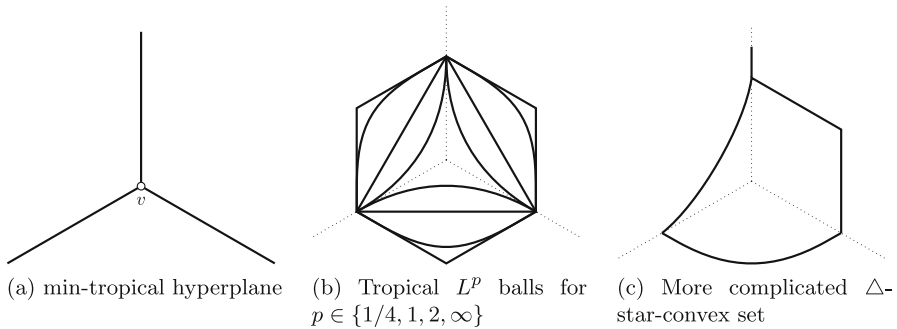
**Proof** If  $i \in \arg \min_j (b_j - a_j)$ , then  $d_{\Delta}(a, b) = \sum_k (b_k - a_k) - n(b_i - a_i)$ . What is more,

$$\begin{aligned} d_{\Delta}(a, x) + d_{\Delta}(x, b) &= \sum_k (x_k - a_k) - n \min_j (x_j - a_j) + \sum_k (b_k - x_k) - n \min_{\ell} (b_{\ell} - x_{\ell}) \\ &= \sum_k (b_k - a_k) + n \max_j (a_j - x_j) + n \max_{\ell} (x_{\ell} - b_{\ell}) \\ &\geq \sum_k (b_k - a_k) + n(a_i - x_i) + n(x_i - b_i) = d_{\Delta}(a, b) \end{aligned}$$

for every  $x \in \mathbb{R}^n / \mathbb{R}\mathbf{1}$  with equality if and only if  $i$  belongs to  $\arg \max_j (a_j - x_j)$  and  $\arg \max_{\ell} (x_{\ell} - b_{\ell})$ . In other words,  $x \in [a, b]_{\Delta}$  if and only if  $x \in S_i^{\min}(a) \cap S_i^{\max}(b)$ .  $\square$

**Remark 8** If we choose representatives  $a$  and  $b$  modulo  $\mathbb{R}\mathbf{1}$  such that  $\min_j (b_j - a_j) = 0$ , then  $[a, b]_{\Delta}$  is the image of  $[a, b]_{\leq}$  in  $\mathbb{R}^n / \mathbb{R}\mathbf{1}$ . Indeed, it can be checked directly using the definition of  $[a, b]_{\leq}$  and Lemma 7.

We move towards geodesical convexity with respect to  $d_{\Delta}$ , which is covered for more general normed spaces in Boltyanski et al. (1997, Chapter 2). In the aforementioned book, the used term is “ $d_{\Delta}$ -convexity,” but we will prefer to use the term “ $\Delta$ -convexity” for better readability and more emphasis on its relationship to the standard simplex. Besides, Lemma 7 shows that  $\Delta$ -convexity can be defined independent of a distance function.



**Fig. 3**  $\Delta$ -star-convex sets

**Definition 9** A  $\Delta$ -star-convex set with kernel  $v$  is a non-empty set  $K \subseteq \mathbb{R}^n / \mathbb{R}\mathbf{1}$  such that for every point  $w \in K$  we have  $[v, w]_{\Delta} \subseteq K$ . We call  $K$  *strictly*  $\Delta$ -star-convex if  $[v, w]_{\Delta} \setminus \{w\}$  belongs to the interior of  $K$  for every  $w \in K$ .

Since  $[v, w]_{\Delta}$  contains the ordinary segment  $[v, w]$ , we conclude that  $\Delta$ -star-convex sets are also star-convex in the ordinary sense. In fact, they are min-tropically convex, which is proven in the Appendix, but we will not need this information later. We focus only on the star-convex variant as there are very few  $\Delta$ -convex sets. They are fully characterized in the Appendix as well.

**Example 10** One can find examples of  $\Delta$ -star-convex sets in Fig. 3. Picture (a) shows a min-tropical hyperplane

$$H_v^{\min} := \{x \in \mathbb{R}^n / \mathbb{R}\mathbf{1} : |\arg \min_j (x_j - v_j)| \geq 2\}$$

which is  $\Delta$ -star-convex with kernel  $v$ , its apex.

Picture (b) displays the unit balls for tropical  $L^p$  norms, which will be defined in Example 12. They are nested increasingly with respect to  $p$ , with the outer one corresponding to the tropical  $L^{\infty}$  norm. This outer unit ball is the only one that is not strictly  $\Delta$ -star-convex. The triangle in the picture represents the unit ball for the asymmetric tropical distance  $d_{\Delta}$ . The min-tropical hyperplane with apex at the origin (the kernel of the  $\Delta$ -star-convex sets) is shown as a dotted line, which can be used to check the  $\Delta$ -star-convexity based on Lemma 7.

Picture (c) depicts a more intricate  $\Delta$ -star-convex set. It is neither convex in the traditional sense nor strictly  $\Delta$ -star-convex.

### 3 Tropically quasiconvex functions

A function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  whose sub-level sets  $L_{\leq \alpha}(f) := \{x : f(x) \leq \alpha\}$  are convex is called *quasiconvex*. While this is a purely geometric definition, some sources define quasiconvex functions as those satisfying  $f(\lambda x + (1 - \lambda)y) \leq \max\{f(x), f(y)\}$  for all  $x, y \in \mathbb{R}^n$  and  $\lambda \in [0, 1]$ . This alternative definition may be more convenient for checking quasiconvexity. For more information, refer to Avriel et al. (2010, Chapter 3).



We are interested in examining a special class of *tropically quasiconvex* functions. To properly introduce these functions, we must first establish some notation. For any function  $\gamma : \mathbb{R}_{\geq 0}^n \rightarrow \mathbb{R}$ , we define the associated function

$$\widehat{\gamma} : \mathbb{R}^n / \mathbb{R}\mathbf{1} \rightarrow \mathbb{R} \quad \text{defined by} \quad \widehat{\gamma}(x) = \gamma(\widehat{x}),$$

where  $\widehat{x} = x - (\min_i x_i) \mathbf{1}$ . It is important to note that  $\widehat{x}$  represents the canonical coordinates of  $x$ .

To extend the concept of  $\Delta$ -convexity from the previous section to functions, we must also consider the natural order on  $\mathbb{R}^n$ . Hence, a function  $\gamma : X \rightarrow \mathbb{R}$  is called *increasing* if  $\gamma(x) \leq \gamma(y)$  for every  $x, y \in X$  such that  $x \leq y$  where  $X$  is a subset of  $\mathbb{R}^n$ . This function is referred to as *strictly increasing* if  $\gamma(x) < \gamma(y)$  for every *distinct*  $x, y \in X$  such that  $x \leq y$ .

**Definition 11** We call a function  $f : \mathbb{R}^n / \mathbb{R}\mathbf{1} \rightarrow \mathbb{R}$   $\Delta$ -star-quasiconvex with kernel  $v$  if  $f(x) = \widehat{\gamma}(x - v)$  for some increasing function  $\gamma : \mathbb{R}_{\geq 0}^n \rightarrow \mathbb{R}$ . Moreover, if  $\gamma$  is strictly increasing, we call  $f$  *strictly*  $\Delta$ -star-quasiconvex.

We will give a geometric interpretation of  $\Delta$ -star-quasiconvex in Theorem 15. However, we prefer the definition above because it is easier to check in practice.

**Example 12** Considering  $\gamma$  a monotonic norm (Bauer et al. 1961),  $f$  measures the distance to the kernel. If  $v = \mathbf{0}$ , then  $f$  is a *gauge*, which is commonly used in convex analysis (Rockafellar 1970) and location science (Nickel and Puerto 2005). Gauges are sometimes referred to as “asymmetric norms” because they satisfy all the properties of a norm, except for the fact that  $f(x)$  need not be equal to  $f(-x)$ .

