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TWO LECTURES ON THE EPISTEMIC FOUNDATIONS OF GAME THEORY

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Abstract

This working paper contains the slides of two invited lectures on the Epistemic Foundations of Game Theory, delivered at the Royal Netherlands Academy of Arts and Sciences (KNAW) on February 8, 2007.

Royal Netherlands Academy of Arts and Sciences (KNAW)
Master Class

Amsterdam, February 8th, 2007

Epistemic Foundations of Game Theory

Lecture 1

Giacomo Bonanno

(<http://www.econ.ucdavis.edu/faculty/bonanno/>)

QUESTION:

What strategies can be chosen by *rational* players who *know* the structure of the game and the preferences of their opponents and who *recognize* each other's rationality and knowledge?

Keywords: knowledge, rationality, recognition of each other's knowledge and rationality

Modular approach

Module 1: representation of belief and knowledge of an individual (Hintikka, 1962; Kripke, 1963).

Module 2: extension to many individuals.
Common belief and common knowledge
("recognition of each other's belief / knowledge")

Module 3: definition of rationality in games
(relationship between choice and beliefs)

QUESTION: what are the implications of rationality and common belief of rationality in games?

Module 1

representation of beliefs and knowledge of an individual

Finite set of states Ω and a binary relation \mathcal{B} on Ω .

$\alpha \mathcal{B} \beta$ means “at state α the individual considers state β possible”

Notation: $\mathcal{B}(\omega) = \{\omega' \in \Omega : \omega \mathcal{B} \omega'\}$ set of states considered possible at ω

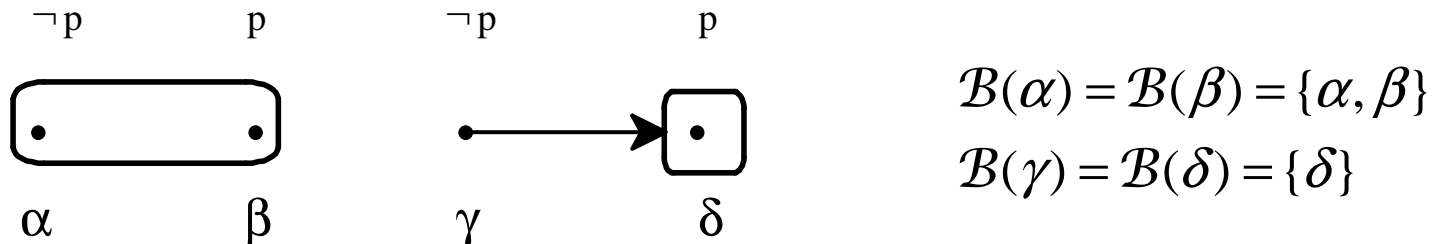
PROPERTIES $\forall \omega, \omega' \in \Omega$,

1. $\mathcal{B}(\omega) \neq \emptyset$ seriality
2. if $\omega' \in \mathcal{B}(\omega)$ then $\mathcal{B}(\omega') \subseteq \mathcal{B}(\omega)$ transitivity
3. if $\omega' \in \mathcal{B}(\omega)$ then $\mathcal{B}(\omega) \subseteq \mathcal{B}(\omega')$ euclidean

Belief operator on events: $B : 2^\Omega \rightarrow 2^\Omega$

For $E \subseteq \Omega$, $\omega \in BE$ if and only if $\mathcal{B}(\omega) \subseteq E$

EXAMPLE:



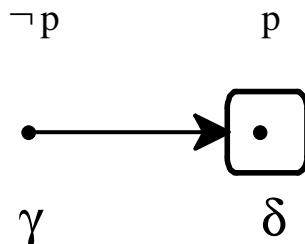
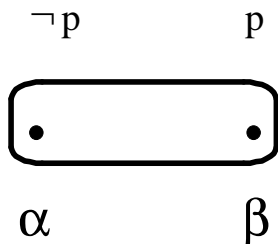
Let $E = \{\beta, \delta\}$: the event that represents the proposition p

Then $BE = \{\gamma, \delta\}$

Properties of the belief operator: $\forall E \subseteq \Omega$

1. $BE \subseteq \neg B\neg E$ (consistency:
follows from seriality of \mathcal{B})
2. $BE \subseteq BBE$ (positive introspection:
follows from transitivity of \mathcal{B})
3. $\neg BE \subseteq B\neg BE$ (negative introspection:
follows from euclideaness of \mathcal{B})

Mistaken beliefs are possible: at γ p is false but the individual believes p



If $E = \{\beta, \delta\}$, then
 $\gamma \notin E$ but $\gamma \in BE = \{\gamma, \delta\}$

KNOWLEDGE

If - in addition to the previous properties - the "doxastic accessibility" relation \mathcal{B} is **reflexive** ($\forall \omega \in \Omega, \omega \in \mathcal{B}(\omega)$) then it is an **equivalence relation** - giving rise to a **partition** of the set of states - and the associated belief operator satisfies the additional property that $\forall E \subseteq \Omega, \mathcal{B}E \subseteq E$ (beliefs are correct). In this case we speak of **knowledge** and the associated operator is denoted by K rather than B



α



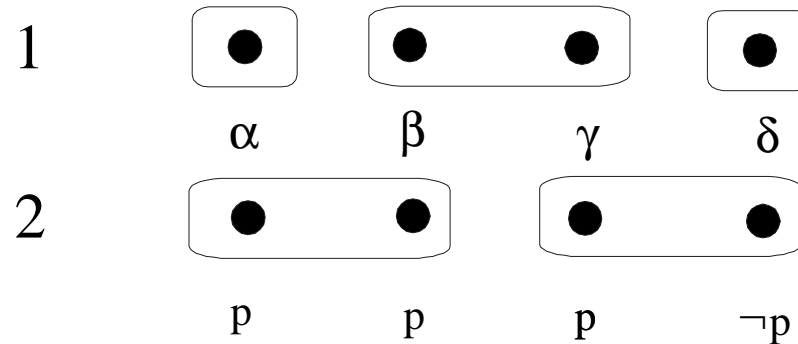
β

γ

Module 2

interactive belief and common belief

Set of individuals N and a binary relation \mathcal{B}_i for every $i \in N$



Let $E = \{\alpha, \beta, \gamma\}$: the event that represents the proposition p

Then $K_1 E = \{\alpha, \beta, \gamma\}$, $K_2 E = \{\alpha, \beta\}$

$K_1 K_2 E = \{\alpha\}$, $K_2 K_1 K_2 E = \emptyset$

An event E is *commonly believed* if (1) everybody believes it, (2) everybody believes that everybody believes it, (3) everybody believes that everybody believes that everybody believes it, etc.

Define the “everybody believes” operator B^e as follows:

$$B^e E = B_1 E \cap B_2 E \cap \dots \cap B_n E$$

The common belief operator B_* is defined as follows:

$$B_* E = B^e E \cap B^e B^e E \cap B^e B^e B^e E \cap \dots$$

Let \mathcal{B}_* be the *transitive closure* of $\mathcal{B}_1 \cup \mathcal{B}_2 \cup \dots \cup \mathcal{B}_n$

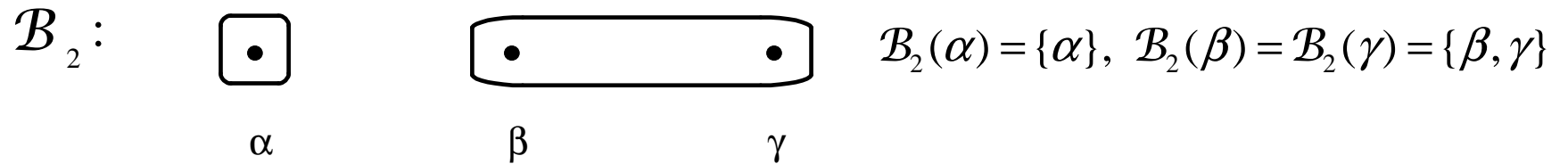
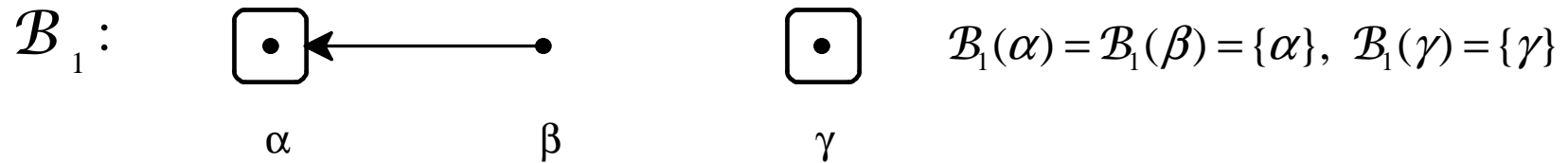
Thus $\omega' \in \mathcal{B}_*(\omega)$ if and only if there exists a sequence

$\langle \omega_1, \dots, \omega_m \rangle$ in Ω such that

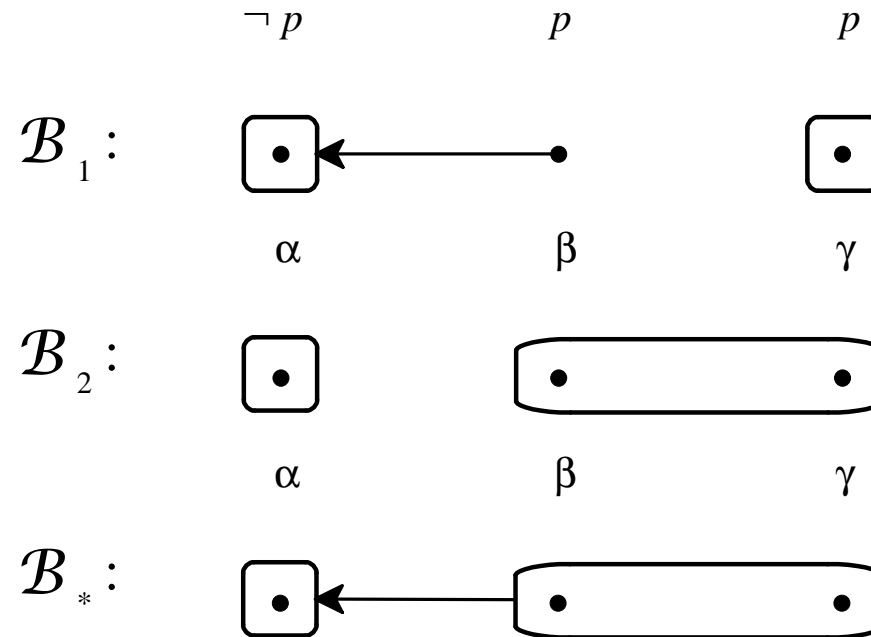
(1) $\omega_1 = \omega$

(2) $\omega_m = \omega'$

(3) for every $j = 1, \dots, m$ there exists an individual $i \in N$ such that $\omega_{j+1} \in \mathcal{B}_i(\omega_j)$



PROPOSITION. $\omega \in B_*E$ if and only if $\mathcal{B}_*(\omega) \subseteq E$.



Let $E = \{\beta, \gamma\}$: the event that represents the proposition p

Then $B_1E = \{\gamma\}$, $B_2E = \{\beta, \gamma\}$, $B_*E = \emptyset$

In fact, while $\gamma \in B_1B_2E = \{\gamma\}$, $\gamma \notin B_2B_1E = \emptyset$

Module 3

Models of games and Rationality

Definition. A finite strategic-form game *with ordinal payoffs* is a quintuple

$$\langle N, \{S_i\}_{i \in N}, O, \{\succeq_i\}_{i \in N}, z \rangle$$

$N = \{1, \dots, n\}$ is a set of *players*

S_i is a finite set of *strategies* or choices of player $i \in N$

O is a set of *outcomes*

\succeq_i is player i 's ordering of O ($o \succeq_i o'$ means that, for player i , outcome o is at least as good as outcome o')

$z : S \rightarrow O$ (where $S = S_1 \times \dots \times S_n$) associates an outcome with every strategy profile $s \in S$

Definition. Given a strategic-form game with ordinal payoffs

$$\langle N, \{S_i\}_{i \in N}, O, \{\geq_i\}_{i \in N}, z \rangle$$

a reduced form of it is a triple

$$\langle N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N} \rangle$$

where $u_i : S \rightarrow \mathbb{R}$ is such that $u_i(s) \geq u_i(s')$ if and only if $z(s) \geq_i z(s')$

↖ player i 's *utility function*

		Player 2		
		e	f	g
Player 1	A	3 , 2	3 , 1	0 , 1
	B	2 , 3	2 , 2	3 , 1
	C	1 , 2	1 , 2	4 , 1
	D	0 , 2	0 , 3	1 , 3

SAME AS

		Player 2		
		e	f	g
Player 1	A	9 , 6	6 , 4	0 , 4
	B	4 , 9	3 , 3	2 , 0
	C	2 , 5	2 , 5	8 , 2
	D	1 , 0	0 , 8	1 , 8

Definition. An *epistemic model* of a strategic-form game is an interactive belief structure together with n functions

$$\sigma_i : \Omega \rightarrow S_i \quad (i \in N)$$

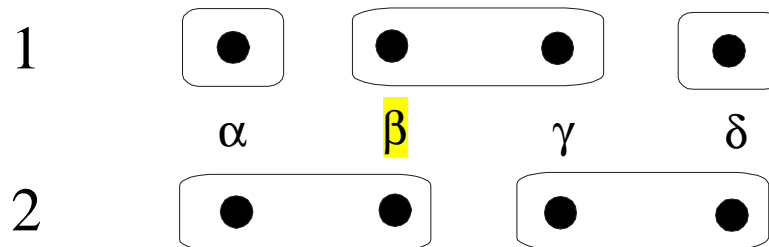
Interpretation: $\sigma_i(\omega)$ is player i 's chosen strategy at state ω

Restriction: if $\omega' \in \mathcal{B}_i(\omega)$ then $\sigma_i(\omega') = \sigma_i(\omega)$

(no player has mistaken beliefs about her own strategy)

EXAMPLE

		Player 2		
		e	f	g
Player 1	A	3 , 2	3 , 1	0 , 1
	B	2 , 3	2 , 2	3 , 1
	C	1 , 2	1 , 2	4 , 1
	D	0 , 2	0 , 3	1 , 3



1's strategy: A **C** C D

2's strategy: f **f** g g

At every state each player knows his own strategy

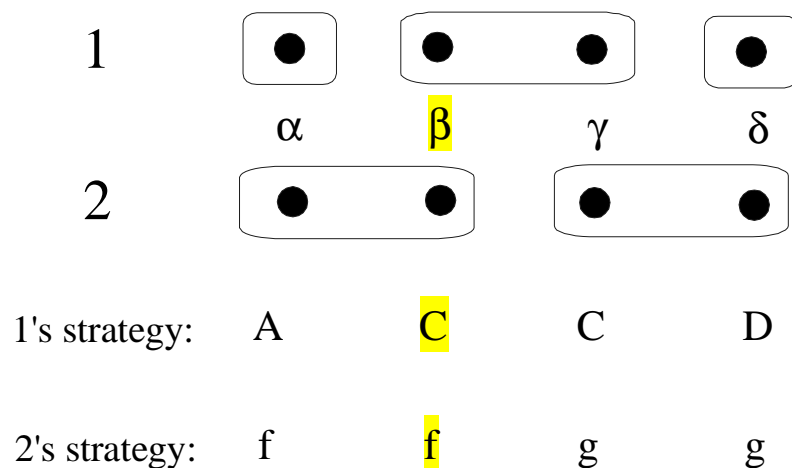
At state β player 1 plays C (he knows this) not knowing whether player 2 is playing f or g and player 2 plays f (she knows this) not knowing whether player 1 is playing A or C

RATIONALITY

Non-probabilistic (no expected utility) and very weak notion of rationality

Definition. Player i is **IRRATIONAL** at state ω if there is a strategy s_i (of player i) which she believes to be better than $\sigma_i(\omega)$ (that is, if she believes that she can do better with another strategy)

Player i is **RATIONAL** at state ω if and only if she is not irrational



		Player 2		
		e	f	g
Player 1	A	3 , 2	3 , 1	0 , 1
	B	2 , 3	2 , 2	3 , 1
	C	1 , 2	1 , 2	4 , 1
	D	0 , 2	0 , 3	1 , 3

Player 1 is rational at state β

Let s_i and t_i be two strategies of player i : $s_i, t_i \in S_i$

$s_i \succ_i t_i$ is interpreted as “strategy s_i is better for player i than strategy t_i ”

$s_i \succ_i t_i$ is true at state ω if $u_i(s_i, \sigma_{-i}(\omega)) > u_i(t_i, \sigma_{-i}(\omega))$

that is, s_i is better than t_i against $\sigma_{-i}(\omega)$

profile of strategies chosen
by the players other than i

			Player 2					
			E F G					
1's strategy:	α	β	γ	P l a y e r 1	A	3 , 2	1 , 1	0 , 1
	●	●	●		B	2 , 3	2 , 2	3 , 1
	A	C	C		C	1 , 2	0 , 2	4 , 1
2's strategy:	E	F	G					

$A \succ_1 B$ $B \succ_1 A$ $C \succ_1 B$
 $A \succ_1 C$ $B \succ_1 C$ $C \succ_1 A$
 $B \succ_1 C$ $A \succ_1 C$ $B \succ_1 A$
 $E \succ_2 F$ $F \succ_2 G$ $F \succ_2 G$ etc.

