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Abstract

This paper provides a simple unified discrete-choice framework for analyzing differentiated duopolies. This framework nests models of horizontal and vertical differentiation, including standard textbook models (Hotelling and Shaked-Sutton). Contrary to these models, it also applies to economic environments where horizontal differentiation coincides with positive correlation of product valuations across consumers, and environments where vertical differentiation coincides with negative correlation. The paper provides an equilibrium characterization that is applicable independently of the type of differentiation and the sign of the valuation correlation.

JEL Codes: D43, L13

Keywords: Duopoly, Differentiated Products, Price Competition

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1 Introduction

Duopoly theory distinguishes between models of horizontal and vertical differentiation. Under horizontal differentiation, each of the two products faces positive demand at equal prices, whereas under vertical differentiation all consumers prefer the same product to the alternative. The simplest textbook models are the Hotelling (1929) model of horizontal differentiation and the Shaked and Sutton (1982) model of vertical differentiation. However, these models also differ from each other in another way: The Hotelling model corresponds to situations with strictly negative correlation between the values of the two products, whereas the correlation is strictly positive for the model of Shaked and Sutton.¹ This linkage between the type of product differentiation and the correlation of valuations appears to have received little attention. This most likely reflects the fact that, presumably for historical reasons, the models are usually presented in very different ways, impeding direct comparison.²

This note presents a simple discrete-choice differentiated duopoly framework that is not only rich enough to nest the two standard textbook models but, in addition, separates the type of differentiation from the type of correlation. In this framework, the *Unified Linear Model* (ULM) of product differentiation, the two firms both face a continuum of consumers. Each consumer is characterized by a two-dimensional valuation vector, with each component corresponding to the willingness-to-pay (WTP) for one of the two products. To maintain the simplicity of the standard textbook models, I focus on the case that (i) the support of the valuation distribution is a straight line in the two-dimensional valuation space, (ii) the distribution is uniform and (iii) marginal costs are constant. Though these assumptions are restrictive, they are compatible with several interesting cases. The standard Hotelling model with linear transportation costs corresponds to the case where the support of the valuation distribution is downward-sloping and intersects the 45-degree line in the space of valuation vectors (\mathcal{R}_+^2). In the Shaked-Sutton model, the support is an upward-sloping line which lies entirely on one side of the 45-degree line (where both valuations are equal). Importantly, however, the framework also applies to circumstances that are not commonly analyzed, even though they do not appear to be less relevant than the two standard textbook cases: Horizontal differentiation with positive correlation can be represented by a valuation distribution with an upward-sloping support that crosses the 45-degree line, while vertical differentiation with negative correlation requires a downward-sloping support that remains on one side of the 45-degree line.

The paper characterizes the unique pure-strategy equilibrium of the ULM for all four cases simultaneously, allowing for substantial cost and demand asymmetries. As is standard in the analysis of the underlying textbook models, parameter restrictions are necessary to guarantee existence of an equilibrium in which (i) both firms sell positive quantities and (ii) each consumer buys one of the two products. With these assumptions in place, one can calculate the equilibrium of the game as a function of four key parameters, namely the marginal costs of the two firms and the difference in the valuations between the two products for the two most extreme consumers.

¹The statement about the Hotelling model refers to the standard case where firms are located at the end of the interval where consumers are located (see Section 3.1)

²In typical textbook treatments such as Tirole (1989) and Belleflamme and Peitz (2010), the horizontal differentiation models of Hotelling (1929) and Salop (1979) are presented as locational models with consumers who face transportation costs, while the models of vertical differentiation are location-free.

In addition to this main result, the paper also shows that cost reductions of a firm and increases in the WTP for all its consumers unambiguously increase a firm's output and market share. For WTP increases that are higher for some consumers than for others, this statement only holds without qualification if the cost differences between the two firms are in an intermediate range.

The framework has the added benefit that it is straightforward to apply to two-stage models (for instance, models of advertising and innovation). The companion paper (Schmutzler, 2025) relies on the current model to analyze process, product and environmental innovation in a duopoly with a brown (polluting) and a green (less polluting) firm. Moreover, the framework is suitable for treating the mode of competition as endogenous: For instance, firms may engage in process or product innovations to turn horizontal differentiation into vertical differentiation. Similarly, under vertical differentiation, a laggard may want to invest to turn the mode of competition into horizontal differentiation. Most standard models of product differentiation do not allow for such considerations as they typically treat the mode of competition as exogenously fixed, either horizontal or vertical differentiation.

