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Temporal Belief Revision Rationalized by Plausibility Orderings

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Abstract

Within the class of structures introduced in [G. Bonanno, Axiomatic characterization of the AGM theory of belief revision in a temporal logic, *Artificial Intelligence*, 171 (2007), 144-160] we consider the subclass satisfying the property that, for every state-instant pair (ω, t) , there is an ordering of the set of states that rationalizes the revised beliefs at every (ω, t') where t' is an immediate successor of t . We characterize this class both semantically and syntactically.

Keywords: plausibility ordering, information, belief revision, branching time, AGM theory.

1 Introduction

In [2] we introduced semantic structures (called \mathbb{L}_{AGM} -frames) and a corresponding temporal logic (called \mathbb{L}_{AGM}) for modeling iterated belief revision over time and explored the relationship between that logic and the AGM postulates for one-stage belief revision introduced in [1]. It is well-known that the AGM postulates can be associated with the existence of a plausibility ordering, in the sense that the revised beliefs correspond to the most plausible formulas compatible with the information received (see, for example, [6]). In this note we consider the subclass of \mathbb{L}_{AGM} -frames where, for every state-instant pair (ω, t) , there is a “plausibility” ordering (total pre-order) of the set of states that rationalizes the revised beliefs at every (ω, t') where t' is an immediate successor of instant t . A frame in this class is called a *plausibility frame*. We provide a necessary and sufficient condition for an \mathbb{L}_{AGM} -frame to be a plausibility frame

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and identify an extension of logic \mathbb{L}_{AGM} that characterizes the class of plausibility frames. We also provide a simpler sufficient condition and identify the corresponding logic. The results are proved by establishing a correspondence between \mathbb{L}_{AGM} -frames and structures studied in the rational choice literature (see [7], [9]).

2 Semantics

We consider the structures introduced in [2] consisting of branching-time frames with the addition of a belief relation and an information relation for every instant t . A *next-time branching frame* is a pair $\langle T, \succ \rangle$ where T is a (possibly infinite) set of instants or dates and \succ is a binary “precedence” relation on T satisfying the following properties: $\forall t_1, t_2, t_3 \in T$, (1) uniqueness: if $t_1 \succ t_3$ and $t_2 \succ t_3$ then $t_1 = t_2$, (2) acyclicity: if $\langle t_1, \dots, t_n \rangle$ is a sequence with $t_i \succ t_{i+1}$, for every $i = 1, \dots, n - 1$, then $t_n \neq t_1$. The interpretation of $t_1 \succ t_2$ is that t_2 is an *immediate successor* of t_1 or t_1 is the *immediate predecessor* of t_2 : every instant has at most a unique immediate predecessor but can have several immediate successors.

A *temporal belief revision frame* is a quintuple $\langle T, \succ, \Omega, \{\mathcal{B}_t\}_{t \in T}, \{\mathcal{I}_t\}_{t \in T} \rangle$ where $\langle T, \succ \rangle$ is a next-time branching frame, Ω is a (possibly infinite) set of states (or possible worlds) and, for every $t \in T$, \mathcal{B}_t and \mathcal{I}_t are binary relations on Ω . The interpretation of $\omega \mathcal{I}_t \omega'$ is that at state ω and time t , according to the information received, it is possible that the true state is ω' . On the other hand, the interpretation of $\omega \mathcal{B}_t \omega'$ is that at state ω and time t , in light of the information received, the individual considers state ω' possible (an alternative expression is “ ω' is a doxastic alternative to ω at time t ”). We shall use the following notation:

$$\mathcal{B}_t(\omega) = \{\omega' \in \Omega : \omega \mathcal{B}_t \omega'\} \text{ and, similarly, } \mathcal{I}_t(\omega) = \{\omega' \in \Omega : \omega \mathcal{I}_t \omega'\}.$$

Thus $\mathcal{B}_t(\omega)$ is the set of states that are reachable from ω according to the relation \mathcal{B}_t and similarly for $\mathcal{I}_t(\omega)$.

Definition 1 *An \mathbb{L}_{AGM} -frame is a temporal belief revision frame that satisfies the following properties: $\forall \omega \in \Omega, \forall t, t_1, t_2 \in T$,*

- (0) *if $t \succ t_1$ and $\mathcal{B}_t(\omega) \cap \mathcal{I}_{t_1}(\omega) \neq \emptyset$ then $\mathcal{B}_{t_1}(\omega) = \mathcal{B}_t(\omega) \cap \mathcal{I}_{t_1}(\omega)$,*
- (1) *$\mathcal{B}_t(\omega) \subseteq \mathcal{I}_t(\omega)$,*
- (2) *if $\mathcal{I}_t(\omega) \neq \emptyset$ then $\mathcal{B}_t(\omega) \neq \emptyset$,*
- (3) *if $t \succ t_1, t \succ t_2, \mathcal{I}_{t_2}(\omega) \subseteq \mathcal{I}_{t_1}(\omega)$ and $\mathcal{I}_{t_2}(\omega) \cap \mathcal{B}_{t_1}(\omega) \neq \emptyset$ then $\mathcal{B}_{t_2}(\omega) = \mathcal{I}_{t_2}(\omega) \cap \mathcal{B}_{t_1}(\omega)$.*

Property 0 was called the Qualitative Bayes Rule in [2]; Property 1 corresponds to the requirement that information be believed, while Property 2

requires beliefs to be consistent if the information received is consistent. Property 3 was referred to as *CAB* in [2] and corresponds to postulates (K*7) and (K*8) of the AGM theory of belief revision [1].

We are interested in the subclass of \mathbb{L}_{AGM} -frames satisfying the property that, at every state-instant pair, belief revision can be rationalized by a “plausibility” ordering of the set of states, in the sense that the states that the agent considers possible are the most plausible among the ones that are compatible with the information received.