Let $\|s_i \succ_i t_i\| = \{\omega \in \Omega : u_i(s_i, \sigma_{-i}(\omega)) > u_i(t_i, \sigma_{-i}(\omega))\}$ event that s_i is better than t_i

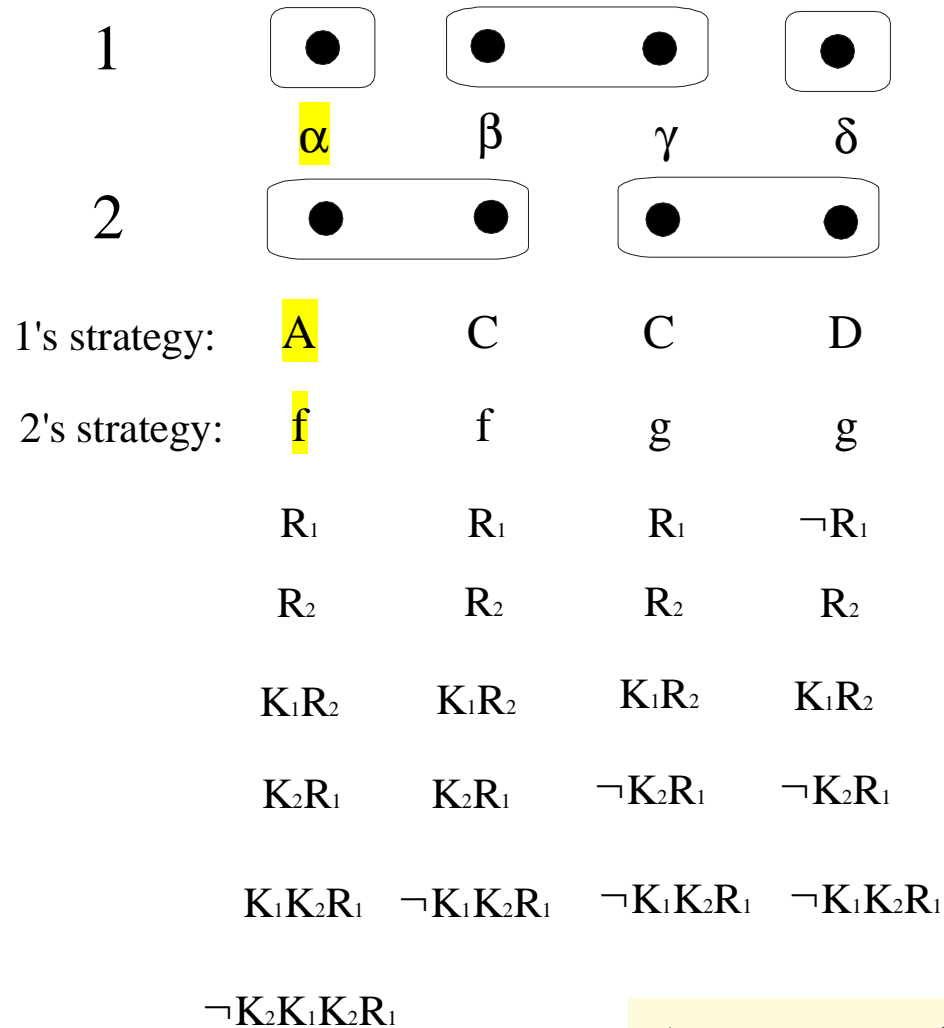
If $s_i \in S_i$, let $\|s_i\| = \{\omega \in \Omega : \sigma_i(\omega) = s_i\}$ event that player i chooses s_i

Let \mathbf{R}_i be the event representing the proposition “player i is rational”

$$\|s_i\| \cap B_i \|t_i \succ_i s_i\| \subseteq \neg \mathbf{R}_i$$

$$\neg \mathbf{R}_i = \bigcup_{s_i \in S_i} \bigcup_{t_i \in S_i} (\|s_i\| \cap B_i \|t_i \succ_i s_i\|)$$

$$\mathbf{R} = \mathbf{R}_1 \cap \dots \cap \mathbf{R}_n \quad \text{all players are rational}$$



		Player 2		
		e	f	g
Player 1	A	3, 2	3, 1	0, 1
	B	2, 3	2, 2	3, 1
	C	1, 2	1, 2	4, 1
	D	0, 2	0, 3	1, 3

$$R_1 = \{\alpha, \beta, \gamma\}, R_2 = \{\alpha, \beta, \gamma, \delta\}$$

$$K_1 R_2 = \{\alpha, \beta, \gamma, \delta\}, K_2 R_1 = \{\alpha, \beta\}$$

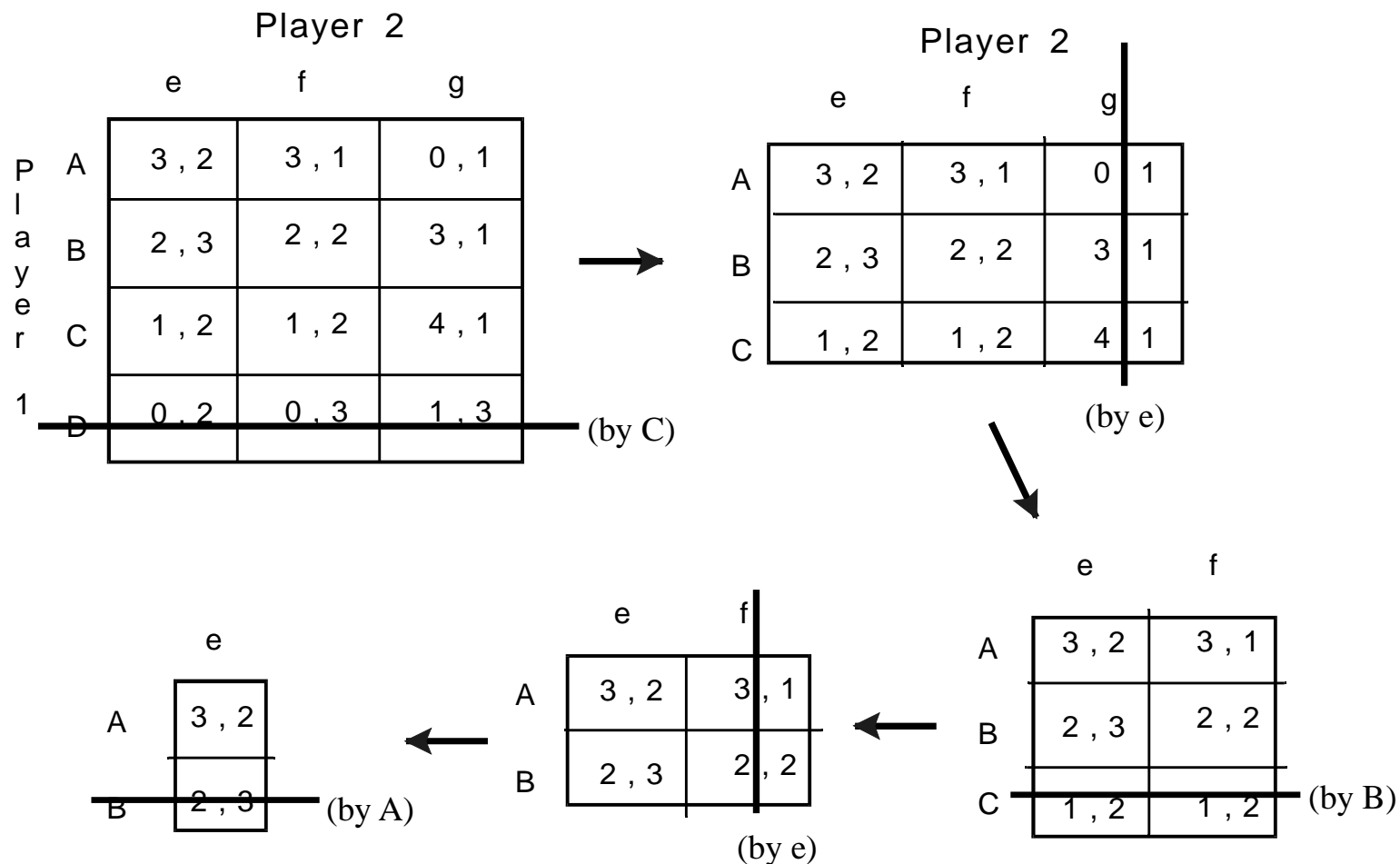
$$K_1 K_2 R_1 = \{\alpha\}, K_2 K_1 K_2 R_1 = \emptyset$$

At state α there is mutual knowledge of rationality but not common knowledge of rationality

Let $S_{-i} = S_1 \times \dots \times S_{i-1} \times S_{i+1} \times \dots \times S_n$ set of strategy profiles of all players except i

Definition. Let $s_i, t_i \in S_i$. We say that t_i is *strictly dominated* by s_i if $u_i(t_i, s_{-i}) < u_i(s_i, s_{-i})$ for all $s_{-i} \in S_{-i}$

ITERATED DELETION OF STRICTLY DOMINATED STRATEGIES



Let G be a strategic-form game with ordinal payoffs and G^∞ be the game obtained after applying the procedure of Iterated Deletion of Strictly Dominated Strategies.

Let S^∞ denote the strategy profiles of game G^∞

Given a model of G , let S^∞ denote the event $\{\omega \in \Omega : \sigma(\omega) \in S^\infty\}$

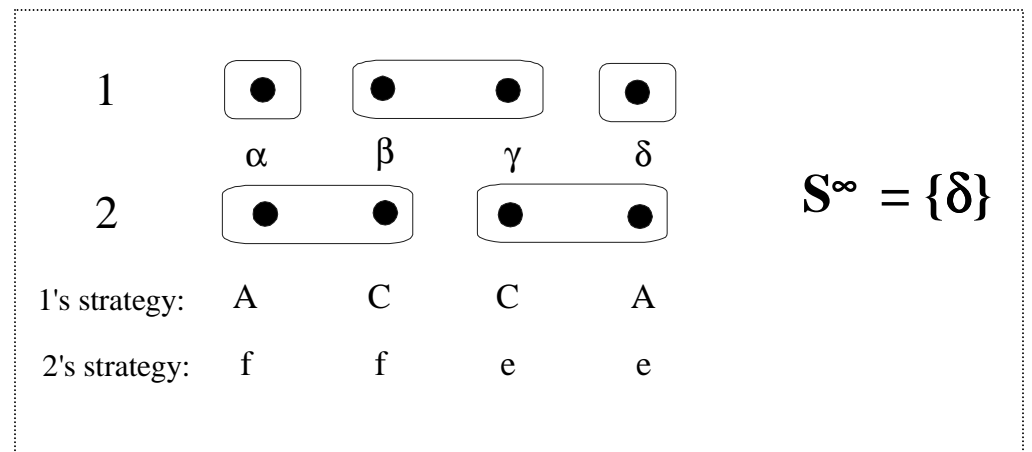
G

		Player 2		
		e	f	g
Player 1	A	3, 2	3, 1	0, 1
	B	2, 3	2, 2	3, 1
	C	1, 2	1, 2	4, 1
	D	0, 2	0, 3	1, 3

G^∞

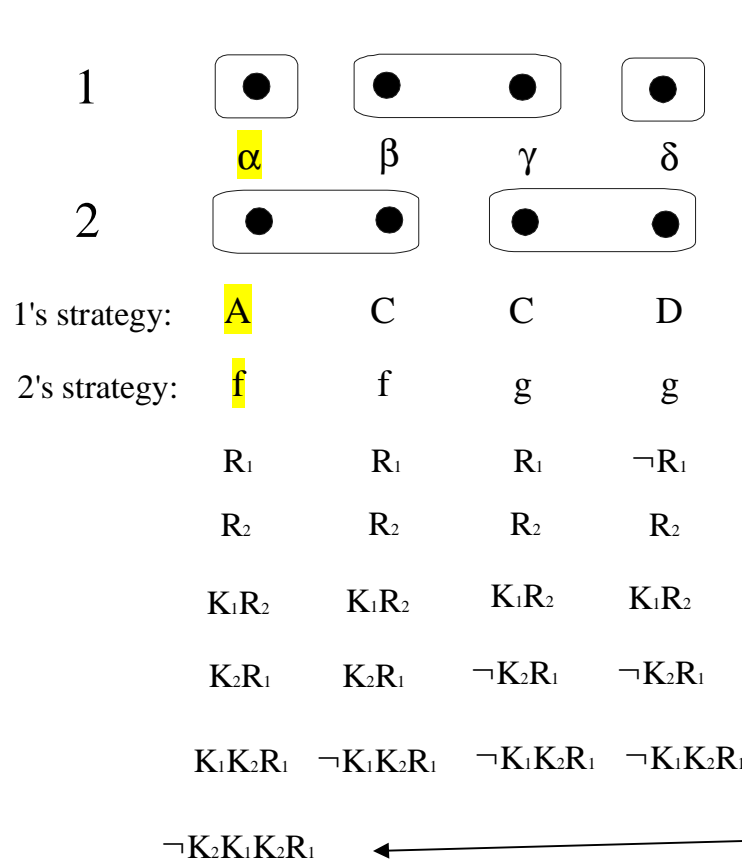
	e
A	3, 2

$$S^\infty = \{(A, e)\}$$



PROPOSITION 1. $B_*R \subseteq S^\infty$

If at a state it is commonly believed that all players are rational, then the strategy profile chosen at that state belongs to the game obtained after applying the iterated deletion of strictly dominated strategies.



		Player 2		
		e	f	g
Player 1	A	3, 2	3, 1	0, 1
	B	2, 3	2, 2	3, 1
	C	1, 2	1, 2	4, 1
	D	0, 2	0, 3	1, 3

At state α there cannot be common knowledge of rationality since $\sigma(\alpha) \neq (A, e)$

Every normal operator B satisfies the property that if $E \subseteq F$ then $BE \subseteq BF$.

B_* is a normal operator. Thus from $B_*R \subseteq S^\infty$

it follows that $B_*B_*R \subseteq B_*S^\infty$.

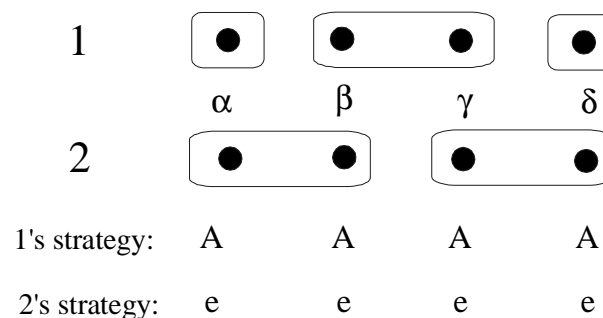
By transitivity of \mathcal{B}_* we have that

$B_*E \subseteq B_*B_*E$ for every event E .

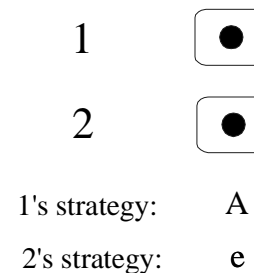
Thus $B_*R \subseteq B_*B_*R$.

It follows that $B_*R \subseteq B_*S^\infty$

		Player 2		
		e	f	g
Player 1	A	3, 2	3, 1	0, 1
	B	2, 3	2, 2	3, 1
	C	1, 2	1, 2	4, 1
	D	0, 2	0, 3	1, 3








Same as:



REMARK. In general it is not true that $S^\infty \subseteq B_*R$

		Player 2		
		e	f	g
Player 1	A	3 , 2	3 , 1	0 , 1
	B	2 , 3	2 , 2	3 , 1
	C	1 , 2	1 , 2	4 , 1
	D	0 , 2	0 , 3	1 , 3

1				
	α	β	γ	δ
2				
1's strategy:	A	C	C	A
2's strategy:	f	f	e	e

$$S^\infty = \{\delta\}$$

$$K_*R = \emptyset$$

$$R_1 = \{\alpha, \delta\}, \quad R_2 = \{\alpha, \beta, \gamma, \delta\}$$

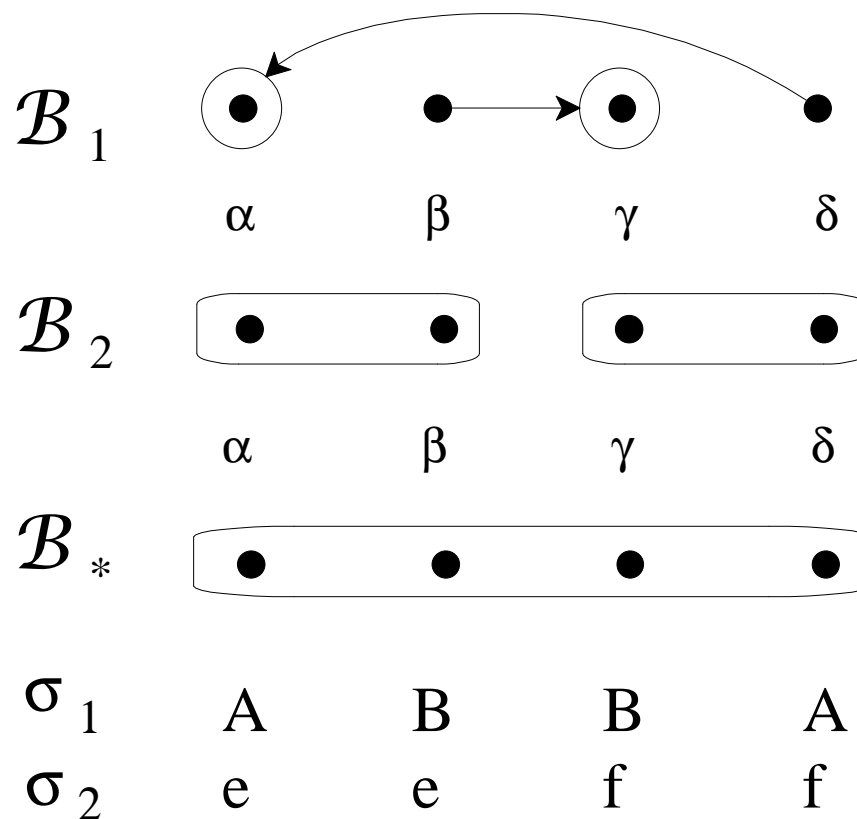
$$K_2R_1 = \emptyset$$

PROPOSITION 2. Fix a strategic-form game with ordinal payoffs G and let $s \in S^\infty$. Then there exists an epistemic model of G and a state ω such that $\sigma(\omega) = s$ and $\omega \in B_*R$.

EXAMPLE

		Player 2	
		e	f
Player 1	A	3 , 2	3 , 3
	B	2 , 3	4 , 2

In this game every strategy profile survives iterative deletion



In this model $R = B_*R = \Omega$ and every strategy profile occurs at some state

REMARK. Given the above notion of rationality, *there is no difference between common belief of rationality and common knowledge of rationality*. The previous two propositions can be restated in terms of knowledge and common knowledge.

PROPOSITION 1'. $K_*R \subseteq S^\infty$

PROPOSITION 2'. Fix a strategic-form game with ordinal payoffs G and let $s \in S^\infty$. Then there exists an epistemic model of G and a state ω such that $\sigma(\omega) = s$ and $\omega \in K_*R$.

STRONGER NOTION OF RATIONALITY

Still non-probabilistic (no expected utility)

Definition. Player i is **IRRATIONAL** at state ω if there is a strategy s_i which she believes to be at least as good as $\sigma_i(\omega)$ and she considers it possible that s_i is better than $\sigma_i(\omega)$

Player i is **RATIONAL** at state ω if and only if she is not irrational

		Player 2		
		e	f	g
Player 1	1	<div><div>●</div><div>α</div></div>	<div><div>●</div><div>β</div><div>●</div></div>	<div><div>●</div><div>γ</div></div>
	2	<div><div>●</div><div>●</div></div>		<div><div>●</div></div>
	1's strategy:	A	C	C
	2's strategy:	f	f	g

Player 1	A	3 , 2	3 , 1	0 , 1
	B	2 , 3	1 , 2	3 , 1
	C	4 , 2	1 , 2	2 , 1
	1			

Player 1 is **irrational** at state β : B is at least as good as C at both β and γ and it is better than C at γ

$$R_1 = \{\alpha\}, R_2 = \emptyset$$

Player i is **IRRATIONAL** at state ω if there is a strategy s_i which she believes to be at least as good as $\sigma_i(\omega)$ and she considers it possible that s_i is better than $\sigma_i(\omega)$

$$\|s_i\| \cap B_i \|t_i \succeq_i s_i\| \cap \neg B_i \neg \|t_i \succ_i s_i\| \subseteq \neg \mathbf{R}_i$$

$$\neg \mathbf{R}_i = \bigcup_{s_i \in S_i} \bigcup_{t_i \in S_i} (\|s_i\| \cap B_i \|t_i \succeq_i s_i\| \cap \neg B_i \neg \|t_i \succ_i s_i\|)$$

$$\mathbf{R} = \mathbf{R}_1 \cap \dots \cap \mathbf{R}_n \quad \text{all players are rational}$$

Definition.