The model is related to Perloff and Salop (1985) who also consider the distribution of consumer valuations as a primitive of the model. Importantly, however, their model assumes independent distributions for different products, while my paper emphasizes the existence of correlations between the valuations for different products.

In Section 2, I introduce the model. Section 3 shows how the assumptions of the model can be derived from familiar textbook models or alternatively from a model where products differ in an objective and a subjective component. Section 4 contains the main results. Section 5 concludes. Proofs and technical details are in the appendix.

2 The Model

Firms $i \in \{A, B\}$ produce differentiated products, also denoted as i . Marginal costs are constant, given as $c_i \geq 0$. The firms simultaneously set prices p_i . Consumers decide which of the two products (if any) to buy. There is a unit mass of consumers who are distributed on the interval $[0, 1]$, with corresponding atomless probability measure μ . Unless stated otherwise, I assume that this distribution is uniform. Consumer $k \in [0, 1]$ values product i at $v_k^i \in \mathbb{R}_+$. If consumer k buys at all at prices (p_i, p_j) , he buys from firm i if $v_k^i - p_i > v_k^j - p_j$ for $j \neq i$. The following notation will be used throughout the paper:

$$\begin{aligned}\Delta_C &:= c_A - c_B \\ \Delta_p &:= p_A - p_B \\ \Delta_k &:= v_k^A - v_k^B\end{aligned}$$

Using the new notation and assuming full coverage, demand functions become³

$$\begin{aligned}x_A(p_A, p_B) &= \mu\{k \in [0, 1] | \Delta_k > \Delta_p\}; \\ x_B(p_B, p_A) &= \mu\{k \in [0, 1] | \Delta_k < \Delta_p\}.\end{aligned}$$

The profits of firm $i \in \{A, B\}$ are

$$\pi_i(p_i, p_j) = (p_i - c_i)x_i(p_i, p_j).$$

³The assumptions make sure that full coverage arises in equilibrium and after unilateral deviations.

The first assumption states that the support of the valuation distribution in \mathbb{R}_+^2 is a straight line from (v_0^A, v_0^B) to (v_1^A, v_1^B) .

Assumption 1. For each $k \in [0, 1]$ and $i \in \{A, B\}$, valuations satisfy

$$v_k^i = v_0^i + k(v_1^i - v_0^i)$$

Using Assumption 1 and uniformity, the distribution of consumer valuations is determined by the valuations of the extreme consumer types, (v_0^A, v_0^B) and (v_1^A, v_1^B) . Moreover,

$$\Delta_k = \Delta_0 + k(\Delta_1 - \Delta_0). \quad (1)$$

The following assumption imposes restrictions on valuations.

Assumption 2. (i) $v_1^A > v_0^A$;

(ii) $\Delta_1 > \Delta_0$ or, equivalently, $v_1^A - v_1^B > v_0^A - v_0^B$;

(iii) $\Delta_1 = v_1^A - v_1^B > 0$.

Part (i) states that $k = 1$ has higher valuation for product A than $k = 0$ so that Assumption 1 implies more generally that v_k^A is increasing in k . Together, part (ii) and Assumption 1 imply the single-crossing condition that $v_k^A - v_k^B$, the difference in WTPs, is increasing in k . (iii) requires that at least the consumer with the strongest valuation for product A prefers it to the alternative B at equal prices.

(i) **Horizontal Differentiation** ($\Delta_0 > 0$) (ii) **Vertical Differentiation** ($\Delta_0 < 0$)

