de Vries, Sven; Raach, Stephen; Vohra, Rakesh V.

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Naval Research Logistics (NRL)

Provided in Cooperation with:
John Wiley & Sons

Suggested Citation: de Vries, Sven; Raach, Stephen; Vohra, Rakesh V. (2024) : On inner independence systems, Naval Research Logistics (NRL), ISSN 1520-6750, John Wiley & Sons, Inc., Hoboken, NJ, Vol. 72, Iss. 1, pp. 133-147, https://doi.org/10.1002/nav.22210

Received: 2 August 2023 | Revised: 13 May 2024 | Accepted: 17 June 2024

DOI: 10.1002/nav.22210

RESEARCH ARTICLE

WILEY

On inner independence systems

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Handling Editor: Sasa Pekec

Abstract

A classic result of Korte and Hausmann [1978] and Jenkyns [1976] bounds the quality of the greedy solution to the problem of finding a maximum value basis of an independence system $(E, \mathcal{I})$ in terms of the rank-quotient. We extend this result in two ways. First, we apply the greedy algorithm to an *inner independence system* contained in $\mathcal{I}$. Additionally, following an idea of Milgrom [2017], we incorporate exogenously given prior information about the set of likely candidates for an optimal basis in terms of a set $\mathcal{O} \subseteq \mathcal{I}$. We provide a generalization of the rank-quotient that yields a tight bound on the worst-case performance of the greedy algorithm applied to the inner independence system relative to the optimal solution in $\mathcal{O}$. Furthermore, we show that for a worst-case objective, the inner independence system approximation may outperform not only the standard greedy algorithm but also the inner matroid approximation proposed by Milgrom [2017]. Second, we generalize the inner approximation framework of independence systems to inner approximations of packing instances in $\mathbb{Z}^n_{\geq 0}$ by inner polymatroids and inner packing instances. We consider the problem of maximizing a separable discrete concave function and show that our inner approximation can be better than the greedy algorithm applied to the original packing instance. Our result provides a lower bound to the generalized rank-quotient of a greedy algorithm to the optimal solution in this more general setting and subsumes Malinov and Kovalyov [1980]. We apply the inner approximation approach to packing instances induced by the FCC incentive auction and by two knapsack constraints.

KEYWORDS

greedy algorithm, independence systems, inner approximation, packing problems

1 | INTRODUCTION

In this paper's first part, we examine the problem of determining a maximum value basis of an independence system. Many well-known optimization problems are a special case of this, such as knapsack, set packing, and matching with side constraints. The problem is, of course, $\mathbb{NP}$-hard, and it is common to resort to heuristics to solve it (or any of its special cases). The most well-known heuristic is the greedy algorithm, in which elements of the ground set with nonnegative value are selected in order of declining value as long as they form an independent set. A classic result of Korte and Hausmann (1978) and Jenkyns (1976) bounds the value of the greedy solution relative to the optimal value in terms of the rank-quotient of the underlying independence system.

We examine an alternative approximation approach. The idea is to find an "inner approximation" of the feasible region of the underlying optimization problem and optimize over that. Milgrom (2017) and de Vries and Vohra (2020) examine this approach in the context of independence systems. Specifically, find an inner matroid that approximates the underlying independence system well. An inner matroid is a matroid with

the property that every one of its basis sets is contained in a basis of the original independence system (but not conversely). Then, the greedy algorithm is applied to the inner matroid. It is well known that this will recover the maximum value basis of the inner matroid.

This paper proposes to approximate the given independence system by an inner independence system (defined similarly to the inner matroid). Then, the greedy algorithm is applied to the inner independence system. At first blush, this appears silly. Indeed, the best inner independence system would be the given independence system itself? Second, the greedy algorithm applied to the inner independence system is itself suboptimal. Hence, one introduces two sources of approximation error, first from the inner independence system and second from the greedy solution itself. Surprisingly, this is not the case. We show that an approximation by inner independence systems may simultaneously outperform the approximation by inner matroids as well as the direct application of the greedy algorithm regarding a worst-case objective.

The basic intuition is this. The greedy algorithm is suboptimal because it gets 'stuck' on a low-rank basis. An inner independence system arises from the given independence system by deleting some of the low-rank bases, that is, removing potential *local* optima.

The second part of this paper extends the inner approximation idea to more general combinatorial objects than independence systems, namely *packing instances*. Such problems arise in combinatorial auctions (see e.g., de Vries & Vohra, 2003). Towards this generalization, the independent sets of the independence system are represented by their characteristic vector. The set of these binary vectors forms an independence system if and only if it is downward closed. Specifically, if x is in the set, any binary vector $y \leq x$ is also in the set. The maximal vectors in this set correspond to the basis sets of the corresponding independence system. Packing instances correspond to finite collections of vectors in $\mathbb{Z}^E_{\geq 0}$ that are downward closed. We consider packing instances where the objective function is nondecreasing, separable, and discrete concave and show that the greedy algorithm's worst-case approximation guarantee can be outperformed by an approximation with inner polymatroids and by inner integer packing instances.

1.1 | A motivating example

Before formally explaining our approach, we start with an example to build up some intuition.

Example 1. Consider a packing problem whose feasible solutions are the set of dots in Figure 1.

A natural (heuristic) approach to solve this problem for arbitrary $c \in \mathbb{R}^2_{\geq 0}$ is the greedy algorithm, which, at each iteration, selects among the variables that can be augmented by 1 without violating feasibility. Among these, the one with the largest objective coefficient value is picked, and its coordinate is increased by 1. If $c^T = (1, 1+\varepsilon)$, for ε arbitrary small then, the greedy algorithm outputs the solution $(0,3)^T$ with value $3+3\varepsilon$, while the optimal solution for this particular choice of c is $(10,0)$ with a value of 10. Thus, in the worst case, the greedy algorithm cannot approximate the optimal solution by a factor better than $\frac{3}{10}$. We restrict the set of feasible solutions to improve this approximation factor, as depicted in Figure 2.

The greedy algorithm applied to this *inner* packing instance outputs the solution $(10,2)^T$, independent of the choice of $c \in \mathbb{R}^2_{\geq 0}$. It is easy to see that the worst-case approximation might occur for $c = (0,1)$. For this particular choice of c, the greedy algorithm applied to the inner packing instance yields a value of 2. In contrast, the optimal value of 3 is achieved at the point $(0,3)$ (which was excluded in the inner packing instance). However, this implies that the greedy algorithm approximates the optimal solution by a factor not worse than $\frac{2}{3}$.

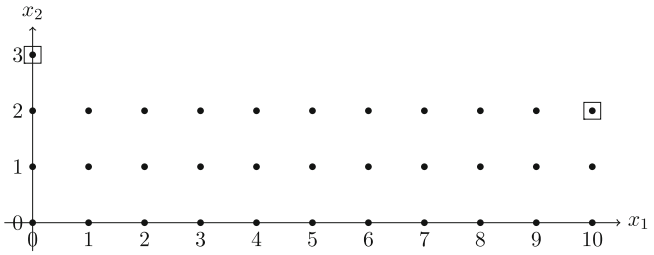

FIGURE 1 Feasible set (dots) of the original packing instance. Its bases are boxed.

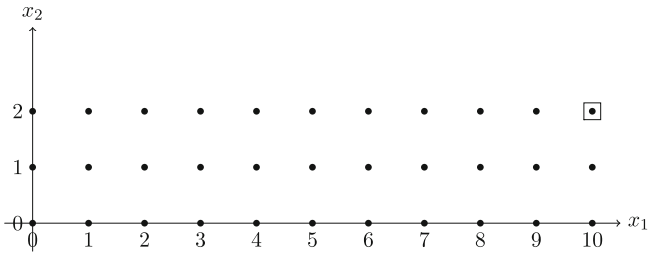

FIGURE 2 Feasible set (dots) of the inner packing instance. Its basis is boxed.

Thus, Example 1 demonstrates that reducing the set of feasible solutions might significantly improve the greedy algorithm's performance. The present paper formalizes this inner approximation approach and provides several examples, demonstrating the possible superiority of inner approximation over plain greedy.

Algorithm 1. Greedy algorithm for independence systems

Input : An independence system $(E, \mathcal{I})$ given by an independence oracle. Values $v_i \in \mathbb{R}_+$

Output: A basis $I_g \in \mathcal{I}$

Order the elements of E in nondecreasing order such that $v_1 \geq v_2 \geq \cdots \geq v_{|E|}$

Set $I = \emptyset$

for $i \leftarrow 1$ **to** $|E|$ **do**

  if $I \cup \{i\} \in \mathcal{I}$ *(oracle call)* **then**

    $I = I \cup \{i\}$

Set $I_g = I$ and return

1.2 | Outline of the paper

Section 2 describes the greedy algorithm for independence systems and presents the approximation guarantee of inner approximation by independence systems. Section 3 compares inner approximation by independence systems with inner approximation by matroids. Section 4 covers the inner approximation of packing instances. Section 5 provides applications of the inner approximation framework. We end with a conclusion and an outlook for further applications in Section 6.

Notice that all deferred proofs and calculations are in the appendix.

2 | GREEDY ALGORITHM AND INNER APPROXIMATION BY INDEPENDENCE SYSTEMS

We recall the definition of an independence system.

> **Definition.** A finite set $E = \{1, \dots, n\}$ and a family of subsets of E denoted $\mathcal{I}$ is called an **independence system** if $\emptyset \in \mathcal{I}$ and $\mathcal{I}$ is **downward closed**, thus, $J \in \mathcal{I}$ implies $I \in \mathcal{I}$ for all $I \subseteq J$. A set $I \in \mathcal{I}$ is called **independent**, and all other subsets of E are called **dependent**. A set $B \in \mathcal{I}$ is called a **basis of** $S \subseteq E$ if $B \subseteq S$ and $B \cup \{s\} \notin \mathcal{I}$ for all $s \in S \setminus B$. A basis of E is called a **basis**. The set of bases of $(E, \mathcal{I})$ is denoted $\mathcal{B}_{\mathcal{I}}$. Any subset $\mathcal{I}$ of 2^E for which $\{i\} \in \mathcal{I}$ for all $i \in E$ is called **normal**.

We associate values $v_i \in \mathbb{R}_+$ with $i \in E$. Then, the problem of finding a maximum value basis of $(E, \mathcal{I})$ is $\max_{I \in \mathcal{I}} v(I) := \max_{I \in \mathcal{I}} \sum_{i \in I} v_i$. Let $(E, \mathcal{I})$ be a normal independence system and $v_i \in \mathbb{R}_+$ for $i \in E$. A standard approach to solve the problem $V^*(\mathcal{I}, v) := \max_{I \in \mathcal{I}} v(I)$ is the greedy algorithm:

Let $V^{greedy}(\mathcal{I}, v)$ denote the value of the *worst* possible (if ties occurs) greedy solution. A common approach to evaluate the quality of an algorithm is to compare the value of its output with the optimal value in the worst-case, here $\min_{v \in \mathbb{R}_+^E \setminus \{\mathbf{0}\}} \frac{V^{greedy}(\mathcal{I}, v)}{V^*(\mathcal{I}, v)}$ if $\mathcal{I}$ is normal.

> **Definition.** The cardinality of the largest basis of a set $T \subseteq E$ is called the **rank** of T, hence $r(T) := \max\{|I| : I \in \mathcal{I}, I \subseteq T\}$, and the size of the smallest basis of a set $T \subseteq E$ is called the **lower rank** of T, that is, $l(T) := \min\{|I| : I \in \mathcal{I}, I \subseteq T, I \cup \{i\} \notin \mathcal{I} \text{ for all } i \in T \setminus I\}$. The **rank-quotient** of the independence system $(E, \mathcal{I})$ is denoted by $q(\mathcal{I}) := \min_{S \subseteq E:\ r(S) \neq 0} \frac{l(S)}{r(S)}$.

Korte and Hausmann (1978) and Jenkyns (1976) showed

$$\min_{v \in \mathbb{R}_+^E \setminus \{\mathbf{0}\}} \frac{V^{greedy}(\mathcal{I}, v)}{V^*(\mathcal{I}, v)} = q(\mathcal{I})$$

for normal independence systems. Motivated by Milgrom (2017), we incorporate exogenously given prior information about the set of likely candidates for an optimal basis, denoted $\mathcal{O} \subseteq \mathcal{I}$.

> **Definition.** We call $\mathcal{O}$ the **acceptable set**. If $\mathcal{O} = \mathcal{I}$, we call this the **zero prior knowledge** case.

In contrast to Milgrom (2017), we do not require $\mathcal{O}$ to be downward closed but will point out where it becomes necessary.

