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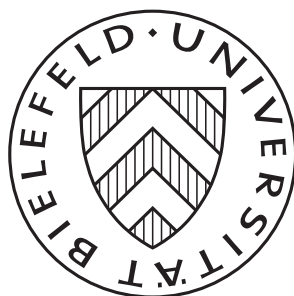
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The Pricing Kernel under Proportional Ambiguity

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Abstract

The pricing kernel is an important tool for understanding asset prices, expected returns, and investor preferences. However, empirical findings often reveal deviations from theoretical predictions, leading to the so-called "pricing kernel puzzle". This article explores the pricing kernel under Knightian uncertainty driven by identifiable business cycles. In a pure exchange economy with a representative agent exhibiting smooth ambiguity preferences, the pricing kernel is derived from equilibrium asset prices. By linking normal variance-mean mixtures with model uncertainty, we account for agents facing uncertainty across a continuum of economic regimes. Our results show that the pricing kernel can either decrease monotonically or exhibit a U-shape, depending on the level of ambiguity aversion. Additionally, we provide economic insights into the conditions that give rise to a U-shaped pricing kernel.

Keywords: Pricing kernel, business cycles, normal variance-mean mixture, model uncertainty, identifiability, ambiguity aversion

JEL Classification: G12, D53, D81

1 Introduction

In the absence of arbitrage, the pricing kernel, also known as stochastic discount factor, enables us to price assets by taking the expectation of the product of the pricing kernel and the asset's payoff. Being derived from equilibrium prices, the pricing kernel expresses information on the investor's preferences across different states, and is therefore of fundamental importance in finance.

In standard portfolio choice models and consumption-based asset pricing models under risk, the pricing kernel is proportional to marginal utility of wealth. Therefore, under risk aversion, the pricing kernel decreases monotonically in wealth. A monotonically decreasing pricing kernel formalizes the economic intuition that the investor values payoffs higher in states of lower wealth than in states of higher wealth.

However, empirical estimates of the pricing kernel in index markets reveal a different picture, documenting the pricing kernel puzzle. For this purpose, empirical risk-neutral

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densities are derived from traded option prices written on an index that serves as an approximation of overall wealth. To find empirical pricing kernels, the risk-neutral densities are divided by "physical distributions", derived from time series of returns of the underlying index. Empirical pricing kernels were first estimated by Ait-Sahalia and Lo (2000), Jackwerth (2000) and Rosenberg and Engle (2002). They all find increasing parts of the pricing kernel, violating the discussed monotonicity property. Cuesdeanu and Jackwerth (2018) provide an overview over empirical shapes of the pricing kernel, stating that tilde- and U-shaped pricing kernels are the most prominent ones discussed in the empirical literature. More recently, Sichert (2023) argues that empirical pricing kernels are consistently U-shaped.

This paper provides a new economic intuition for U-shaped pricing kernels by investigating the pricing kernel under Knightian uncertainty about identifiable regimes of the economy. In particular, mean growth and volatility of the regime are unknown to the investor at the point of decision-making. In contrast to the existing literature, we consider a continuum of possible regimes instead of only a discrete set. We fit our model to index return data and find that the pricing kernel either decreases monotonically or exhibits a U-shape, depending on the level of ambiguity aversion.

Knightian uncertainty (or ambiguity), as opposed to risk, describes a situation where there is no probabilistic information available to the investor. Thus, under uncertainty, the investor is not able to assign exact probabilities to states. Instead, he takes a set of probability distributions into consideration. Additionally, we require this set of possible probability distributions to be identifiable. Identifiability of probability distributions, a concept in statistics, allows one to learn the true value of the distribution's underlying parameters. Hence, probability distributions (or just models) become objective descriptions that can be identified from observed events. Therefore, realized states also provide information on the probabilistic law that the investor needs to take into account.

To derive the pricing kernel from equilibrium prices, we employ a pure exchange economy with a representative investor who has smooth ambiguity preferences such as in Klibanoff et al., (2005). Smooth ambiguity preferences allow for a separation of beliefs and ambiguity attitudes, enabling us to analyze the impact of the level of ambiguity aversion on the shape of the pricing kernel. Denti and Pomatto (2022) places smooth ambiguity preferences in the framework that we consider, of identifiable models.

We model business cycles or regimes of the economy as probability distributions of the states. As the pricing kernel reflects the investor's preferences across states, this modeling approach conceptualizes the intuition that the valuation of payoffs should incorporate the investor's expectations regarding the economic regime and how it is evaluated. Identifiability of models provides a foundation for this approach. The investor observes a particular state and, by identifiability, this state informs about the prevailing business cycle. Thus, the investor can anticipate the economic regime, which influences his valuation of payoffs in the observed state.

Our approach can be seen as a static version of regime-switching under ambiguity. In regime-switching models, a framework often applied in macro-finance, distributions of states switch according to the prevailing regime. Regime-switching models were first discussed in Hamilton (1989), where the distribution switches between a high growth and a low growth phase. In an ambiguity setting, Ju and Miao (2012) also consider switching between two regimes with different means – a recession and a boom. However, we allow for different volatilities and assume that an expansive regime is associated with a higher mean and lower volatility and a recessive regime with a lower mean and higher volatility,

respectively.

Since the literature on regime-switching models focuses on a discrete set of regimes, it falls short of accounting for regimes between recessions and expansions that are hard to distinguish. Especially the prominent modeling of only two regimes cannot grasp the complexity accordingly. Therefore, instead of discussing only a discrete set of regimes, we consider a continuum of possible regimes. To do so, we make use of normal variance-mean mixtures, first discussed by Barndorff-Nielsen et al., (1982). A random variable R , describing the distribution of possible states, follows a normal variance-mean mixture if

$$R \sim \mathcal{N}(\alpha + \beta M, M), \text{ with } \alpha, \beta \in \mathbb{R},$$

conditional on a distribution M on $(0, \infty)$. Thus, each regime is a normal distribution with unknown mean and unknown variance, with both being in a proportional relationship to each other. We purposely let $\beta < 0$ such that the set of models

$$\mathcal{P} = \{\mathcal{N}(\alpha + \beta m, m) \mid m \in (0, \infty)\}$$

can be interpreted as a continuum of possible regimes of the economy.

For tractability, we consider possible returns of an index that correlates with aggregate wealth as states and discuss the pricing kernel as a function of return. We show that the pricing kernel is logarithmically convex, which allows for either a monotonically decreasing or a U-shaped pricing kernel. We state characterizing conditions for these two shapes and find that the pricing kernel is U-shaped if the effect of ambiguity attitude dominates the effect of risk attitude. Consequently, a higher ambiguity aversion causes a more pronounced U-shape, providing another explanation of U-shaped pricing kernels. It is the anticipation of a recessive regime when observing a high index return that drives the valuation of payoffs upward. The magnitude of the increase in the valuation of payoffs crucially depends on the level of ambiguity aversion.

This article contributes to the literature on trying to explain the pricing kernel puzzle. Hens and Reichlin (2013) discuss possible solutions of the pricing kernel puzzle by weakening the assumptions of complete markets, risk-averse agents and correct beliefs. They find that, as one weakens at least one assumption, the pricing kernel admits locally increasing parts. However, the boundary behavior will still be decreasing. On the other hand, Christoffersen et al., (2013) argue in favor of U-shaped pricing kernels by allowing the pricing kernel to depend on the variance. Bakshi et al., (2010) study the implications of U-shaped pricing kernels and find broad support in index option data. The pricing kernel in the presence of Knightian uncertainty is also discussed in Hara et al., (2024), Gollier (2011) and Chen et al., (2025). These papers have in common that there is uncertainty driven by two possible scenarios and the investor forms a binomial belief over these scenarios.

The paper is structured as follows. First, Section 2 introduces the formal model and derives equilibrium prices, from which we infer the pricing kernel. Section 3 discusses the modeling of a continuum of different business cycles and its consequences on the shape of the pricing kernel. Furthermore, we fit the pricing kernel to index return data and investigate the impact of the level of ambiguity aversion on the pricing kernel. Finally, Section 4 concludes.

2 The Model: A Pure Exchange Economy under Identifiable Model Uncertainty

2.1 Setting and Preliminaries

Consider a state space Ω and denote by $\Delta(\Omega)$ the set of probability distributions on Ω . Let $\mathcal{P} \subset \Delta(\Omega)$ constitute the set of possible regimes of the economy that the agent deems possible.