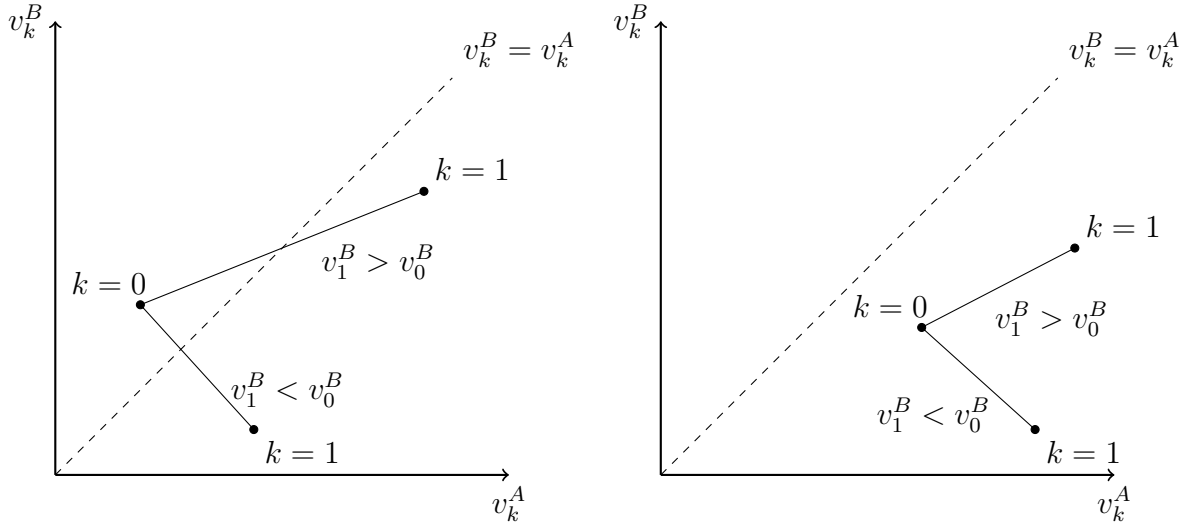


Figure 1: Illustration of Assumption 1. In each panel, the upward-sloping line captures positively correlated valuation for the two products, whereas the downward-sloping line captures negatively correlated valuations.

The assumptions made so far leave substantial flexibility regarding the nature of competitive interaction. They are compatible with the following cases.

Definition 1. (i) The market is described by *horizontal differentiation* if $v_0^A < v_0^B$ and by *vertical differentiation* if $v_0^A > v_0^B$.
(ii) Preferences for the two products are *positively correlated* if $v_1^B > v_0^B$, *negatively correlated* if $v_1^B < v_0^B$.

Figure 1 illustrates these cases. With horizontal differentiation (Figure 1(i)), consumers with sufficiently small k prefer product B to A at equal prices, whereas with vertical differentiation all consumers prefer A .⁴ Under horizontal differentiation, Assumption 2 is essentially just a notational convention. To see this, note that Assumptions 2(i) and 2(iii) can be made to hold for arbitrary labeling of products as A and B by suitably indexing consumers.⁵ Then one labeling of the two products will also satisfy Assumption 2(ii). Thus, under horizontal differentiation, Assumption 2 always holds. Under vertical differentiation, Assumption 2 is more restrictive. For Assumption 2(iii) to hold, A must be the better product. Consumers can always be ordered so that Assumption 2(i) holds. However, Assumption 2(ii), the single-crossing condition, has bite: It requires that the support line is flatter than the 45-degree line, as depicted in Figure 1(b). This is an actual restriction, implying that, as k increases, the additional WTP increase for the better product dominates the increase in WTP for the competing product.

The remaining assumptions jointly assure the existence of a full coverage equilibrium where all consumers are served and both firms sell a positive output.

Assumption 3. (i) $\Delta_1 - 2\Delta_0 + \Delta_C > 0$

(ii) $2\Delta_1 - \Delta_0 - \Delta_C > 0$

(iii) (a) $v_0^A > \frac{2c_A + c_B - \Delta_0 + 2\Delta_1}{3}$ and (b) $\min\{v_0^B, v_1^B\} > \frac{\Delta_1 - 2\Delta_0 + c_A + 2c_B}{3}$

Together, parts (i) and (ii) will be shown to guarantee that, in the proposed equilibrium, the indifferent consumer k^* is located in $(0, 1)$, the interior of the type space. Part (iii) is a sufficient conditions making sure that the market remains covered if at most one firm deviates from the equilibrium price. Specifically, (iii)(a) can be shown to guarantee that all consumers (including those who buy B in equilibrium) would obtain a positive net valuation from consuming good A at the equilibrium price. (iii)(b) is the corresponding statement for product B . Condition (iii) obviously holds as long as all valuations are high enough and costs sufficiently low.

The following result shows that it is not only possible to simultaneously satisfy all assumptions, but that all four constellations depicted in Figure 1 arise for suitable parameter vectors satisfying the assumptions. To see this, it suffices to show that this can be guaranteed for parameter vectors with zero marginal costs.

Proposition 1. Suppose $c_A = c_B = 0$. Each combination of differentiation types (horizontal or vertical) and WTP correlations (positive or negative) is satisfied for a non-degenerate set of parameter vectors $(v_0^A, v_1^A, v_0^B, v_1^B) \in \mathcal{R}_+^4$ for which Assumption 2 and 3 hold.

Figure 2 illustrates this result. The two textbook cases described in the introduction and the alternatives of horizontal differentiation combined with positive correlation and vertical differentiation combined with negative correlation all arise.

⁴W.l.o.g. Assumption 2(iii) rules out the possibility that all consumers prefer B at equal prices.