Given a game $G = \langle N, \{S_i\}_{i \in N}, O, \{\succeq\}_{i \in N}, z \rangle$, a subset of strategy profiles $X \subseteq S$ and a strategy profile $x \in X$, we say that x is **inferior relative to X** if there exist a player i and a strategy $s_i \in S_i$ of player i (thus s_i need not belong to the projection of X onto S_i) such that:

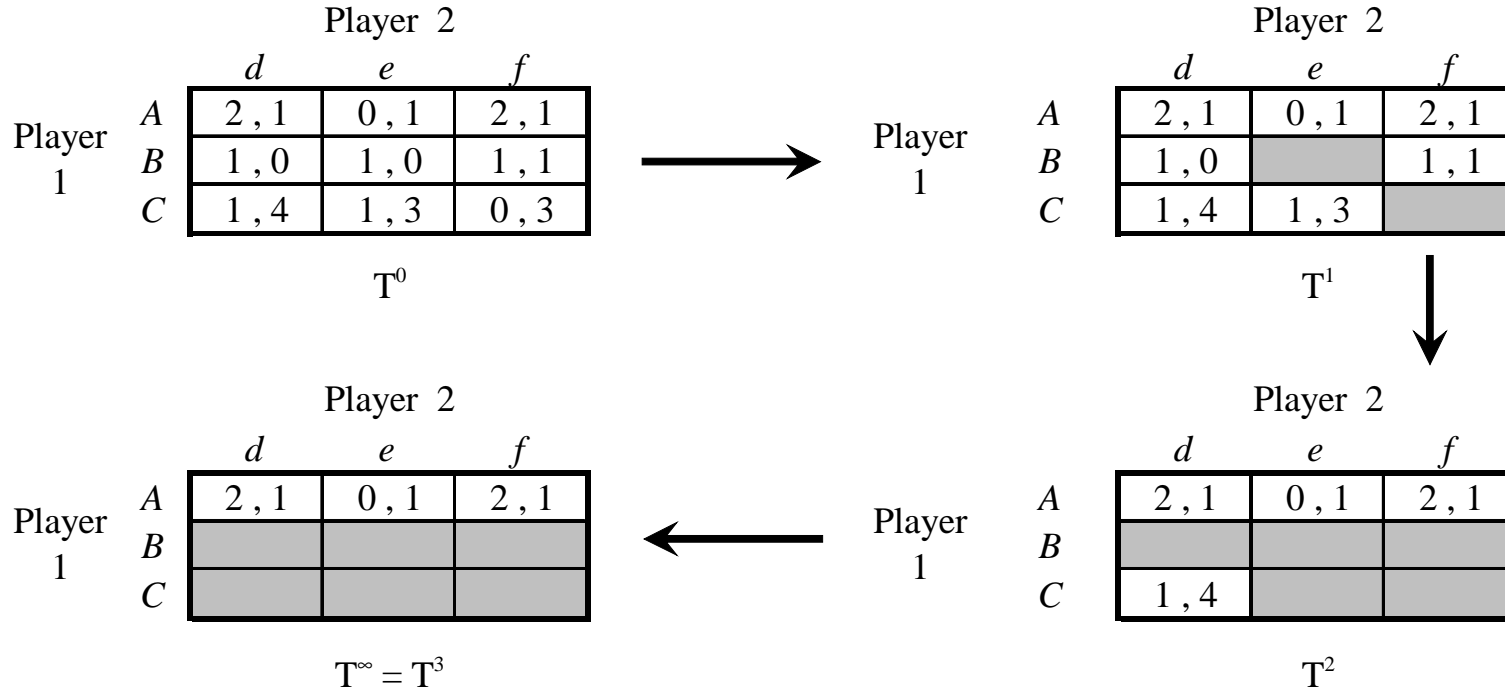
1. $z(s_i, x_{-i}) \succ_i z(x_i, x_{-i})$ and
2. for all $s_{-i} \in S_{-i}$, if $(x_i, s_{-i}) \in X$ then $z(s_i, s_{-i}) \succeq_i z(x_i, s_{-i})$.

Iterated Deletion of Inferior Profiles : for $m \in \mathbb{N}$ define

$T^m \subseteq S$ recursively as follows: $T^0 = S$ and, for $m \geq 1$,

$T^m = T^{m-1} \setminus I^{m-1}$, where $I^{m-1} \subseteq T^{m-1}$ is the set of strategy profiles

that are inferior relative to T^{m-1} . Let $T^\infty = \bigcap_{m \in \mathbb{N}} T^m$.



$T^0 = S = \{(A,d), (A,e), (A,f), (B,d), (B,e), (B,f), (C,d), (C,e), (C,f)\}$, $I^0 = \{(B,e), (C,f)\}$ (the elimination of (B,e) is done through player 2 and strategy f , while the elimination of (C,f) is done through player 1 and strategy B);

$T^1 = \{(A,d), (A,e), (A,f), (B,d), (B,f), (C,d), (C,e)\}$, $I^1 = \{(B,d), (B,f), (C,e)\}$ (the elimination of (B,d) and (B,f) is done through player 1 and strategy A , while the elimination of (C,e) is done through player 2 and strategy d);

$T^2 = \{(A,d), (A,e), (A,f), (C,d)\}$, $I^2 = \{(C,d)\}$ (the elimination of (C,d) is done through player 1 and strategy A);

$T^3 = \{(A,d), (A,e), (A,f)\}$, $I^3 = \emptyset$; thus $T^\infty = T^3$.

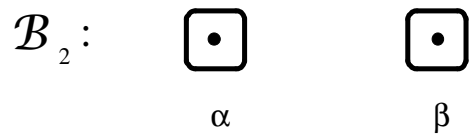
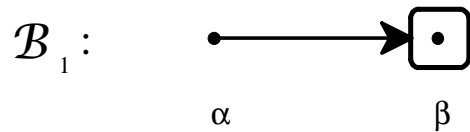
PROPOSITION 3. $K_*R \subseteq T^\infty$

If at a state it is commonly **known** that all players are rational, then the strategy profile chosen at that state belongs to the game obtained after applying the iterated deletion of Inferior strategy profiles.

PROPOSITION 4. Fix a strategic-form game with ordinal payoffs G and let $s \in T^\infty$. Then there exists an epistemic model of G and a state ω such that $\sigma(\omega) = s$ and $\omega \in K_*R$.

NOT TRUE if we replace common knowledge with common belief

		Player 2	
		c	d
Player 1	A	1, 1	1, 0
	B	1, 1	0, 1



$\sigma_1 :$ B B

$\sigma_2 :$ d c

$$R_1 = \{\alpha, \beta\}, \quad R_2 = \{\alpha, \beta\}$$

There is common belief of rationality at every state and yet at state α the strategy profile played is (B, d) which is inferior

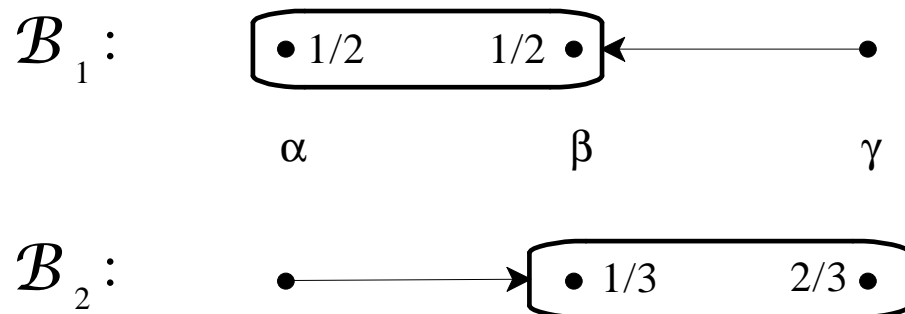
$$T^\infty = \{(A, c), (B, c)\}$$

$$S^\infty = \{(A, c), (A, d), (B, c), (B, d)\}$$

PROBABILISTIC BELIEFS

Definition. A *Bayesian frame* is an interactive belief frame together with a collection $\{p_{i,\omega}\}_{i \in N, \omega \in \Omega}$ of probability distributions on Ω such that

- (1) if $\omega' \in \mathcal{B}_i(\omega)$ then $p_{i,\omega'} = p_{i,\omega}$
- (2) $p_{i,\omega}(\omega') > 0$ if and only if $\omega' \in \mathcal{B}_i(\omega)$
(the support of $p_{i,\omega}$ coincides with $\mathcal{B}_i(\omega)$)



Definition. A strategic-form game *with von Neumann-Morgenstern payoffs* is a quintuple

$$\langle N, \{S_i\}_{i \in N}, O, \{U_i\}_{i \in N}, z \rangle$$

where

$N = \{1, \dots, n\}$ is a set of *players*

S_i is the set of *strategies* of player $i \in N$

O is a set of *outcomes*

$U_i: O \rightarrow \mathbb{R}$ is player i 's von Neumann-Morgenstern utility function

$z: S \rightarrow O$ (where $S = S_1 \times \dots \times S_n$) associates an outcome with every strategy profile $s \in S$

Its reduced form is a triple $\langle N, \{S_i\}_{i \in N}, \{\pi_i\}_{i \in N} \rangle$ where $\pi_i(s) = U_i(z(s))$.

An *epistemic model* of a strategic-form game is a Bayesian frame together with n functions

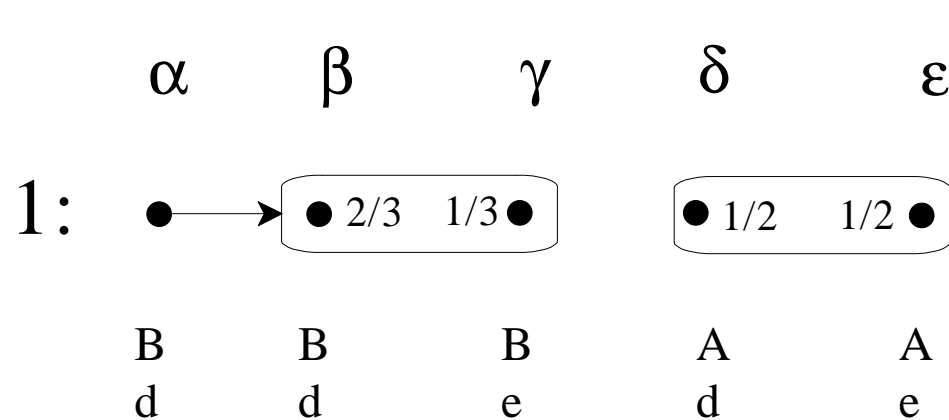
$$\sigma_i : \Omega \rightarrow S_i \quad (i \in N)$$

such that if $\omega' \in \mathcal{B}_i(\omega)$ then $\sigma_i(\omega') = \sigma_i(\omega)$

Stronger definition of Rationality than the previous ones

Player i is **RATIONAL** at state α if her choice at α maximizes her expected payoff, given her beliefs at α : for all $t_i \in S_i$

$$\sum_{\omega \in \mathcal{B}_i(\alpha)} \pi_i(\sigma_i(\alpha), \sigma_{-i}(\omega)) p_{i,\alpha}(\omega) \geq \sum_{\omega \in \mathcal{B}_i(\alpha)} \pi_i(t_i, \sigma_{-i}(\omega)) p_{i,\alpha}(\omega)$$



		Player 2	
		d	e
Player 1	A	3 , 0	0 , 1
	B	1 , 0	2 , 2
	C	0 , 3	3 , 2

$$R_1 = \{\delta, \varepsilon\}$$

Player 1 is not rational at α because her expected payoff is $\frac{2}{3} 1 + \frac{1}{3} 2 = \frac{4}{3}$

while if she had chosen strategy A her payoff would have been $\frac{2}{3} 3 + \frac{1}{3} 0 = 2$

On the other hand, Player 1 *is* rational at δ because her expected payoff is $\frac{1}{2} 3 + \frac{1}{2} 0 = \frac{3}{2}$

and if she had chosen strategy B her payoff would have been $\frac{1}{2} 1 + \frac{1}{2} 2 = \frac{3}{2}$

and if she had chosen strategy C her payoff would have been $\frac{1}{2} 0 + \frac{1}{2} 3 = \frac{3}{2}$

What are the implications of Common Belief of this stronger notion of rationality?

Definition. A mixed strategy of player i is a probability distribution over S_i . The set of mixed strategies of player i is denoted by $\Delta(S_i)$.

Let $t_i \in S_i$ and $\nu_i \in \Delta(S_i)$. We say that t_i is *strictly dominated* by ν_i if, for every $s_{-i} \in S_{-i}$, $\pi_i(t_i, s_{-i}) < \sum_{s_i \in S_i} \nu_i(s_i) \pi_i(s_i, s_{-i})$

		Player 2	
		d	e
Player 1	A	3 , 0	0 , 1
	B	0 , 0	2 , 2
	C	0 , 3	3 , 2

In this game strategy B of player 1 is

strictly dominated by the mixed strategy $\begin{pmatrix} A & C \\ \frac{1}{6} & \frac{5}{6} \end{pmatrix}$

ITERATIVE
DELETION
OF PURE
STRATEGIES
THAT ARE
STRICTLY
DOMINATED
BY (POSSIBLY
MIXED)
STRATEGIES

		Player 2		
		e	f	g
Player 1	A	3, 0	1, 0	0, 1
	B	1, 1	0, 2	1, 1
	C	0, 0	4, 1	2, 2
	D	0, 3	1, 0	3, 2

(a) The game G

B is strictly dominated by $(1/2 A, 1/2 D)$

		Player 2		
		e	f	g
Player 1	A	3, 0	1, 0	0, 1
	C	0, 0	4, 1	2, 2
	D	0, 3	1, 0	3, 2

(b) The game G^1

Now f is strictly dominated by g

		Player 2	
		e	g
Player 1	A	3, 0	0, 1
	C	0, 0	2, 2
	D	0, 3	3, 2

(c) The game G^2

Now C is strictly dominated by $(1/6 A, 5/6 D)$

		Player 2	
		e	g
Player 1	A	3, 0	0, 1
	D	0, 3	3, 2

(d) The game $G^3 = G^\infty$

No strategy is strictly dominated

Let G be a strategic-form game with von Neumann-Morgenstern payoffs and G^∞ be the game obtained after applying the procedure of Iterated Deletion of Pure Strategies that are Strictly Dominated by Possibly Mixed Strategies.

Let S_m^∞ denote the pure-strategy profiles of game G^∞

Given a model of G , let S_m^∞ be the event $\{\omega \in \Omega : \sigma(\omega) \in S_m^\infty\}$

PROPOSITION 5. $B_*R \subseteq S_m^\infty$

PROPOSITION 6. Fix a strategic-form game with von Neumann-Morgenstern payoffs G and let $s \in S_m^\infty$. Then there exists a Bayesian model of G and a state ω such that $\sigma(\omega) = s$ and $\omega \in B_*R$.

Given this stronger notion of rationality, *there is a difference between common belief of rationality and common knowledge of rationality*. The implications of common knowledge of rationality are stronger.

With knowledge, a player's beliefs are always correct and are believed to be correct by every other player. Thus there is *correctness and common belief of correctness* of everybody's beliefs.

Definition. Given a strategic-form game with von Neumann-Morgenstern payoffs G , a pure-strategy profile $x \in X \subseteq S$ is *inferior relative to X* if there exists a player i and a (possibly mixed) strategy ν_i of player i (whose support can be any subset of S_i , not necessarily the projection of X onto S_i) such that:

- (1) $\pi_i(x_i, x_{-i}) < \sum_{s_i \in S_i} \pi_i(s_i, x_{-i}) \nu_i(s_i)$ (ν_i yields a higher expected payoff than x_i against x_{-i})
- (2) for all $s_{-i} \in S_{-i}$ such that $(x_i, s_{-i}) \in X$, $\pi_i(x_i, s_{-i}) \leq \sum_{s_i \in S_i} \pi_i(s_i, s_{-i}) \nu_i(s_i)$

		Player 2		
		D	E	F
Player 1	A	2 , 0	2 , 2	0 , 2
	B	2 , 2	1 , 2	5 , 1
	C	2 , 0	1 , 0	1 , 5

Here (C, F) is inferior relative to S (for player 1, B weakly dominates C and is strictly better than C against F)

and (A, D) is inferior relative to S (for player 2, E weakly dominates D and is strictly better than D against A)

ITERATED
DELETION
OF
INFERIOR
PURE
STRATEGY
PROFILES

		Player 2		
		D	E	F
Player 1	A	2 , 0	2 , 2	0 , 2
	B	2 , 2	1 , 2	5 , 1
	C	2 , 0	1 , 0	1 , 5

(a)

$$S_s^0 = S, D_s^0 = \{(A, D), (C, F)\}$$

		Player 2		
		D	E	F
Player 1	A		2 , 2	0 , 2
	B	2 , 2	1 , 2	
	C	2 , 0		

(c)

$$S_s^2 = \{(A, E), (A, F), (B, D), (B, E), (C, D)\}, \\ D_s^2 = \{(B, E)\}.$$

		Player 2		
		D	E	F
Player 1	A		2 , 2	0 , 2
	B	2 , 2	1 , 2	5 , 1
	C	2 , 0	1 , 0	

(b)

$$S_s^1 = \{(A, E), (A, F), (B, D), (B, E), \\ (B, F), (C, D), (C, E)\} \\ D_s^1 = \{(C, E), (B, F)\}$$

		Player 2		
		D	E	F
Player 1	A		2 , 2	0 , 2
	B	2 , 2		
	C	2 , 0		

(d)

$$S_s^3 = S_s^\infty = \{(A, E), (A, F), (B, D), \\ (C, D)\}, D_s^3 = \emptyset.$$

Let G be a strategic-form game with von Neumann-Morgenstern payoffs and G^∞ be the game obtained after applying the procedure of Iterated Deletion of Inferior Pure-Strategy Profiles.

Let S_s^∞ denote the pure-strategy profiles of game G^∞

Given a model of G , let S_s^∞ be the event $\{\omega \in \Omega : \sigma(\omega) \in S_s^\infty\}$

PROPOSITION 7. $K_*R \subseteq S_s^\infty$

PROPOSITION 8. Fix a strategic-form game with von Neumann-Morgenstern payoffs G and let $s \in S_s^\infty$. Then there exists a Bayesian model of G and a state ω such that $\sigma(\omega) = s$ and $\omega \in K_*R$.

		Player 2		
		D	E	F
Player 1	A	2 , 0	2 , 2	0 , 2
	B	2 , 2	1 , 2	5 , 1
	C	2 , 0	1 , 0	1 , 5

In this game $S^\infty = S_m^\infty = S$

while $S_s^\infty = \{(A, E), (A, F), (B, D), (C, D)\}$

Thus every strategy profile is compatible with *common belief* of rationality while only (A, E) , (A, F) , (B, D) and (C, D) are compatible with *common knowledge* of rationality

CREDITS

The link between the iterated deletion of strictly dominated strategies and the informal notion of common belief of rationality was first shown by Bernheim (1984) and Pearce (1984)

The first explicit epistemic characterization was provided by Tan and Werlang (1998) using a universal type space.

The state space formulation used in Propositions 5 and 6 is due to Stalnaker (1994), but it was implicit in Brandenburger and Dekel (1987).

Propositions 7 and 8 are due to Stalnaker (1994) (with a correction given in Bonanno and Nehring, 1996b).

To my knowledge, Propositions 1, 2, 3 and 4 have not been explicitly stated before.

References and further details can be found in

Battigalli, Pierpaolo and Bonanno Giacomo, “Recent results on belief, knowledge and the epistemic foundations of game theory”, *Research in Economics*, 53 (2), June 1999, pp. 149-225.