One famous class of monotonic norms is the  $L^p$  norms. These norms give rise to  $\Delta$ -star-quasiconvex gauges, which can be expressed as

$$\gamma_p(x) = \begin{cases} \sqrt[p]{\sum_{i \in [n]} (x_i - \min_{j \in [n]} x_j)^p} & \text{if } p \in [1, \infty) \\ \max_{i \in [n]} x_i - \min_{j \in [n]} x_j & \text{if } p = \infty. \end{cases}$$

We call them *tropical  $L^p$  norms*. They appeared in the work of Luo (2018) under the name “ $B^p$ -pseudonorms.” We recall that their unit balls, i.e. the sets

$$B_{\gamma_p} := \{x \in \mathbb{R}^n / \mathbb{R}\mathbf{1} : \gamma_p(x) \leq 1\},$$

are displayed in Fig. 3b. We remark that tropical  $L^p$  norms can be defined also for  $p \in (0, 1)$  and are strictly  $\Delta$ -star-quasiconvex, but they are no longer convex.

One can identify the tropical  $L^\infty$  norm as the tropical norm defined in Hampe (2015, Sect. 5). The connection to the  $L^\infty$  norm is stressed in Hampe (2015, Lemma 5.2.1). The tropical  $L^1$  norm leads to the asymmetric tropical distance  $d_\Delta$ , which is mentioned in Comănesci and Joswig (2024, Sect. 6).

**Remark 13** The function  $\widehat{\gamma}$  depends only on the values on the boundary of the non-negative orthant,  $\partial \mathbb{R}_{\geq 0}^n$ , so we could have considered only  $\partial \mathbb{R}_{\geq 0}^n$  as the domain of  $\gamma$ . However, this does not increase the generality since every (strictly) increasing function

defined on  $\partial\mathbb{R}_{\geq 0}^n$  can be extended to a (strictly) increasing function on  $\mathbb{R}_{\geq 0}^n$ , as shown in the following lemma.

**Lemma 14** *Every (strictly) increasing function  $\gamma : \partial\mathbb{R}_{\geq 0}^n \rightarrow \mathbb{R}$  can be extended to a (strictly) increasing function  $\tilde{\gamma} : \mathbb{R}_{\geq 0}^n \rightarrow \mathbb{R}$ . Moreover, if  $\gamma$  is continuous, then the extension can also be made continuous.*

**Proof** We construct the extension

$$\tilde{\gamma}(x) := \max_{i \in [n]} \gamma(x_{-i}, 0_i) + x_1 x_2 \dots x_n, \quad (1)$$

where the vector  $(x_{-i}, 0_i)$  is obtained from  $x$  by replacing the  $i$ th entry with 0.

Clearly,  $\tilde{\gamma}$  is continuous if  $\gamma$  is, as being a composition of continuous functions. Moreover,  $\tilde{\gamma}(x) = \gamma(x)$  for every  $x \in \partial\mathbb{R}_{\geq 0}^n$ , due to the monotonicity of  $\gamma$  and the fact that  $x_1 x_2 \dots x_n = 0$  for  $x \in \partial\mathbb{R}_{\geq 0}^n$ .

Since the entry-wise product and maximum are increasing functions combined with the monotonicity of  $\gamma$ , we immediately see from (1) that the extension  $\tilde{\gamma}$  is also increasing.

If  $\gamma$  is strictly increasing, then  $\tilde{\gamma}$  is also strictly increasing as  $x \mapsto \max_{i \in [n]} \gamma(x_{-i}, 0_i)$  is strictly increasing over  $\partial\mathbb{R}_{\geq 0}^n$  while  $x \mapsto x_1 \dots x_n$  is strictly increasing over  $\mathbb{R}_{\geq 0}^n \setminus \partial\mathbb{R}_{\geq 0}^n$ .  $\square$

The following result explains why the functions from Definition 11 deserve the name “ $\Delta$ -star- quasiconvex.”

**Theorem 15** *Let  $f : \mathbb{R}^n / \mathbb{R}\mathbf{1} \rightarrow \mathbb{R}$  be a continuous function. If  $f$  is (strictly)  $\Delta$ -star- quasiconvex then all of its non-empty sub-level sets are (strictly)  $\Delta$ -star convex with the same kernel. Conversely, if all non-empty sub-level sets of  $f$  are  $\Delta$ -star-convex with the same kernel, then  $f$  is  $\Delta$ -star- quasiconvex.*

**Proof** After an eventual translation, we can assume that the kernel is  $\mathbf{0}$ .

Firstly, assume  $f$  is  $\Delta$ -star- quasiconvex and let  $\alpha \in \mathbb{R}^n$  arbitrary such that  $L_{\leq \alpha}(f)$  is non-empty. Let  $\gamma : \mathbb{R}^n \rightarrow \mathbb{R}$  increasing such that  $f(x) = \widehat{\gamma}(x)$ .

Let  $w \in L_{\leq \alpha}(f)$  and choose  $i \in [n]$  such that  $w \in S_i^{\min}$ . Since  $\gamma$  is increasing, the points  $x \in \mathbb{R}^n$  satisfying  $\mathbf{0} \leq x \leq \widehat{w}$  belong to  $L_{\leq \alpha}(\gamma)$ . This set projects onto  $[\mathbf{0}, w]_{\Delta}$  showing that  $[\mathbf{0}, w]_{\Delta} \subseteq L_{\leq \alpha}(f)$ . Since  $w$  was selected arbitrarily,  $L_{\leq \alpha}(f)$  must be  $\Delta$ -star convex with kernel  $\mathbf{0}$ .

If  $f$  is strictly  $\Delta$ -star- quasiconvex, then the points satisfying  $\mathbf{0} \leq x \leq \widehat{w}$  different from  $\widehat{w}$  actually belong to  $L_{< \alpha}(f)$ . Due to the continuity of  $f$ , this is included in the interior of  $L_{\leq \alpha}(f)$ . This shows that  $L_{\leq \alpha}(f)$  is strictly  $\Delta$ -star-convex.

Conversely, assume that  $L_{\leq \alpha}(f)$  is  $\Delta$ -star-convex with kernel  $\mathbf{0}$  for every  $\alpha \geq f(\mathbf{0})$ . Take  $\gamma : \partial\mathbb{R}_{\geq 0}^n \rightarrow \mathbb{R}$  defined as  $\gamma(x) = f(\widehat{x})$  for  $x \in \partial\mathbb{R}_{\geq 0}^n$ . Using Lemma 14 it is enough to show that  $\gamma$  is increasing.

Let  $x$  and  $y$  arbitrary points of  $\partial\mathbb{R}_{\geq 0}^n$  such that  $x \leq y$ . The order segment  $[\mathbf{0}, y]_{\leq}$  projects onto  $[\mathbf{0}, y]_{\Delta}$  which is included in  $L_{\leq f(y)}(f)$ . Due to the  $\Delta$ -star-convexity of sub-level sets, we obtain  $\gamma(x) = f(x) \leq f(y) = \gamma(y)$ .  $\square$

**Remark 16** The continuity of  $f$  is only relevant for strictly  $\Delta$ -star-quasiconvex functions. For example, consider the function  $\gamma(x, y, z) := x + y + z + \xi(y, z)$  where  $\xi$  takes the value 1 everywhere except at  $(0, 0)$ , where it is 0. The function  $\widehat{\gamma}$  is strictly  $\Delta$ -star-quasiconvex with kernel  $\mathbf{0}$ , but the sub-level set  $L_{\leq 1}(\widehat{\gamma})$  is not strictly  $\Delta$ -star-convex. Instead, it is the segment from  $\mathbf{0}$  to  $(1, 0, 0)$ .

Similar to ordinary quasiconvex functions, the strict  $\Delta$ -star-convexity of the sub-level sets alone is not enough for  $f$  to be strictly  $\Delta$ -star-quasiconvex. This can be seen in Avriel et al. (2010, Proposition 3.28) and Avriel et al. (2010, Example 3.3). For instance, a constant function can serve as a counterexample. Further technical conditions between different sub-level sets are necessary to obtain a converse.