⁵For each product, by horizontal differentiation, one of the two most extreme consumers prefers it to the other one. Thus, if we denote the extreme consumer who prefers product A as $k = 1$, Assumptions 2(i) and 2(iii) both hold. The argument for product B is similar.

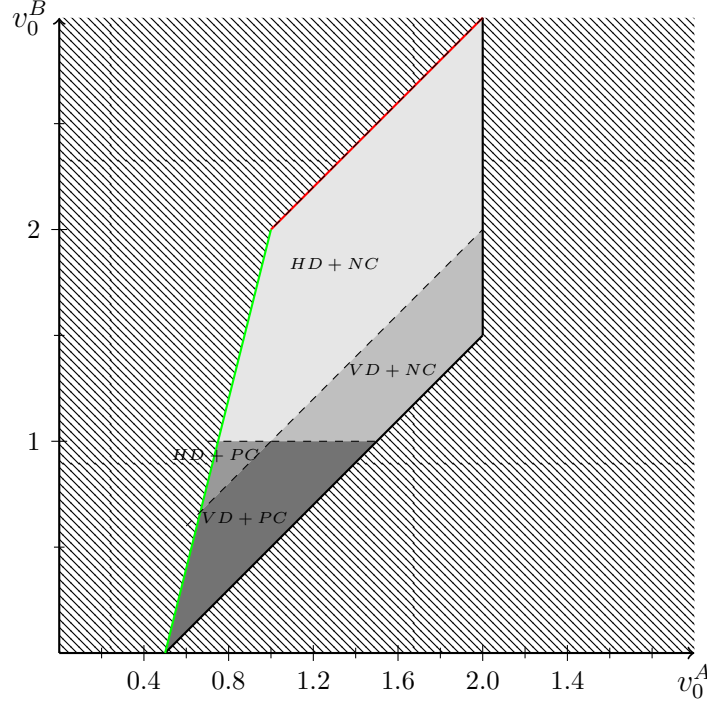


Figure 2: Illustration of Proposition 1. The figure depicts the values of v_0^A and v_0^B for which each combination of differentiation and correlation exists. HD (VD) stand for horizontal (vertical) differentiation, PC (NC) for positive (negative) correlation. The figure is drawn for $v_1^A = 2$ and $v_1^B = 1$.

3 Examples

The primitives of the model are the two cost parameters and the valuation distribution—nothing else matters for the equilibrium of the price game. It is nonetheless instructive to see that there are multiple ways in which the properties of the valuation distribution can be derived from underlying properties of the economic environment.

3.1 Textbook Models

As mentioned above, two simple models of differentiated duopoly fit the framework.

The Hotelling Model with Linear Transportation Cost In line with the standard Hotelling setting, interpret the type k of a consumer on the unit interval as his location. Firms A and B are located at $k = 1$ and $k = 0$, respectively. With linear transportation cost $t > 0$ and gross valuations $v > 0$, the valuations are therefore $v_k^A = v - t(1 - k)$ and $v_k^B = v - tk$. Therefore, the valuation vectors are $(v_0^A, v_0^B) = (v - t, v)$ for consumer $k = 0$ and $(v_1^A, v_1^B) = (v, v - t)$ for $k = 1$. Thus, the support slopes downward from $(v - t, v)$ to $(v, v - t)$, and it is symmetric at the diagonal. Assumptions 1 and 2 always hold. If the firms have identical and constant marginal costs, Assumption 3 merely requires that v is large enough. More generally, the framework can cater for cost and demand asymmetries as long as they are sufficiently small.⁶

⁶With quadratic transportation costs, the support would be convex to the origin, thus violating Assumption 1.

The Shaked-Sutton Model of Vertical Differentiation In line with standard models of vertical differentiation, assume that products have qualities $s_A > s_B > 0$.⁷ Further suppose that consumers have types $\theta \in [\underline{\theta}, \bar{\theta}]$, where $\underline{\theta} > 0$ and $\bar{\theta} := \underline{\theta} + 1$, and the type distribution is uniform. Moreover, suppose that type θ values product i at θs_i . The model fits into the framework of Section 2 with $v_k^i = (\underline{\theta} + k)s_i$, and Assumptions 1 and 2 hold without further restrictions. Clearly, the model satisfies the conditions for vertical differentiation ($v_0^A > v_0^B$) and positive correlation ($v_1^B > v_0^B$). Graphically, in Figure 1, the support is upward-sloping and does not intersect the diagonal. Finally, focusing for simplicity on the case of symmetric costs ($c_A = c_B$), Assumption 3 holds if one assumes in addition that $\underline{\theta} < 1$ and $c + \frac{(2\underline{\theta}+1)(s^A-s^B)}{3} < \underline{\theta}s^A$.⁸

In these models, horizontal differentiation coincides with negative correlation and vertical differentiation with positive correlation so that only two of the four cases depicted in Figure 1 arise. In the following, I will discuss an alternative setting where the type of differentiation is independent of the type of correlation.

