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# Exploiting complete linear descriptions for decentralized power market problems with integralities

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## Abstract

It is well known that linear prices supporting a competitive equilibrium exist in the case of convex markets, however, in the presence of integralities this is open and hard to decide in general. We present necessary and sufficient conditions for the existence of such prices for decentralized market problems where market participants have integral decision variables and their feasible sets are given in complete linear description. We utilize total unimodularity and the aforementioned conditions to show that such linear prices exist and present some applications. Furthermore, we compute competitive equilibria for two classes of decentralized market problems arising in energy markets and show that competitive equilibria may exist regardless of integralities.

**Keywords** Mixed-integer programming · Competitive equilibrium · Linear prices · Total unimodularity

## 1 Introduction

In energy markets decentralized market problems are commonly modeled via producers who maximize their profit and consumers who maximize their utility (O'Neill et al. 2005). Furthermore it is assumed that producers and consumers regard prices as independent of their own choices (Arrow and Debreu 1954). If the markets are

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convex, i.e., the market participants' problems are convex and the market clearing constraint is linear, there are linear prices, meaning uniform and constant per quantity, leading to a competitive equilibrium, also shown by Arrow and Debreu (1954); Gale (1955); McKenzie (1959), as the dual variables associated with the market clearing constraints can be interpreted as market clearing prices (Azizan et al. 2019). In the case of nonconvexities, however, linear prices may not exist (Wolsey 1981).

Nonconvexities caused by integralities arise in energy markets from, e.g., min-up/min-down constraints (Lee et al. 2004; Hua and Baldick 2016) or startup costs (Vyve 2011; Ruiz et al. 2012). A classical example for nonconvexities in energy markets is given by Scarf (1994). The market participants' problems are often modeled as mixed-integer problems. In the case that the corresponding linear relaxation does not solve the mixed-integer problem, linear prices may not exist (O'Neill et al. 2005).

There are various methods for finding prices supporting a competitive equilibrium, regardless of integralities, via side payments, called *uplifts*. A standard approach was introduced in O'Neill et al. (2005) referred to as *IP pricing*. This approach was introduced using the example of a power market. The main idea of the approach is to pay not only for the generated output, but also for integral decisions of market participants. In order to compute prices for those integral decisions, in a first step a mixed-integer problem minimizing the cost of meeting the demand is solved, and in a second step the corresponding linear problem with all integral variables fixed to their optimal values, as computed in the first step. The dual variables corresponding to the additional equations are interpreted as prices. Because the uplifts of IP pricing can be volatile and unnecessarily high (Hogan and Ring 2003), a pricing scheme was proposed with less volatility (Bjørndal and Jörnsten 2010) and with minimal uplifts (Hogan and Ring 2003; Gribik et al. 2007).

In power markets the market clearing is ensured via such side-payments — for example the New York Independent System Operator also uses mixed-integer programming in order to compute unit commitment and economic dispatch and linear programming to compute prices (New York Independent System Operator 2017). Likewise the Midwest Independent Transmission System Operator uses a model incorporating mixed-integer decision variables such as minimum uptime and minimum downtime constraints (Carlson et al. 2012).

In the literature the existence of market clearing prices is investigated for the following three important special cases of decentralized market problems. In Bikhchandani and Mamer (1997), where a result regarding the existence of linear prices, supporting a competitive equilibrium, is introduced for the special case of an exchange economy. Also in Hatfield et al. (2019) the existence of competitive equilibria in trading networks is being investigated. Here there are buyers and sellers and a certain set of possible contracts, which these buyers and sellers can conclude with each other. Another special case is investigated by Bikhchandani et al. (2002) who analyze a package assignment model with various price functions.

Recently, Harks (2019) has independently shown similar results to ours within a different context.

In this article, we consider decentralized market problems, where the market participants solve mixed-integer linear problems and the coupling condition is linear. So far, to the best of our knowledge, there has not been established a proposition, stating

under which necessary and sufficient conditions there exist linear prices leading to a competitive equilibrium of these problems. We state a sufficient condition under which linear prices, supporting a competitive equilibrium, exist. The condition is utilized to show that such prices exist for decentralized market problems where the market participants solve integer problems with totally unimodular constraint matrices and integral right-hand sides. In those decentralized market problems the market participants are coupled via one variable each. We then give applications for decentralized market problems fulfilling these properties.

Finally we give a computational study in which we exemplify that checking whether a competitive equilibrium exists is worthwhile because competitive equilibria may exist regardless of integralities. Additionally we show that how the decentralized market problems are modeled heavily influences the ability to find competitive equilibria. For the computations we both consider decentralized market problems arising through min-up/min-down constraints and decentralized market problems arising through Scarf's example. These computations to Scarf's example show that it is useful to check for the existence of competitive equilibria even though there are integralities in the market. Furthermore, we show that, in order to check for the existence of competitive equilibria, it is essential to use a certain problem formulation which we discuss later in the paper. The importance of using this formulation is illustrated by the min-up/min-down example where we compare two formulations of the arising problems.

## 2 Existence of linear prices

In this section the main goal is to prove Theorem 1, which states when a solution to a mixed-integer linear decentralized market problem exists. Furthermore we show that this Theorem 1 can not be generalized to cover convex mixed-integer nonlinear decentralized market problems. Let  $\mathcal{N} := \{1, \dots, N\}$  be a set of market participants where each participant  $n$  has variables for production, as entries of  $p_n \in \mathbb{Z}^{T_I} \times \mathbb{R}^{T_C}$ . We denote the dimension of  $p_n$  with  $T = T_C + T_I$ . Each market participant may have further auxiliary variables, as entries of  $x_n \in \mathbb{Z}^{m_I} \times \mathbb{R}^{m_C}$ , for modeling further aspects besides production. Let  $m = m_C + m_I$  take the dimensions of  $x_n$  which we suppose to be independent of  $n$ . An example for such variables are the binary variables in (Gribik et al. 2007, Chapter 'Electricity Market Model') which model whether a machine is turned on or off. Every market participant solves a mixed-integer problem of the form

$$\max_{p_n, x_n} -c_n^{\text{aux}\top} x_n - c_n^{\text{prod}\top} p_n + \pi^\top p_n \quad (1a)$$

$$\text{s.t. } A_n^{\text{aux}} x_n + A_n^{\text{prod}} p_n = b_n \quad (1b)$$

$$x_n \in \mathbb{Z}^{m_I} \times \mathbb{R}^{m_C} \quad (1c)$$

$$p_n \in \mathbb{Z}^{T_I} \times \mathbb{R}^{T_C} \quad (1d)$$

$$x_n, p_n \geq 0, \quad (1e)$$

where the auxiliary costs are the entries of  $c_n^{\text{aux}} \in \mathbb{R}^m$ , variable costs are the entries of  $c_n^{\text{prod}} \in \mathbb{R}^T$  and the market prices are the entries of  $\pi \in \mathbb{R}^q$ . The matrices  $A_n^{\text{aux}} \in \mathbb{R}^{r_n \times m}$ ,  $A_n^{\text{prod}} \in \mathbb{R}^{r_n \times T}$  and the vector  $b_n \in \mathbb{R}^{r_n}$  determine the linear constraints of the problem, where the constraints are given as the complete linear description of the feasible set. We assume that the constraints are given such that the market participants are coupled via a linear coupling constraint

$$\sum_{n=1}^N p_n = d. \quad (2)$$

This constraint ensures that a certain demand  $d \in \mathbb{R}^T$  is met and is called the market clearing constraint. The formulation of Problem (1) and Eq. (2) is equivalent to the formulation in Gribik et al. (2007) and similar to O'Neill et al. (2005) where in Eq. (2) the equality is replaced by an inequality.