We will see that convexity, in the ordinary sense, will also be helpful for our applications. We give a simple criterion for checking when a  $\Delta$ -star-quasiconvex function is convex.

**Lemma 17** *If  $\gamma$  is increasing and convex, then  $\widehat{\gamma}$  is convex.*

**Proof.** Let  $x, y \in \mathbb{R}^n$  and  $\lambda \in [0, 1]$  arbitrary. Because the entry-wise minimum is a concave function, we obtain

$$\lambda x + (1 - \lambda)y - \min_j (\lambda x_j + (1 - \lambda)y_j) \mathbf{1} \leq \lambda(x - \min_i x_i \mathbf{1}) + (1 - \lambda)(y - \min_i y_i \mathbf{1}).$$

Since  $\gamma$  is increasing and convex, we get the convexity of  $\widehat{\gamma}$  by the inequalities:

$$\begin{aligned} \widehat{\gamma}(\lambda x + (1 - \lambda)y) &\leq \gamma\left(\lambda(x - \min_i x_i \mathbf{1}) + (1 - \lambda)(y - \min_i y_i \mathbf{1})\right) \\ &\leq \lambda \gamma(x - \min_i x_i \mathbf{1}) + (1 - \lambda) \gamma(y - \min_i y_i \mathbf{1}) \\ &= \lambda \widehat{\gamma}(x) + (1 - \lambda) \widehat{\gamma}(y). \end{aligned}$$

□

## 4 Tropically convex location problems

We will consider some input points  $v_1, \dots, v_m$  in  $\mathbb{R}^n/\mathbb{R}\mathbf{1}$ . We measure the distance (or dissimilarity) from  $x \in \mathbb{R}^n/\mathbb{R}\mathbf{1}$  to a point  $v_i$  using a  $\Delta$ -star-quasiconvex function  $f_i$  having kernel  $v_i$ . We consider increasing functions  $\gamma_i : \mathbb{R}^n \rightarrow \mathbb{R}$  such that  $f_i(x) = \widehat{\gamma}_i(x - v_i)$ . Without loss of generality, we assume  $\gamma_i(\mathbf{0}) = 0$ , so that all dissimilarities are non-negative.

The aim of location problems is to find a point as close (or similar) as possible to the input points. Formally, it means that we have to find a minimum of the vector function

$$x \mapsto (f_1(x), \dots, f_m(x)) \quad (2)$$

where we take the pointwise order  $\leq$  on the codomain  $\mathbb{R}_{\geq 0}^m$ . In this way, it is a multi-objective optimization problem and we are searching for *Pareto optima*, i.e. points of  $x \in \mathbb{R}^n/\mathbb{R}\mathbf{1}$  such that there is no  $y \in \mathbb{R}^n/\mathbb{R}\mathbf{1}$  such that  $f_i(x) \geq f_i(y)$  for all  $i \in [m]$  and there is at least a strict inequality (Durier 1990; Jahn 2011).

Nonetheless, location theory primarily deals with scalar objective functions, thus requiring a *scalarization* of (2) in the context of multi-objective programming (Jahn 2011, Chapter 5). For the scalarization, we need a function  $g : \mathbb{R}_{\geq 0}^m \rightarrow \mathbb{R}$  to obtain the scalar objective  $h : \mathbb{R}^n/\mathbb{R}\mathbf{1} \rightarrow \mathbb{R}$  defined as

$$h(x) = g(f_1(x), \dots, f_m(x)). \quad (3)$$

We further assume that  $g$  is increasing since we want the minima of  $h$  to be Pareto optima of (2).

For most location problems, we would have a distance  $d$  on  $\mathbb{R}^n/\mathbb{R}\mathbf{1}$  and set  $f_i(x) = d(x, v_i)$ , but some studies focus on variable distances, e.g., in Durier and Michelot (1985). Then the choice of  $g$  gives rise to various studied problems:

- $g(x) = x_1 + \dots + x_m$  leads to Fermat–Weber problems (Durier 1990);
- $g(x) = x_1^2 + \dots + x_m^2$  leads to Fréchet means (Fréchet 1948);
- $g(x) = \max_{i \in [m]} x_i$  leads to center problems (Laporte et al. 2019, Chapter 3).

Nevertheless, our analysis works for (2) so also for every location problem with  $g$  increasing. This approach is actually taken by Wendell and Hurter (1973) in linking the optimal location to the (ordinary) convex hull of the input points.

In the following result, we assume that  $h$  has a minimum, which happens, for example, when  $h$  is lower semi-continuous. One could also state a similar result for Pareto optima of (2) with a similar proof. Still, we prefer the version below as it is commonly encountered in practice.

**Theorem 18** *Let  $h$  be an objective function of a location problem as in (3). Then there is a minimum of  $h$  belonging to  $\text{tconv}^{\max}(v_1, \dots, v_m)$ . Moreover, if  $g$  is strictly increasing and at least one of  $f_1, \dots, f_m$  is strictly  $\Delta$ -star- quasiconvex, then all the minima of  $h$  are contained in  $\text{tconv}^{\max}(v_1, \dots, v_m)$ .*

**Proof** Consider  $x \notin \text{tconv}^{\max}(v_1, \dots, v_m)$  which is a minimum of  $h$ . Proposition 4 implies the existence of an index  $k \in [n]$  such that  $k \notin \arg \min_j (x_j - v_{ij})$  for all  $i \in [m]$ . We set

$$\delta_i := x_k - v_{ik} - \min_j (x_j - v_{ij})$$

for all  $i$ , and  $\delta = \min_i \delta_i$ , which is strictly positive by the consideration of  $k$ . Note that

$$\begin{aligned} f_i(x - \delta e_k) &= \gamma_i \left( x - v_i - \delta e_k - \min_j (x_j - v_{ij}) \mathbf{1} \right) \\ &\leq \gamma_i (x - v_i - \min_j (x_j - v_{ij}) \mathbf{1}) = f_i(x) \end{aligned}$$

for all  $i \in [n]$ . Hence  $h(x - \delta e_k) \leq h(x)$ .

Note that the inequality above is strict if  $g$  and some  $\gamma_\ell$  are strictly increasing. Indeed, in that case, we must have  $f_\ell(x - \delta e_k) < f_\ell(x)$  and we use the strict increase of  $g$  in the  $\ell$ th entry to obtain a better optimum. That would contradict the optimality of  $x$ , thus proving the second statement of the theorem.

For the first statement, we can only infer that  $x - \delta e_k$  is also a minimum of  $h$ . Hence, we can find an optimum of  $h$  in  $\text{tconv}^{\max}(v_1, \dots, v_m)$  by moving  $x$  in directions  $-e_k$  for indices  $k$  as above. These will be the elementary descent direction we will focus on and we identify them by their index  $k \in [n]$ .

To be more precise, we collect in  $D(x)$  the common elementary descent directions from  $x$ . Formally, due to Proposition 4, the possible descent directions are described by

$$D(x) := \bigcap_{i \in [m]} [n] \setminus \arg \min_{j \in [n]} (x_j - v_{ij}).$$

Notice that  $k \in D(x)$ , but  $k \notin D(x - \delta e_k)$ . Moreover,  $D(x - \delta e_k) \subsetneq D(x)$ , as the  $\arg \min$  functions only increase by our move in a descent direction. Thus, replacing  $x$  by  $x - \delta e_k$ , we find a minimum with smaller set  $D(x)$  of common elementary descent directions. We can repeat the procedure to construct a minimum  $x^*$  of  $h$  with  $D(x^*) = \emptyset$ . The last condition is equivalent to  $x^* \in \text{tconv}^{\max}(v_1, \dots, v_m)$  due to Proposition 4.  $\square$

The proof of Theorem 18 shows, in fact, that there are Pareto optima of (2) in  $\text{tconv}^{\max}(v_1, \dots, v_m)$ . When all distances are strictly  $\Delta$ -star-quasiconvex, we actually obtain that all Pareto optima are contained in  $\text{tconv}^{\max}(v_1, \dots, v_m)$ . By studying the Fermat–Weber problem with  $d_\Delta$ , Cox and Curiel (2023) showed also a converse: every point of  $\text{tconv}^{\max}(v_1, \dots, v_m)$  is a Pareto optimum of

$$x \mapsto (d_\Delta(v_1, x), \dots, d_\Delta(v_m, x)).$$

This gives a metric characterization of tropical convexity.