For any acceptable set $\mathcal{O}$ we define $\mathcal{O}^{\subseteq} := \{I \subseteq O : O \in \mathcal{O}\}$. The optimal value with respect to the acceptable set $\mathcal{O}$ is denoted $V^*(\mathcal{O}, v) := \max_{I \in \mathcal{O}} v(I)$. We consider an independence system $(E, \mathcal{J})$ inside the independence system $(E, \mathcal{I})$ (but not necessarily containing all elements of $\mathcal{O}$) and apply the greedy algorithm to find an optimal value basis in $\mathcal{J}$. We call an independence system $(E, \mathcal{J})$ an **inner independence system** of $(E, \mathcal{I})$ if $\mathcal{J} \subseteq \mathcal{I}$ and propose that the greedy solution to $(E, \mathcal{J})$ be used as a solution to the problem of finding a maximum value basis in $\mathcal{O}$.

We want to compare the quality of the greedy solution of the inner independence system to $V^*(\mathcal{O}, v)$.

> **Definition.** We abbreviate the condition $S \subseteq E, S \neq \emptyset$ by $S \sqsubseteq E$.

> **Assumption.** For an independence system $(E, \mathcal{I})$, an inner independence system $(E, \mathcal{J})$ and acceptable set $\mathcal{O}$ we assume that $\mathcal{O}^{\subseteq}$ is normal, hence for every $i \in E$ there exists $O \in \mathcal{O}$ with $i \in O$.

We define the **generalized rank-quotient** as

$$\omega(\mathcal{I}, \mathcal{O}, \mathcal{J}) := \min_{S \sqsubseteq E} \frac{l_{\mathcal{J}}(S)}{\max\{|O| : O \subseteq S, O \in \mathcal{O}^{\subseteq}\}} = \min_{S \sqsubseteq E} \frac{l_{\mathcal{J}}(S)}{r_{\mathcal{O}^{\subseteq}}(S)}.$$

The normality assumption of $\mathcal{O}^{\subseteq}$ is needed to ensure that $\omega(\mathcal{I}, \mathcal{O}, \mathcal{J})$ is well defined. Otherwise, there exists $i \in E$ with $\{i\} \notin \mathcal{O}^{\subseteq}$ such that $\frac{l_{\mathcal{J}}(\{i\})}{\max\{|O| : O \subseteq \{i\}, O \in \mathcal{O}^{\subseteq}\}}$ is undefined.

The definition of the generalized rank-quotient is similar to the rank-quotient and depends only indirectly on $\mathcal{I}$. We provide a bound on the quality of the solution $V^{greedy}(\mathcal{J}, v)$ in terms of $\omega(\mathcal{I}, \mathcal{O}, \mathcal{J})$.

Theorem 2. *Let $(E, \mathcal{I})$ be an independence system, $(E, \mathcal{J})$ an inner independence system of $(E, \mathcal{I})$, and $\mathcal{O}$ the acceptable set. Then,*

$$\min_{v \in \mathbb{R}_+^E \setminus \{\mathbf{0}\}} \frac{V^{greedy}(\mathcal{J}, v)}{V^*(\mathcal{O}, v)} = \omega(\mathcal{I}, \mathcal{O}, \mathcal{J}).$$

Proof. Let $S^* \in \arg\min_{S \sqsubseteq E} \frac{l_{\mathcal{J}}(S)}{r_{\mathcal{O}^{\subseteq}}(S)} = \omega(\mathcal{I}, \mathcal{O}, \mathcal{J})$. Then, with $v = \mathbf{1}_{S^*}$ holds $l_{\mathcal{J}}(S^*) = V^{\text{greedy}}(\mathcal{J}, \mathbf{1}_{S^*})$ and $r_{\mathcal{O}^{\subseteq}}(S^*) = V^*(\mathcal{O}, \mathbf{1}_{S^*})$ and $\min_{v \in \mathbb{R}_+^E \setminus \{\mathbf{0}\}} \frac{V^{greedy}(\mathcal{J}, v)}{V^*(\mathcal{O}, v)} \leq \omega(\mathcal{I}, \mathcal{O}, \mathcal{J})$.

To prove $\min_{v \in \mathbb{R}_+^E \setminus \{\mathbf{0}\}} \frac{V^{greedy}(\mathcal{J}, v)}{V^*(\mathcal{O}, v)} \geq \min_{S \sqsubseteq E} \frac{l_{\mathcal{J}}(S)}{r_{\mathcal{O}^{\subseteq}}(S)}$ we argue similar to Korte and Hausmann (1978). Let $v \in \mathbb{R}_+^E \setminus \{\mathbf{0}\}$, and X_g a worst possible greedy solution for $(E, \mathcal{J})$, and O the optimal solution for $\mathcal{O}$. Let E be ordered such that $v_1 \geq v_2 \geq \ldots \geq v_n \geq v_{n+1} := 0$. Define $E_i := \{1, \ldots, i\}$. Then, for $F \in \mathcal{I}$ we can rewrite

$$v(F) = \sum_{i=1}^{n} |F \cap E_i| \cdot (v_i - v_{i+1}). \tag{1}$$

Note that $O \cap E_i \subseteq O$, and therefore $O \cap E_i \in \mathcal{O}^{\subseteq}$, hence $r_{\mathcal{O}^{\subseteq}}(E_i) \geq |O \cap E_i|$ for all $i \in E$. Furthermore, $X_g \cap E_i$ is a basis of E_i for $\mathcal{J}$ for all $i \in E$, since the greedy algorithm generates it. That implies $|X_g \cap E_i| \geq l_{\mathcal{J}}(E_i)$ for all $i \in E$. It follows

$$\begin{aligned}
\frac{v(X_g)}{v(O)} &= \frac{\sum_{i=1}^{n} |X_g \cap E_i| \cdot (v_i - v_{i+1})}{\sum_{i=1}^{n} |O \cap E_i| \cdot (v_i - v_{i+1})} \\
&\geq \frac{\sum_{i=1}^{n} l_{\mathcal{J}}(E_i) \cdot \frac{|O \cap E_i|}{r_{\mathcal{O}^{\subseteq}}(E_i)} \cdot (v_i - v_{i+1})}{\sum_{i=1}^{n} |O \cap E_i| \cdot (v_i - v_{i+1})} \\
&= \frac{\sum_{i=1}^{n} |O \cap E_i| \cdot (v_i - v_{i+1}) \cdot \frac{l_{\mathcal{J}}(E_i)}{r_{\mathcal{O}^{\subseteq}}(E_i)}}{\sum_{i=1}^{n} |O \cap E_i| \cdot (v_i - v_{i+1})} \\
&\geq \frac{\sum_{i=1}^{n} |O \cap E_i| \cdot (v_i - v_{i+1}) \cdot \min_{S \sqsubseteq E} \frac{l_{\mathcal{J}}(S)}{r_{\mathcal{O}^{\subseteq}}(S)}}{\sum_{i=1}^{n} |O \cap E_i| \cdot (v_i - v_{i+1})} \\
&= \frac{\sum_{i=1}^{n} |O \cap E_i| \cdot (v_i - v_{i+1})}{\sum_{i=1}^{n} |O \cap E_i| \cdot (v_i - v_{i+1})} \cdot \min_{S \sqsubseteq E} \frac{l_{\mathcal{J}}(S)}{r_{\mathcal{O}^{\subseteq}}(S)} \\
&= \min_{S \sqsubseteq E} \frac{l_{\mathcal{J}}(S)}{r_{\mathcal{O}^{\subseteq}}(S)}.
\end{aligned}$$

■

The classic result of Hausmann et al. (1980) is a special case of Theorem 2 for $\mathcal{O} = \mathcal{I}$:

Corollary 3. *For an independence system $(E, \mathcal{I})$ holds $\omega(\mathcal{I}, \mathcal{I}, \mathcal{I}) = q(\mathcal{I})$.*

Proof. It holds that $\omega(\mathcal{I}, \mathcal{I}, \mathcal{I}) = \min_{S \sqsubseteq E} \frac{l_{\mathcal{I}}(S)}{\max\{|O| : O \subseteq S, O \in \mathcal{I}\}} = \min_{S \sqsubseteq E} \frac{l_{\mathcal{I}}(S)}{r_{\mathcal{I}}(S)} = q(\mathcal{I})$. ■

We give an example that shows that an inner independence system approximation can outperform the direct application of the greedy algorithm. For an independence system $(E, \mathcal{I})$ and acceptable set $\mathcal{O}$ call any element of

$$\operatorname*{argmax}_{\mathcal{J} \subseteq \mathcal{I} :\ (E, \mathcal{J}) \text{ is independence system}} \omega(\mathcal{I}, \mathcal{O}, \mathcal{J})$$

a **best inner independence system**.

We need a technical definition to represent the next example simply:

Definition. For any pair of sets A, B of sets we define $A \vee B := \{s : s = a \cup b, a \in A, b \in B\}$.

Example 4. Consider the independence system $(E, \mathcal{I})$ with $E = \{a, b\} \cup C$ with $C := \{c_1, \ldots, c_{10}\}$ defined via its set of bases $\mathcal{B}_{\mathcal{I}} = \{\{a, b\}, C\} \cup \left(\{\{a\}\} \vee \binom{C}{5}\right) \cup \left(\{\{b\}\} \vee \binom{C}{5}\right)$ with $\binom{C}{5} := \{S \subseteq C : |S| = 5\}$. Let $\mathcal{O} = \mathcal{I}$ so that the acceptable set coincides with the set of all independent sets. The ground set has a basis C of cardinality 10, and $\{a, b\}$ is a low rank basis of cardinality 2, therefore, $q(\mathcal{I}) \leq \frac{1}{5}$. We calculate (see Appendix A.2) for $\mathcal{J} := \mathcal{I} \setminus \{\{a, b\}\}$ that $\omega(\mathcal{I}, \mathcal{O}, \mathcal{J}) = \frac{1}{2}$, which yields a better worst-case approximation guarantee than $q(\mathcal{I})$.

One can extend Example 4 to show that there exist cases where the inner independence system approximation yields an arbitrarily better worst-case approximation guarantee than the original independence system:

Example 5. For $i \in \mathbb{N}$ define the independence system $(E^{2i}, \mathcal{I}^{2i})$ with $E^{2i} := \{a, b\} \cup C^{2i}$ and $C^{2i} := \{c_1, \ldots, c_{2i}\}$ via its set of bases $\mathcal{B}_{\mathcal{I}} = \{\{a, b\}, C^{2i}\} \cup (\{\{a\}\} \vee \binom{C^{2i}}{i}) \cup (\{\{b\}\} \vee \binom{C^{2i}}{i})$. Let $\mathcal{O}^{2i} = \mathcal{I}^{2i}$ and $\mathcal{J}^{2i} := \mathcal{I}^{2i} \setminus \{\{a, b\}\}$. An analogous argument as in Example 4 shows that $q(\mathcal{I}^{2i}) = \frac{2}{2i}$ and $\omega(\mathcal{I}^{2i}, \mathcal{I}^{2i}, \mathcal{J}^{2i}) = \frac{1}{2}$ and therefore $\lim_{i \to \infty} \frac{\omega(\mathcal{I}^{2i}, \mathcal{I}^{2i}, \mathcal{J}^{2i})}{q(\mathcal{I}^{2i})} = \lim_{i \to \infty} \frac{i}{2} \to \infty$.

3 | COMPARISON TO INNER MATROID APPROXIMATION

To compare our result with the inner matroid approximation proposed by Milgrom (2017) and de Vries and Vohra (2020), we recall some definitions from matroid theory:

Definition. An independence systems $(E, \mathcal{I})$ can also be characterized via a **rank function**: Let $r : 2^E \to \mathbb{N}_0$ be such that

(R1) $r(\emptyset) = 0$,

(R2) $r(S) \leq r(S \cup \{i\}) \leq r(S) + 1$ for all $S \subset E, i \in E \setminus S$,

then, the pair $(E, \{I \subseteq E : r(I) = |I|\})$ is an independence system. If not clear from context, we denote the rank function of the independence system $(E, \mathcal{I})$ by $r_{\mathcal{I}}$. An independence system $(E, \mathcal{I})$ is called a **matroid** if $q(\mathcal{I}) = 1$. We present several equivalent characterizations of matroids used later. A matroid is an independence system $(E, \mathcal{M})$ with rank function r for which one of the following holds:

(Submodularity) $r(S) + r(S \cup \{i,j\}) \leq r(S \cup \{i\}) + r(S \cup \{j\})$ for all $i, j \in E, S \subseteq E \setminus \{i,j\}$.

(Basis Exchange) For every pair of bases $B_1, B_2 \in \mathcal{B}_{\mathcal{I}}$ and $i \in B_1 \setminus B_2$ there exists a $j \in B_2 \setminus B_1$ such that $(B_1 \cup \{j\}) \setminus \{i\}$ is a basis.

A standard example of a matroid is the **uniform matroid** U_n^k, which consists of all subsets of an n-element set of cardinality at most k, where $0 \leq k \leq n$.