Now we define a special case of a competitive equilibrium with fixed demand, which is supported by linear prices. This means there are prices that are constant per quantity and are uniform, i.e., the market prices are the same for all market participants.

**Definition 1** (*Decentralized market problem* (O'Neill et al. 2005)) We define a decentralized market problem as a set  $\mathcal{N}$  of market participants, where each market participant  $n \in \mathcal{N}$  solves a mixed-integer problem (1) depending on the market price  $\pi$ , and the market clearing condition (2).

Thus we define the decentralized market problem as O'Neill et al. (2005) such that no market participant has the power to influence the other market participants. This is a common assumption in energy market modeling, see Grimm et al. (2019) within a model for the European entry exit gas market. In pricing problems for electricity markets this assumption is often present as well. An overview over such pricing schemes can be found in (Liberopoulos and Andrianesis 2016). More precisely, O'Neill et al. (2005) make this assumption when introducing the well-known IP-pricing scheme, which is adapted by Bjørndal and Jörnsten (2008, 2010) in order generate less volatile prices. The assumption is also used in the minimum uplift pricing scheme (Hogan and Ring 2003; Gribik et al. 2007; Hua and Baldick 2016). Additional pricing schemes are given, e.g., by Ruiz et al. (2012); Araoz and Jörnsten (2011); Vyve (2011). Furthermore Grübel et al. (2021) use this assumption while investigating the existence of competitive equilibria in gas and power markets.

As we introduced the decentralized market problem we can define a solution to the problem, i.e., a competitive equilibrium.

**Definition 2** (*Solution to a decentralized market problem*) Given a decentralized market problem defined by Definition 1, we say there is a solution  $\hat{\pi}, \hat{x}, \hat{p}$  to the decentralized market problem if and only if there are optimal solutions  $\hat{x}_n, \hat{p}_n$  to all  $n \in \mathcal{N}$  market participants' problems (1), which satisfy the coupling constraint (2). We call a solution  $\hat{\pi}, \hat{x}, \hat{p}$  to the decentralized market problem a competitive equilibrium.

The main differences in the modeling compared to the paper by Bikhchandani and Mamer (1997) is that they allow only for integral goods and an additional continu-

ous variable which denotes the participants' wealth. However, in the case of energy markets divisible goods, i.e., continuous variables, play a role, as exemplified by the computational study in Sect. 4. Furthermore, they assume the reservation value functions of the participants to be weakly increasing. In our context this would imply monotonicity of the cost functions. We do not assume this and thus allow for more freedom in the modeling. This is necessary, e.g., in gas markets to model the cost of reducing the gas flow from a gas field (Tomasgard et al. 2007). In addition, they say a solution exists in case demand for all goods does not exceed supply. In our case demand has to equal supply, which is an assumption we address later on. Additionally, we assume the demand to be fix and not dependent on the choices of consumers.

For an in-depth discussion of more general decentralized market problems see see MasCollell, Whinston, and Green (1995, Part 3). The following problem minimizes the overall cost, while fulfilling the coupling constraint:

$$\min_{p, x} \sum_{n=1}^N c_n^{\text{aux}} \top x_n + c_n^{\text{prod}} \top p_n \quad (3a)$$

$$\text{s.t. } A_n^{\text{aux}} x_n + A_n^{\text{prod}} p_n = b_n \quad \text{for all } n \in \mathcal{N} \quad (3b)$$

$$x_n, p_n \geq 0 \quad \text{for all } n \in \mathcal{N} \quad (3c)$$

$$x_n \in \mathbb{Z}^{m_I} \times \mathbb{R}^{m_C} \quad \text{for all } n \in \mathcal{N} \quad (3d)$$

$$p_n \in \mathbb{Z}^{T_I} \times \mathbb{R}^{T_C} \quad \text{for all } n \in \mathcal{N} \quad (3e)$$

$$\sum_{n=1}^N p_n = d, \quad (3f)$$

which we call the *central planner's problem*.

Recall that we obtain for a general mixed-integer program, e.g., as in (1), the LP-relaxation by neglecting the integrality conditions of the integer variables. In addition we say that a MIP is given in complete linear description if for its solution space holds

$$\text{conv}(\{x \in \mathbb{Z}^n \times \mathbb{R}^m \mid Ax \leq b\}) = \{x \in \mathbb{R}^{n+m} \mid Ax \leq b\}.$$

Note that this implies that the LP-relaxation to a problem given in complete linear description has a mixed-integer optimal solution if a solution exists.

Furthermore note that for linear problems the dual variables corresponding to an optimal solution exist (Solodov 2011, Chap. 1).

As the main goal of this chapter is to prove Theorem 1, we introduce two lemmas which directly prove Theorem 1.

**Theorem 1** Assume we are given a decentralized market problem (Definition 1) where each of the market participants' problems (1) is given in a complete linear description. Then this problem has a solution  $\hat{\pi}, \hat{x}, \hat{p}$  if and only if there is an optimal solution  $x^*, p^*$  to the LP-relaxation of the central planner's problem (3) that satisfies the integrality condition  $x_n \in \mathbb{Z}^{m_I} \times \mathbb{R}^{m_C}, p_n \in \mathbb{Z}^{T_I} \times \mathbb{R}^{T_C}$ .

As already mentioned in the introduction, Harks (2019) has shown similar results independently as part of more general results. More precisely, our Theorem 1 corresponds to Theorem 3.11 in his article. We only refer to this theorem even though the entire paper by Harks (2019) examines more settings and contains more results because these are not relevant for the setting that we consider in this paper. There are little differences between the theorems from a theoretical standpoint and from a modeling standpoint. In the following we categorize and specify these differences. Furthermore, we like to point out that our Theorem 1 can be applied more easily in computations that arise when checking for the existence of decentralized market problems in energy markets, which we do in the last section. In addition we show that Theorem 1, in contrast to Theorem 3.11 by (Harks 2019), can be directly applied to check the existence of competitive equilibria in energy markets without reformulations.

One of the differences between the theorems is that Harks (2019) describes the feasible sets as the convex hull of finitely many points, while we use the outer description. Of course this makes no difference in theory, however, in the case of energy markets we are usually given the outer description and therefore it makes a difference from the computational side, as computing the inner description is complex.

Another difference is that formally his results are more general, because we use linear objective functions, not concave ones as he does. However, in Theorem 3.11 by Harks (2019) the central planner is reformulated as a linear problem using the fact that minimizing over a concave function yields an optimal solution in a vertex of the polyhedral feasible set and thus eliminating the nonlinearity.

Furthermore, there are differences that are pure modeling aspects that we address in the following. We model auxiliary variables explicitly, e.g., for min-up/min-down constraints. We give the market clearing constraint (3f) in the central planner's problem (3) with an equation rather than an inequality. This allows us to directly model negative market prices without reformulations, which is useful in the context of electricity markets, because too much energy in electricity grids is harmful and therefore negative market prices often arise.

Summarized, our Theorem 1 — in theory — can be straight forwardly derived from Theorem 3.11 by Harks (2019). Conversely, when the concave market participants are linearized in the aforementioned way, the existence of competitive equilibria can be analogously be checked for the setting in (Harks 2019, Theorem 3.11). Hence, the differences are the computational tractability and the direct applicability without reformulations to energy markets.