## 4.1 Examples

Here we review the tropical location problems from literature that fall in our category, i.e. an optimum belongs to the tropical convex hull of the input.

**Example 19** (*Tropical Fermat–Weber and Fréchet problems*) To the best of our knowledge, the first one-point location problems in tropical geometry were proposed by Lin et al. (2017). In this paper, they suggest studying Fermat–Weber points and Fréchet means using the symmetric tropical distance  $d_{\text{trop}}$ , which is equivalent to the tropical  $L^\infty$  distance. The main objective of this study was to establish a connection between these points and tropical convexity, with potential applications in phylogenetics.

However, it has been observed that tropical Fermat–Weber points may fall outside the tropical convex hull of the input points, resulting in medians that are not easily

interpretable in biological applications (Lin et al. 2017, Example 27). Fortunately, Theorem 18 states that it is possible to locate an optimum within the tropical convex hull. This was previously noted for tropical Fermat–Weber points (Page et al. 2020, Lemma 3.5), but it remained an open problem for tropical Fréchet means; see (Yoshida 2022, Sect. 5.2).

**Example 20 (Tropical center)** Consider the center problem with the asymmetric tropical distance, i.e.  $f_i(x) = d_{\Delta}(v_i, x)$  and  $g(y) = \max(y_1, \dots, y_m)$ . This can be interpreted as finding the center of the smallest tropical  $L^1$  ball that contains the points  $v_1, \dots, v_m$ . The tropical center is mentioned in Comănesci and Joswig (2024, Example 23), but the specific details are not provided.

If we choose representatives of the input points in  $\mathcal{H} = \{x \in \mathbb{R}^n : x_1 + \dots + x_n = 0\}$ , the optimum can be obtained by solving the linear program:

$$\begin{aligned} & \text{minimize } n \cdot t \\ & \text{subject to } v_{ij} - x_j \leq t, \quad \text{for } i \in [m] \text{ and } j \in [n] \\ & \quad x_1 + \dots + x_n = 0. \end{aligned} \quad (4)$$

Note that the  $x$ -coordinates of the optimal solutions are equal, modulo  $\mathbb{R}\mathbf{1}$ , to the  $x$ -coordinates of the optimum of the linear program

$$\begin{aligned} & \text{minimize } n \cdot t + \sum_{j=1}^n x_j \\ & \text{subject to } v_{ij} - x_j \leq t, \quad \text{for } i \in [m] \text{ and } j \in [n]. \end{aligned} \quad (5)$$

Let  $(t^*, x^*)$  an optimal solution of (5). For any feasible solution  $(t, x)$  of (5) we have  $t + x_j \geq \max_{i \in [m]} v_{ij} =: V_j$ . In particular,  $x^*$  will have the smallest entries if we actually have equality:  $t^* + x_j^* = V_j$ , otherwise we could replace  $x^*$  by some  $x^* - \varepsilon e_i$  to minimize the objective function. This implies  $x^* = V$  modulo  $\mathbb{R}\mathbf{1}$ ; in particular, the solution is unique in  $\mathbb{R}^n / \mathbb{R}\mathbf{1}$ .

Even if we do not have  $g$  strictly increasing, the uniqueness and Theorem 18 ensures that the optimum is in the tropical convex hull. However, this could have been noticed from the closed form  $V = \bigvee_i v_i$  for  $v_1, \dots, v_m \in \mathcal{H}$ .

**Example 21 (Transportation problems)** Consider  $\lambda_1, \dots, \lambda_n > 0$  and  $\Delta(\lambda)$  the simplex in  $\mathbb{R}^n / \mathbb{R}\mathbf{1}$  whose vertices are  $e_i / \lambda_i$ . Then  $\gamma_{\Delta(\lambda)}(x) = \sum_i \lambda_i x_i - (\sum_i \lambda_i) \min_j x_j$  is the gauge on  $\mathbb{R}^n / \mathbb{R}\mathbf{1}$  whose unit ball is  $\Delta(\lambda)$ .

The (weighted) Fermat–Weber problem with input  $v_1, \dots, v_m$  is

$$\text{minimize}_x \sum_{i \in [m]} w_i \gamma_{\Delta(\lambda)}(x - v_i)$$

and is equivalent to a transportation problem. In fact, every transportation problem can be reduced to this case. To better understand this, we can rewrite it as a linear program after scaling by weights  $w_i$  such that  $\sum_i w_i = \sum_j \lambda_j$ . This change does not influence the optimum. We will not go into the details here, as they follow the same steps as

outlined in Comănesci and Joswig (2024), where the authors specifically focused on the case where  $\lambda_1 = \dots = \lambda_n$ . The optimal solution in this case is referred to as a *tropical median* in the cited work.

The optimal point is referred to as a  $\lambda$ -splitter by Tokuyama and Nakano (1995), but they do not provide a metric interpretation. These authors gave a condition for partitioning the space in  $n$  region in an equal fashion depending on some weights coming from  $\lambda$  and  $w$ . This can be seen as a reinterpretation of the first-order optimality condition for the corresponding Fermat–Weber problem. The concept of a  $\lambda$ -splitter is a specific instance of Minkowski partition problems (Aurenhammer et al. 1998).

**Example 22** (*Locating tropical hyperplanes*) The tropical hyperplanes are parametrized by points of  $\mathbb{R}^n/\mathbb{R}\mathbf{1}$  through the identification with their apex. Moreover, we have  $d_{\text{trop}}(a, H_x^{\max}) = (x - a)_{(2)} - (x - a)_{(1)}$ . For a vector  $y$ , we denote by  $y_{(k)}$  the  $k$ th smallest entry, also known as the  $k$ th order statistic. Note that the aforementioned distance is  $\Delta$ -star-quasiconvex with apex  $a$ . The easiest way to see this is by noticing that the second order statistic is increasing. Therefore, our general location problems also cover the case of locating tropical hyperplanes.

The best-fit tropical hyperplane with  $L^1$  error, i.e.  $g$  is the  $L^1$  norm, was considered by Yoshida, Zhang, and Zhang as part of *tropical principal component analysis* (Yoshida et al. 2019).

The case of  $L^\infty$  error was considered by Akian et al. (2023) for applications to auction theory and called *tropical linear regression*. They also show that the problem is polynomial-time equivalent to mean-payoff games (Akian et al. 2023, Corollary 4.15). Additionally, they show that it is the dual of the problem of finding the largest inscribed ball in the tropical convex hull of the input points (Akian et al. 2023, Theorem 4.6).

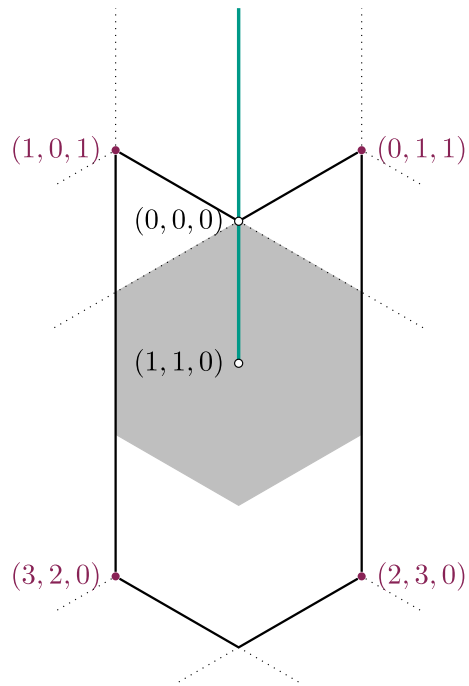
To end this subsection, we compute the optimal location from the examples above for specific input points. We consider the points from Akian et al. (2023, Sect. 4) which are given by the columns of the matrix

$$V = \begin{pmatrix} 0 & 1 & 3 & 2 \\ 1 & 0 & 2 & 3 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$

For this input, there is a unique tropical Fréchet point,  $(1, 1, 0)$ , but the set of tropical Fermat–Weber points is a hexagon, marked with grey in Fig. 4. We remark that  $V$  has two axes of symmetry and  $(1, 1, 0)$  is their intersection.