We assume that the set of models \mathcal{P} is point-identifiable, i.e., the regime that operates can be inferred from realizations of Ω . By Denti and Pomatto (2022), identifiability of models implies that there exists a kernel function $k : \Omega \rightarrow \mathcal{P}$ such that for all $P \in \mathcal{P}$ we have $P[\{\omega \mid k(\omega) = P\}] = 1$. The kernel function k , assigning observed states to models, therefore captures the idea of an expert committee that announces the true model after a given time.

For our purposes, we can rewrite the state space in the product form $\Omega = \mathbb{R} \times \mathcal{P}$, as suggested by Hara et al., (2024). Each regime $P \in \mathcal{P}$ is a probability distribution on the set of possible index returns \mathbb{R} . Hence, a state of the world $\omega = (r, P)$ consists of a realized return $r \in \mathbb{R}$ and the regime from which r is drawn. The true parameter that characterizes the regime is identified by the kernel function k such that $k(r, P) = P$ for all $r \in \mathbb{R}$.

To illustrate this, think of the NBER's Business Cycle Dating Committee observing, next to index return data, economic indicators such as the unemployment rate or the GDP, and, based on these observations, announcing the business cycle that operates after a given time. At the point of decision making, the current business cycle is unknown, but impacts the index return distribution, and therefore aggregate wealth of the economy.

2.2 A Pure Exchange Economy and its Representative Agent

We consider a pure exchange economy with a representative agent that allocates ambiguous aggregate wealth $\bar{X} : \Omega \rightarrow \mathbb{R}_+$, similar to Hara et al., (2024). As the agent does not know which state $\omega \in \Omega$ will materialize, she faces risk regarding the index return value $r \in \mathbb{R}$. In addition to that, she is uncertain which model $P \in \mathcal{P}$ is the true one, i.e., which business cycle is currently operating. These models govern the probability distribution of the return r .

As the models share the same support, the range of possible realizations of the return (and therefore aggregate wealth) does not depend on the regime that operates. Thus, ambiguity of aggregate wealth is generated by different distributions under different regimes, but these distributions do not exclude any possible realization of the return.

The representative agent evaluates aggregate wealth according to identifiable smooth preferences, i.e., the smooth model of decision making under ambiguity by Klibanoff et al., (2005) restricted to identifiable models. Identifiable smooth preferences were introduced and axiomatized by Cerreia-Vioglio et al., (2013).¹ Denote by \mathbb{E}^P the expectation operator for random variables defined on \mathbb{R} under the probability measure P . Denote by $u : \text{Im}(\bar{X}) \rightarrow \mathbb{R}$ a twice continuously differentiable, strictly increasing and strictly concave Bernoulli utility function that governs risk attitude. Furthermore, let $\phi : \text{Im}(u) \rightarrow \mathbb{R}$

¹Denti and Pomatto (2022) show that the utility function u , ambiguity index ϕ and the prior μ are uniquely determined from preferences.

be twice continuously differentiable, strictly increasing and concave and let $\mu \in \Delta(\Delta(\Omega))$ be a probability distribution over models. Then, the agent's utility functional is given by

$$V(\bar{X}) = \int_{\mathcal{P}} \phi(\mathbb{E}^P[u(\bar{X})]) \mu(dP). \quad (1)$$

The (identifiable) smooth model allows for a separation of ambiguity attitudes and beliefs. The representative agent's ambiguity attitude is captured by ϕ . A strictly concave ϕ describes ambiguity aversion, whereas ϕ being linear indicates ambiguity neutrality. The prior belief μ can be thought of as an expert's assessment, and the degree of trust in this assessment is driven by the degree of ambiguity attitude.

We can also rewrite (1) in terms of certainty equivalents,² defined by $c_u^P(\bar{X}) = u^{-1}(\mathbb{E}^P[u(\bar{X})])$, instead of expected utilities. By introducing $v : \text{Im}(\bar{X}) \rightarrow \mathbb{R}$ such that $\phi = v \circ u^{-1}$, we have

$$V(\bar{X}) = \int_{\mathcal{P}} v(c_u^P(\bar{X})) \mu(dP). \quad (2)$$

Subsequently, we assume that the utility functions u and v of the representative agent admit constant relative risk aversion (CRRA).³

Assumption 1. *The Bernoulli utility functions u and v of the representative agent with parameters ρ and γ are given by*

$$u(x) = \frac{x^{1-\rho}}{1-\rho} \text{ and } v(x) = \frac{x^{1-\gamma}}{1-\gamma}.$$

Observe that the agent is ambiguity averse if and only if $\gamma > \rho$.

2.3 Equilibrium Asset Prices

To study the pricing kernel of the economy, suppose that there is a complete set of contingent claims available. As discussed, we assume that each model $P \in \mathcal{P}$ is a normal distribution R on \mathbb{R} . By identifiability, each model has to be uniquely identified by its characterizing parameter $m \in \mathbb{R}_{++}$. We denote by $p_{R|M}(r | m)$ the probability density of return r under regime R , characterized by m . Similarly, the prior belief μ is a continuous probability distribution M on \mathbb{R}_{++} , whose density is given by $p_M(m)$. Lastly, $p_{M|R}(m | r)$ denotes the density⁴ of the regime's characterizing parameter m conditional on an observed return r .

According to Hara et al., (2024), the equilibrium price density is given by a function $\Psi : \mathbb{R} \times \mathcal{P} \rightarrow \mathbb{R}_{++}$ with

$$\Psi(r, m) = \phi'(\mathbb{E}^m[u(\bar{X}(r))]) p_{R|M}(r | m) u'(\bar{X}(r)). \quad (3)$$

²This has the advantage that one can define the ambiguity attitude in terms of two Bernoulli utility functions u and v .

³By Hara et al., (2024), this corresponds to an economy consisting of homogeneously ambiguity averse agents. Heterogeneity of ambiguity attitudes across agents corresponds to v exhibiting decreasing relative risk aversion (DRRA). In this case, the pricing kernel cannot be computed explicitly. However, numerical computations revealed that the overall shape of the pricing kernel does not depend on whether there is homogeneity or heterogeneity of ambiguity attitudes across agents. This is also supported by the findings in Hara et al., (2024).

⁴Each density is defined with respect to the Lebesgue measure on \mathbb{R} and \mathbb{R}_{++} , respectively.

As the payoff of contingent claims written only on the return does not depend on the regime R , its price density is given by the integral of (3) over all regimes

$$\Psi(r) = u'(\bar{X}(r)) \int_0^\infty \phi'(\mathbb{E}^m[u(\bar{X}(r))]) p_{R|M}(r | m) p_M(m) dm. \quad (4)$$

Note that (4) is our economies' risk-neutral density of the underlying index return.

To obtain the pricing kernel, the risk-neutral density has to be divided by the "physical density" of the return. In our case, this is given by the μ -average of the probability density of return r under each regime

$$p_\mu(r) = \int_0^\infty p_{R|M}(r | m) p_M(m) dm.$$

Hence, the pricing kernel is given by a function $\psi : \mathbb{R} \rightarrow \mathbb{R}_{++}$ with

$$\begin{aligned} \psi(r) &= \int_0^\infty \frac{p_{R|M}(r | m) p_M(m)}{p_\mu(r)} \phi'(\mathbb{E}^m[u(\bar{X}(r))]) u'(\bar{X}(r)) dm \\ &= u'(\bar{X}(r)) \int_0^\infty \frac{v'(c_u^m(\bar{X}))}{u'(c_u^m(\bar{X}))} p_{M|R}(m | r) dm. \end{aligned} \quad (5)$$

We see that the pricing kernel splits into two effects. On the one hand, we have the effect of risk attitude given by marginal risk utility $u'(\bar{X}(r))$. On the other hand, there is the effect of ambiguity attitude⁵

$$\int_0^\infty \frac{v'(c_u^m(\bar{X}))}{u'(c_u^m(\bar{X}))} p_{M|R}(m | r) dm,$$

a weighted mean of marginal ambiguity utility $\frac{v'(c_u^m(\bar{X}))}{u'(c_u^m(\bar{X}))}$, weighted by the posterior distribution $p_{M|R}(m | r)$. While the effect of risk attitude is monotonically decreasing in r , the effect of ambiguity attitude could cause non-monotone behavior.