In addition we like to emphasize that the prerequisite of complete linear description is crucial for Theorem 1.

The relaxation that we consider concentrates on the feasible sets of the market participants. As we consider mixed-integer problems we relax the decentralized market problems by LP-relaxing the market participants. Thus, as already mentioned in the introduction O'Neill et al. (2005) state that in case optimal solutions to Problem (3) and its LP-relaxation coincide there is a solution to the decentralized market problem (1), i.e., a solution to the relaxed decentralized market problem is a solution to the decentralized market problem as well. The other direction does not hold, i.e., in case there is an integrality gap in Problem (3) there might still be a solution to the decentralized market problem.

This is exemplified in Sect. 4.3 where we also consider decentralized market problems where the feasible sets of the market participants are not given in complete linear description. Here simply considering the LP-relaxation to the central planner's problem (3) fails to accurately determine whether a solution exists. To be more precise, in case one is given a decentralized market problem (1) with integrality constraints but the market participants' problems (1) are not given in complete linear description — applying Theorem 1 may indicate that there is no solution even though there is a solution.

The idea behind the following Lemma 1 is to use the Karush-Kuhn-Tucker systems to the LP-relaxations of the market participants' problems with the assumption of complete linear description. By stacking these KKT-systems and adding the market clearing constraint the system arises with additional integrality constraints. Solving this system is equivalent to fulfilling the prerequisites of Definition 2 and therefore is equivalent to finding a competitive equilibrium. This system is the KKT-system corresponding to the LP-relaxation of the central planner's problem with additional integrality constraints. Thus only an optimal mixed-integer solution to the LP-relaxation of the central planner solves this system of equations.

**Lemma 1** *Let the market participants' problems (1) be given in complete linear description, then the decentralized market problem has a solution  $\hat{\pi}, \hat{x}, \hat{p}$  if and only if the following system of equations and inequalities has a solution  $\bar{\pi}, \bar{x}, \bar{p}, \bar{\lambda}^{\text{prod}}, \bar{\mu}^{\text{prod}}, \bar{\lambda}^{\text{aux}}, \bar{\mu}^{\text{aux}}$ .*

$$c_n^{\text{prod}} + A_n^{\text{prod}^\top} \mu_n^{\text{prod}} - \lambda_n^{\text{prod}} - \pi = 0 \quad \text{for all } n \in \mathcal{N} \quad (4a)$$

$$c_n^{\text{aux}} + A_n^{\text{aux}^\top} \mu_n^{\text{aux}} - \lambda_n^{\text{aux}} = 0 \quad \text{for all } n \in \mathcal{N} \quad (4b)$$

$$A_n^{\text{aux}} x_n + A_n^{\text{prod}} p_n = b_n \quad \text{for all } n \in \mathcal{N} \quad (4c)$$

$$p_n^\top \lambda_n^{\text{prod}} = 0 \quad \text{for all } n \in \mathcal{N} \quad (4d)$$

$$x_n^\top \lambda_n^{\text{aux}} = 0 \quad \text{for all } n \in \mathcal{N} \quad (4e)$$

$$x_n, p_n, \lambda_n^{\text{aux}}, \lambda_n^{\text{prod}} \geq 0 \quad \text{for all } n \in \mathcal{N} \quad (4f)$$

$$\sum_{n=1}^N p_n = d \quad [-\pi] \quad (4g)$$

$$x_n \in \mathbb{Z}^{m_I} \times \mathbb{R}^{m_C} \quad \text{for all } n \in \mathcal{N} \quad (4h)$$

$$p_n \in \mathbb{Z}^{T_I} \times \mathbb{R}^{T_C} \quad \text{for all } n \in \mathcal{N} \quad (4i)$$

**Proof** First, we show that the existence of a solution to System (4) implies the existence of a solution to the decentralized market problem. If there is a solution  $\bar{\pi}, \bar{x}, \bar{p}, \bar{\lambda}^{\text{prod}}, \bar{\mu}^{\text{prod}}, \bar{\lambda}^{\text{aux}}, \bar{\mu}^{\text{aux}}$  to System (4), then this solution also solves the KKT-systems corresponding to the market participants' problems (1), thus  $\bar{x}, \bar{p}$  also is an optimal solution to the LP-relaxed market participants' problems (1) and with the integrality constraints (4h), (4i) it also is a solution to the non-relaxed problems (1). Furthermore Eq. (4g) in System (4) implies that the coupling condition (2) is fulfilled

and therefore  $\bar{\pi} = \hat{\pi}$ ,  $\bar{x} = \hat{x}$ ,  $\bar{p} = \hat{p}$  is a solution to the decentralized market problem. This implies we can directly use the solution to System (4) as a solution to the decentralized market problem.

Next, we show that the existence of a solution to the decentralized market problem implies the existence of a solution to System (4). If there is a solution  $\hat{\pi}$ ,  $\hat{x}$ ,  $\hat{p}$  to the decentralized market problem then this solution is optimal for the market participants' problems (1). Because of the complete linear description there are dual variables  $\widehat{\mu}_n^{\text{aux}}$ ,  $\widehat{\mu}_n^{\text{prod}}$ ,  $\widehat{\lambda}_n^{\text{aux}}$ ,  $\widehat{\lambda}_n^{\text{prod}}$  such that  $\hat{\pi}$ ,  $\hat{x}_n$ ,  $\hat{p}_n$ ,  $\widehat{\mu}_n^{\text{aux}}$ ,  $\widehat{\mu}_n^{\text{prod}}$ ,  $\widehat{\lambda}_n^{\text{aux}}$ ,  $\widehat{\lambda}_n^{\text{prod}}$  solves the KKT-systems to the market participants' problems (1).

By Definition 2, which defines a solution to a decentralized market problem, this solution also fulfills the coupling condition (3f) and thus also Eq. (4g). The solution is mixed-integer because of the feasibility of the market participants' problems (1) therefore the integrality constraints (4h), (4i) are also fulfilled. Thus we can set  $\bar{\pi} = \hat{\pi}$ ,  $\bar{x} = \hat{x}$ ,  $\bar{p} = \hat{p}$ ,  $\overline{\mu}_n^{\text{aux}} = \widehat{\mu}_n^{\text{aux}}$ ,  $\overline{\mu}_n^{\text{prod}} = \widehat{\mu}_n^{\text{prod}}$ ,  $\overline{\lambda}_n^{\text{aux}} = \widehat{\lambda}_n^{\text{aux}}$ ,  $\overline{\lambda}_n^{\text{prod}} = \widehat{\lambda}_n^{\text{prod}}$ . So here we can use the solution to the decentralized market problem and compute the dual variables corresponding to this solution in order to receive a solution to System (4).  $\square$

The intuition behind Lemma 2 is very similar to the proof of Lemma 1 in the sense that we also show that any solution to the system introduced in the preceding Lemma 1 yields an optimal solution to the LP-relaxed central planner's problem that fulfills the integrality condition and vice versa.