The point  $(1, 1, 0)$  is also the tropical center of  $V$ , while the tropical median is  $(0, 0, 0)$ . The latter point is also the unique apex of the best-fit tropical hyperplane with  $L^1$  error of Yoshida et al. (2019). It is also a solution of the tropical linear regression, but not the unique one. The apices of the best-fit tropical hyperplanes with  $L^\infty$  error are of the form  $(\lambda, \lambda, 0)$  with  $\lambda \leq 1$ . These points are represented in green in Fig. 4.

**Fig. 4** Input points (purple) with their convex hull (black boundary) and various locations from the examples of Sect. 4.1; see the discussion after the examples



## 4.2 Regularization

In certain situations, it is not always reasonable to assume that the function  $g$  will be strictly increasing, or that all dissimilarity functions  $f_i$  will be strictly  $\Delta$ -star-quasiconvex. Therefore, in cases where there are multiple solutions, a minimization algorithm may return a point that lies outside the max-tropical convex hull of the given points. In this subsection, we will demonstrate how we can attempt to find a solution within  $\text{tconv}^{\max}(v_1, \dots, v_m)$  by using a regularized formulation.

The idea of regularization is to consider a small parameter  $\lambda > 0$  and a nicely behaved function  $f_{m+1} : \mathbb{R}^n / \mathbb{R}\mathbf{1} \rightarrow \mathbb{R}_{\geq 0}$  and try to solve the optimization problem

$$\text{minimize } g(f_1(x), \dots, f_m(x)) + \lambda f_{m+1}(x).$$

For our purposes,  $f_{m+1}$  is nicely behaved if it is strictly  $\Delta$ -star-quasiconvex with a kernel from  $\text{tconv}^{\max}(v_1, \dots, v_m)$ . An easy choice for  $v$  is the tropical center from Example 20.

This is also a location problem with  $g_\lambda : \mathbb{R}_{\geq 0}^{m+1} \rightarrow \mathbb{R}$  given by

$$g_\lambda(x_1, \dots, x_m, x_{m+1}) = g(x_1, \dots, x_m) + \lambda x_{m+1}$$

and the optimality criterion is the function  $h_\lambda : \mathbb{R}^n / \mathbb{R}\mathbf{1} \rightarrow \mathbb{R}$  given by

$$h_\lambda(x) = g_\lambda(f_1(x), \dots, f_{m+1}(x)).$$



Note that  $g_\lambda$  is strictly increasing in the  $(m + 1)$ st entry for every  $\lambda > 0$ .

Checking more carefully the proof of Theorem 18, the second statement holds if  $f_\ell$  is strictly  $\Delta$ -star-quasiconvex and  $g$  strictly increasing in its  $\ell$ th entry. We use this property for the regularization. Therefore, we obtain the following direct consequence of Theorem 18.

**Corollary 23** *For every  $\lambda > 0$ , all the minima of  $h_\lambda$  lie in  $\text{tconv}^{\max}(v_1, \dots, v_m)$ .*

When  $h_\lambda$  is continuous for each  $\lambda \geq 0$ , Berge's maximum theorem (Berge 1997, p. 116) implies the existence of a family  $(x_\lambda^*)_{\lambda > 0}$  of minima of  $h_\lambda$  which admits a limit  $x_0^*$  as  $\lambda$  goes to 0. This point  $x_0^*$  will be also a minimum of  $h$ . Additionally, since tropical polytopes are compact, Corollary 23 implies that  $x_0^* \in \text{tconv}^{\max}(v_1, \dots, v_m)$ . In this way, we obtain a minimum of  $h$  in the tropical convex hull of the input points.

## 5 Location problems with tropically convex sites

Location problems can also arise when facilities are regions within the ambient space, rather than just points. In this case, we are considering a generalization where the sites are tropically convex sets. An example of this can be found in Breen-McKay (2022, Chapter 6).

In the previous section, we used different distances to the input points. Here, we will measure our dissimilarities in a uniform way, by fixing an increasing function  $\gamma : \mathbb{R}_{\geq 0}^n \rightarrow \mathbb{R}$  and considering  $d_\gamma(x, y) = \widehat{\gamma}(y - x)$ . We then say that  $d_\gamma$  is  $\Delta$ -star-quasiconvex; if  $\gamma$  is strictly increasing we say that  $d_\gamma$  is strictly  $\Delta$ -star-quasiconvex. This allows for a precise definition of the distance from a region to a point:  $d_\gamma(A, x) := \inf_{y \in A} d_\gamma(y, x)$ .

For a closed max-tropical cone  $K \subseteq \mathbb{R}^n$  we define the projection  $\pi_K : \mathbb{R}^n \rightarrow K$  as

$$\pi_K(x) = \max\{y \in K : y \leq x\}. \quad (6)$$

The projection  $\pi_K$  is well-defined since  $K$  is a join semilattice, i.e. closed under taking maxima, and always satisfies  $\min_i [x_i - \pi_K(x)_i] = 0$ .

We note that  $\pi_K(x + \lambda \mathbf{1}) = \pi_K(x) + \lambda \mathbf{1}$  for every  $x \in \mathbb{R}^n$  and  $\lambda \in \mathbb{R}$ , so it induces a well-defined function  $\pi_{K/\mathbb{R}\mathbf{1}} : \mathbb{R}^n/\mathbb{R}\mathbf{1} \rightarrow K/\mathbb{R}\mathbf{1}$  called the *tropical projection* onto the max-tropically convex set  $K/\mathbb{R}\mathbf{1}$ .

The following lemma provides an explicit formula for the tropical projection and characterizes it as the closest point under the  $d_\gamma$  metric. The proof is omitted as it is a well-known result, demonstrated for the tropical distance in Cohen et al. (2004, Sect. 3) and for a general tropical  $L^p$  norm in Luo (2018, Theorem 4.6).

**Lemma 24** *Let  $A$  be a closed max-tropically convex set. Then the tropical projection  $\pi_A(x)$  of a point  $x$  has the entries*

$$\pi_A(x)_i = \max_{a \in A} \left( a_i + \min_{j \in [n]} (x_j - a_j) \right). \quad (7)$$

Moreover,  $d_\gamma(A, x) = d_\gamma(\pi_A(x), x)$  and  $\pi_A(x)$  is the unique point whose distance to  $x$  equals  $d_\gamma(A, x)$  if  $d_\gamma$  is strictly  $\Delta$ -star-quasiconvex.

From now on, our given sites are closed max-tropically convex sites  $A_1, \dots, A_m$  in  $\mathbb{R}^n/\mathbb{R}\mathbf{1}$ . Similar to Sect. 4, the objective function is  $h(x) := g(d_\gamma(A_1, x), \dots, d_\gamma(A_m, x))$ , where  $g : \mathbb{R}_{\geq 0}^m \rightarrow \mathbb{R}_{\geq 0}$  is increasing.

**Theorem 25** *There exists an minimum of  $h$  lying in the tropical convex hull of the input  $\text{tconv}^{\max}(A_1 \cup \dots \cup A_m)$ . Moreover, if  $g$  and  $\gamma$  are strictly increasing, then all the minima of  $h$  lie in  $\text{tconv}^{\max}(A_1 \cup \dots \cup A_m)$ .*

**Proof** If  $x \notin \text{tconv}^{\max}(A_1 \cup \dots \cup A_m)$ , then Proposition 4 entails the existence of an index  $\ell \in [n]$  such that  $\min_{j \neq \ell} (x_j - a_j) < x_\ell - a_\ell$  for every  $a \in \text{tconv}^{\max}(A_1 \cup \dots \cup A_m)$ . Since  $A_1, \dots, A_m$  are closed sets, then there exists an open ball around  $x$  not intersecting the union of these sets. Thus, for  $\delta > 0$  sufficiently small and  $y = x - \delta e_\ell$  we have

$$\min_j (y_j - a_j) = \min_j (x_j - a_j) \text{ for every } a \in \text{tconv}^{\max}(A_1 \cup \dots \cup A_m).$$

Therefore, Eq. (7) implies  $\pi_{A_i}(y) = \pi_{A_i}(x)$  for all  $i \in [m]$ .

