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Semimartingale price systems in models with transaction costs beyond efficient friction

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Abstract

A standing assumption in the literature on proportional transaction costs is efficient friction. Together with robust no free lunch with vanishing risk, it rules out strategies of infinite variation as they usually appear in frictionless markets. In this paper, we show how the models with and without transaction costs can be unified.

The bid and ask prices of a risky asset are given by càdlàg processes which are locally bounded from below and may coincide at some points. In a first step, we show that if the bid–ask model satisfies “no unbounded profit with bounded risk” for simple strategies, then there exists a semimartingale lying between the bid and ask price processes.

In a second step, under the additional assumption that the zeros of the bid–ask spread are either starting points of an excursion away from zero or inner points from the right, we show that for every bounded predictable strategy specifying the amount of risky assets, the semimartingale can be used to construct the corresponding self-financing risk-free position in a consistent way. Finally, the set of most general strategies is introduced, which also provides a new view on the frictionless case.

Keywords Proportional transaction costs · No unbounded profit with bounded risk · Strategies of infinite variation · Semimartingales · Stochastic integration

Mathematics Subject Classification (2020) 91G10 · 60H05 · 26A42 · 60G40

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1 Introduction

In frictionless markets, asset price processes have to be semimartingales unless they allow an “unbounded profit with bounded risk” (UPBR) with simple strategies (see

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Delbaen and Schachermayer [13]). With semimartingale price processes, the powerful tools of stochastic calculus can be used to construct the gains from dynamic trading. A trading strategy specifying the amounts of shares an investor holds in her portfolio is a predictable process that is integrable with respect to the vector-valued price process. Strategies can be of infinite variation since in the underlying limiting procedure, one directly considers the (book) profits made rather than the portfolio rebalancings.

On the other hand, under arbitrarily small transaction costs, also non-semimartingales can lead to markets without “approximate arbitrage opportunities”. Guasoni [19] and Guasoni et al. [21] derive the sufficient condition of “conditional full support” of the mid-price process, that is satisfied e.g. by a fractional Brownian motion, and arbitrarily small constant proportional costs. Guasoni et al. [22] derive a fundamental theorem of asset pricing for a family of transaction costs models.

Under the assumptions of efficient friction, i.e., nonvanishing bid–ask spreads, and the existence of a strictly consistent price system, Kabanov and Stricker [28] and Campi and Schachermayer [4] show for continuous and càdlàg processes, respectively, that a finite credit line implies that the variation of the trading strategies is bounded in probability. A similar assertion is shown in Guasoni et al. [20] under the condition of “robust no free lunch with vanishing risk”. An important consequence for hedging and portfolio optimisation is that the set of portfolios that are attainable with strategies of finite variation is Fatou-closed. For a detailed discussion, we refer to the monograph of Kabanov and Safarian [27, Sect. 3.6].

In this paper, we consider càdlàg bid and ask price processes that are not necessarily different. The ask price is bigger than or equal to the bid price. The spread, which models the transaction costs, can vary in time and can even vanish. The contribution of this paper is twofold. First, we show that if the bid–ask model satisfies “no unbounded profit with bounded risk” (NUPBR) for simple long-only strategies, then there exists a semimartingale lying between the bid and ask price processes. This generalises Delbaen and Schachermayer [13, Theorem 7.2] for the frictionless case. The proof in [13] is very intuitive. Roughly speaking, it first shows that an explosion of the quadratic increments of the price process along stopping times would lead to a UPBR. Then it considers a discrete-time Doob decomposition of the asset price process and shows that an explosion of the drift part as the mesh of the grid tends to zero would lead to a UPBR. This already yields that under NUPBR, the asset price process has to be a good integrator and thus a semimartingale by the Bichteler–Dellacherie theorem. More recently, Beiglböck et al. [2] provide an alternative proof of the Bichteler–Dellacherie theorem combining these no-arbitrage arguments with Komlós-type arguments. Kardaras and Platen [31] follow a quite different approach that only requires long investments. They construct supermartingale deflators as dual variables in suitable utility maximisation problems under a variation of NUPBR for simple long-only strategies. Bálint and Schweizer [1] assume that asset prices are expressed in a possibly nontradable accounting unit. In their setting, there need not exist an asset with a strictly positive price process that can be used as a numéraire. They show that if there exists a portfolio with strictly positive value process, then under a discounting-invariant form of absence of arbitrage, which generalises the condition used in Kardaras and Platen [31], the asset prices discounted by the portfolio value

are semimartingales. Since in transaction costs' models it is natural to start with the relative prices of the tradable assets, there is no obvious analogy of discounting by a portfolio value. In our model, we implicitly assume the existence of an asset with strictly positive price process that serves as a reference asset.

In the bid–ask model, we consider a Dynkin zero-sum stopping game in which the lower payoff process is the bid price and the upper payoff process the ask price. The Doob decomposition of the dynamic value of the discrete-time game along arbitrarily fine grids is used to identify smart investment opportunities. The crucial point is that the drift of the Dynkin value can be earned by trading in the bid–ask market. This we combine with the brilliant idea in Delbaen and Schachermayer [13, Lemma 4.7] to control the martingale part. We complete the proof by showing that under the assumptions above, the continuous-time Dynkin value must be a local quasimartingale.

In the second part of the paper, we show how a semimartingale between the bid and ask processes can be used to define the self-financing condition of the model beyond efficient friction. Without efficient friction, strategies of infinite variation can make sense since they do not necessarily produce infinite trading costs. This of course means that we cannot use them as integrators without major hesitation. In the first step, we only consider bounded amounts of risky assets. Thus the trading gains charged in the semimartingale are finite. Then we add the costs caused by the fact that the trades are carried out at the less favorable bid–ask prices. Roughly speaking, if the spread is away from zero, the costs are a Riemann–Stieltjes integral similarly as in Guasoni et al. [20]. Then we exhaust the costs when the spread is away from zero. The crucial point is that these costs are always nonnegative, and the semimartingale gains are finite. Especially, infinite costs cannot be compensated and lead to ruin. Under a rather mild additional assumption on the behaviour of the spread at zero (see Assumption 3.18), an assumption that goes at least far beyond the frictionless case and the case of efficient friction, this approach leads to a well-founded self-financing condition. Especially, the self-financing riskless position does not depend on the choice of the semimartingale we use in the construction (see Corollary 3.22).

A self-financing condition for general strategies has to be justified by suitable approximations with simple strategies. With transaction costs, this is a delicate issue. Indeed, under pointwise convergence of the strategies alone, one should not expect that portfolio processes converge. By the strict Fatou-type inequality (see Guasoni et al. [20, Theorem A.9 (iv)]), some variation/costs can disappear in the limit. Thus roughly speaking, we postulate the following: first, the limit strategy is better than all (almost) pointwise converging simple strategies and second, for each strategy, there exists a special sequence of approximating simple strategies such that the wealth processes converge (see Theorem 3.19).

In the second step, we extend the self-financing condition from bounded strategies to the maximal set of strategies for which it can be defined in a “reasonable” way. In the special case of a frictionless market, this maximal set coincides with the set of predictable processes which are integrable with respect to the semimartingale price process in the classic sense (see e.g. Jacod and Shiryaev [26, Definition III.6.17]). Thus we also provide a further characterisation of this ubiquitous set.

In no-arbitrage theory, the need for general strategies is already proved in the special case of frictionless markets. Indeed, Delbaen and Schachermayer [13, Lemmas 7.9 and 7.10] provide an example with a bounded asset price process showing

that no free lunch with vanishing risk (NFLVR) for simple strategies does not imply the existence of an equivalent martingale measure (EMM). Consequently, under transaction costs, general strategies can become an important tool to guarantee the existence of a consistent price system (CPS), which plays a similar role as an EMM in the frictionless theory, under an appropriate no-arbitrage condition. On the other hand, a CPS in general need not exist even when NFLVR for multivariate portfolio processes is satisfied. This can already be seen in discrete time (see Schachermayer [38, Example 3.1]) with the observation that general strategies as described in Definition 4.1 coincide with simple strategies if time is discrete.

In a nutshell, we provide a well-founded self-financing condition for models beyond efficient friction by relating the original trading gains under transaction costs with the gains in a fictitious frictionless market defined by a semimartingale and subtracting the appropriate costs. The idea of relating markets under transaction costs with fictitious frictionless markets is not new. It is already widely used in the theory of portfolio optimisation. Here, shadow price processes, i.e., fictitious frictionless pricing systems that lead to the same *optimal* decisions and trading gains as under transaction costs, are utilised to determine optimal trading strategies. The existence of shadow prices and their relationship with a suitable dual problem go back to Cvitanic and Karatzas [7]. In discrete time, Kallsen and Muhle-Karbe [29] show that on finite probability spaces shadow price processes always exist as long as the original problem has a solution, and Czichowsky et al. [8] provide counterexamples on infinite probability spaces. Conditions for the existence of a shadow price process in a semimartingale model are established by Czichowsky et al. [12], and starting with Kallsen and Muhle-Karbe [30], various explicit constructions of shadow price processes have been given in Black–Scholes-type models. Even in non-semimartingale models, this *dual* approach is successfully applied (see e.g. Czichowsky and Schachermayer [10, 11] and Czichowsky et al. [9]) under efficient friction. In the proof of Theorem 4.5, we provide a direct connection between our work and shadow price processes for particular optimisation problems.

The paper is organised as follows. In Sect. 2, we show the existence of a semimartingale price system (Theorem 2.7). In Sect. 3, we construct the cost process which allows us to introduce the self-financing condition for bounded strategies, which is justified by Theorem 3.19 and Corollary 3.22. In Sect. 4, the extension to unbounded strategies is established (Proposition 4.2). In addition, the special case of a frictionless market is considered (Proposition 4.3), and the separate convergence of trading gains and cost terms of the approximating bounded strategies is discussed (Theorem 4.5). Technical proofs are postponed to Sect. 5 and Appendix A.

2 Existence of a semimartingale price system

Throughout the paper, we fix a terminal time $T \in \mathbb{R}_+$ and a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$ satisfying the usual conditions. The predictable σ -algebra on $\Omega \times [0, T]$ is denoted by \mathcal{P} , the set of bounded predictable processes starting at zero by \mathbf{bP} . To simplify the notation, a stopping time τ is allowed to take the value ∞ , but $[\tau] := \{(\omega, t) \in \Omega \times [0, T] : t = \tau(\omega)\}$. Especially, we use

the notation τ_A , $A \in \mathcal{F}_\tau$, for the stopping time that coincides with τ on A and is infinite otherwise; $\text{Var}_a^b(X)$ denotes the pathwise variation of a process X on the interval $[a, b]$. A process X is called *l\`agl\`ad* if all its paths possess finite left and right limits (but they can have double jumps). We set $\Delta^+X := X_+ - X$ and $\Delta X := \Delta^-X := X - X_-$, where $X_{t+} := \lim_{s \downarrow t} X_s$ and $X_{t-} := \lim_{s \uparrow t} X_s$. We use the notation $X_t^* := \sup_{s \in [0, t]} |X_s|$ and $X^* := X_T^*$. For a random variable Y , we set $Y^+ := \max(Y, 0)$ and $Y^- := \max(-Y, 0)$. If not stated otherwise, (in)equalities between stochastic processes are understood “up to evanescence”, i.e., up to a global \mathbb{P} -null set not depending on time. In Sect. 2, apart from (2.4), exceptional \mathbb{P} -null sets only occur at dyadic times, and we dispense with writing “ \mathbb{P} -a.s.” there.

The financial market consists of one risk-free asset or bank account that does not pay interest and one risky asset with bid price \underline{S} and ask price \overline{S} expressed in units of the risk-free asset. **Throughout the paper**, we make the following assumption.

Assumption 2.1 $\underline{S} = (\underline{S}_t)_{t \in [0, T]}$ and $\overline{S} = (\overline{S}_t)_{t \in [0, T]}$ are adapted processes with c\`adl\`ag paths. In addition, $\underline{S} \leq \overline{S}$ and \underline{S} is locally bounded from below.

In this section, we only consider simple trading strategies in the following sense.

Definition 2.2 A *simple trading strategy* is a stochastic process $(\varphi_t)_{t \in [0, T]}$ of the form

$$\varphi = \sum_{i=1}^n Z_{i-1} \mathbb{1}_{[T_{i-1}, T_i]},$$

where $n \in \mathbb{N}$, $0 = T_0 \leq T_1 \leq \dots \leq T_n = T$ are stopping times and Z_i is \mathcal{F}_{T_i} -measurable for all $i = 0, \dots, n-1$.

The strategy φ specifies the amount of risky assets in the portfolio. The next definition corresponds to the self-financing condition of the model. It specifies the holdings in the bank account given a simple trading strategy.

Definition 2.3 Let $(\varphi_t)_{t \in [0, T]}$ be a simple trading strategy. The *corresponding position in the bank account* $(\varphi_t^0)_{t \in [0, T]}$ is given by

$$\varphi_t^0 := \sum_{0 \leq s < t} (\underline{S}_s(\Delta^+ \varphi_s)^- - \overline{S}_s(\Delta^+ \varphi_s)^+), \quad t \in [0, T]. \quad (2.1)$$

Definition 2.4 Let $(\varphi_t)_{t \in [0, T]}$ be a simple trading strategy. The *liquidation value process* $(V_t^{\text{liq}}(\varphi))_{t \in [0, T]}$ is given by

$$V_t^{\text{liq}}(\varphi) := \varphi_t^0 + (\varphi_t)^+ \underline{S}_t - (\varphi_t)^- \overline{S}_t, \quad t \in [0, T].$$

If φ is clear from the context, we write $(V_t^{\text{liq}})_{t \in [0, T]}$ instead of $(V_t^{\text{liq}}(\varphi))_{t \in [0, T]}$.

We adapt the notion of an unbounded profit with bounded risk (UPBR) to the present setting of simple long-only trading strategies.

Definition 2.5 We say that $(\underline{S}, \overline{S})$ admits an *unbounded profit with bounded risk (UPBR)* for simple long-only strategies if there exists a sequence of simple trading strategies $(\varphi^n)_{n \in \mathbb{N}}$ with $\varphi^n \geq 0$ such that

- (i) $V^{\text{liq}}(\varphi^n) \geq -1$ for all $n \in \mathbb{N}$.
- (ii) The sequence $(V_T^{\text{liq}}(\varphi^n))_{n \in \mathbb{N}}$ is unbounded in probability, i.e.,

$$\lim_{m \rightarrow \infty} \sup_{n \in \mathbb{N}} \mathbb{P}[V_T^{\text{liq}}(\varphi^n) \geq m] > 0. \quad (2.2)$$

If no such sequence exists, we say that the bid–ask process $(\underline{S}, \overline{S})$ satisfies the *no unbounded profit with bounded risk (NUPBR)* condition for simple long-only strategies.

Remark 2.6 The admissibility condition (i) is rather restrictive e.g. compared to Guasoni et al. [20, Definition 4.4], which means that the present version of NUPBR is a weak condition. But for the following first main result of the paper, it is already sufficient.

Theorem 2.7 Let $(\underline{S}, \overline{S})$ satisfy Assumption 2.1 and the NUPBR condition for simple long-only strategies. Then there exists a semimartingale $S = (S_t)_{t \in [0, T]}$ such that

$$\underline{S} \leq S \leq \overline{S}. \quad (2.3)$$

A semimartingale S satisfying (2.3) is called a *semimartingale price system*. The remaining part of this section is devoted to the proof of Theorem 2.7. As a first step, we show that it is actually sufficient to prove the following seemingly weaker version of the result.