Definition 2 A plausibility ordering of the set of states is a binary relation $R \subseteq \Omega \times \Omega$ which is reflexive ($\forall \omega \in \Omega, (\omega, \omega) \in R$), connected ($\forall \omega, \omega' \in \Omega$ with $\omega \neq \omega'$, either $(\omega, \omega') \in R$ or $(\omega', \omega) \in R$) and transitive ($\forall \omega, \omega', \omega'' \in \Omega$, if $\{(\omega, \omega'), (\omega', \omega'')\} \subseteq R$ then $(\omega, \omega'') \in R$). Given a plausibility ordering of Ω and a subset $E \subseteq \Omega$, let

$$best_R E = \{\omega \in E : (\omega, \omega') \in R, \forall \omega' \in E\}.$$

Definition 3 A plausibility frame is an \mathbb{L}_{AGM} -frame that satisfies the following property: $\forall \omega \in \Omega, \forall t \in T$, there exists a plausibility ordering $R_{\omega, t}$ of Ω such that

$$\text{for every } t' \in T \text{ such that } t \succ t', \mathcal{B}_{t'}(\omega) = best_{R_{\omega, t}} \mathcal{I}_{t'}(\omega).$$

Not every \mathbb{L}_{AGM} -frame is a plausibility frame, as the example illustrated in Figure 1 shows. For simplicity, we have assumed that the information relations \mathcal{I}_t are equivalence relations (whose equivalence classes are denoted by rectangles) and the belief relations \mathcal{B}_t are serial, transitive and euclidean¹ (we represent this fact by enclosing states in ovals and, within an equivalence class of \mathcal{I}_t , we have that, for every two states ω and ω' , $\omega' \in \mathcal{B}_t(\omega)$ if and only if ω' belongs to an oval).² For example, in the \mathbb{L}_{AGM} -frame of Figure 1 we have that $\mathcal{I}_{t_1}(\beta) = \{\alpha, \beta, \delta\}$, $\mathcal{B}_{t_1}(\beta) = \{\beta\}$, $\mathcal{I}_{t_2}(\beta) = \{\beta, \gamma, \delta\}$, $\mathcal{B}_{t_2}(\beta) = \{\beta, \gamma\}$, $\mathcal{I}_{t_3}(\beta) = \{\alpha, \beta, \gamma\}$ and $\mathcal{B}_{t_3}(\beta) = \{\gamma\}$.

Consider the state-instant pair (β, t_0) in Figure 1 and suppose that R is a plausibility ordering of Ω that rationalizes belief revision at (β, t_0) . Then, considering (β, t_2) , we have that $best_R \{\beta, \gamma, \delta\} = \{\beta, \gamma\}$, so that $\{(\beta, \beta), (\beta, \gamma)\} \subseteq R$. Considering (β, t_1) , we have that $best_R \{\alpha, \beta, \delta\} = \{\beta\}$, so that $(\beta, \alpha) \in R$. Thus $\{(\beta, \alpha), (\beta, \beta), (\beta, \gamma)\} \subseteq R$, implying that $\beta \in best_R \{\alpha, \beta, \gamma\}$, but this contradicts what is the case at (β, t_3) , namely that $best_R \{\alpha, \beta, \gamma\} = \{\gamma\}$.

¹ \mathcal{B}_t is serial if, $\forall \omega \in \Omega, \mathcal{B}_t(\omega) \neq \emptyset$; it is transitive if $\omega' \in \mathcal{B}_t(\omega)$ implies that $\mathcal{B}_t(\omega') \subseteq \mathcal{B}_t(\omega)$; it is euclidean if $\omega' \in \mathcal{B}_t(\omega)$ implies that $\mathcal{B}_t(\omega) \subseteq \mathcal{B}_t(\omega')$.

² Note, however, that the results below do *not* require \mathcal{I}_t to be an equivalence relation, nor do they require \mathcal{B}_t to be serial, transitive and euclidean.

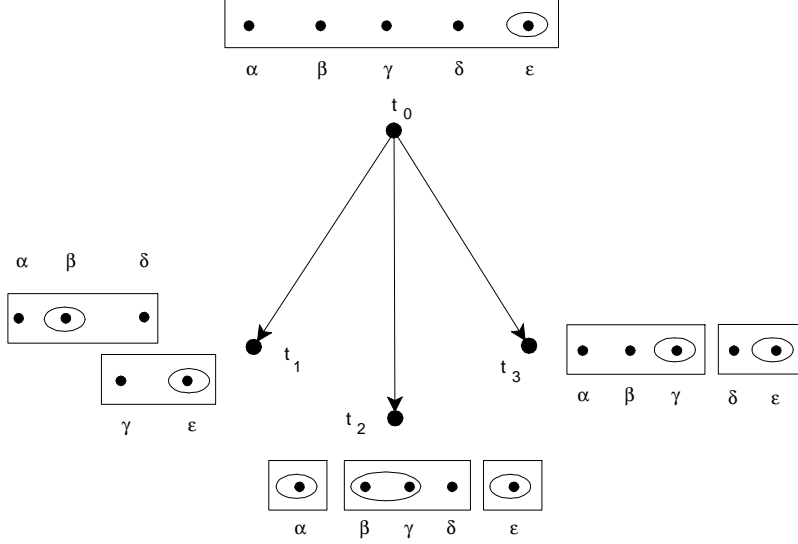


Figure 1

The proofs of all the propositions are given in the Appendix.