3.2 Objective and Subjective Quality Differentiation

Many products can be regarded as multi-dimensional objects, where the overall valuation of a consumer for the product is a function of his assessment of the valuation for each individual characteristic. Some dimensions, such as durability, have strong objective aspects, others, such as style, are intrinsically subjective. This is a natural setting in which all four combinations of differentiation type and WTP correlation can arise.

To illustrate the point, I assume that the value of each product $i \in \{A, B\}$ can be decomposed into a *quality* component $\omega^i \in \mathcal{R}^+$ that is fully objective and a *style* component $\sigma_k^i \in \mathcal{R}$ that is subjective so that

$$v_k^i = \omega^i + \sigma_k^i. \quad (2)$$

I further assume that, for all $k \in [0, 1]$,

$$\sigma_k^i = \sigma_0^i + k(\sigma_1^i - \sigma_0^i).$$

Thus, clearly, Assumption 1 holds. Assumption 2 requires

$$\begin{aligned} \sigma_1^A &> \sigma_0^A; \\ \sigma_1^A - \sigma_0^A &> \sigma_1^B - \sigma_0^B; \\ \omega^A + \sigma_1^A &> \omega^B + \sigma_1^B. \end{aligned}$$

These conditions all hold if σ_1^A is large enough. Horizontal differentiation applies whenever

$$\omega^A + \sigma_0^A < \omega^B + \sigma_0^B, \quad (3)$$

while vertical differentiation corresponds to

$$\omega^A + \sigma_0^A > \omega^B + \sigma_0^B. \quad (4)$$

⁷The following treatment adapts Section 7.5 in Tirole (1989) to the setting of Section 2.

⁸These two conditions correspond to parts (i) and (iii) of Assumption 2, and they immediately imply Assumptions (ii) and (iv), respectively. The first condition is equivalent to Assumption 1 in Tirole (1989); the latter condition implies Assumption 2 in Tirole (1989).

Finally, positive (negative) correlation applies if $\sigma_1^B > (<) \sigma_0^B$.

By applying Proposition 1, I show in Section A.3 in the appendix that, for suitable parameter vectors, all four cases in Figure 1 can arise, rather than only the two that were discussed in the previous subsection. For instance, horizontal differentiation is consistent with positive correlation if product A has a better objective component, but consumers with low k place much higher subjective value on B than on A .⁹ Similarly, vertical differentiation can go hand in hand with negative correlation if product A has a sufficiently strong objective advantage over product B , but there is disagreement about the subjective component so that consumers with high valuation σ_k^A at the same time having low valuation σ_k^B for product B . Importantly, however, even for type $k = 1$, the agreement about the objective component captured by $\omega^A - \omega^B$ must dominate the disagreement about the subjective component captured by $\sigma_0^B - \sigma_0^A$ so that (4) holds.

4 Analysis of the Price Game

I now analyze the price game, returning to the general setting of Section 2. The main result identifies the unique second-stage equilibrium.

Proposition 2. *Suppose that Assumptions 1-3 hold. Then the game has an interior equilibrium with full coverage. Equilibrium prices are*

$$p_A^* = \frac{c_B + 2c_A - \Delta_0 + 2\Delta_1}{3}$$

$$p_B^* = \frac{c_A + 2c_B + \Delta_1 - 2\Delta_0}{3}$$

Importantly, this equilibrium applies no matter whether differentiation is horizontal or vertical and whether correlation is positive or negative. These properties of preferences and products are only relevant for the equilibrium to the extent that they affect the WTP differences Δ_1 and Δ_0 . Both prices are increasing in Δ_1 , but decreasing in Δ_0 . The difference reflects the intuition that an increase in Δ_1 (the WTP of the consumer with the strongest preferences for product A) increases the heterogeneity in consumer valuations for the two products, thereby softening competition, whereas an increase in Δ_0 reduces heterogeneity, thereby intensifying competition. Finally, both prices are increasing in costs, with the price effects of own cost increases dominating those of competitor cost increases.