**Lemma 2** *System (4) has a solution  $\bar{\pi}$ ,  $\bar{x}$ ,  $\bar{p}$ ,  $\overline{\mu}_n^{\text{prod}}$ ,  $\overline{\mu}_n^{\text{aux}}$ ,  $\overline{\lambda}_n^{\text{aux}}$ ,  $\overline{\lambda}_n^{\text{prod}}$  if and only if there is an optimal solution  $x^*$ ,  $p^*$  to the LP-relaxed central planner's problem (3) that fulfills the integrality condition  $x_n^* \in \mathbb{Z}^{m_I} \times \mathbb{R}^{m_C}$ ,  $p_n^* \in \mathbb{Z}^{T_I} \times \mathbb{R}^{T_C}$  for all  $n \in \mathcal{N}$ .*

**Proof** System (4) is the KKT-system to the central planner's problem (3) with added integrality constraints (4h), (4i). This implies that finding a solution  $\bar{\pi}$ ,  $\bar{x}$ ,  $\bar{p}$ ,  $\overline{\mu}_n^{\text{prod}}$ ,  $\overline{\mu}_n^{\text{aux}}$ ,  $\overline{\lambda}_n^{\text{aux}}$ ,  $\overline{\lambda}_n^{\text{prod}}$  to the System (4) is equivalent to finding an optimal solution to the LP-relaxation of the central planner's problem (3) because solving the KKT-system is equivalent to finding an optimal solution to the LP-relaxation. With the added integrality constraints (4h), (4i) this implies that if a solution exists for either problem we can set  $\bar{x} = x^*$ ,  $\bar{p} = p^*$ . Therefore the existence of a solution to the decentralized market problem is equivalent to the existence of an optimal solution to the LP-relaxation of the central planner's problem (3) that fulfills the integrality condition  $x_n \in \mathbb{Z}^{m_I} \times \mathbb{R}^{m_C}$ ,  $p_n \in \mathbb{Z}^{T_I} \times \mathbb{R}^{T_C}$ . We can thus use the solution to System (4) as a solution to the central planner's problem (3) and in order to find a solution to System (4) we need to compute the duals to the optimal solution  $x^*$ ,  $p^*$  to the LP-relaxation to the central planner's problem (3).  $\square$

Lemma 1 and Lemma 2 then directly prove Theorem 1.

Note that the existence of a solution to the decentralized market problem is completely independent of the existence of a fractional optimal solution for the LP-relaxed central planner's problem. Such a solution is not feasible for the market participants and thus is not an issue. Also note that the prerequisite of complete linear description

is only needed to show that the existence of a competitive equilibrium leads to an optimal solution to the LP-relaxed central planner's problem (3) that fulfills the integrality constraint.

To show Theorem 1 one can also utilize the minimum uplift scheme: The minimal uplifts are the duality gap of the Lagrangian relaxation of the coupling condition (Vyve 2011; Gribik et al. 2007; Hua and Baldick 2016; Borokhov 2016) and therefore the sum of the minimal uplifts is zero if and only if the optimal values of the mixed integer solution and the LP-relaxation coincide. This follows from the equality of the Lagrangian dual and the LP-relaxation (Geoffrion 1974).

As mentioned in the beginning of this section we show that Theorem 1 does not hold in case the market participants are modeled by convex mixed-integer nonlinear problems. Consider the following example of a decentralized market problem with two identical producers. The producers' problems are given as

$$\begin{aligned} \max_{p_n, x_n} \quad & -\frac{1}{2}2p_n^2 + 1p_n - 3x_n + \pi p_n \\ \text{s.t.} \quad & 0 \leq p_n \leq x_n \\ & x_n \in \{0, 1\}. \end{aligned}$$

The coupling condition is  $p_1 + p_2 = 1$ , which leads to the following central planner's problem

$$\begin{aligned} \min_{p_n, x_n} \quad & \sum_{n \in \{1, 2\}} \frac{1}{2}2p_n^2 - 1p_n + 3x_n \\ \text{s.t.} \quad & 0 \leq p_n \leq x_n && \text{for all } n \in \{1, 2\} \\ & x_n \in \{0, 1\} && \text{for all } n \in \{1, 2\} \\ & p_1 + p_2 = 1. \end{aligned}$$

If we set the market price  $\pi = 3$  then the producers are indifferent between producing  $p_n = 0$  or  $p_n = 1$ . Therefore this is a market clearing price and a competitive equilibrium exists. In the central planner's problem we set w.l.o.g.  $p_1 = 1, p_2 = 0, x_1 = 1, x_2 = 0$ . This leads to an objective value of 3 and is the optimal value of the central planner's MINLP. In the relaxation of the central planner's problem the optimal solution is  $p_1 = 0.5, p_2 = 0.5, x_1 = 0.5, x_2 = 0.5$  with an objective value of 2.5, which is less than 3 and therefore an integrality gap exists but a competitive equilibrium exists regardless of the gap. This implies that Theorem 1 can not be generalized to include mixed-integer nonlinear problems as we need the linearity of the central planner's problem (3) and therefore we also assume the coupling condition (2) to be linear.

The problem defined in Bikhchandani and Mamer (1997, Chap. 3) can be seen as a special case of Problem (3) for an exchange economy. And the problem defined in Bikhchandani, Ostroy, et al. (2002, Chap. 4) for a package assignment model with linear prices can also be seen as a special case of Problem (3). The statements on the existence of competitive equilibria can be transferred like in Theorem 1.

We utilize Theorem 1 in the next section to show the existence of competitive equilibria for a special case of decentralized market problems.

### 3 A special case of decentralized market problems for which solutions exist

In this chapter we introduce a special case of decentralized market problems. We utilize total unimodularity to show that there exists a solution for this special case. First, we introduce an example for such a special case for which a solution exists. This example utilizes integral flow problems.

**Example 1** (Flow) Let the market participants be modeled via integral minimum cost flow problems. Let the capacities on the arcs be integral. The participants are coupled via an arc, which is shared among all participants.

The resulting integer problem for market participant  $n$  can be written as

$$\min_{x_n} c_n^\top x_n + \pi x_n^1 \quad (7a)$$

$$\text{s.t. } E_n x_n \leq u_n \quad (7b)$$

$$M_n x_n = b_n \quad (7c)$$

$$x_n \geq 0 \quad (7d)$$

$$x_n \in \mathbb{Z}^{w_n}. \quad (7e)$$

Here  $M_n \in \mathbb{R}^{v_n \times w_n}$  denotes the incidence matrix of the graph of market participant  $n$ ,  $b_n \in \mathbb{Z}^{v_n}$  the balance of the nodes,  $u_n \in \mathbb{Z}^{w_n}$  the capacity on the arcs and  $x_n^1 \in \mathbb{Z}$  the variable, which pertains to the shared arc. Matrix  $E_n \in \mathbb{R}^{w_n \times w_n}$  is the identity matrix.

The coupling condition is

$$\sum_{n=1}^N x_n^1 = d. \quad (8)$$

Therefore the capacity on the shared arc is saturated.

In the following we show that the preceding example has a solution.

We consider the special case of decentralized market problems where the market participants solve integer problems with totally unimodular constraint matrices and integral right-hand sides. The participants are coupled via one variable each such that the sum of these variables amounts to an integer. We utilize Theorem 1 to show that solutions to these decentralized market problems exist. Then we give examples for decentralized market problems with these properties.

Recall that an  $m \times n$  integral matrix  $A$  is totally unimodular if the determinant of each square submatrix of  $A$  is equal to 0, 1, or  $-1$  (Nemhauser and Wolsey 1988), and, observe the following well known property of totally unimodular matrices.