Theorem 2.8 Suppose that $0 \leq \underline{S} \leq \overline{S} \leq 1$ and that NUPBR for simple long-only strategies holds. Then there exists a semimartingale $S = (S_t)_{t \in [0, T]}$ such that

$$\underline{S} \leq S \leq \overline{S}.$$

Proposition 2.9 Theorem 2.8 implies Theorem 2.7.

Proof We assume that Theorem 2.8 holds true.

Step 1: Let \underline{S} be locally bounded from below, $\overline{S} \leq 1$ and $(\underline{S}, \overline{S})$ satisfy NUPBR for simple long-only strategies. Thus there is an increasing sequence $(\sigma_n)_{n \in \mathbb{N}}$ of stopping times with $\mathbb{P}[\sigma_n = \infty] \rightarrow 1$ such that $\underline{S}1_{\{\sigma_n > 0\}} \geq -n$ on $\llbracket 0, \sigma_n \rrbracket$ for all $n \in \mathbb{N}$. As $(\underline{S}, \overline{S})$ satisfies NUPBR for simple long-only strategies, the market $((\underline{S}^{\sigma_n} + n)1_{\{\sigma_n > 0\}}/(n+1), (\overline{S}^{\sigma_n} + n)1_{\{\sigma_n > 0\}}/(n+1))$ satisfies NUPBR for simple long-only strategies, too. By Theorem 2.8, for each $n \in \mathbb{N}$, there is a semimartingale S^n such that

$$(\underline{S}^{\sigma_n} + n)1_{\{\sigma_n > 0\}}/(n+1) \leq S^n \leq (\overline{S}^{\sigma_n} + n)1_{\{\sigma_n > 0\}}/(n+1).$$

Therefore, the process $S := \sum_{n=1}^{\infty} \mathbb{1}_{\llbracket \sigma_{n-1}, \sigma_n \rrbracket} ((n+1)S^n - n)$, with $\sigma_0 := 0$, lies between \underline{S} and \overline{S} ; this S is a local semimartingale and thus a semimartingale. Consequently, Theorem 2.8 holds true under the weaker condition that \underline{S} is only locally bounded from below instead of being nonnegative.

Step 2: Let \underline{S} be locally bounded from below and (\underline{S}, \bar{S}) satisfy NUPBR for simple long-only strategies. Consider the stopping times $\tau_n := \inf\{t \geq 0 : \bar{S}_t > n\}$ for all $n \in \mathbb{N}$. One has that $\mathbb{P}[\tau_n = \infty] = \mathbb{P}[\bar{S}_t \leq n, \forall t \in [0, T]] \rightarrow 1$ as $n \rightarrow \infty$. With short-selling constraints, liquidation value processes that are attainable by trading in the bid–ask model $((\underline{S}^{\tau_n}/n) \wedge 1, (\bar{S}^{\tau_n}/n) \wedge 1)$ can be dominated by those in (\underline{S}, \bar{S}) . Indeed, for $t < \tau_n$, one has $(\bar{S}_t^{\tau_n}/n) \wedge 1 = \bar{S}_t/n$, $(\underline{S}_t^{\tau_n}/n) \wedge 1 = \underline{S}_t/n$, and a purchase at time τ_n cannot generate a profit in the bid–ask model $((\underline{S}^{\tau_n}/n) \wedge 1, (\bar{S}^{\tau_n}/n) \wedge 1)$. Thus $((\underline{S}^{\tau_n}/n) \wedge 1, (\bar{S}^{\tau_n}/n) \wedge 1)$ satisfies NUPBR with simple long-only strategies, and by Step 1 there exist semimartingales S^n with $(\underline{S}^{\tau_n}/n) \wedge 1 \leq S^n \leq (\bar{S}^{\tau_n}/n) \wedge 1$ for all $n \in \mathbb{N}$. Then $S := \sum_{n=1}^{\infty} \mathbb{1}_{[\tau_{n-1}, \tau_n]} n S^n$, with $\tau^0 := 0$, shows the assertion. \square

For the remainder of the section, we work under the assumptions of Theorem 2.8. More specifically we assume the following.

Assumption 2.10 We assume $0 \leq \underline{S} \leq \bar{S} \leq 1$ and that (\underline{S}, \bar{S}) satisfies NUPBR for simple long-only strategies for the remainder of the section.

In addition, we set without loss of generality $T = 1$. We now proceed with the proof of Theorem 2.8. The candidate for the semimartingale is the value process of a Dynkin zero-sum stopping game played on the bid and ask prices, i.e., let $S = (S_t)_{t \in [0,1]}$ be the right-continuous version of

$$\begin{aligned} S_t &= \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{t,1}} \operatorname{ess\,inf}_{\sigma \in \mathcal{T}_{t,1}} \mathbb{E}[\underline{S}_\tau \mathbb{1}_{\{\tau \leq \sigma\}} + \bar{S}_\sigma \mathbb{1}_{\{\tau > \sigma\}} | \mathcal{F}_t] \\ &= \operatorname{ess\,inf}_{\sigma \in \mathcal{T}_{t,1}} \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{t,1}} \mathbb{E}[\underline{S}_\tau \mathbb{1}_{\{\tau \leq \sigma\}} + \bar{S}_\sigma \mathbb{1}_{\{\tau > \sigma\}} | \mathcal{F}_t] \quad \mathbb{P}\text{-a.s.}, \end{aligned} \quad (2.4)$$

where $\mathcal{T}_{t,1}$ is the set of $[t, 1]$ -valued stopping times for $t \in [0, 1]$. The existence of such a process and the non-trivial equality in (2.4) is guaranteed by Lepeltier and Maingueneau [34, Theorems 7 and 9 and Corollary 12]. Obviously, S satisfies $\underline{S} \leq S \leq \bar{S}$. Thus we only have to show that NUPBR for simple long-only trading strategies implies that S is a semimartingale. We note that all arguments remain valid for a different terminal value of the game between \underline{S}_1 and \bar{S}_1 .

The arguments below also provide a financial interpretation of the value process S of this Dynkin game. In the special case that the terminal bid and ask prices coincide, a discrete time approximation of S can be interpreted as a shadow price for a utility maximisation problem with a risk-neutral investor and the constraint that her dynamic stock position has to take values in $[-1, 1]$. Put differently, in the bid–ask market, an investor can earn the same expected profit as via an optimal strategy in the frictionless market with price process S (besides a finite deviation caused by different liquidation values).

Definition 2.11 Let $X = (X_t)_{t \in [0,1]}$ be an adapted process such that $\mathbb{E}[|X_t|] < \infty$ for all $t \in [0, 1]$. Given a deterministic partition $\pi = \{0 = t_0 < t_1 < \dots < t_n = 1\}$ of $[0, 1]$, we define the *mean variation of X along π* as

$$MV(X, \pi) := \mathbb{E} \left[\sum_{t_i \in \pi \setminus \{1\}} |\mathbb{E}[X_{t_i} - X_{t_{i+1}} | \mathcal{F}_{t_i}]| \right], \quad (2.5)$$

and the mean variation of X as

$$MV(X) := \sup_{\pi} MV(X, \pi).$$

Finally, X is called a *quasimartingale* if $MV(X) < \infty$.

For notational convenience, we removed the term $\mathbb{E}[|X_1|]$ that usually appears on the right-hand side of (2.5). Next, we recall Rao's theorem (see e.g. Protter [37, Chap. 3, Theorem 17] or Beiglböck and Siorpaes [3, Theorem 3.1]).

Theorem 2.12 *Let X be an adapted right-continuous process. Then X is a quasimartingale if and only if X has a decomposition $X = Y - Z$, where Y and Z are positive right-continuous supermartingales. In this case, the paths of X are a.s. càdlàg.*

Remark 2.13 Usually, Rao's theorem is formulated for an adapted càdlàg process X . However, to show that X can be written as the difference of two right-continuous supermartingales, the existence of the finite left limits of X is not needed (see the proofs of He et al. [23, Theorem 8.13] or Protter [37, Chap. 3, Theorem 14]). On the other hand, right-continuous supermartingales possess a.s. finite left limits (see e.g. Dellacherie and Meyer [14, Theorem VI.3]). This means that the theorem can be formulated for an a priori only right-continuous quasimartingale that then turns out to be càdlàg.

If we can show that the right-continuous process S is a local quasimartingale, Rao's theorem (in the version of Theorem 2.12) yields that S can locally be written as the difference of two supermartingales, and it admits a càdlàg modification. Thus S is a semimartingale by the Doob–Meyer theorem (case without class D) in Protter [37, Chap. 3, Theorem 16]. Hence we now want to show that S is a local quasimartingale.

For this, we consider a discrete time approximation $S^n = (S_t^n)_{t \in D_n}$ of S on the set $D_n := \{0, 1/2^n, \dots, (2^n - 1)/2^n, 1\}$ of dyadic numbers, defined by $S_1^n := \underline{S}_1$ and

$$S_t^n := \min(\bar{S}_t, \max(\underline{S}_t, \mathbb{E}[S_{t+1/2^n}^n | \mathcal{F}_t])), \quad t \in D_n, t < 1.$$

Indeed, it is well known (see e.g. Neveu [36, Proposition VI-6-9]) that

$$\begin{aligned} S_t^n &= \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{t,1}^n} \operatorname{ess\,inf}_{\sigma \in \mathcal{T}_{t,1}^n} \mathbb{E}[\underline{S}_\tau \mathbb{1}_{\{\tau \leq \sigma\}} + \bar{S}_\sigma \mathbb{1}_{\{\tau > \sigma\}} | \mathcal{F}_t] \\ &= \operatorname{ess\,inf}_{\sigma \in \mathcal{T}_{t,1}^n} \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{t,1}^n} \mathbb{E}[\underline{S}_\tau \mathbb{1}_{\{\tau \leq \sigma\}} + \bar{S}_\sigma \mathbb{1}_{\{\tau > \sigma\}} | \mathcal{F}_t], \quad t \in D_n, \end{aligned}$$

where $\mathcal{T}_{t,1}^n$ denotes the set of all $\{t, t + 1/2^n, \dots, 1\}$ -valued stopping times. The following proposition generalises Kifer [32, Proposition 3.2] from continuous to right-continuous processes.

Proposition 2.14 *Let $m \in \mathbb{N}$ and $t \in D_m$. Then we have*

$$\lim_{n \rightarrow \infty} S_t^n = S_t \quad \mathbb{P}\text{-a.s.}$$

Proof Let $n \in \mathbb{N}$ with $n \geq m$ and $t \in D_m$. The pair of $\{t, t + 1/2^n, \dots, 1\}$ -valued stopping times

$$\begin{aligned}\tau_t^n &:= \inf\{s \geq t : s \in D_n, S_s^n = \underline{S}_s\}, \\ \sigma_t^n &:= \inf\{s \geq t : s \in D_n, S_s^n = \overline{S}_s\}\end{aligned}$$

is a Nash equilibrium of the discrete-time game started at time t , i.e.,

$$\mathbb{E}[R(\tau, \sigma_t^n) | \mathcal{F}_t] \leq S_t^n \leq \mathbb{E}[R(\tau_t^n, \sigma) | \mathcal{F}_t] \quad \text{for all } \tau, \sigma \in \mathcal{T}_{t,T}^n, \quad (2.6)$$

where $R(\tau, \sigma) := \underline{S}_\tau \mathbb{1}_{\{\tau \leq \sigma\}} + \overline{S}_\sigma \mathbb{1}_{\{\tau > \sigma\}}$. This follows from Neveu [36, Proposition VI-6-9] and its proof with the observation that in *finite* discrete time, the assertion also holds for $\varepsilon = 0$ by dominated convergence. We define $d_n(\tau) := \inf\{t \geq \tau : t \in D_n\}$ for $\tau \in \mathcal{T}_{t,T}$ and

$$\eta_n(\tau)(\omega) := \sup_{s \in (\tau(\omega), \tau(\omega) + 1/2^n)} \max(|\underline{S}_s(\omega) - \underline{S}_\tau(\omega)|, |\overline{S}_s(\omega) - \overline{S}_\tau(\omega)|), \quad \omega \in \Omega.$$

This yields the estimates

$$R(\tau, d_n(\sigma)) - \eta_n(\tau) \leq R(d_n(\tau), d_n(\sigma)) \leq R(d_n(\tau), \sigma) + \eta_n(\sigma) \quad (2.7)$$

for all $\tau, \sigma \in \mathcal{T}_{0,T}$. Let $\varepsilon > 0$. For the continuous-time game, the pair of stopping times

$$\begin{aligned}\tau_t^\star &:= \inf\{s \geq t : S_s \leq \underline{S}_s + \varepsilon\}, \\ \sigma_t^\star &:= \inf\{s \geq t : S_s \geq \overline{S}_s - \varepsilon\}\end{aligned}$$

is an ε -Nash equilibrium, i.e.,

$$\mathbb{E}[R(\tau, \sigma_t^\star) | \mathcal{F}_t] - \varepsilon \leq S_t \leq \mathbb{E}[R(\tau_t^\star, \sigma) | \mathcal{F}_t] + \varepsilon \quad \text{for all } \tau, \sigma \in \mathcal{T}_{t,T}. \quad (2.8)$$

This is shown in Lepeltier and Maingueneau [34, Corollaire 12 and its proof]. Combining the first inequality in (2.6) with $\tau = d_n(\tau_t^\star)$, the first inequality in (2.7) and the second inequality in (2.8) yields

$$\begin{aligned}S_t^n &\geq \mathbb{E}[R(d_n(\tau_t^\star), \sigma_t^n) | \mathcal{F}_t] \\ &\geq \mathbb{E}[R(\tau_t^\star, \sigma_t^n) | \mathcal{F}_t] - \mathbb{E}[\eta_n(\tau_t^\star) | \mathcal{F}_t] \\ &\geq S_t - \varepsilon - \mathbb{E}[\eta_n(\tau_t^\star) | \mathcal{F}_t].\end{aligned}$$

Similarly, applying the second inequality (2.6) with $\sigma = d_n(\sigma_t^\star)$, the second inequality in (2.7) and the first inequality in (2.8) yields the corresponding upper estimate on S_t^n . Putting together, we get

$$S_t + \varepsilon + \mathbb{E}[\eta_n(\sigma_t^\star) | \mathcal{F}_t] \geq S_t^n \geq S_t - \varepsilon - \mathbb{E}[\eta_n(\tau_t^\star) | \mathcal{F}_t].$$

Since $\eta_n(\tau_t^*) \rightarrow 0$ and $\eta_n(\sigma_t^*) \rightarrow 0$ a.s. for $n \rightarrow \infty$ (by the right-continuity of \underline{S} and \overline{S}), the dominated convergence theorem for conditional expectations implies

$$S_t + \varepsilon \geq \limsup_{n \rightarrow \infty} S_t^n \geq \liminf_{n \rightarrow \infty} S_t^n \geq S_t - \varepsilon \quad \mathbb{P}\text{-a.s.},$$

which is the assertion as $\varepsilon > 0$ is arbitrary. \square

In the following, we consider the discrete-time Doob decomposition of the processes $(S^n)_{n \in \mathbb{N}}$, i.e., we write $S_t^n = S_0^n + M_t^n + A_t^n$ with

$$\begin{aligned} A_t^n &:= \sum_{t_i \in D_n, 0 < t_i \leq t} \mathbb{E}[S_{t_i}^n - S_{t_{i-1}}^n | \mathcal{F}_{t_{i-1}}], \\ M_t^n &:= \sum_{t_i \in D_n, 0 < t_i \leq t} (S_{t_i}^n - S_{t_{i-1}}^n - \mathbb{E}[S_{t_i}^n - S_{t_{i-1}}^n | \mathcal{F}_{t_{i-1}}]) \end{aligned} \quad (2.9)$$

for $t \in D_n$. In particular, we have

$$\begin{aligned} MV(S^n, D_n) &:= \mathbb{E} \left[\sum_{t_i \in D_n \setminus \{1\}} |\mathbb{E}[S_{t_{i+1}}^n - S_{t_i}^n | \mathcal{F}_{t_i}]| \right] \\ &= \mathbb{E} \left[\sum_{t_i \in D_n \setminus \{1\}} |A_{t_{i+1}}^n - A_{t_i}^n| \right]. \end{aligned} \quad (2.10)$$

The following observation is at the core of why our approach works.