Using Proposition 2, it is simple to derive equilibrium outputs (see Appendix A.5). To show how outputs (and hence market shares) depend on primitives, it is useful to consider three different parameter regimes, as in Figure 3. This figure distinguishes parameter constellations according to whether firm A has relatively low (L), intermediate (M), or high (H) costs.

Corollary 1. *Market Shares*

- (i) *A reduction in Δ_C or a uniform increase in Δ_0 and Δ_1 increases the market share of firm A .*
- (ii) *In parameter constellation L, increases in Δ_0 increase the market share of firm A ; while increases in Δ_1 reduce it; in constellation M, increases in Δ_0 and Δ_1 both*

⁹From condition (3), the requirement is $0 < \omega^A - \omega^B < \sigma_0^B - \sigma_0^A$.

increase the market share of firm A; in constellation H, increases in Δ_0 reduce the market share of firm A; whereas increases in Δ_1 increase it.

Result (i) shows that, if a firm gets stronger in the sense of having lower costs or higher WTP for all consumers, then its market share increases. Result (ii) shows that, by contrast, the effects of biased changes of WTP differences depend on relative costs: A clear-cut result as in case (i) only arises in the intermediate constellation (M). The complexities reflect the interplay between the clear direct effects of WTP changes and potentially countervailing effects resulting from price effects. Note that in the special case that $\Delta_C = 0$ only constellations L and M are consistent with Assumption 2. Then, L corresponds to vertical differentiation, whereas M corresponds to horizontal differentiation.

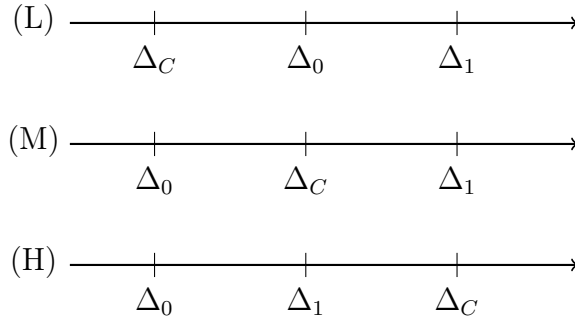


Figure 3: Parameter Constellations. The figure describes the parameter regimes that are used in Corollary 1 to analyze the effects of WTP changes on equilibrium market shares. In regime L, $\Delta_C < \Delta_0$. In regime M, $\Delta_0 < \Delta_C < \Delta_1$. In regime H, $\Delta_1 < \Delta_C$.

5 Conclusions

This note provides a simple framework for analyzing standard models of horizontal and vertical differentiation in a unified way. The framework uses the willingness-to-pay distribution of consumers as the primitive of the model, and it expresses the equilibrium as a function of these distributions. The approach allows us to separate the role of the type of differentiation (horizontal or vertical) from the type of correlation (positive or negative). It turns out that the former distinction plays a critical role for some comparative statics results, whereas the latter does not. This framework has been chosen in the simplest possible way to incorporate standard textbook models of differentiation. Obvious candidates for generalization include (i) the consumer distribution (for fixed support), (ii) the support (which could be a non-linear curve or a two-dimensional subset of \mathcal{R}_+^2) and the number of firms.

A Appendix

This appendix contains proofs and calculations required for the above results.

A.1 Proof of Proposition 1

It suffices to show that the desired conditions can be fulfilled even in the special case that $c_A = c_B = 0$, in which the conditions of Assumption 2(i)-(iii) and 3(i)-(iii) can be rewritten as follows.

$$v_1^A > v_0^A \quad (5)$$

$$v_1^A > v_0^A + v_1^B - v_0^B \quad (6)$$

$$v_1^A > v_1^B \quad (7)$$

$$v_1^A > v_1^B + 2v_0^A - 2v_0^B \quad (8)$$

$$v_1^A > v_1^B + \frac{v_0^A - v_0^B}{2} \quad (9)$$

$$v_1^B + 2v_0^A - \frac{v_0^B}{2} > v_1^A \quad (10)$$

$$4v_1^B + 2v_0^A - 2v_0^B > v_1^A \quad (11)$$

$$v_1^B + 2v_0^A + v_0^B > v_1^A \quad (12)$$

In the following, I will show that, for each combination of differentiation and correlation, all conditions (5)-(12) can be satisfied simultaneously. As (10) implies (12), the latter condition can be ignored below.