**Theorem 2** (Ghouila-Houri (1962)) *The following statements are equivalent.*

- $A \in \mathbb{R}^{m \times n}$  is totally unimodular.
- For every  $J \subseteq N = \{1, \dots, n\}$ , there exists a partition  $J_1, J_2$  of  $J$  such that

$$|\sum_{j \in J_1} a_{ij} - \sum_{j \in J_2} a_{ij}| \leq 1 \quad \text{for } i = 1, \dots, m \quad (9)$$

Utilizing this equivalence we can prove a theorem that leads to the existence of solutions to the decentralized market problems with the aforementioned properties. For that we need the following statement.

**Lemma 3** *The matrix  $A$*

$$A := \begin{bmatrix} A_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & A_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & A_3 & \cdots & 0 & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & A_{p-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & A_p \\ h_1 & h_2 & h_3 & \cdots & h_{p-1} & h_p \end{bmatrix},$$

where  $A_i \in \mathbb{R}^{m_i \times n_i}$  are totally unimodular matrices and  $h_i \in \mathbb{R}^{n_i}$  are row vectors of the form  $(1, 0, \dots, 0)$ , is totally unimodular.

**Proof** We are given a set  $J$  of columns of  $A$ . We have to show that there exists a partition  $J_1, J_2$  of  $J$  such that condition (9) holds for all rows  $i$  of  $A$ . Let  $i_c$  be the coupling row in  $A$ . Let  $J_c$  be the set of all indices  $j \in J$  such that the entry  $A_{i_c j} = 1$ . We start constructing  $J_1$  and  $J_2$  by inserting the first  $\lceil |J_c|/2 \rceil$  indices in  $J_c$  into  $J_1$  and the other indices of  $J_c$  into  $J_2$ . Consider the blocks  $A_k$  in turn. We have to distinguish two cases: Either no column of  $A_k$  is in  $J_c$ , then we can, using Theorem 2, partition the columns of  $A_k$  in  $J$  into two sets  $J_1^k, J_2^k$  such that condition (9) is satisfied. Otherwise, exactly one column  $j$  of  $A_k$  is in  $J_c$ . Then  $j$  is already either in  $J_1$  or  $J_2$ . We may assume that it is in  $J_1$ . Then, again using Theorem 2, partition the columns of  $A_k$  in  $J$  into two sets  $J_1^k, J_2^k$  such that condition (9) is satisfied and such that the column  $j$  is in  $J_1^k$ . In both cases, we can add  $J_1^k$  to  $J_1$  and  $J_2^k$  to  $J_2$ . As every column occurs in exactly one of the blocks  $A_k$ , this assigns every column to exactly one of the sets  $J_1$  or  $J_2$ .

Now, we consider an arbitrary row  $i$ . Every row is either the coupling row or occurs in exactly one of the blocks  $A_k$ . If row  $i$  is the coupling row, then condition (9) holds as the number of ones in  $J_1$  and  $J_2$  in this row differs by at most one. If row  $i$  occurs in block  $A_k$  then the only nonzero entries in condition (9) are the entries in  $J_1^k$  and  $J_2^k$ . It follows that condition (9) is satisfied by the construction of  $J_1^k$  and  $J_2^k$ .  $\square$

Next we show that it is not possible to add another nonzero entry in the coupling vector  $h$  or couple the market participants via more than one variable even if there are more columns per block than coupled variables, and still guarantee total unimodularity

of matrix  $A$  and thus the existence of a solution to the corresponding decentralized market problem.

Consider the following three matrices

$$A_1 := \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, A_2 := \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, A_3 := \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix},$$

all not being totally unimodular, as the first two matrices have a determinant not equal to  $-1$ ,  $0$  or  $1$  and the third matrix has a submatrix, counting out rows 4, 8, 9, 10 and columns 3, 4, 7, 8, which is not unimodular.

The first matrix  $A_1$  shows that it is not possible to add another nonzero entry in vector  $h$ , and still guarantee total unimodularity. Matrix  $A_2$  lets us infer that it is not possible to add another coupling condition, and still guarantee total unimodularity. The last matrix  $A_3$  shows that total unimodularity is not preserved by more columns per block than coupling rows. As mentined above observe that for index sets  $J \subseteq N = \{3, 4, 7, 8\}$  in rows 4, 8, 9, 10 the conditions of Theorem 2 are the same as for matrix  $A_2$ . This implies that, in general, degrees of freedom in the coupling rows do not preserve total unimodularity. Thus we have shown that we can only guarantee total unimodularity in the case of at most one coupling row and one nonzero entry per block in this row.

In the following we show an application of Theorem 1. Let  $\mathcal{N}$  be a set of  $N$  market participants, whose problems are given as Problems (1) where we suppose in addition that  $A_n = (A_n^{\text{aux}}, A_n^{\text{prod}}) \in \{0, 1\}^{q \times m_n}$  is a totally unimodular matrix,  $b_n \in \mathbb{Z}^p$  and a single market price  $\pi \in \mathbb{R}$ . The coupling condition in this decentralized market problem is the same Eq. (2) with  $d \in \mathbb{Z}$ .

**Theorem 3** *Let a decentralized market problem, as described above, be given with totally unimodular market participants' problems (1) and coupling constraint (2). Then there is a solution to this decentralized market problem if there are feasible solutions for all market participants' problems (1) that satisfy constraint (2).*

**Proof** The corresponding central planner's problem has a constraint matrix of the form  $A$ , as defined in Lemma 3, which leads to the feasible set of the central planner's problem being an integral polyhedron (Hoffman and Kruskal 1956). This implies that an integral optimal solution to the LP-relaxation of the central planner's problem exists, and therefore by Theorem 1 a solution to the decentralized market problem exists.  $\square$

Next we show that Example 1 has a solution.

**Lemma 4** *Problem (7) can be written as*

$$\max_{p_n, x_n} -c_n^{\text{aux}\top} x_n - c_n^{\text{prod}\top} p_n + \pi p_n \quad (10a)$$

$$\text{s.t. } A_n^{\text{aux}} x_n + A_n^{\text{prod}} p_n = b_n \quad (10b)$$

$$x_n \in \mathbb{Z}^{m_I} \quad (10c)$$

$$p_n \in \mathbb{Z} \quad (10d)$$

$$x_n, p_n \geq 0, \quad (10e)$$

with the matrix  $A_n = (A_n^{\text{aux}} A_n^{\text{prod}})$  being totally unimodular.

**Proof** The matrix

$$A_n := \begin{pmatrix} E_n & E_n \\ M_n & 0 \end{pmatrix}$$

with the first row of  $A_n$  being  $A_n^{\text{prod}}$  and the rest of the matrix being  $A_n^{\text{aux}}$  is totally unimodular, as  $M_n$  is totally unimodular (Bertsimas and Weismantel 2005), because it is the incidence matrix of a directed graph for all  $n \in \mathcal{N}$ , and concatenating identity matrices and totally unimodular matrices preserves total unimodularity (Bertsimas and Weismantel 2005).  $\square$

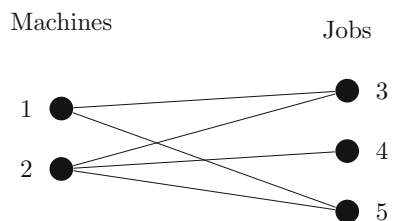
**Corollary 1** *Given a decentralized market problem defined as in Example 1. Then there is a solution to the decentralized market problem if  $d \in \mathbb{Z}$  and there are feasible flows for all market participants such that Constraint (8) is fulfilled.*

**Proof** The preceding lemma shows that the market participants' problems can be modeled such that the prerequisites of Theorem 3 are fulfilled and thus there is a solution to the decentralized market problem if there are feasible solutions which satisfy Constraint (8).  $\square$

The second application to Theorem 3 utilizes an assignment problem.

