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ORIGINAL ARTICLE



Complexity of the multiobjective minimum weight minimum stretch spanner problem

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Abstract

In this paper, we take an in-depth look at the complexity of a hitherto unexplored *multiobjective minimum weight minimum stretch spanner* problem; or in short *multiobjective spanner (MSp)* problem. The MSp is a multiobjective generalization of the well-studied minimum t-spanner problem. This multiobjective approach allows to find solutions that offer a viable compromise between cost and utility—a property that is usually neglected in singleobjective optimization. Thus, the MSp can be a powerful modeling tool when it comes to, e.g., the planning of transportation or communication networks. This holds especially in disaster management, where both responsiveness and practicality are crucial. We show that for degree-3 bounded outerplanar instances the MSp is intractable and computing the non-dominated set is **BUCO**-hard. Additionally, we prove that if $\mathbf{P} \neq \mathbf{NP}$, the set of extreme points cannot be computed in output-polynomial time, for instances with unit costs and arbitrary graphs. Furthermore, we consider the directed versions of the cases above.

Keywords Multiobjective optimization · Graph spanners · Output-sensitive complexity · Extreme points · Parametric optimization

1 Introduction

Natural disasters require a significant logistical effort to provide relief to victims and to distribute equipment, humanitarian goods, and information. The efficient design of emergency infrastructure therefore forms the basis for initial responses, as well as for long-term measures taken to stabilize affected communities. An important part of this

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infrastructure is the construction of makeshift communication networks connecting cut-off areas. These makeshift networks can be modeled as *spanners*, a subset of connections that connect all of the nodes in the network. As time and resources are scarce, spanners must offer a balanced combination of cost-effectiveness and utility. Their utility can be measured in their stretch factor-the maximum factor by which the length of the shortest path between any two nodes is extended in the spanner compared to the original network. Spanners are an important tool in a wide range of fields. Therefore, their efficient construction is examined in a variety of papers (Sigurd and Zachariasen 2004; Ahmed et al. 2019). Clearly, cost-effectiveness and a low stretch-factor are two inherently conflicting objectives. In a literature review, Caunhye et al. (2012) state that in emergency logistics, many optimization models only focus on one of the aspects, marginalising the other as a constraint or even ignoring it completely. This is done as for many problems, multiobjective approaches are perceived as too difficult to solve in practice. Albeit, Caunhye et al. (2012) conclude that these singleobjective models may hamper relief services by causing an oversupply of resources leading to difficulty with coordination, greater traffic, and complex scheduling. This motivates the consideration of the Multiobjective Minimum Weight Minimum Stretch Spanner problem; or in short Multiobjective Spanner (MSp) problem-a multiobjective generalization of the Minimum *t*-Spanner problem.

Given a connected, simple graph G = (V, E) where every edge has a cost and length of 1, a subset of edges S is a t-spanner of G if for every pair of vertices $u, v \in V, \frac{d^S(u,v)}{d^E(u,v)} \leq t$ holds, with $d^S(u, v)$ being the distance from u to v in S and $d^{E}(u, v)$ their respective distance in E (Peleg and Schäffer 1989). For a given graph, the problem of finding the cheapest t-Spanner, with regard to the sum over all edge-costs, is commonly known as the *Minimum t-Spanner* problem. It is well-known to be NPhard (Cai 1994). We refer to t as the stretch factor. The MSp generalizes the Minimum t-Spanner problem by introducing two edge-weight functions, allowing us to assign each edge a cost independent of its length. Furthermore, in contrast to the Minimum t-Spanner problem, the goal of the MSp is not to find a minimum weight spanner for a given stretch factor. Instead, the stretch factor is another objective we aim to minimize. The stretch factor is an interesting objective function in itself that is to be minimized in, e.g., the Minimum Max-Stretch Spanning Tree (MMST) problem (Cai and Corneil 1995). Feasible solutions for the MSp are defined less restrictive than t-Spanners. We define a spanner of a connected, undirected graph G = (V, E), as a subset of edges $S \subseteq E$, such that G' = (V, S) is a connected subgraph. Since the MSp is a Multiobjective Combinatorial Optimization (MOCO) problem, solutions are mapped to a value vector instead of a single value. Due to conflicts among objectives, there does not necessarily exist a solution that achieves the best value in all objective functions simultaneously. Instead, we look for value vectors for which there are no other value vectors that dominate them. This set of value vectors is called *non-dominated set* (or Pareto-front) \mathcal{Y}_N . See below for formal definitions of these notions. The reason we look at the value vectors instead of the solutions they are associated with is that we consider two different solutions that produce the same value vectors to be equivalent. As is also the case in singleobjective optimization: we usually require algorithms to produce only one optimal solution, even if more than one exist.

Definition 1 (*Multiobjective minimum weight minimum stretch spanner* (*MSp*) problem) The input is a connected, undirected graph G = (V, E) and edge-weight functions $c_1: E \to \mathbb{Z}$ and $c_2: E \to \mathbb{N}_+$. Feasible solutions are spanners S of G and are assessed based on the two objective functions

$$f_1(S) = \sum_{e \in S} c_1(e) \text{ and } f_2(S) = \max_{u,v \in V} \frac{d_{c_2}^S(u,v)}{d_{c_2}^E(u,v)},$$

with $d_{c_2}^S(u, v)$ and $d_{c_2}^E(u, v)$ being the length of the shortest u-v-path in S and E respectively, regarding the c_2 -length. We consider an instance of the MSp to be solved if we output its non-dominated set \mathcal{Y}_N .

The concept of t-Spanners and their related problems were first introduced by Peleg, Schäffer, and Ullman in the context of synchronization in distributed systems and communication networks (Peleg and Schäffer 1989; Peleg and Ullman 1989) and have since been explored in a variety of publications. A greedy algorithm with one of the best cost-guarantees was developed by Althöfer et al. (1993). For a graph G and a stretch factor of t = 2k - 1 ($k \in \mathbb{N}_{>1}$), it creates a *t*-Spanner *S* of *G* containing $\mathcal{O}(n^{1+1/k})$ edges in time $\mathcal{O}(m(n^{1+1/k} + n \log n))$. This algorithm can even be applied to the Weighted Minimum t-Spanner problem, where every edge is assigned an arbitrary positive cost. Then the algorithm additionally guarantees that S has a cost of at most $\mathcal{O}(n/k)$ times the cost of the minimum spanning tree of G. For undirected Minimum 2-Spanners, Kortsarz and Peleg (1994) published a $\mathcal{O}(\log(m/n))$ -approximation with a theoretical running time of $\mathcal{O}(m^2n^2\log(n^2/m))$. For a weighted, undirected graph, Baswana and Sen (2007) give a method that computes a t = 2k - 1 spanner S that contains at most $\mathcal{O}(kn^{1+1/k})$ edges in an expected running time of $\mathcal{O}(km)$, but with no cost guarantee. For unweighted graphs the size of *S* is bounded by $O(n^{1+1/k} + kn)$. Cai and Keil (1994) focused on the complexity of the Minimum t-Spanner problem for degree bounded graphs and showed, among others, that if the maximum degree of the graph is at most 4, the Minimum 2-Spanner problem can be solved in linear time, whereas the problem is NP-hard even if the maximum degree is at most 9. A recent paper by Kobayashi (2018) focuses on the complexity of the Minimum t-Spanner problem in planar graphs and, as a byproduct, improves the degree bounds for **NP**-hardness found by Cai and Keil.