Horizontal Differentiation and Negative Correlation: It is straightforward to show that, if $v_0^A < v_0^B$, the requirement for horizontal differentiation, and $v_0^B > v_1^B$, the requirement for negative correlation, both hold, then all conditions for Assumption 2 and 3 (5)-(12) hold whenever (5), (7), (10) and (11) do. These four conditions as well as $v_0^A < v_0^B$ and $v_0^B > v_1^B$ obviously hold strictly if $v_0^B = v_1^A =: v$ and $v_1^B = v_0^A = \lambda v$ for some $\lambda \in (\frac{1}{2}, 1)$. Small perturbations of the parameter vector do not change this.

Horizontal Differentiation and Positive Correlation: It is straightforward to show that if the assumptions for horizontal differentiation ($v_0^A < v_0^B$) and positive correlation ($v_0^B < v_1^B$) hold, then equations (5)-(12) all hold if conditions (7) and (10) both hold as well. To see that it is possible to fulfill (7),(10), $v_0^A < v_0^B$ and $v_0^B < v_1^B$ simultaneously, first choose $v_0^A < v_0^B < v_1^B$, but such that $4v_0^A > v_0^B$. Then $v_1^B + 2v_0^A - \frac{v_0^B}{2} > v_1^B$, so that by appropriate choice of v_1^A , conditions (7) and (10) can both be made to hold.

Vertical Differentiation and Positive Correlation: It is straightforward to show that the assumptions for vertical differentiation ($v_0^A > v_0^B$) and positive correlation ($v_0^B < v_1^B$) and the conditions for Assumption 2 and 3 (equations (5)-(12)) all hold if

$$v_1^B > v_0^B \quad (13)$$

$$v_1^A > v_0^A > v_0^B \quad (14)$$

$$v_1^A > v_1^B + 2v_0^A - 2v_0^B \quad (15)$$

$$v_1^B + 2v_0^A - \frac{v_0^B}{2} > v_1^A \quad (16)$$

Conditions (13) and $v_0^A > v_0^B$ imply that the right-hand side of (15) is bounded below by v_0^A . Thus, $v_1^A > v_0^A$ can be ignored in (14). To jointly fulfill the remaining inequalities, first choose v_0^A , v_0^B and v_1^B so that (13) holds and $v_0^A > v_0^B$. Then (15) and (16) hold if and only if

$$v_1^A \in \left(v_1^B + 2v_0^A - 2v_0^B, v_1^B + 2v_0^A - \frac{v_0^B}{2} \right), \quad (17)$$

which is a non-empty interval.

Vertical Differentiation and Negative Correlation: It is straightforward to show that the conditions for Assumption 2 and 3 (equations (5)-(12)) as well as those for vertical differentiation ($v_0^A > v_0^B$) and negative correlation ($v_0^B > v_1^B$) hold if and only if

$$v_1^A > v_0^A > v_0^B > v_1^B \quad (18)$$

$$v_1^A > v_1^B + 2v_0^A - 2v_0^B \quad (19)$$

$$v_1^B + 2v_0^A - \frac{v_0^B}{2} > v_1^A \quad (20)$$

Choose $v_0^A > v_0^B > v_1^B$ so that $v_0^A - v_1^B$ is sufficiently small. As

$$v_1^B + 2v_0^A - 2v_0^B < 2v_0^A + v_1^B - \frac{v_0^B}{2},$$

the three requirements on v_1^A can be satisfied by choosing v_1^A slightly larger than v_0^A .

A.2 Derivations for Figure 2

The figure is depicted for $v_1^A = 2$ and $v_1^B = 1$. Inserting these values into equations (5)-(12), one can show that the equations all hold provided the following conditions do:

$$2 > v_0^A \quad (21)$$

$$v_0^B > v_0^A - \frac{1}{2} \quad (22)$$

$$v_0^B < v_0^A + 1 \quad (23)$$

$$4v_0^A > 2 + 2v_0^B. \quad (24)$$

For a parameter vector given by v_0^A , v_0^B and $v_1^A = 2$, $v_1^B = 1$, this corresponds to the parameter area described by the rectangle in Figure 2. The four regimes within the rectangle reflect the signs of $v_0^A - v_0^B$ and $v_1^B - v_0^B$, respectively. They thus correspond to the four different combinations of differentiation type and correlation.

A.3 Derivations for Section 3

Objective and Subjective Quality Differentiation: To show that, in this model, all four combinations of differentiation and correlation type can be generated, it suffices to show that every valuation vector $(v_0^A, v_1^A, v_0^B, v_1^B)$ can be constructed by suitable choices of (ω_A, ω_B) and $(\sigma_0^A, \sigma_1^A, \sigma_0^B, \sigma_1^B)$, because then Proposition 1 immediately implies the result. This corresponds to choosing the six variables (ω_A, ω_B) and $(\sigma_0^A, \sigma_1^A, \sigma_0^B, \sigma_1^B)$ so as to solve the four equations (2) for any fixed valuation vector $(v_0^A, v_1^A, v_0^B, v_1^B)$ satisfying Assumptions 2 and 3, which is trivial.