**Example 2** (Assignment) Let the market participants assign jobs to machines. Every machine has the capacity to accept a certain integral number of jobs and every job has linear costs. The participants share one machine and therefore the capacity of this machine is shared across all participants. This machine must operate at full capacity. Every participant has  $j_n$  jobs and  $m_n$  machines.

**Fig. 1** Bipartite graph of an assignment problem



The market participants' problems can be modeled as

$$\min_{x_n} c_n^\top x_n + \pi x_n^1 \quad (11a)$$

$$\text{s.t. } M_n^{\text{shared}} x_n = 0 \quad (11b)$$

$$M_n^{\text{machines}} x_n = b_n \quad (11c)$$

$$M_n^{\text{jobs}} x_n = 1 \quad (11d)$$

$$x_n \geq 0 \quad (11e)$$

$$x_n \in \mathbb{Z}^m. \quad (11f)$$

Here  $x_n \in \{0, 1\}^{w_n}$ ,  $M_n^{\text{shared}} \in \{0, 1\}^{1 \times w_n}$ ,  $M_n^{\text{machines}} \in \{0, 1\}^{m_n \times w_n}$ ,  $b_n \in \mathbb{Z}^{w_n}$ ,  $M_n^{\text{jobs}} \in \{0, 1\}^{j_n \times w_n}$ ,  $\pi \in \mathbb{R}$  and  $c_n \in \mathbb{R}^{w_n}$ . Eq. (11b) models the jobs the shared machine takes, as  $x_n^1$ , the first entry of  $x_n$ , equals the number of jobs the shared machine takes. Constraint (11c) formulates the capacity of jobs the machines can accept by utilizing the incidence matrix and adding slack variables. Constraint (11d) ensures all jobs are assigned.

The coupling constraint (12) ensures the shared machine is assigned  $d \in \mathbb{Z}$  jobs.

$$\sum_{n=1}^N x_n^1 = d \quad (12)$$

See Fig. 1 for an example of the assignment problem of a player  $n$ . Here three jobs have to be assigned to two machines, where machine one is shared among the players.

This leads to the following integer program:

$$\min_{x_n} (0, 2, 1, 3, 1, 4, 0, 0) x_n + \pi x_n^1 \quad (13a)$$

$$\text{s.t. } (-1 \ 1 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0) x_n = 0 \quad (13b)$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \end{pmatrix} x_n = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \quad (13c)$$

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix} x_n = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad (13d)$$

$$x_n \geq 0 \quad (13e)$$

$$x_n \in \mathbb{Z}^8 \quad (13f)$$

In this example Constraint (13b) ensures  $x_n^1$  equals the number of jobs machine one accepts. Constraint (13c) models that every machine can take at most two jobs, and Constraint (13d) ensures that every job is done.

**Lemma 5** Problem (11) can be written as

$$\max_{p_n, x_n} -c_n^{\text{aux}\top} x_n - c_n^{\text{prod}\top} p_n + \pi p_n \quad (14a)$$

$$\text{s.t. } A_n^{\text{aux}} x_n + A_n^{\text{prod}} p_n = b_n \quad (14b)$$

$$x_n \in \mathbb{Z}^{m_l} \quad (14c)$$

$$p_n \in \mathbb{Z} \quad (14d)$$

$$x_n, p_n \geq 0, \quad (14e)$$

with the matrix  $A_n = (A_n^{\text{aux}} A_n^{\text{prod}})$  being totally unimodular.

**Proof** The constraint matrices of problems (11) are composed of incidence matrix  $M_n$  of an undirected bipartite graph, unit vectors  $e_i$  and duplicating the first row  $m_n$  of matrix  $M_n$ . The matrix can thus be written as

$$A_n = \begin{pmatrix} m_n & 0 & \cdots & 0 & -1 \\ M_n & e_1 & \cdots & e_{1+p} & 0 \end{pmatrix}$$

with

$$A_n^{\text{aux}} = \begin{pmatrix} m_n & 0 & \cdots & 0 \\ M_n & e_1 & \cdots & e_{1+p} \end{pmatrix}$$

and

$$A_n^{\text{prod}} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

We show that this matrix is totally unimodular.

Matrix  $M_n$  is totally unimodular, as incidence matrices of undirected bipartite graphs are totally unimodular (Korte and Vygen 2012). Matrix  $B_n = \begin{pmatrix} m_n \\ M_n \end{pmatrix}$  is totally unimodular, because every partition of  $J$ , as in Theorem 2 that is valid for  $M_n$ , is also valid for  $B_n$ . Concatenating identity matrices and totally unimodular matrices leads to a totally unimodular matrix (Bertsimas and Weismantel 2005), thus  $(B_n \ E)$  is totally unimodular, where  $E$  is an identity matrix. Interchanging two rows or multiplying a column with  $-1$  preserves total unimodularity (Nemhauser and Wolsey 1988). This infers that we can move the first column of  $E$  to the back and multiply it with  $-1$ . This implies that matrix  $A_n$  is totally unimodular.  $\square$

**Corollary 2** *Given a decentralized market problem defined by problems (11) and Constraint (12). Then there is a solution to the decentralized market problem if the market participants' problems (11) have feasible solutions which satisfy Constraint (12).*

**Proof** The preceding lemma shows that the market participants' problems can be modeled such that the prerequisites of Theorem 3 are fulfilled and thus there is a solution to the decentralized market problem if there are feasible solutions which satisfy Constraint (12).  $\square$

## 4 Computational experiments

In this section we apply Theorem 1 to two classes of decentralized market problems with integralities. We show that solutions to decentralized market problems (Definition 1) may exist even in the case that the market participants' problems (1) contain integrality constraints (1c) and (1d). Further we show that the choice of formulation of the problems makes a difference. To be more precise, Theorem 1 requires the market participants' problems (1) to be given in complete linear description. In case the complete linear description is not used, some solutions to decentralized market problems may not be found.

The first class is motivated by the classical Scarf's example (Scarf 1994) which was adapted by Hogan and Ring (2003). In this class it can be seen that competitive equilibria often exist even though that may be unexpected — as this implies that an optimal solution to the central planner's problem (3) coincides with an optimal solution to its LP-relaxation. Therefore this implies that checking whether a competitive equilibrium exists regardless of integralities is worthwhile.

The second class consists of decentralized market problems that contain min-up/min-down constraints (Rajan and Takriti 2005). Here we compare two formulations of the market participants' problems where one formulation is in complete linear description and the other is not. The results show that in case the market participants' problems (1) are not given in complete linear description competitive equilibria may not be found even though they exist. Thus, we emphasize the importance of using the complete linear description.

### 4.1 Computational setup

In order to be able to apply Theorem 1 to check the existence of competitive equilibria computationally we give the following lemma.