Lemma 2.15 *Let $n \in \mathbb{N}$ and $t \in D_n \setminus \{1\}$. Then we have*

$$\begin{aligned} \{A_{t+1/2^n}^n - A_t^n > 0\} &\subseteq \{S_t^n = \overline{S}_t\}, \\ \{A_{t+1/2^n}^n - A_t^n < 0\} &\subseteq \{S_t^n = \underline{S}_t\}. \end{aligned}$$

Proof From the definition (2.9), we get $\mathbb{E}[S_{t+1/2^n}^n | \mathcal{F}_t] - S_t^n = A_{t+1/2^n}^n - A_t^n$, which together with $S_t^n = \min(\overline{S}_t, \max(\underline{S}_t, \mathbb{E}[S_{t+1/2^n}^n | \mathcal{F}_t]))$ yields the assertion. \square

We now start to establish a uniform bound on (2.10) (after some stopping).

Lemma 2.16 *Let Assumption 2.10 hold. Then the set*

$$\left\{ \sup_{t \in D_n} |M_t^n| : n \in \mathbb{N} \right\}$$

is bounded in probability.

Proof First, we roughly sketch the idea of the proof. If $\{\sup_{t \in D_n} |M_t^n| : n \in \mathbb{N}\}$ failed to be bounded in probability, the same would hold in some sense for the sequence $(A^n)_{n \in \mathbb{N}}$. Indeed, this is a consequence of $S^n = S_0^n + M^n + A^n$ and the fact that

$|S^n| \leq 1$. Keeping Lemma 2.15 in mind, we show that by suitable long-only investments in the bid–ask market, one can earn the increasing parts of A^n without suffering from the decreasing parts. In doing so, we would achieve a UPBR since the gains from A^n are of a higher order than the potential losses from the martingale part M^n . The proof of the latter relies on the brilliant ideas of Delbaen and Schachermayer [13, Lemma 4.7], which we adapt to the present setting. The present setting is easier than in [13, Lemma 4.7] since the jumps of S^n are uniformly bounded.

Step 1: Assume that the claim does not hold true, i.e., there are a subsequence $(\sup_{t \in D_{m_n}} |M_t^{m_n}|)_{n \in \mathbb{N}}$ and $\alpha \in (0, 1/10)$ such that

$$\mathbb{P}\left[\sup_{t \in D_{m_n}} |M_t^{m_n}| \geq n^3\right] > 10\alpha, \quad n \in \mathbb{N}.$$

In the following, we write $(\sup_{t \in D_n} |M_t^n|)_{n \in \mathbb{N}}$ instead of $(\sup_{t \in D_{m_n}} |M_t^{m_n}|)_{n \in \mathbb{N}}$ in order to simplify the notation. For this, it is important to note that from now on, we do not use properties of M^n that do not hold for M^{m_n} . Let $T_n := \inf\{t \in D_n : |M_t^n| \geq n^3\}$ and define the process $(\tilde{S}_t^n)_{t \in D_n}$ by $\tilde{S}_t^n := \frac{1}{n^2} S_{t \wedge T_n}^n$. Note that the (discrete-time) Doob decomposition of \tilde{S}^n is given by

$$\tilde{S}_t^n = \tilde{S}_0^n + \tilde{M}_t^n + \tilde{A}_t^n = \frac{1}{n^2} S_0^n + \frac{1}{n^2} M_{t \wedge T_n}^n + \frac{1}{n^2} A_{t \wedge T_n}^n, \quad t \in D_n,$$

where $(\tilde{M}_t^n)_{t \in D_n} = (\frac{1}{n^2} M_{t \wedge T_n}^n)_{t \in D_n}$ and $(\tilde{A}_t^n)_{t \in D_n} = (\frac{1}{n^2} A_{t \wedge T_n}^n)_{t \in D_n}$ are the martingale part and the predictable part, respectively. In addition, we have

$$\mathbb{P}\left[\sup_{t \in D_n} |\tilde{M}_t^n| \geq n\right] > 10\alpha, \quad |\tilde{S}_t^n - \tilde{S}_{t-1/2^n}^n| \leq \frac{1}{n^2}, \quad t \in D_n \setminus \{0\}.$$

Next, we define $T_{n,0} := 0$ and recursively

$$T_{n,i} := \inf\{t \geq T_{n,i-1} : t \in D_n, |\tilde{M}_t^n - \tilde{M}_{T_{n,i-1}}^n| \geq 1\}, \quad i \in \mathbb{N}.$$

Since $|A_t^n - A_{t-1/2^n}^n| \leq 1$ and thus

$$|M_t^n - M_{t-1/2^n}^n| \leq |S_t^n - S_{t-1/2^n}^n| + |A_t^n - A_{t-1/2^n}^n| \leq 2$$

for all $t \in D_n \setminus \{0\}$, we get

$$|\tilde{M}_{T_{n,i} \wedge 1}^n - \tilde{M}_{T_{n,i-1} \wedge 1}^n| \leq 1 + |\tilde{M}_{T_{n,i} \wedge 1}^n - \tilde{M}_{(T_{n,i-1} \wedge 1) \wedge 1}^n| \leq 1 + 2/n^2 \leq 3 \quad (2.11)$$

for all $n, i \in \mathbb{N}$. Equation (2.11) implies that

$$\mathbb{P}[T_{n,i} < \infty] > 10\alpha \quad \text{for } n \in \mathbb{N} \text{ and } i = 0, \dots, k_n, \quad (2.12)$$

where $k_n := \lfloor (n-1)/3 \rfloor$ denotes the integer part of $(n-1)/3$.

In a next step, we establish a lower bound on $(\tilde{M}_{T_{n,i} \wedge 1}^n - \tilde{M}_{T_{n,i-1} \wedge 1}^n)^-$ in $L^0(\mathbb{P})$ for $i = 1, \dots, k_n$. The martingale property of \tilde{M}^n together with (2.12) implies that

$$\mathbb{E}[(\tilde{M}_{T_{n,i} \wedge 1}^n - \tilde{M}_{T_{n,i-1} \wedge 1}^n)^-] = \frac{1}{2} \mathbb{E}[|\tilde{M}_{T_{n,i} \wedge 1}^n - \tilde{M}_{T_{n,i-1} \wedge 1}^n|] \geq \frac{1}{2} \mathbb{P}[T_{n,i} < \infty] > 5\alpha.$$

For $B_{n,i} := \{(\tilde{M}_{T_{n,i} \wedge 1}^n - \tilde{M}_{T_{n,i-1} \wedge 1}^n) \geq 2\alpha\}$, we get

$$\mathbb{E}[(\tilde{M}_{T_{n,i} \wedge 1}^n - \tilde{M}_{T_{n,i-1} \wedge 1}^n)^- \mathbb{1}_{B_{n,i}}] \geq \mathbb{E}[(\tilde{M}_{T_{n,i} \wedge 1}^n - \tilde{M}_{T_{n,i-1} \wedge 1}^n)^-] - 2\alpha > 3\alpha$$

and thus by (2.11),

$$\mathbb{P}[B_{n,i}] > \alpha \quad \text{for } n \in \mathbb{N} \text{ and } i = 0, \dots, k_n. \quad (2.13)$$

We now turn our attention to the increments $(\tilde{A}_{T_{n,i} \wedge 1}^n - \tilde{A}_{T_{n,i-1} \wedge 1}^n)_{i=1, \dots, k_n}$ for $n \in \mathbb{N}$. Since $|\tilde{S}_{T_{n,i} \wedge 1}^n - \tilde{S}_{T_{n,i-1} \wedge 1}^n| \leq 1/n^2$, (2.13) implies that

$$\mathbb{P}[\tilde{A}_{T_{n,i} \wedge 1}^n - \tilde{A}_{T_{n,i-1} \wedge 1}^n \geq \alpha] \geq \mathbb{P}[\tilde{A}_{T_{n,i} \wedge 1}^n - \tilde{A}_{T_{n,i-1} \wedge 1}^n \geq 2\alpha - \frac{1}{n^2}] \geq \mathbb{P}[B_{n,i}] > \alpha$$

for all $n \geq \sqrt{\alpha}$ and $i = 1, \dots, k_n$. In particular, if we define $(\tilde{A}_t^{n,\uparrow})_{t \in D_n}$ by

$$\tilde{A}_t^{n,\uparrow} := \sum_{t_i \in D_n, 0 < t_i \leq t} (\tilde{A}_{t_i}^n - \tilde{A}_{t_{i-1}}^n)^+, \quad t \in D_n,$$

we also get

$$\mathbb{P}[\tilde{A}_{T_{n,i} \wedge 1}^{n,\uparrow} - \tilde{A}_{T_{n,i-1} \wedge 1}^{n,\uparrow} \geq \alpha] > \alpha \quad (2.14)$$

for all $n \geq 1/\sqrt{\alpha}$ and $i = 1, \dots, k_n$.

Step 2: In the second part of the proof, we construct a UPBR by placing smart bets on the process $(\tilde{A}_t^{n,\uparrow})_{t \in D_n}$. This is similar to the second part of Delbaen and Schachermayer [13, Lemma 4.7] with the major difference that we cannot invest directly into S^n , but only in the bid–ask market (\underline{S}, \bar{S}) . We define two sequences of $D_n \cup \{\infty\}$ -valued stopping times $(\sigma_k^n)_{k=1}^{2^n}$ and $(\tau_k^n)_{k=1}^{2^n}$ by

$$\begin{aligned} \sigma_1^n &:= \inf\{t \in D_n^* : A_{t+1/2^n}^n - A_t^n > 0\}, \\ \tau_1^n &:= \inf\{t > \sigma_1^n : t \in D_n^*, A_{t+1/2^n}^n - A_t^n < 0\} \end{aligned}$$

with $D_n^* := D_n \setminus \{1\}$ and recursively

$$\begin{aligned} \sigma_k^n &:= \inf\{t > \tau_{k-1}^n : t \in D_n^*, A_{t+1/2^n}^n - A_t^n > 0\}, \\ \tau_k^n &:= \inf\{t > \sigma_k^n : t \in D_n^*, A_{t+1/2^n}^n - A_t^n < 0\} \end{aligned}$$

for $k = 2, 3, \dots, 2^n$. Next, define a sequence of simple trading strategies $(\varphi^n)_{n \in \mathbb{N}}$ by

$$\varphi^n := \left(\sum_{k=1}^{2^n} \frac{1}{n^2} \mathbb{1}_{[\sigma_k^n, \tau_k^n]} \right) \mathbb{1}_{[0, T_{n,k_n}]}. \quad (2.15)$$

By Lemma 2.15, the strategies φ^n only buy if $S_t^n = \bar{S}_t$ and sell if $S_t^n = \underline{S}_t$, despite a possible liquidation at T_{n,k_n} . Together with $S_{t_i}^n - \underline{S}_t \leq 1$ for all $t_i \in D_n$, $t \in [0, 1]$, this

implies that $V_t^{\text{liq}}(\varphi^n)$ can be bounded from below by

$$\begin{aligned}
 V_t^{\text{liq}}(\varphi^n) &\geq \sum_{t_i \in D_n, 0 < t_i \leq t} \varphi_{t_i}^n(S_{t_i}^n - S_{t_{i-1}}^n) - \frac{1}{n^2} \\
 &= \tilde{A}_{\lfloor 2^n t \rfloor / 2^n \wedge T_{n, k_n}}^{n, \uparrow} + \sum_{t_i \in D_n, 0 < t_i \leq t} \varphi_{t_i}^n(M_{t_i}^n - M_{t_{i-1}}^n) - \frac{1}{n^2} \\
 &\geq \sum_{t_i \in D_n, 0 < t_i \leq t} \varphi_{t_i}^n(M_{t_i}^n - M_{t_{i-1}}^n) - \frac{1}{n^2} \\
 &= \sum_{t_i \in D_n, 0 < t_i \leq t} (n^2 \varphi_{t_i}^n)(\tilde{M}_{t_i}^n - \tilde{M}_{t_{i-1}}^n) - \frac{1}{n^2}, \quad t \in [0, 1]. \quad (2.15)
 \end{aligned}$$

This means that the strategy allows us to invest in $\tilde{A}^{n, \uparrow}$, but we still do not know if it actually allows a UPBR as we need to get some control on the martingale part in (2.15). Therefore notice that

$$\begin{aligned}
 &\left\| \sum_{t_i \in D_n, 0 < t_i \leq T_{n, k_n}} (n^2 \varphi_{t_i}^n)(\tilde{M}_{t_i}^n - \tilde{M}_{t_{i-1}}^n) \right\|_{L^2(\mathbb{P})} \\
 &\leq \|\tilde{M}_{T_{n, k_n} \wedge 1}\|_{L^2(\mathbb{P})} \leq \sqrt{\sum_{i=1}^{k_n} \|\tilde{M}_{T_{n, i} \wedge 1}^n - \tilde{M}_{T_{n, i-1} \wedge 1}^n\|_{L^2(\mathbb{P})}^2} \leq 3\sqrt{k_n}.
 \end{aligned}$$

Thus Doob's maximal inequality yields

$$\left\| \sup_{t \in D_n, t \leq T_{n, k_n}} \left| \sum_{t_i \in D_n, 0 < t_i \leq t} (n^2 \varphi_{t_i}^n)(\tilde{M}_{t_i}^n - \tilde{M}_{t_{i-1}}^n) \right| \right\|_{L^2(\mathbb{P})} \leq 6\sqrt{k_n}.$$

Consequently, we get the estimate

$$\begin{aligned}
 &\mathbb{P}\left[\inf_{t \in [0, T_{n, k_n} \wedge 1]} V_t^{\text{liq}}(\varphi^n) \leq -k_n^{3/4} n^{-1/8} - n^{-2}\right] \\
 &\leq \mathbb{P}\left[\sup_{t \in D_n, t \leq T_{n, k_n}} \left| \sum_{t_i \in D_n, 0 < t_i \leq t} (n^2 \varphi_{t_i}^n)(\tilde{M}_{t_i}^n - \tilde{M}_{t_{i-1}}^n) \right| \geq k_n^{3/4} n^{-1/8}\right] \\
 &\leq \frac{36n^{1/4}}{\sqrt{k_n}} \quad (2.16)
 \end{aligned}$$

by Tschebyscheff's inequality. Thus let us define the stopping times

$$U_n := \inf\{t \geq 0 : V_t^{\text{liq}}(\varphi^n) \leq -k_n^{3/4} n^{-1/8} - n^{-2}\} \wedge T_{n, k_n},$$

which satisfy $\mathbb{P}[U_n < T_{n, k_n}] \leq 36n^{1/4}/\sqrt{k_n}$. We now pass to the strategy

$$\tilde{\varphi}^n := (k_n)^{-3/4} \varphi^n \mathbb{1}_{[0, U_n]}.$$

The left and right jumps of $V^{\text{liq}}(\tilde{\varphi}^n)$ are bounded from below by $-k_n^{-3/4}n^{-2}$, which is a direct consequence of $0 \leq \underline{S} \leq \overline{S} \leq 1$. We obtain

$$\inf_{t \in [0, T_{n,k_n} \wedge 1]} V_t^{\text{liq}}(\tilde{\varphi}^n) \geq -n^{-1/8} - 2k_n^{-3/4}n^{-2} \rightarrow 0 \quad \text{for } n \rightarrow \infty. \quad (2.17)$$