As many decisions require the consideration of multiple goals and conflicting demands, MOCO problems are an important modeling tool in a variety of fields. Practical applications include routing problems in public transport (Delling et al. 2015; Wagner and Zündorf 2017), the planning of radiotherapy (Hamacher and Küfer 1999; Thieke et al. 2007; Giantsoudi et al. 2013) and the determination of control strategies for vaccine administration in COVID-19 pandemic treatment (Libotte et al. 2020). As mentioned before, MOCO problems are often perceived as too difficult to solve in practice. However, a theoretical base can substantiate this perception. In the multi-objective context, a problem is called *intractable*, if there is no algorithm capable of solving it in polynomial time (Ehrgott 2005). Due to the exponential size of their non-dominated sets, many interesting MOCO problems are intractable, e.g., multiobjective variants of the Traveling Salesperson (Emelichev and Perepelitsa 1992), Shortest Path

(Hansen 1980), or Spanning Tree (Hamacher and Ruhe 1994) problem. Therefore, it makes sense to consider a complexity class that distinguishes between problems that cannot be solved in polynomial time due to the size of their output and the ones that are genuinely hard to solve. This motivates an output-sensitive analysis: We say a MOCO problem O is solvable in *output-polynomial time* if there is an algorithm that, for any given instance I of O, outputs every $y \in \mathcal{Y}_N$ exactly once and runs in polynomial time depending on the size of the input I and the output \mathcal{Y}_N (Johnson et al. 1988). Such an algorithm is called *output-polynomial*. We denote the class of problems for which an output-polynomial algorithm exists as **OP**. There are many different gradations of output-sensitivity, but for our purposes we exclusively focus on **OP**. This is done as it is the most basic class. Thus, showing that a problem is not in **OP** directly implies negative results for all of the more restrictive classes, such as Polynomial Time Delay.

Three observations further motivate the investigation of this enumerative approach. First, in experimental studies for other MOCO problems, the non-dominated sets are much smaller than in the worst case (cf., e.g., Bökler and Chimani 2020). Second, there is also a theoretical reason for this behavior: In a smoothed analysis setting, Brunsch and Röglin (2015) showed that the expected size of the non-dominated set is at most polynomial in the input size for each fixed number of objectives. Third, although in theory, for some MOCOs an approximation with quality guarantees can be computed efficiently (Papadimitriou and Yannakakis 2000). However, to the best of our knowledge, there currently is no practical implementation capitalizing on this property and thus is competitive to exact algorithms except in special cases (see, e.g., Bökler and Chimani 2020).

An interesting subset of the non-dominated set is the set of so called extreme points \mathcal{Y}_X (for a definition see Sect. 5). Determining \mathcal{Y}_X is a *parametric optimization* problem. The extreme points of a MOCO problem instance are exactly the points that are needed to solve any weighted sum scalarization (WSS) of the instance. In a WSS all objective functions are combined into a singleobjective linear scalar (or preference) function, where each objective function is weighted according to its importance. If every decision maker has a linear preference function, computing the extreme points suffices. Note, however, that in general not all non-dominated points can be found in this way. As every extreme point is non-dominated, while not every non-dominated value vector is an extreme point, solving the MSp could be hard, while the problem of only computing the set of extreme points could be in **OP**. This is the case for, e.g., *Multiobjective* Shortest Path (Bökler and Mutzel 2015; Bökler et al. 2017). It is important to note, however, that determining weights accurately reflecting the preferences of the decision makers is not trivial and that scalar approaches do not give as much information about trade-offs between solutions as the enumerative approach. For more information on MOCOs and related topics cf. the book by Ehrgott (2005).

Contribution and organisation. In the remainder of this paper, we first give some definitions and establish basic concepts and results in Sect. 2. In Sect. 3, we study the classic tractability of MSp.

Theorem 1 *MSp is intractable even on degree-3 bounded outerplanar graphs.*

This is an interesting result as there are non-trivial stretch factors for which the Minimum *t*-Spanner problem is solvable in linear time under such restrictions. In Sect. 4,

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we first consider the output-sensitive complexity of computing the non-dominated set for unweighted instances of the MSp, where each edge has a cost and length of 1. In Corollary 8 we infer that, unless $\mathbf{P} \neq \mathbf{NP}$, such computations cannot be done in output-polynomial time for graphs on which the *t*-Spanner^{DEC} problem is **NP**-hard. This includes, e.g., degree-6 bounded planar graphs (Kobayashi 2018). Afterwards, we consider the *BUCO* problem (cf. Definition 2 below) that can be interpreted as an unrestricted version of the Knapsack problem and discuss the output-sensitive complexity of computing the non-dominated set for degree bounded outerplanar instances of the MSp. While BUCO appears to be a straight forward problem, it is currently unknown whether it can be solved in output-polynomial time (Bökler et al. 2020). However, it has been shown that there are other problems of unknown output-sensitive complexity that the BUCO problem can be reduced to. This motivated the introduction of the complexity-class of **BUCO**-hard problems (Bökler 2018). If we showed that one of the **BUCO**-hard problems can be solved in output-polynomial time, we would prove that the BUCO problem can be solved in output-polynomial time.

Theorem 2 MSp is BUCO-hard even on degree-3 bounded outerplanar graphs.

Note that this result holds for a very restricted class of graphs to which currently Corollary 8 cannot be applied, as it is as of now unclear whether there is a stretch factor *t* such that the Minimum *t*-Spanner problem is NP-hard on degree-3 bounded outerplanar graphs (Kobayashi 2018). Hence, Theorem 2 improves on Corollary 8 in the degree requirement as well as the planarity requirement. However, the precondition that BUCO \notin OP is weaker than $\mathbf{P} \neq \mathbf{NP}$, as $\mathbf{P} = \mathbf{NP}$ implies BUCO \in OP. Moreover, this theorem implies that if there is a polynomial time algorithm for the Minimum *t*-Spanner problem on degree-3 bounded outerplaner graphs where t > 1 is part of the input then BUCO can be solved in output polynomial time.

As Corollary 8 states that we cannot compute the entire non-dominated set of unweighted MSp instances in output-polynomial time, in Sect. 5 we define the problem of computing the set of extreme points for instances of the MSp (MSp^{YEx}) and show its hardness with regard to output-sensitive complexity.

Theorem 3 If $P \neq NP$, then $MSp^{YEx} \notin OP$, even for unweighted instances.

Finally, Sect. 6 has concluding remarks. Note that we also define a directed version of the MSp (diMSp), for which we prove the same results. The only exemption being Theorem 2. The diMSp is **BUCO**-hard, even for degree-4 bounded outerplanar instances. It is clear that the definition of the MSp can be extended by adding further arbitrary objective functions. However, keep in mind that the basic biobjective MSp remains a special case of these inflated problem variants. Thus, all the hardness results mentioned above still apply.

2 Preliminaries

We denote $\mathbb{N} = \{0, 1, 2, ...\}, \mathbb{N}_+ := \mathbb{N} \setminus \{0\}$ and non-negative real numbers as \mathbb{R}_{\geq} . For $n \in \mathbb{N}$, we denote the set $\{1, ..., n\}$ as [n]. For a graph G = (V, E), an edge $\{u, v\} \subseteq E$

and an edge-weight function $c_i : E \to \mathbb{Z}, i \in [2]$ we abbreviate $c_i(\{u, v\})$ by $c_i(u, v)$. Furthermore, for a set of edges $S \subseteq E$, we denote $c_i(S) = \sum_{e \in S} c_i(e)$. In order to simplify the input of the (di)MSp, we sometimes combine the edge-weight functions c_1 and c_2 into one function $c : E \to \mathbb{Z} \times \mathbb{N}_+$ with $c(e) = (c_1(e), c_2(e))^T$. Similarly, for an instance of the (di)MSp and a feasible (directed) spanner *S*, we combine the two objective functions $f_1(S)$ and $f_2(S)$ into one single function $f(S) = (f_1(S), f_2(S))^T$ that directly maps *S* to its value vector. We sometimes refer to $c : E \to \{(1, 1)^T\}$ with $c(e) = (c_1(e), c_2(e))^T = (1, 1)^T$ for all $e \in E$ as the *trivial edge-weight function* and call instances of the MSp with these edge-weight functions *unweighted*.