3 The Pricing Kernel

3.1 Normal Variance-Mean Mixtures: A Continuum of Regimes

In the macro-finance literature, it is quite common to model aggregate wealth via a log-normal distribution, that is, we have $\bar{X} = \exp(R)$, where R follows a normal distribution.⁶ Recall that R is supposed to be the probability distribution of returns of an index that correlates with overall wealth in an economy. However, the mean and the variance of the return distribution are uncertain, and they depend on the current business cycle that operates.

To rationalize the dependence of return distributions on the economic regime, let us discuss some empirical studies. Most studies focus on stock market volatility during recessions. Hamilton and Lin (1996) find that stock market volatility significantly

⁵Note that, under ambiguity neutrality, ϕ' is a constant such that the effect of ambiguity attitude vanishes.

⁶See for example Cecchetti et al., (1990) and Kandel and Stambaugh (1991).

increases in recessive regimes. Regarding the mean, DeStefano (2004) points out that returns decline at the onset of recessions, reaching their minimum, before beginning to increase during the later part of recessions. Additionally, Kroencke (2022) investigates stock prices during recessions. Within the first quarter of recessions, stock prices drop sharply, explaining highly negative returns at the beginning of recessions. However, a stylized fact in the finance literature is asymmetric volatility, i.e., stock returns are negatively correlated with the volatility of subsequent returns, as, e.g., Bae et al., (2007) point out. Therefore, we assume that in a more expansive regime of the economy, the mean is high and the variance is low, whereas a more recessive regime is characterized by a low mean and a high variance.

We implement normal variance-mean mixtures, first discussed by Barndorff-Nielsen et al., (1982), to model a continuum of different regimes.

Definition 1. *A normal variance-mean mixture is the continuous probability distribution of a random variable R given by*

$$R = \alpha + \beta M + \sigma \sqrt{M} X,$$

where $\alpha, \beta \in \mathbb{R}$, and $\sigma > 0$, X and M are independent random variables with $X \sim \mathcal{N}(0, 1)$ and M being continuously distributed on $(0, \infty)$.

Thus, conditional on a realized variance m , we have that

$$R \sim \mathcal{N}(\alpha + \beta m, \sigma^2 m).$$

Normal variance-mean mixtures are prominent models in financial engineering as they give access to the flexible class of generalized hyperbolic distributions while still maintaining the convenience of Gaussian environments.⁷

However, we explore normal variance-mean mixtures in a different context. There is uncertainty about variance and mean of the normal distribution, while both parameters are in a proportional relationship to each other. For simplicity, we let $\sigma = 1$ and assume the following.

Assumption 2. *Aggregate wealth \bar{X} follows a lognormal distribution*

$$\bar{X} = \exp(R) \text{ with } R \sim \mathcal{N}(\alpha + \beta m, m),$$

where m is unknown and $\beta < 0$.

Letting $\beta < 0$ ensures that a higher variance comes with a lower mean.

As Barndorff-Nielsen (1997) points out, the conjugate prior in a normal variance-mean mixture is the generalized inverse Gaussian distribution, which leads to our next assumption.

Assumption 3. *The prior belief M follows a generalized inverse Gaussian distribution with parameters $\delta > 0$, $\kappa > 0$ and $\lambda \in \mathbb{R}$. Its density is given by*

$$p_M(m) = \frac{\left(\frac{\delta}{\kappa}\right)^{\lambda/2}}{2K_\lambda\left(\sqrt{\delta\kappa}\right)} m^{\lambda-1} \exp\left(-\frac{1}{2}\left(\delta m + \frac{\kappa}{m}\right)\right). \quad (6)$$

⁷For examples see Bingham et al., (2003) and Luciano and Semeraro (2010).

For specifications of the posterior parameters see Remark 1 in the Appendix. Note that $K_\lambda(\cdot)$ in the normalizing factor in (6) denotes the modified Bessel function of the second kind of order λ .

We can compute the certainty equivalent conditional on the variance m as

$$c_u^m(\bar{X}) = \exp\left(\alpha + \beta m + \frac{1}{2}m(1 - \rho)\right),$$

and thus

$$\frac{v'(c_u^m(\bar{X}))}{u'(c_u^m(\bar{X}))} = \exp\left(-(\gamma - \rho)(\alpha + \beta m) - (\gamma - \rho)(1 - \rho)\frac{1}{2}m\right).$$

By (5), the pricing kernel is given by

$$\begin{aligned} \psi(r) = & \exp(-\rho r) \\ & \int_0^\infty \exp\left(-(\gamma - \rho)(\alpha + \beta m) - (\gamma - \rho)(1 - \rho)\frac{1}{2}m\right) m^{\lambda - \frac{3}{2}} \\ & \frac{\left(\frac{\delta + \beta^2}{\kappa + (r - \alpha)^2}\right)^{\frac{2\lambda - 1}{4}}}{2K_{\lambda - \frac{1}{2}}\left(\sqrt{(\delta + \beta^2)(\kappa + (r - \alpha)^2)}\right)} \exp\left(-\frac{1}{2}\left((\delta + \beta^2)m + \frac{\kappa + (r - \alpha)^2}{m}\right)\right) dm, \end{aligned}$$

which admits a solution only if the following condition is met.

Lemma 1. *The pricing kernel ψ is well-defined only if*

$$2\beta(\gamma - \rho) + (\gamma - \rho)(1 - \rho) + \delta + \beta^2 > 0.$$

Given the condition in Lemma 1, one can solve the pricing kernel analytically in terms of modified Bessel functions of the second kind. In particular, the following integral representation is useful for our purposes.

Lemma 2. *The modified Bessel function of the second kind admits the integral representation*

$$K_\lambda(\sqrt{\eta\zeta}) = \frac{1}{2}\sqrt{\frac{\eta}{\zeta}} \int_0^\infty z^{\lambda-1} \exp\left(-\frac{1}{2}\left(\eta z + \frac{\zeta}{z}\right)\right) dz \quad (7)$$

for $\eta, \zeta > 0$.

We apply Lemma 2 to the formula for the pricing kernel.

Lemma 3. *The pricing kernel is given by*

$$\psi(r) = C \exp(-\rho r) \frac{K_{\lambda - \frac{1}{2}}\left(\sqrt{\kappa + (r - \alpha)^2} \sqrt{2\beta(\gamma - \rho) + (\gamma - \rho)(1 - \rho) + \delta + \beta^2}\right)}{K_{\lambda - \frac{1}{2}}\left(\sqrt{\kappa + (r - \alpha)^2} \sqrt{\delta + \beta^2}\right)},$$

where C is a constant.

That is, the effect of ambiguity attitude is given by a ratio of modified Bessel functions of the second kind with the same order.

3.2 The Shape of the Pricing Kernel

Bakshi et al., (2010) characterize U-shaped pricing kernels as convex functions of the return that are decreasing up to a point and increasing afterwards. Akin to that, we aim to investigate the convexity of the pricing kernel. In particular, we even find that the pricing kernel is logarithmically convex, a stronger notion than convexity.

Proposition 1. *Let $\lambda \in \mathbb{R} \setminus (0, 1)$.⁸ If $2\beta(\gamma - \rho) + (\gamma - \rho)(1 - \rho) < 0$, then the pricing kernel ψ is logarithmically convex.*

Proof. First, recall that the product of logarithmically convex functions is logarithmically convex. Moreover, note that the composition of an increasing, (log-)convex function and a convex function is again (log-)convex (see, e.g., Bertsekas et al., (2003)). As the function

$$\mathbb{R} \rightarrow \mathbb{R}_{++}, r \mapsto \sqrt{\kappa + (r - \alpha)^2}$$

is convex, we are left to show that the function

$$\mathbb{R}_{++} \rightarrow \mathbb{R}_{++}, x \mapsto \frac{K_\lambda(C_1 x)}{K_\lambda(C_2 x)} \quad (8)$$

is logarithmically convex and increasing for $\lambda \notin (-\frac{1}{2}, \frac{1}{2})$ and constants $C_2 > C_1$. Let us first prove logarithmic convexity. We have

$$\log \left(\frac{K_\lambda(C_1 x)}{K_\lambda(C_2 x)} \right)' = C_1 \frac{K'_\lambda(C_1 x)}{K_\lambda(C_1 x)} - C_2 \frac{K'_\lambda(C_2 x)}{K_\lambda(C_2 x)}$$

and

$$\begin{aligned} & \log \left(\frac{K_\lambda(C_1 x)}{K_\lambda(C_2 x)} \right)'' \\ &= C_1 \frac{C_1 K''_\lambda(C_1 x) K_\lambda(C_1 x) - C_1 K'_\lambda(C_1 x)^2}{K_\lambda(C_1 x)^2} - C_2 \frac{C_2 K''_\lambda(C_2 x) K_\lambda(C_2 x) - C_2 K'_\lambda(C_2 x)^2}{K_\lambda(C_2 x)^2} \\ &= C_1^2 \left(\frac{K'_\lambda(\hat{x})}{K_\lambda(\hat{x})} \right)' \Big|_{\hat{x}=C_1 x} - C_2^2 \left(\frac{K'_\lambda(\hat{x})}{K_\lambda(\hat{x})} \right)' \Big|_{\hat{x}=C_2 x}. \end{aligned}$$