**Lemma 6** *A decentralized market problem as in Definition 1 where all market participants' problems (1) are given in complete linear description has a solution if and only if the optimal values of the central planner's problem (3) and its corresponding LP-relaxation coincide.*

From now on we denote the proportion of the integrality gap with  $\frac{c^{\text{MIP}} - c^{\text{LP}}}{c^{\text{MIP}}}$ , where  $c^{\text{MIP}}$  is the optimal value of the mixed-integer problem (3) and  $c^{\text{LP}}$  the optimal value of its LP-relaxation. In our implementation we assume a competitive equilibrium exists if the integrality gap proportion is lower than  $10^{-05}$ . The computations are done on the Woody cluster of the RRZE-HPC Regionales Rechenzentrum Erlangen (2021) with four Xeon E3-1240 v5 CPUs running with 32 GB of RAM and 3.50 GHz. The optimization problems in the algorithm are solved via Gurobi 9.1 (Gurobi Optimization 2021) while the algorithm itself is implemented in Python 3.7 and the random parameters are generated using the random package of Python.

**Table 1** Costs for power plants

Type	Variable costs	Capacities	Startup costs	Minimal outputs
Smokestack	3	16	53	0
High tech	2	7	30	0
Med Tech	7	6	0	2

## 4.2 Scarf's example

First, we investigate the existence of solutions to the decentralized market problem posed by Scarf (1994) where there are two types of plants one type called “smokestack” — representing an older type of plant — and a type called “high tech”.

This example is also used in the literature, see O'Neill et al. (2005) while there is a slightly modified version introduced by Hogan and Ring (2003) where also a so called “med tech” type of plant was introduced into the example. Here the number of plants are limited to 6, 5 and 5 for smokestack, high tech and med tech respectively. This example is also used by Bjørndal and Jörnsten (2008); Azizan et al. (2019).

Then the corresponding central planner's problem is

$$\begin{aligned}
 & \min_{x,p} 3p_{\text{smoke}} + 2p_{\text{high}} + 7p_{\text{med}} + 53x_{\text{smoke}} + 30x_{\text{high}} + 0x_{\text{med}} \\
 & \text{s.t. } 0 \leq p_{\text{smoke}} \leq 16x_{\text{smoke}} \\
 & \quad 0 \leq p_{\text{high}} \leq 7x_{\text{high}} \\
 & \quad 2x_{\text{med}} \leq p_{\text{med}} \leq 6x_{\text{med}} \\
 & \quad p_{\text{smoke}} + p_{\text{high}} + p_{\text{med}} = d \\
 & \quad 0 \leq x_{\text{smoke}} \leq 6 \\
 & \quad 0 \leq x_{\text{high}} \leq 5 \\
 & \quad 0 \leq x_{\text{med}} \leq 5 \\
 & \quad x \in \mathbb{Z}^3 \\
 & \quad p \in \mathbb{R}^3,
 \end{aligned}$$

where the production is modeled by  $p$  and the investments by  $x$ . The demand is given by  $d \in \mathbb{Z}$ . Here the prerequisite of complete linear description for the market participants is fulfilled — therefore Lemma 6 can be applied directly.

The maximal production here is 161 units thus we compute solutions to the decentralized market problems with integral demands from 1 to 161 units.

In the decentralized market problems SC arising via Scarf (1994) without limits on the numbers of plants used and without the med tech type of plants there are 22 solutions while in HR, the setting arising through Hogan and Ring (2003), there are 40 solutions as depicted in Table 2.

As stated in the beginning of the section, there are competitive equilibria, even though normally one would not expect that, as — in general — it is rare that optimal

**Table 2** Results Scarf's example: With the number of problems containing solutions and the integrality gap proportion with standard deviation, percentiles and maximum

# Sol		Integrality gap proportion					
		Mean	Std	25%	50%	75%	Max
SC	22	0.0307	0.1011	0.0010	0.0026	0.0100	0.8036
HR	40	0.0143	0.0657	0.0002	0.0037	0.0076	0.8036

solutions to mixed-integer problems and optimal solutions to the corresponding LP-relaxations coincide. However, here this is the case because the market participants' problems are given in complete linear description and therefore only the market clearing condition (2) prevents the central planner's problem (3) from being in complete linear description as well and therefore for these problems the influence of the market clearing condition is small. In the following example, on the other hand, the influence of is higher, leading to fewer competitive equilibria. Thus, we could demonstrate our claim in the beginning of the section that competitive equilibria may exist regardless of integrality.

### 4.3 Min-up/min-down constraints

Second, we investigate power market problems with min-up/min-down constraints (Lee et al. 2004; Hua and Baldick 2016), startup costs (Vyve 2011; Ruiz et al. 2012) and variable costs, which were already mentioned in the introduction. Here we demonstrate that it is important to use the complete linear description of the market participants' problems. To do this we implement the model introduced by Rajan and Takriti (2005), which gives a complete linear description for the aforementioned class of problems and another formulation, which does not give the complete linear description. We then compare the ability to find competitive equilibria for both models and make clear that in case the formulation, that is not in complete linear description, is chosen some decentralized market problems that have a solution might not be identified.

**Table 3** Costs for power plants with  $c^{\text{var}}$  variable costs,  $b$  upper bound on production and  $c^{\text{start}}$  startup costs

Type	$c^{\text{var}}$	$b$	$c^{\text{start}}$
Nuclear	5.00	12068.00	423224.76
Lignite new	15.77	9364.40	261266.76
Lignite old	20.58	11528.10	321633.99
Hard coal new	31.29	6769.90	302208.34
Hard coal old	40.86	20104.10	897447.03
Gas CCGT	62.17	5365.30	407494.54
Gasturbine new	117.24	8228.70	196419.07
Gasturbine old	140.68	5313.20	126826.08
Mineral oil new	222.63	831.60	15733.87
Mineral oil old	267.15	2619.10	49553.37

The parameters are chosen from Table 3 where the data stems from Braun (2020) and is given in Euro per MWh.

Here the optimization problem for the central planner reads

$$\min \sum_{n=1}^N \sum_{t=1}^T c_n^{\text{var}} p_{t,n} + c_n^{\text{start}} v_{t,n} \quad (15a)$$

$$\text{s.t.} \quad \sum_{i=t-L+1}^t v_{i,n} \leq u_{t,n} \quad \text{for all } t \in [L+1, T], n \in [1, N] \quad (15b)$$

$$\sum_{i=t-l+1}^t v_{i,n} \leq 1 - u_{t,n} \quad \text{for all } t \in [l+1, T], n \in [1, N] \quad (15c)$$

$$v_{t,n} \geq u_{t,n} - u_{t-1,n} \quad \text{for all } t \in [2, T], n \in [1, N] \quad (15d)$$

$$0 \leq qb_n u_{t,n} \leq p_{t,n} \leq b_n u_{t,n} \quad \text{for all } t \in [1, T], n \in [1, N] \quad (15e)$$

$$u_{t,n} - u_{t-1,n} \leq u_{i,n} \quad \text{for all } i \in [t+1, \min\{t+L, T\}], \quad (15f)$$

$$t \in [2, T], n \in [1, N]$$

$$u_{t-1,n} - u_{t,n} \leq 1 - u_{i,n} \quad \text{for all } i \in [t+1, \min\{t+l, T\}], \quad (15g)$$

$$t \in [2, T], n \in [1, N]$$

$$\sum_{t=1}^N p_{t,n} = d_t \quad \text{for all } t \in [1, T] \quad (15h)$$

$$v_{t,n}, u_{t,n} \in \{0, 1\} \quad \text{for all } t \in [1, T], n \in [1, N], \quad (15i)$$

where the production bounds are given by  $b_n \in \mathbb{R}$  as the upper bound and  $qb_n$  with  $q \in [0, 1]$  the lower bound. Furthermore  $p_{t,n} \in \mathbb{R}$  is the production,  $v_{t,n}$  models the startup of a plant and  $u_{t,n}$  models whether a plant is turned on. The minimum uptime and downtime are  $L \in \mathbb{N}$  and  $l \in \mathbb{N}$  respectively with  $T \in \mathbb{N}$  being the number of time steps considered. The number of market participants is given by  $N \in \mathbb{N}$  and the startup costs by  $c_n^{\text{start}} \in \mathbb{R}$ . The variable costs are modeled by  $c_n^{\text{var}} \in \mathbb{R}$ .