A.4 Proof of Proposition 2

Proof. I show that, under Assumptions 1-3, the price game has an equilibrium with full coverage, where consumers with k below a cut-off k^* buy product B and those with $k > k^*$ buy product A . For prices (p_A, p_B) , consumer k is indifferent between the products if

$$v_0^B + k(v_1^B - v_0^B) - p_B = v_0^A + k(v_1^A - v_0^A) - p_A. \quad (25)$$

This condition for the cut-off is equivalent with

$$k = \frac{\Delta_p - \Delta_0}{\Delta_1 - \Delta_0}. \quad (26)$$

This value lies strictly between 0 and 1 if and only if

$$\Delta_0 < \Delta_p < \Delta_1. \quad (27)$$

Using the uniformity of the consumer distribution and assuming full coverage, (26) immediately gives demand functions for prices satisfying (27) as

$$x_A(p_A, p_B) = \frac{\Delta_1 - \Delta_p}{\Delta_1 - \Delta_0} \quad (28)$$

$$x_B(p_B, p_A) = \frac{\Delta_p - \Delta_0}{\Delta_1 - \Delta_0} \quad (29)$$

Suppose prices are such that both firms have positive demand and that the market is covered. Then, using (26), Firm B's profits are $(p_B - c_B) \frac{(p_A - p_B) - \Delta_0}{\Delta_1 - \Delta_0}$. Solving the FOC gives the reaction function

$$p_B = \frac{p_A + c_B - \Delta_0}{2} \quad (30)$$

Firm A's profits are given as $(p_A - p_A) \left(\frac{\Delta_1 - (p_A - p_B)}{\Delta_1 - \Delta_0} \right)$, yielding

$$p_A = \frac{p_B + p_A + \Delta_1}{2} \quad (31)$$

Solving the system of equations given by (30) and (31) yields p_B^* and p_A^* as stated in Proposition 2 as a candidate equilibrium.

Assumptions 2 and 3 guarantee that the FOCs define an equilibrium. To see this, first note that the second-order conditions obviously hold for both profit functions. Assumptions 3(i) and (ii) make sure that both firms have positive market shares and margins in the

proposed equilibrium and thus earn positive profits. Assumption 3(iii) guarantees that all consumers obtain a positive net surplus from buying either of the two products at the equilibrium price. Thus, no matter how a firm deviates, each consumer buys from one of the two firms, and the deviating firm's demand will be determined by the indifferent consumer (as in (26)) unless it deviates so much that demand becomes 0 or 1. First- and second-order conditions jointly guarantee that deviations where both firms share the market are not profitable. They also imply that lowering prices so strongly that demand becomes 1 cannot be profitable. A fortiori, lowering prices even further cannot be profitable. Finally, increasing prices so much that demand becomes zero is obviously not profitable. \square

A.5 Equilibrium Outputs and Proof of Corollary 1

Equilibrium outputs result from inserting equilibrium prices into demand functions:

$$x_A^* = x_A^*(\Delta_0, \Delta_1, \Delta_C) = \frac{2\Delta_1 - \Delta_0 - \Delta_C}{3(\Delta_1 - \Delta_0)} \quad (32)$$

$$x_B^* = x_B^*(\Delta_0, \Delta_1, \Delta_C) = \frac{\Delta_1 - 2\Delta_0 + \Delta_C}{3(\Delta_1 - \Delta_0)} \quad (33)$$

Using Assumption 2, Corollary 1(i) is an immediate implication of these formulas. (ii) holds because

$$\frac{\partial}{\partial \Delta_0} \frac{2\Delta_1 - \Delta_0 - \Delta_C}{3(\Delta_1 - \Delta_0)} = \frac{\Delta_1 - \Delta_C}{3(\Delta_0 - \Delta_1)^2} > 0 \text{ if and only if } \Delta_1 > \Delta_C$$

and

$$\frac{\partial}{\partial \Delta_1} \frac{2\Delta_1 - \Delta_0 - \Delta_C}{3(\Delta_1 - \Delta_0)} = -\frac{\Delta_0 - \Delta_C}{3(\Delta_0 - \Delta_1)^2} > 0 \text{ if and only if } \Delta_0 < \Delta_C.$$

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