As $K_\lambda(x)$ is symmetric in λ , we will focus on $\lambda \geq 0$. By Yang and Zheng (2017) (Lemma 2.2), we have that

$$\begin{aligned} \frac{K'_\lambda(x)}{K_\lambda(x)} &= -\frac{\lambda}{x} - \frac{K_{\lambda-1}(x)}{K_\lambda(x)} = -\frac{\lambda}{x} - \frac{4}{\pi^2} \int_0^\infty \frac{x}{x^2 + t^2} \frac{1}{t(J_\lambda^2(t) + Y_\lambda^2(t))} dt \\ &= -\frac{\lambda}{x} - \frac{4}{\pi^2} \int_0^\infty \frac{x}{x^2 + t^2} \Phi_\lambda(t) dt, \end{aligned}$$

where $\Phi_\lambda(t) = \frac{1}{t(J_\lambda^2(t) + Y_\lambda^2(t))}$. Here, J_λ and Y_λ denote the Bessel functions of the first and second kind, respectively. Differentiation yields

$$\left(\frac{K'_\lambda(x)}{K_\lambda(x)} \right)' = \frac{\lambda}{x^2} - \frac{4}{\pi^2} \int_0^\infty \frac{t^2 - x^2}{(x^2 + t^2)^2} \Phi_\lambda(t) dt.$$

⁸We have to exclude $\lambda \in (0, 1)$ due to technical issues. In particular, the reason is that $K_\lambda(x)$ is diverging too fast for $x \rightarrow 0$ for $\lambda \in (-\frac{1}{2}, \frac{1}{2})$.

Thus,

$$\begin{aligned} \log \left(\frac{K_\lambda(C_1x)}{K_\lambda(C_2x)} \right)'' &= C_1^2 \left(\frac{\lambda}{C_1^2x^2} - \frac{4}{\pi^2} \int_0^\infty \frac{t^2 - C_1^2x^2}{(C_1^2x^2 + t^2)^2} \Phi_\lambda(t) dt \right) \\ &\quad - C_2^2 \left(\frac{\lambda}{C_2^2x^2} - \frac{4}{\pi^2} \int_0^\infty \frac{t^2 - C_2^2x^2}{(C_2^2x^2 + t^2)^2} \Phi_\lambda(t) dt \right) \\ &= \frac{4}{\pi^2} \int_0^\infty \left(C_2^2 \frac{t^2 - C_2^2x^2}{(C_2^2x^2 + t^2)^2} - C_1^2 \frac{t^2 - C_1^2x^2}{(C_1^2x^2 + t^2)^2} \right) \Phi_\lambda(t) dt. \end{aligned}$$

We make use of two results provided by Yang and Tian (2024). Lemma 2.2 states that

$$\int_a^b f(t)g(t)dt \geq g(t_0) \int_a^b f(t)dt \geq 0$$

for two functions f and g such that

1. f, g are continuous on $[a, b]$;
2. there is a $t_0 \in (a, b)$ such that $f(t) < 0$ for $t \in (a, t_0)$ and $f(t) > 0$ for $t \in (t_0, b)$;
3. $g(t)$ is nonnegative and increasing on $[a, b]$;
4. $\int_a^b f(t)dt \geq 0$.

Moreover, Lemma 2.1 states that $\Phi_\lambda(t)$ is continuous, increasing on $(0, \infty)$ for $\lambda \geq \frac{1}{2}$, and decreasing for $0 \leq \lambda < \frac{1}{2}$. Thus, we are only left to show 1., 2. and 4. for the function

$$f(t) := C_2^2 \frac{t^2 - C_2^2x^2}{(C_2^2x^2 + t^2)^2} - C_1^2 \frac{t^2 - C_1^2x^2}{(C_1^2x^2 + t^2)^2},$$

which is postponed to Lemma 4 in Appendix A.1. We are left to prove that (8) is increasing. To this end, we need to show that

$$\left(\frac{K_\lambda(C_1x)}{K_\lambda(C_2x)} \right)' = \frac{C_1 K'_\lambda(C_1x) K_\lambda(C_2x) - C_2 K'_\lambda(C_2x) K_\lambda(C_1x)}{K_\lambda(C_2x)^2} > 0,$$

which holds true if and only if

$$C_1 \frac{K'_\lambda(C_1x)}{K_\lambda(C_1x)} - C_2 \frac{K'_\lambda(C_2x)}{K_\lambda(C_2x)} > 0.$$

Again, by Yang and Zheng (2017), we infer

$$C_1 \frac{K'_\lambda(C_1x)}{K_\lambda(C_1x)} - C_2 \frac{K'_\lambda(C_2x)}{K_\lambda(C_2x)} = \frac{4}{\pi^2} \int_0^\infty \left(\frac{C_2^2x}{C_2^2x^2 + t^2} - \frac{C_1^2x}{C_1^2x^2 + t^2} \right) \Phi_\lambda(t) dt.$$

We have that $\frac{C_2^2x}{C_2^2x^2 + t^2} - \frac{C_1^2x}{C_1^2x^2 + t^2} > 0$ for all $t \in (0, \infty)$ since $C_2 > C_1$. As $\Phi_\lambda(t)$ is also nonnegative for all $t \in (0, \infty)$ and all $\lambda \in \mathbb{R}$, the proof is complete. \square

Note that, assuming ambiguity aversion, $\rho > 1$ is a sufficient condition for the pricing kernel to be logarithmically convex. Based on this observation, the pricing kernel can take on two different shapes, either monotonically decreasing or U-shaped.

Theorem 1. Let $\lambda \in \mathbb{R} \setminus (0, 1)$ and $2\beta(\gamma - \rho) + (\gamma - \rho)(1 - \rho) < 0$. Then,

1. the pricing kernel is U-shaped if and only if

$$\sqrt{\delta + \beta^2} - \sqrt{2\beta(\gamma - \rho) + (\gamma - \rho)(1 - \rho) + \delta + \beta^2} > \rho \quad (9)$$

2. the pricing kernel is monotonically decreasing if and only if

$$\sqrt{\delta + \beta^2} - \sqrt{2\beta(\gamma - \rho) + (\gamma - \rho)(1 - \rho) + \delta + \beta^2} \leq \rho. \quad (10)$$

Proof. By Proposition 1, the pricing kernel is logarithmically convex. Therefore, it is sufficient to look at the behavior for large absolute values of r to investigate monotonicity properties. This enables us to use the asymptotic expansion of the modified Bessel function of the second kind, which is, e.g., given in Abramowitz and Stegun (1968) as

$$K_\lambda(x) \approx \sqrt{\frac{\pi}{2x}} \exp(-x) \text{ with relative error } \mathcal{O}\left(\frac{1}{x}\right).$$

Then, for large absolute values of r , the pricing kernel can be approximated by

$$f(r) := \hat{C} e^{-\rho r + \sqrt{\kappa + (r - \alpha)^2} (\sqrt{\delta + \beta^2} - \sqrt{2\beta(\gamma - \rho) + (\gamma - \rho)(1 - \rho) + \delta + \beta^2})}.$$

Differentiation yields

$$f'(r) = f(r) \left(-\rho + \frac{r - \alpha}{\sqrt{\kappa + (r - \alpha)^2}} \left(\sqrt{\delta + \beta^2} - \sqrt{2\beta(\gamma - \rho) + (\gamma - \rho)(1 - \rho) + \delta + \beta^2} \right) \right).$$

Clearly, for $r \leq \alpha$, $f'(r) < 0$. Otherwise, note that $\frac{r - \alpha}{\sqrt{\kappa + (r - \alpha)^2}} \nearrow 1$ for $r \rightarrow \infty$. Hence, if inequality (9) is satisfied, there will be a $r^* \in \mathbb{R}_{++}$ such that $f'(r) > 0$ for all $r \geq r^*$. If inequality (10) holds, $f'(r) < 0$ for all $r \in \mathbb{R}$.