It remains to show (2.2). First notice that using (2.14) in conjunction with Delbaen and Schachermayer [13, Corollary A1.3] yields

$$\mathbb{P}\left[\tilde{A}_{T_{n,k_n} \wedge 1}^{n, \uparrow} \geq \frac{\alpha^2}{2}\right] > \frac{\alpha}{2}.$$

It follows that

$$\begin{aligned} \mathbb{P}\left[(k_n)^{-3/4} \tilde{A}_{T_{n,k_n} \wedge \lfloor \frac{2^n U_n}{2^n} \wedge 1}^{n, \uparrow} \geq k_n^{1/4} \frac{\alpha^2}{2}\right] &> \frac{\alpha}{2} - \mathbb{P}[U_n < T_{n,k_n}] \\ &\geq \frac{\alpha}{2} - \frac{36n^{1/4}}{\sqrt{k_n}}. \end{aligned} \quad (2.18)$$

Putting (2.15)–(2.18) together yields that $(\tilde{\varphi}^n)_{n \in \mathbb{N}}$ provides UPBR. \square

Lemma 2.17 *Let Assumption 2.10 hold. For each $\varepsilon > 0$, there exist a constant $C > 0$ and a sequence of $D_n \cup \{\infty\}$ -valued stopping times $(\tau_n)_{n \in \mathbb{N}}$ such that $\mathbb{P}[\tau_n < \infty] < \varepsilon$ and the stopped processes $S^{n, \tau_n} = (S_{t \wedge \tau_n}^n)_{t \in D_n}$, $A^{n, \tau_n} = (A_{t \wedge \tau_n}^n)_{t \in D_n}$ satisfy*

$$\sum_{t_i \in D_n \setminus \{0\}} |A_{t_i}^{n, \tau_n} - A_{t_{i-1}}^{n, \tau_n}| \leq C$$

and, consequently,

$$MV(S^{n, \tau_n}, D_n) = \mathbb{E}\left[\sum_{t_i \in D_n \setminus \{0\}} |A_{t_i}^{n, \tau_n} - A_{t_{i-1}}^{n, \tau_n}|\right] \leq C.$$

Proof The idea of the proof is akin to the proofs of Proposition 3.1 and Lemma 3.4 in Beiglböck et al. [2]. Thus we only give a sketch of the proof and leave the details to the reader. We first claim that

$$\left\{ \sum_{t_i \in D_n \setminus \{0\}} (A_{t_i}^n - A_{t_{i-1}}^n)^+ : n \in \mathbb{N} \right\} \quad (2.19)$$

is bounded in probability. We proceed by contraposition, i.e., we suppose otherwise and want to show that this leads to a UPBR. Using Lemma 2.15, we can analogously to the previous proof construct a sequence $(\varphi^n)_{n \in \mathbb{N}}$ of simple trading strategies with $0 \leq \varphi^n \leq 1$ such that φ^n invests in $\sum_{t_i \in D_n \setminus \{0\}} (A_{t_i}^n - A_{t_{i-1}}^n)^+$ while only making potential losses in the martingale part M^n and at liquidation. Indeed, similarly as in

Step 2 of the proof of Lemma 2.16, it can be shown that the associated liquidation values can be bounded from below by

$$V_t^{\text{liq}}(\varphi^n) \geq \sum_{t_i \in D_n, 0 < t_i \leq t} (A_{t_i}^n - A_{t_{i-1}}^n)^+ + \sum_{t_i \in D_n, 0 < t_i \leq t} \varphi_{t_i}^n (M_{t_i}^n - M_{t_{i-1}}^n) - 1. \quad (2.20)$$

By Lemma 2.16 and some stopping, there is no loss of generality in assuming that $(M^n)_{n \in \mathbb{N}}$ is uniformly bounded. Hence by Doob's maximal inequality, the pathwise maxima of the martingale parts in (2.20) are bounded in L^2 . Thus by further stopping (cf. the arguments used in Beiglböck et al. [2, second half of the proof of Lemma 3.4]), we may assume that the right-hand side of (2.20) is uniformly bounded from below. On the other hand, by the assumption on (2.19), for $t = 1$, the right-hand side of (2.20) is unbounded in probability from above. Thus the obtained sequence of strategies yields UPBR with long-only strategies (after multiplying it by a positive real number such that the right-hand side of (2.20) is bounded from below by -1), and we arrive at a contradiction. Consequently, (2.19) must be bounded in probability. Since the martingale parts are also bounded in probability by Lemma 2.16, the same holds for $\{\sum_{t_i \in D_n \setminus \{0\}} (A_{t_i}^n - A_{t_{i-1}}^n)^- : n \in \mathbb{N}\}$, and we are done. \square

In order to complete the proof of Theorem 2.8, we still need a couple of auxiliary results which give us some more information about $MV(S^n, D_n)$ in comparison to $MV(S^m, D_m)$. Given a partition $\pi = \{0 = t_0 < t_1 < \dots < t_n = 1\}$ of $[0, 1]$ and a stopping time τ , we use the notation $p(\tau) := \inf\{t \in \pi : t \geq \tau\}$. Recall the following useful result from Beiglböck and Siorpaes [3, Lemma 3.2].

Lemma 2.18 *Let Assumption 2.10 hold. Then*

$$MV(S^{p(\tau)}, \pi) = \mathbb{E} \left[\sum_{t_i \in \pi \setminus \{1\}} \mathbb{1}_{\{t_i < \tau\}} |\mathbb{E}[S_{t_{i+1}} - S_{t_i} | \mathcal{F}_{t_i}]| \right]$$

and $|MV(S^{p(\tau)}, \pi) - MV(S^\tau, \pi)| \leq 1$.

Compared to the frictionless case with $S^n = \underline{S} = \bar{S}$, the analysis is complicated by the fact that in general $S_t^m \neq S_t^n$ for $t \in D_n$. We have, nevertheless, the following monotonicity result.

Lemma 2.19 *Let Assumption 2.10 hold. In addition, let $n, m \in \mathbb{N}$ with $m > n$ and let τ_m be a $D_m \cup \{\infty\}$ -valued stopping time. For any $s \in D_n^* = D_n \setminus \{1\}$, we have*

$$\begin{aligned} & \mathbb{E} \left[\sum_{t_i \in D_n^*, t_i \geq s} \mathbb{1}_{\{t_i < \tau_m\}} |\mathbb{E}[S_{t_{i+1}}^n - S_{t_i}^n | \mathcal{F}_{t_i}]| \middle| \mathcal{F}_s \right] \\ & \leq \mathbb{E} \left[\sum_{t_i \in D_m^*, t_i \geq s} \mathbb{1}_{\{t_i < \tau_m\}} |\mathbb{E}[S_{t_{i+1}}^m - S_{t_i}^m | \mathcal{F}_{t_i}]| \middle| \mathcal{F}_s \right] + (2 - |S_s^n - S_s^m|) \mathbb{1}_{\{s < \tau_m\}}. \end{aligned}$$

In particular, for $s = 0$, this yields

$$MV(S^{n, d_n(\tau_m)}, D_n) \leq MV(S^{m, \tau_m}, D_m) + 2.$$

In addition, we have

$$\text{MV}(S^{m,d_n(\tau)}, D_n) \leq \text{MV}(S^{m,d_m(\tau)}, D_m) + 1 \quad (2.21)$$

for all $[0, 1] \cup \{\infty\}$ -valued stopping times τ .

Proof Step 1: In a first step, we keep the grid D_n and estimate the term with S^n from above by that with S^m . Thus we want to show that

$$\begin{aligned} & \mathbb{E} \left[\sum_{t_i \in D_n^*, t_i \geq s} \mathbb{1}_{\{t_i < \tau_m\}} |\mathbb{E}[S_{t_{i+1}}^n - S_{t_i}^n | \mathcal{F}_{t_i}]| \middle| \mathcal{F}_s \right] \\ & \leq \mathbb{E} \left[\sum_{t_i \in D_n^*, t_i \geq s} \mathbb{1}_{\{t_i < \tau_m\}} |\mathbb{E}[S_{t_{i+1}}^m - S_{t_i}^m | \mathcal{F}_{t_i}]| \middle| \mathcal{F}_s \right] + (1 - |S_s^n - S_s^m|) \mathbb{1}_{\{s < \tau_m\}}. \end{aligned} \quad (2.22)$$

We start by showing the one-step estimate

$$\begin{aligned} & |\mathbb{E}[S_{s+1/2^n}^n - S_s^n | \mathcal{F}_s]| \\ & = |\mathbb{E}[S_{s+1/2^n}^n - S_s^m | \mathcal{F}_s]| - |S_s^n - S_s^m| \\ & \leq |\mathbb{E}[S_{s+1/2^n}^m - S_s^m | \mathcal{F}_s]| + \mathbb{E}[|S_{s+1/2^n}^m - S_{s+1/2^n}^n| | \mathcal{F}_s] - |S_s^n - S_s^m| \end{aligned} \quad (2.23)$$

for all $s = 1 - 1/2^n, 1 - 2/2^n, \dots, 0$. The equality in (2.23) can be checked separately on the \mathcal{F}_s -measurable sets $B_1 := \{\mathbb{E}[S_{s+2^{-n}}^n | \mathcal{F}_s] > \overline{S}_s\}$, $B_2 := \{\mathbb{E}[S_{s+2^{-n}}^n | \mathcal{F}_s] < \underline{S}_s\}$ and $B_3 := \{\underline{S}_s \leq \mathbb{E}[S_{s+2^{-n}}^n | \mathcal{F}_s] \leq \overline{S}_s\}$. By the definition of S^n , $B_1 \subseteq \{S_s^n = \overline{S}_s\}$. On the other hand, we have $S_s^m \leq \overline{S}_s$ which implies the equality on B_1 . On the set $B_2 \subseteq \{S_s^n = \underline{S}_s\}$, the situation is symmetric. Finally, on $B_3 = \{S_s^n = \mathbb{E}[S_{s+2^{-n}}^n | \mathcal{F}_s]\}$, the equality is obvious. The inequality in (2.23) follows from Jensen's inequality for conditional expectations and the triangle inequality.

Now, we show (2.22) by a backward induction on $s = 1 - 1/2^n, 1 - 2/2^n, \dots, 0$. For the initial step $s = 1 - 1/2^n$, we only have to multiply (2.23) for $s = 1 - 1/2^n$ by $\mathbb{1}_{\{1-2^{-n} < \tau_m\}}$ and use that $|S_1^m - S_1^n| \leq 1$.

Induction step $s + 1/2^n \rightsquigarrow s$: by the induction hypothesis, one has

$$\begin{aligned} & \mathbb{E} \left[\sum_{t_i \in D_n^*, t_i \geq s+1/2^n} \mathbb{1}_{\{t_i < \tau_m\}} |\mathbb{E}[S_{t_{i+1}}^n - S_{t_i}^n | \mathcal{F}_{t_i}]| \middle| \mathcal{F}_s \right] \\ & \leq \mathbb{E} \left[\sum_{t_i \in D_n^*, t_i \geq s+1/2^n} \mathbb{1}_{\{t_i < \tau_m\}} |\mathbb{E}[S_{t_{i+1}}^m - S_{t_i}^m | \mathcal{F}_{t_i}]| \middle| \mathcal{F}_s \right] \\ & \quad + \mathbb{1}_{\{s < \tau_m\}} \mathbb{E}[1 - |S_{s+1/2^n}^n - S_{s+1/2^n}^m| | \mathcal{F}_s], \end{aligned} \quad (2.24)$$

where we take on both sides of (2.22) for $s + 1/2^n$ the conditional expectation given \mathcal{F}_s and use that $\{s + 1/2^n < \tau_m\} \subseteq \{s < \tau_m\}$. Multiplying (2.23) by $\mathbb{1}_{\{s < \tau_m\}}$ and adding (2.24) yields (2.22).

Step 2: We still need to pass from D_n to D_m for the process S^m , i.e., we now want to show that

$$\begin{aligned} & \mathbb{E} \left[\sum_{t_i \in D_n^*, t_i \geq s} \mathbb{1}_{\{t_i < \tau_m\}} \left| \mathbb{E}[S_{t_{i+1}}^m - S_{t_i}^m | \mathcal{F}_{t_i}] \right| \right] \\ & \leq \mathbb{E} \left[\sum_{t_i \in D_m^*, t_i \geq s} \mathbb{1}_{\{t_i < \tau_m\}} \left| \mathbb{E}[S_{t_{i+1}}^m - S_{t_i}^m | \mathcal{F}_{t_i}] \right| \right] + \mathbb{1}_{\{s < \tau_m\}}. \end{aligned} \quad (2.25)$$

This is less tricky: for $\tau_m = 1$, it directly follows from the triangle inequality together with Jensen's inequality for conditional expectations, and the second summand on the right-hand side is not needed. However, in the general case, there is the problem that τ_m can stop in $D_m \setminus D_n$. Thus for every $i \in \{s2^n, s2^n + 1, \dots, 2^n - 1\}$, we have to make the calculations

$$\begin{aligned} & \mathbb{1}_{\{i/2^n < \tau_m\}} \left| \mathbb{E}[S_{(i+1)/2^n}^m - S_{i/2^n}^m | \mathcal{F}_{i/2^n}] \right| \\ & = \mathbb{1}_{\{i/2^n < \tau_m\}} \left| \mathbb{E} \left[\sum_{j=i2^{m-n}}^{(i+1)2^{m-n}-1} (S_{(j+1)/2^m}^m - S_{j/2^m}^m) \middle| \mathcal{F}_{i/2^n} \right] \right| \\ & \leq \mathbb{E} \left[\sum_{j=i2^{m-n}}^{(i+1)2^{m-n}-1} \mathbb{1}_{\{j/2^m < \tau_m\}} \left| \mathbb{E}[S_{(j+1)/2^m}^m - S_{j/2^m}^m | \mathcal{F}_{j/2^m}] \right| \middle| \mathcal{F}_{i/2^n} \right] \\ & \quad + \left| \mathbb{E} \left[\mathbb{1}_{\{i/2^n < \tau_m\}} \sum_{j=i2^{m-n}}^{(i+1)2^{m-n}-1} \mathbb{1}_{\{j/2^m \geq \tau_m\}} (S_{(j+1)/2^m}^m - S_{j/2^m}^m) \middle| \mathcal{F}_{i/2^n} \right] \right|. \end{aligned} \quad (2.26)$$

For the second summand, we can use the estimate

$$\begin{aligned} & \left| \mathbb{1}_{\{i/2^n < \tau_m\}} \sum_{j=i2^{m-n}}^{(i+1)2^{m-n}-1} \mathbb{1}_{\{j/2^m \geq \tau_m\}} (S_{(j+1)/2^m}^m - S_{j/2^m}^m) \right| \\ & = \left| \sum_{j=i2^{m-n}+1}^{(i+1)2^{m-n}-1} \mathbb{1}_{\{(j-1)/2^m < \tau_m \leq j/2^m\}} (S_{(i+1)/2^n}^m - S_{j/2^m}^m) \right| \\ & \leq \sum_{j=i2^{m-n}+1}^{(i+1)2^{m-n}-1} \mathbb{1}_{\{(j-1)/2^m < \tau_m \leq j/2^m\}} \leq \mathbb{1}_{\{i/2^n < \tau_m \leq (i+1)/2^n\}}, \end{aligned} \quad (2.27)$$

where we use $0 \leq S_{t_i}^m \leq 1$ for all $t_i \in D_m$. Putting (2.26) and (2.27) together and summing up over all i , we arrive at (2.25). Together with (2.22), this yields the main assertion. Inequality (2.21) is just (2.25) for $s = 0$. \square

For the convenience of the reader, we recall the following result from Beiglböck and Siorpaes [3, Lemma 4.2].