The degree of a vertex in an undirected graph is the number of vertices it is adjacent to. The degree of a vertex in a directed graph is the sum of the number of its in- and out-going edges. For $\delta \in \mathbb{N}$, we call any graph G = (V, E) degree- δ bounded if for all $v \in V$ their degree is less than or equal to δ . We call graphs *outerplanar* if they have a planar drawing, in which every vertex lies on the boundary of the outer face. We call undirected graphs *connected* if they are non-empty and any two of their vertices are linked by a path. A directed graph is called *weakly connected* if replacing all of its arcs with undirected edges results in a connected (undirected) graph. See also Diestel (2017); Bang-Jensen and Gutin (2008). For an instance of a MOCO problem with an objective function f, we denote the set of all its value vectors as \mathcal{Y} . For unequal value vectors $y, y' \in \mathcal{Y}$, we say y is *dominated* by y' if y' is component wise less than or equal to y. Analogously, for feasible solutions S, S' we say S is *dominated* by S' if f(S) is dominated by f(S'). If a value vector is not dominated by any value vector, we call it *non-dominated*; the associated solution is called *Pareto-optimal*.

For a weakly connected, directed graph G = (V, A), we call a subset of arcs $S \subseteq A$ a *directed spanner* of G, if G' = (V, S) is a subgraph such that, for every pair of vertices $u, v \in V$, if there is a directed u-v-path in A, there is one in S as well.

Note that the definition of a (directed) spanner does not require the resulting subgraph to be acyclic. Analogously to the MSp, we define the *directed multiobjective minimum weight minimum stretch spanner* (diMSp) problem. The only differences being that the input is a weakly connected, directed graph, solutions are now directed spanners and that in the second objective function, we only consider pairs of vertices that are connected in the initial graph. This guarantees the well-definedness of the objective function values.

Lemma 4 For a set of (di)MSp instances \mathcal{I} , if there is a polynomial $p: \mathbb{N} \to \mathbb{N}$, such that for every instance $I \in \mathcal{I}$ and its set of solutions $\mathcal{S}: |f_i(\mathcal{S})| \le p(|I|)$ for i = 1 or i = 2, then $|\mathcal{Y}_N| \le p(|I|)$.

Proof Without loss of generality, we can assume that f_1 only has polynomially many different values in its image. For every $a \in f_1(S)$, there is one $s' \in S$ with $f_1(s') = a$ and $f_2(s') \leq f_2(s)$ for all $s \in S$ with $f_1(s) = a$. Hence, $(a, f_2(s'))^T$ dominates $(a, f_2(s))^T$ for all $s \in S$ with $f_1(s) = a$. Thus, for each $a \in f_1(S)$ there is only one non-dominated value vector.

Observation 1 It is clear that adding edges to a spanner never increases its stretch factor and that therefore, for every non-dominated value vector $y \in \mathcal{Y}_N$ there is a spanner *S* with f(S) = y and $e \in S$ for all $e \in E$ with $c_1(e) = 0$.

We call the decision problem corresponding to the Minimum *t*-Spanner problem *t*-Spanner^{DEC}. For it, verifying the stretch factor *t* of a spanner only requires considering pairs of vertices that are adjacent in the underlying graph (Peleg and Schäffer 1989). In case of the MSp a similar statement can be made.

Lemma 5 Let $I = (G = (V, E), c_1, c_2)$ be an MSp instance with a connected, undirected graph G and edge-weight functions c_1 and c_2 . For any spanner S of G, $f_2(S) = \max_{u,v \in V} \frac{d_{c_2}^S(u,v)}{d_{c_2}^E(u,v)} = \max_{\{u,v\} \in E} \frac{d_{c_2}^S(u,v)}{d_{c_2}^E(u,v)}$ holds.

Proof Let *S* be a spanner of *G* and assume $f_2(S) = \frac{d_{c_2}^S(r,z)}{d_{c_2}^E(r,z)}$ with $\{r, z\} \notin E$. Let $\{r = u_0, u_1\}, \{u_1, u_2\}, ..., \{u_{m-1}, u_m = z\}$ be the shortest r-z-path in *E*. Denote the set of edge $\{u_i, u_{i+1}\}, 0 \le i \le m-1$ as *U*. Observe that for every $0 \le i \le m-1$, $\frac{d_{c_2}^S(u_i, u_{i+1})}{d_{c_2}^E(u_i, u_{i+1})} \le \max_{\{u_k, u_{k+1}\} \in U} \frac{d_{c_2}^S(u_k, u_{k+1})}{d_{c_2}^E(u_k, u_{k+1})}$ holds. We get

$$\frac{d_{c_2}^{S}(r,z)}{d_{c_2}^{E}(r,z)} \leq \frac{\sum_{i=0}^{m-1} d_{c_2}^{S}(u_i, u_{i+1})}{\sum_{i=0}^{m-1} d_{c_2}^{E}(u_i, u_{i+1})} \leq \max_{\{u_k, u_{k+1}\} \in U} \frac{d_{c_2}^{S}(u_k, u_{k+1})}{d_{c_2}^{E}(u_k, u_{k+1})} \cdot \frac{\sum_{i=0}^{m-1} d_{c_2}^{E}(u_i, u_{i+1})}{\sum_{i=0}^{m-1} d_{c_2}^{E}(u_i, u_{i+1})} \\
= \max_{\{u_i, u_{i+1}\} \in U} \frac{d_{c_2}^{S}(u_i, u_{i+1})}{d_{c_2}^{E}(u_i, u_{i+1})}.$$

It is clear that the same arguments hold for the directed case.

3 Intractability

We begin by proving that no algorithm is capable of solving the MSp in polynomial time, even if we restrict the considered graphs to be both degree-3 bounded and outerplanar. We do this by showing that there is a family of instances, complying to these restrictions, for which the size of the non-dominated set \mathcal{Y}_N is exponential in the size of the instance, proving the intractability of the MSp.

Consider the following family of instances for $2 \le n \in \mathbb{N}$, of connected, undirected graphs G = (V, E) and edge-weight functions $c_1 : E \to \mathbb{Z}$ and $c_2 : E \to \mathbb{N}_+$. For every $i \in [n]$, we create vertices v_i, v'_i and w_i , and add edges $\{v_i, w_i\}$ with weights $(2^i, 2^i)^T$, as well as edges $\{v_i, v'_i\}$ and $\{v'_i, w_i\}$ with respective weights of $(0, 2^i)^T$. Furthermore, we introduce edges $\{w_i, v_{i+1}\}$ with weights $(0, 1)^T$, for $i \in [n - 1]$. Finally, we define $v_1 := s$ and $w_n := t$ and add the edge $\{s, t\}$ with weights $(2^{n+1}, 1)^T$. An example of this construction can be seen in Fig. 1. Note that every so constructed graph is degree-3 bounded and outerplanar.

Note that the graph *G* contains at least 2^n spanners that do not contain the edge $\{s, t\}$. With Observation 1, we know that for every non-dominated value vector *y*, there is a Pareto-optimal spanner *S* of *G* that contains every $e \in E$ with $c_1(e) = 0$, with f(S) = y. Define *X* as the set of all feasible spanners *S* of *G*, that contain every



Fig. 1 The family of instances constructed for Fig. 1. Thick edges hold weights $(0, 1)^{T}$

 $e \in E$ with $c_1(e) = 0$ and do not contain the edge $\{s, t\}$. We now simplify the second objective function $f_2(S)$, for every $S \in X$, using Lemma 5.