As ψ is logarithmically convex, the monotonicity properties derived for f also hold for the pricing kernel. \square

Observe that the inequalities (9) and (10) express whether the effect of ambiguity attitude or the effect of risk attitude dominates. Since both effects work in the same direction for $r < \alpha$, the inequalities matter only for $r > \alpha$. Thus, the pricing kernel is initially decreasing, regardless of which effect dominates. Having a closer look at the effect of ambiguity attitude, encapsulated by

$$\sqrt{\delta + \beta^2} - \sqrt{2\beta(\gamma - \rho) + (\gamma - \rho)(1 - \rho) + \delta + \beta^2},$$

gives insights into the conditions that provoke a U-shaped pricing kernel. First, assuming risk aversion of $\rho > 1$, a high ambiguity aversion γ causes a U-shape. Second, the impact of a high ambiguity aversion multiplies with β . The smaller β , the lower the means that are associated with higher volatilities. Hence, intuitively, the fear of a recessive regime is even more pronounced when the anticipated recession is more severe.

In the next section, we fit the pricing kernel to index return data and discuss in more detail why the preceding arguments cause a U-shaped pricing kernel.

3.3 Plots and Comparative Statics

To fit the pricing kernel to index return data, we aim to fit the marginal distribution of r to a return distribution of an index and take advantage of the properties of normal variance-mean mixtures mentioned earlier. Recall that the marginal distribution represents the physical distribution of an underlying index return that is used when estimating an empirical pricing kernel. As is well known, the distribution of a normal variance-mean mixture when the generalized inverse Gaussian distribution serves as a mixing distribution is a generalized hyperbolic (GH) distribution, introduced by Barndorff-Nielsen (1977).

Eberlein and Prause (2002) study applications of the GH distribution in finance. In particular, they fit return data from the German DAX from 1993 to 1997 to the GH distribution, and we adopt the parameter values (see Table I).

Parameter	λ	β	δ	κ	α
Value	-2.018	-24.91 ⁹	1571.6	0.0003	0.0034

Table I: Fitted parameter values, DAX 1993 to 1997 (Eberlein and Prause, 2002)

To give a better insight into the GH distribution and its properties, we additionally perform a parameter fitting to more recent data. In particular, we fit the GH distribution to daily log return observations from the S&P 500 from January 2020 to December 2023. The corresponding parameter values to the GH fit in Figure 1 are given in Table II and the process of parameter fitting is described in Appendix A.2.¹⁰

Parameter	λ	β	δ	κ	α
Value	-2.017	-19.36	971.3	0.0003	0.0030

Table II: Fitted parameter values, S&P 500 2020 to 2023

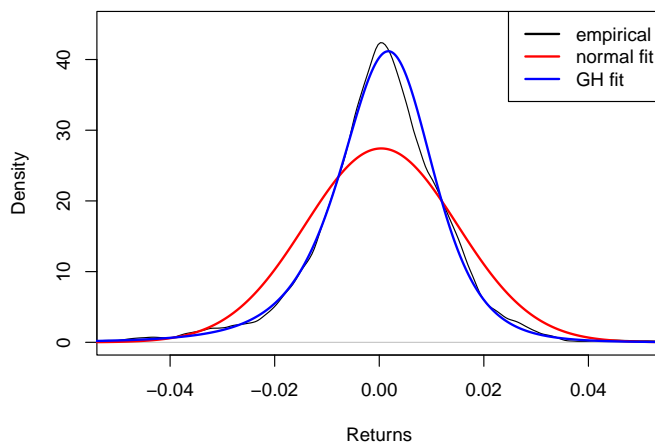
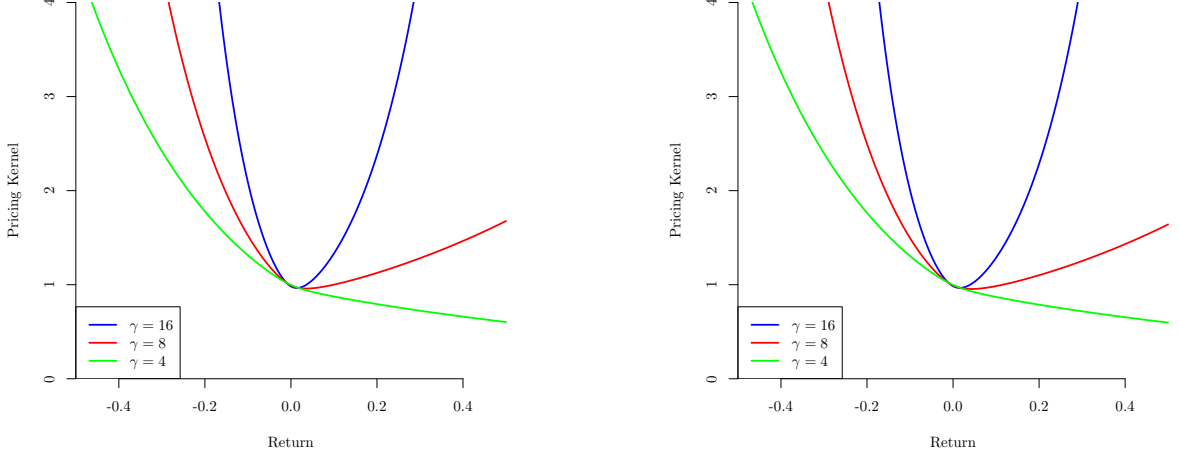


Figure 1: Fitting of the GH and normal distribution to S&P 500 daily return data from January 2020 to December 2023

⁹Note that β is a skewness parameter of the GH distribution. A negative β indicates a negatively skewed return distribution, i.e. a long left tail with negative returns.

¹⁰For a more detailed and sophisticated approach to fitting the GH distribution to return data see Prause (1999).

Figure 1 illustrates the advantages of the GH distribution. Having a more pronounced peak in the center and having more mass in the tails, the GH distribution is a much better fit to return data than, e.g., a normal distribution. We plot both pricing kernels in Figure 2 for a fixed parameter of risk aversion ρ and different values for the parameter of ambiguity aversion γ . Observe that both pricing kernels are pretty similar, with a higher



(a) Pricing kernel for $\rho = 2$ and different values for γ , DAX 1993 to 1997

(b) Pricing kernel for $\rho = 2$ and different values for γ , S&P 500 2020 to 2023

Figure 2: Pricing kernels

ambiguity aversion implying a more pronounced U-shape. For higher values of ambiguity aversion the pricing kernels match empirically observed U-shaped pricing kernels, as found by, e.g., Cuesdeanu and Jackwerth (2018) and Sichert (2023), quite well. An ambiguity aversion of $\gamma = 4$ is not large enough to dominate the effect of risk attitude.

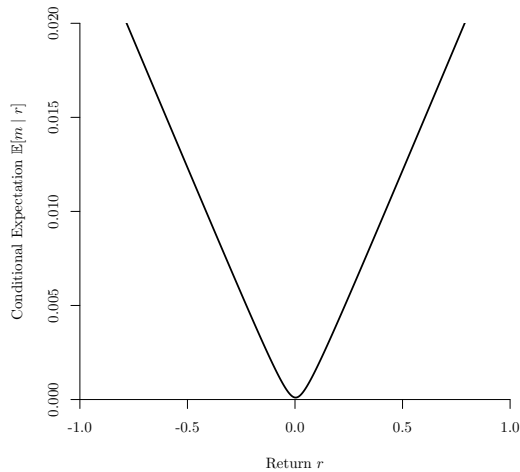


Figure 3: Expected variance m conditional on observed return r (for parameter values from Table II)

Let us now investigate the economic intuition underlying the shape of the pricing

kernel in our model. Recall the effect of ambiguity aversion given by a mean of marginal ambiguity utility, weighted by the posterior distribution. As can be seen in Figure 3, the posterior distribution assigns a greater probability mass to higher variances for returns that deviate from zero. That is, observing a high or low index return, the agent believes that the current variance of the index returns is high. As a result of the normal variance-mean mixture with $\beta < 0$, a high variance indicates a recessive regime. The anticipation of a bad regime is assessed by the agent's utility function $\phi = v \circ u^{-1}$. By ambiguity aversion, ϕ is a concave function such that marginal ambiguity utility decreases. However, the agent evaluates marginal ambiguity utility at expected utility, which decreases as we move to more recessive regimes of the economy. Hence, marginal ambiguity utility is an increasing function of the variance. Since greater ambiguity aversion results in ϕ being more concave, a greater ambiguity aversion causes a more pronounced U-shape of the pricing kernel. Otherwise, if ambiguity aversion is sufficiently small, the effect of risk attitude dominates and the pricing kernel will be monotonically decreasing, as suggested by Theorem 1.