We compare our result with the inner matroid approximation proposed by Milgrom (2017) and de Vries and Vohra (2020). Milgrom (2017) considers an **inner matroid** $(E, \mathcal{M})$ of the independence system $(E, \mathcal{I})$ with $\mathcal{M} \subseteq \mathcal{I}$ and proposes the greedy solution to $(E, \mathcal{M})$ be used as a solution to the problem of finding a maximum value basis in the *downward closed* acceptable set $\mathcal{O} \subseteq \mathcal{I}$. He calls the term $\min_{S \in \mathcal{O} \setminus \{\emptyset\}} \frac{r_{\mathcal{M}}(S)}{|S|} =: \rho(\mathcal{I}, \mathcal{O}, \mathcal{M})$ the **substitutability index** of $\mathcal{M}$ for $\mathcal{I}$ with respect to the acceptable set $\mathcal{O} \subseteq \mathcal{I}$. Note that $\min_{S \in \mathcal{O} \setminus \{\emptyset\}} \frac{r_{\mathcal{M}}(S)}{|S|} = \min_{S \in \mathcal{O} \setminus \{\emptyset\}} \max_{M \in \mathcal{M} : M \subseteq S} \frac{|M|}{|S|}$. Consequently, he defines the **best inner matroid** by

$$(E, \mathcal{M}^*) \in \operatorname*{argmax}_{\mathcal{M} \subseteq \mathcal{I} : (E, \mathcal{M}) \text{ is matroid}} \min_{S \in \mathcal{O}^{\subseteq} \setminus \{\emptyset\}} \max_{M \in \mathcal{M} : M \subseteq S} \frac{|M|}{|S|}.$$

We show that the substitutability index is unsuitable when applied to an inner independence system instead of inner matroids.

Example 6. Let $N := \{1,2,3\}$ be an independence system $(E, \mathcal{I})$ whose bases are $\mathcal{B}_{\mathcal{I}} := \{\{1,2\}, \{3\}\}$, thus $q(\mathcal{I}) = \frac{1}{2}$. Let the inner independence system $(E, \mathcal{J})$ coincide with $(E, \mathcal{I})$, and the acceptable set is $\mathcal{O} = \mathcal{I}$. Clearly, $\rho(\mathcal{I}, \mathcal{I}, \mathcal{J}) = 1$. The greedy algorithm, however, may fail to find an optimal basis. Assume $v_1 = v_2 = 1, v_3 = 1 + \varepsilon$. Clearly, $V^*(\mathcal{J}, v) = 2 = V^*(\mathcal{O}, v)$. The greedy algorithm carried out on $\mathcal{J}$ gets stuck with the low rank basis $\{3\}$ with $V^{greedy}(E, \mathcal{J}) = 1 + \varepsilon$ and therefore $\frac{V^{greedy}(E, \mathcal{J})}{V^*(\mathcal{I}, v)} = \frac{1+\varepsilon}{2}$. Hence, the greedy algorithm performs worse than $\rho(\mathcal{I}, \mathcal{I}, \mathcal{J})$ suggests.

However, the substitutability index of $\mathcal{M}$ coincides with the generalized rank-quotient $(\mathcal{I}, \mathcal{O}, \mathcal{M})$ in the case of $(E, \mathcal{M})$ being an inner matroid and acceptable set $\mathcal{O}$ being the independence set of a normal independence system $(E, \mathcal{I})$.

Theorem 7. *Let $(E, \mathcal{M})$ an inner matroid of the independence system $(E, \mathcal{I})$ and $\mathcal{O} \subseteq \mathcal{I}$ an acceptable set such that $(E, \mathcal{O})$ is a normal independence system. Then, it holds that $\omega(\mathcal{I}, \mathcal{O}, \mathcal{M}) = \rho(\mathcal{I}, \mathcal{O}, \mathcal{M})$.*

The proof of Theorem 7 can be found in Appendix A.1.

As reported in Milgrom (2017) and de Vries and Vohra (2020), the substitutability index can be used to bound the approximation quality of an inner matroid $(E, \mathcal{M})$ if $\mathcal{O}$ is a normal independence system. It follows directly from Theorems 2 and 7.

Corollary 8 (de Vries & Vohra, 2020; Milgrom, 2017). *Let $(E, \mathcal{M})$ be an inner matroid of the independence system $(E, \mathcal{I})$. If $\mathcal{O}$ is a normal independence system, then*

$$\min_{v \in \mathbb{R}_+^E \setminus \{\mathbf{0}\}} \frac{V^*(\mathcal{M}, v)}{V^*(\mathcal{O}, v)} = \min_{S \in \mathcal{O} \setminus \{\emptyset\}} \frac{r_{\mathcal{M}}(S)}{|S|}.$$

Remark 9. Note that in de Vries and Vohra (2020), the condition that $\mathcal{O}$ is an independence system is wrongly omitted. For an example, see Remark 22 in the appendix.

It is natural to ask if the inner independence system approximation outperforms the best inner matroid approximation. This is not the case in Example 4, as the following shows.

Example 10. Let $(E, \mathcal{I})$ be defined as in Example 4 and $(E, \mathcal{M})$ be defined by its bases $\mathcal{B}_{\mathcal{M}} := \binom{C}{6} \cup (\{\{a\}\} \vee \binom{C}{5}) \cup (\{\{b\}\} \vee \binom{C}{5})$. Verification of the basis exchange axiom (see Appendix A.1) would confirm that $(E, \mathcal{M})$ is a matroid. An analogous argument, as in Example 4, yields

$$\omega(\mathcal{I}, \mathcal{I}, \mathcal{M}) = \frac{1}{2}.$$

Despite this, there exist examples, even in the zero prior knowledge case $\mathcal{O} = \mathcal{I}$, where the approximation guarantee of the *best* inner independence system strictly dominates the approximation guarantee of the best inner matroid. First, we provide some intuition for why the later inequality might occur before giving Example 11.

Suppose for some independence system $(E, \mathcal{I})$ we know a nonmatroidal best approximating independence system $(E, \mathcal{J})$. Assume now that we want to construct an inner matroid $(E, \mathcal{M})$ contained in $(E, \mathcal{J})$ with the same approximation guarantee. Recall that for any pair of rank function $r_{\mathcal{M}}$ and lower rank function $l_{\mathcal{M}}$ of a matroid, it has to be that $l_{\mathcal{M}}(S) = r_{\mathcal{M}}(S)$ for every $S \subseteq E$. As $(E, \mathcal{I})$ is nonmatroidal, there exists $S^* \subseteq E$ such that $l_{\mathcal{J}}(S^*) < r_{\mathcal{J}}(S^*)$ with F

lower rank basis of S^* and G upper rank basis of S^*. Therefore, to construct an inner matroid from $(E, \mathcal{J})$ one either has to reduce the rank of F to eliminate the low-rank basis F and make the newly independent subsets of F contained in a higher rank basis of S^* or one has to reduce the rank of G to obtain $r_{\mathcal{M}}(S^*) = r_{\mathcal{M}}(F) = r_{\mathcal{M}}(G)$. Note that both ideas can be incorporated by adding circuits to $\mathcal{J}$. If we follow the idea to reduce the rank of F it might occur that $\frac{r'_{\mathcal{M}}(F')}{r_{\mathcal{O}\subseteq}(F')} < \min_{S\subseteq E} \frac{l_{\mathcal{J}}(S)}{r_{\mathcal{O}\subseteq}(S)}$ for some $F' \subseteq F$. Conversely, in the case that we reduce the rank of G it could happen that we implicitly reduce the rank of some $A \supset G$ such that $\frac{r'_{\mathcal{M}}(A)}{r_{\mathcal{O}\subseteq}(A)} < \min_{S\subseteq E} \frac{l_{\mathcal{J}}(S)}{r_{\mathcal{O}\subseteq}(S)}$.

This explanation tacitly assumes that the best inner matroid is contained in the best inner independence system. This assumption need not be true because the best inner matroid could be a *superset* of *some* best inner independence system. To see this, consider the independence system $(E, \mathcal{I})$ and the best inner matroid $(E, \mathcal{M})$ defined as in Example 10. The inner independence system $(E, \mathcal{J})$ given via its bases $\mathcal{B}_{\mathcal{J}} := \mathcal{B}_{\mathcal{M}} \setminus \{\{c_1, c_2, c_3, c_4, c_5, c_6\}\}$ is a best inner independence system but $\mathcal{M} \supset \mathcal{J}$.

In the following example, we exhibit an independence system $\mathcal{I}$, which is a best inner independence system that contains a best inner matroid with a worse generalized rank quotient.

Example 11. Let $E := \{1, 2, 3, 4, 5\}$ and the independence system $(E, \mathcal{I})$ be defined via its bases $\mathcal{B}_{\mathcal{I}} := \{\{1, 2, 3\}, \{1, 4\}, \{2, 5\}\}$. We verify (see Appendix A.2) that $q(\mathcal{I}) = \frac{1}{2}$ and $\omega(\mathcal{I}, \mathcal{I}, \mathcal{M}) = \frac{1}{3}$ for the best inner matroid $(E, \mathcal{M})$ with $\mathcal{M} = U_5^1$.

Definition. An independence system $(E, \mathcal{I})$ arises from a knapsack problem if there exists $W \in \mathbb{R}_+$ and $w : E \to \mathbb{R}_+$ such that $\mathcal{I} = \{I \subseteq E : \sum_{x\in I} w(x) \leq W\}$.

We remark that the independence system in Example 11, unlike that in Example 4, cannot arise from a knapsack problem.[1] In a knapsack instance, either $\{1, 5\} \in \mathcal{I}$ or $\{2, 4\} \in \mathcal{I}$ would be necessary, since for the weight function w it has to be that $w(1) \leq w(2)$ or $w(2) \leq w(1)$. Therefore, we conjecture a connection between knapsack-like independence systems and the equivalence in the generalized rank-quotient of inner matroids and inner independence systems.

Conjecture 1. For every knapsack instance the underlying independence system $(E, \mathcal{I})$ fulfills

$$\max_{\mathcal{J}\subseteq\mathcal{I}:\ (E,\mathcal{J}) \text{ independence system}} \omega(\mathcal{I}, \mathcal{I}, \mathcal{J}) = \max_{\mathcal{M}\subseteq\mathcal{I}:\ (E,\mathcal{M}) \text{ matroid}} \omega(\mathcal{I}, \mathcal{I}, \mathcal{M}).$$

The converse of Conjecture 1 is false. To see this, consider the independence system $(E, \mathcal{I})$ on the ground set $E = \{1, 2, 3, 4\}$ given by its bases $\mathcal{B}_{\mathcal{I}} := \{\{1, 2\}, \{3, 4\}\}$. No knapsack instance can induce the independence system $(E, \mathcal{I})$: Assume w.l.o.g. that $w_1 \leq w_i$ for $i \in \{2, 3, 4\}$. Then, it has to hold that $\{1, 3\}, \{1, 4\} \in \mathcal{B}_{\mathcal{I}}$ since $w_1 + w_3 \leq w_3 + w_4$ and $w_1 + w_4 \leq w_3 + w_4$. Nevertheless, it holds that $\max_{\mathcal{J}\subseteq\mathcal{I}:(E,\mathcal{J}) \text{ is independence system}} \omega(\mathcal{I}, \mathcal{I}, \mathcal{J}) = q(\mathcal{I}) = \frac{1}{2} = \max_{\mathcal{M}\subseteq\mathcal{I}:(E,\mathcal{M}) \text{ is matroid}} \omega(\mathcal{I}, \mathcal{I}, \mathcal{M}) = \omega(\mathcal{I}, \mathcal{I}, U_4^1)$.

3.1 | An example demonstrating superiority of inner independence system approximation

We have seen in Example 11 that the best inner independence system may outperform the best inner matroid. However, the approximation guarantee of the best inner independence system does not improve over the standard greedy algorithm. Example 4 and Remark 10 show that the best inner independence system may have the same approximation guarantee as the best inner matroid but outperforms the standard greedy algorithm. To provide an example where the best inner independence system outperforms not only the best inner matroid but the standard greedy algorithm as well, we need the following Lemma

Lemma 12. *Let $(E_1, \mathcal{I}_1)$ and $(E_2, \mathcal{I}_2)$ be independence systems with $E_1 \cap E_2 = \emptyset$ and $(E, \mathcal{I}) := (E_1 \dot\cup E_2, \mathcal{I}_1 \vee \mathcal{I}_2)$, where $E_1 \dot\cup E_2$ denotes the disjoint union of E_1 and E_2. Furthermore, let $(E_1, \mathcal{J}_1)$ and $(E_2, \mathcal{J}_2)$ be inner independence systems (matroids) and $\mathcal{O}_1, \mathcal{O}_2$ acceptable sets.*

Then, it holds that $q(\mathcal{I}) = \min\{q(\mathcal{I}_1), q(\mathcal{I}_2)\}$, and for the inner independence system (matroid) $(E, \mathcal{J}) := (E_1 \dot\cup E_2, \mathcal{J}_1 \vee \mathcal{J}_2\}$ and acceptable set $\mathcal{O} := \mathcal{O}_1 \dot\cup \mathcal{O}_2$ we have $\omega(\mathcal{I}, \mathcal{O}, \mathcal{J}) = \min\{\omega(\mathcal{I}_1, \mathcal{O}_1, \mathcal{J}_1), \omega(\mathcal{I}_2, \mathcal{O}_2, \mathcal{J}_2)\}$.