Lemma 2.20 Assume that $(\tau_n)_{n \in \mathbb{N}}$ is a sequence of $[0, 1] \cup \{\infty\}$ -valued stopping times such that $\mathbb{P}[\tau_n = \infty] \geq 1 - \varepsilon$ for some $\varepsilon > 0$ and all $n \in \mathbb{N}$. Then there exist a stopping time τ and for each $n \in \mathbb{N}$ convex weights $\mu_n^n, \dots, \mu_{N_n}^n$, i.e., $\mu_k^n \geq 0$, $k = n, \dots, N_n$, and $\sum_{k=n}^{N_n} \mu_k^n = 1$, such that $\mathbb{P}[\tau = \infty] \geq 1 - 3\varepsilon$ and

$$\mathbb{1}_{[0, \tau]} \leq 2 \sum_{k=n}^{N_n} \mu_k^n \mathbb{1}_{[0, \tau_k]}, \quad n \in \mathbb{N}.$$

We are now in the position to prove Theorem 2.8.

Proof of Theorem 2.8 Let Assumption 2.10 hold. Let $\varepsilon > 0$, $(\tau_n)_{n \in \mathbb{N}}$ and $C > 0$ be as in Lemma 2.17. In addition, let τ be as in Lemma 2.20. We have

$$\begin{aligned} MV(S^{n, d_n(\tau)}, D_n) &= \mathbb{E} \left[\sum_{t_i \in D_n^*} \mathbb{1}_{\{t_i < \tau\}} \left| \mathbb{E}[S_{t_{i+1}}^n - S_{t_i}^n | \mathcal{F}_{t_i}] \right| \right] \\ &\leq 2 \mathbb{E} \left[\sum_{t_i \in D_n^*} \sum_{k=n}^{N_n} \mu_k^n \mathbb{1}_{\{t_i < \tau_k\}} \left| \mathbb{E}[S_{t_{i+1}}^n - S_{t_i}^n | \mathcal{F}_{t_i}] \right| \right] \\ &= 2 \sum_{k=n}^{N_n} \mu_k^n MV(S^{n, d_n(\tau_k)}, D_n) \\ &\leq 2 \sum_{k=n}^{N_n} \mu_k^n (MV(S^{k, \tau_k}, D_k) + 2) \leq 2C + 4, \quad n \in \mathbb{N}. \end{aligned} \quad (2.28)$$

Indeed, both equalities hold by Lemma 2.18. The first inequality is due to Lemma 2.20, and the second follows from Lemma 2.19. The third inequality holds by Lemma 2.17. Next, let us show that for all $n \in \mathbb{N}$,

$$\begin{aligned} MV(S^{d_n(\tau)}, D_n) &= \lim_{m \rightarrow \infty} MV(S^{m, d_n(\tau)}, D_n) \\ &\leq \limsup_{m \rightarrow \infty} MV(S^{m, d_m(\tau)}, D_m) + 1 \leq 2C + 5, \end{aligned}$$

where S is the value process of the continuous-time game. Indeed, the equality follows from Proposition 2.14 and the dominated convergence theorem. The first inequality is (2.21) and the second follows from (2.28). Together with Lemma 2.18, we arrive at

$$MV(S^\tau, D_n) \leq 2C + 6, \quad n \in \mathbb{N}. \quad (2.29)$$

Finally, by the right-continuity of S^τ and (2.29), we get

$$MV(S^\tau) = \lim_{n \rightarrow \infty} MV(S^\tau, D_n) \leq 2C + 6.$$

Together with $\mathbb{P}[\tau < \infty] \leq 3\varepsilon$, this establishes that the right-continuous process S is a local quasimartingale and thus a semimartingale by Rao's theorem (in the version

of Theorem 2.12) and the Doob–Meyer decomposition; see e.g. Protter [37, Chap. 3, Theorem 16]. \square

Proof of Theorem 2.7 Having shown that Theorem 2.8 holds, the assertion follows directly by Proposition 2.9. \square

Remark 2.21 The arguments presented here rely heavily on the two-dimensional setting. However, Theorem 2.7 can be directly applied to a model with a bank account and finitely many risky assets since in this case it is sufficient to have a semimartingale price system for each risky asset separately (cf. also Delbaen and Schachermayer [13, Theorem 7.2]). On the other hand, it seems that the approach cannot be adapted to the general Kabanov model (cf. Kabanov and Safarian [27, Sect. 3.6]) in which there need not exist a bank account that is involved in every transaction.

3 The self-financing condition

As already discussed in the introduction, we use a semimartingale price system to define the self-financing condition in the bid–ask model for general strategies. A self-financing condition can be identified with an operator $\varphi \mapsto \Pi(\varphi)$ that maps each amount of risky assets to the corresponding position in the riskless bank account (if the latter exists). Here, we assume that the initial position and the riskless interest rate are zero. In addition, **for the rest of the paper, we assume that there exists a semimartingale price system S** , i.e., S is a semimartingale such that $\underline{S} \leq S \leq \bar{S}$. By Theorem 2.7, this assumption holds under NUPBR for simple long-only strategies. In the following, S is some arbitrary semimartingale price system. It need not coincide with the semimartingale price system constructed in Sect. 2. Moreover, the NUPBR condition for simple long-only strategies need not hold. The aim is to define $\Pi(\varphi)$ as $\varphi \bullet S - \varphi S$ – “costs”, where the process $\varphi \bullet S$ denotes the stochastic integral. At this stage, the process φ is bounded (see Sect. 4 for the extension to unbounded strategies). The costs are caused by the approach that stock positions are evaluated by S , but trades are carried out at the less favorable bid–ask prices. Since the gains in the semimartingale S are modelled by a finite-valued stochastic integral, they cannot compensate infinite costs, and the latter lead to infinite debts in the bank account. The “costs” that we construct below depend on the choice of the semimartingale price system S , but the self-financing riskless position $\Pi(\varphi)$ does not (see Corollary 3.22). The latter means that the operator Π only depends on (\underline{S}, \bar{S}) . Moreover, for a simple trading strategy φ , $\Pi_t(\varphi)$ coincides with φ_t^0 in (2.1).

3.1 Construction of the cost term

We construct the cost associated to a strategy $\varphi \in \mathbf{bP}$ path by path, i.e., **in the following, $\omega \in \Omega$ is fixed and $\varphi, \underline{S}, \bar{S}, S$ are identified with functions of time.**

We follow a two-step procedure. First, we calculate the costs on intervals in which the left limit of the spread is bounded away from zero by means of a modified Riemann–Stieltjes integral. The integral turns out to always exist (but it can take

the value $+\infty$). In the second step, we exhaust the set of points with positive spread by finite unions of such intervals and define the total costs as the supremum of the costs along these unions. One may see a vague analogy between the second step and the way a Lebesgue integral is constructed.

This approach leads to a well-founded self-financing condition under the additional Assumption 3.18 on the behaviour of the spread at zero. Very roughly speaking, there should not occur costs if the investor builds up positions at times the spread is zero and the positions are already closed *before* the spread reaches any given positive value (cf. Example 3.23 for a counterexample). Since for the construction of our cost process itself, the assumption is not needed, we introduce it later on.

In order to introduce the integral, we need the following notation.

Definition 3.1 Let $I = [a, b] \subseteq [0, T]$ with $a < b$.

- (i) A collection $P = \{t_0, \dots, t_n\}$ of points $t_i \in [a, b]$ for $n \in \mathbb{N}$ and $i = 0, \dots, n$ with $a = t_0 < t_1 < \dots < t_n = b$ is called a *partition of I* .
- (ii) A partition $P' = \{t'_0, \dots, t'_m\}$ with $P' \supseteq P$ is called a *refinement of P* .
- (iii) If P, P' are two partitions of I , the *common refinement* $P \cup P'$ is the partition obtained by ordering the points of $\{t_0, \dots, t_n\} \cup \{t'_0, \dots, t'_m\}$ in increasing order.
- (iv) Given a partition $P = \{t_0, \dots, t_n\}$ of I , a collection $\lambda = \{s_1, \dots, s_n\}$ of points $s_i \in [t_{i-1}, t_i]$ for $i = 1, \dots, n$ is called a *modified intermediate subdivision of P* .
- (v) Let $\varphi \in \mathbf{bP}$, $P = \{t_0, \dots, t_n\}$ be a partition of I and $\lambda = \{s_1, \dots, s_n\}$ a modified intermediate subdivision of P . Then the *modified Riemann–Stieltjes sum* is defined by

$$R(\varphi, P, \lambda) := \sum_{i=1}^n (\bar{S}_{s_i} - S_{s_i})(\varphi_{t_i} - \varphi_{t_{i-1}})^+ + \sum_{i=1}^n (S_{s_i} - \underline{S}_{s_i})(\varphi_{t_i} - \varphi_{t_{i-1}})^-.$$

Definition 3.2 Let $\varphi \in \mathbf{bP}$ and $I = [a, b] \subseteq [0, T]$ with $a < b$. The *cost term of φ on I* exists and equals $C(\varphi, I) \in \mathbb{R}_+ \cup \{\infty\}$ if for all $\varepsilon > 0$, there is a partition P_ε of I such that for all refinements P of P_ε and all modified intermediate subdivisions λ of P , the following is satisfied:

- (i) In the case of $C(\varphi, I) < \infty$, we have $|R(\varphi, P, \lambda) - C(\varphi, I)| < \varepsilon$.
- (ii) In the case of $C(\varphi, I) = \infty$, we have $|R(\varphi, P, \lambda)| > \frac{1}{\varepsilon}$.

In addition, we set $C(\varphi, \{a\}) := 0$ for all $a \in [0, T]$ and $C(\varphi, \emptyset) := 0$.

The next proposition establishes the existence of the cost term on an interval I where the spread is bounded away from zero.

Proposition 3.3 Let $\varphi \in \mathbf{bP}$ and $I = [a, b] \subseteq [0, T]$ be an interval with $a < b$ such that $\inf_{t \in [a, b]} (\bar{S}_t - \underline{S}_t) > 0$. Then the cost term $C(\varphi, I)$ in Definition 3.2 exists and is unique. In addition, we have

$$\begin{cases} C(\varphi, I) < \infty, & \text{if } \text{Var}_a^b(\varphi) < \infty, \\ C(\varphi, I) = \infty, & \text{if } \text{Var}_a^b(\varphi) = \infty, \end{cases}$$

where $\text{Var}_a^b(\varphi)$ denotes the pathwise variation of φ on the interval $[a, b]$.

We postpone the technical proof of Proposition 3.3 to Appendix A.

Remark 3.4 First note that a priori φ need not be of finite variation. Thus we cannot decompose the strategy into its increasing part φ^\uparrow and decreasing part φ^\downarrow to define $\int_a^b (\bar{S}_s - S_s) d\varphi_s^\uparrow + \int_a^b (S_s - \underline{S}_s) d\varphi_s^\downarrow =: C(\varphi^\uparrow, [a, b]) + C(\varphi^\downarrow, [a, b])$. Instead, we consider the increasing and decreasing parts of φ along grids and weigh them with the corresponding prices before passing to the limit.

However, if $\text{Var}_a^b(\varphi) < \infty$, it can be seen by an inspection of the proof of Proposition 3.3 that $C(\varphi^\uparrow, [a, b]) + C(\varphi^\downarrow, [a, b]) = C(\varphi, [a, b])$. In addition, the condition $\inf_{t \in [a, b]} (\bar{S}_t - \underline{S}_t) > 0$ can be dropped if $\text{Var}_a^b(\varphi) < \infty$.

Remark 3.5 Definition 3.2 (i) only requires that the cost term exists in the Moore–Pollard–Stieltjes sense (see e.g. Hildebrandt [24, Sect. 4] and Mikosch and Norvaiša [35, Sect. 2.3]), i.e., as the limit of the net $R(\varphi, \cdot, \cdot)$ indexed by the directed set of tuples (P, λ) with the partial order $(P, \lambda) \geq (P', \lambda')$ if and only if P is a refinement of P' . This is weaker than the existence in the norm-sense, i.e., as the limit of the net $R(\varphi, \cdot, \cdot)$ indexed by the tuples (P, λ) with the partial order $(P, \lambda) \geq (P', \lambda')$ if and only if $\max_{i=1, \dots, n} (t_i - t_{i-1}) \leq \max_{i=1, \dots, m} (t'_i - t'_{i-1})$, that is required for the usual Riemann–Stieltjes integral with a continuous integrator of finite variation. A straightforward adaptation of the existence in the norm-sense of the usual Riemann–Stieltjes integral to the present context would read as follows:

The cost term is said to exist and equal $C(\varphi, I) \in \mathbb{R}_+$ if for each $\varepsilon > 0$, there is $\delta > 0$ such that $|C(\varphi, I) - R(\varphi, P, \lambda)| < \varepsilon$ for all partitions $P = \{t_0, \dots, t_n\}$ with $\max_{i=1, \dots, n} (t_i - t_{i-1}) < \delta$ and all subdivisions $\lambda = \{s_1, \dots, s_n\}$ with $s_i \in [t_{i-1}, t_i]$.

But the following example, similar to Guasoni et al. [20, Example A.3], shows that $C(\varphi, I)$ in general does not exist in the norm-sense: let $T = 2$, $\bar{S} - S = \mathbb{1}_{[1, 2]}$ and $\varphi = \mathbb{1}_{(1, 2]}$. Then if $t_i = 1$ is not included in the partition P , $R(\varphi, P, \lambda)$ can oscillate between 0 and 1.

The example shows that the points of common discontinuities of integrator and integrand are critical to calculate the costs. Thus they have to be included in the partition, which is guaranteed by the Moore–Pollard–Stieltjes approach.

Remark 3.6 The restriction that the point s_i of the intermediate subdivision λ must lie in the interval $[t_{i-1}, t_i]$, and not only in $[t_{i-1}, t_i]$, has a clear financial interpretation.

If an investor buys $\varphi_s - \varphi_{s-}$ shares at time s , she pays $(\varphi_s - \varphi_{s-})\bar{S}_{s-}$ monetary units. Consequently, if she updates her position between t_{i-1} and t_i , only the stock prices on the time interval $[t_{i-1}, t_i]$ need to be considered. In the limit, the choice of the price in $[t_{i-1}, t_i]$ does not matter. Indeed, a well-known way to guarantee the existence of Riemann–Stieltjes integrals in the case of simultaneous jump discontinuities on the same side of integrator and integrand is to exclude the boundary points (see Hildebrandt [24, Sect. 6]).