To wrap up, observing a high return of the index, the agent believes that the variance is high, which hints at a recessive business cycle. As an ambiguity averse agent values payoffs in recessions highly, marginal ambiguity utility is high. The greater the ambiguity aversion, the higher payoffs are valued in recessive business cycles.

4 Conclusion

This paper studied the pricing kernel in a pure exchange economy under identifiable model uncertainty. Uncertainty is driven by the agents' lack of awareness of the business cycle that is in play at the point of decision making. However, the existing literature covers the modeling of a discrete set of regimes, mostly either recession or expansion. By linking normal variance-mean mixtures of Barndorff-Nielsen et al., (1982) with uncertainty, we were able to model uncertainty about a continuum of different business cycles.

This led us to give an explanation of U-shaped pricing kernels. Observing a high index return, the agent believes that the economy is in a recessive regime. As an ambiguity averse agent values payoffs highly in recessions, the pricing kernel increases. The more ambiguity averse the agent, the more pronounced the U-shape of the pricing kernel.

References

- Abramowitz, Milton and Stegun, Irene A (1968). *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, Vol. 55. US Government printing office.
- Ait-Sahalia, Yacine and Lo, Andrew W (2000). "Nonparametric risk management and implied risk aversion", *Journal of Econometrics*, Vol. 94 No. 1-2, pp. 9–51.
- Bae, Jinho, Kim, Chang-Jin, and Nelson, Charles R (2007). "Why are stock returns and volatility negatively correlated?", *Journal of Empirical Finance*, Vol. 14 No. 1, pp. 41–58.
- Bakshi, Gurdip, Madan, Dilip, and Panayotov, George (2010). "Returns of claims on the upside and the viability of U-shaped pricing kernels", *Journal of Financial Economics*, Vol. 97 No. 1, pp. 130–154.

- Barndorff-Nielsen, Ole (1977). “Exponentially decreasing distributions for the logarithm of particle size”, *Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences*, Vol. 353 No. 1674, pp. 401–419.
- (1997). “Normal inverse Gaussian distributions and stochastic volatility modelling”, *Scandinavian Journal of Statistics*, Vol. 24 No. 1, pp. 1–13.
- Barndorff-Nielsen, Ole, Kent, John, and Sørensen, Michael (1982). “Normal variance-mean mixtures and z distributions”, *International Statistical Review*, pp. 145–159.
- Bertsekas, Dimitri, Nedic, Angelia, and Ozdaglar, Asuman (2003). *Convex analysis and optimization*, Vol. 1. Athena Scientific.
- Bingham, Nicholas H, Kiesel, Rüdiger, and Schmidt, Rafael (2003). “A semi-parametric approach to risk management”, *Quantitative Finance*, Vol. 3 No. 6, p. 426.
- Cecchetti, Stephen G, Lam, Pok-sang, and Mark, Nelson C (1990). “Mean Reversion in Equilibrium Asset Prices”, *American Economic Review*, Vol. 80, pp. 398–418.
- Cerreia-Vioglio, Simone, Maccheroni, Fabio, Marinacci, Massimo, and Montrucchio, Luigi (2013). “Ambiguity and robust statistics”, *Journal of Economic Theory*, Vol. 148 No. 3, pp. 974–1049.
- Chen, An, Vanduffel, Steven, and Wilke, Morten (2025). “Optimal payoffs under smooth ambiguity”, *European Journal of Operational Research*, Vol. 320 No. 3, pp. 754–764.
- Christoffersen, Peter, Heston, Steven, and Jacobs, Kris (2013). “Capturing option anomalies with a variance-dependent pricing kernel”, *The Review of Financial Studies*, Vol. 26 No. 8, pp. 1963–2006.
- Cuesdeanu, Horatio and Jackwerth, Jens Carsten (2018). “The pricing kernel puzzle: Survey and outlook”, *Annals of Finance*, Vol. 14, pp. 289–329.
- Denti, Tommaso and Pomatto, Luciano (2022). “Model and predictive uncertainty: A foundation for smooth ambiguity preferences”, *Econometrica*, Vol. 90 No. 2, pp. 551–584.
- DeStefano, Michael (2004). “Stock returns and the business cycle”, *Financial Review*, Vol. 39 No. 4, pp. 527–547.
- Eberlein, Ernst (2001). “Application of generalized hyperbolic Lévy motions to finance”, *Lévy processes: theory and applications*. Springer, pp. 319–336.
- Eberlein, Ernst and Prause, Karsten (2002). “The generalized hyperbolic model: financial derivatives and risk measures”, *Mathematical Finance—Bachelier Congress 2000: Selected Papers from the First World Congress of the Bachelier Finance Society, Paris, June 29–July 1, 2000*. Springer, pp. 245–267.
- Gollier, Christian (2011). “Portfolio choices and asset prices: The comparative statics of ambiguity aversion”, *The Review of Economic Studies*, Vol. 78 No. 4, pp. 1329–1344.
- Hamilton, James D (1989). “A new approach to the economic analysis of nonstationary time series and the business cycle”, *Econometrica*, pp. 357–384.
- Hamilton, James D and Lin, Gang (1996). “Stock market volatility and the business cycle”, *Journal of Applied Econometrics*, Vol. 11 No. 5, pp. 573–593.
- Hara, Chiaki, Mukerji, Sujoy, Riedel, Frank, and Tallon, Jean-Marc (2024). “Sharing Model Uncertainty”, Working Paper. URL: <https://shs.hal.science/halshs-04598577>.
- Hens, Thorsten and Reichlin, Christian (2013). “Three solutions to the pricing kernel puzzle”, *Review of Finance*, Vol. 17 No. 3, pp. 1065–1098.
- Jackwerth, Jens Carsten (2000). “Recovering risk aversion from option prices and realized returns”, *The Review of Financial Studies*, Vol. 13 No. 2, pp. 433–451.

- Ju, Nengjiu and Miao, Jianjun (2012). “Ambiguity, learning, and asset returns”, *Econometrica*, Vol. 80 No. 2, pp. 559–591.
- Kandel, Shmuel and Stambaugh, Robert F (1991). “Asset returns and intertemporal preferences”, *Journal of Monetary Economics*, Vol. 27 No. 1, pp. 39–71.
- Klibanoff, Peter, Marinacci, Massimo, and Mukerji, Sujoy (2005). “A smooth model of decision making under ambiguity”, *Econometrica*, Vol. 73 No. 6, pp. 1849–1892.
- Kroencke, Tim A (2022). “Recessions and the stock market”, *Journal of Monetary Economics*, Vol. 131, pp. 61–77.
- Luciano, Elisa and Semeraro, Patrizia (2010). “A generalized normal mean-variance mixture for return processes in finance”, *International Journal of Theoretical and Applied Finance*, Vol. 13 No. 03, pp. 415–440.
- Powell, Michael JD (1964). “An efficient method for finding the minimum of a function of several variables without calculating derivatives”, *The Computer Journal*, Vol. 7 No. 2, pp. 155–162.
- Prause, Karsten (1999). *The generalized hyperbolic model: Estimation, financial derivatives, and risk measures*, PhD thesis. Citeseer.
- Rosenberg, Joshua V and Engle, Robert F (2002). “Empirical pricing kernels”, *Journal of Financial Economics*, Vol. 64 No. 3, pp. 341–372.
- Sichert, Tobias (2023). “The Pricing Kernel is U-shaped”, Working Paper. URL: <https://ssrn.com/abstract=3095551>.
- Yang, Zhen-Hang and Tian, Jing-feng (2024). “High order monotonicity of a ratio of the modified Bessel function with applications”, *Hacettepe Journal of Mathematics and Statistics*, pp. 1–15.
- Yang, Zhen-Hang and Zheng, Shen-Zhou (2017). “The monotonicity and convexity for the ratios of modified Bessel functions of the second kind and applications”, *Proceedings of the American Mathematical Society*, Vol. 145 No. 7, pp. 2943–2958.