We provide an independence system $(E, \mathcal{I})$, a best inner matroid $(E, \mathcal{M})$, and a best inner independence system $(E, \mathcal{J})$ for which hold $q(\mathcal{I}) < \omega(\mathcal{I}, \mathcal{I}, \mathcal{M}\} < \omega(\mathcal{I}, \mathcal{I}, \mathcal{J})$ by joining Examples 4 and 11.

Example 13. Let $(E, \mathcal{I})$ be the direct sum of the independence systems in Example 4 and Theorem 11, hence $E_1 = \{a, b, c_1, .., c_{10}\}$ with $(E_1, \mathcal{I}_1)$ given by the bases $\mathcal{B}_{\mathcal{I}_1} := \{\{a, b\}, C\} \cup (\{\{a\}\} \vee \binom{C}{5}) \cup (\{\{b\}\} \vee \binom{C}{5})$ with $C := \{c_1, \ldots, c_{10}\}$ and $\binom{C}{5} := \{S \subset C : |S| = 5\}$ and $E_2 = \{1, 2, 3, 4, 5\}$ with $(E_2, \mathcal{I}_2)$ defined by its bases $\mathcal{B}_{\mathcal{I}_2} := \{\{1, 2, 3\}, \{1, 4\}, \{2, 5\}\}$ and $(E, \mathcal{I}) := (E_1 \dot\cup E_2, \mathcal{I}_1 \vee \mathcal{I}_2)$, hence

$$\mathcal{B}_{\mathcal{I}} := \left(\{\{a, b\}, C\} \cup \left(\{\{a\}\} \vee \binom{C}{5}\right) \cup \left(\{\{b\}\} \vee \binom{C}{5}\right)\right) \vee \{\{1, 2, 3\}, \{1, 4\}, \{2, 5\}\}.$$

Then, it can be verified (see Appendix A.2) that

$$q(\mathcal{I}) < \omega(\mathcal{I}, \mathcal{I}, \mathcal{M}) < \omega(\mathcal{I}, \mathcal{I}, \mathcal{J}),$$

where $(E, \mathcal{J})$ and $(E, \mathcal{M})$ are the best inner independence system and best inner matroid, respectively.

We have demonstrated that an inner independence system can improve the performance guarantee of the greedy algorithm. Next, we extend our findings to a more general structure than independence systems.

4 | INNER APPROXIMATIONS TO PACKING INSTANCES

In the previous section, we approximated independence systems by inner independence systems. Here, we consider inner approximations of more general structures. Every independence system $(E, \mathcal{I})$ can be interpreted as a subset of $\{0,1\}^n$ by identifying independent sets $I \in \mathcal{I}$ with their characteristic vectors χ^I. We allow for multiplicities and generalize independence systems:

Definition. A finite set $D \subset \mathbb{Z}^n_{\geq 0}$ is called a **packing instance** if $x \in D$ implies $y \in D$ for all $y \in \mathbb{Z}^n_{\geq 0}$ such that $y \leq x$ (component wise). Consequently, $G \subseteq D$ is an **inner packing instance** of D if G is a packing instance. Call a packing instance **normal** if it contains all unit vectors.

Although one can define packing instances over arbitrary posets, here we focus on the special case of the poset $(\mathbb{Z}^n_{\geq 0}, \leq)$.

For each $i \in E := \{1, \dots, n\}$ let $f : \mathbb{Z}_{\geq 0} \to \mathbb{R}_+$ be a nondecreasing and discrete concave function (hence, $f_i(x+1) - f_i(x) \leq f_i(y+1) - f_i(y)$ if $y \leq x$). Then the function $f : \mathbb{Z}^n_{\geq 0} \to \mathbb{R}_+, f := \sum_{i=1}^n f_i$ is called **nondecreasing, separable and discrete concave**. We are interested in the problem $\max_{x \in D} f(x)$. Note, that this is more general than $\max_{x \in D} \sum_{i=1}^n x_i \cdot v_i$ with values v_i associated to $i \in E$.

Definition. For any packing instance $D \subset \mathbb{Z}^n_{\geq 0}$, an element $x \in D$ is called **maximal** if $x + e^i \notin D$ for all $i \in \{1, \dots, n\}$ where e^i denotes the i-th unit vector. Let D^+ denote the **set of maximal elements** of D and $r(D) := \max\{\sum_{i \in E} x_i : x \in D^+\}$ the **maximal height** and $l(D) := \min\{\sum_{i \in E} x_i : x \in D^+\}$ the **minimal height** of D.

In the following, assume D is normal and for $\alpha \in \mathbb{Z}^n_{\geq 0}$ define the **α-truncated** packing instance $D_\alpha := \{x \in D : x_i \leq \alpha_i \text{ for } i \in E\}$ and the **height-quotient of** D as

$$\rho(D) := \min_{\alpha \in \mathbb{Z}^n_{\geq 0} \setminus \{\mathbf{0}\}} \frac{l(D_\alpha)}{r(D_\alpha)}.$$

Clearly, $\rho(D) = \min_{\alpha \in \mathbb{Z}^n_{\geq 0} \setminus \{\mathbf{0}\} : \alpha \leq \beta} \frac{l(D_\alpha)}{r(D_\alpha)}$ with $\beta_i := \max_{x \in D} x_i$, hence $\rho(D)$ is well defined.The following notion is equivalent to Dunstan & Welsh (1973).

Algorithm 2. Greedy algorithm for integer packing instances

Input : A packing instance $D \subset \mathbb{Z}^n_{\geq 0}$ given by a membership oracle. Nondecreasing separable discrete concave function $f : \mathbb{Z}^n_{\geq 0} \to \mathbb{R}_+$

Output: A maximal element $x_g \in D^+$

```
Set x = 0
while x is not maximal (oracle call) do
    choose j ∈ arg max_{i∈E : x+e^i∈D (oracle call)} f(x + e^i)
    x = x + e^j
Set x_g = x and return
```

Definition. The packing instance D is called a **polymatroid** if $\rho(D) = 1$, and an **inner polymatroid** of a packing instance D is a polymatroid $P \subseteq D$.

Notice that this polymatroid definition coincides with the *integer-restriction* of the standard definition of an integer polymatroid, as for $D \subset \mathbb{Z}^n_{\geq 0}$ with $\rho(D) = 1$ the function $r_D : 2^{\{1,\dots,n\}} \to \mathbb{Z}_{\geq 0}, r_D(S) := \max_{x \in D} \sum_{i \in S} x_i$ turns out to be nondecreasing submodular and $D = \{x \in \mathbb{Z}^n_{\geq 0} : \sum_{i \in S} x_i \leq r_D(S) \text{ fo all } S \subseteq E\}$ (see e.g., Theorem 44.5, Schrijver, 2003). However, since we are only interested in subsets of $\mathbb{Z}^n_{\geq 0}$, for convenience, we regard polymatroids as consisting only of *integer-valued members*.

The solution of $\max_{x \in D} f(x)$ of a packing instance D can be approached by the greedy algorithm generalizing Algorithm 1 for independence systems, which yields an exact solution if D is a polymatroid.

Let $V^*(D,f) := \max_{x \in D} \sum_{i=1}^n f(x)$ and $V^{greedy}(D,f)$ be the worst possible (if tie-breaking occurs) solution obtained by Algorithm 2. Malinov and Kovalyov (1980) show the height-quotient $\rho(D)$ to be the worst-case approximation guarantee.

Theorem 14 (Malinov & Kovalyov, 1980). *Let $D \subset \mathbb{Z}^n_{\geq 0}$ be a normal packing instance and $f(x) := \sum_{i=1}^n f_i(x_i)$ with $f_i : \mathbb{Z}_{\geq 0} \to \mathbb{R}_+$ being nondecreasing and discrete concave. Then, $\frac{V^{greedy}(D,f)}{V^*(D,f)} \geq \rho(D)$. Furthermore, for any normal packing instance $D \subseteq \mathbb{Z}^n_{\geq 0}$, there exists a nondecreasing, separable, and discrete concave function f such that*

$$\frac{V^{greedy}(D,f)}{V^*(D,f)} = \rho(D). \tag{2}$$

We want to approximate packing instances by inner packing instances and inner polymatroids for the economy of exposition. We assume that the whole packing instance forms the acceptable set, so $\mathcal{O} = D$. Our results are easily generalized to other acceptable downward closed subsets of the packing instance D. To achieve a better performance than Equation (2), we first search for an inner packing instance G

contained in D, and then apply Algorithm 2 to G. We aim to find G such that the worst-case performance of Algorithm 2 on G is better than the worst-case performance on D.

Definition. For a normal packing instance $D \subset \mathbb{Z}^n_{\geq 0}$ and a packing instance $G \subseteq D$ we define the **generalized height-quotient** as

$$\min_{\alpha \in \mathbb{Z}^n_+ \setminus \{\mathbf{0}\}} \frac{l(G_\alpha)}{r(D_\alpha)} =: \omega(D, G).$$

Analogously to independence systems, for any packing instance $D \subset \mathbb{Z}^n_{\geq 0}$, we call any packing instance contained in

$$\operatorname*{argmax}_{G \subseteq D:\ G \text{ is packing instance}} \omega(D, G)$$

a **best inner packing instance,** and any polymatroid contained in

$$\operatorname*{argmax}_{G \subseteq D:\ G \text{ is polymatroid}} \omega(D, G)$$

a **best inner polymatroid**.

Remark 15. Consider a normal packing instance $D \subset \mathbb{Z}^n_{\geq 0}$. The unique maximal integer-valued nondecreasing submodular function r' weakly below r_D induces a best inner polymatroid.

However, there are other ways to construct best inner polymatroids.

The next result generalizes Theorem 2 and Equation (2).

Theorem 16. *Let $D, G \subset \mathbb{Z}^n_{\geq 0}$ be packing instances such that D is normal, $G \subseteq D$, and*

$$C := \{f : \mathbb{Z}^n_{\geq 0} \to \mathbb{R}_+ : f \text{ is nondecreasing, separable, and discrete concave}\}.$$

Then, holds

$$\min_{f \in C} \frac{V^{\text{greedy}}(G,f)}{V^*(D,f)} = \omega(D, G).$$

The proof of Theorem 16 is in Appendix A.1.

Note that Theorem 16 also implies the Theorems 14 and 2, and the classic result of Hausmann et al. (1980).

4.1 | An example demonstrating superiority of inner packing instance approximation

As seen in Example 13, there exist cases in which the inner independence system approximation simultaneously outperforms the inner matroid approximation and the greedy algorithm performed on the original independence system. For completeness, by combining Examples 1 and 11, we provide an additional, genuine packing instance, in which the inner packing instance approximation simultaneously outperforms the direct application of Algorithm 2 to the original packing instance and the inner polymatroid approximation. We make use of an analog to Lemma 12.

Lemma 17. *Let $E \subset \mathbb{Z}^n_{\geq 0}$ and $F \subset \mathbb{Z}^m_{\geq 0}$ be packing instances, and $H \subseteq E$ and $I \subseteq F$ inner packing instance and $D := E \times F$ and $G := H \times I$. Then, $\omega(D, G) = \min\{\omega(E, H), \omega(F, I)\}$.*

Proof. Let $\alpha^n \in \operatorname{argmin} \omega(E, H)$ and $\alpha^m \in \operatorname{argmin} \omega(F, I)$. Let w.l.o.g. $\omega(E, H) \leq \omega(F, I)$. For any $\beta := (\beta^n, \beta^m) \in \mathbb{Z}^n_{\geq 0} \times \mathbb{Z}^m_{\geq 0}$ it is $\frac{l_{(\beta^n,\beta^m)}(G)}{r_{(\beta^n,\beta^m)}(D)} = \frac{l_{\beta^n}(H)+l_{\beta^m}(I)}{r_{\beta^n}(E)+r_{\beta^m}(F)} \geq \frac{l_{\alpha^n}(H)+l_{\beta^m}(I)}{r_{\alpha^n}(E)+r_{\beta^m}(F)} \geq \frac{l_{\alpha^n}(H)+l_{\alpha^m}(I)}{r_{\alpha^n}(E)+r_{\alpha^m}(F)} \geq \frac{l_{\alpha^n}(H)}{r_{\alpha^n}(E)} = \omega(E, H)$. Obviously, $\omega(E, H) \geq \omega(D, G)$, therefore $\omega(E, H) = \omega(D, G)$ and this concludes the proof. ∎

Note that Lemma 17 also implies $\rho(D) = \min\{\rho(E), \rho(F)\}$ since $\rho(X) = \omega(X, X)$ for any packing instance X.