For a syntactic version of Propositions 1, 2, 3 and 4 see
Giacomo Bonanno, A syntactic approach to rationality in games, Working Paper,
University of California, Davis (<http://www.econ.ucdavis.edu/faculty/bonanno/PDF/CBR.pdf>)

Royal Netherlands Academy of Arts and Sciences (KNAW)
Master Class

Amsterdam, February 8th, 2007

Epistemic Foundations of Game Theory

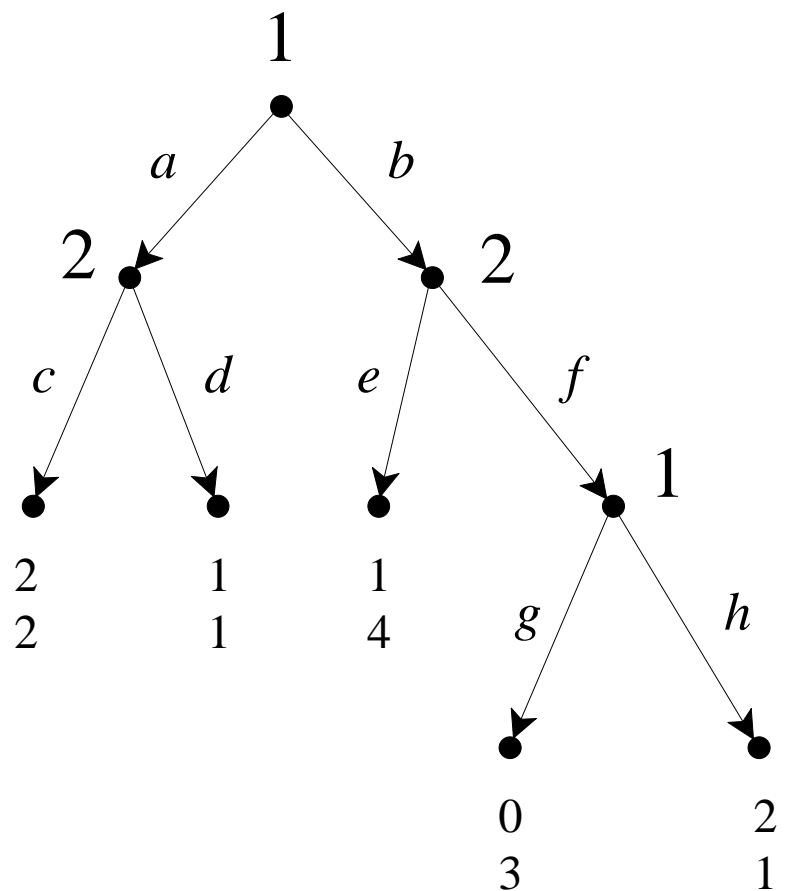
Lecture 2

Giacomo Bonanno

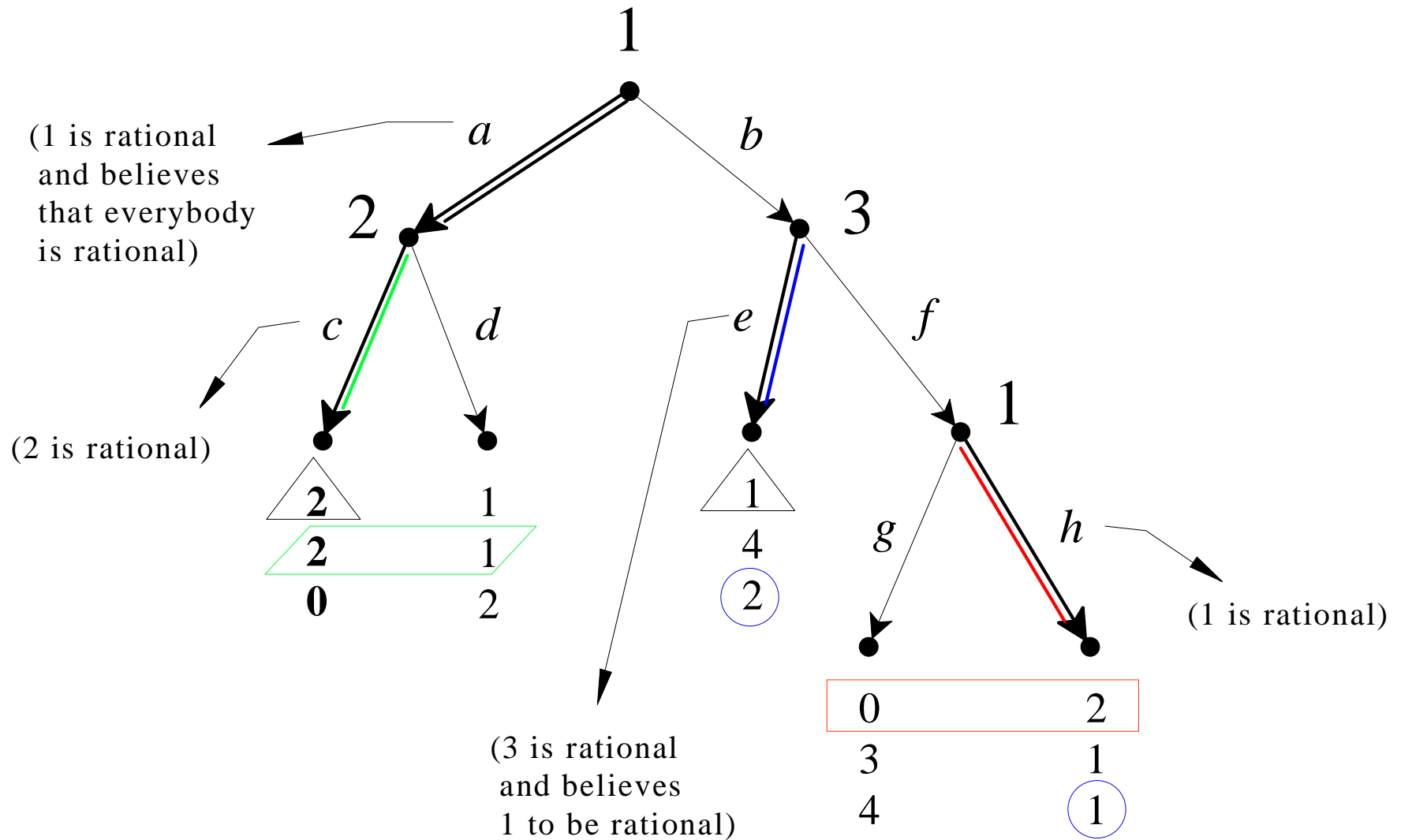
(<http://www.econ.ucdavis.edu/faculty/bonanno/>)

EXTENSIVE GAMES WITH PERFECT INFORMATION

- tree
- n players
- assignment of one player to every non-terminal node
- assignment of an *ordinal* payoff to every player at every terminal node



BACKWARD-INDUCTION SOLUTION



STRATEGIES IN PERFECT-INFORMATION GAMES

Non-terminal nodes are called *decision nodes*

X : set of decision nodes

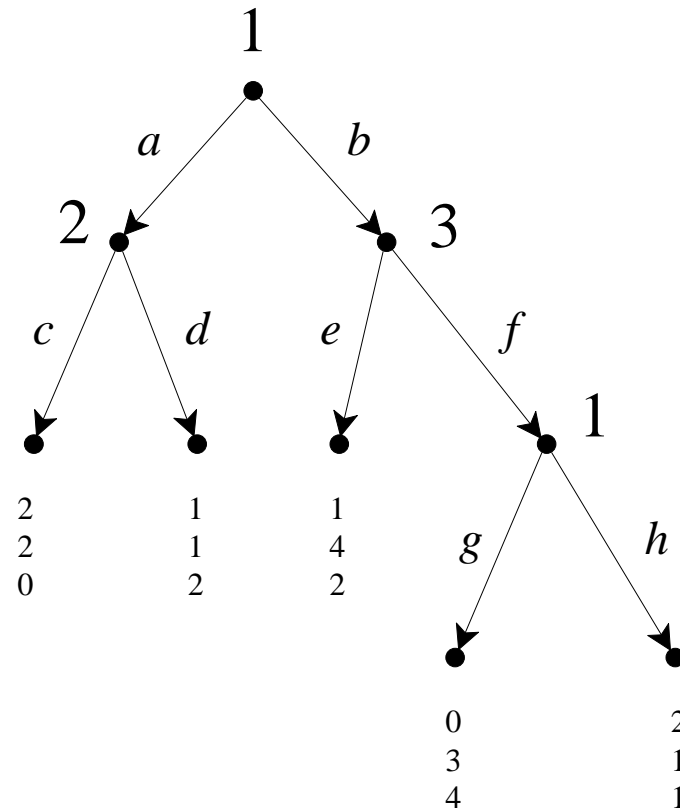
X_i : set of decision nodes assigned to player i

Definition. A strategy of player i is a function that assigns to every

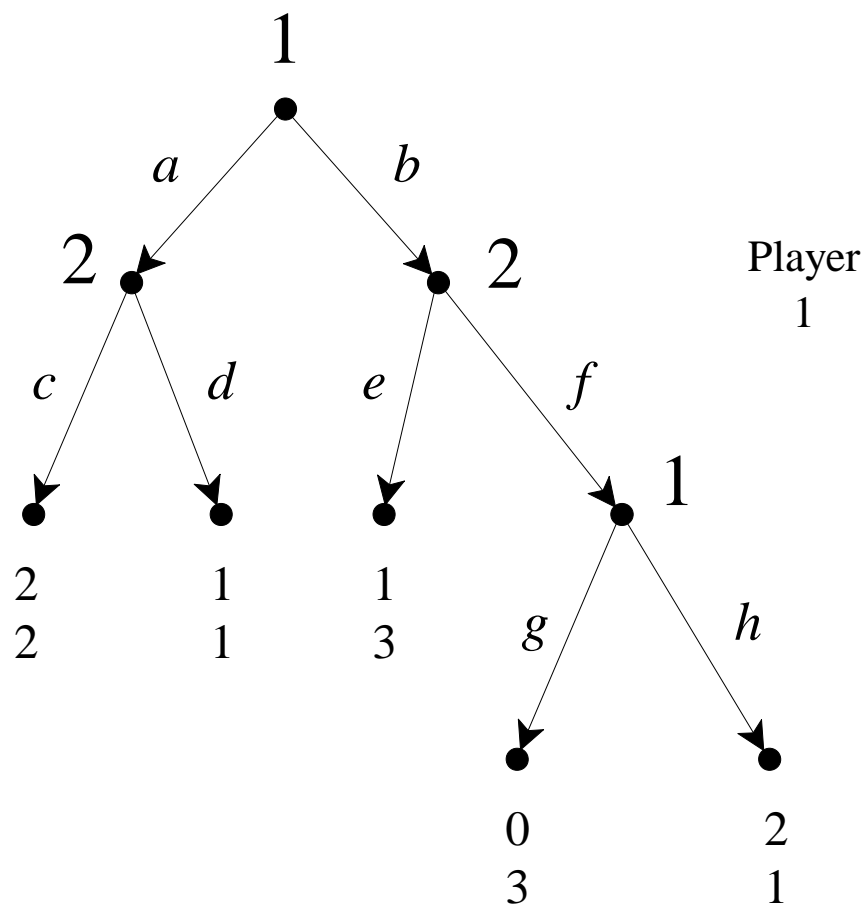
$x \in X_i$ a choice at x

Player 1's strategies:

(a,g) , (a,h) , (b,g) and (b,h)



THE STRATEGIC FORM OF A PERFECT-INFORMATION GAME



	Player 2			
	ce	cf	de	df
ag	2 , 2	2 , 2	1 , 1	1 , 1
ah	2 , 2	2 , 2	1 , 1	1 , 1
bg	1 , 3	0 , 3	1 , 3	0 , 3
bh	1 , 3	2 , 1	1 , 3	2 , 1

EPISTEMIC MODEL OF A PERFECT-INFORMATION GAME

(Knowledge based)

- Set of states Ω
- Equivalence relation \mathcal{K}_i on Ω for every player i
- For every player i a function $\sigma_i : \Omega \rightarrow S_i$ satisfying
if $\omega' \in \mathcal{K}_i(\omega)$ then $\sigma_i(\omega') = \sigma_i(\omega)$

Thus a standard epistemic model for the associated strategic form

Recall from Lecture 1:

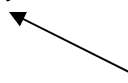
Let s_i and t_i be two strategies of player i : $s_i, t_i \in S_i$

$s_i \succ_i t_i$ is interpreted as “strategy s_i is better for player i than strategy t_i ”

$s_i \succ_i t_i$ is true at state ω if $u_i(s_i, \sigma_{-i}(\omega)) > u_i(t_i, \sigma_{-i}(\omega))$

that is, s_i is better than t_i against $\sigma_{-i}(\omega)$

profile of strategies chosen
by the players other than i



Let $\|s_i \succ_i t_i\| = \{\omega \in \Omega : u_i(s_i, \sigma_{-i}(\omega)) > u_i(t_i, \sigma_{-i}(\omega))\}$ event that s_i is better than t_i

If $s_i \in S_i$, let $\|s_i\| = \{\omega \in \Omega : \sigma_i(\omega) = s_i\}$ event that player i chooses s_i

Let \mathbf{R}_i^{EA} be the event representing the proposition “player i is *ex ante* rational”

$$\|s_i\| \cap K_i \|t_i \succ_i s_i\| \subseteq \neg \mathbf{R}_i^{EA}$$

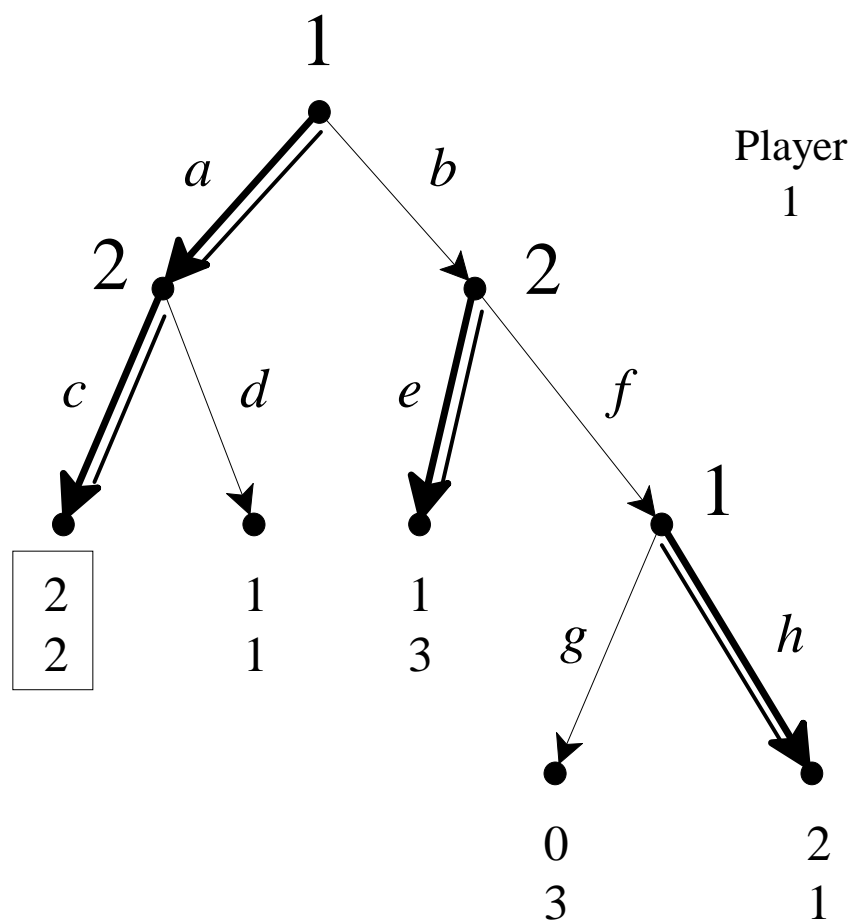
$$\neg \mathbf{R}_i^{EA} = \bigcup_{s_i \in S_i} \bigcup_{t_i \in S_i} (\|s_i\| \cap K_i \|t_i \succ_i s_i\|)$$

$$\mathbf{R}^{EA} = \mathbf{R}_1^{EA} \cap \dots \cap \mathbf{R}_n^{EA} \quad \text{all players are rational}$$

Recall from Lecture 1:

PROPOSITION: if at a state there is common knowledge of *ex ante* rationality then the strategy profile chosen at that state belongs to the game obtained by applying the iterated deletion of strictly dominated strategies; conversely, for every such strategy profile there is a model and a state where (1) the strategy profile is chosen and (2) there is common knowledge of *ex ante* rationality.

This notion of rationality is not sufficient
to yield backward induction



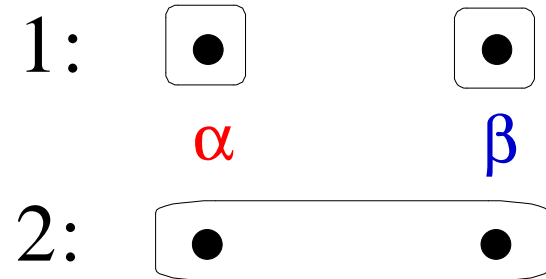
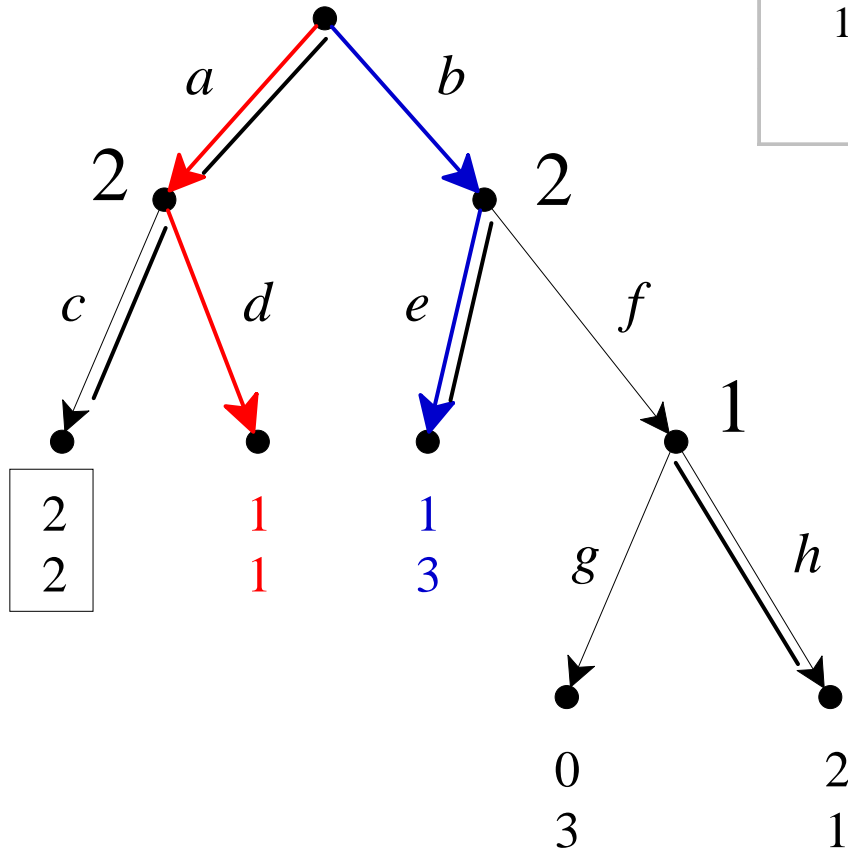
	Player 2			
	<i>ce</i>	<i>cf</i>	<i>de</i>	<i>df</i>
<i>ag</i>	2 , 2	2 , 2	1 , 1	1 , 1
<i>ah</i>	2 , 2	2 , 2	1 , 1	1 , 1
<i>bg</i>	1 , 3	0 , 3	1 , 3	0 , 3
<i>bh</i>	1 , 3	2 , 1	1 , 3	2 , 1

Here there are no strictly
dominated strategies

Thus every strategy profile is
consistent with common
belief/knowledge of *ex ante*
rationality

For example:

		Player 2			
		<i>ce</i>	<i>cf</i>	<i>de</i>	<i>df</i>
Player 1	<i>ag</i>	2 , 2	2 , 2	1 , 1	1 , 1
	<i>ah</i>	2 , 2	2 , 2	1 , 1	1 , 1
	<i>bg</i>	1 , 3	0 , 3	1 , 3	0 , 3
	<i>bh</i>	1 , 3	2 , 1	1 , 3	2 , 1



1's strategy: *ah* *bh*

2's strategy: *de* *de*

(For 2 *ce* better than *de* at α but not at β , thus at α she does not know that *ce* is better.)

Here: *ex ante* rationality and common knowledge of *ex ante* rationality at both states.