Proposition 4 *Let $\mathcal{F} = \langle T, \succ, \Omega, \{\mathcal{B}_t\}_{t \in T}, \{\mathcal{I}_t\}_{t \in T} \rangle$ be an \mathbb{L}_{AGM} -frame. Then \mathcal{F} is a plausibility frame if and only if it satisfies the following property: $\forall \omega \in \Omega, \forall t, t_1, \dots, t_n \in T$ with $t \succ t_k$ for all $k = 1, \dots, n$,*

$$\begin{aligned} & \text{if } \mathcal{I}_{t_k}(\omega) \cap \mathcal{B}_{t_{k+1}}(\omega) \neq \emptyset, \forall k = 1, \dots, n-1, \text{ and } \mathcal{I}_{t_n}(\omega) \cap \mathcal{B}_{t_1}(\omega) \neq \emptyset \\ & \text{then } \mathcal{I}_{t_k}(\omega) \cap \mathcal{B}_{t_{k+1}}(\omega) = \mathcal{B}_{t_k}(\omega) \cap \mathcal{I}_{t_{k+1}}(\omega), \forall k = 1, \dots, n-1. \end{aligned} \quad (PLS)$$

For example, in the \mathbb{L}_{AGM} -frame of Figure 1 we have that $\mathcal{I}_{t_1}(\beta) \cap \mathcal{B}_{t_2}(\beta) = \{\beta\} \neq \emptyset$, $\mathcal{I}_{t_2}(\beta) \cap \mathcal{B}_{t_3}(\beta) = \{\gamma\} \neq \emptyset$ and $\mathcal{I}_{t_3}(\beta) \cap \mathcal{B}_{t_1}(\beta) = \{\beta\} \neq \emptyset$, so that the antecedent of *PLS* is satisfied. However, the consequent is violated since $\mathcal{I}_{t_2}(\beta) \cap \mathcal{B}_{t_3}(\beta) = \{\gamma\}$ which is different from $\mathcal{B}_{t_2}(\beta) \cap \mathcal{I}_{t_3}(\beta) = \{\beta, \gamma\}$.

While the above proposition gives a necessary and sufficient condition for rationalizability by plausibility orderings, the following proposition gives a simpler sufficient condition which, however, is not necessary.

Proposition 5 *Let $\mathcal{F} = \langle T, \succ, \Omega, \{\mathcal{B}_t\}_{t \in T}, \{\mathcal{I}_t\}_{t \in T} \rangle$ be an \mathbb{L}_{AGM} -frame. Then \mathcal{F} is a plausibility frame if it satisfies the following property: $\forall \omega \in \Omega, \forall t_1, t_2, t_3 \in T$,*

$$\begin{aligned} & \text{if } t_1 \succ t_2 \text{ and } t_1 \succ t_3 \text{ then there exists a } t_4 \in T \text{ such that} \\ & t_1 \succ t_4 \text{ and } \mathcal{I}_{t_4}(\omega) = \mathcal{I}_{t_2}(\omega) \cup \mathcal{I}_{t_3}(\omega). \end{aligned} \quad (P_{un})$$

We now turn to a syntactic characterization of plausibility frames.

3 A temporal logic for plausibility

We briefly review the logic proposed in [2]. It is a propositional language with five modal operators: the next-time operator \bigcirc and its inverse \bigcirc^{-1} , the belief operator B , the information operator I and the “all state” operator A . The intended interpretation is as follows:

$\bigcirc\phi$:	“at every next instant it will be the case that ϕ ”
$\bigcirc^{-1}\phi$:	“at every previous instant it was the case that ϕ ”
$B\phi$:	“the agent believes that ϕ ”
$I\phi$:	“the agent is informed that ϕ ”
$A\phi$:	“it is true at every state that ϕ ”.

Given a temporal belief revision frame $\langle T, \succ, \Omega, \{\mathcal{B}_t\}_{t \in T}, \{\mathcal{I}_t\}_{t \in T} \rangle$ one obtains a *model based on it* by adding a function $V : S \rightarrow 2^\Omega$ (where S is the set of atomic propositions and 2^Ω denotes the set of subsets of Ω) that associates with every atomic proposition q the set of states at which q is true. Given a model, a state ω , an instant t and a formula ϕ , we write $(\omega, t) \models \phi$ to denote that ϕ is true at state ω and time t . Let $\|\phi\|$ denote the truth set of ϕ , that is, $\|\phi\| = \{(\omega, t) \in \Omega \times T : (\omega, t) \models \phi\}$ and let $[\phi]_t \subseteq \Omega$ denote the set of states at which ϕ is true *at time* t , that is, $[\phi]_t = \{\omega \in \Omega : (\omega, t) \models \phi\}$. Truth at (ω, t) is defined as usual for $\neg\phi$ and $\phi \vee \psi$. For the modal formulas we have

$(\omega, t) \models \bigcirc\phi$	if and only if $(\omega, t') \models \phi$ for every t' such that $t \succ t'$.
$(\omega, t) \models \bigcirc^{-1}\phi$	if and only if $(\omega, t'') \models \phi$ for every t'' such that $t'' \succ t$.
$(\omega, t) \models B\phi$	if and only if $\mathcal{B}_t(\omega) \subseteq [\phi]_t$
$(\omega, t) \models I\phi$	if and only if $\mathcal{I}_t(\omega) = [\phi]_t$
$(\omega, t) \models A\phi$	if and only if $[\phi]_t = \Omega$.

Note that, while the truth condition for the operator B is the standard one, the truth condition for the operator I is non-standard: instead of simply requiring that $\mathcal{I}_t(\omega) \subseteq [\phi]_t$ we require equality: $\mathcal{I}_t(\omega) = [\phi]_t$ (for an explanation see [2]).

A formula ϕ is *valid in a model* if $\|\phi\| = \Omega \times T$, that is, if ϕ is true at every state-instant pair (ω, t) . A formula ϕ is *valid in a frame* if it is valid in every model based on it. A property of frames *characterizes* (or is *characterized by*) an axiom if the axiom is valid in every frame that satisfies the property and, conversely, if the frame violates the property then there is a model based on that frame and a state-instant pair at which the axiom is falsified.