Lemma 6 For all spanners $S \in X$, $f_2(S) = \max_{u,v \in V} \frac{d_{c_2}^S(u,v)}{d_{c_2}^E(u,v)} = \frac{d_{c_2}^S(s,t)}{d_{c_2}^E(s,t)}$.

Proof Let $S \in X$ be a spanner. With Lemma 5, we know that we only have to consider pairs of vertices $u, v \in V$ with $\{u, v\} \in E$ and $\{u, v\} \notin S$ in order to determine $f_2(S)$. We know that $\{s, t\} \notin S$ holds. Thus,

$$\frac{d_{c_2}^S(s,t)}{d_{c_2}^E(s,t)} \ge \frac{\left(\sum_{i=1}^n c_2(v_i,w_i)\right) + \left(\sum_{i=1}^{n-1} c_2(w_i,v_{i+1})\right)}{1} = 2^{n+1} + n - 3.$$

The only other pairs of vertices $u, v \in V$ with $\{u, v\} \in E$ for which $\{u, v\} \notin S$ might hold are v_i, w_i , for every $i \in [n]$. We get

$$\frac{d_{c_2}^S(v_i, w_i)}{d_{c_2}^E(v_i, w_i)} \le \frac{c_2(v_i, v_i') + c_2(v_i', w_i)}{c_2(v_i, w_i)} = \frac{2 \cdot 2^i}{2^i} = 2 < 2^{n+1} + n - 3 \le \frac{d_{c_2}^S(s, t)}{d_{c_2}^E(s, t)}.$$

With the next Lemma, we estimate how many of the spanners in X are associated to a non-dominated value vector contained in \mathcal{Y}_N .

Lemma 7 For all $S, S' \in X$ with $S \neq S'$, S and S' do not dominate each other and have different value vectors.

Proof Let $S, S' \in X$ be two different spanners and assume S dominates S'. Therefore, either $f_1(S) < f_1(S')$ or $f_1(S) = f_1(S')$ holds. We begin by considering the first case. Let $j \in [n]$ be the greatest index at which the shortest s-t-paths in S and S' differ. Since $f_1(S) < f_1(S')$ holds, S must not contain the edge $\{v_j, w_j\}$ while S' has to contain it. Let P be the remaining path that is identical for S and S'. Thus, with Lemma 6,

$$f_2(S') = d_{c_2}^S(s, t)$$

$$\leq \left(\sum_{i=1}^{j-1} c_2(v_i, v'_i) + c_2(v'_i, w_i) + c_2(w_i, v_{i+1})\right) + c_2(v_j, w_j) + c_2(P)$$

$$= \left(\sum_{i=1}^{j-1} 2 \cdot 2^{i} + 1\right) + 2^{j} + c_{2}(P) = \left(\sum_{i=2}^{j} 2^{i} + 1\right) + 2^{j} + c_{2}(P)$$

$$< \left(\sum_{i=1}^{j-1} 2^{i} + 1\right) + 2^{j} + 2^{j} + c_{2}(P)$$

$$= \left(\sum_{i=1}^{j-1} c_{2}(v_{i}, w_{i}) + c_{2}(w_{i}, v_{i+1})\right) + c_{2}(v_{j}, v_{j}') + c_{2}(v_{j}', w_{j}) + c_{2}(P)$$

$$\leq d_{c_{2}}^{S}(s, t) = f_{2}(S).$$

This contradicts the assumed domination.

Let us now consider the second case, in which $f_1(S) = f_1(S')$ holds. Then, in order for *S* to dominate *S'*, $f_2(S) < f_2(S')$ must hold as well. By design of the c_1 edge-weights, we know that in order for $f_1(S) = f_1(S')$ to hold, it is true for every edge $\{v_i, w_i\} \in E$ that $\{v_i, w_i\} \in S \Leftrightarrow \{v_i, w_i\} \in S'$. This claim can be verified by considering that every edge $\{v_i, w_i\}$ has a unique c_1 -cost that cannot be reproduced by any combination of edges $e \in E \setminus \{v_i, w_i\}$. Consequently, the shortest s-t-paths in *S* and *S'* are exactly the same and therefore $f_2(S) = f_2(S')$ holds, which contradicts the assumed domination. \Box

Finally, consider that no $S \in X$ can be dominated by any $\hat{S} \notin X$. This is clearly the case, due to the high c_1 -cost of the edge $\{s, t\}$. Concluding, we have shown that the set X contains 2^n spanners, all of which have different value vectors that are not dominated. Thus, $|\mathcal{Y}_N| \ge |X| = 2^n$ and, consequently, Theorem 1 hold.

Note that an analogous proof can be conducted for the diMSp. The only difference to the undirected case lies in the construction of the family of instances. We turn the family of MSp instances into a family of diMSp instances. For every $i \in [n]$, we replace edges $\{v_i, w_i\}$ with arcs (v_i, w_i) , edges $\{v_i, v'_i\}$ with arcs (v_i, v'_i) , edges $\{v'_i, w_i\}$ with arcs (v'_i, w_i) . In addition, we replace edges $\{w_i, v_{i+1}\}$ with arcs (w_i, v_{i+1}) , for $i \in [n-1]$. Finally, we replace $\{s, t\}$ with (s, t). Every arc holds the same edge-weights as the undirected edge it replaced.

4 Non-dominated set

In this section, we first consider the output-sensitive complexity of computing the non-dominated set of unweighted (di)MSp instances. Afterwards, we prove that if the (di)MSp can be solved by an output-polynomial algorithm, one can also solve the BUCO problem in output-polynomial time. Therefore, the (di)MSp is *BUCO-hard*.

Observation 2 As the trivial edge-weight function only allows linear many values in the range of either of the two objective functions, with Lemma 4, we can infer that the non-dominated set of unweighted MSp instances is only polynomially sized.

This observation directly infers that the non-dominated set of the (di)MSp cannot be computed in output-polynomial time as this would enable us to solve the (directed) t-Spanner^{DEC} in polynomial time. Thus, Corollary 8 holds.

Corollary 8 If $P \neq NP$, then $MSp \notin OP$, even for unweighted instances.

We now study the output-sensitive complexity of degree bounded outerplanar instances. For a set of vectors M, min M refers to its non-dominated subset.

Definition 2 (Biobjective Unconstrained Combinatorial Optimization (BUCO) Problem (Bökler 2018)) The input are vectors $c^1, c^2 \in \mathbb{N}^n$. A feasible solution is an element of $\{0, 1\}^n$. The goal is to find the set of the non-dominated vectors

$$\mathcal{Y}_N = \min\left\{ \begin{pmatrix} -c^{1^{\mathsf{T}}} \\ c^{2^{\mathsf{T}}} \end{pmatrix} x \ \middle| \ x \in \{0, 1\}^n \right\}.$$

The BUCO problem can be interpreted as an unrestricted Knapsack problem. Without loss of generality, we can assume $c_i^1 > 0$ for every $i \in [n]$, since any item that does not contribute value is never part of a viable solution. Similarly, we can assume $c_i^2 > 0$ for every $i \in [n]$.

We prove that the MSp is **BUCO**-hard by showing that if there is an outputpolynomial algorithm \mathcal{A} for the MSp, we could use it to solve the BUCO problem in output-polynomial time. We start by constructing an algorithm that transforms any BUCO instance *I* into a valid MSp instance *I'* in polynomial time. Subsequently, we show that the set of non-dominated value vectors of the constructed MSp instance *I'*, that can be found using the algorithm \mathcal{A} , can be transformed into the set of non-dominated value vectors of the BUCO instance *I*, using an output-polynomial filter-algorithm.

This approach is an output-sensitive version of the classic Turing Reduction and therefore works similar to the more commonly known *Cook Reduction* (Cook 1971).