Note that  $y - \pi_{A_i}(y) = x - \pi_{A_i}(x) - \delta e_\ell \leq x - \pi_{A_i}(x)$ . Since  $\gamma$  is increasing, we have

$$d_\gamma(A_i, y) = \gamma(y - \pi_{A_i}(y)) \leq \gamma(x - \pi_{A_i}(x)) = d_\gamma(A_i, x)$$

for every  $i \in [m]$ . Moreover, if  $\gamma$  is strictly increasing we get  $d_\gamma(A_i, y) < d_\gamma(A_i, x)$ .

In other words, going from  $x$  in the direction  $-e_\ell$  we obtain a decrease in all the distances  $d_\gamma(A_i, x)$ ; in particular, a decrease of  $h$ . Using this observation, the rest of the proof is identical to the proof of Theorem 18.  $\square$

## 6 Tropically convex consensus methods

In this section, we will be discussing applications of our methods to phylogenetics, which is the study of evolutionary history of species (Felsenstein 2003; Semple and Steel 2003). The information is typically represented in the form of an evolutionary tree, also known as a phylogeny, where the leaves are labeled with the names of the species. For the purposes of this paper, we will be focusing on trees that depict the evolution from a common ancestor and have a molecular clock. In other words, the ancestral relationships are depicted through a rooted tree with edge lengths. Since the labels are not important for our methods, we will assume that the species correspond to numbers from  $[n] := \{1, \dots, n\}$ .

To a rooted phylogeny  $T$  we associate a distance matrix  $D \in \mathbb{R}^{n \times n}$  where the entry  $D_{ij}$  represents the distance between the leaves  $i$  and  $j$  in  $T$ . We will focus only on *equidistant trees*, namely rooted trees having the same distance from the root to its leaves. It is known that  $T$  is equidistant if and only if  $D$  is ultrametric (Semple and

Steel 2003, Theorem 7.2.5), i.e.

$$D_{ij} \leq \max(D_{ik}, D_{kj}) \quad \forall i, j, k \in [n]. \quad (8)$$

Hence, we will not differentiate between equidistant trees and ultrametric matrices for the remainder of the paper.

Because  $D$  is symmetric and has zero entries on the diagonal, we can see it as a point of  $\mathbb{R}^{\binom{n}{2}}$ . We define the *tree space*  $\mathcal{T}_n$  as the image of the space of all ultrametrics in  $\mathbb{R}^{\binom{n}{2}}/\mathbb{R}\mathbf{1}$ . According to Ardila and Klivans (2006, Proposition 3), this space is homeomorphic to the BHV space defined in Billera et al. (2001). It is worth noting that the ultrametric condition (8) implies that  $\mathcal{T}_n$  is max-tropically convex.

We are interested in *consensus methods*, which involve taking multiple phylogenies in  $\mathcal{T}_n$  as input and finding an evolutionary tree that is as similar as possible to the input trees. This is a common problem in evolutionary biology, as multiple distinct trees can arise from different statistical procedures or methods used to reconstruct phylogenies from various data sources. For more information, refer to Bryant (2003) or Felsenstein (2003, Chapter 30).

A consensus method can be viewed as a location statistic in the tree space. As the tree space is max-tropically convex, there have been numerous attempts to exploit this geometric structure to gather pertinent information (Comănesci and Joswig 2024; Lin et al. 2017; Lin and Yoshida 2018; Page et al. 2020). Our focus is on tropically convex consensus methods, as defined in Comănesci and Joswig (2024).

**Definition 26** A consensus method  $c$  is *tropically convex* if it satisfies:

$$c(T_1, \dots, T_m) \in \text{tconv}^{\max}(T_1, \dots, T_m)$$

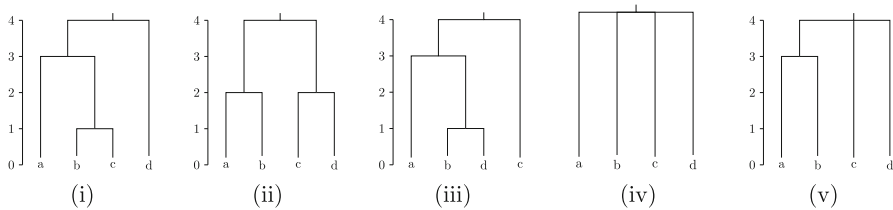
for all  $m \geq 1$  and  $T_1, \dots, T_m \in \mathcal{T}_n$ .

The previous section discussed location problems, which lead to the development of tropically convex consensus methods. It is important to note that there is no need to impose the restriction that the optimum must lie in  $\mathcal{T}_n$ . This requirement is automatically satisfied due to the tropical convexity of  $\mathcal{T}_n$  and Theorem 18. This observation guarantees that tropical median consensus methods can be quickly computed (Comănesci and Joswig 2024, Sect. 5.3).

Tropically convex consensus methods are also of particular interest because they preserve relationships from the input trees. To clarify, we must first introduce the terminology from Adams III (1986): two subsets of taxa  $A$  and  $B$  form a *nesting* in  $T$ , and we denote it by  $A < B$ , if the most recent common ancestor (MRCA) of  $A$  in  $T$  is a strict descendant of the MRCA of  $A \cup B$ .<sup>2</sup>

The collection of all nestings of a tree fully characterizes the phylogeny and Adams argued that they are better at revealing common features among different trees. Indeed, he points out that taxa having inconsistent positions in the tree might affect negatively

<sup>2</sup> Adams III (1986) restricted the definition to the case  $A \subset B$ , but we prefer this more relaxed version which occurs in Wilkinson et al. (2007).



**Fig. 5** Phylogenies of four taxa, namely  $a$ ,  $b$ ,  $c$ , and  $d$ . The first three trees are used as input for a consensus method. Tree (iv) represents their majority-rule consensus, while their tropical median consensus is (v)

methods based on clusters,<sup>3</sup> such as the majority-rule consensus of Margush and McMorris (1981). This observation was confirmed in practice by Wilkinson (1996) who called the inconsistent taxa “rogues.”

**Example 27** Consider the trees (i), (ii), and (iii) shown in Fig. 5 for taxa  $\{a, b, c, d\}$ . These three trees display the nesting  $\{a, b\} < \{c, d\}$  because the MRCA of  $a$  and  $b$  is never the root in any of those trees, while the root always represents the MRCA of all four taxa. It is worth noting that the placement of  $c$  and  $d$  in these trees varies, indicating that there is no consistent pattern. That would classify them as rogue taxa.

Tree (iv) is the majority-rule consensus of (i)–(iii). It is the simplest tree, having a single non-trivial cluster. This happens because no other non-trivial cluster occurs in more than one of the input trees. But the presence of the nesting  $\{a, b\} < \{c, d\}$  in all the input trees indicates that the taxa  $a$  and  $b$  have a stronger bond than shown in this consensus. On the other hand, the tropical median consensus tree (Comănesci and Joswig 2024) of (i)–(iii) is (v), which displays this nesting.

Since we focus on the matrix representation of a tree, we proceed to describing nestings in this representation. If  $D$  is the ultrametric associated to  $T$ , then we can write the condition  $A < B$  as

$$\max_{i,j \in A} D_{ij} < \max_{k,\ell \in A \cup B} D_{k\ell}. \quad (9)$$

We say that a consensus method  $c$  is *Pareto on nestings* if  $c(T_1, \dots, T_m)$  displays the nesting  $A < B$  whenever  $A < B$  appears in all input trees  $T_1, \dots, T_m$ . The consensus method  $c$  is called *co-Pareto on nestings* if  $c(T_1, \dots, T_m)$  does not display the nesting  $A < B$  unless  $A < B$  appears in some input tree  $T_i$ . These conditions are desirable for consensus methods (Bryant 2003; Wilkinson et al. 2007). Note that preserving nestings increases robustness against rogue taxa by maintaining relationships between more consistent taxa.