Define $\diamond\phi = \neg\bigcirc\neg\phi$, and $\diamond^{-1}\phi = \neg\bigcirc^{-1}\neg\phi$. The basic logic \mathbb{L}_0 has the following axioms: (1) all tautologies, (2) axiom K for all the modal axioms, (3) the temporal axioms: $\phi \rightarrow \bigcirc\diamond^{-1}\phi$ and $\phi \rightarrow \bigcirc^{-1}\diamond\phi$, (4) the backward-uniqueness axiom: $\diamond^{-1}\phi \rightarrow \bigcirc^{-1}\phi$, (5) S5 axioms for A : $A\phi \rightarrow \phi$ and $\neg A\phi \rightarrow A\neg A\phi$, (6) the inclusion axiom for B : $A\phi \rightarrow B\phi$ and (7) axioms to capture the non-standard semantics for I : $(I\phi \wedge I\psi) \rightarrow A(\phi \leftrightarrow \psi)$ and $A(\phi \leftrightarrow \psi) \rightarrow (I\phi \leftrightarrow I\psi)$. The rules of inference are Necessitation for all the modal operators *except* I and Modus Ponens.

The logic \mathbb{L}_{AGM} is obtained by adding to \mathbb{L}_0 the following axioms, where ϕ

and ψ are restricted to be Boolean (that is, non-modal) formulas.³

$$\begin{aligned}
I\phi &\rightarrow B\phi && \text{(A)} \\
(\neg B\neg\phi \wedge B\psi) &\rightarrow \bigcirc(I\phi \rightarrow B\psi) && \text{(ND)} \\
\neg B\neg(\phi \wedge \neg\psi) &\rightarrow \bigcirc(I\phi \rightarrow \neg B\psi) && \text{(NA)} \\
(I\phi \wedge \neg A\neg\phi) &\rightarrow (B\psi \rightarrow \neg B\neg\psi) && \text{(WC)} \\
\Diamond(I(\phi \wedge \psi) \wedge B\chi) &\rightarrow \bigcirc(I\phi \rightarrow B((\phi \wedge \psi) \rightarrow \chi)) && \text{(K7)} \\
\Diamond(I\phi \wedge \neg B\neg(\phi \wedge \psi) \wedge B(\psi \rightarrow \chi)) &\rightarrow \bigcirc(I(\phi \wedge \psi) \rightarrow B\chi) && \text{(K8)}
\end{aligned}$$

If ϕ_1, \dots, ϕ_m are formulas we denote by $\bigwedge_{j=1, \dots, m} \phi_j$ the conjunction $(\phi_1 \wedge \dots \wedge \phi_m)$.

Proposition 6 *Within the class of \mathbb{L}_{AGM} -frames property PLS is characterized by the following axiom, where ϕ_j and χ_j are Boolean formulas, for every $j = 1, \dots, n$,*

$$\begin{aligned}
&\Diamond(I\phi_1 \wedge \neg B\neg\phi_n \wedge B\chi_1) \wedge \left(\bigwedge_{j=2, \dots, n} \Diamond(I\phi_j \wedge \neg B\neg\phi_{j-1} \wedge B\chi_j) \right) \rightarrow \\
&\left(\bigwedge_{j=2, \dots, n} \bigcirc(I\phi_j \rightarrow B(\phi_{j-1} \rightarrow \chi_{j-1})) \right) \wedge \left(\bigwedge_{j=1, \dots, n-1} \bigcirc(I\phi_j \rightarrow B(\phi_{j+1} \rightarrow \chi_{j+1})) \right).
\end{aligned} \tag{K9}$$

Define \mathbb{L}_{PLS} to be the logic obtained by adding axiom *K9* to logic \mathbb{L}_{AGM} . It was proved in [2] that logic \mathbb{L}_{AGM} is sound with respect to the class of \mathbb{L}_{AGM} -frames. That result was extended to a characterization result in [4]. Putting together the latter result and Proposition 6 we get the following.

Proposition 7 *Logic \mathbb{L}_{PLS} is characterized by the class of plausibility frames, that is, all the theorems of \mathbb{L}_{PLS} are valid in every plausibility frame and, conversely, if a temporal belief revision frame is not a plausibility frame then there is a model based on that frame and an axiom of \mathbb{L}_{PLS} that is falsified at some state-instant pair in that model.*

We conclude with a logic that corresponds to the class of \mathbb{L}_{AGM} -frames that satisfy Property P_{un} (see Proposition 5).

Proposition 8 *Property P_{un} of temporal belief revision frames is characterized by the following axiom*

$$\Diamond I\phi \wedge \Diamond I\psi \rightarrow \Diamond I(\phi \vee \psi) \tag{UN}$$

Let \mathbb{L}_{UN} be the logic obtained by adding axiom *UN* to logic \mathbb{L}_{AGM} . The following proposition follows from Propositions 5, 7 and 8.

³For an explanation of these axioms and the restriction to Boolean formulas see [2].

Proposition 9 (a) Logic \mathbb{L}_{UN} is characterized by the class of \mathbb{L}_{AGM} -frames that satisfy property P_{un} ,

(b) Logic \mathbb{L}_{PLS} is sound with respect to the class of \mathbb{L}_{AGM} -frames that satisfy property P_{un} .

4 Concluding remarks

In general, in a plausibility frame the ordering $R_{\omega,t}$ that rationalizes belief revision at state-instant pair (ω, t) is different from the ordering $R_{\omega,t'}$ that rationalizes belief revision at (ω, t') with $t \succ t'$. In the literature on iterated belief revision several proposals have been made concerning the restrictions that should be placed on how different the new ordering can be from the earlier one: see, for example, [5] and [8]. The analysis proposed in this paper allows one to frame the discussion both semantically (with conditions such as property PLS) and syntactically (in terms of axioms such as axiom $K9$).

Alternatively, one might be interested in the existence of an ordering that rationalizes belief revision not at (some state ω and) all the immediate successors of an instant t but along a particular history, that is, along a sequence $\langle (\omega, t_0), (\omega, t_1), \dots, (\omega, t_n) \rangle$ with $t_i \succ t_{i+1}$ for all $i = 1, \dots, n-1$.⁴ A partial analysis along these lines is offered in [3] where it is shown that such an ordering exists whenever information gets refined over time, in the sense that later information implies earlier one.