Definition 3 (Output-Sensitive Turing Reduction) Let P_1 , P_2 be two algorithmic problems and $p, q : \mathbb{N} \to \mathbb{N}$ be two polynomials. Additionally, let L(J) be the solution of an instance J of P_1 , e.g., its non-dominated set. We say P_1 is *output-sensitive reducible* to P_2 , if there is an algorithm \mathcal{A}' for P_1 , that may use a (hypothetical) algorithm \mathcal{A} for P_2 with the worst-case runtime of \mathcal{A}' and the number of uses in \mathcal{A}' of \mathcal{A} being in $\mathcal{O}(p(|J|, |L(J)|))$ and $\mathcal{O}(q(|J|, |L(J)|))$.

Note that this is a generalization of the *Output-Sensitive Many-One Reduction* as described by (Bökler 2018, Definition 6.2). Thus, all previously proven reduction relations concerning the many-one reduction also apply to the Turing reduction described above. This includes, e.g., the **BUCO**-hardness of the Multiobjective Spanning Tree problem (Bökler 2018).

Let *I* be an instance of the BUCO problem given by $c^1, c^2 \in \mathbb{N}_+^n$. We construct an instance *I'* of the MSp with a connected, undirected graph *G*, and edge-weight functions $c_1: E \to \mathbb{Z}$ and $c_2: E \to \mathbb{N}_+$. We define the constant $C := \sum_{i=1}^n c_i^1$ and construct the graph G = (V, E) in the following way: Create vertices v_i, w_i and v'_i



Fig. 2 Showing the reduction for Theorem 2. Thick edges hold weights $(0, 1)^{\mathsf{T}}$

for $i \in [n]$ and connect them with edges $\{v_i, w_i\}$ with weights $(0, c_i^2 + 2)^T$, edges $\{v_i, v_i'\}$ with weights $(0, 1)^T$ and edges $\{v_i', w_i\}$ with weights $(c_i^1, 1)^T$. Furthermore, we add edges $\{w_i, v_{i+1}\}$ with weights $(0, 1)^T$ for $i \in [n - 1]$. We define $v_1 := s$ and $w_n := t$. Finally, we add the edge $\{s, t\}$ with weights $(C + 1, 1)^T$. An example of this construction can be seen in Fig. 2.

We observe that every so constructed instance is a valid MSp instance and that all steps can be performed in polynomial time in the size of the instance *I*. Clearly, the constructed graph *G* is degree-3 bounded and outerplanar. In order to show that the reduction is correct, we now have to prove that the non-dominated set $\mathcal{Y}_N^{\text{MSp}}$ of the constructed MSp instance can be transformed into the non-dominated set $\mathcal{Y}_N^{\text{BUCO}}$ of the initial BUCO instance in output-polynomial time with regard to the instance *I*.

Let $x \in \{0, 1\}^n$ be a solution of a BUCO instance given by $c^1, c^2 \in \mathbb{N}^n_+$ and let $z = (-c^{1^T}x, c^{2^T}x)^T$ be the value vector that x is mapped to. There is a spanner S^x of G with the following properties: For every $i \in [n]$, S^x contains edges $\{v_i, w_i\}$ and $\{v_i, v'_i\}$. Furthermore, S^x contains the edge $\{w_i, v_{i+1}\}$, for every $i \in [n-1]$. In addition, if $x_i = 0$: S^x contains the edge $\{v'_i, w_i\}$. We observe that for every $x \in \{0, 1\}^n$ the resulting set of edges S^x is a feasible spanner of G and that none of these spanners contains the edge $\{s, t\}$. Denote the set of all the spanners generated this way as X. Clearly, analogously to Lemma 6, for every spanner $S \in X$, $f_2(S) = \max_{u,v \in V} \frac{d_{c_2}^S(u,v)}{d_{c_2}^E(u,v)} = \frac{d_{c_2}^S(s,t)}{d_{c_2}^E(s,t)}$ holds.

We now examine how the value vector z of a BUCO solution $x \in \{0, 1\}^n$ is connected to the value vector $f(S^x)$ of the corresponding spanner S^x .

Lemma 9 For any solution $x \in \{0, 1\}^n$ of a BUCO instance and its associated value vector $z = (-c^{1^T}x, c^{2^T}x)^T$, for the constructed corresponding spanner $S^x \in X$, $f_1(S^x) = C + z_1$ and $f_2(S^x) = z_2 + 3n - 1$ hold.

Proof Let $x \in \{0, 1\}^n$ be a solution of a BUCO instance and let $z = (-c^{1^T}x, c^{2^T}x)^T$ be its associated value vector. Let $S^x \in X$ be the spanner in the constructed MSp instance that is based on x. Consider the two objective functions.

$$f_1(S^x) = \left(\sum_{i=1}^n c_1(v'_i, w_i) \cdot (1 - x_i)\right) = \left(\sum_{i=1}^n c_i^1 \cdot (1 - x_i)\right) = C + \sum_{i=1}^n -c_i^1 x_i$$
$$= C + z_1$$

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Note that the shortest s-t-path in *E* has length 1 and uses the edge $\{s, t\}$. Furthermore, observe that for every $i \in [n]$ the shortest s-t-path in S^x uses the edge $\{v_i, w_i\}$ if $x_i = 1$. If $x_i = 0$, S^x uses the edges $\{v_i, v'_i\}$ and $\{v'_i, w_i\}$. Additionally, for every $i \in [n-1]$, S^x uses the edge $\{w_i, v_{i+1}\}$.

$$f_2(S^x) = \left(\sum_{i=1}^n c_2(v_i, w_i) \cdot x_i + (c_2(v_i, v'_i) + c_2(v'_i, w_i))(1 - x_i)\right) + \sum_{i=1}^{n-1} c_2(w_i, v_{i+1})$$
$$= \left(\sum_{i=1}^n (c_i^2 + 2)x_i + (1 + 1)(1 - x_i)\right) + (n - 1) \cdot 1$$
$$= \left(\sum_{i=1}^n c_i^2 x_i + 2\right) + n - 1 = z_2 + 3n - 1$$

Let us now consider the relationship between spanners $S^x \in X$ and spanners S with $\{s, t\} \in S$.

Lemma 10 If $x \in \{0, 1\}^n$ is a Pareto-optimal solution for the BUCO instance, its associated spanner S^x is a Pareto-optimal solution for the constructed MSp instance.

Proof Let $x \in \{0, 1\}^n$ be a Pareto-optimal solution for the BUCO instance and let $z = (-c^{1^T}x, c^{2^T}x)^T$ be the corresponding value vector. Let $S^x \in X$ be the spanner associated to x. Assume there is a feasible spanner \widehat{S} that dominates S^x . It is clear that due to the c_1 -cost of the edge $\{s, t\}$, if $\{s, t\} \in \widehat{S}$ holds, \widehat{S} does not dominate S^x . Therefore, we can assume that $\{s, t\} \notin \widehat{S}$ holds. Furthermore, with Observation 1 and w.l.o.g., we can assume that \widehat{S} contains the edges $\{v_i, v'_i\}$, $\{v_i, w_i\}$ for all $i \in [n]$ and $\{w_i, v_{i+1}\}$ for $i \in [n-1]$. Based on these observations, we can say that \widehat{S} meets the specifications for a spanner that corresponds to a BUCO solution. Let $\widehat{x} \in \{0, 1\}^n$ be this BUCO solution such that $f(\widehat{S^x}) = f(\widehat{S})$. Let \widehat{z} be the value vector associated to \widehat{x} . Assume that $\widehat{S^x}$. Then, by Lemma 9, we obtain that

$$\widehat{z}_1 = f_1(S^x) - C \le f_2(S^x) - C = z_1$$

and

$$\widehat{z}_2 = f_2(S^{\widehat{x}}) - 3n + 1 \le f_2(S^x) - 3n + 1 = z_2.$$

Since at least one of these inequalities is strict \hat{x} dominates x, which is a contradiction to the Pareto-optimality of x.