Let $\mathbf{R}_i^{EA/S}$ be the event representing the proposition “player i is *ex ante* rational in a strong sense”

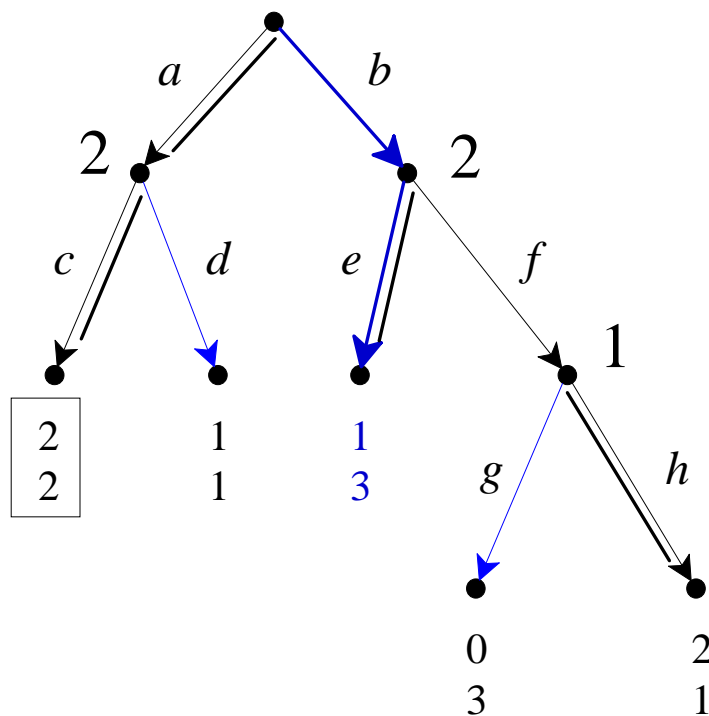
$$\|s_i\| \cap K_i \|t_i \succeq_i s_i\| \cap \neg K_i \neg \|t_i \succ_i s_i\| \subseteq \neg \mathbf{R}_i^{EA/S}$$

$$\neg \mathbf{R}_i^{EA/S} = \bigcup_{s_i \in S_i} \bigcup_{t_i \in S_i} \left(\|s_i\| \cap K_i \|t_i \succeq_i s_i\| \cap \neg K_i \neg \|t_i \succ_i s_i\| \right)$$

$$\mathbf{R}^{EA/S} = \mathbf{R}_1^{EA/S} \cap \dots \cap \mathbf{R}_n^{EA/S} \quad \text{all players are rational in a strong sense}$$

Recall from Lecture 1:

PROPOSITION: if at a state there is common knowledge of *ex ante* rationality in a strong sense then the strategy profile chosen at that state belongs to the set T^∞ of strategy profiles that survive the iterated deletion of inferior profiles; conversely, for every such strategy profile there is a model and a state where (1) the strategy profile is chosen and (2) there is common knowledge of *ex ante* rationality in a strong sense.



1: ●
 α
 2: ●

1's strategy: bg
 2's strategy: de

	ce	cf	de	df
ag	2, 2	2, 2	1, 1	1, 1
ah	2, 2	2, 2	1, 1	1, 1
bg	1, 3	0, 3	1, 3	0, 3
bh	1, 3	2, 1	1, 3	2, 1

player 1 using ah
 player 2 using ce

	ce	cf	de
ag	2, 2	2, 2	1, 1
ah	2, 2	2, 2	1, 1
bg			1, 3
bh	1, 3		1, 3

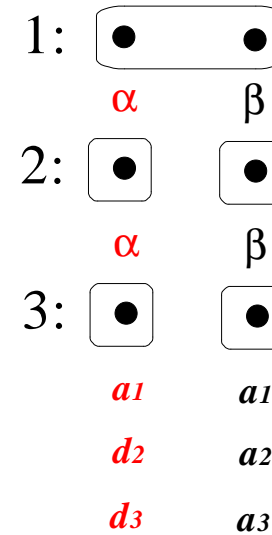
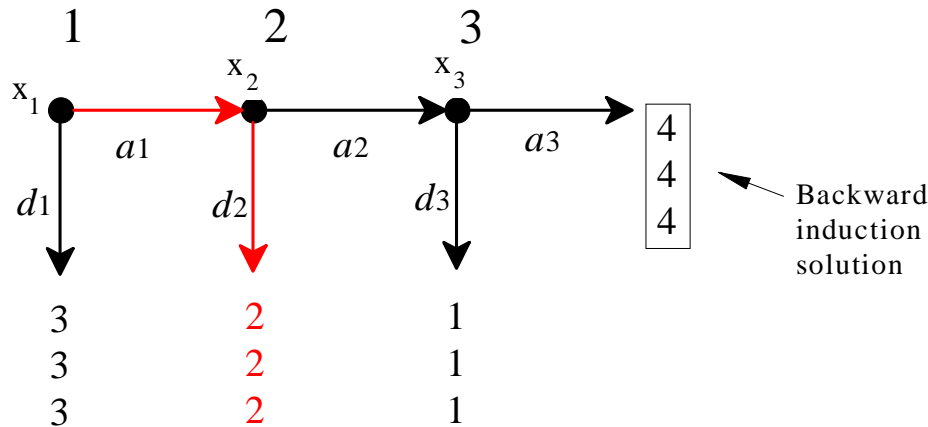
player 2 using cf
 player 1 using ah

	ce	cf	de
ag	2, 2	2, 2	
ah	2, 2	2, 2	
bg			1, 3
bh			1, 3

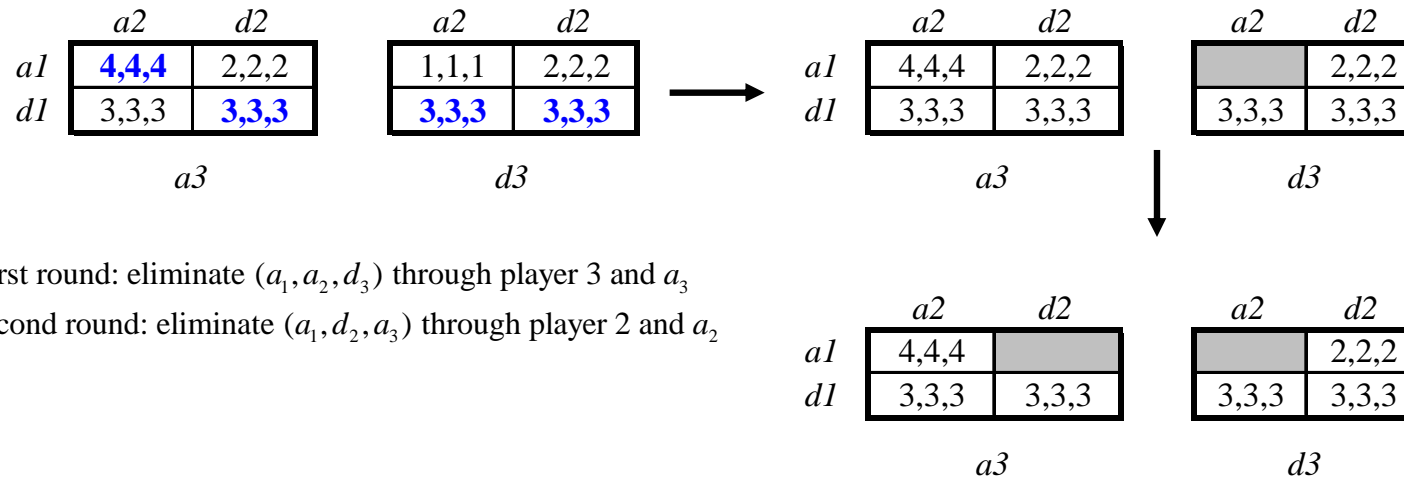
Thus even common knowledge of *ex ante* rationality in a strong sense is not sufficient to yield backward induction

In this example all the strategy profiles in T^∞ are Nash equilibria. Is it the case that common knowledge of *ex ante* rationality in the strong sense gives Nash equilibrium **play** in perfect information games?

The answer is NO!



There is no Nash equilibrium that yields the play $a_1 d_2$ (the Nash equilibria are marked in blue)

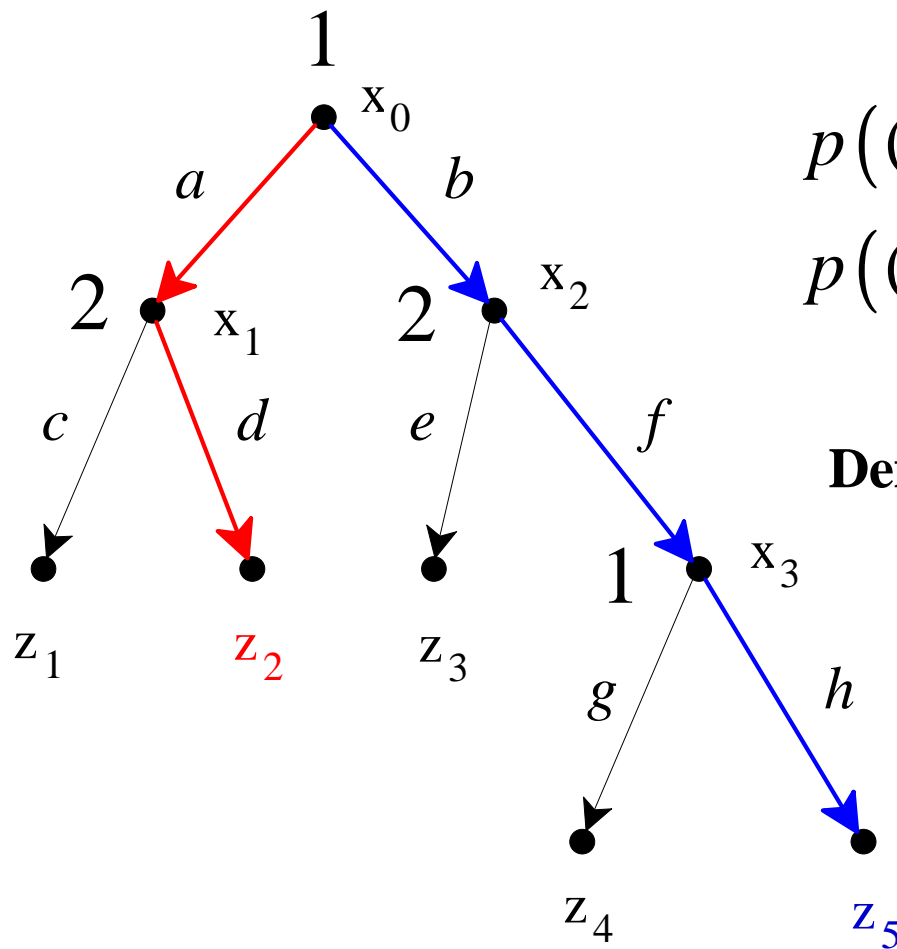


First round: eliminate (a_1, a_2, d_3) through player 3 and a_3

second round: eliminate (a_1, d_2, a_3) through player 2 and a_2

Going beyond *ex ante* rationality

Given a strategy profile s , let $p(s)$ be the associated play



$$p((ag, df)) = x_0 x_1 z_2$$

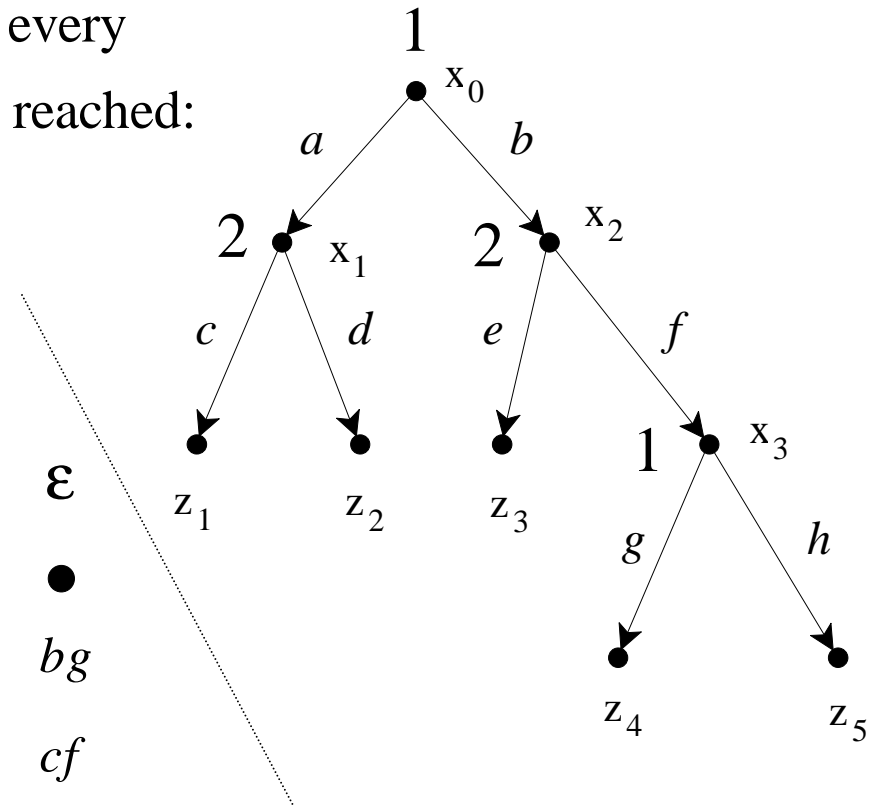
$$p((bh, df)) = x_0 x_2 x_3 z_5$$

Definition. At state ω node x is *reached* if and only if $x \in p(\sigma(\omega))$.

Definition. Given an epistemic model, for every node x , let $\|x\|$ be the event that node x is reached:

$$\|x\| = \{\omega \in \Omega : x \in p(\sigma(\omega))\}$$

	α	β	γ	δ	ε
	●	●	●	●	●
1's strategy:	ag	bh	bg	bh	bg
2's strategy:	df	df	ce	de	cf



$$\|x_1\| = \{\alpha\}, \quad \|x_2\| = \{\beta, \gamma, \delta, \varepsilon\}$$

$$\|x_3\| = \{\beta, \varepsilon\}, \quad \|z_1\| = \emptyset, \quad \|z_2\| = \{\alpha\}, \quad \text{etc.}$$

Let $E, F \subseteq \Omega$ be two events.

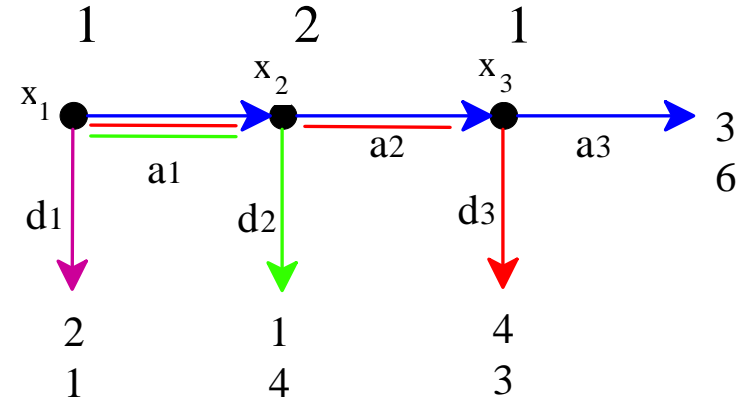
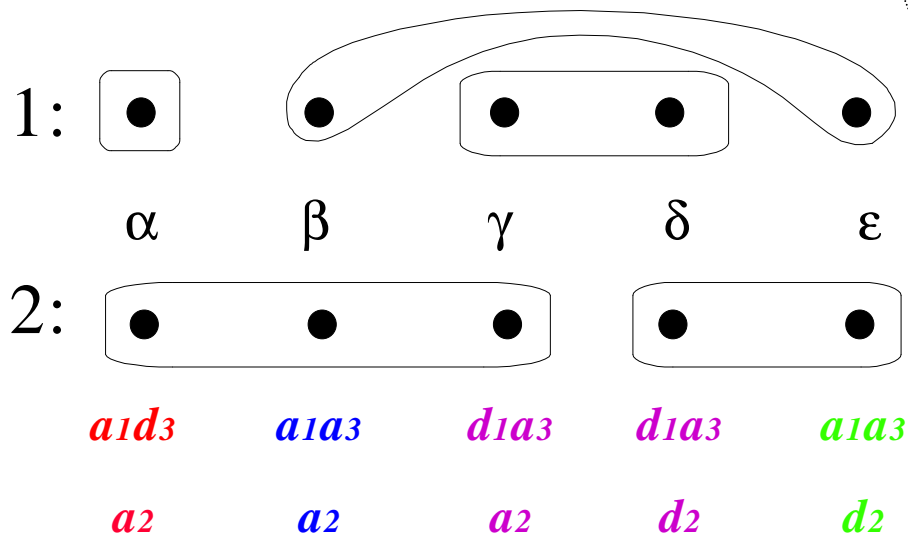
Denote by $E \rightarrow F$ the event $\neg E \cup F$ (if E then F)

Let \mathbf{R}_i^{RN} be the event representing the proposition “player i is rational *at reached nodes*”

$$\text{if } x \in X_i \quad \|x\| \cap \|s_i\| \cap K_i(\|x\| \rightarrow \|t_i \succ_i s_i\|) \subseteq \neg \mathbf{R}_i^{RN}$$

$$\neg \mathbf{R}_i^{RN} = \bigcup_{x \in X_i} \bigcup_{s_i \in S_i} \bigcup_{t_i \in S_i} (\|s_i\| \cap K_i(\|x\| \rightarrow \|t_i \succ_i s_i\|) \cap \|x\|)$$

$$\mathbf{R}^{RN} = \mathbf{R}_1^{RN} \cap \dots \cap \mathbf{R}_n^{RN} \quad \text{all players are rational at reached nodes}$$



$$\|d_2 \succ_2 a_2\| = \{\alpha\} \quad \|x_2\| = \{\alpha, \beta, \epsilon\} \quad \neg \|x_2\| \cup \|d_2 \succ_2 a_2\| = \{\alpha, \gamma, \delta\}$$

$K_2(\|x_2\| \rightarrow \|d_2 \succ_2 a_2\|) = \emptyset$ Thus player 2 is rational at nodes α and β and trivially at γ .

$$\|a_2 \succ_2 d_2\| = \{\beta, \epsilon\} \quad \|x_2\| = \{\alpha, \beta, \epsilon\} \quad \neg \|x_2\| \cup \|a_2 \succ_2 d_2\| = \{\beta, \gamma, \delta, \epsilon\}$$

$$K_2(\|x_2\| \rightarrow \|a_2 \succ_2 d_2\|) = \{\delta, \epsilon\} \quad \|x_2\| \cap \|d_2\| \cap K_2(\|x_2\| \rightarrow \|a_2 \succ_2 d_2\|) = \{\epsilon\}$$

Thus **player 2 is** trivially rational at state δ , and **irrational at ϵ** .

$$K_* \mathbf{R} = \emptyset$$

Backward Induction terminating games

Definition. A *BI terminating game* is a perfect information game where

- (1) at each decision node there is a choice that terminates the game (it leads to a terminal node) and (2) the backward-induction solution prescribes a terminating choice at every decision node.