A Appendix: proofs

In order to prove Propositions 4 and 5 we first consider structures $\langle \Omega, \mathcal{E}, f \rangle$ where Ω is a non-empty set, $\mathcal{E} \subseteq 2^\Omega$ (2^Ω denotes the set of subsets of Ω) and $f : \mathcal{E} \rightarrow 2^\Omega$ is a function that associates with every $E \in \mathcal{E}$ a subset $f(E) \subseteq \Omega$.

Definition 10 $\langle \Omega, \mathcal{E}, f \rangle$ is a choice structure if it satisfies the following properties: $\forall E, F \in \mathcal{E}$,

- (1) $f(E) \subseteq E$,
- (2) if $E \neq \emptyset$ then $f(E) \neq \emptyset$,
- (3) if $E \subseteq F$ and $f(F) \cap E \neq \emptyset$ then $f(E) = f(F) \cap E$.

The following result is due to Hansson ([7], Theorem 7, p. 455).

Proposition 11 . Let $\langle \Omega, \mathcal{E}, f \rangle$ be a choice structure. Then the following are equivalent:

- (a) there is an ordering $R \subseteq \Omega \times \Omega$ such that, $\forall E \in \mathcal{E}$, $f(E) = \text{best}_R E$,
- (b) $\forall E_1, \dots, E_n \in \mathcal{E}$, if $E_n \cap f(E_1) \neq \emptyset$ and $E_k \cap f(E_{k+1}) \neq \emptyset$, $\forall k = 1, \dots, n-1$, then $E_k \cap f(E_{k+1}) = f(E_k) \cap E_{k+1}$, $\forall k = 1, \dots, n-1$.

⁴That is, given a history $h = \langle (\omega, t_0), \dots, (\omega, t_n) \rangle$ there exists an ordering $R_h \subseteq \Omega \times \Omega$ such that, for very $i = 1, \dots, n$, $\mathcal{B}_{t_i}(\omega) = \text{best}_{R_h} \mathcal{I}_{t_i}(\omega)$.

Proof of Proposition 4. First we show that with every state-instant pair in an \mathbb{L}_{AGM} -frame we can associate a choice structure. Fix an \mathbb{L}_{AGM} -frame and arbitrary $\omega_0 \in \Omega$ and $t_0 \in T$. Let $T_0 = \{t \in T : t_0 \rightsquigarrow t\}$ be the set of immediate successors of t_0 . Construct the following choice structure $\langle \Omega, \mathcal{E}, f \rangle$: $\mathcal{E} = \{\mathcal{I}_t(\omega_0) : t \in T_0\}$ and, for every $\mathcal{I}_t(\omega_0) \in \mathcal{E}$, $f(\mathcal{I}_t(\omega_0)) = \mathcal{B}_t(\omega_0)$. By Property 1 of \mathbb{L}_{AGM} -frames (see Definition 1), f satisfies Property 1 of choice structures (see Definition 10), while Property 2 of \mathbb{L}_{AGM} -frames guarantees that f satisfies Property 2 of Definition 10. To see that Property 3 of Definition 10 is also satisfied, fix arbitrary $E, F \in \mathcal{E}$ and suppose that $E \subseteq F$ and $f(F) \cap E \neq \emptyset$. Then there exist $t_1, t_2 \in T_0$ such that $E = \mathcal{I}_{t_2}(\omega_0)$ and $F = \mathcal{I}_{t_1}(\omega_0)$ (so that $f(E) = \mathcal{B}_{t_2}(\omega_0)$ and $f(F) = \mathcal{B}_{t_1}(\omega_0)$), $\mathcal{I}_{t_2}(\omega_0) \subseteq \mathcal{I}_{t_1}(\omega_0)$ and $\mathcal{B}_{t_1}(\omega_0) \cap \mathcal{I}_{t_2}(\omega_0) \neq \emptyset$. Then, by Property 3 of \mathbb{L}_{AGM} -frame $\mathcal{B}_{t_2}(\omega_0) = \mathcal{B}_{t_1}(\omega_0) \cap \mathcal{I}_{t_2}(\omega_0)$, that is, $f(E) = f(F) \cap E$. Next we show that for the choice structure so constructed Property *b* of Proposition 11 is equivalent to Property *PLS*. Fix $E_1, \dots, E_n \in \mathcal{E}$, such that $E_n \cap f(E_1) \neq \emptyset$ and $E_k \cap f(E_{k+1}) \neq \emptyset$, $\forall k = 1, \dots, n-1$. Then there exist $t_1, \dots, t_n \in T_0$ such that, for all $k = 1, \dots, n$, $E_k = \mathcal{I}_{t_k}(\omega_0)$ and thus $f(E_k) = \mathcal{B}_{t_k}(\omega_0)$. By Property *PLS*, $\mathcal{I}_{t_k}(\omega_0) \cap \mathcal{B}_{t_{k+1}}(\omega_0) = \mathcal{B}_{t_k}(\omega_0) \cap \mathcal{I}_{t_{k+1}}(\omega_0)$, $\forall k = 1, \dots, n-1$, that is, $E_k \cap f(E_{k+1}) = f(E_k) \cap E_{k+1}$, $\forall k = 1, \dots, n-1$. Conversely, fix $t_1, \dots, t_n \in T_0$ and suppose that $\mathcal{I}_{t_n}(\omega_0) \cap \mathcal{B}_{t_1}(\omega_0) \neq \emptyset$ and, for all $k = 1, \dots, n-1$, $\mathcal{I}_{t_k}(\omega_0) \cap \mathcal{B}_{t_{k+1}}(\omega_0) \neq \emptyset$. Letting $E_k = \mathcal{I}_{t_k}(\omega_0)$ and $f(E_k) = \mathcal{B}_{t_k}(\omega_0)$, for all $k = 1, \dots, n$, it follows from Property *b* of Proposition 11 that $E_k \cap f(E_{k+1}) = f(E_k) \cap E_{k+1}$, $\forall k = 1, \dots, n-1$, that is, $\mathcal{I}_{t_k}(\omega_0) \cap \mathcal{B}_{t_{k+1}}(\omega_0) = \mathcal{B}_{t_k}(\omega_0) \cap \mathcal{I}_{t_{k+1}}(\omega_0)$, $\forall k = 1, \dots, n-1$. Proposition 4 now follows from Proposition 11. ■

The following proposition is a corollary of a result due to Hansson ([7], Theorem 4, p. 448).