In order to complete the verification of this reduction, we now prove that there are only polynomially many non-dominated value vectors in the non-dominated set of I'that are not based on a Pareto-optimal solution of the BUCO instance. Consider that the only way a spanner S can divert from the form of a BUCO solution based

spanner, without its value vector being dominated, is by containing the edge $\{s, t\}$. Hence, combining Lemma 4 with the following Lemma proves the aforementioned claim.

Lemma 11 For every Pareto-optimal spanner S with $\{s, t\} \in S$ and $l \in [n]$ being the index of the BUCO item with the greatest c_2 -weight for which $\{v'_l, w_l\} \notin S$ holds, $f_2(S) = \frac{d_{c_2}^S(v'_l, w_l)}{d_{c_2}^E(v'_l, w_l)} = d_{c_2}^S(v'_l, w_l).$

Proof Let *S* with $\{s, t\} \in S$ be a Pareto-optimal spanner and let $l \in [n]$ be the index of the BUCO item with the greatest c_2 -weight for which $\{v'_l, w_l\} \notin S$ holds. With Lemma 5, and since $\{s, t\} \in S$ and Observation 1 hold, we know that in order to determine $f_2(S)$, we only have to consider edges $\{v'_i, w_i\}$, for all $i \in [n]$. Thus,

$$f_2(S) = \max_{u,v \in V} \frac{d_{c_2}^S(u,v)}{d_{c_2}^E(u,v)} = \max_{v'_i,w_i \in V} \frac{d_{c_2}^S(v'_i,w_i)}{d_{c_2}^E(v'_i,w_i)} = \max_{v'_i,w_i \in V} \frac{d_{c_2}^S(v'_i,w_i)}{1} = d_{c_2}^S(v'_i,w_l).$$

Note that there is an additional edge-case, in which the spanner *S* contains every $e \in E$. Hence, there are only n + 1 possible values in the range of the second objective function if the considered spanner contains the edge $\{s, t\}$.

With Lemma 4, we infer there are only n + 1 non-dominated value vectors, in the non-dominated set of the constructed MSp instance, that do not correspond to a Pareto-optimal BUCO solution. Finally, we describe the algorithm that solves any BUCO instance in output-polynomial time, assuming that there is an output-polynomial algorithm \mathcal{A} capable of solving any MSp instance.

Let I be a BUCO instance, and let A be an algorithm capable of solving an MSp instance in output-polynomial time. We begin by using the algorithm described above to transform I into the corresponding MSp instance I'. Subsequently, we solve I'using the algorithm \mathcal{A} and receive the set of non-dominated value vectors \mathcal{Y}_{N}^{MSp} . We know that for every Pareto-optimal solution $x \in \{0, 1\}^n$ of the BUCO instance, the constructed MSp instance contains a corresponding Pareto-optimal spanner S^{x} . Therefore, we can assume that for every such x, $f(S^x) = (f_1(S^x), f_2(S^x))^{\mathsf{T}} \in \mathcal{Y}_N^{\mathsf{MSp}}$ holds. Now, we have to filter out all the non-dominated value vectors that do not correspond to a feasible BUCO solution. We do this by inspecting the y_1 value for each $y \in \mathcal{Y}_N^{MSp}$. If $y_1 \ge C + 1$ holds, then the spanner corresponding to y contains the edge $\{s, t\}$ and consequently is not based on a feasible BUCO solution. If $y_1 \leq t$ C holds, we transform y according to Lemma 9, so that its values match the ones of the corresponding BUCO solution. We construct $z = (y_1 - C, y_2 - 3n + 1)$ and add it to the set of non-dominated value vectors of the initial BUCO instance $\mathcal{Y}_{N}^{\text{BUCO}}$. All of these steps are output-polynomial with regard to the BUCO instance I and therefore, the existence of an output-polynomial algorithm for the MSp directly implies the existence of an output-polynomial algorithm for the BUCO problem. Thus, Theorem 2 holds.

An analogous reduction can be conducted for the diMSp, by replacing every undirected edge with an arc. This transformation works similar to the one conducted at the end of Sect. 3. Note that the diMSp instances require an additional arc (v'_i, v_i) with edge-weights $(0, 1)^T$, for every $i \in [n]$. These arcs ensure that the directed spanners constructed during the reduction are feasible. Observe that the resulting diMSp instances remain outerplanar but are only degree-4 bounded.

5 Extreme points

In this section, we define the problem of determining the set of extreme points of a given (di)MSp instance and consider its output-sensitive complexity.

For all $y' \in \mathcal{Y}$, define W(y') as the set of vectors $\lambda \in \mathbb{R}^d_{\geq}$, $\lambda \neq 0$, so that $\min_{y \in \mathcal{Y}} \lambda^T y = \lambda^T y'$. A value vector $y' \in \mathcal{Y}$ is called an *extreme point* if there is a $\lambda \in \mathbb{R}^d_{\geq}$, $\lambda \neq 0$ with $\lambda \in W(y')$ and $\forall y \in \mathcal{Y}_N \setminus \{y'\}$: $\lambda \notin W(y)$ (Ehrgott 2005, Definition 8.7). For an instance of the (di)MSp, we define $(di)MSp^{YEx}$ to be the problem of computing its set of extreme points \mathcal{Y}_X .

We now show that if $\mathbf{P} \neq \mathbf{NP}$, even the unweighted MSp^{YEx} cannot be solved in output-polynomial time. With Corollary 8, we know that we cannot compute the entire non-dominated set of MSp instances in output-polynomial time. However, this does not imply the hardness of computing their set of extreme points.

We do this by conducting an indirect reduction from 3SAT. Consider Cai's reduction from 3SAT to *t*-Spanner^{DEC} (Cai 1994) for every $2 \le t \in \mathbb{N}$. We turn Cai's constructed 2-Spanner^{DEC} instances into MSp^{YEx} instances and show that the initial 3SAT instance is a yes-instance iff the yes-witness for the 2-Spanner^{DEC} instance creates an extreme point in the corresponding MSp^{YEx} instance. In consideration of Observation 2, any output-polynomial algorithm capable of solving unweighted MSp^{YEx} instances in output-polynomial time could solve 3SAT in polynomial time. We begin with a quick summary of Cai's proof for the special case of t = 2.

Revisiting Cai's proof. Cai transforms 3SAT to 2-Spanner^{DEC}. Let an instance I = (U, C) of 3SAT consisting of a set U of n distinct variables and a collection C of m 3-element clauses over U be given. They construct a 2-Spanner^{DEC} instance \hat{I} , with a graph G = (V, E) and a positive integer $K \in \mathbb{N}_+$, such that G contains a 2-spanner with at most K edges if and only if C is satisfiable. They define a 2-path as a path with 2 edges. One can force an edge to be in any minimum 2-spanner of a graph by the addition of two distinct 2-paths between the two ends of the edge (Cai 1994, Lemma 3). This operation is called *forcing an edge*. Such an edge is called a *forced edge* and the two 2-paths are called *forcing paths*. Edges lying on forcing paths are called *forcing edges*.

They construct the *truth-setting component* T as follows: They take five vertices z, the literal vertices x and \bar{x} , and the y-type vertices y and y'; and join z to each of the remaining four vertices by an edge. Finally, they add forced edges $\{x, \bar{x}\}, \{x, y\}, \{x, y'\}, \{\bar{x}, y\}, \{\bar{x}, y'\}$. An example of such a component can be seen in Fig. 3.