Finally, we mention that in the case of $\text{Var}_a^b(\varphi) < \infty$, the integrals are the same as in Guasoni et al. [20, Sect. A.2]. But besides considering different processes, we introduce the integrals in a different way.

The next proposition states that the cost term is additive with regard to the underlying interval. Its proof is obvious from the definition.

Proposition 3.7 *Let $\varphi \in \mathbf{bP}$, $I = [a, b] \subseteq [0, T]$ such that $\inf_{t \in [a, b]} (\bar{S}_t - \underline{S}_t) > 0$ and $c \in [a, b]$. Then we have*

$$C(\varphi, [a, b]) = C(\varphi, [a, c]) + C(\varphi, [c, b]).$$

Having defined the costs for all intervals $I = [a, b] \subseteq [0, T]$ which satisfy the inequality $\inf_{t \in [a, b]} (\bar{S}_t - \underline{S}_t) > 0$, we now proceed to define the accumulated costs as a process. To that end, we let

$$\mathcal{I} := \left\{ \bigcup_{i=1}^n [a_i, b_i] : \begin{array}{l} n \in \mathbb{N}, \\ 0 \leq a_1 \leq b_1 \leq a_2 \leq \dots \leq a_n \leq b_n \leq T, \\ \inf_{t \in [a_i, b_i]} (\bar{S}_t - \underline{S}_t) > 0, \quad i = 1, \dots, n \end{array} \right\} \cup \{\emptyset\}. \quad (3.1)$$

We now extend the cost term to \mathcal{I} . Given $\varphi \in \mathbf{bP}$ and $J = \bigcup_{i=1}^n [a_i, b_i] \in \mathcal{I}$, we define *the costs along J* by

$$C(\varphi, J) := \sum_{i=1}^n C(\varphi, [a_i, b_i]), \quad (3.2)$$

where the cost terms $C(\varphi, [a_i, b_i])$ for $i = 1, \dots, n$ are defined in Definition 3.2. By Proposition 3.7, the right-hand side of (3.2) does not depend on the representation of J . Thus the cost term $C(\varphi, J)$ is well defined for all $J \in \mathcal{I}$.

Definition 3.8 Let $\varphi \in \mathbf{bP}$. Then the *cost process* $(C_t(\varphi))_{t \in [0, T]}$ is defined by

$$C_t(\varphi) := \sup_{J \in \mathcal{I}} C(\varphi, J \cap [0, t]) \in [0, \infty], \quad t \in [0, T].$$

(Note that $\{0\} \in \mathcal{I}$ with $C(\varphi, \{0\}) = 0$ so that the supremum is nonnegative.) If φ is clear from the context, we also write $(C_t)_{t \in [0, T]}$ for the cost process associated to φ .

Proposition 3.9 *Let $\varphi \in \mathbf{bP}$. The cost process $(C_t(\varphi))_{t \in [0, T]}$ is $[0, \infty]$ -valued, increasing and consequently $\text{l\`a} \text{g} \text{l\`a} \text{d}$ (if finite). In addition, the following assertions hold:*

- (i) *For any $0 \leq s \leq t \leq T$, we have $C_t(\varphi) = C_s(\varphi) + \sup_{J \in \mathcal{I}} C(\varphi, J \cap [s, t])$.*
- (ii) *For any $0 \leq s \leq t \leq T$ with $\inf_{u \in [s, t]} (\bar{S}_u - \underline{S}_u) > 0$, we have*

$$C_t(\varphi) = C_s(\varphi) + C(\varphi, [s, t]).$$

- (iii) *For any $0 \leq s \leq t \leq T$, we have*

$$C_t(\varphi) \leq C_s(\varphi) + \sup_{u \in [s, t]} (\bar{S}_u - \underline{S}_u) \text{Var}_s^t(\varphi).$$

The assertions above follow directly from Definitions 3.2 and 3.8. Thus we leave the easy proof to the reader.

The next proposition determines sequences of partitions whose corresponding Riemann–Stieltjes sums converge to the cost term on an interval where the spread is

bounded away from zero. This will be crucial to show that the cost term is predictable. For this purpose, recall that the oscillation $\text{osc}(f, I)$ of a function $f : [0, T] \rightarrow \mathbb{R}$ on an interval $I \subseteq [0, T]$ is defined by $\text{osc}(f, I) := \sup\{|f(t) - f(s)| : s, t \in I\}$.

Proposition 3.10 Take $\varphi \in \mathbf{bP}$ and $I = [a, b] \subseteq [0, T]$ with $a < b$ and such that $\inf_{t \in [a, b]} (\bar{S}_t - \underline{S}_t) > 0$, and let $(P_n)_{n \in \mathbb{N}}$ be a refining sequence of partitions of I , i.e., $P_n = \{t_0^n, \dots, t_{m_n}^n\}$ with $a = t_0^n < t_1^n < \dots < t_{m_n}^n = b$ and $P_{n+1} \supseteq P_n$, such that

- (i) $\lim_{n \rightarrow \infty} \max \left(\sup_{i=1, \dots, m_n} \text{osc}(\bar{S} - S, [t_{i-1}^n, t_i^n]), \sup_{i=1, \dots, m_n} \text{osc}(S - \underline{S}, [t_{i-1}^n, t_i^n]) \right) = 0.$
- (ii) $\lim_{n \rightarrow \infty} \sum_{i=1}^{m_n} |\varphi_{t_i^n} - \varphi_{t_{i-1}^n}| = \text{Var}_a^b(\varphi).$

Then for any sequence $\lambda_n = \{s_1^n, \dots, s_{m_n}^n\}$ of modified intermediate subdivisions, we have

$$R(\varphi, P_n, \lambda_n) \longrightarrow C(\varphi, [a, b]) \quad \text{as } n \rightarrow \infty.$$

In addition, a sequence $(P_n)_{n \in \mathbb{N}}$ as above always exists.

The proof of Proposition 3.10 is closely related to the proof of Proposition 3.3. Thus we also postpone it to Appendix A. We now conclude the subsection with a first approximation result.

Proposition 3.11 Let $\varphi, \varphi^n \in \mathbf{bP}$, $n \in \mathbb{N}$, $t \in [0, T]$ and $J \in \mathcal{I}$. Then we have the implication

$$\varphi^n \rightarrow \varphi \text{ pointwise} \implies \liminf_{n \rightarrow \infty} C(\varphi^n, J \cap [0, t]) \geq C(\varphi, J \cap [0, t]). \quad (3.3)$$

Proof Let $\varphi^n \rightarrow \varphi$ pointwise and $t \in [0, T]$. We start by noting that the claim is trivial if $J = \{a\}$ for some $a \in [0, T]$ or if $J = \emptyset$.

Step 1. We now treat the special case $J = [a, b] \in \mathcal{I}$ with $a < b$. In this case, we have $C(\varphi, J \cap [0, t]) = C(\varphi, [a, b \wedge t])$ and $C(\varphi^n, J \cap [0, t]) = C(\varphi^n, [a, b \wedge t])$ for all $n \in \mathbb{N}$, where we use the convention $[c, d] = \emptyset$ if $d < c$. In addition, by the preceding observation, we may assume $t > a$.

We only consider the case $C(\varphi, [a, b \wedge t]) < \infty$ since the case $C(\varphi, [a, b \wedge t]) = \infty$ is analogous. Let $\varepsilon > 0$. There is a partition $P_\varepsilon = \{t_0, \dots, t_m\}$ of $[a, b \wedge t]$ such that

$$\begin{aligned} & \sum_{i=1}^m \inf_{s \in [t_{i-1}, t_i]} (\bar{S}_s - S_s)(\varphi_{t_i} - \varphi_{t_{i-1}})^+ + \sum_{i=1}^m \inf_{s \in [t_{i-1}, t_i]} (S_s - \underline{S}_s)(\varphi_{t_i} - \varphi_{t_{i-1}})^- \\ & \geq C(\varphi, [a, b \wedge t]) - \varepsilon. \end{aligned}$$

Using the pointwise convergence of $(\varphi^n)_{n \in \mathbb{N}}$, we can find $N \in \mathbb{N}$ such that for all $n \geq N$, we have

$$\begin{aligned} & \sum_{i=1}^m \inf_{s \in [t_{i-1}, t_i)} (\bar{S}_s - S_s)(\varphi_{t_i}^n - \varphi_{t_{i-1}}^n)^+ + \sum_{i=1}^m \inf_{s \in [t_{i-1}, t_i)} (S_s - \underline{S}_s)(\varphi_{t_i}^n - \varphi_{t_{i-1}}^n)^- \\ & \geq C(\varphi, [a, b \wedge t]) - 2\varepsilon. \end{aligned} \quad (3.4)$$

Keeping this in mind, for each n , we choose a partition \bar{P}_n such that for all refinements P of \bar{P}_n and subdivisions λ of P , we have $C(\varphi^n, [a, b \wedge t]) \geq R(\varphi^n, P, \lambda) - \varepsilon$. Now let $P_n := P_\varepsilon \cup \bar{P}_n$ and write $P_n = \{t_0^n, \dots, t_{m_n}^n\}$. Denoting the points of P_n in between t_{i-1} and t_i by $t_{i-1} = t_{i_1}^n < t_{i_2}^n < \dots < t_{i_j}^n = t_i$, we have

$$\sum_{k=2}^j (\varphi_{t_{i_k}^n}^n - \varphi_{t_{i_{k-1}}^n}^n)^+ \geq (\varphi_{t_i}^n - \varphi_{t_{i-1}}^n)^+ \quad \text{and} \quad \sum_{k=2}^j (\varphi_{t_{i_k}^n}^n - \varphi_{t_{i_{k-1}}^n}^n)^- \geq (\varphi_{t_i}^n - \varphi_{t_{i-1}}^n)^-.$$

Together with (3.4), this yields

$$C(\varphi^n, [a, b \wedge t]) \geq R(\varphi^n, P_n, \lambda_n) - \varepsilon \geq C(\varphi, [a, b \wedge t]) - 3\varepsilon$$

for all $n \geq N$ and intermediate subdivisions λ_n of P_n . Hence we have

$$\liminf_{n \rightarrow \infty} C(\varphi^n, [a, b \wedge t]) \geq C(\varphi, [a, b \wedge t]) - 3\varepsilon,$$

which is tantamount to the claim as $\varepsilon \downarrow 0$.

Step 2. Finally, let $J = \bigcup_{i=1}^m [a_i, b_i] \in \mathcal{I}$. Then, using the nonnegativity of the sequences $(C(\varphi^n, [a_i, b_i \wedge t]))_{n \in \mathbb{N}}$ for $i = 1, \dots, m$, we have

$$\begin{aligned} \liminf_{n \rightarrow \infty} C(\varphi^n, J \cap [0, t]) &= \liminf_{n \rightarrow \infty} \sum_{i=1}^m C(\varphi^n, [a_i, b_i \wedge t]) \\ &\geq \sum_{i=1}^m \liminf_{n \rightarrow \infty} C(\varphi^n, [a_i, b_i \wedge t]). \end{aligned}$$

Thus (3.3) follows from Step 1 and the observation at the start of the proof. \square

3.2 The cost term as a stochastic process

Until now we kept $\omega \in \Omega$ fixed, i.e., the construction up to here is path by path. To show some measurability properties of the cost term, we now consider it as a stochastic process.

Proposition 3.12 *Let $\varphi \in \mathbf{bP}$. The cost process $C(\varphi) = (C_t(\varphi))_{t \in [0, T]}$ coincides with a predictable process up to evanescence.*

In order to prove Proposition 3.12, we need the following lemma, whose proof relies on some deep results of Doob [15] and it is postponed to Appendix A.

Lemma 3.13 *Let $\varphi \in \mathbf{bP}$ and $\sigma \leq \tau$ be two stopping times such that*

$$\inf_{\sigma(\omega) \leq t < \tau(\omega)} (\bar{S}_t(\omega) - \underline{S}_t(\omega)) > 0 \quad \text{for all } \omega \in \Omega.$$

Then the process $C(\varphi, [\sigma \wedge \cdot, \tau \wedge \cdot])$ coincides with a predictable process up to evanescence.

We start to prepare the proof of Proposition 3.12 (assuming that Lemma 3.13 holds true). Let us approximate the supremum in Definition 3.8 in a measurable way. For this, we define for each $n \in \mathbb{N}$ a sequence of stopping times by $\tau_0^n := 0$ and

$$\tau_k^n := \begin{cases} \inf\{t \geq \tau_{k-1}^n : \bar{S}_t - \underline{S}_t \leq 2^{-(n+1)}\}, & k \text{ odd,} \\ \inf\{t > \tau_{k-1}^n : \bar{S}_t - \underline{S}_t \geq 2^{-n}\}, & k \text{ even,} \end{cases} \quad \text{for } k \in \mathbb{N}. \quad (3.5)$$

Note that there are only finitely many $\tau_k^n(\omega) < \infty$ as the process $\bar{S} - \underline{S}$ has càdlàg sample paths, $\tau_{2k}^n < \tau_{2k+1}^n$ on $\{\tau_{2k}^n < \infty\}$ and

$$\inf_{\tau_{2k}^n(\omega) \leq t < \tau_{2k+1}^n(\omega)} (\bar{S}_t(\omega) - \underline{S}_t(\omega)) \geq 2^{-(n+1)} \quad \text{for } k \in \mathbb{N}_0, \omega \in \Omega.$$

In particular, this means that the process $C^n(\varphi) = (C_t^n(\varphi))_{t \in [0, T]}$ given by

$$C_t^n(\varphi) := \sum_{k=0}^{\infty} C(\varphi, [\tau_{2k}^n \wedge t, \tau_{2k+1}^n \wedge t])$$

is well defined and coincides with a predictable process up to evanescence for each $n \in \mathbb{N}$ by Lemma 3.13.

Lemma 3.14 *Let $\varphi \in \mathbf{bP}$ and $(C^n(\varphi))_{n \in \mathbb{N}}$ as above. Then $C^n(\varphi) \rightarrow C(\varphi)$ pointwise.*

Proof We write C^n instead of $C^n(\varphi)$ to not overburden the notation. In addition, we fix $(\omega, t) \in \Omega \times [0, T]$. For $C_t(\omega) < \infty$, we claim that for each $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that

$$C_t(\omega) - \varepsilon \leq C_t^n(\omega) \leq C_t(\omega) \quad \text{for all } n \geq N. \quad (3.6)$$

To prove (3.6), note first that $C_t(\omega) \geq C_t^n(\omega)$ for all $n \in \mathbb{N}$ by Definition 3.8. For the other inequality, let $\varepsilon > 0$ and choose $0 \leq a_1 < b_1 \leq a_2 < \dots \leq a_m < b_m \leq t$ such that $\inf_{u \in [a_i, b_i]} (\bar{S}_u(\omega) - \underline{S}_u(\omega)) > 0$ for $i = 1, \dots, m$ and

$$C_t(\omega) - \varepsilon \leq \sum_{i=1}^m C(\varphi(\omega), [a_i, b_i]). \quad (3.7)$$

Let $\delta := \min_{i=1, \dots, m} \inf_{u \in [a_i, b_i]} (\bar{S}_u(\omega) - \underline{S}_u(\omega)) > 0$ and choose $N \in \mathbb{N}$ such that $2^{-N} < \delta$. Then it follows from the definition of the stopping times (3.5) that

$$\bigcup_{i=1}^m [a_i, b_i] \subseteq \bigcup_{k=0}^{\infty} [\tau_{2k}^n(\omega) \wedge t, \tau_{2k+1}^n(\omega) \wedge t] \quad \text{for all } n \geq N.$$

Combining this with Proposition 3.7 and (3.7), the left inequality in (3.6) is proved. Of course, for $C_t(\omega) = \infty$, the arguments are completely analogous. We note that by Proposition 3.9 (i), N could be chosen independently of t if $C_T(\omega) < \infty$. \square

Proof of Proposition 3.12 Applying Lemma 3.13, we find that C^n coincides with a predictable process up to evanescence. Together with Lemma 3.14, this yields that the same holds for C . \square

Next, we calculate the cost of an “almost simple” trading strategy (cf. Guasoni et al. [20] for a detailed discussion).