The best-known example is the **centipede game** (n is the number of decision nodes)

$$\begin{aligned}
 u_1(z_1) &= 2 & \text{for } 1 < k \leq n \\
 u_2(z_1) &= 1 & u_1(z_k) = u_2(z_{k-1}) \\
 & & u_2(z_k) = u_1(z_{k-1}) + 2
 \end{aligned}$$

If n is even

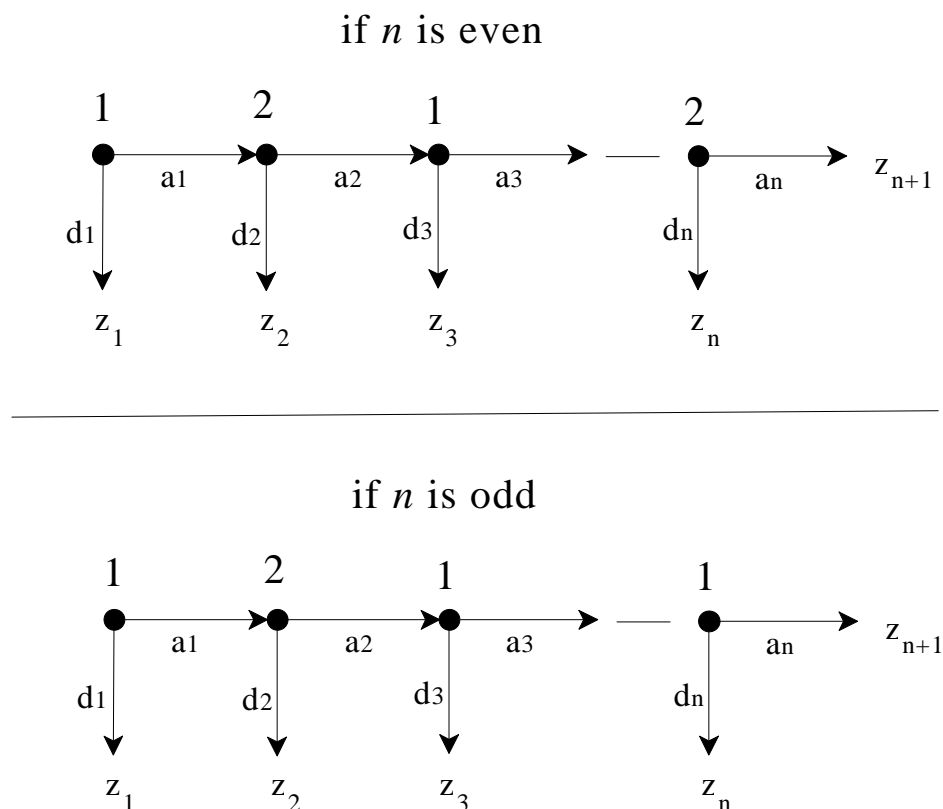
$$u_1(z_{n+1}) = u_1(z_n) + 1$$

$$u_2(z_{n+1}) = u_2(z_n) - 1$$

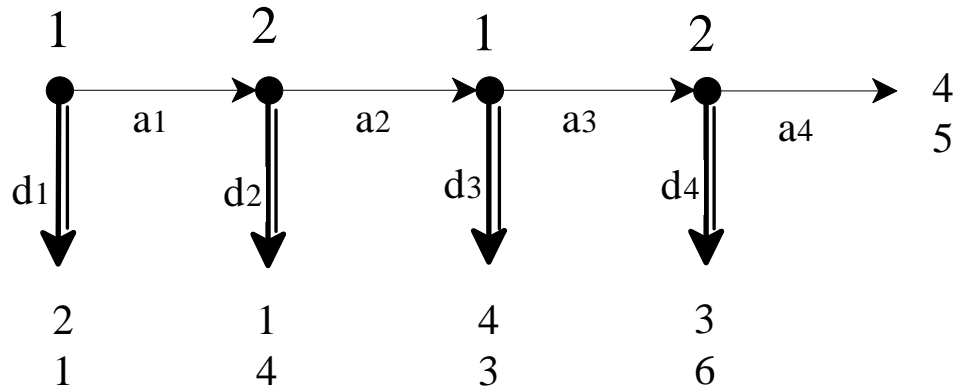
If n is odd

$$u_1(z_{n+1}) = u_1(z_n) - 1$$

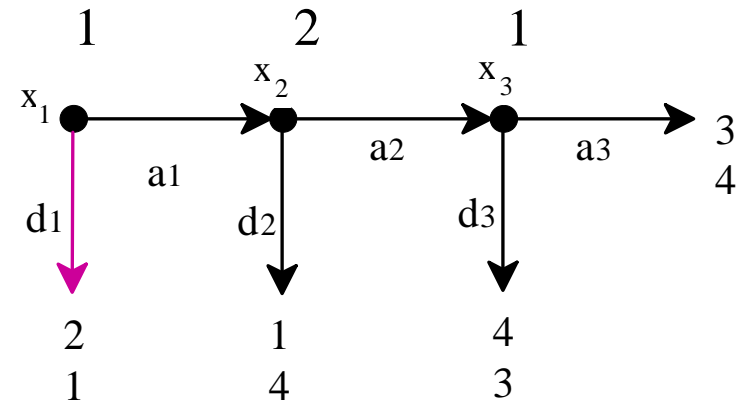
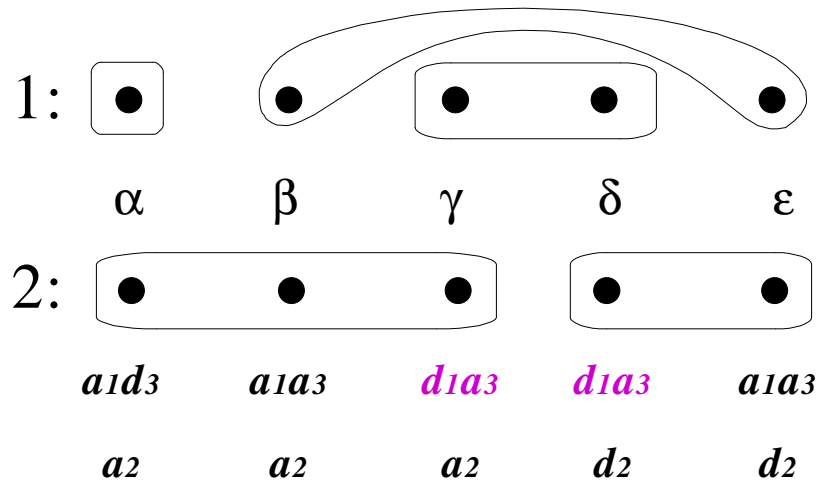
$$u_2(z_{n+1}) = u_2(z_n) + 1$$



$$n = 4$$



Definition. Given an epistemic model of a **BI** terminating game, let **BI** be the event that the backward-induction **play** obtains, that is, $\mathbf{BI} = \{\omega \in \Omega : p(\sigma(\omega)) = x_1 z_1\}$



$$\mathbf{BI} = \{\gamma, \delta\}$$

PROPOSITION 1. In every BI terminating game, $K_*R^{RN} \subseteq BI$

PROPOSITION 2. For every BI terminating game, there is a model of it where $K_*R^{RN} \neq \emptyset$

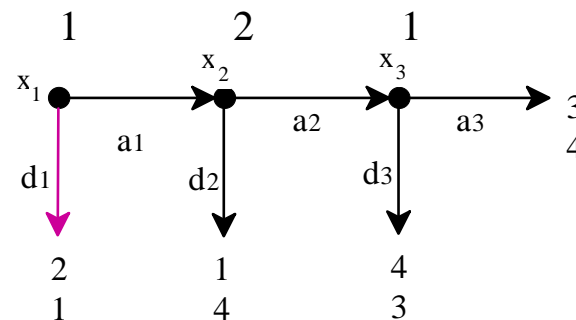
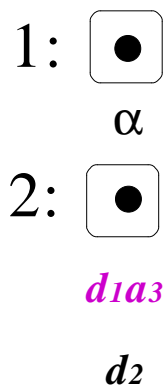
Aumann, R., A note on the centipede game, *Games and Economic Behavior*, 1998, 23: 97-105.

Broome, J. and W. Rabinowicz, Backwards induction in the centipede game, *Analysis*, 1999, 59:237-242.

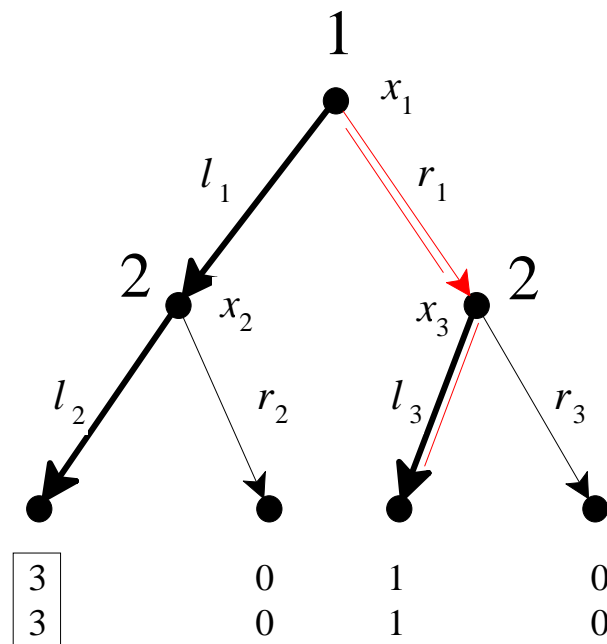
Rabinowicz, W., Grappling with the centipede, *Economics and Philosophy*, 1998, 14: 95-126.

Sugden, R., Rational choice: a survey of contributions from economics and philosophy, *Economic Journal*, 1991, 101:751-785.

Note: it is not necessarily the case that if $\omega \in \Omega$ is such that at ω there is common knowledge of rationality then $\sigma(\omega)$ coincides with the backward-induction **strategy profile**. What is true is that player 1's strategy assigns the terminating choice to the root.



In general perfect-information games common knowledge of Rationality at Reached Nodes does **not** yield the backward-induction play.



1:

α

2:

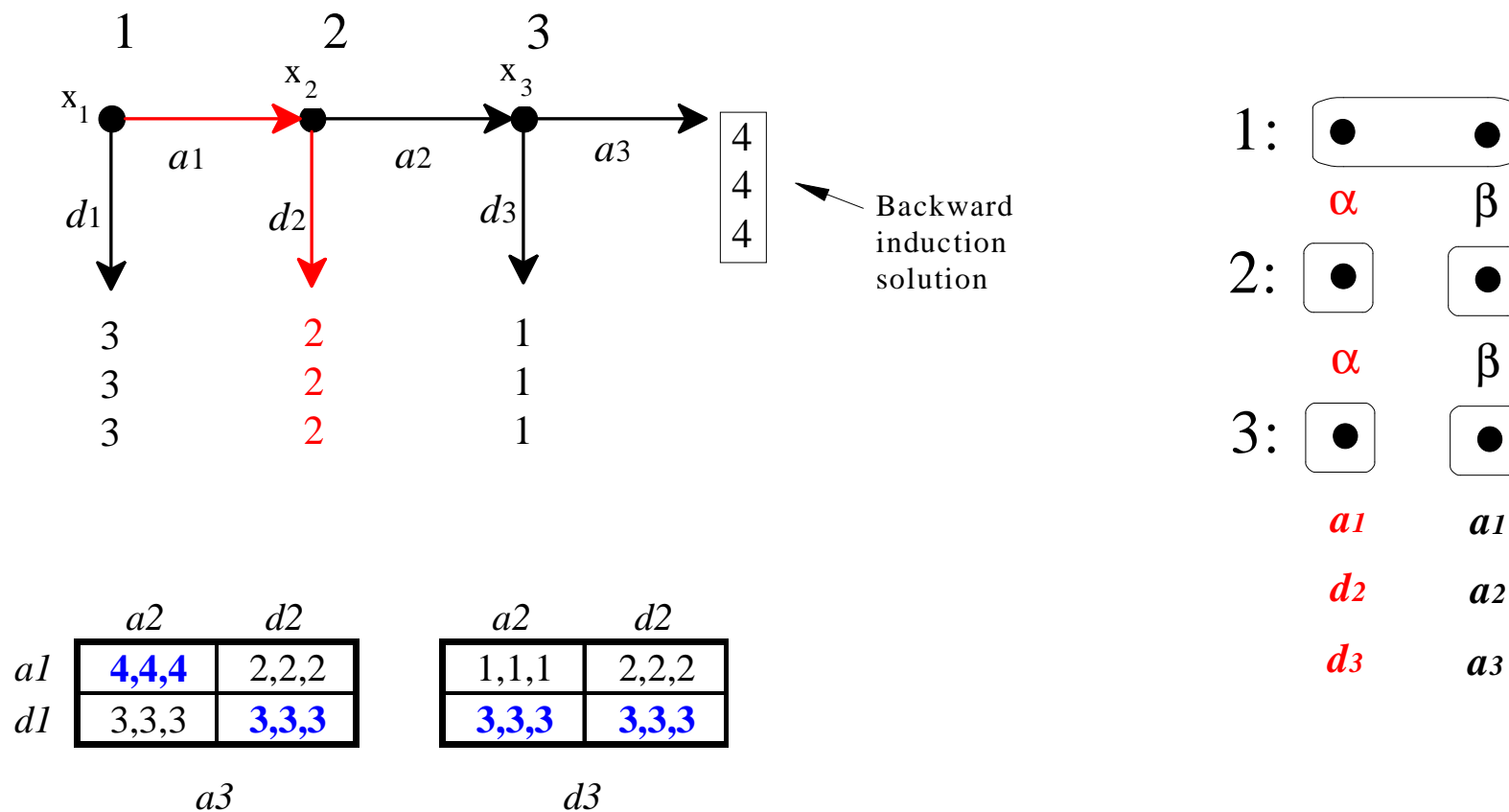
r_1

$r_2 l_3$

The backward induction play is $l_1 l_2$ while in this model we get $r_1 l_3$

$(r_1, r_2 l_3)$ is a Nash equilibrium. Does common knowledge of Rationality at Reached Nodes at least yield a play that can be sustained by a Nash equilibrium?

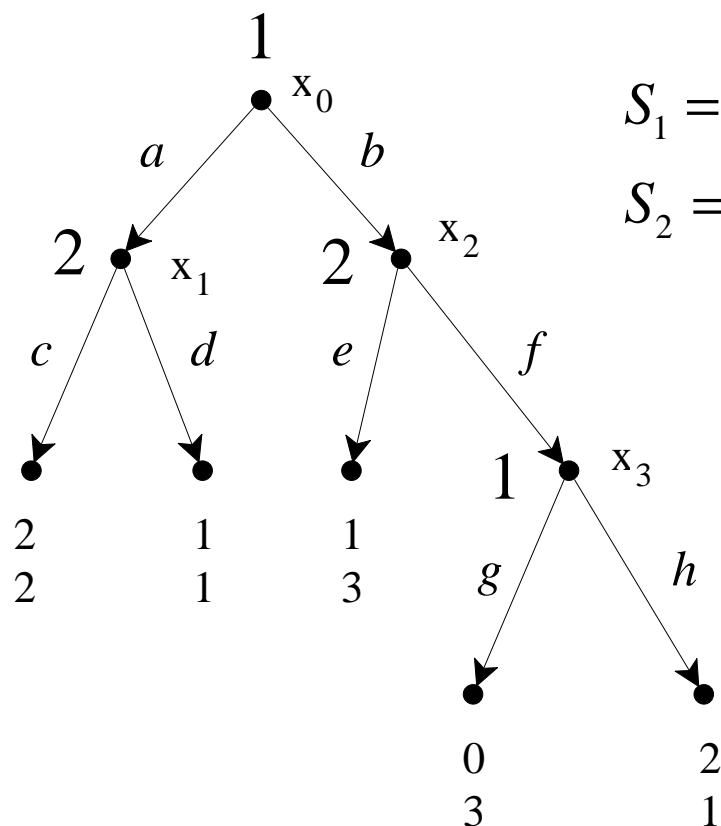
NO! In general, common knowledge of Rationality at Reached Nodes does not yield Nash equilibrium play



The Nash equilibria are marked in blue

Dealing with general perfect-information games

Let $x \in X_i$ be a decision node of player i . Denote by S_i^x the set of player i 's strategies in the subgame that starts at node x .



$$S_1 = \{ag, ah, bg, bh\}, \quad S_1^{x_3} = \{g, h\}$$

$$S_2 = \{cd, cf, de, df\}, \quad S_2^{x_1} = \{c, d\}, \quad S_2^{x_2} = \{e, f\}$$

Let x be a decision node of player i and let $s_i^x, t_i^x \in S_i^x$
be two strategies of player i in the subgame that starts at node x

$s_i^x \succ_i t_i^x$ is interpreted as "for player i , strategy s_i^x is better than strategy t_i^x
in the subgame that starts at node x "

$s_i^x \succ_i t_i^x$ is true at state ω if, **starting from node x ,**
 s_i^x gives a higher payoff to player i than t_i^x against $\sigma_{-i}(\omega)$

Let $\|s_i^x \succ_i t_i^x\|$ be the event that $s_i^x \succ_i t_i^x$ is true.

If x is a node of player i , let $\sigma_i(\omega)|_x$ denote the restriction of
 $\sigma_i(\omega)$ to the subgame that starts at x

If $s_i^x \in S_i^x$, let $\|s_i^x\| = \{\omega \in \Omega : s_i^x = \sigma_i(\omega)|_x\}$

SUBSTANTIVE RATIONALITY (Aumann, GEB 1995)

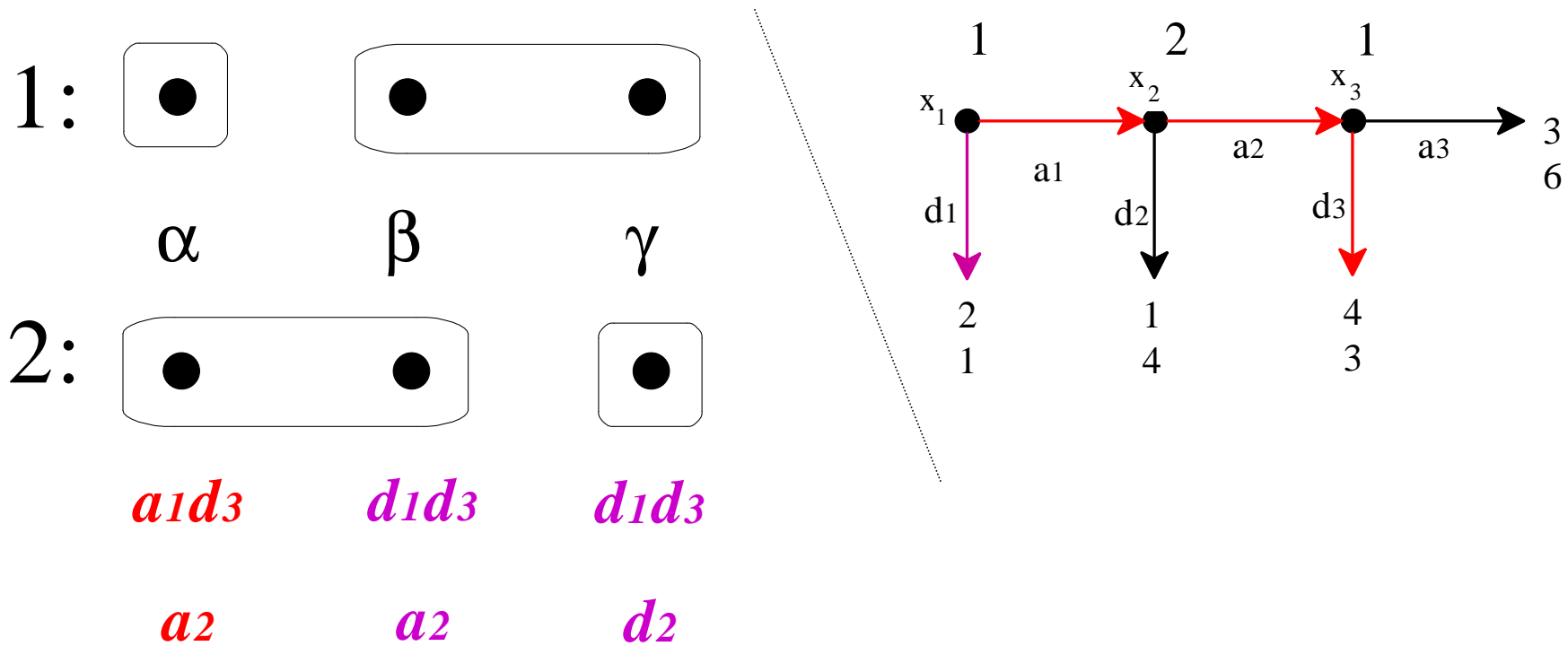
Recall that if $E, F \subseteq \Omega$, $E \rightarrow F$ is the event $\neg E \cup F$ (if E then F)

Let \mathbf{R}_i^{SR} be the event representing the proposition “player i is substantively rational”

$$\text{if } x \in X_i \quad \left\| s_i^x \right\| \cap K_i \left(\left\| t_i^x \succ_i s_i^x \right\| \right) \subseteq \neg \mathbf{R}_i^{SR}$$

$$\neg \mathbf{R}_i^{SR} = \bigcup_{x \in X_i} \bigcup_{s_i \in S_i^x} \bigcup_{t_i \in S_i^x} \left(\left\| s_i^x \right\| \cap K_i \left(\left\| t_i^x \succ_i s_i^x \right\| \right) \right)$$

$$\mathbf{R}^{SR} = \mathbf{R}_1^{SR} \cap \dots \cap \mathbf{R}_n^{SR} \quad \text{all players are substantively rational}$$



$R_2^{EA} = \{\alpha, \beta, \gamma\}$ (*ex ante* rationality)


$R_2^{RN} = \{\beta, \gamma\}$ (rationality at reached nodes)

$R_2^{SR} = \{\gamma\}$ (substantive rationality)

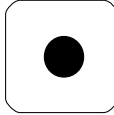
PROPOSITION 3. In every perfect information game, $K_*\mathbf{R}^{SR} \subseteq \mathbf{BI}$

PROPOSITION 4. For every perfect information game, there is a model of it where $K_*\mathbf{R}^{SR} \neq \emptyset$

Aumann, R., Backward induction and common knowledge of rationality, *Games and Economic Behavior*, 1995, 8: 6-19.