Proposition 12 *Let $\langle \Omega, \mathcal{E}, f \rangle$ be a choice structure that satisfies the following property: if $E, F \in \mathcal{E}$, then $(E \cup F) \in \mathcal{E}$. Then there is an ordering $R \subseteq \Omega \times \Omega$ such that, $\forall E \in \mathcal{E}$, $f(E) = \text{best}_R E$.*

Proof of Proposition 5. Fix an \mathbb{L}_{AGM} -frame that satisfies Property P_{un} . Then the associated choice structure $\langle \Omega, \mathcal{E}, f \rangle$ defined in the proof of Proposition 4 satisfies the property that if $E, F \in \mathcal{E}$, then $(E \cup F) \in \mathcal{E}$. Thus Proposition 5 follows from Proposition 12. ■

Proof of Proposition 6. Fix an \mathbb{L}_{AGM} -frame that satisfies property *PLS*, an arbitrary model based on it, arbitrary Boolean formulas ϕ_1, \dots, ϕ_n and χ_1, \dots, χ_n and arbitrary $\omega_0 \in \Omega$ and $t_0 \in T$ and suppose that

$$(\omega_0, t_0) \models \diamond (I\phi_1 \wedge \neg B \neg \phi_n \wedge B\chi_1) \wedge \left(\bigwedge_{j=2, \dots, n} \diamond (I\phi_j \wedge \neg B \neg \phi_{j-1} \wedge B\chi_j) \right) \quad (1)$$

Then there exist $t_1, \dots, t_n \in T$ such that $t_0 \rightsquigarrow t_j$ for all $j = 1, \dots, n$ and

$$\begin{aligned} \text{(a)} \quad & (\omega_0, t_1) \models I\phi_1 \wedge \neg B \neg \phi_n \wedge B\chi_1 \quad \text{and} \\ \text{(b)} \quad & (\omega_0, t_j) \models I\phi_j \wedge \neg B \neg \phi_{j-1} \wedge B\chi_j \quad \text{for all } j = 2, \dots, n. \end{aligned} \quad (2)$$

Thus

$$\begin{aligned}
& \text{(a) } \mathcal{I}_{t_j}(\omega_0) = \lceil \phi_j \rceil_{t_j} \quad \text{for all } j = 1, \dots, n, \\
& \text{(b) } \mathcal{B}_{t_j}(\omega_0) \cap \mathcal{I}_{t_{j-1}}(\omega_0) \neq \emptyset \quad \text{for all } j = 2, \dots, n, \\
& \text{(c) } \mathcal{B}_{t_1}(\omega_0) \cap \mathcal{I}_{t_n}(\omega_0) \neq \emptyset \\
& \text{(d) } \mathcal{B}_{t_j}(\omega_0) \subseteq \lceil \chi_j \rceil_{t_j} \quad \text{for all } j = 1, \dots, n.
\end{aligned} \tag{3}$$

Fix an arbitrary $t \in T$ with $t_0 \succ t$ and suppose that, for some $j \in \{1, \dots, n\}$, $(\omega_0, t) \models I\phi_j$, that is, $\mathcal{I}_t(\omega_0) = \lceil \phi_j \rceil_t$. Since ϕ_j is a Boolean formula, by Proposition 5 in ([2] p. 148), $\lceil \phi_j \rceil_t = \lceil \phi_j \rceil_{t_j}$, so that $\mathcal{I}_t(\omega_0) = \mathcal{I}_{t_j}(\omega_0)$. It follows from this and property (3) of \mathbb{L}_{AGM} -frames (see Definition 1), that $\mathcal{B}_t(\omega_0) = \mathcal{B}_{t_j}(\omega_0)$. Thus without loss of generality we can take $t = t_j$.

We need to show that if $j \in \{2, \dots, n\}$ then $(\omega_0, t_j) \models B(\phi_{j-1} \rightarrow \chi_{j-1})$ and if $j \in \{1, \dots, n-1\}$ then $(\omega_0, t_j) \models B(\phi_{j+1} \rightarrow \chi_{j+1})$. By (b) and (c) of (3) and property *PLS* (letting $k = j-1$) we have that

$$\mathcal{I}_{t_{j-1}}(\omega_0) \cap \mathcal{B}_{t_j}(\omega_0) = \mathcal{B}_{t_{j-1}}(\omega_0) \cap \mathcal{I}_{t_j}(\omega_0), \quad \forall j = 2, \dots, n. \tag{4}$$

It follows from (4) and (d) of (3) that if $j \in \{2, \dots, n\}$ then $(\omega_0, t_j) \models B(\phi_{j-1} \rightarrow \chi_{j-1})$ and if $j \in \{1, \dots, n-1\}$ then $(\omega_0, t) \models B(\phi_{j+1} \rightarrow \chi_{j+1})$.⁵

Conversely, fix an \mathbb{L}_{AGM} -frame that violates property *PLS*. Then there exist $\omega_0 \in \Omega$, $t_0, t_1, \dots, t_n \in T$ with $t_0 \succ t_k$, for all $k = 1, \dots, n$, and a $k^* \in \{1, \dots, n-1\}$ such that

$$\begin{aligned}
& \text{(a) } \mathcal{I}_{t_k}(\omega_0) \cap \mathcal{B}_{t_{k+1}}(\omega_0) \neq \emptyset, \quad \forall k = 1, \dots, n-1, \\
& \text{(b) } \mathcal{B}_{t_1}(\omega_0) \cap \mathcal{I}_{t_n}(\omega_0) \neq \emptyset, \\
& \text{(c) } \mathcal{I}_{t_{k^*}}(\omega_0) \cap \mathcal{B}_{t_{k^*+1}}(\omega_0) \neq \mathcal{B}_{t_{k^*}}(\omega_0) \cap \mathcal{I}_{t_{k^*+1}}(\omega_0).
\end{aligned} \tag{5}$$