Example 18. Let $D \subset \mathbb{Z}^7_+$ be given by its set of maximal elements

$$D^+ := \{(10, 2), (0, 3)\} \times \{(1, 1, 1, 0, 0), (1, 0, 0, 1, 0), (0, 1, 0, 0, 1)\}.$$

Then, D is the cartesian product of the packing instances in Examples 1 and 11. It follows by Lemma 17, Example 1, and Theorem 11 that $P^+ := \{(10, 2)\} \times \{x \in \mathbb{R}^5 : x = e^i \text{ for } 1 \leq i \leq 5\}$ is the set of maximal elements of a best inner polymatroid, $G^+ := \{(10, 2)\} \times \{(1, 1, 1, 0, 0), (1, 0, 0, 1, 0), (0, 1, 0, 0, 1)\}$ is the set of maximal elements of a best inner packing instance and

$$\begin{aligned} \rho(D) &\leq \frac{1}{4} \\ &< \max_{P \subseteq D:\, P \text{is polymatroid}} \omega(D, P) = \frac{1}{3} \\ &< \max_{G \subseteq D:\, G \text{is packinginstance}} \omega(D, G) = \frac{1}{2}. \end{aligned}$$

5 | APPLICATIONS

5.1 | Applications in auctions

We give an application of the inner approximation concept to the recent incentive auction run by the FCC (see Milgrom & Segal, 2020). The auction was conducted to reallocate radio spectrum for wireless broadband services. In the procurement phase, spectrum licenses were acquired by the FCC from the current spectrum holders of channels 38–51 through a descending auction. Any spectrum licenses not acquired were reassigned to other channels. The FCC purchased licenses that were not reassigned.

The problem of deciding which license holders to reassign is an inner packing instance. Roughly speaking, holders who demanded a high price would be candidates for reassignment. However, the set of holders who are reassigned needs to satisfy various packing constraints. Hence, a maximum-weight set of licenses that satisfied all the packing constraints would

be reassigned. Under the assumption that bidders bid truthfully, the descending auction implemented by the FCC used a greedy algorithm for selecting which subset of license holders to reassign, that is, Algorithm 2 applies.

Assume each holder owns a single license, and there are two types of spectrum holders. In fact, 11 of type A and three of type B. The dots in Figure 3 represent all feasible combinations of the two types of holders that can be reassigned. For example, it is feasible to reassign all type A holders. It is also feasible to reassign all type B holders. It is feasible to reassign 4 of the type A holders *and* 2 of the type B holders.

Every type A holder values their license at \$1.1, and each type B holder values their license at \$1. The greedy algorithm would result in the FCC reassigning all type B holders and purchasing all licenses from type A holders and no licenses from type B holders for a total expenditure of \$12.1. The total value of the reassigned spectrum would be \$3.3. Instead, suppose the set of feasible combinations of license holders that can be reassigned is artificially *restricted* by the FCC as shown in Figure 4. This set corresponds to an inner packing instance for Figure 3.

If we apply the greedy algorithm with the same valuations to this instance, ten type A holders and two type B holders would be reassigned. The FCC would purchase one license from type A and one from type B holders for a total expenditure of \$2.1, lower than before. The total value of the reassigned spectrum would be \$12.2, higher than before.

The inner approximation could notably lower the FCC's expenditures needed to clear channels 38–51.

5.2 | Application to two-dimensional knapsack

Next, we present an application of the inner approximation framework to variations of the knapsack problem. Recall that in Example 4, the best inner independence system yields

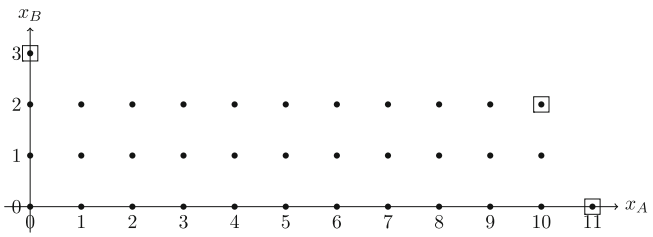

FIGURE 3 Set of feasible reassigned holders (dots) with bases boxed.

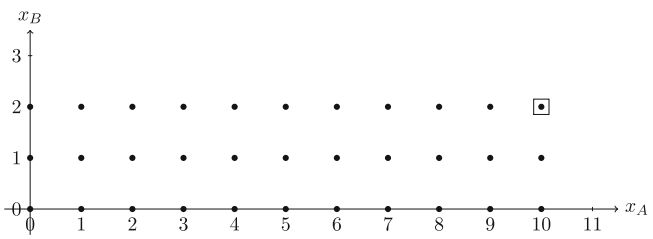

FIGURE 4 Restricted set of feasible reassigned holders (dots) with bases boxed.

a worst-case approximation guarantee of $\frac{1}{2}$, whereas the direct application of the greedy algorithm only guarantees $\frac{1}{5}$. However, as previously noted, Example 4 represents a knapsack problem. For knapsack problems, there is a classic algorithm that always guarantees worst-case approximation of $\frac{1}{2}$: Given a knapsack instance $(E, \mathcal{I})$ with capacity $W \in \mathbb{R}_+$ and item weights $w : E \to \mathbb{R}_+$ and values v_e for $e \in E$, let the elements in Line 1 of Algorithm 2 be ordered decreasingly according to the *ratio* $\frac{v_e}{w_e}$ and denote the worst-case solution of Algorithm 2 (if ties occur) by V^{ratio}. Further, let $V^{\text{best of}} := \max\{V^{\text{ratio}}(\mathcal{I}, v), \max_{e\in E} v_e\}$ be the best of the ratio-greedy solution and the maximum value item. Then, it is well-known (see e.g., Dantzig, 1957) that $\frac{V^{\text{best of}}}{V^*(\mathcal{I},v)} \geq \frac{1}{2}$. Thus, in particular, for the knapsack problem in Example 4, an inner approximation is not stronger than this classic algorithm. Therefore, Example 4 fails to demonstrate the superiority of the inner approximation framework in knapsack-induced independence systems over that classic algorithm.

However, the situation looks quite different if more than one knapsack constraint induces the independence system. Then, there does not exist an analog to the aforementioned classic algorithm that yields a constant factor approximation (see e.g., Kellerer et al., 2013).

We examine packing instances generated by two knapsack constraints, each associated with three types of items: one restricted by the first knapsack constraint, another by the second knapsack constraint, and a third constrained by both. In this scenario, we demonstrate in Example 19 that inner approximation might outperform the basic greedy algorithm, which might be perceived as the intuitive choice in such a setting. We introduce a two-person game where the inner approximations surpass the greedy algorithm.

Example 19. Consider a class of packing instances in $\mathbb{Z}^3_+$ given by the following two knapsack constraints:

$$k_1 \cdot x_1 + 0 \cdot x_2 + 10 \cdot x_3 \leq 100, \tag{3}$$

$$\begin{aligned} &0 \cdot x_1 + k_2 \cdot x_2 + 10 \cdot x_3 \leq 100, \\ &x_i \in \mathbb{Z}_{\geq 0}, \quad i \in \{1, 2, 3\}, \end{aligned} \tag{4}$$

with $k := (k_1, k_2), k_1, k_2 \in [10, 20] \cap \mathbb{Z}$. For such k, let D^k denote the associated packing instance. Then, in Appendix A.2 we demonstrate that for any $k \in ([10, 20] \cap \mathbb{Z})^2$ there exists $G^k \subsetneq D_k$ such that

$$\omega(D^k, G^k) \geq \rho(D^k),$$

and for a majority of possible choices of k this inequality is strict.

In most of the packing instances considered in Example 19, the worst-case approximation guarantee of the greedy algorithm applied to a well-chosen inner independence system improves over the direct application of the greedy

algorithm. However, there still exist (only a few) instances in which inner approximation only matches the worst-case approximation guarantee of the greedy algorithm. Next, we provide a more generic application of the inner independence system approach in which inner approximation dominates the direct application of the greedy algorithm.

Consider the following two-person game: A set of normal packing instances of the same dimension is revealed to the first player. The first player has to propose a simple generic algorithm to determine a maximal element of any packing instance of this set. Afterward, the second player chooses an instance with the worst possible approximation ratio obtained by the algorithm from the family of packing instances. Thus, the goal of the first player is to maximize his worst-case approximation guarantee over the worst instance of the given set of packing instances. In this setting, we show that inner approximation outperforms the standard greedy algorithm for the set of instances of Example 19. Notice that our game resembles the min-max regret knapsack problem introduced by Furini et al. (2015), in which the known knapsack constraints are fixed, but the coefficients of the linear objective function are variable. Further, a similar game to ours was considered in the context of approximating perfect matchings in bipartite graphs by Eden et al. (2022).

Example 20. Consider the set of packing instances given in Example 19, denote $K := ([10,20] \cap \mathbb{Z}) \times ([10,20] \cap \mathbb{Z})$, and recall that

$$\min_{k \in K} \rho(D^k) = \rho(D^{(20,20)}) = \frac{l(D^{(20,20)}_{10,0,10})}{r(D^{(20,20)}_{10,0,10})} = \frac{\|(5,0,0)\|_1}{\|(0,0,10)\|_1} = \frac{1}{2}.$$

Now, for $k \in K$ define G^k by adding the constraints $x_1 \leq \lfloor \frac{100}{k_1} \rfloor - 2$, $x_2 \leq \lfloor \frac{100}{k_2} \rfloor - 2$ and $x_3 \leq 7$ to D^k. Then, it holds by Example 19 that

$$\min_{k \in K} \omega(D^k, G^k) = \omega(D^{(20,20)}, G^{(20,20)}) = \frac{l(D^{(20,20)}_{10,0,0})}{r(D^{(20,20)}_{10,0,0})} = \frac{3}{5} > \frac{1}{2}.$$

and thus the worst-case approximation guarantee of inner approximation over $\bigcup_{k_1,k_2 \in [10,20] \cap \mathbb{Z}} D^{(k_1,k_2)}$ is strictly better than the approximation guarantee of the greedy algorithm.

One might think that the superiority of inner approximation in Example 20 is due to the zero coefficients in the knapsack constraints. However, increasing the class of packing instances by allowing the first type of items to be also constrained by the second constraint and the second type of items to be also constrained by the first constraint, hence, allowing the zeros in the Inequalities 3 and 4 to be small natural numbers, actually increases the power of the inner approximation, as shown in the following example.

Example 21. Consider a set of packing instances in $\mathbb{Z}^3_+$ given by the following two knapsack constraints:

$$\begin{aligned} k_1 \cdot x_1 + l_1 \cdot x_2 + 10 \cdot x_3 &\leq 100, \\ k_2 \cdot x_1 + l_2 \cdot x_2 + 10 \cdot x_3 &\leq 100, \\ x_i \in \mathbb{Z}_{\geq 0}, \quad i &\in \{1,2,3\}, \end{aligned}$$

where $k_1, l_2 \in [10,20] \cap \mathbb{Z}$ and $k_2, l_1 \in [0,5] \cap \mathbb{Z}$. For $k = (k_1, k_2)$, $l = (l_1, l_2)$ let $D^{(k,l)}$ denote the associated packing instance. It holds that

$$\begin{aligned} &\min_{(k,l) \in (([10,20] \times [0,5]) \cap \mathbb{Z}^2) \times (([0,5] \times [10,20]) \cap \mathbb{Z}^2)} \rho(D^{(k,l)}) \\ &= \rho(D^{(20,1),(1,10)}) = \frac{l(D^{(20,1),(1,10)}_{10,10,0})}{r(D^{(20,0),(0,10)}_{10,10,10})} \\ &= \frac{\|(5,0,0)\|_1}{\|(4,9,0)\|_1} = \frac{5}{13}, \end{aligned}$$

and therefore, greedy performs even worse than in Example 20.

Now, for $(k,l) \in (([10,20] \times [0,5]) \cap \mathbb{Z}^2) \times (([0,5] \times [10,20]) \cap \mathbb{Z}^2)$ define $G^{(k,l)}$ by adding analogous constraints to $D^{(k,l)}$ as in Example 20, namely $x_1 \leq \left\lfloor \frac{100}{k_1} \right\rfloor - 2$, $x_2 \leq \left\lfloor \frac{100}{l_2} \right\rfloor - 2$, $x_3 \leq 7$. Then, it is

$$\begin{aligned} &\min_{(k,l) \in (([10,20] \times [0,5]) \cap \mathbb{Z}^2)} \omega(D^{(k,l)}, G^{(k,l)}) \\ &= \omega(D^{((20,0),(5,10))}, G^{((20,0),(5,10))}) \\ &= \frac{l(G^{((20,0),(5,10))}_{(10,0,0)})}{r(D^{((20,0),(5,10))}_{(10,0,0)})} = \frac{\|(3,0,0)\|}{\|(5,0,0)\|} \\ &= \frac{3}{5} > \frac{5}{13}. \end{aligned}$$

Therefore, relaxing the condition that the constraints (3) and (4) must have a zero-coefficient increases the worst-case superiority of the inner approximation approach.