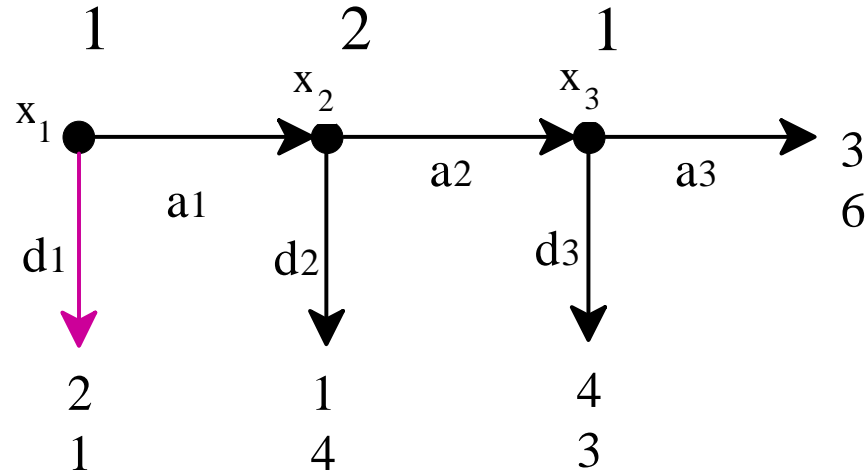
1: 

α

2: 

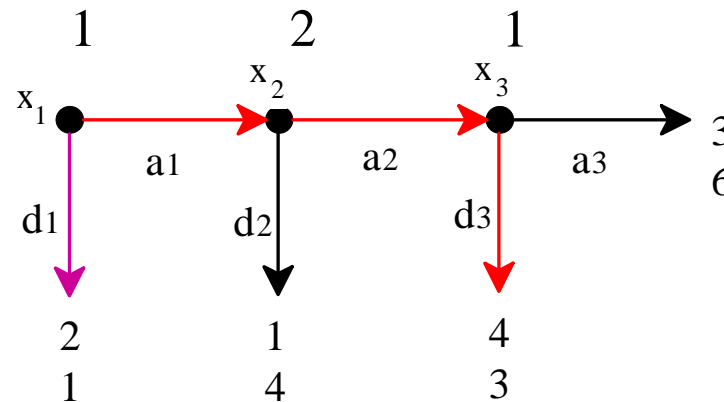
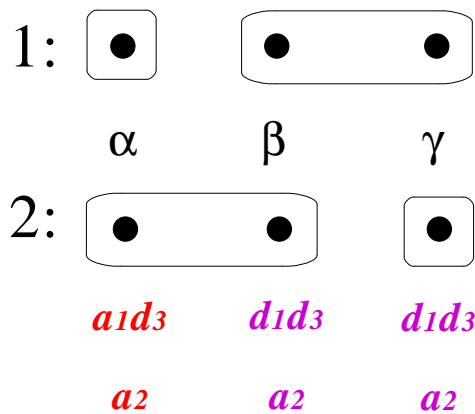
$d_1 a_3$

d_2



Why is player 2 substantively irrational at state α ? What is true at state α that makes player 2 substantively irrational?

At state α player 2 is not taking any actions, because her node x_2 is not reached. In fact, at state α player 2 *knows* that her node is not reached. So what makes her irrational (according to the notion of substantive rationality) must be her *plan* to choose d_2 *if her decision node were to be reached*. This is a *counterfactual* statement.



The association of a strategy profile with every state gives rise to *two types of counterfactuals*:

- (1) An objective statement about what the relevant player would do at a node that is not reached.
 - (2) (With the help of the partitions) a subjective statement about what a player believes would happen if he were to take a different action from the one he is actually taking.
- (1) Thus at state γ it is true that **player 2** would take action a_2 if her node x_2 were to be reached (although it is not in fact reached and she knows that it is not reached)
 - (2) At states β and γ **player 1** knows that if he were to take action a_1 instead of d_1 at the root (he knows that he is taking d_1) then his payoff would be 4 (the payoff associated with $a_1a_2d_3$)

Modeling counterfactuals indirectly through strategies is not satisfactory. We have abandoned the modular approach suggested in Lecture 1, since there exists a module that deals with counterfactuals.

Modeling Counterfactuals

For every $\omega \in \Omega$, let \mathcal{P}_ω be a relation on Ω satisfying, $\forall \alpha, \beta \in \Omega$,

- (1) either $\alpha \in \mathcal{P}_\omega(\beta)$ or $\beta \in \mathcal{P}_\omega(\alpha)$ (completeness)
- (2) if $\beta \in \mathcal{P}_\omega(\alpha)$ then $\mathcal{P}_\omega(\beta) \subseteq \mathcal{P}_\omega(\alpha)$ (transitivity)
- (3) if $\alpha \in \mathcal{P}_\omega(\beta)$ and $\beta \in \mathcal{P}_\omega(\alpha)$ then $\alpha = \beta$ (antisymmetry)
- (4) $\omega' \in \mathcal{P}_\omega(\omega)$, for all $\omega' \in \Omega$ (centeredness)

The interpretation of $\beta \in \mathcal{P}_\omega(\alpha)$ or $\alpha \mathcal{P}_\omega \beta$ is that state α is at least as close to state ω as state β is. Thus, for every state ω , the closeness relation \mathcal{P}_ω determines a strict ordering of the set of states based on closeness to ω , with ω itself being the closest state.

$\mathcal{P}_\omega(\alpha)$ = set of states that are not closer to ω than α is.

REPRESENTATION

$$\Omega = \{\alpha, \beta, \gamma, \delta\}$$

	α	β	γ	δ	
	●	●	●	●	
closest	α	β	γ	δ	} ordering
	γ	α	δ	β	
	β	δ	α	γ	
farthest	δ	γ	β	α	

Given a state ω and an event E , denote by $\min(\omega, E)$ the closest state to ω that belongs to event E . Thus if $\omega \in E$, then $\min(\omega, E) = \omega$.

In the above example, if $E = \{\beta, \delta\}$ then $\min(\alpha, E) = \beta$

Recall that, if $E, F \subseteq \Omega$ are two events, $E \rightarrow F$ denotes the event $\neg E \cup F$ (if E then F). Thus $\omega \in E \rightarrow F$ if either $\omega \notin E$ or $\omega \in E \cap F$.

\rightarrow represents the material conditional, which is true whenever the antecedent is false

We use the symbol \rightsquigarrow to denote the counterfactual conditional.

Thus $E \rightsquigarrow F$ is interpreted as “if E were the case then F would be the case”

Definition. $E \rightsquigarrow F = \{ \omega \in \Omega : \min(\omega, E) \in F \}$

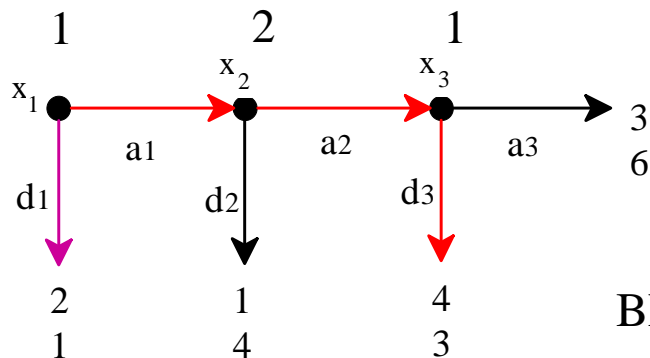
	α	β	γ	δ	
	●	●	●	●	
closest	α	β	γ	δ	If $E = \{\beta, \delta\}$ and $F = \{\alpha, \gamma, \delta\}$ then $E \rightsquigarrow F = \{\gamma, \delta\}$ while $E \rightarrow F = \{\alpha, \gamma, \delta\}$
	γ	α	δ	β	
	β	δ	α	γ	
farthest	δ	γ	β	α	

Note that, for all $E, F \subseteq \Omega$, $E \rightsquigarrow F \subseteq E \rightarrow F$

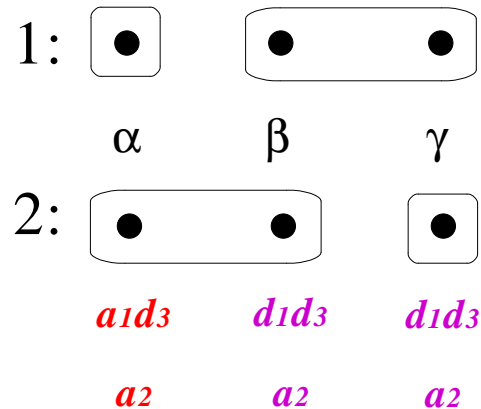
MODELING STRATEGIES WITH COUNTERFACTUALS

Given a perfect information game define an epistemic model of it as before, but with the following changes:

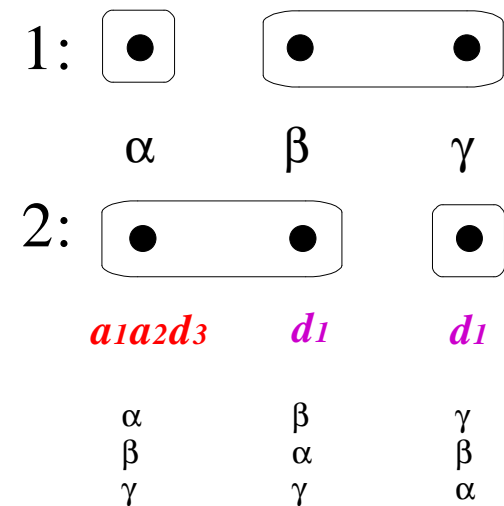
- (1) replace the n functions $\sigma_i : \Omega \rightarrow S$ with a single function $d : \Omega \rightarrow P$ where P is the set of plays of the game written in terms of actions taken,
- (2) add a set of closeness relations $\{\mathcal{P}_\omega\}_{\omega \in \Omega}$



BEFORE

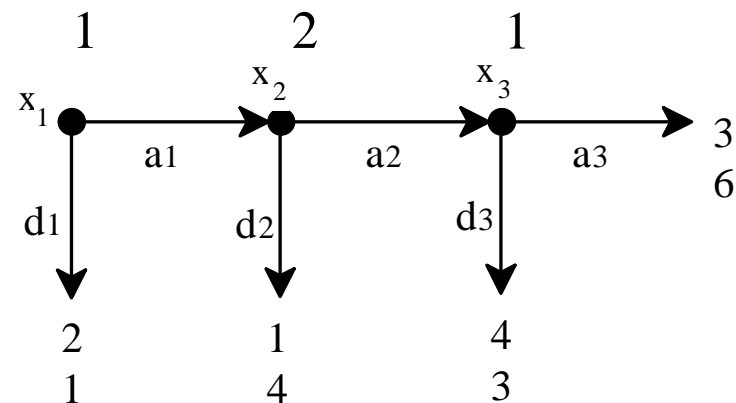
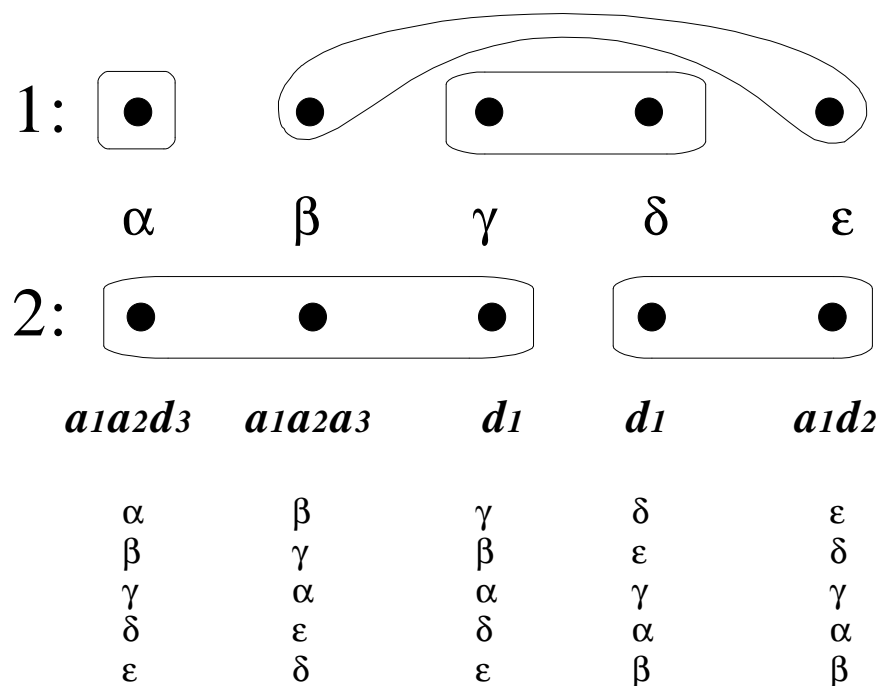


NOW



We add two more requirements:

- (3) for every play there is at least one state where that play is realized
- (4) if, at a state, node x of player i is reached and he takes action a there, then he knows that if x is reached he takes action a : $\|a\| \subseteq K_i(\|x\| \rightarrow \|a\|)$



$$\|x_2\| = \{\alpha, \beta, \epsilon\}, \quad \|a_2\| = \{a, \beta\}$$

$$\|x_2\| \rightarrow \|a_2\| = \neg\|x_2\| \cup \|a_2\| = \{\alpha, \beta, \gamma, \delta\}$$

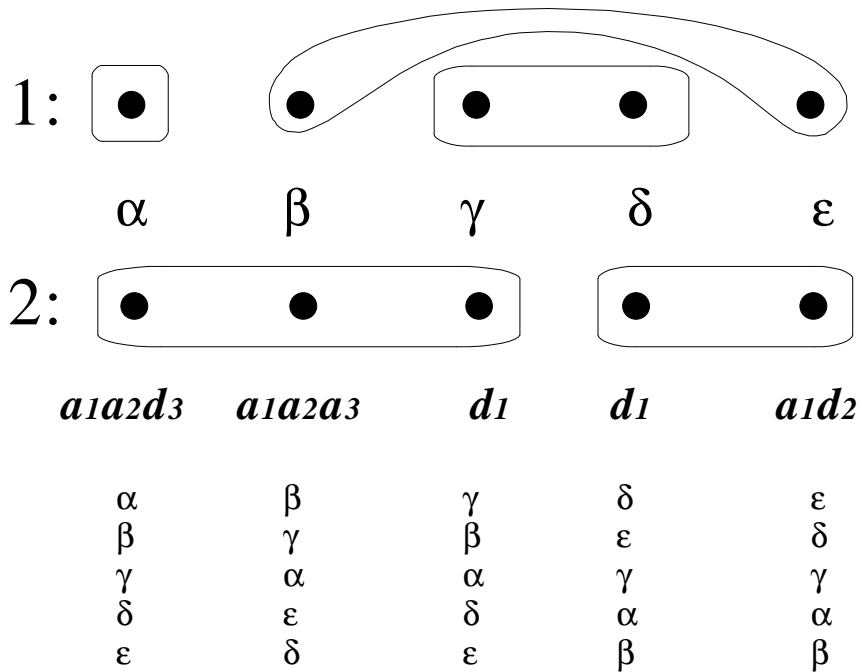
$$K_2(\|x_2\| \rightarrow \|a_2\|) = \{\alpha, \beta, \gamma\}$$

EXTRACTING STRATEGIES FROM A MODEL

Given a model we can extract a strategy profile at every state as follows.

If s_i is a strategy of player i and x_i is a decision node of player i , denote by $s_i(x_i)$ the choice prescribed by s_i at x_i .

Define $\sigma_i(\omega)$ as follows: $\sigma_i(\omega)(x_i) = c_i$ if and only if $\omega \in \|x_i\| \rightsquigarrow \|c_i\|$



$$\sigma_1(\alpha) = a_1d_3, \quad \sigma_1(\beta) = a_1a_3$$

$$\sigma_1(\gamma) = d_1a_3 \quad (\text{for node } x_3 \text{ we use state } \beta)$$

$$\sigma_1(\delta) = d_1d_3 \quad (\text{for node } x_3 \text{ we use state } \alpha)$$

$$\sigma_1(\varepsilon) = a_1d_3 \quad (\text{for node } x_3 \text{ we use state } \alpha)$$

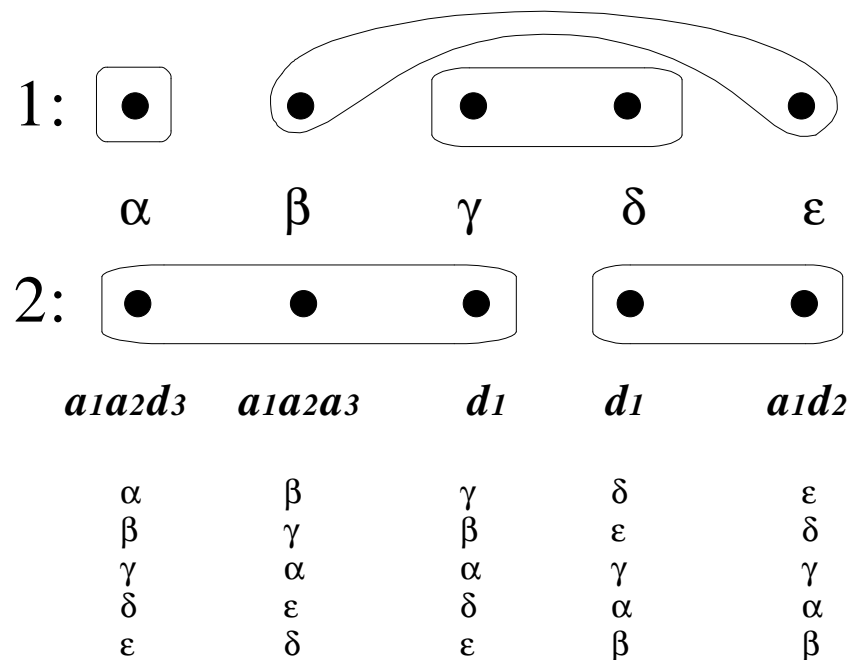
$$\sigma_2(\alpha) = a_2, \quad \sigma_2(\beta) = a_2$$

$$\sigma_2(\gamma) = a_2 \quad (\text{for node } x_2 \text{ we use state } \beta)$$

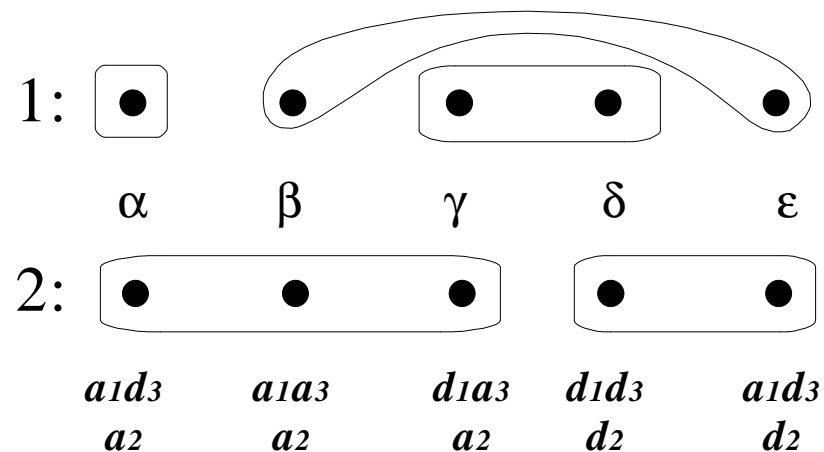
$$\sigma_2(\delta) = d_2 \quad (\text{for node } x_2 \text{ we use state } \varepsilon)$$

$$\sigma_2(\varepsilon) = d_2$$

From



We get



In this model it is not true that players know their own strategies. E.g. player 1 at state γ

In order for a counterfactual model to give rise to a standard model based on strategies, we need to impose a further condition:

$$(5) \quad (\|x_i\| \Vdash \|c_i\|) \rightarrow K_i (\|x_i\| \Vdash \|c_i\|)$$

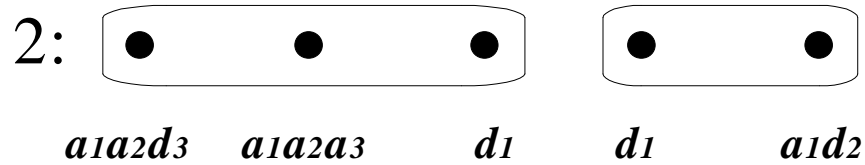
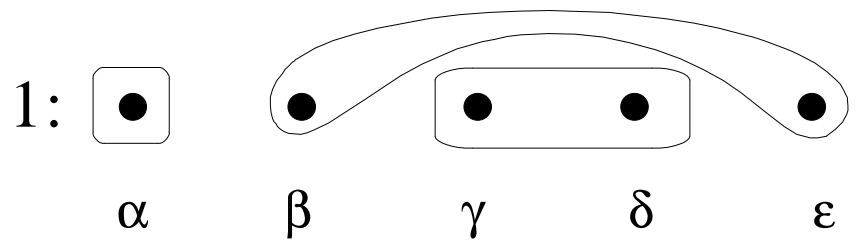
RE-DEFINING RATIONALITY AT REACHED NODES

Let x_i be a decision node of player i and c_i and c_i' be two choices of player i at x_i .