Definition 3.15 A predictable stochastic process φ of finite variation is called an *almost simple strategy* if there is a sequence of stopping times $(\tau_n)_{n \geq 0}$ with $\tau_n < \tau_{n+1}$ on $\{\tau_n < \infty\}$ and $\#\{n : \tau_n(\omega) < \infty\} < \infty$ for all $\omega \in \Omega$ such that

$$\varphi = \sum_{n=0}^{\infty} (\varphi_{\tau_n} \mathbb{1}_{\llbracket \tau_n \rrbracket} + \varphi_{\tau_n+} \mathbb{1}_{\llbracket \tau_n, \tau_{n+1} \rrbracket}).$$

Proposition 3.16 Let φ be an almost simple strategy. Then for all $t \in [0, T]$,

$$\begin{aligned} C_t(\varphi) &= \sum_{n=0}^{\infty} \mathbb{1}_{\{\tau_n \leq t\}} ((\bar{S}_{\tau_n} - S_{\tau_n-})(\varphi_{\tau_n} - \varphi_{\tau_n-})^+ + (S_{\tau_n} - \underline{S}_{\tau_n-})(\varphi_{\tau_n} - \varphi_{\tau_n-})^-) \\ &\quad + \sum_{n=0}^{\infty} \mathbb{1}_{\{\tau_n < t\}} ((\bar{S}_{\tau_n} - S_{\tau_n})(\varphi_{\tau_n+} - \varphi_{\tau_n})^+ + (S_{\tau_n} - \underline{S}_{\tau_n})(\varphi_{\tau_n+} - \varphi_{\tau_n})^-). \end{aligned}$$

Proof For $\omega \in \Omega$ fixed, there is some $n \in \mathbb{N}_0$ with $\tau_0(\omega) < \dots < \tau_{n-1}(\omega) \leq T$ and $\tau_n(\omega) = \infty$. Now, it is sufficient to consider partitions containing the points $\tau_i(\omega) - \delta$ and $\tau_i(\omega)$ if $(\bar{S}_{\tau_i-}(\omega) - S_{\tau_i-}(\omega)) \wedge (S_{\tau_i-}(\omega) - \underline{S}_{\tau_i-}(\omega)) > 0$, and the points $\tau_i(\omega)$ and $\tau_i(\omega) + \delta$ if $(\bar{S}_{\tau_i}(\omega) - S_{\tau_i}(\omega)) \wedge (S_{\tau_i}(\omega) - \underline{S}_{\tau_i}(\omega)) > 0$ for $i = 0, \dots, n-1$ and $\delta > 0$ small. We leave the details to the reader. \square

Finally, we show how a $\varphi \in \mathbf{bP}$ which incurs finite cost on a stochastic interval where the spread is bounded away from zero can be approximated by almost simple strategies on this interval such that the cost terms converge as well.

Proposition 3.17 Let $\varphi \in \mathbf{bP}$ and $\sigma \leq \tau$ be two stopping times such that

$$\inf_{\sigma(\omega) \leq t < \tau(\omega)} (\bar{S}_t(\omega) - \underline{S}_t(\omega)) > 0$$

for all $\omega \in \Omega$ and $C(\varphi, [\sigma \wedge T, \tau \wedge T]) < \infty$ a.s. Then there exists a uniformly bounded sequence $(\varphi^n)_{n \in \mathbb{N}}$ such that $\varphi^n \mathbb{1}_{\llbracket \sigma, \tau \rrbracket}$ is almost simple with $\varphi^n_\sigma = \varphi_\sigma$ on $\{\sigma < \infty\}$ and $|\varphi - \varphi^n| \leq 1/n$ on $\llbracket \sigma, \tau \rrbracket$ (up to evanescence) for all $n \in \mathbb{N}$, and such that

$$\sup_{t \in [0, T]} |C(\varphi^n, [\sigma \wedge t, \tau \wedge t]) - C(\varphi, [\sigma \wedge t, \tau \wedge t])| \longrightarrow 0 \quad \mathbb{P}\text{-a.s.} \quad (3.8)$$

The proof is postponed to Appendix A.

3.3 Definition and characterisation

For the remainder of the paper, we make the following assumption on the bid–ask spread.

Assumption 3.18 For every $(\omega, t) \in \Omega \times [0, T)$ with $\bar{S}_t(\omega) = \underline{S}_t(\omega)$, there exists an $\varepsilon > 0$ such that either $\bar{S}_s(\omega) = \underline{S}_s(\omega)$ for all $s \in (t, (t + \varepsilon) \wedge T)$ or $\bar{S}_s(\omega) > \underline{S}_s(\omega)$ for all $s \in (t, (t + \varepsilon) \wedge T)$.

Assumption 3.18 means that each zero of the path $t \mapsto \bar{S}_t(\omega) - \underline{S}_t(\omega)$ is either an inner point from the right of the zero set of $\bar{S}(\omega) - \underline{S}(\omega)$ or a starting point of an excursion away from zero. This excludes e.g. Brownian behaviour of the spread, which is exploited in Example 3.23 where we show what can go wrong without this assumption.

For the rest of the paper, we work with the predictable versions of the cost processes (cf. Proposition 3.12) and identify processes that coincide up to evanescence. Given a semimartingale S , we define the operator Π that maps a bounded, predictable strategy φ starting at zero, i.e., $\varphi \in \mathbf{bP}$, to the associated $[-\infty, \infty)$ -valued riskless position (also starting at zero) by

$$\Pi_t(\varphi) := \varphi \bullet S_t - \varphi_t S_t - C_t(\varphi), \quad t \in [0, T],$$

which coincides with $\varphi \bullet S_t - \varphi_t S_t - C_t(\varphi)$. Throughout the paper, $\varphi \bullet S$ denotes the *standard* stochastic integral as defined in Jacod and Shiryaev [26, Definition III.6.17]. If stock positions are evaluated by the semimartingale S , the wealth process is given by

$$V_t(\varphi) := \varphi \bullet S_t - C_t(\varphi) = \Pi_t(\varphi) + \varphi_t S_t. \quad (3.9)$$

If there is ambiguity about the semimartingale S used in the construction, we write $C^S(\varphi)$, $\Pi^S(\varphi)$, $V^S(\varphi)$ instead of $C(\varphi)$, $\Pi(\varphi)$, $V(\varphi)$.

We still have to introduce a measure that gives some information about the convergence of integrals with respect to S . There exists a probability measure $Q \approx \mathbb{P}$ such that the semimartingale S possesses a decomposition $S = M + A$, where M is a Q -square-integrable martingale and A is a process of Q -integrable variation (see Dellacherie and Meyer [14, Theorem VII.58]). We introduce the finite measure

$$\mu^S(B) := \mathbb{E}_Q[\mathbb{1}_B \bullet \langle M, M \rangle_T] + \mathbb{E}_Q[\mathbb{1}_B \bullet \text{Var}_T(A)], \quad B \in \mathcal{P}, \quad (3.10)$$

where $\langle M, M \rangle$ denotes the predictable quadratic variation of M (see e.g. Jacod and Shiryaev [26, Theorem I.4.2]).

The following theorem characterises the process $V(\varphi)$ as the limit of wealth processes associated with suitable almost simple strategies. For almost simple strategies, Proposition 3.16 shows that C is the cost term that one intuitively expects. Thus for almost simple strategies, V as defined above does not need further justification.

Theorem 3.19 Let $\varphi \in \mathbf{bP}$ and let μ be a σ -finite measure on the predictable σ -algebra with $\mu^S \ll \mu$.

(i) For all $\{0, 1\}$ -valued nonincreasing predictable processes A and all uniformly bounded sequences $(\varphi^n)_{n \in \mathbb{N}}$ of predictable processes, we have the implication

$$\begin{aligned} \varphi^n \rightarrow \varphi \text{ pointwise on } \{\bar{S}_- > \underline{S}_-\} \cap \{A = 1\} \mu^S\text{-a.e. on } \{\bar{S}_- = \underline{S}_-\} \cap \{A = 1\} \\ \implies \liminf_{n \rightarrow \infty} V(\varphi^n) \leq V(\varphi) \text{ on } \{A = 1\} \text{ up to evanescence.} \end{aligned}$$

(ii) There exists a uniformly bounded sequence $(\varphi^n)_{n \in \mathbb{N}}$ of almost simple strategies such that

$$\begin{aligned} \varphi^n \longrightarrow \varphi \text{ pointwise on } \{\bar{S}_- > \underline{S}_-\} \cap \{C(\varphi) < \infty\}, \\ \mu\text{-a.e. on } \{\bar{S}_- = \underline{S}_-\} \cap \{C(\varphi) < \infty\}, \end{aligned}$$

and for all $K \in \mathbb{N}$,

$$\sup_{t \in [0, T]} |V_t(\varphi^n) - V_t(\varphi)| \mathbb{1}_{\{C_t(\varphi) \leq K\}} \longrightarrow 0 \text{ in probability for } n \rightarrow \infty.$$

Remark 3.20 In the special case $C(\varphi) < \infty$, which is equivalent to $V(\varphi) > -\infty$, setting $A = 1$ yields the following characterisation of the wealth process of a bounded strategy: (i) The wealth of the strategy exceeds the limiting wealth of (almost) pointwise converging simple strategies, and (ii) there exists a special approximating sequence such that the wealth processes converge.

On the set $\{V(\varphi) = -\infty\} = \{C(\varphi) = \infty\}$, one cannot expect the existence of a sequence of simple strategies that converge pointwise to φ on $\{\bar{S}_- > \underline{S}_-\}$. Nevertheless, Theorem 3.19 (i) provides a motivation for $V(\varphi) = -\infty$.

For the proof of Corollary 3.22, we need the theorem in this general form, covering the case of infinite costs, since it is not clear a priori that the latter property does not depend on the choice of S .

Remark 3.21 In Theorem 3.19 (i), one cannot expect convergence “uniformly in probability” as in the frictionless case. Indeed, consider $S = 1$, $\bar{S} = 2$ and $\varphi^n = \mathbb{1}_{[1/n, 1]}$ which converges pointwise to $\varphi = \mathbb{1}_{[0, 1]}$, but $V(\varphi^n) - V(\varphi) = \mathbb{1}_{[0, 1/n]}$.

Corollary 3.22 Let $\varphi \in \mathbf{bP}$. The self-financing condition, i.e., the riskless position $\Pi(\varphi)$, does not depend on the choice of the semimartingale price system up to evanescence.

Proof Let $\varphi \in \mathbf{bP}$ and S, \tilde{S} be semimartingale price systems. Of course, the measure Q in (3.10) can be chosen jointly for S and \tilde{S} , and for ease of notation, without loss of generality, $Q = \mathbb{P}$. We set $\mu := \mu^S + \mu^{\tilde{S}}$. Let us fix $K \in \mathbb{N}$ and show that

$$\Pi^{\tilde{S}}(\varphi) \geq \Pi^S(\varphi) \quad \text{on } \{C^S(\varphi) \leq K\} \text{ up to evanescence.} \quad (3.11)$$

Observe that (3.11) for all $K \in \mathbb{N}$ implies that $\Pi^{\tilde{S}}(\varphi) \geq \Pi^S(\varphi)$ up to evanescence since $\Pi^S(\varphi) = -\infty$ on $\{C^S(\varphi) = \infty\} = (\Omega \times [0, T]) \setminus \bigcup_{K \in \mathbb{N}} \{C^S(\varphi) \leq K\}$. Then the assertion of the corollary follows by symmetry. Thus it is sufficient to show (3.11).

For this, let $(\varphi^n)_{n \in \mathbb{N}}$ be a sequence of almost simple strategies satisfying the properties in Theorem 3.19 (ii) for the semimartingale S and μ given above. According to Theorem 3.19 (ii), we may suppose that

$$\sup_{t \in [0, T]} |V_t^S(\varphi^n) - V_t^S(\varphi)| \mathbb{1}_{\{C_t^S(\varphi) \leq K\}} \longrightarrow 0 \quad \mathbb{P}\text{-a.s.} \quad (3.12)$$

by passing to a subsequence. On the other hand, by applying Theorem 3.19 (i) with respect to the semimartingale \tilde{S} and $A := \mathbb{1}_{\{C^S(\varphi) \leq K\}}$, we get

$$\liminf_{n \rightarrow \infty} V^{\tilde{S}}(\varphi^n) \leq V^{\tilde{S}}(\varphi) \quad \text{on } \{C^S(\varphi) \leq K\} \text{ up to evanescence.} \quad (3.13)$$

In addition, Proposition 3.16 and elementary calculations yield the assertion of the corollary for almost simple strategies, i.e., in view of (3.9),

$$V^{\tilde{S}}(\varphi^n) = V^S(\varphi^n) + \varphi^n(\tilde{S} - S), \quad n \in \mathbb{N}. \quad (3.14)$$

It remains to analyse $(\varphi^n - \varphi)(\tilde{S} - S)$, especially on $\{\bar{S}_- = \underline{S}_-\} \cap \{\bar{S} > \underline{S}\}$. If a sequence of càdlàg processes converges to zero uniformly in probability, the same holds for the associated jump processes. Thus the choice of μ and the same arguments as in the proof of Theorem 3.19 (i) yield that in probability for $n \rightarrow \infty$,

$$\begin{aligned} \sup_{t \in [0, T]} |\varphi_t^n \Delta S_t - \varphi_t \Delta S_t| \mathbb{1}_{\{\bar{S}_{t-} - \underline{S}_{t-} = 0, C_t^S(\varphi) < \infty\}} &\longrightarrow 0, \\ \sup_{t \in [0, T]} |\varphi_t^n \Delta \tilde{S}_t - \varphi_t \Delta \tilde{S}_t| \mathbb{1}_{\{\bar{S}_{t-} - \underline{S}_{t-} = 0, C_t^S(\varphi) < \infty\}} &\longrightarrow 0. \end{aligned} \quad (3.15)$$

By passing to a further subsequence (again denoted by $(\varphi^n)_{n \in \mathbb{N}}$), we can and do assume that the convergence in (3.15) holds for \mathbb{P} -a.e. $\omega \in \Omega$. Thus on the set $\{\bar{S}_- = \underline{S}_-, C^S(\varphi) < \infty\}$, we have

$$\varphi^n(\tilde{S} - S) = \varphi^n(\tilde{S}_- - S_-) + \varphi^n(\Delta \tilde{S} - \Delta S) = \varphi^n(\Delta \tilde{S} - \Delta S)$$

which converges to $\varphi(\Delta \tilde{S} - \Delta S) = \varphi(\tilde{S} - S)$ up to evanescence. In addition, Theorem 3.19 (ii) yields $\varphi^n(\tilde{S} - S) \rightarrow \varphi(\tilde{S} - S)$ on $\{\bar{S}_- > \underline{S}_-, C^S(\varphi) < \infty\}$, i.e., we have $\varphi^n(\tilde{S} - S) \rightarrow \varphi(\tilde{S} - S)$ on $\{C^S(\varphi) < \infty\}$ up to evanescence. Combining this with (3.12)–(3.14) yields

$$\begin{aligned} \Pi^{\tilde{S}}(\varphi) - \Pi^S(\varphi) &= V^{\tilde{S}}(\varphi) - \varphi \tilde{S} - (V^S(\varphi) - \varphi S) \\ &\geq \liminf_{n \rightarrow \infty} (V^{\tilde{S}}(\varphi^n) - V^S(\varphi^n) - \varphi^n(\tilde{S} - S)) \\ &= 0 \quad \text{on } \{C^S(\varphi) \leq K\} \text{ up to evanescence,} \end{aligned}$$

and we are done. We note that the differences above are well defined since $\Pi^S(\varphi)$ and $V^S(\varphi)$ are finite on $\{C^S(\varphi) \leq K\}$. \square

The following example shows that our approach does not work without Assumption 3.18.

is Cauchy in $(\mathcal{L}, d_{\text{up}})$ and such that for all sequences $(\tilde{\varphi}^n)_{n \in \mathbb{N}} \subseteq (\mathbf{bP})^\Pi$ satisfying (i), there exists a deterministic subsequence $(n_k)_{k \in \mathbb{N}}$ such that

$$(V^S(\tilde{\varphi}^{n_k}) - V^S(\varphi^{n_k}))^+ \longrightarrow 0 \quad \text{as } k \rightarrow \infty, \text{ up to evanescence.} \quad (4.1)$$

The requirement (ii) means that in the limit, the approximation with $(\varphi^n)_{n \in \mathbb{N}}$ is better than all other pointwise approximations $(\tilde{\varphi}^n)_{n \in \mathbb{N}}$ if the stock position is evaluated by the same semimartingale. In (4.1), we cannot expect uniform convergence in time, but exceptional \mathbb{P} -nullsets can be chosen independently of time. By Corollary 3.24, we have $(\mathbf{bP})^\Pi \subseteq L$.