They assign each variable $u_i \in U$, $i \in [n]$ a distinct copy T_i of T and identify all vertices z_i into a single vertex z to form a subgraph T' of G. To finish the construction of G, they create a new vertex v_i for each clause $c_i \in C$, $i \in [m]$, join it to vertex z



Fig.3 A truth-setting component T and its symbolic representation. Thick edges indicate forced edges. For clarity, forcing paths have been omitted from the figure



Fig. 4 The graph G for $C = \{\{\bar{u_1}, u_2, u_3\}, \{u_1, u_2, \bar{u_3}\}\}$. Thick edges indicate forced edges. For clarity, forcing paths have been omitted from the figure

with an edge and add a forced edge between v_i and each of the three literal vertices in T' corresponding to the three literals of c_i . An example of the finished construction can be seen in Fig. 4.

They finish the construction of the 2-Spanner^{DEC} instance, by setting K = 16n+9m. For the constructed graph *G*, it holds that any minimum 2-spanner *S* of *G* contains at least *K* edges. Furthermore, if *S* contains exactly *K* edges, then for each T_i , $i \in$ [*n*] exactly one of the two literal edges $\{z, x_i\}$ and $\{z, \bar{x_i}\}$ belongs in *S* (Cai 1994, Lemma 4).

Now, suppose that *C* is satisfiable and let ϕ be a satisfying truth assignment for *C*. They construct a yes-witness-spanner S^w as follows: put every forced edge in S^w . For each forcing path, put one of the two edges in S^w . Finally, for each variable $u_i \in U$, if *u* is "true" under ϕ then put edge $\{z, x_i\}$, in S^w else put edge $\{z, \bar{x_i}\}$ in S^w . The complete proof that this is a correct reduction goes beyond the scope of this paper and can be found in the original paper (Cai 1994). Instead, let us now construct an equivalent MSp^{YEx} instance and prove that if the initial 2-Spanner^{DEC} instance is a yes-instance, the value vector of the yes-witness-spanner is an extreme point.

The Associated MSp^{YEx} **Instance.** First, let *I* be a 3SAT instance and let $\hat{I} = (G = (V, E), K)$ be the associated 2-Spanner^{DEC} instance, constructed according to the algorithm described in Cai's proof. We turn \hat{I} into an instance I' = (G, c) of the unweighted MSp^{YEx} by copying *G* and adding the trivial edge-weight function $c: E \rightarrow \{(1, 1)^T\}, c(e) = (1, 1)^T$ for all $e \in E$. Clearly, this can be done in polynomial time. Note that I' is a valid, unweighted MSp^{YEx} instance. Now, let \hat{I} be a

yes-instance and let S^w be the yes-witness-spanner. It is clear that the same spanner exists in I' and that

$$f_1(S^w) = \sum_{e \in S^w} c_1(e) = \sum_{e \in S^w} 1 = |S^w| = K \text{ and } f_2(S^w) = \max_{u,v \in V} \frac{d_{c_2}^{S^-}(u,v)}{d_{c_2}^E(u,v)} = 2$$

hold. We now show that the value vector of the yes-witness-spanner $f(S^w)$ is an extreme point by finding the non-dominated value vectors \mathcal{Y}_N and showing that there is a $\lambda \in \mathbb{R}^d_{\geq}$, $\lambda \neq 0$, so that $\lambda \in W(f(S^w))$ and $\forall y \in \mathcal{Y}_N \setminus \{f(S^w)\}$: $\lambda \notin W(y)$ hold.

Note that $f(S^w)$ dominates or is equal to every value vector $y \in \mathcal{Y}$ with $y_1 \ge K$ and $y_2 \ge 2$. Let us first find the non-dominated value vectors $y \in \mathcal{Y}_N$ with $y_2 < 2$. We begin by constructing the spanner $S^1 \subseteq E$ that contains the fewest edges for which $f_2(S^1) < f_2(S^w) = 2$ holds. Consider $S^1 = E$ and observe that if we remove any edge $e \in E$ from S^1 , $f_2(S^1 \setminus \{e\}) = 2$ holds. Hence, $S^1 = E$ and we conclude that for every feasible spanner $S \subseteq E$ with $K < f_1(S) < |E|$, $f(S^w)$ dominates f(S). Consider the amount of edges in G. For every truth-setting component T_i , $i \in [n]$ there are 20 forcing edges, 5 forced edges and 4 edges connecting to the vertex z. Furthermore, for each clause $c_i \in C$, $i \in [m]$ there are 12 forcing edges, 3 forced edges and one edge connecting to z. Hence,

$$f_1(S^1) = |E| = 29n + 16m$$
 and $f_2(S^1) = 1$.

We now construct a hypothetical value vector y^h that either dominates or is equal to every value vector f(S) of feasible spanners S with $f_1(S) < K$. We begin by constructing y_1^h . Consider the number of vertices in G. For every truth-setting component T_i , $i \in [n]$ there are 10 vertices that are part of forcing paths and 4 vertices $x_i, \bar{x_i}, y_i, y'_i$. Furthermore, for each clause $c_i \in C$, $i \in [m]$ there is one vertex v_i and 6 vertices that are part of forcing paths. Finally, there is the vertex z. Thus, there are 14n + 7m + 1 vertices in the G. Therefore, for every feasible spanner S of G, $f_1(S) = |S| \ge 14n + 7m$. Hence, we set $y_1^h = 14n + 7m$. Let us now focus on y_2^h . We know that there are no spanners S that contain fewer edges than S^w for which $f_2(S) \le 2$ hold. Hence, we set $y_2^h = 3$. Thus,

$$y_1^h = 14n + 7m$$
 and $y_2^h = 3$.

Clearly, for every value vector f(S) of a feasible spanner S with $f_1(S) < K$, $y_1^h \leq f_1(S)$ and $y_2^h \leq f_2(S)$ hold. Hence, y^h either dominates or is equal to every such value vector and thus, for every $\lambda \in \mathbb{R}^d_{\geq}$, $\lambda \neq 0$, $\lambda^T y^h \leq \lambda^T \cdot f(S)$ holds.

Lemma 12 The value vector of the yes-witness-spanner is an extreme point.

Proof Consider the vector $\lambda = (2, 15n + 9m)^{\mathsf{T}} \in \mathbb{R}^d_{\geq}, \lambda \neq 0$. We show that $\lambda \in W(f(S^w))$ holds and that for all $y \in \mathcal{Y}_N \setminus \{f(S^w)\}: \lambda \notin W(y)$. Begin by considering $\lambda^{\mathsf{T}} \cdot f(S^1)$ and $\lambda^{\mathsf{T}} y^h$.

$$\lambda^{\mathsf{T}} \cdot f(S^1) = (2\ 15n + 9m) \cdot {\binom{29n + 16m}{1}} = 73n + 41m$$

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Fig. 5 The value vectors y^h , $f(S^w)$, $f(S^1)$ and slopes visualized

$$= \left(2\ 15n+9m\right) \cdot \binom{14n+7m}{3} = \lambda^{\mathsf{T}} y^{h}$$

Now, consider $\lambda^{\mathsf{T}} \cdot f(S^w)$. We get

$$\lambda^{\mathsf{T}} \cdot f(S^w) = (2 \ 15n + 9m) \cdot \binom{K}{2} = (2 \ 15n + 9m) \cdot \binom{16n + 9m}{2}$$
$$= 62n + 36m < 73n + 41m = \lambda^{\mathsf{T}} \cdot f(S^1) = \lambda^{\mathsf{T}} y^h,$$

for n, m > 0. Hence, $\lambda \in W(f(S^w))$ and for all $y \in \mathcal{Y}_N \setminus \{f(S^w)\} \colon \lambda \notin W(y)$. \Box

A sketch of the relevant value vectors can be seen in Fig. 5.