If m is a number, let $\|\pi_i = m\|$ be the event that player i 's payoff is m .

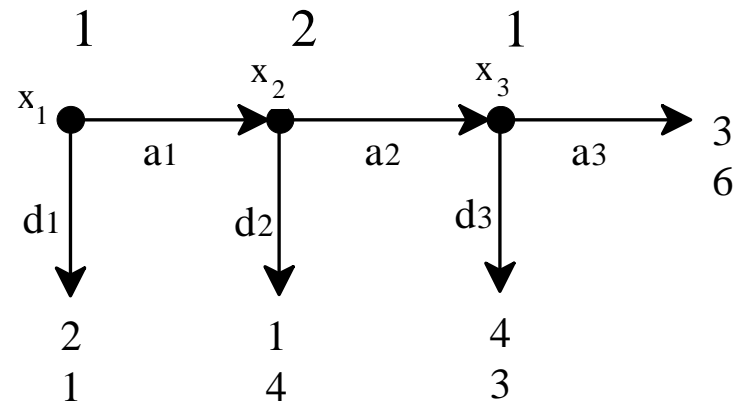
If k and ℓ are numbers, let $\|k > \ell\| = \Omega$ if $k > \ell$ and $\|k > \ell\| = \emptyset$ otherwise.

$$\|c_i\| \cap \|\pi_i = k\| \cap K_i \left(\|x_i\| \rightarrow \left(\|c_i'\| \rightsquigarrow \|\pi_i = \ell\| \right) \right) \cap \|\ell > k\| \subseteq \neg R_i^{RN}$$

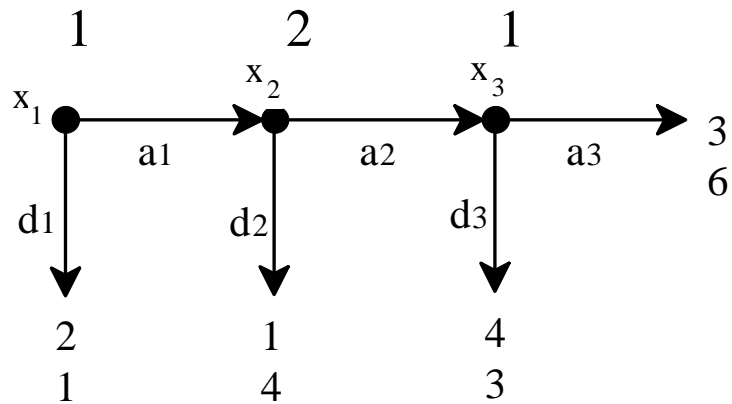
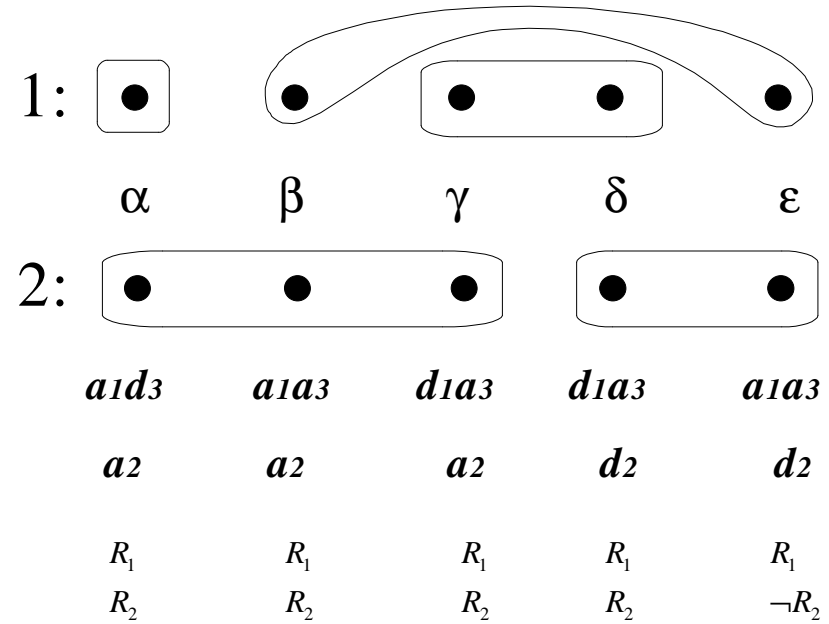
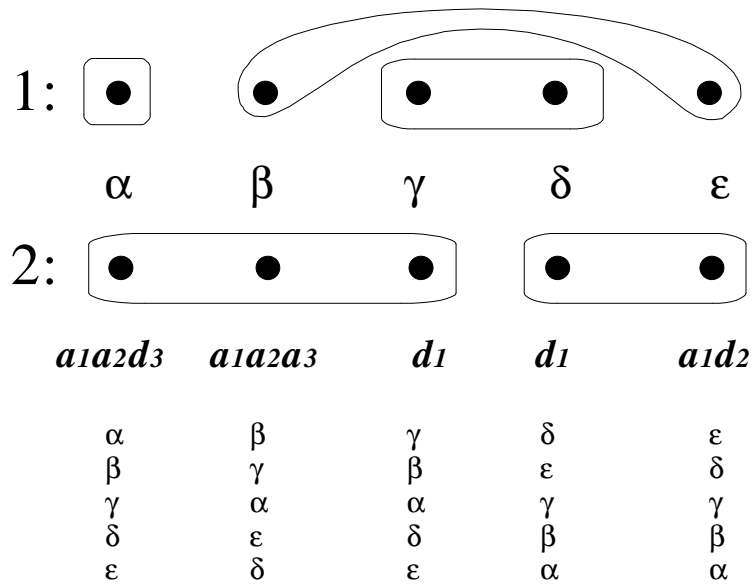


α	β	γ	δ	ϵ
β	γ	β	ϵ	δ
γ	α	α	γ	γ
δ	ϵ	δ	β	β
ϵ	δ	ϵ	α	α

$\pi_1 = 4$	$\pi_1 = 3$	$\pi_1 = 2$	$\pi_1 = 2$	$\pi_1 = 1$
$d_1 \rightsquigarrow \pi_1 = 2$	$d_1 \rightsquigarrow \pi_1 = 2$	$a_1 \rightsquigarrow \pi_1 = 3$	$a_1 \rightsquigarrow \pi_1 = 1$	$d_1 \rightsquigarrow \pi_1 = 2$
$\pi_2 = 3$	$\pi_2 = 6$	$\pi_2 = 1$	$\pi_2 = 1$	$\pi_2 = 4$
$d_2 \rightsquigarrow \pi_2 = 4$	$d_2 \rightsquigarrow \pi_2 = 4$	no choices	no choices	$a_2 \rightsquigarrow \pi_2 = 6$
R_1	R_1	by 2	by 2	R_1
R_2	R_2	R_1	R_1	$\neg R_2$
		R_2	R_2	
		use β	use ϵ	use β



Thus no common knowledge of rationality at any state.



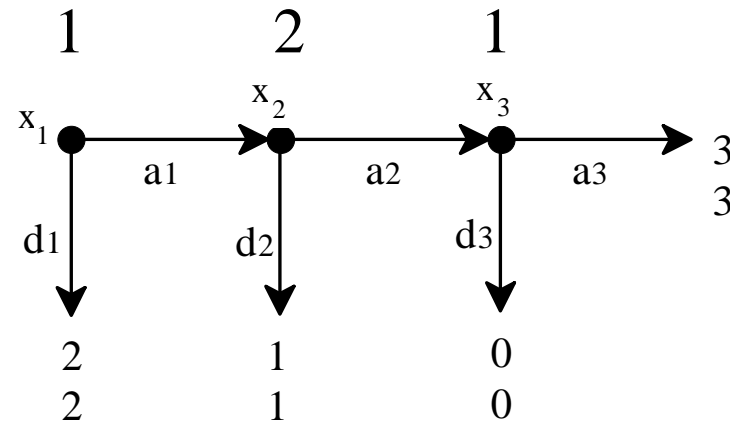
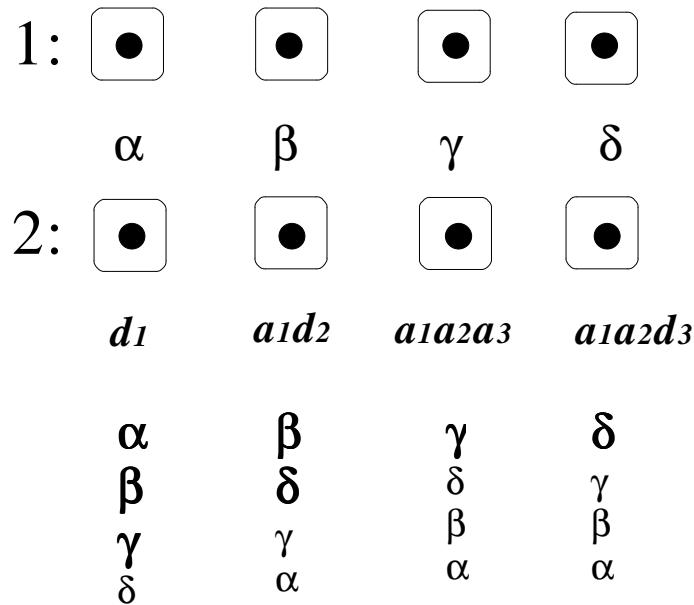
The corresponding strategy-based model

Redefining substantive rationality (Stalnaker's notion)

$$\mathbf{R}_i^{SR} = \bigcap_{x_i \in X_i} \left(\|x_i\| \rightsquigarrow \mathbf{R}_i^{RN} \right)$$

rationality at all nodes: reached and un-reached

Does common knowledge of substantial rationality so defined imply the backward-induction play?



$$R_1^{RN} = \{\alpha, \gamma\}$$

$$R_2^{RN} = \{\alpha, \beta, \gamma\}$$

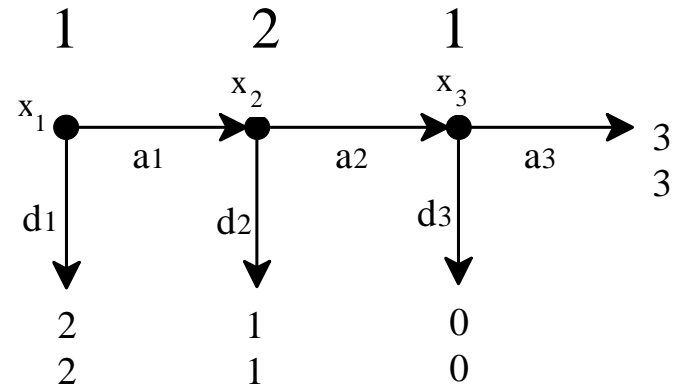
At state α there is common knowledge of substantive rationality. The following is true at α :

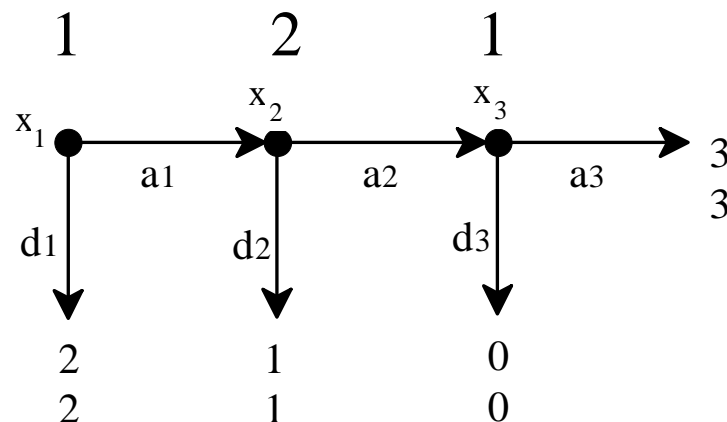
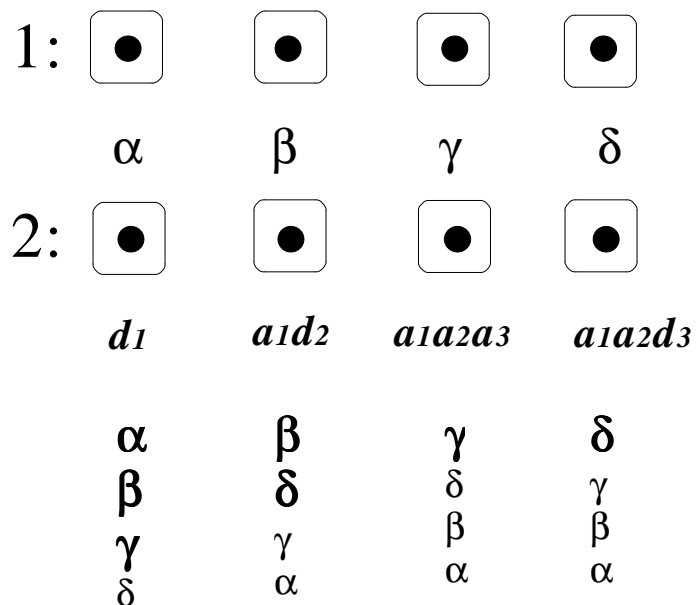
- (1) 1 is materially rational at x_1 : 1 knows that if he played a_1 then 2 would play d_2 . [**state β**]
- (2) 2 is materially rational (does not do anything) but also substantively rational: if x_2 were reached [**state β**] then player 2 would be materially rational (she would play d_2 knowing that if she played a_2 then 1 would play d_3) [**state δ**].
- (3) 1 is substantively rational at x_3 : if x_3 were reached he would play a_3 [**state γ**].

Stalnaker (1998 p. 48)

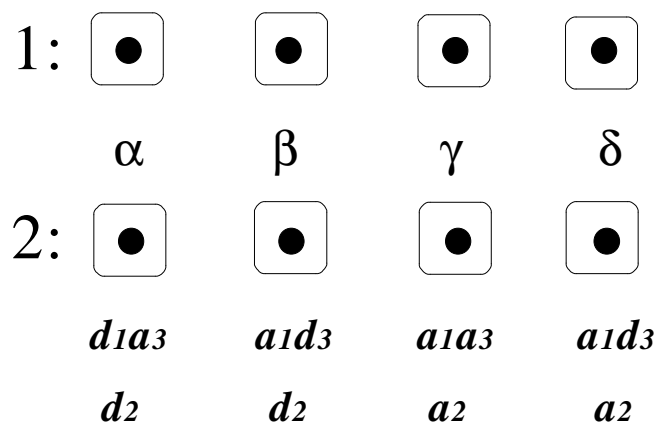
Player 2 has the following initial belief: player 1 would choose a_3 on her second move *if* she had a second move. This is a causal ‘if’ – an ‘if’ used to express 2’s opinion about 1’s *disposition to act* in a situation that they both know will not arise. Player 2 knows that since player 1 is rational, if she somehow found herself at her second node, she would choose a_3 . But to ask what player 2 would believe about player 1 *if* he learned that he was wrong about 1’s first choice is to ask a completely different question – this ‘if’ is epistemic; it concerns player 2’s belief revision policies, and not player 1’s disposition to be rational. No assumption about player 1’s substantive rationality, or about player 2’s knowledge of her substantive rationality, can imply that player 2 should be disposed to maintain his belief that she will act rationally on her second move even were he to learn that she acted irrationally on her first.

1:				
	α	β	γ	δ
2:				
	d_1	a_1d_2	$a_1a_2a_3$	$a_1a_2d_3$
	α	β	γ	δ
	β	δ	δ	γ
	γ	γ	β	β
	δ	α	α	α

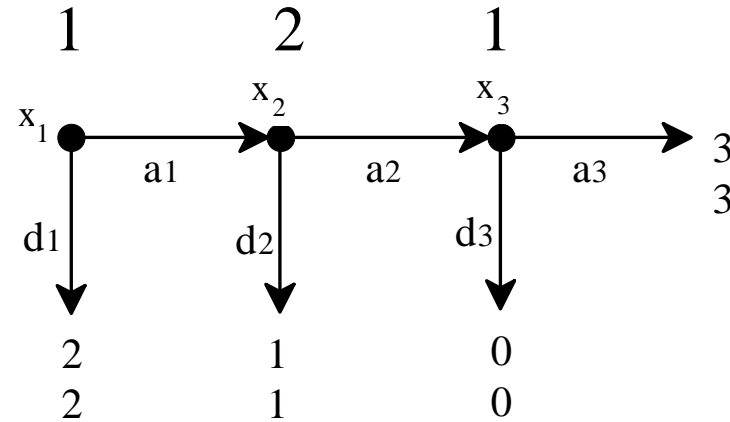
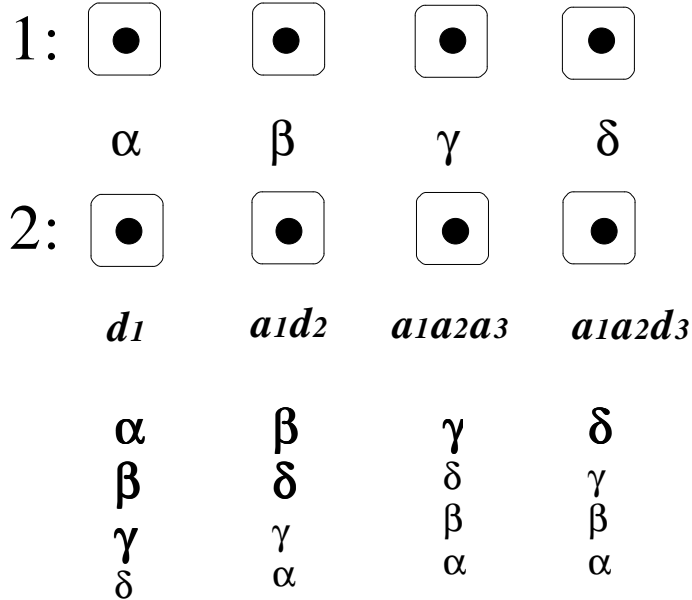




The corresponding strategy-based model is:



According to Aumann, player 2 is not substantively rational at α : player 2 is planning to play d_2 knowing that player 1 would play a_3 .



$$\alpha \in K_2 \left(\|x_3\| \rightsquigarrow a_3 \right) \text{ and also } \alpha \in \|x_2\| \rightsquigarrow K_2 \left(\|x_3\| \rightsquigarrow d_3 \right)$$

Thus what player 2 believes about player 1's behavior in the hypothetical world where node x_3 is reached changes going from node x_1 (where the game ends without node x_2 being reached) to the hypothetical world where x_2 is reached. *If one imposes the constraint that such changes cannot happen, then common knowledge of substantive rationality implies the backward-induction play.*

ADDITIONAL REFERENCES

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