**Remark 28** It would be beneficial to approach these properties from a geometric perspective. Specifically, let us examine the subset  $\mathcal{T}_n(A < B)$  of  $\mathcal{T}_n$ , which comprises trees that exhibit the nesting  $A < B$ . This subset can be defined using (9). We also

<sup>3</sup> A *cluster* is the set of taxa marking the leaves of a connected component in a phylogeny, not containing the root, obtained after removing one branch.

make the notation  $\mathcal{T}_n(A \not< B)$  for the complement  $\mathcal{T}_n \setminus \mathcal{T}_n(A < B)$ , which is the set of trees not displaying  $A < B$ .

Then  $c$  is Pareto on nestings if and only if for every nesting  $A < B$  and trees  $T_1, \dots, T_m \in \mathcal{T}_n(A < B)$  we have  $c(T_1, \dots, T_m) \in \mathcal{T}_n(A < B)$ . We also note that  $c$  is co-Pareto on nestings if and only if for every nesting  $A < B$  and trees  $T_1, \dots, T_m \in \mathcal{T}_n(A \not< B)$  we have  $c(T_1, \dots, T_m) \in \mathcal{T}_n(A \not< B)$ .

The next result shows that tropically convex consensus methods have both Pareto and co-Pareto properties, being an improved version of Comănesci and Joswig (2024, Proposition 22). Thus, we have a large class of consensus methods satisfying both properties. This is remarkable, as no such consensus method is listed in the surveys Bryant (2003); Wilkinson et al. (2007).

**Proposition 29** *Tropically convex consensus methods are Pareto and co-Pareto on nestings.*

**Proof** For every nesting  $A < B$ , the set  $\mathcal{T}_n(A < B)$  is max-tropically convex as (9) describes an open max-tropical halfspace. Whence, Remark 28 implies that tropically convex consensus methods are Pareto on nestings.

Similarly, the set  $\mathcal{T}_n(A \not< B)$  is max-tropically convex as it is the intersection of  $\mathcal{T}_n$  with the tropical halfspace defined by the inequality  $\max_{i,j \in A} D_{ij} \geq \max_{k,\ell \in A \cup B} D_{k\ell}$ . Remark 28 implies also the co-Pareto property.  $\square$

The Pareto property states that any nesting present in all trees will also be present in the consensus tree. However, it is worth considering if this rule can be relaxed, as in (super)majority-rule consensus trees, which were developed for the unweighted case. These consensus methods are denoted  $M_\ell$  by Felsenstein in Felsenstein (2003, Chapter 30), where  $\ell$  is the proportion of majority needed for the occurrence of a cluster. Indeed, one can find a similar rule for tropical medians (Comănesci and Joswig 2024).

**Proposition 30** *A nesting appears in the tropical median consensus tree if it appears in a proportion of the input trees greater than  $1 - 1/\binom{n}{2}$ . Moreover, a nesting will not appear in the tropical median consensus tree if it occurs in a proportion less than  $1/\binom{n}{2}$  of the input trees.*

**Proof** The tropical median corresponds to the Fermat–Weber problem whose gauge distance is given by the regular simplex. Therefore, the elementary hull of a finite set  $A$  defined in Comănesci and Plastria (2023) coincides with the max-tropical convex hull of  $A$ . Then the conclusion follows from (Comănesci and Plastria 2023, Proposition 5.6) and Remark 28, as in the proof of Proposition 29.  $\square$

**Remark 31** Note that a consensus method is not well-defined when there are multiple minimum points. This is especially problematic when there are different possible tree topologies, as it is unclear how to resolve incompatible optimum trees. However, this is not an issue when the set of optimal locations is convex (Comănesci and Joswig 2022, Proposition 6): separating the tree space into cones of trees with different topologies creates a convexly disjoint collection, as defined in Hacking et al., (2009, Definition 1.15).

Nonetheless, the aforementioned proposition applies when the set of all optima in  $\mathbb{R}^{(n)} / \mathbb{R}\mathbf{1}$  is contained in  $\mathcal{T}_n$ ; guaranteed for strictly  $\Delta$ -star-quasiconvex dissimilarities. Otherwise, one might still have problems in defining consistently a consensus method; see (Comănesci and Joswig 2024, Example 24) for the symmetric tropical Fermat–Weber problem. For this reason, one has to consider regularized versions as discussed in Sect. 4.2.

## 7 Conclusion and future perspectives

We have developed a comprehensive set of location estimators that are contained within the max-tropical convex hull of the input. These estimators are designed to produce consensus methods with strong properties. The first direction would be to obtain methods to obtain the optima efficiently. Furthermore, exploring additional properties of specific location problems could prove beneficial for practical applications. More information on this topic is provided below.

### 7.1 Comparison to consensus methods based on the BHV distance

We have exploited tropical convexity to obtain consensus methods with good properties. More precisely, we focused on (co-)Pareto properties that can be interpreted in a purely geometric way. The associated spaces are also max-tropically convex so the aforesaid properties are immediate for the tropical approach.

Although the BHV geometry of the tree space has been studied more extensively than its tropical counterpart, there are currently few consensus methods proposed for this geometry. The first proposal was given in the pioneering paper by Billera et al. (2001), but it has been noted that there are some drawbacks, such as the fact that doubling every input tree changes the output. Another approach, based on Fréchet means, was proposed by Miller et al. (2015) and Bačák (2014). This method is also Pareto and co-Pareto on splits (Miller et al. 2015, Lemma 5.1), but the result is more involved. The same properties hold for Fermat–Weber and center problems in the BHV space (Botte 2019, Chapter 3). The approach is again analytical, but it is similar for all cases. However, it may be worth considering a geometric approach, as in the tropical case, as it could potentially lead to a faster identification of self-consistent properties for consensus methods.

### 7.2 Majority rules in consensus methods

Proposition 30 provides a supermajority rule for tropical median consensus with respect to nestings. This is a crucial step towards comprehending the connection between median weighted trees and the commonly used majority-rule consensus for unweighted trees. In fact, it has been suggested that the majority-rule consensus can be viewed as a type of median (Margush and McMorris 1981), but it remains uncertain whether this can be applied to weighted phylogenies.

Proposition 30 states that tropical median consensus trees have a high threshold for a majority rule, suggesting that they are conservative. This is due to the low breakdown point of the tropical median, which is caused by its asymmetry. For more information, refer to Comănesci and Plastria (2023). Therefore, exploring location estimators with a higher breakdown point may lead to a stronger connection with the majority-rule consensus.

### 7.3 Compositional data

A different application of our location estimators could be to compositional data (Pawlowsky-Glahn et al. 2015). In this case, the data can be viewed as points within a simplex, and our methods would be applied to the centered logratio transform of the input. It should be noted that  $\Delta$ -star-quasiconvex sets are defined based on specific directions, which correspond to the vertices of the simplex.

What is more, the motivation of Tokuyama and Nakano in studying algorithms for transportation problem came from splitting the points from a simplex in multiple regions (Tokuyama and Nakano 1995). Moreover, Nielsen and Sun analyzed clustering methods with the symmetric tropical distance on compositional data showing a better performance than other more commonly used dissimilarity measures (Nielsen and Sun 2019). These results suggest that using  $\Delta$ -star-quasiconvex dissimilarities may be beneficial in compositional data analysis.

## Appendix: $\Delta$ -convexity

In our paper, we defined the tropical semiring  $\mathbb{T} = (\mathbb{R} \cup \{-\infty\}, \vee, +)$  using the max convention. However, some sources focus on the min convention:  $\mathbb{T} = (\mathbb{R} \cup \{+\infty\}, \wedge, +)$  where  $x \wedge y := \min(x, y)$ . The tropical cones and tropical convex sets were defined based on the choice of  $\vee$  as the tropical addition, but equivalent definitions hold for the min setting. In fact, the two semirings are isomorphic under the map  $x \mapsto -x$ . Further relationship between this duality is explored in Joswig (2021, Chapter 6).