This formulation of min-up/min-down constraints is in complete linear description, i.e., it is a tight formulation, for the market participants' problems (Rajan and Takriti 2005). However, omitting Constraints (15b) and (15c) yields an equivalent mixed-integer problem, but the market participants are not in complete linear description any more, i.e., it is a loose formulation, and therefore Lemma 6 may not yield a solution via this problem formulation. We investigate the differences in the following.

Here we present three settings for which we investigate the existence of competitive equilibria. The parameters chosen are depicted in Table 4.

First, we assume that both the minimum-uptime as the minimum-downtime are set to 3 hours while we take into account a time-horizon of 24 hours. Furthermore we compute two models of decentralized market problems where the first one assumes exactly one power plant of each type is available while the second model is more modern where neither the old types nor the nuclear power plant are available. We denote the modern model in complete linear description, with MT and the model that

**Table 4** Parameters in min-up/min-down:  $T$  Number of time steps,  $L$  minimum uptime,  $l$  minimum downtime,  $q$  proportion for lower bound

Setting	$T$	$L$	$l$	$q$
First	24	3	3	0.2
Second	24	3	3	0.3
Third	24	3	7	0.1

**Table 5** Results min-up/min-down First setting: With the number of problems containing solutions and the integrality gap proportion with standard deviation, percentiles, minimum and maximum

Model	# Sol	Integrality gap proportion						
		Mean	Std	Min	25%	50%	75%	Max
MT	4	0.0069	0.0052	0.0000	0.0034	0.0061	0.0101	0.0277
ML	4	0.0074	0.0055	0.0000	0.0036	0.0064	0.0104	0.0281
OT	0	0.0165	0.0094	0.0005	0.0088	0.0153	0.0210	0.0435
OL	0	0.0174	0.0100	0.0005	0.0096	0.0158	0.0230	0.0454

is not in complete linear description with ML. The old models are denoted OT and OL respectively.

For both modern and old models we set the minimal production  $b$  to 20% of the respective maximal capacities and solve 100 decentralized market problems for each model and each setting First, Second and Third. For each of these problems we draw random uniformly distributed floats differing at most by 20% from the parameters given in Table 1. The demand is chosen randomly between the minimal production while operating all plants and the maximal production while operating all plants in order to ensure feasibility.

We solve 100 decentralized market problems for each setting and here in the First setting there are four decentralized market problems in MT and ML that have a solution. On the other hand there are no competitive equilibria in OT and OL. This can be seen in Table 5. Also Table 5 shows that the lowest proportion of the integrality gaps is at 0.0005, and therefore the setting is close to have a competitive equilibrium.

In case we raise the minimal production capacity to 30% of the maximal production capacity in the Second setting there is only one decentralized market problem with a solution in MT and there is no decentralized market problem with a solution in the old setting OT and OL. However, the solution found in the new setting exemplifies that using the complete linear description for the market participants is important.

More competitive equilibria exist in the Third case where the minimal production is set to 10% of the maximal production capacity while the minimal uptime is set to 3 hours and the minimum downtime is set to 7 hours. Here there are 11 decentralized market problems with solutions in MT and ML while in the old setting there are 7 decentralized market problems with solutions found in OT where only 4 of them are found in the OL computations.

The statistical data for all three settings shows that the standard deviation is very low overall, while it still is higher in the old models than in the new models. Furthermore the

**Table 6** Results min-up/min-down *Second* setting: With the number of problems containing solutions and the integrality gap proportion with standard deviation, percentiles, minimum and maximum

Model	# Sol	Integrality gap proportion						
		Mean	Std	Min	25%	50%	75%	Max
MT	1	0.0099	0.0061	0.0000	0.0051	0.0089	0.0134	0.0263
ML	0	0.0109	0.0066	0.0003	0.0056	0.0102	0.0152	0.0273
OT	0	0.0216	0.0115	0.0010	0.0131	0.0199	0.0268	0.0530
OL	0	0.0230	0.0122	0.0010	0.0140	0.0208	0.0295	0.0553

**Table 7** Results min-up/min-down *Third* setting: With the number of problems containing solutions and the integrality gap proportion with standard deviation, percentiles, minimum and maximum

Model	# Sol	Integrality gap proportion						
		Mean	Std	Min	25%	50%	75%	Max
MT	11	0.0021	0.0024	0.0000	0.0003	0.0013	0.0028	0.0107
ML	11	0.0022	0.0027	0.0000	0.0003	0.0014	0.0031	0.0141
OT	7	0.0022	0.0026	0.0000	0.0003	0.0013	0.0027	0.0105
OL	4	0.0037	0.0046	0.0000	0.0006	0.0022	0.0051	0.0266

25% percentile in the first two settings is not close to have a competitive equilibrium. However in the *Third* setting for MT, ML, OT, OL the 25% percentile is smaller than  $10^{-03}$ . Therefore in *Third* there are a lot of instances very close to having a competitive equilibrium. Another observation is that in the old models the integrality gap proportions are higher than in the new models. This is because the startup costs for the old types of plants are, with the exception of gas plants, a lot higher. Thus, the integrality gaps make more of a difference in this setting as the corresponding costs are higher and thus the integrality gap widens.

In the beginning of this section we highlighted the importance of using the complete linear description for the market participants' problems (1). This is illustrated by the preceding computations — some decentralized market problems with competitive equilibria are found even though we did not use the complete linear descriptions, but some are not. This implies that in order to ensure that decentralized market problems with solutions are correctly identified it is thus crucial to use the complete linear description.

## 5 Conclusion

In convex markets linear prices, leading to a competitive equilibrium, exist, however, in markets with nonconvexities such a competitive equilibrium may not exist. For the case where nonconvexities arise because of integrality gaps, we give sufficient conditions under which a competitive equilibrium, supported by linear prices, exists. The resulting

Theorem 1 can be used to identify decentralized market problems to which such a solution exists. As a consequence we show that a competitive equilibrium, supported by linear prices, exists for a special case of decentralized market problems, where inter alia the constraint matrices of the market participants' problems are totally unimodular. We then give examples for decentralized market problems fulfilling the properties of this special case. Finally, we apply Theorem 1 to decentralized market problems arising in energy markets. There, first we show that competitive equilibria may exist in energy markets regardless of integrality constraints even though they do not exist in general. This can especially be seen in the classical example by Scarf. Therefore this shows that checking whether a competitive equilibrium exists in decentralized market problems with integralities may lead to finding competitive equilibria. Second, we show that it is important to choose the right model, because otherwise only part of the solutions might be found. This is exemplified by decentralized market problems arising in energy markets with min-up/min-down constraints where we compare two problem formulations. One of these formulations is in complete linear description and the other one is not. Here it can be seen that in case a formulation is chosen that is not in complete linear description there are some decentralized market problems that have solution where no competitive equilibrium can be found using this formulation.

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