Let $p_1, \dots, p_n, q_1, \dots, q_n$, be atomic propositions and construct a model where, for every $k = 1, \dots, n$, $\|p_k\| = \mathcal{I}_{t_k}(\omega_0) \times T$ and $\|q_k\| = \mathcal{B}_{t_k}(\omega_0) \times T$. Then, by (a) and (b) of (5) (letting $j = k+1$)

⁵Proof. Suppose first that $j \in \{2, \dots, n\}$. Fix an arbitrary $\omega \in \mathcal{B}_{t_j}(\omega_0)$. If $(\omega, t_j) \models \neg\phi_{j-1}$ then $(\omega, t_j) \models (\phi_{j-1} \rightarrow \chi_{j-1})$. If $(\omega, t_j) \models \phi_{j-1}$ then $\omega \in \lceil \phi_{j-1} \rceil_{t_j}$. Since ϕ_{j-1} is a Boolean formula, by Proposition 5 in [2], $\lceil \phi_{j-1} \rceil_{t_j} = \lceil \phi_{j-1} \rceil_{t_{j-1}}$ and by (a) of (3) $\lceil \phi_{j-1} \rceil_{t_{j-1}} = \mathcal{I}_{t_{j-1}}(\omega_0)$. Thus $\omega \in \mathcal{I}_{t_{j-1}}(\omega_0)$, so that $\omega \in \mathcal{I}_{t_{j-1}}(\omega_0) \cap \mathcal{B}_{t_j}(\omega_0)$. By (4) $\mathcal{I}_{t_{j-1}}(\omega_0) \cap \mathcal{B}_{t_j}(\omega_0) \subseteq \mathcal{B}_{t_{j-1}}(\omega_0)$ and by (d) of (3), $\mathcal{B}_{t_{j-1}}(\omega_0) \subseteq \lceil \chi_{j-1} \rceil_{t_{j-1}}$. Since χ_{j-1} is a Boolean formula, by Proposition 5 in [2], $\lceil \chi_{j-1} \rceil_{t_{j-1}} = \lceil \chi_{j-1} \rceil_{t_j}$. Thus $\omega \in \lceil \chi_{j-1} \rceil_{t_j}$, that is, $(\omega, t_j) \models \chi_{j-1}$ and, therefore, $(\omega, t_j) \models (\phi_{j-1} \rightarrow \chi_{j-1})$. Thus, since $\omega \in \mathcal{B}_{t_j}(\omega_0)$ was chosen arbitrarily, $(\omega_0, t_j) \models B(\phi_{j-1} \rightarrow \chi_{j-1})$.

Next we consider the case where $j = 1$ and show that $(\omega_0, t_1) \models B(\phi_2 \rightarrow \chi_2)$. Fix an arbitrary $\omega \in \mathcal{B}_{t_1}(\omega_0)$. If $(\omega, t_1) \models \neg\phi_2$ then $(\omega, t_1) \models (\phi_2 \rightarrow \chi_2)$. If $(\omega, t_1) \models \phi_2$ then $\omega \in \lceil \phi_2 \rceil_{t_1}$. Since ϕ_2 is a Boolean formula, by Proposition 5 in [2], $\lceil \phi_2 \rceil_{t_1} = \lceil \phi_2 \rceil_{t_2}$ and by (a) of (3) $\lceil \phi_2 \rceil_{t_2} = \mathcal{I}_{t_2}(\omega_0)$. Thus $\omega \in \mathcal{B}_{t_1}(\omega_0) \cap \mathcal{I}_{t_2}(\omega_0)$. By (4) $\mathcal{B}_{t_1}(\omega_0) \cap \mathcal{I}_{t_2}(\omega_0) \subseteq \mathcal{B}_{t_2}(\omega_0)$ and by (d) of (3), $\mathcal{B}_{t_2}(\omega_0) \subseteq \lceil \chi_2 \rceil_{t_2}$. Since χ_2 is a Boolean formula, by Proposition 5 in [2], $\lceil \chi_2 \rceil_{t_2} = \lceil \chi_2 \rceil_{t_1}$. Thus $\omega \in \lceil \chi_2 \rceil_{t_1}$, that is, $(\omega, t_1) \models \chi_2$ and, therefore, $(\omega, t_1) \models (\phi_2 \rightarrow \chi_2)$. Thus $(\omega_0, t_1) \models B(\phi_2 \rightarrow \chi_2)$.

The proof that if $j \in \{2, \dots, n-1\}$ then $(\omega_0, t_j) \models B(\phi_{j+1} \rightarrow \chi_{j+1})$ is similar.

$$(\omega_0, t_0) \models \diamond (Ip_1 \wedge \neg B \neg p_n \wedge Bq_1) \wedge \left(\bigwedge_{j=2, \dots, n} \diamond (Ip_j \wedge \neg B \neg p_{j-1} \wedge Bq_j) \right). \quad (6)$$

By (c) of (5), either

(A) there is an $\alpha \in \mathcal{I}_{t_{k^*}}(\omega_0) \cap \mathcal{B}_{t_{k^*+1}}(\omega_0)$ such that $\alpha \notin \mathcal{B}_{t_{k^*}}(\omega_0) \cap \mathcal{I}_{t_{k^*+1}}(\omega_0)$
or

(B) there is a $\beta \in \mathcal{B}_{t_{k^*}}(\omega_0) \cap \mathcal{I}_{t_{k^*+1}}(\omega_0)$ such that $\beta \notin \mathcal{I}_{t_{k^*}}(\omega_0) \cap \mathcal{B}_{t_{k^*+1}}(\omega_0)$.