Dobson (1982) proposes a variant of the greedy algorithm which selects elements based on the ratio $\frac{v_i}{\sum_{j \in \{1,2\}} w_{i,j}}$, where $w_{i,1}$ is the weight of item i in the first constraint and $w_{i,2}$ is the weight of item i in the second constraint. It is easy to see that inner approximation also outperforms this ratio-greedy in the Examples 20 and 21, since for $k_1 = k_2 = 20$ all items have the same weight and therefore the worst-case approximation guarantee of this ratio-greedy algorithm applied to $D^{(20,20)}$ coincides with the height-quotient $\rho(D^{(20,20)})$, hence, equals $\frac{1}{2} < \frac{3}{5}$.

6 | CONCLUSION

We studied the problem of maximizing a nondecreasing, separable, and discrete concave function over a packing instance, particularly the important special case of finding a maximum value basis of an independence system.

The concepts of inner independence systems and inner packing instances were introduced, and generalizations of the rank-quotient of independence systems and the height-quotient of packing instances were given to provide a bound on the approximation quality of the greedy algorithm applied to inner independence systems and inner packing instances compared to the optimal solution. We demonstrated that the generalized rank-quotient and the height-quotient provide tight bounds on the worst-case performance of our algorithms. Furthermore, we provided examples where our approach simultaneously outperforms the standard greedy algorithm for independence systems, as well as the method proposed by Milgrom (2017) and de Vries and Vohra (2020) of approximating independence systems by inner matroids, in terms of worst-case approximation guarantee. Similarly, we demonstrated that our algorithm may outperform the greedy algorithm when applied to maximize a nondecreasing, separable, and discrete concave function over a packing instance. Lastly, we presented genuine applications of the inner approximation framework.

ACKNOWLEDGMENTS

We thank Roberto Saitto, Paul Milgrom, the editors, and two anonymous referees for their helpful comments.

DATA AVAILABILITY STATEMENT

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

ENDNOTE

1 To see that the independence system in Example 4 is a knapsack problem set $v_{c_i} = 1$ for all $i \in \{1, \dots, 10\}$, $v_a = v_b = 5$ and the capacity to 10.

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How to cite this article: de Vries, S., Raach, S., & Vohra, R. V. (2025). On inner independence systems. *Naval Research Logistics (NRL)*, *72*(1), 133–147. https://doi.org/10.1002/nav.22210

APPENDIX A

A.1 | Omitted arguments

Proof of Theorem 7. Clearly, for $S \in \mathcal{O} \setminus \{\emptyset\}$ it holds that $\frac{l_{\mathcal{M}}(S)}{|r_{\mathcal{O}^{\subseteq}}(S)|} = \frac{r_{\mathcal{M}}(S)}{|S|}$ and therefore $\omega(\mathcal{I}, \mathcal{O}, \mathcal{M}) \geq \rho(\mathcal{I}, \mathcal{O}, \mathcal{M})$. Suppose $S' \in \arg\min_{S \sqsubseteq E} \frac{l_{\mathcal{M}}(S)}{r_{\mathcal{O}^{\subseteq}}(S')}$ but $S' \notin \mathcal{O}^{\subseteq}$. Let F be upper rank basis of S' for $\mathcal{O}^{\subseteq}$, then $F \subset S'$ and $r_{\mathcal{O}^{\subseteq}}(S') = r_{\mathcal{O}^{\subseteq}}(F)$. But since $(E, \mathcal{M})$ is a matroid it holds that $l_{\mathcal{M}}(F) = r_{\mathcal{M}}(F) \leq r_{\mathcal{M}}(S') = l_{\mathcal{M}}(S')$ and therefore $\frac{l_{\mathcal{M}}(F)}{r_{\mathcal{O}^{\subseteq}}(F)} \leq \frac{l_{\mathcal{M}}(S')}{r_{\mathcal{O}^{\subseteq}}(S')}$, hence $F \in \arg\min_{S \sqsubseteq E} \frac{l_{\mathcal{M}}(S)}{r_{\mathcal{O}^{\subseteq}}(S)}$ and $\omega(\mathcal{I}, \mathcal{O}, \mathcal{M}) \leq \rho(\mathcal{I}, \mathcal{O}, \mathcal{M})$, and the claim follows. ∎

Remark 22. The following explains why as in Remark 9, the condition that $\mathcal{O}$ is an independence system is wrongly omitted in de Vries and Vohra (2020). Let $E = \{1, 2, 3, 4, 5\}$ and $\mathcal{I} = 2^{\{1,2,3\}} \cup 2^{\{1,4,5\}}$ and the bases of the inner matroid be given by $\mathcal{B}_{\mathcal{M}} = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}\}$ and $\mathcal{O} = \{\{1, 2, 3\}, \{1, 4, 5\}\}$. It is easy to see that $\min_{S \in \mathcal{O} \setminus \{\emptyset\}} \frac{r_{\mathcal{M}}(S)}{|S|} = \frac{2}{3}$. However, setting $v_1 = v_2 = v_3 = 0$ and $v_4 = v_5 = \frac{1}{2}$ yields

$V^*(\mathcal{M}, v) = \frac{1}{2}$ and $V^*(\mathcal{O}, v) = 1$, and therefore $\min_{v \in \mathbb{R}^E_+ \setminus \{0\}} \frac{V^*(\mathcal{M},v)}{V^*(\mathcal{O},v)} \leq \frac{1}{2} \overset{!}{<} \frac{2}{3}$.

Proof of Lemma 12. Let $S \in \arg\min_{T \subseteq E} \frac{l_I(T)}{r_I(T)}$ with $S_1 = S \cap E_1, S_2 = S \cap E_2$, and w.l.o.g. let $q(\mathcal{I}_1) \leq q(\mathcal{I}_2)$. It holds that

$$\begin{aligned} q(\mathcal{I}) = \frac{l_I(S)}{r_I(S)} &= \frac{l_{I_1}(S_1)+l_{I_2}(S_2)}{r_{I_1}(S_1)+r_{I_2}(S_2)} \\ &\geq \min\left\{ \frac{l_{I_1}(S_1)}{r_{I_1}(S_1)}, \frac{l_{I_2}(S_2)}{r_{I_2}(S_2)} \right\} \\ &\geq \min\{q(\mathcal{I}_1), q(\mathcal{I}_2)\}, \end{aligned}$$

and since clearly $q(\mathcal{I}) \leq \min\{q(\mathcal{I}_1), q(\mathcal{I}_2)\}$ it follows that $q(\mathcal{I}) = \min\{q(\mathcal{I}_1), q(\mathcal{I}_2)\}$.

Analogously, let $F \in \arg\min_{S \subseteq E} \frac{l_\mathcal{J}(S)}{r_{\mathcal{O}^\subseteq}(S)}$ and $T \in \arg\max\{|O| : O \subseteq F, O \in \mathcal{O}^\subseteq\}$ with $F_1 = F \cap E_1, F_2 = F \cap E_2, T_1 = T \cap E_1$ and $T_2 = T \cap E_2$, and w.l.o.g. $\omega(\mathcal{I}_1, \mathcal{O}_1, \mathcal{J}_1) \leq \omega(\mathcal{I}_2, \mathcal{O}_2, \mathcal{J}_2)$. It holds that

$$\begin{aligned} \omega(\mathcal{I}, \mathcal{O}, \mathcal{J}) = \frac{l_\mathcal{J}(F)}{|T|} &= \frac{l_{\mathcal{J}_1}(F_1) + l_{\mathcal{J}_2}(F_2)}{|T_1| + |T_2|} \\ &\geq \min\left\{ \frac{l_{\mathcal{J}_1}(F_1)}{|T_1|}, \frac{l_{\mathcal{J}_2}(F_2)}{|T_2|} \right\} \\ &\geq \min\{\omega(\mathcal{I}_1, \mathcal{O}_1, \mathcal{J}_1), \omega(\mathcal{I}_2, \mathcal{O}_2, \mathcal{J}_2)\}, \end{aligned}$$

and since trivially holds $\omega(\mathcal{I}, \mathcal{O}, \mathcal{J}) \leq \min\{\omega(\mathcal{I}_1, \mathcal{O}_1, \mathcal{J}_1), \omega(\mathcal{I}_2, \mathcal{O}_2, \mathcal{J}_2)\}$ the claim follows. ∎

Proof of Theorem 16. Let $\beta \in \arg\min_{\alpha \in \mathbb{Z}^n_{\geq 0} \setminus \{\mathbf{0}\}} \frac{l(G_\alpha)}{r(D_\alpha)}$ and $u \in \arg\max\{\|x\|_1 : x \in D^+_\beta\}$, hence $\|u\|_1 = r(D_\beta)$ and $v \in \arg\min\{\|x\|_1 : x \in G^+_\beta\}$, hence $\|v\|_1 = l(G_\beta)$.

Define $g_i(x_i) := \frac{1}{\|u\|_1} \cdot x_i$ for $x_i \leq \beta_i$ and $g_i(x_i) := \frac{\beta_i}{\|u\|_1}$ for $x_i > \beta_i$ and $g(x) := \sum_i^n g_i(x_i)$. Hence, g_i is a piecewise linear function in x_i that has slope $\frac{1}{\|u\|_1}$ between 0 and β_i and constant value $\frac{\beta_i}{\|u\|}$ starting from β_i. Observe that for $y \in D \setminus D_\beta$ holds $g(y) = \max_{x \leq y:\ x \in D^+_\beta} g(x)$, hence $\max_{x \in D}\{g(x)\} = \max_{x \in D_\beta}\{g(x)\}$. For every $x \in D_\beta$ it holds that $g(x) = \frac{\sum_{i=1}^n x_i}{\|u\|_1} \leq \frac{\sum_{i=1}^n u_i}{\|u\|_1}$. Hence, $V^*(D, g) = \sum_{i=1}^n g_i(u_i) = \sum_{i=1}^n \frac{u_i}{\|u\|_1} = 1$. The worst greedy solution that Algorithm 2 may produce on G is $V^{\text{greedy}}(G, g) = g(v) = \sum_{i=1}^n \frac{v_i}{\|u\|} = \frac{\|v\|_1}{\|u\|_1}$. In total, we get $\frac{V^{\text{greedy}}(G,g)}{V^*(D,g)} = \frac{\|v\|_1}{\|u\|_1}$ and therefore $\min_{f \in \mathcal{C}} \frac{V^{\text{greedy}}(G,f)}{V^*(D,f)} \leq \frac{V^{\text{greedy}}(G,g)}{V^*(D,g)} = \frac{\|v\|_1}{\|u\|_1} = \min_{\alpha \in \mathbb{Z}^n_{\geq 0} \setminus \{\mathbf{0}\}} \frac{l(G_\alpha)}{r(D_\alpha)}$.

Conversely, let $f \in \mathcal{C}$. Denote the marginal value of $f_i(\cdot)$ of j by $\Delta f_i(j) := f_i(j) - f_i(j-1)$ for $i \in E, 1 \leq j \leq k$ and totally order $F := \{(i,j) : 1 \leq i \leq n, 1 \leq j \leq k)\}$ such that (i,j) is ordered before (u,w) if $\Delta f_i(j) > \Delta f_u(w)$ and (i,j) is ordered before $(i,j+1)$ if $\Delta f_i(j) = \Delta f_i(j+1)$. Hence, F is ordered such that the higher the marginal value of an additional unit of i added to $(j-1)$, the earlier (i,j) occurs. Let $p : F \to \{1, \dots, n \cdot m\}$ be such that $p((i,j))$ is the position of (i,j) in the ordered sequence. For $1 \leq t \leq n \cdot m$ and $i \in E$ define $q_i(t) := |\{(i,j) : p((i,j)) \leq t\}|$, which can be interpreted as the total number of units of i among the t most valuable units, and set $q(t) := (q_1(t), \dots, q_n(t))$. Furthermore, for $x \in \mathbb{Z}^n_{\geq 0}$ we set $(q(t) \wedge x)_i := \min\{x_i, q_i(t)\}$, hence $(q(t) \wedge x)_i$ counts how many of the t best elements of F are among the x_i best of $i \in \{1, \dots, n\}$. Consequently, $q(t) \wedge x = (q^1(t) \wedge x, \dots, q^n(t) \wedge x)$ and $\|q(t) \wedge x\|_1 = \sum_{i=1}^n (q(t) \wedge x)_i$. For convenience, we define $p^{-1}(m \cdot n + 1) := 0$ and $h : F \to \mathbb{R}_+, h((i,j)) := \Delta f_i(j)$.Thus, for $x \in \mathbb{Z}^n_{\geq 0}$, we can rewrite analogous to Equation (1)

$$f(x) = \sum_{t=1}^{m \cdot n} (h(p^{-1}(t)) - h(p^{-1}(t+1))) \cdot \|q(t) \wedge x\|_1.$$