A Appendix

A.1 Additional Lemmas and Proofs

Remark 1. Suppose we have a normal variance-mean mixture R given by

$$R = \alpha + \beta M + \sqrt{M}X,$$

where M follows a generalized inverse Gaussian distribution with parameters $\delta > 0$, $\kappa > 0$ and $\lambda \in \mathbb{R}$. Then the conditional distribution of M given an observation r is also generalized inverse Gaussian with parameters

$$\delta + \beta^2, \kappa + (r - \alpha)^2 \text{ and } \lambda - \frac{1}{2}.$$

Proof of Lemma 1. The exponential terms in the integral in the formula for the pricing kernel read as

$$\exp \left(-(\gamma - \rho)(\alpha + \beta m) - (\gamma - \rho)(1 - \rho)\frac{1}{2}m - \frac{1}{2} \left((\delta + \beta^2)m + \frac{\kappa + (r - \alpha)^2}{m} \right) \right).$$

Dropping all terms independent of m and rearranging terms yield

$$\exp \left(- \left(2\beta(\gamma - \rho) + (\gamma - \rho)(1 - \rho) + \delta + \beta^2 \right) m - \frac{\kappa + (r - \alpha)^2}{2m} \right). \quad (11)$$

Since we only allow for $\kappa > 0$, we have that $\kappa + (r - \alpha)^2 > 0$ such that

$$\int_0^\infty \exp\left(-\frac{\kappa + (r - \alpha)^2}{m}\right) dm < \infty.$$

Moreover, from (11) it is obvious to see that

$$2\beta(\gamma - \rho) + (\gamma - \rho)(1 - \rho) + \delta + \beta^2 > 0$$

must hold since $\int_0^\infty \exp(cm) dm = \infty$ for any $c > 0$. \square

Proof of Lemma 2. By Eberlein (2001), the modified Bessel function of the second kind possesses the integral representation

$$K_\lambda(z) = \frac{1}{2} \int_0^\infty x^{\lambda-1} \exp\left(-\frac{z}{2}\left(x + \frac{1}{x}\right)\right) dx. \quad (12)$$

In (12), substitute $x = \sqrt{\frac{\eta}{\zeta}}z$. Then,

$$\begin{aligned} K_\lambda(\sqrt{\eta\zeta}) &= \frac{1}{2} \int_0^\infty \sqrt{\frac{\eta}{\zeta}}^{\lambda-1} z^{\lambda-1} \exp\left(-\frac{\sqrt{\eta\zeta}}{2}\left(\sqrt{\frac{\eta}{\zeta}}z + \frac{\sqrt{\frac{\eta}{\zeta}}}{z}\right)\right) \sqrt{\frac{\eta}{\zeta}} dz \\ &= \frac{1}{2} \sqrt{\frac{\eta}{\zeta}}^\lambda \int_0^\infty z^{\lambda-1} \exp\left(-\frac{1}{2}\left(\eta z + \frac{\zeta}{z}\right)\right) dz. \end{aligned}$$

\square

Proof of Lemma 3. By Lemma 2 and choosing

$$\eta = 2\beta(\gamma - \rho) + (\gamma - \rho)(1 - \rho) + \delta + \beta^2 \text{ and } \zeta = \kappa + (r - \alpha)^2,$$

we infer

$$\begin{aligned} \psi(r) &= \exp(-\rho r - \alpha(\gamma - \rho)) (\delta + \beta^2)^{\frac{2\lambda-1}{4}} (2\beta(\gamma - \rho) + (\gamma - \rho)(1 - \rho) + \delta + \beta^2)^{\frac{1-2\lambda}{4}} \\ &\quad \frac{K_{\lambda-\frac{1}{2}}\left(\sqrt{\kappa + (r - \alpha)^2} \sqrt{2\beta(\gamma - \rho) + (\gamma - \rho)(1 - \rho) + \delta + \beta^2}\right)}{K_{\lambda-\frac{1}{2}}\left(\sqrt{(\delta + \beta^2)(\kappa + (r - \alpha)^2)}\right)}. \end{aligned}$$

Substituting all constant terms by $C > 0$,

$$\psi(r) = C \exp(-\rho r) \frac{K_{\lambda-\frac{1}{2}}\left(\sqrt{\kappa + (r - \alpha)^2} \sqrt{2\beta(\gamma - \rho) + (\gamma - \rho)(1 - \rho) + \delta + \beta^2}\right)}{K_{\lambda-\frac{1}{2}}\left(\sqrt{\kappa + (r - \alpha)^2} \sqrt{\delta + \beta^2}\right)}.$$

\square

Lemma 4. *Define*

$$f(t) := C_2^2 \frac{t^2 - C_2^2 x^2}{(C_2^2 x^2 + t^2)^2} - C_1^2 \frac{t^2 - C_1^2 x^2}{(C_1^2 x^2 + t^2)^2}$$

for $t > 0$ and $C_2, C_1 \in \mathbb{R}_{++}$ with $C_2 > C_1$. Then, f is continuous on $(0, \infty)$ and there exists exactly one solution to $f(t) = 0$, given by

$$t_0 = \frac{x}{\sqrt{2}} \sqrt{C_1^2 + C_2^2 + \sqrt{C_1^4 + 14C_1^2C_2^2 + C_2^4}},$$

and $f(t) < 0$ for all $t \in (0, t_0)$ and $f(t) > 0$ for all $t \in (t_0, \infty)$. Moreover, we have

$$\int_0^\infty f(t)dt = 0.$$

Proof. Continuity is obvious. For the second part, we have to solve

$$C_2^2 \frac{t^2 - C_2^2 x^2}{(C_2^2 x^2 + t^2)^2} = C_1^2 \frac{t^2 - C_1^2 x^2}{(C_1^2 x^2 + t^2)^2},$$

which can be rewritten as

$$t^6(C_2^2 - C_1^2) + t^4(x^2(C_1^4 - C_2^4)) + t^2(3x^4(C_1^4C_2^2 - C_2^4C_1^2)) = 0.$$

As $t > 0$, we have

$$t^4(C_2^2 - C_1^2) + t^2(x^2(C_1^4 - C_2^4)) + 3x^4(C_1^4C_2^2 - C_2^4C_1^2) = 0.$$

Substituting $t^2 = u$ and since $C_2 \neq C_1$, we have

$$u^2 - ux^2 \frac{C_2^4 - C_1^4}{C_2^2 - C_1^2} + 3x^4 \frac{C_1^4C_2^2 - C_2^4C_1^2}{C_2^2 - C_1^2} = 0.$$

This yields

$$\begin{aligned} u_{1/2} &= \frac{x^2}{2}(C_2^2 + C_1^2) \pm \sqrt{\frac{\left(x^2 \frac{C_2^4 - C_1^4}{C_2^2 - C_1^2}\right)^2}{4} - 3x^4 \frac{C_1^4C_2^2 - C_2^4C_1^2}{C_2^2 - C_1^2}} \\ &= \frac{x^2}{2} \left((C_2^2 + C_1^2) \pm \frac{\sqrt{(C_1^4 - C_2^4)^2 - 12(C_1^4C_2^2 - C_2^4C_1^2)(C_2^2 - C_1^2)}}{C_2^2 - C_1^2} \right) \\ &= \frac{x^2}{2} \left((C_2^2 + C_1^2) \pm \frac{\sqrt{(C_1^2 - C_2^2)^2(C_1^4 + 14C_1^2C_2^2 + C_2^4)}}{C_2^2 - C_1^2} \right) \\ &= \frac{x^2}{2} \left((C_2^2 + C_1^2) \pm \sqrt{C_1^4 + 14C_1^2C_2^2 + C_2^4} \right). \end{aligned}$$

As $C_2^2 + C_1^2 < \sqrt{C_1^4 + 14C_1^2C_2^2 + C_2^4}$, resubstituting yields the unique (real) solution

$$t_0 = \frac{x}{\sqrt{2}} \sqrt{C_1^2 + C_2^2 + \sqrt{C_1^4 + 14C_1^2C_2^2 + C_2^4}}.$$