Consider Case A first. Since $\alpha \in \mathcal{B}_{t_{k^*+1}}(\omega_0)$ and, by property (1) of \mathbb{L}_{AGM} -frames (see Definition 1), $\mathcal{B}_{t_{k^*+1}}(\omega_0) \subseteq \mathcal{I}_{t_{k^*+1}}(\omega_0)$, it must be that $\alpha \notin \mathcal{B}_{t_{k^*}}(\omega_0)$, so that $(\alpha, t) \models \neg q_{k^*}$, for every $t \in T$. Since $\alpha \in \mathcal{I}_{t_{k^*}}(\omega_0)$, $(\alpha, t) \models p_{k^*}$, for every $t \in T$. Thus $(\alpha, t) \models \neg(p_{k^*} \rightarrow q_{k^*})$, for every $t \in T$, in particular $(\alpha, t_{k^*+1}) \models \neg(p_{k^*} \rightarrow q_{k^*})$. Since $\alpha \in \mathcal{B}_{t_{k^*+1}}(\omega_0)$, it follows that $(\omega_0, t_{k^*+1}) \models \neg B(p_{k^*} \rightarrow q_{k^*})$, so that, since $(\omega_0, t_{k^*+1}) \models Ip_{k^*+1}$, $(\omega_0, t_{k^*+1}) \models \neg(Ip_{k^*+1} \rightarrow B(p_{k^*} \rightarrow q_{k^*}))$. It follows from this and the fact that $t_0 \succ t_{k^*+1}$ that $(\omega_0, t_0) \models \neg \bigcirc (Ip_{k^*+1} \rightarrow B(p_{k^*} \rightarrow q_{k^*}))$. This, together with (6) falsifies axiom *K9* at (ω_0, t_0) .

Now consider Case B. Since $\beta \in \mathcal{B}_{t_{k^*}}(\omega_0)$ and $\mathcal{B}_{t_{k^*}}(\omega_0) \subseteq \mathcal{I}_{t_{k^*}}(\omega_0)$, it must be that $\beta \notin \mathcal{B}_{t_{k^*+1}}(\omega_0)$, so that $(\beta, t) \models \neg q_{k^*+1}$, for every $t \in T$. Since $\beta \in \mathcal{I}_{t_{k^*+1}}(\omega_0)$, $(\beta, t) \models p_{k^*+1}$, for every $t \in T$. Thus $(\beta, t) \models \neg(p_{k^*+1} \rightarrow q_{k^*+1})$, for every $t \in T$, in particular $(\beta, t_{k^*}) \models \neg(p_{k^*+1} \rightarrow q_{k^*+1})$. Since $\beta \in \mathcal{B}_{t_{k^*}}(\omega_0)$, it follows that $(\omega_0, t_{k^*}) \models \neg B(p_{k^*+1} \rightarrow q_{k^*+1})$, so that, since $(\omega_0, t_{k^*}) \models Ip_{k^*}$, $(\omega_0, t_{k^*}) \models \neg(Ip_{k^*} \rightarrow B(p_{k^*+1} \rightarrow q_{k^*+1}))$. It follows from this and the fact that $t_0 \succ t_{k^*}$ that $(\omega_0, t_0) \models \neg \bigcirc (Ip_{k^*} \rightarrow B(p_{k^*+1} \rightarrow q_{k^*+1}))$. This, together with (6) falsifies axiom *K9* at (ω_0, t_0) . ■

Proof of Proposition 8. Assume Property P_{un} and suppose that $(\omega, t_1) \models \diamond I\phi \wedge \diamond I\psi$. Then there exist $t_2, t_3 \in T$ such that $t_1 \succ t_2$, $t_1 \succ t_3$, $\mathcal{I}_{t_2}(\omega) = [\phi]_{t_2}$ and $\mathcal{I}_{t_3}(\omega) = [\psi]_{t_3}$. Then, by P_{un} , there exists a $t_4 \in T$ such that $t_1 \succ t_4$ and $\mathcal{I}_{t_4}(\omega) = \mathcal{I}_{t_2}(\omega) \cup \mathcal{I}_{t_3}(\omega)$. Since ψ is Boolean, $[\psi]_{t_3} = [\psi]_{t_2}$. Furthermore, $[\phi \vee \psi]_{t_2} = [\phi]_{t_2} \cup [\psi]_{t_2}$. Thus $\mathcal{I}_{t_4}(\omega) = [\phi \vee \psi]_{t_2}$ and, since $(\phi \vee \psi)$ is Boolean, $[\phi \vee \psi]_{t_2} = [\phi \vee \psi]_{t_4}$. Thus $(\omega, t_4) \models I(\phi \vee \psi)$ so that $(\omega, t_1) \models \diamond I(\phi \vee \psi)$.

Conversely, suppose that the property is violated, that is, there exist $t_1, t_2, t_3 \in T$ and $\omega \in \Omega$ such that $t_1 \succ t_2$ and $t_1 \succ t_3$ and, for every $t \in T$ if $t_1 \succ t$ then $\mathcal{I}_t(\omega) \neq \mathcal{I}_{t_2}(\omega) \cup \mathcal{I}_{t_3}(\omega)$. Let p and q be sentence letters and construct a model where $\|p\| = \mathcal{I}_{t_2}(\omega) \times T$ and $\|q\| = \mathcal{I}_{t_3}(\omega) \times T$. Then $(\omega, t_2) \models Ip$ and $(\omega, t_3) \models Iq$ so that $(\omega, t_1) \models \diamond Ip \wedge \diamond Iq$. Fix an arbitrary $t \in T$ such that $t_1 \succ t$. By hypothesis $\mathcal{I}_t(\omega) \neq \mathcal{I}_{t_2}(\omega) \cup \mathcal{I}_{t_3}(\omega)$, so that, since $[p \vee q]_t = \mathcal{I}_{t_2}(\omega) \cup \mathcal{I}_{t_3}(\omega)$, $(\omega, t) \not\models I(p \vee q)$. Thus $(\omega, t_1) \models \bigcirc \neg I(p \vee q)$, that is, $(\omega, t_1) \models \neg \diamond I(p \vee q)$. ■

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