Let x^g be a worst-case greedy solution obtained by Algorithm 2 on G and x^* an optimal solution on D. Note that for $x \in D$ or $x \in G$, we can interpret $q(t) \wedge x$ as the quantity of i in x which also is contained in $D_{q(t)}$ and $G_{q(t)}$, respectively. Then, it holds that $q(t) \wedge x^g \in G_{q(t)}$ and $q(t) \wedge x^g + e^i \notin G_{q(t)}$ due to the definition of $q^g_i(t)$. Hence, $q(t) \wedge x^g \in G^+_{q(t)}$, which yields the simple inequality $l(G_{q(t)}) \leq \|q(t) \wedge x^g\|_1$. Analogously, it holds that $q(t) \wedge x^* \in D_{q(t)}$ and $q(t) \wedge x^* + e^i \notin D_{q(t)}$, therefore $q(t) \wedge x^* \in D^+_{q(t)}$, which yields $r(D_{q(t)}) \geq \|q(t) \wedge x^*\|_1$. We use these two inequalities to show

$$\begin{aligned} \frac{f(x^g)}{f(x^*)} &= \frac{\sum_{t=1}^{m \cdot n} (h(p^{-1}(t)) - h(p^{-1}(t+1))) \cdot \|q(t) \wedge x^g\|_1}{\sum_{t=1}^{m \cdot n} (h(p^{-1}(t)) - h(p^{-1}(t+1))) \cdot \|q(t) \wedge x^*\|_1} \\ &\geq \frac{\sum_{t=1}^{m \cdot n} (h(p^{-1}(t)) - h(p^{-1}(t+1))) \cdot l(G_{q(t)}) \cdot \frac{\|q(t) \wedge x^*\|_1}{r(D_{q(t)})}}{\sum_{t=1}^{m \cdot n} (h(p^{-1}(t)) - h(p^{-1}(t+1))) \cdot \|q(t) \wedge x^*\|_1} \\ &= \frac{\sum_{t=1}^{m \cdot n} (h(p^{-1}(t)) - h(p^{-1}(t+1))) \cdot \|q(t) \wedge x^*\|_1 \cdot \frac{l(G_{q(t)})}{r(D_{q(t)})}}{\sum_{t=1}^{m \cdot n} (p^{-1}(t)' - p^{-1}(t+1)') \cdot \|q(t) \wedge x^*\|_1} \\ &\geq \frac{\sum_{t=1}^{m \cdot n} (h(p^{-1}(t)) - h(p^{-1}(t+1))) \cdot \|q(t) \wedge x^*\|_1 \cdot \min_{1 \leq s \leq m \cdot n} \frac{l(G_{q(s)})}{r(D_{q(s)})}}{\sum_{t=1}^{m \cdot n} (h(p^{-1}(t)) - h(p^{-1}(t+1))) \cdot \|q(t) \wedge x^*\|_1} \end{aligned}$$

$$= \frac{\sum_{t=1}^{m\cdot n}(h(p^{-1}(t)) - h(p^{-1}(t+1))) \cdot \|q(t) \wedge x^*\|_1}{\sum_{t=1}^{m\cdot n}(h(p^{-1}(t)) - h(p^{-1}(t+1))) \cdot \|q(t) \wedge x^*\|_1} \cdot \min_{1\le s\le m\cdot n} \frac{l(G_{q(s)})}{r(D_{q(s)})}$$

$$= \min_{1\le s\le m\cdot n} \frac{l(G_{q(s)})}{r(D_{q(s)})} = \frac{l(G_\beta)}{r(D_\beta)}$$

and the claim follows. ∎

A.2 | Omitted calculations

Calculations in Example 4. It is

$$\mathcal{B}_\mathcal{J} = \{C\} \cup \left(\{\{a\}\} \vee \binom{C}{5}\right) \cup \left(\{\{b\}\} \vee \binom{C}{5}\right).$$

For $\{a,b\}$ it is $\frac{l_\mathcal{J}(\{a,b\})}{r_\mathcal{I}(\{a,b\})} = \frac{1}{2}$ since $\{a\}, \{b\} \in \mathcal{J} \not\ni \{a,b\}$. Now, let $S \subseteq E$, $S_1 = S \cap \{a,b\}$ and $S_2 = S \cap C$. If $S_1 = \emptyset$ then $\mathcal{B}_\mathcal{J}(S) = \{S_2\}$ and therefore $\frac{l_\mathcal{J}(S)}{r_\mathcal{I}(S)} = 1$. If $S_1 = \{a\}$ (note that $S_1 = \{b\}$ is treated analogously) and $|S_2| \le 5$ it is $\mathcal{B}_\mathcal{J}(S) = \{\{a\}\} \vee \{S_2\}$ and therefore $\frac{l_\mathcal{J}(S)}{r_\mathcal{I}(S)} = 1$ and if $S_1 = \{a\}$ and $|S_2| > 5$ it is $\mathcal{B}_\mathcal{J}(S) = (\{\{a\}\} \vee \binom{C}{5}) \cup \{S_2\}$ and therefore $\frac{l_\mathcal{J}(S)}{r_\mathcal{I}(S)} \ge \frac{5}{r_\mathcal{I}(S)} \ge \frac{5}{10} = \frac{1}{2}$. The case remains that $S_1 = \{a,b\}$. Then, if $|S_2| < 5$ it is $\mathcal{B}_\mathcal{J}(S) = (\{\{a\}\} \vee \{S_2\}) \cup (\{\{b\}\} \vee \{S_2\})$ and therefore $\frac{l_\mathcal{J}(S)}{r_\mathcal{I}(S)} \ge \frac{1+|S_2|}{2+|S_2|} \ge \frac{1}{2}$. If $|S_2| \ge 5$ it is $B_\mathcal{J}(S) = (\{\{a\}\} \vee \binom{C}{5}) \cup (\{\{b\}\} \vee \binom{C}{5}) \cup \{S_2\}$ and therefore $\frac{l_\mathcal{J}(S)}{r_\mathcal{I}(S)} \ge \frac{5}{10} = \frac{1}{2}$. In total, it follows

$$\omega(\mathcal{I},\mathcal{I},\mathcal{J}) = \frac{1}{2} > q(\mathcal{I}) = \frac{1}{5}.$$

It is evident that $(E,\mathcal{J})$ is a best inner independence system: For any inner independence system $(E,\mathcal{J}')$ with $\{a,b\} \in \mathcal{J}'$ holds $\omega(\mathcal{I},\mathcal{I},\mathcal{J}') \le \frac{l_{\mathcal{J}'}(E)}{r_\mathcal{I}(E)} = \frac{2}{10}$ and any independence system $\mathcal{J}' \subset \mathcal{J}$ cannot improve the approximation guarantee since $\omega(\mathcal{I},\mathcal{I},\mathcal{J}') \le \frac{l_{\mathcal{J}'}(\{a,b\})}{r_\mathcal{I}(\{a,b\})} \le \frac{1}{2}$.

Calculation that $(E,\mathcal{M})$ is a matroid in Example 10. Each of $\binom{C}{6}$, $\{\{a\}\} \vee \binom{C}{5}$ and $\{\{b\}\} \vee \binom{C}{5}$ is the set of bases of a matroid of rank 6. Therefore, it suffices to consider any pair of bases Y, Z that is not contained in the same matroid to prove the basis exchange axiom for matroids. Note that $\{\{a\}\} \vee \binom{C}{5}$ and $\{\{b\}\} \vee \binom{C}{5}$ are isomorphic. Let $Y \in \binom{C}{6}$ and $Z \in \{\{a\}\} \vee \binom{C}{5}$. Then, it holds that $a \in Z \setminus Y$ and for each $y \in Y \setminus Z$ it is $(Y \setminus \{y\}) \cup \{a\} \in \{\{a\}\} \vee \binom{C}{5}$. Conversely, for every $y \in Y \setminus Z$ it is $(Y \setminus \{a\}) \cup \{y\} \in \binom{C}{6}$ and for $z \in Z \setminus y$ and $y \in Y \setminus Z$ it is $(Z \setminus \{z\}) \cup \{y\} \in \{\{a\}\} \vee \binom{C}{5}$. It remains $Y \in \{\{a\}\} \vee \binom{C}{5}$ and $Z \in \{\{b\}\} \vee \binom{C}{5}$. Clearly, it holds that $(Y \setminus \{a\}) \cup \{b\} \in \{\{b\}\} \vee \binom{C}{5}$ and for $y \in Y \setminus Z$, $y \ne a$ and $z \in (Z \setminus \{b\}) \setminus (Y \setminus \{a\})$ it holds that $(Y \setminus \{y\}) \cup \{z\} \in \{\{a\}\} \vee \binom{C}{6}$.

Calculations in Example 11. Claim: $q(\mathcal{I}) = \frac{1}{2}$.

> *Proof of claim.* Since $\{l_\mathcal{I}(S) : S \sqsubseteq E\} = \{r_\mathcal{I}(S) : S \sqsubseteq E\} = \{1,2,3\}$ it has to hold $q(\mathcal{I}) \in \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}\right\}$. It is $q(\mathcal{I}) \le \frac{l(\{1,3,5\})}{r(\{1,3,5\})} = \frac{1}{2}$. Suppose there were $S \subseteq E$ with $r_\mathcal{I}(S) = 3$, then it has to hold $S \supseteq \{1,2,3\}$. It is $l_\mathcal{I}(\{1,2,3\}) = 3$ and $l_\mathcal{I}(\{1,2,3,4\}) = l_\mathcal{I}(\{1,2,3,5\}) = l_\mathcal{I}(\{1,2,3,4,5\}) = 2$. Therefore, there cannot exist $S \subseteq E$ with, $\frac{l(E)}{r(E)} = \frac{1}{3}$ and it follows $q(\mathcal{I}) = \frac{1}{2}$. ∎

Claim: $\omega(\mathcal{I},\mathcal{I},\mathcal{J}) = \frac{1}{2}$ for a best inner independence system $(E,\mathcal{J})$.

> *Proof of claim.* Since $\mathcal{I}$ is an independence system $\mathcal{J} := \mathcal{I}$ itself is an inner independence system with $\omega(\mathcal{I},\mathcal{I},\mathcal{J}) = \frac{1}{2}$. Assume a better approximation of inner independent system $(E,\mathcal{J}')$ exists. It either has to hold that the only basis of $\mathcal{J}'$ containing 4 is $\{4\}$ or $\{1,4\}$. Either way, it is $\frac{l_{\mathcal{J}'}(\{2,4,5\})}{\max\{|O| : O \subseteq \{2,4,5\}, O \in \mathcal{I}\}} \le \frac{1}{2}$ and therefore $\omega(\mathcal{I},\mathcal{I},\mathcal{J}') \le \frac{1}{2}$. ∎

Claim: $\omega(\mathcal{I},\mathcal{I},\mathcal{M}) = \frac{1}{3}$ for a best inner matroid $(E,\mathcal{M})$.

> *Proof of claim.* We argue that the uniform matroid U_5^1 is the best inner matroid. A best inner matroid $(E,\mathcal{M})$ has to be contained in $(E,\mathcal{I})$. Consider $S = \{1,4,5\}$ and note that $\{5\}$ is a low rank basis of E in $(E,\mathcal{I})$. In order to construct $r_\mathcal{M}$ out of $r_\mathcal{J}$, we cannot reduce the rank of $\{5\}$ since this would make $\{5\}$ inapproximable. Therefore, $r_\mathcal{M}(\{1,4,5\}) = 1$, hence $r_\mathcal{M}(\{1,4\}) = 1$. Since $\{1,4\} \notin \mathcal{M}$ it must be that $r_\mathcal{M}(\{4\}) = 1$ because setting $r_\mathcal{M}(\{4\}) = 0$ would would make $\{4\}$ inapproximable. Now, since $\{2,4\} \notin \mathcal{I}$ and $\{3,4\} \notin \mathcal{I}$ it follows that $r_\mathcal{M}(\{1,2,3,4\}) = r_\mathcal{M}(\{4\}) = 1$. Then, as $\{4,5\} \notin \mathcal{I}$ it has to hold that $\{4\}$ is a basis of $\{1,2,3,4,5\}$ from which follows $\mathcal{M} = U_5^1$. This yields $\omega(\mathcal{I},\mathcal{I},\mathcal{M}) = \frac{l_\mathcal{M}(\{1,2,3,4,5\})}{r_\mathcal{I}\{1,2,3,4,5\},} = \frac{l_\mathcal{M}(\{1\})}{r_\mathcal{I}(\{1,2,3\})} = \frac{1}{3}$. ∎

In sum, we can conclude

$$\max_{\mathcal{M}\subseteq\mathcal{I}:(E,\mathcal{M}) \text{ is matroid}} \omega(\mathcal{I},\mathcal{I},\mathcal{M}) = \frac{1}{3} <^{!} \frac{1}{2} = \max_{\mathcal{J}\subseteq\mathcal{I}:(E,\mathcal{J}) \text{ is indpendence system}} \omega(\mathcal{I},\mathcal{I},\mathcal{J}).$$