Define $t_- := \frac{x}{\sqrt{2}}C_1 < t_0$. Then

$$f(t_-) = C_2^2 \frac{\frac{x^2C_1^2}{2} - C_2^2x^2}{(C_2^2x^2 + \frac{x^2C_1^2}{2})^2} - C_1^2 \frac{\frac{x^2C_1^2}{2} - C_1^2x^2}{(C_1^2x^2 + \frac{x^2C_1^2}{2})^2} = \frac{18C_2^2(C_1^2 - 2C_2^2) + 2(2C_2^2 + C_1^2)^2}{9x^2(2C_2^2 + C_1^2)^2},$$

which is negative if and only if $18C_2^2(C_1^2 - 2C_2^2) + 2(2C_2^2 + C_1^2)^2$ is negative. This term can be rewritten as

$$28C_2^2(C_1^2 - C_2^2) + 2(C_1^4 - C_1^2C_2^2),$$

which is clearly negative since $C_2 > C_1$. Hence $f(t) < 0$ for all $t \in (0, t_0)$. But since $\int_0^\infty f(t)dt = 0$ and f crosses the t -axis only once, we have that $f(t) > 0$ for all $t \in (t_0, \infty)$. For the integral, consider

$$\int_0^\infty f(t)dt = \int_0^\infty C_2^2 \frac{t^2 - C_2^2 x^2}{(C_2^2 x^2 + t^2)^2} dt - \int_0^\infty C_1^2 \frac{t^2 - C_1^2 x^2}{(C_1^2 x^2 + t^2)^2} dt.$$

We focus on

$$\int_0^\infty C_2^2 \frac{t^2 - C_2^2 x^2}{(C_2^2 x^2 + t^2)^2} dt = C_2^2 \left(\int_0^\infty \frac{1}{t^2 + C_2^2 x^2} dt - 2C_2^2 x^2 \int_0^\infty \frac{1}{(t^2 + C_2^2 x^2)^2} dt \right).$$

We calculate the first integral by substituting $u = \frac{t}{C_2 x}$. Then

$$\int \frac{1}{t^2 + C_2^2 x^2} dt = \int \frac{1}{u^2 C_2^2 x^2 + C_2^2 x^2} C_2 x du = \frac{1}{C_2 x} \int \frac{1}{u^2 + 1} du = \frac{\arctan\left(\frac{t}{C_2 x}\right)}{C_2 x}.$$

For the second integral, consider $t = C_2 x \tan(u)$ such that $dt = C_2 x \frac{1}{\cos^2(u)} du$. Then

$$\begin{aligned} \int \frac{1}{(t^2 + C_2^2 x^2)^2} dt &= \int \frac{C_2 x \frac{1}{\cos^2(u)}}{(C_2^2 x^2 \tan^2(u) + C_2^2 x^2)^2} du = \int \frac{C_2 x \frac{1}{\cos^2(u)}}{C_2^4 x^4 \left(\frac{1}{\cos^2(u)}\right)^2} du \\ &= \frac{1}{C_2^3 x^3} \int \cos^2(u) du \\ &= \frac{\frac{1}{2}u + \frac{1}{4}\sin(2u)}{C_2^3 x^3}. \end{aligned}$$

Resubstituting yields

$$\int \frac{1}{(t^2 + C_2^2 x^2)^2} dt = \frac{\arctan(\frac{t}{C_2 x})}{2C_2^3 x^3} + \frac{\sin(2 \arctan(\frac{t}{C_2 x}))}{4C_2^3 x^3}.$$

All in all,

$$\begin{aligned} &\int_0^\infty C_2^2 \frac{t^2 - C_2^2 x^2}{(C_2^2 x^2 + t^2)^2} dt \\ &= C_2^2 \left[\frac{\arctan\left(\frac{t}{C_2 x}\right)}{C_2 x} - 2C_2^2 x^2 \left(\frac{\arctan(\frac{t}{C_2 x})}{2C_2^3 x^3} + \frac{\sin(2 \arctan(\frac{t}{C_2 x}))}{4C_2^3 x^3} \right) \right]_{t=0}^{t=\infty} \\ &= C_2^2 \left[-\frac{\sin(2 \arctan(\frac{t}{C_2 x}))}{2C_2 x} \right]_{t=0}^{t=\infty} = C_2 \left(-\frac{\sin(\pi)}{2x} + \frac{\sin(0)}{2x} \right) = 0, \end{aligned}$$

which completes the proof. □

A.2 Parameter Fitting

By Lemma 2, the marginal distribution $p_\mu(r)$ is given by

$$\frac{\left(\frac{\delta}{\kappa}\right)^{\frac{\lambda}{2}}}{\sqrt{2\pi}K_\lambda\left(\sqrt{\delta\kappa}\right)}e^{\beta(r-\alpha)}\left(\frac{\kappa+(r-\alpha)^2}{\beta^2+\delta}\right)^{\frac{\lambda}{2}-\frac{1}{4}}K_{\lambda-\frac{1}{2}}\left(\sqrt{\beta^2+\delta}\sqrt{\kappa+(r-\alpha)^2}\right), \quad (13)$$

which is the density of a generalized hyperbolic distribution with parameters

$$\lambda, \beta, \delta, \kappa \text{ and } \alpha.$$

However, to fit the GH distribution to daily log return data, we use a different parametrization, provided by Prause (1999). It allows for an easier interpretation of the parameters and is given by

$$g(r \mid \lambda, \bar{\alpha}, \bar{\beta}, \sigma, \mu) = \frac{(\bar{\alpha} - \bar{\beta})^{\frac{\lambda}{2}}}{\sqrt{2\pi}\bar{\alpha}^{\lambda-\frac{1}{2}}\sigma K_\lambda\left(\sqrt{\bar{\alpha}^2 - \bar{\beta}^2}\right)} \left(1 + \left(\frac{r - \mu}{\sigma}\right)^2\right)^{\frac{\lambda}{2}-\frac{1}{4}} K_{\lambda-\frac{1}{2}}\left(\bar{\alpha}\sqrt{1 + \left(\frac{r - \mu}{\sigma}\right)^2}\right) \exp\left(\bar{\beta}\left(\frac{r - \mu}{\sigma}\right)\right).$$

Note that there is a clear interpretation for μ being the location and σ being the scale parameter, next to the others being shape parameters. We assume independent observations and derive the log likelihood function, given by

$$\begin{aligned} \mathcal{L}(\lambda, \bar{\alpha}, \bar{\beta}, \sigma, \mu \mid r_1, \dots, r_n) = & n \log \left(\frac{(\bar{\alpha} - \bar{\beta})^{\frac{\lambda}{2}}}{\sqrt{2\pi}\bar{\alpha}^{\lambda-\frac{1}{2}}\sigma K_\lambda\left(\sqrt{\bar{\alpha}^2 - \bar{\beta}^2}\right)} \right) \\ & + \left(\frac{\lambda}{2} - \frac{1}{4}\right) \sum_{i=1}^n \log \left(1 + \left(\frac{r_i - \mu}{\sigma}\right)^2 \right) \\ & + \sum_{i=1}^n \log \left(K_{\lambda-\frac{1}{2}}\left(\bar{\alpha}\sqrt{1 + \left(\frac{r_i - \mu}{\sigma}\right)^2}\right) \right) + \bar{\beta} \left(\frac{r_i - \mu}{\sigma}\right). \end{aligned}$$

In order to avoid taking derivatives with respect to λ , we use an optimization algorithm that does not make use of derivatives. In particular, we use the method by Powell (1964). To select suitable starting values, we combine the interpretation of μ and σ as well as Eberlein and Prause (2002)'s values, which leads to the starting values in Table III.

Parameter	λ	$\bar{\alpha}$	$\bar{\beta}$	σ	μ
Value	-2.018	$46.82 \cdot \text{sd}(\text{data})$	$-24.91 \cdot \text{sd}(\text{data})$	$\text{sd}(\text{data})$	$\text{mean}(\text{data})$

Table III: Starting values for the ML algorithm

We fit daily log return data from the S&P 500 from 01.01.2020 to 31.12.2023, provided by *FRED*. We compute log returns as

$$s_t = \log \left(\frac{\text{Close}_t}{\text{Close}_{t-1}} \right).$$

The algorithm converged after 19 iterations to the parameter values given in Table II, with a log likelihood value of 2981.15.