Finally, let us consider the entire reduction. Suppose there is an algorithm \mathcal{A} capable of solving unweighted MSp^{YEx} instances in output-polynomial time. We know that the initial 3SAT instance I is a yes-instance iff the 2-Spanner^{DEC} instance \hat{I} constructed according to Cai's proof is a yes-instance too. Therefore, the yes-witness-spanner S^w exists in \hat{I} and thus, also in the associated MSp^{YEx} instance I'. Since the value vector $f(S^w)$ of S^w is an extreme point, it is therefore part of the output \mathcal{Y}_X of algorithm \mathcal{A} , when applied to I'. In consideration of $\mathcal{Y}_X \subseteq \mathcal{Y}_N$ and Lemma 4 we infer that solving I' with \mathcal{A} and checking whether $f(S^w) \in \mathcal{Y}_X$ holds is possible in polynomial time in the size of I. In conclusion, if \mathcal{A} existed, we could solve 3SAT in polynomial time. Thus, Theorem 3 holds.

Similarly, based on Cai's reduction from 3SAT to directed 2-Spanner^{DEC} (Cai 1994, Section 3), we can show the same results for the diMSp^{YEx}. The only difference to the undirected case lies in the construction of the directed 2-Spanner^{DEC} instance by Cai. These differences in turn cause slightly different values in the objective functions of the considered spanners in the diMSp^{YEx} instance that we construct analogously to the undirected case. It is easy to see that these differences have no influence on the validity of the statement. Due to the analogy of the proofs, we leave the details to the reader.

6 Conclusion

In this paper, we introduced the Multiobjective Minimum Weight Minimum Stretch Spanner(MSp) Problem, as a multiobjective generalization of the Minimum t-Spanner problem, and analyzed its complexity. We showed that for degree-3 bounded out-erplanar instances the MSp is intractable and computing the non-dominated set is

BUCO-hard. This contrasts the complexity of the Minimum t-Spanner problem, as there are non-trivial stretch factors for which the Minimum t-Spanner problem is solvable in linear time under such restrictions. Additionally, we proved that if $\mathbf{P} \neq \mathbf{NP}$, the set of extreme points cannot be computed in output-polynomial time, for instances with unit costs and arbitrary graphs. Furthermore, we showed similar results for the directed version of the MSp.

What remains open is the output-sensitive complexity of computing the set of extreme points for degree-3 bounded outerplanar instances as it is currently unknown whether there is a stretch factor t, such that the t-Spanner^{DEC} problem is **NP**-hard under these restrictions. Thus, such a proof requires a different approach than the one used in Sect. 5. Another open question is whether the MSp^{YEx} is intractable. Future work might include the development of approximation techniques for the MSp and related problems, as well as investigating what existing approaches can be applied to them.

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Declarations

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References

- Ahmed R, Hamm K, Latifi Jebelli MJ et al (2019) Approximation algorithms and an integer program for multi-level graph spanners. In: Kotsireas I, Pardalos P, Parsopoulos KE et al (eds) Analysis of experimental algorithms. Springer, Cham, pp 541–562
- Althöfer I, Das G, Dobkin D et al (1993) On sparse spanners of weighted graphs. Discrete Comput Geom 9(1):81–100
- Bang-Jensen J, Gutin G (2008) Digraphs: theory, algorithms and applications, 2nd edn. Springer monographs in mathematics. Springer, London
- Baswana S, Sen S (2007) A simple and linear time randomized algorithm for computing sparse spanners in weighted graphs. Random Struct Algorithms 30(4):532–563
- Bökler F (2018) Output-sensitive complexity of multiobjective combinatorial optimization with an application to the multiobjective shortest path problem. PhD thesis, Technische Universität Dortmund
- Bökler F, Ehrgott M, Morris C et al (2017) Output-sensitive complexity of multiobjective combinatorial optimization. J Multi-Criteria Decis Anal 24(1–2):25–36
- Bökler F, Ehrgott M, Rui Figueira J et al (2020) The output-sensitive complexity of the BUCO problem. Dagstuhl Rep 10(1):76–78. https://doi.org/10.4230/DagRep.10.1.52

- Bökler F, Mutzel P (2015) Output-sensitive algorithms for enumerating the extreme nondominated points of multiobjective combinatorial optimization problems. Algorithms—ESA 2015. Springer, Berlin, pp 288–299
- Bökler F, Chimani M (2020) Approximating multiobjective shortest path in practice. In: SIAM ALENEX, pp 120–133
- Brunsch T, Röglin H (2015) Improved smoothed analysis of multiobjective optimization. J ACM 62(1):4:1-4:58
- Cai L (1994) NP-completeness of minimum spanner problems. Discrete Appl Math 48(2):187–194. https:// doi.org/10.1016/0166-218X(94)90073-6
- Cai L, Keil M (1994) Spanners in graphs of bounded degree. Networks 24(4):233–249. https://doi.org/10. 1002/net.3230240406
- Cai L, Corneil D (1995) Tree spanners. SIAM J Discrete Math 8(3):359–387. https://doi.org/10.1137/ S08954801922374033
- Caunhye A, Nie X, Pokharel S (2012) Optimization models in emergency logistics: a literature review. Socioecon Plann Sci 46(1):4–13
- Cook S (1971) The complexity of theorem-proving procedures. In: ACM STOC, pp 151–158. https://doi. org/10.1145/800157.805047
- Delling D, Pajor T, Werneck R (2015) Round-based public transit routing. Transp Sci 49(3):591-604
- Diestel R (2017) Graph theory, vol 173. Graduate texts in mathematics. Springer, Berlin
- Ehrgott M (2005) Multicriteria optimization. Springer, Berlin
- Emelichev V, Perepelitsa V (1992) On cardinality of the set of alternatives in discrete many-criterion problems. Discrete Math Appl 2(5):461–471
- Giantsoudi D, Grassberger C, Craft D et al (2013) Linear energy transfer-guided optimization in intensity modulated proton therapy: feasibility study and clinical potential. Int J Radiat Oncol Biol Phys 87(1):216–222
- Hamacher H, Ruhe G (1994) On spanning tree problems with multiple objectives. Ann Oper Res 52(4):209–230
- Hamacher H, Küfer K (1999) Inverse radiation therapy planning: a multiple objective optimisation approach. Monitoring evaluating. Planning health services. World Scientific, Singapore, pp 177–189
- Hansen P (1980) Bicriterion path problems. Multiple criteria decision making theory and application, vol 177. Springer, Berlin, pp 109–127
- Johnson DS, Yannakakis M, Papadimitriou CH (1988) On generating all maximal independent sets. Inf Process Lett 27(3):119–123
- Kobayashi Y (2018) NP-hardness and fixed-parameter tractability of the minimum spanner problem. Theoret Comput Sci 746:88–97. https://doi.org/10.1016/j.tcs.2018.06.031
- Kortsarz G, Peleg D (1994) Generating sparse 2-spanners. J Algorithms 17(2):222-236
- Libotte G, Lobato F, Platt G et al (2020) Determination of an optimal control strategy for vaccine administration in COVID-19 pandemic treatment. Comput Methods Programs Biomed 196:105664
- Papadimitriou C, Yannakakis M (2000) On the approximability of trade-offs and optimal access of web sources. In: 41st IEEE FoCS, pp 86–92. https://doi.org/10.1109/SFCS.2000.892068
- Peleg D, Schäffer A (1989) Graph spanners. J Graph Theory 13(1):99-116
- Peleg D, Ullman J (1989) An optimal synchronizer for the hypercube. SIAM J Comput 18(4):740-747
- Sigurd M, Zachariasen M (2004) Construction of minimum-weight spanners. Algorithms—ESA 2004. Springer, Berlin, pp 797–808
- Thieke C, Küfer K, Monz M et al (2007) A new concept for interactive radiotherapy planning with multicriteria optimization: first clinical evaluation. Radiother Oncol 85(2):292–298
- Wagner D, Zündorf T (2017) Public transit routing with unrestricted walking. In: ATMOS 2017. Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik

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