Calculations in Example 13. Let $\mathcal{B}_{\mathcal{J}_1} := \{C\} \cup (\{\{a\}\} \vee \binom{C}{5}) \cup (\{\{b\}\} \vee \binom{C}{5})$ and $\mathcal{B}_{\mathcal{J}_2} := \{\{1,2,3\},\{1,4\},\{2,5\}\}$ be the set of bases of a best inner independence systems regarding $(E, \mathcal{I}_1)$ and $(E, \mathcal{I}_2)$ (due to Example 4 and Theorem 11). Then, since E_1 and E_2 are disjoint, it holds that $\mathcal{B}_{\mathcal{J}} := \left(\{C\} \cup (\{\{a\}\} \vee \binom{C}{5}) \cup (\{\{b\}\} \vee \binom{C}{5})\right) \vee \{\{1,2,3\},\{1,4\},\{2,5\}\}$ is the set of bases of a best inner independence system regarding $(E, \mathcal{I})$ by Lemma 12. Analogously, $\mathcal{B}_{\mathcal{M}_1} := \binom{C}{6} \cup (\{\{a\}\} \vee \binom{C}{5}) \cup (\{\{b\}\} \vee \binom{C}{5})$ and $\mathcal{B}_{\mathcal{M}_2} := \mathcal{B}_{U_5^1}$ are best inner matroids regarding $(E, \mathcal{I}_1)$ respectively $(E, \mathcal{I}_2)$ due to Remark 10 and Theorem 11. Then, again by Lemma 12, it holds that $\mathcal{B}_{\mathcal{M}} := (\binom{C}{6} \cup (\{\{a\}\} \vee \binom{C}{5})) \cup (\{\{b\}\} \vee \binom{C}{5})) \vee \mathcal{B}_{U_5^1}$ is the set of bases of a best inner matroid regarding $(E, \mathcal{I})$. It follows directly from Lemma 12 that

$$q(\mathcal{I}) = \frac{1}{5} < \frac{1}{3} = \omega(\mathcal{I}, \mathcal{I}, \mathcal{M}\} < \frac{1}{2} = \omega(\mathcal{I}, \mathcal{I}, \mathcal{J}).$$

Calculations in Example 19. It holds that $\max_{x \in D^k} x_i \leq 10$ for $i \in \{1,2,3\}$. Let $\alpha_i \in [0,10] \cap \mathbb{Z}$, $i \in \{1,2,3\}$ and define $\beta_i := \min\{\alpha_i, \lfloor \frac{100}{k_1} \rfloor\}$ for $i \in \{1,2\}$. It holds that

$$\begin{aligned} l(D^k_{(\alpha_1,\alpha_2,\alpha_3)}) = \min \Bigg\{ & \alpha_3 + \min\left\{\alpha_1, \left\lfloor \frac{100 - 10 \cdot \alpha_3}{k_1} \right\rfloor\right\} \\ & + \min\left\{\alpha_2, \left\lfloor \frac{100 - 10 \cdot \alpha_3}{k_2} \right\rfloor\right\}, \\ & \beta_1 + \beta_2 + \min\left\{\alpha_3, \left\lfloor \frac{100 - \max\{\beta_1 \cdot k_1, \beta_2 \cdot k_2\}}{10} \right\rfloor\right\} \Bigg\} \end{aligned}$$

and

$$\begin{aligned} r(D^k_{(\alpha_1,\alpha_2,\alpha_3)}) = \max \Bigg\{ & \alpha_3 + \min\left\{\alpha_1, \left\lfloor \frac{100 - 10 \cdot \alpha_3}{k_1} \right\rfloor\right\} \\ & + \min\left\{\alpha_2, \left\lfloor \frac{100 - 10 \cdot \alpha_3}{k_2} \right\rfloor\right\}, \\ & \beta_1 + \beta_2 + \min\left\{\alpha_3, \left\lfloor \frac{100 - \max\{\beta_1 \cdot k_1, \beta_2 \cdot k_2\}}{10} \right\rfloor\right\} \Bigg\}. \end{aligned}$$

Due to symmetry, we can assume w.l.o.g. $k_1 \geq k_2$. Then, it is easy to see that $\frac{l(D^k_{(\alpha_1,\alpha_2,\alpha_3)})}{r(D^k_{(\alpha_1,\alpha_2,\alpha_3)})}$ is minimized either for $\alpha = (10,10,10)$ or for $\alpha = (10,0,10)$ (depending on k). (The ratio of lower and higher rank is minimized if either the lower rank is obtained by packing only items of type 3 into the knapsack and the higher rank is obtained by packing as many items as possible of type 1 and 2 into the knapsack, or the lower rank is obtained by excluding the items of type 2 and packing only a few items (possibly only 5) of type 1 into the knapsack and the higher rank is obtained by packing as many as possible items of type 3 (possibly up to 10) into the knapsack.) Therefore, it holds that

$$\begin{aligned} \rho(D^k) &= \min\left\{ \frac{l(D^k_{(10,10,10)})}{r(D^k_{(10,10,10)})}, \frac{l(D^k_{(10,0,10)})}{r(D^k_{(10,0,10)})} \right\} \\ &= \min\left\{ \frac{\|(0,0,10)\|_1}{\|\left(\left\lfloor \frac{100}{k_1} \right\rfloor, \left\lfloor \frac{100}{k_2} \right\rfloor, \min\left\{\left\lfloor \frac{100 - k_1 \cdot \left\lfloor \frac{100}{k_1} \right\rfloor}{10} \right\rfloor, \left\lfloor \frac{100 - k_2 \cdot \left\lfloor \frac{100}{k_2} \right\rfloor}{10} \right\rfloor\right\}\right)\|_1}, \frac{\|\left(\left\lfloor \frac{100}{k_1} \right\rfloor, 0, \left\lfloor \frac{100 - \left\lfloor \frac{100}{k_1} \right\rfloor}{10} \right\rfloor\right)\|_1}{\|(0,0,10)\|_1} \right\} \\ &= \min\left\{ \frac{10}{\left\lfloor \frac{100}{k_1} \right\rfloor + \left\lfloor \frac{100}{k_2} \right\rfloor + \min\left\{\left\lfloor \frac{100 - k_1 \cdot \left\lfloor \frac{100}{k_1} \right\rfloor}{10} \right\rfloor, \left\lfloor \frac{100 - k_2 \cdot \left\lfloor \frac{100}{k_2} \right\rfloor}{10} \right\rfloor\right\}}, \frac{\left\lfloor \frac{100}{k_1} \right\rfloor + \left\lfloor \frac{100 - \left\lfloor \frac{100}{k_1} \right\rfloor}{10} \right\rfloor}{10} \right\}. \end{aligned}$$

We distinguish the following cases:

(i) $k_1 \leq 12$,
(ii) $k_1 \in \{13,14\}$ and $k_2 \in \{10,11,12\}$,
(iii) $13 \leq k_2 \leq k_1 \leq 14$,
(iv) $k_1 \in \{16,19,20\}$,

(v) $k_1 = 15$ and $k_2 \in \{10, 11\}$,
(vi) $k_1 = 17$,
(vii) $(k_1, k_2) \in (\{15\} \times \{12, 13, 14, 15\}) \cup (\{18\} \times ([10, 20] \cap \mathbb{Z}))$.

In the cases (i) and (ii) consider the inner packing instance G^k defined by adding the constraint $x_3 \leq 7$ to D^k. It holds that

$$\omega(D^k, G^k) = \frac{l(G^k_{(10,10,10)})}{r(D^k_{(10,10,10)})} = \frac{\|\left(\left\lfloor \frac{30}{k_1} \right\rfloor, \left\lfloor \frac{30}{k_2} \right\rfloor, 7\right)\|_1}{\|\left(\left\lfloor \frac{100}{k_1} \right\rfloor, \left\lfloor \frac{100}{k_2} \right\rfloor, 0\right)\|_1} \geq \frac{\|(2,2,7)\|_1}{\left\lfloor \frac{100}{k_1} \right\rfloor + \left\lfloor \frac{100}{k_2} \right\rfloor} = \frac{11}{\left\lfloor \frac{100}{k_1} \right\rfloor + \left\lfloor \frac{100}{k_2} \right\rfloor}$$
$$> \frac{10}{\left\lfloor \frac{100}{k_1} \right\rfloor + \left\lfloor \frac{100}{k_2} \right\rfloor} = \frac{10}{\left\lfloor \frac{100}{k_1} \right\rfloor + \left\lfloor \frac{100}{k_2} \right\rfloor + \min\left\{ \left\lfloor \frac{100 - k_1 \cdot \left\lfloor \frac{100}{k_1} \right\rfloor}{10} \right\rfloor, \left\lfloor \frac{100 - k_2 \cdot \left\lfloor \frac{100}{k_2} \right\rfloor}{10} \right\rfloor \right\}} = \rho(D^k).$$

In cases (iii) and (iv), consider the inner packing instance G^k defined by adding the constraints $x_1 \leq \lfloor \frac{100}{k_1} \rfloor - 2$ and $x_2 \leq \lfloor \frac{100}{k_2} \rfloor - 2$. We have

$$\omega(D^k, G^k) = \frac{l(G^k_{(10,0,0)})}{r(D^k_{(10,0,0)})} \text{and} \rho(D^k) = \frac{l(D^k_{(10,0,10)})}{r(D^k_{(10,0,10)})},$$

and thus

$$\omega(D^k, G^k) = \frac{l(G^k_{(\left\lfloor \frac{100}{k_1} \right\rfloor - 2, 0, 0)})}{r(D^k_{(10,0,0)})} = \frac{\left\lfloor \frac{100}{k_1} \right\rfloor - 2}{\left\lfloor \frac{100}{k_1} \right\rfloor} > \frac{\left\lfloor \frac{100}{k_1} \right\rfloor}{10} = \rho(D^k).$$

In the case (v), define the inner packing instance G^k by adding the the constraints $x_1 \leq \left\lfloor \frac{100}{k_1} \right\rfloor - 1 = 5$, $x_2 \leq \left\lfloor \frac{100}{k_2} \right\rfloor - 2$ and $x_3 \leq 7$. It is

$$\rho(D^k) = \frac{l(D^k_{(10,10,10)})}{r(D^k_{(10,10,10)})} = \frac{\|(0,0,10)\|_1}{\|(6, \left\lfloor \frac{100}{k_2} \right\rfloor, 0)\|_1} = \frac{10}{6 + \left\lfloor \frac{100}{k_2} \right\rfloor} \leq \frac{10}{15} < \frac{7}{10} = \min\left\{ \frac{7}{10}, \frac{11}{15} \right\}$$
$$\leq \min\left\{ \frac{\|(5,0,2)\|_1}{\|(0,0,10)\|_1} \frac{\left\| \left(2, \left\lfloor \frac{30}{k_2} \right\rfloor, 7\right) \right\|_1}{\left\| \left(6, \left\lfloor \frac{100}{k_2} \right\rfloor, 0\right) \right\|_1} \right\} = \frac{l(D^k_{(0,0,10)})}{r(D^k_{(0,0,10)})} = \omega(D^k, G^k).$$

In the case (vi), define the inner packing instance G^k by adding the the constraints $x_1 \leq \left\lfloor \frac{100}{k_1} \right\rfloor - 1$ and $x_2 \leq \left\lfloor \frac{100}{k_2} \right\rfloor - 1$. It is

$$\rho(D^k) = \min\left\{ \frac{l(D^k_{(10,10,10)})}{r(D^k_{(10,10,10)})}, \frac{l(D^k_{(10,0,10)})}{r(D^k_{(10,0,10)})} \right\} = \min\left\{ \frac{\|(0,0,10)\|_1}{\left\| \left(5, \left\lceil \frac{100}{k_2} \right\rceil, \left\lceil \frac{100 - \left\lceil \frac{100}{k_2} \right\rceil \cdot k_2}{100} \right\rceil \right) \right\|_1}, \frac{\|(5,0,1)\|_1}{\|(0,0,10)\|_1} \right\} = \frac{6}{10}$$
$$< \min\left\{ \frac{10}{15}, \frac{7}{10} \right\} = \min\left\{ \frac{\|(0,0,10)\|_1}{\|(5,10,0)\|_1}, \frac{\|(4,0,3)\|_1}{\|(0,0,10)\|_1} \right\} \leq \min\left\{ \frac{l(G^k_{(5, \lfloor \frac{100}{k_2} \rfloor - 1, 10)})}{r(D^k_{(6, \lfloor \frac{100}{k_2} \rfloor, 10)})}, \frac{l(G^k_{(5,0,10)})}{r(D^k_{(6,0,10)})} \right\}$$
$$= \min\left\{ \frac{l(G^k_{(10,10,10)})}{r(D^k_{(10,10,10)})}, \frac{l(G^k_{(10,0,10)})}{r(D^k_{(10,0,10)})} \right\} = \omega(D^k, G^k).$$

In the case (vii), there exists no inner packing instance G^k for which the generalized height-quotient $\omega(D^k, G^k)$ is better than the height-quotient $\rho(D^k)$. However, there exists a proper inner packing instance for which the generalized height-quotient is not worse than the height-quotient, for example, adding $x_1 \leq \left\lfloor \frac{100}{k_1} \right\rfloor - 1$ to D^k yields one.