Below, we need show that  $\Delta$ -star-convexity is related to min-tropical convexity and not the max analogue. For this, it will be useful to define the min-tropical segment between  $a$  and  $b$ , denoted  $[a, b]_{\min}$ , as the min-tropical convex hull of two point  $a$  and  $b$ . Then, a subset  $S$  of  $\mathbb{R}^n/\mathbb{R}\mathbf{1}$  is min-tropically convex if and only if  $[a, b]_{\min} \subseteq S$  for every  $a, b \in S$ .

**Lemma 32** *The set  $[a, b]_{\Delta}$  contains the ordinary segment  $[a, b]$  as well as the min- and max-tropical segments between  $a$  and  $b$ . Further, we have  $[a, b]_{\min} \subseteq [c, a]_{\Delta} \cup [c, b]_{\Delta}$  for every  $c \in \mathbb{R}^n/\mathbb{R}\mathbf{1}$ .*

**Proof** The first statement is a direct consequence of Lemma 7. For the last statement, we take arbitrary representatives modulo  $\mathbb{R}\mathbf{1}$  for  $a$  and  $b$  and show that  $a \wedge b \in [c, a]_{\Delta} \cup [c, b]_{\Delta}$ . Let  $i \in \arg \min_j [(a_j \wedge b_j) - c_j]$ . Without loss of generality, we can assume that  $a_i \wedge b_i = a_i$ . Thus,  $a \wedge b \in S_i^{\min}(c) \cap S_i^{\max}(a) = [c, a]_{\Delta}$ .  $\square$

**Proposition 33** *Any  $\Delta$ -star-convex set is min-tropically convex.*

**Proof** Let  $K$  be a  $\Delta$ -star-convex set with kernel  $v$  and  $a, b$  arbitrary points in  $K$ . According to Lemma 32, we have  $[a, b]_{\min} \subseteq [v, a]_{\Delta} \cup [v, b]_{\Delta}$ . The latter set is contained in  $K$  due to its  $\Delta$ -star-convexity.  $\square$

However,  $\Delta$ -star-convex sets might not be max-tropically convex. For example, the image of the regular simplex  $\Delta_n = \text{conv}\{e_1, \dots, e_n\}$  in  $\mathbb{R}^n/\mathbb{R}\mathbf{1}$  is  $\Delta$ -star-convex but not max-tropically convex.

### $\Delta$ -convex sets

In this subsection, we are interested in analyzing  $\Delta$ -convex sets. In other words, we want to characterize a subset  $S$  of  $\mathbb{R}^n/\mathbb{R}\mathbf{1}$  such that  $[v, w]_{\Delta} \subseteq S$  for every  $v, w \in S$ . The interesting situation will be for  $n \geq 3$ , as the case  $n = 2$  reduces to a line and the geodesic segment  $[v, w]_{\Delta}$  coincides with the ordinary segment  $[v, w]$ . So we will assume below that  $n \geq 3$  and  $S$  is  $\Delta$ -convex.

In the following,  $e_i$  is the standard basis vector in  $\mathbb{R}^n$  with 1 on the  $i$ th position. Without loss of generality, we will use the notation  $e_i$  also for its image in  $\mathbb{R}^n/\mathbb{R}\mathbf{1}$ .

**Lemma 34** *If  $\mathbf{0} \in S$  and  $\varepsilon e_i \in S$  for some  $\varepsilon > 0$ , then  $2\varepsilon e_i \in S$ .*

**Proof** Since the natural action of symmetric group  $\Sigma_n$  on  $\mathbb{R}^n/\mathbb{R}\mathbf{1}$  preserves  $\Delta$ -convexity, we can assume that  $i = n$ . Furthermore, since  $\Delta$ -convexity is preserved under positive scalings, we can also assume that  $\varepsilon = 1$ .

Note that  $-e_1 \in [\mathbf{0}, e_n]_{\Delta} \subseteq S$ . Therefore, we also have  $[-e_1, e_n]_{\Delta} \subseteq S$ . Since  $n \geq 3$ , we have  $[-e_1, e_n]_{\Delta} = S_2^{\min}(-e_1) \cap S_2^{\max}(e_n) \ni e_n - e_1$ . Using the  $\Delta$ -convexity of  $S$  again, we get  $[e_n, e_n - e_1]_{\Delta} \subseteq S$ . But we have  $2e_n \in S_1^{\min}(e_n) \cap S_1^{\max}(e_n - e_1) = [e_n, e_n - e_1]_{\Delta}$ . Thus,  $2e_n \in S$ .  $\square$

**Corollary 35** *If  $\mathbf{0} \in S$  and  $\varepsilon e_i \in S$  for some  $\varepsilon > 0$ , then*

$$\text{pos}(e_i) := \{\lambda e_i : \lambda \geq 0\} \subseteq S.$$

**Proof** Using the Lemma 34 repeatedly, we obtain  $2^\ell \varepsilon e_i \in S$  for every  $\ell \geq 0$ . Since  $S$  is convex in the ordinary sense, we obtain the conclusion.  $\square$

**Lemma 36** *If  $\mathbf{0} \in S$  and  $|S| \geq 2$ , then there exist  $i \in [n]$  and  $\varepsilon > 0$  such that  $\varepsilon e_i \in S$ .*

**Proof** Let  $v \in S \setminus \{\mathbf{0}\}$ ,  $k \in \arg \min_j v_j$  and  $i \notin \arg \min_j v_j$ . Then

$$\hat{v}_i e_i \in S_k^{\min}(\mathbf{0}) \cap S_k^{\max}(v) = [\mathbf{0}, v]_{\Delta} \subseteq S$$

where  $\hat{v}$  are the canonical coordinates of  $v$ .  $\square$

**Theorem 37** *If  $n \geq 3$  and  $S$  is  $\Delta$ -convex, then  $S$  is a singleton or  $S = \mathbb{R}^n/\mathbb{R}\mathbf{1}$ .*



**Proof.** Since translations preserve  $\Delta$ -convexity, we can assume that  $\mathbf{0} \in S$ . We further consider the case when  $|S| \geq 2$ . From Lemma 36 there exist  $\varepsilon > 0$  and  $i \in [n]$  such that  $\varepsilon e_i \in S$ . What is more, Corollary 35 implies that  $\text{pos}(e_i) \subseteq S$ .

Using the  $\Delta$ -convexity of  $S$ , we obtain

$$-e_j \in S_i^{\min}(e_i) \cap S_i^{\max}(\mathbf{0}) = [e_i, \mathbf{0}]_{\Delta} \subseteq S$$

for every  $j \neq i$ . Taking any index  $k \neq j$ , we obtain also

$$e_k \in S_j^{\min}(\mathbf{0}) \cap S_j^{\max}(-e_j) = [\mathbf{0}, -e_j]_{\Delta} \subseteq S.$$

In other words, we obtained  $e_k \in S$  for every  $k \in [n] \setminus \{j\}$ . But  $j$  was an arbitrary index different from  $i$  and  $n \geq 3$ , so a similar argument shows that  $e_j$  also belongs to  $S$ .

Using Corollary 35 again, we obtain  $\text{pos}(e_j) \subseteq S$  for every  $j \in [n]$ . But we also know that  $S$  is convex in the ordinary sense, so

$$\mathbb{R}^n / \mathbb{R}\mathbf{1} = \text{conv} \left( \bigcup_{j \in [n]} \text{pos}(e_j) \right) \subseteq S.$$

□

This shows that there are not many  $\Delta$ -convex sets. In fact, for  $n \geq 3$ , we have just proven that they fall in two categories: singletons and the entire space. This is very restrictive, but not uncommon for geodesically convex sets when the metric is polyhedral. Such examples can be seen in Boltyanski et al. (1997, Chapter 2), especially in Exercise 6 from Boltyanski et al. (1997, §9).

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## Declarations

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