Proposition 4.2 *Let $\varphi \in L$. If $(\varphi^n)_{n \in \mathbb{N}} \subseteq (\mathbf{bP})^\Pi$ is a sequence satisfying the requirements of Definition 4.1 for φ with respect to a semimartingale S , and $(\tilde{\varphi}^n)_{n \in \mathbb{N}} \subseteq (\mathbf{bP})^\Pi$ is another sequence satisfying the same requirements for φ with respect to a semimartingale \tilde{S} , then we have*

$$\text{up-lim}_{n \rightarrow \infty} V^S(\varphi^n) - \varphi S = \text{up-lim}_{n \rightarrow \infty} V^{\tilde{S}}(\tilde{\varphi}^n) - \varphi \tilde{S}$$

up to evanescence.

We now can extend the operator Π to L by setting

$$\Pi(\varphi) := \text{up-lim}_{n \rightarrow \infty} V^S(\varphi^n) - \varphi S, \quad \varphi \in L, \quad (4.2)$$

where $(\varphi^n)_{n \in \mathbb{N}}$ is any sequence satisfying the requirements of Definition 4.1 with respect to the semimartingale S . By Proposition 4.2, Π is well defined on L , i.e., it does not depend on the choice of the approximating sequence and the semimartingale.

Proof of Proposition 4.2 Let $(\varphi^n)_{n \in \mathbb{N}}$ and $(\tilde{\varphi}^n)_{n \in \mathbb{N}}$ be sequences that satisfy the assumptions of the proposition. Corollary 3.22 states that the process $\Pi(\tilde{\varphi}^n)$ does not depend on the semimartingale, i.e., we have

$$V^S(\tilde{\varphi}^n) - \tilde{\varphi}^n S = V^{\tilde{S}}(\tilde{\varphi}^n) - \tilde{\varphi}^n \tilde{S} \quad \text{up to evanescence for all } n \in \mathbb{N}, \quad (4.3)$$

and thus

$$\begin{aligned} & \left(V^{\tilde{S}}(\tilde{\varphi}^n) - \tilde{\varphi}^n \tilde{S} - (V^S(\varphi^n) - \varphi^n S) \right)^+ \\ &= (V^S(\tilde{\varphi}^n) - V^S(\varphi^n) + (\varphi^n - \tilde{\varphi}^n)S)^+ \\ &\leq (V^S(\tilde{\varphi}^n) - V^S(\varphi^n))^+ + ((\varphi^n - \tilde{\varphi}^n)S)^+ \end{aligned} \quad (4.4)$$

up to evanescence for all $n \in \mathbb{N}$. We have $\varphi^n \rightarrow \varphi$ and $\tilde{\varphi}^n \rightarrow \varphi$ pointwise as $n \rightarrow \infty$. We may pass to a subsequence such that $((V^S(\tilde{\varphi}^n) - V^S(\varphi^n))^+)_{n \in \mathbb{N}}$ converges to zero pointwise up to evanescence by (4.1). In addition, we may further pass to subsequences such that $(V^{\tilde{S}}(\tilde{\varphi}^n))_{n \in \mathbb{N}}, (V^S(\varphi^n))_{n \in \mathbb{N}}$ converge pointwise up to evanescence. Thus by symmetry, (4.4) yields the assertion. \square

4.1 Frictionless markets

We now turn towards the frictionless case, i.e., $\overline{S} = \underline{S} = S$, and show that L equals the set $L(S)$ of S -integrable processes.

Proposition 4.3 *Let $\underline{S} = \overline{S} = S$ be a semimartingale. Then we have $L = L(S)$ and $\Pi(\varphi) = \varphi \bullet S - \varphi S$ for all $\varphi \in L$.*

The set $L(S)$ as given in Jacod and Shiryaev [26, Definition III.6.17] was introduced by Jacod [25], but there are equivalent definitions that may look a bit smarter and that are based on $\mathbf{bP} \subseteq L(S)$. For this, recall that the space of equivalence classes \mathbb{S} of semimartingales (identifying processes that coincide up to evanescence) endowed with the semimartingale topology defined by the metric

$$d_{\mathbb{S}}(X, Y) := \sup_{H \in \mathbf{bP}, \|H\|_{\infty} \leq 1} \|H \bullet (X - Y)\|_{\text{up}}, \quad X, Y \in \mathbb{S},$$

is a complete metric space by Émery [17, Theorem 1]. The following characterisation of S -integrability is effectively due to Chou et al. [5].

Note 4.4 Let S be a semimartingale and φ a predictable process. The following assertions are equivalent:

- (i) $\varphi \in L(S)$.
- (ii) There exists a sequence $(\varphi^n)_{n \in \mathbb{N}} \subseteq \mathbf{bP}$ with $\varphi^n \rightarrow \varphi$ pointwise, $(\varphi^n)^+ \leq \varphi^+$, $(\varphi^n)^- \leq \varphi^-$ for all $n \in \mathbb{N}$, and such that $(\varphi^n \bullet S)_{n \in \mathbb{N}}$ is Cauchy in $(\mathbb{S}, d_{\mathbb{S}})$.
- (iii) For all sequences $(\varphi^n)_{n \in \mathbb{N}} \subseteq \mathbf{bP}$ with $\varphi^n \rightarrow \varphi$ pointwise and $|\varphi^n| \leq |\varphi|$ for all $n \in \mathbb{N}$, the sequence $(\varphi^n \bullet S)_{n \in \mathbb{N}}$ is Cauchy in $(\mathbb{S}, d_{\mathbb{S}})$.

In this case, the integral $\varphi \bullet S$ is given by the limit of any such sequence $(\varphi^n \bullet S)_{n \in \mathbb{N}}$ with regard to $d_{\mathbb{S}}$.

Proof of Note 4.4 Chou et al. [5, first definition] (see also Dellacherie and Meyer [14, VIII.75]) introduce the special approximating sequence $\varphi^n := \varphi \mathbb{1}_{\{|\varphi| \leq n\}}$ for some predictable process φ . Later on, the only properties of the sequence $(\varphi^n)_{n \in \mathbb{N}}$ they use is that $\varphi^n \in \mathbf{bP}$ for $n \in \mathbb{N}$, $|\varphi^n| \leq |\varphi|$ for $n \in \mathbb{N}$, and $\varphi^n \rightarrow \varphi$ pointwise. Thus the note is just a reformulation of their results; see Chou et al. [5, Properties b), c), d) and Théorème 1] (see also [14, VIII.74–77]). \square

A similar characterisation is provided in Eberlein and Kallsen [16, equation after (3.35)], by

$$L(S) = \{\varphi \text{ predictable} : \exists \text{ semimartingale } Z \text{ such that}$$

$$(\varphi \mathbb{1}_{\{|\varphi| \leq n\}}) \bullet S = \mathbb{1}_{\{|\varphi| \leq n\}} \bullet Z \text{ for all } n \in \mathbb{N}\}.$$

It emphasises the maximality of $L(S)$ if one *requires* that the integral $\varphi \bullet S := Z$ itself is a semimartingale. By contrast, in our characterisation from Definition 4.1, the semimartingale property can be seen more as a result since it is stated with the up-metric and not with the semimartingale metric.

Proof of Proposition 4.3 $L(S) \subseteq L$: This follows from (i) \Rightarrow (ii) \Rightarrow (iii) in Note 4.4.

$L \subseteq L(S)$: Let $\varphi \in L$. Thus there exists $(\varphi^n)_{n \in \mathbb{N}} \subseteq \mathbf{bP}$ satisfying Definition 4.1 (i) and (ii). In particular, the sequence $(V^S(\varphi^n))_{n \in \mathbb{N}} = (\varphi^n \bullet S)_{n \in \mathbb{N}}$ is Cauchy with regard to d_{up} . To show by contradiction that the sequence is also Cauchy in $(\mathbb{S}, d_{\mathbb{S}})$, we assume there exist $\varepsilon > 0$, a sequence $(H^n)_{n \in \mathbb{N}}$ of predictable processes with $0 \leq H^n \leq 1$ for all $n \in \mathbb{N}$ and a subsequence $(m_n)_{n \in \mathbb{N}}$ with $m_n \geq n$ such that

$$\mathbb{P}\left[\left((H^n(\varphi^n - \varphi^{m_n})) \bullet S\right)_T^* > \varepsilon\right] > \varepsilon, \quad \forall n \in \mathbb{N}. \quad (4.5)$$

We note that in (4.5), it is assumed that H^n is $[0, 1]$ -valued and not only $[-1, 1]$ -valued, since otherwise it can be decomposed into its positive and its negative part. Next, we define the strategies $\psi^n := H^n \varphi^n + (1 - H^n) \varphi^{m_n} \in \mathbf{bP}$ for $n \in \mathbb{N}$ and

$$\theta^n := (1 - H^n) \varphi^n + H^n \varphi^{m_n} \in \mathbf{bP} \quad \text{for } n \in \mathbb{N}.$$

The strategies satisfy $\psi^n \rightarrow \varphi$, $\theta^n \rightarrow \varphi$ pointwise and

$$(\psi^n)^+ \vee (\theta^n)^+ \leq \varphi^+, \quad (\psi^n)^- \vee (\theta^n)^- \leq \varphi^-,$$

i.e., they satisfy Definition 4.1 (i).

Consider the two stopping times $\sigma^n := \inf\{t \geq 0 : \psi^n \bullet S_t - \varphi^n \bullet S_t > \varepsilon/2\}$ and $\tau^n := \inf\{t \geq 0 : \theta^n \bullet S_t - \varphi^n \bullet S_t > \varepsilon/2\}$ for each $n \in \mathbb{N}$. There is an $N \in \mathbb{N}$ such that $\mathbb{P}[(\varphi^n - \varphi^{m_n}) \bullet S_T^* > \varepsilon/2] < \varepsilon/2$ for all $n \geq N$, as $(\varphi^n - \varphi^{m_n}) \bullet S \rightarrow 0$ uniformly in probability by Definition 4.1 (ii). Thus we have

$$\begin{aligned} \mathbb{P}[\sigma^n \wedge \tau^n \leq T] &\geq \mathbb{P}\left[\left((H^n(\varphi^n - \varphi^{m_n})) \bullet S\right)_T^* > \varepsilon\right] - \mathbb{P}[(\varphi^n - \varphi^{m_n}) \bullet S_T^* > \varepsilon/2] \\ &> \varepsilon/2, \quad \forall n \geq N. \end{aligned}$$

Next, we define the adjusted strategies $\tilde{\psi}^n := \psi^n \mathbb{1}_{[0, \sigma^n]} + \varphi^n \mathbb{1}_{[\sigma^n, T]}$ and similarly $\tilde{\theta}^n := \theta^n \mathbb{1}_{[0, \tau^n]} + \varphi^n \mathbb{1}_{[\tau^n, T]}$ which still satisfy Definition 4.1 (i). Thus together with

$$\begin{aligned} &\mathbb{P}[\{\tilde{\psi}^n \bullet S_T - \varphi^n \bullet S_T > \varepsilon/2\} \cup \{\tilde{\theta}^n \bullet S_T - \varphi^n \bullet S_T > \varepsilon/2\}] \\ &\geq \mathbb{P}[\sigma^n \wedge \tau^n \leq T] > \varepsilon/2 \end{aligned}$$

for all $n \geq N$, we arrive at a contradiction to (4.1). Thus $(\varphi^n \bullet S)_{n \in \mathbb{N}}$ is Cauchy in $(\mathbb{S}, d_{\mathbb{S}})$ and the assertion follows by (ii) \Rightarrow (i) in Note 4.4. \square

One of the referees raised the following interesting question that can be considered as a generalisation of Proposition 4.3 to markets with friction. Does $\varphi \in L$ imply that there exists a semimartingale price system S such that $\varphi \in L(S)$? This would mean that if stock positions are evaluated by S , the trading gains and the cost term of the approximating bounded strategies converge separately (and not only their sum).

Under additional assumptions, the following theorem gives a positive answer to this question. Note especially that the considered model is deterministic: see Remark 4.6 below for a discussion.

Theorem 4.5 Let $\Omega = \{\omega\}$ and \bar{S}, \underline{S} be continuous. If $\varphi \in L$, $\varphi > 0$ on $(0, T]$ and φ is lower semicontinuous at all $t \in [0, T]$ with $\bar{S}_t > \underline{S}_t$, then there exists a semimartingale price system S with $\varphi \in L(S)$.

Proof We choose an arbitrary semimartingale price system \tilde{S} (whose existence is assumed at the beginning of Sect. 3). We note that the semimartingale price system S with $\varphi \in L(S)$ which we construct in this proof depends in general on the choice of \tilde{S} .

Step 1: Let us show that

$$\sup_{\psi \text{ bounded}, 0 \leq \psi \leq \varphi} V_T^{\tilde{S}}(\psi) < \infty. \quad (4.6)$$

Assume by contradiction that there exist bounded strategies ψ^n , $n \in \mathbb{N}$, such that $0 \leq \psi^n \leq \varphi$ and $V_T^{\tilde{S}}(\psi^n) \rightarrow \infty$. On the other hand, since $\varphi \in L$ and by (4.3), there exist bounded φ^n , $n \in \mathbb{N}$, with $0 \leq \varphi^n \leq \varphi$, $\varphi^n \rightarrow \varphi$ and $V_T^{\tilde{S}}(\varphi^n) \rightarrow V_T^{\tilde{S}}(\varphi) \in \mathbb{R}$. Thus there is a null sequence $(\varepsilon_n)_{n \in \mathbb{N}} \subseteq (0, 1)$ such that

$$V_T^{\tilde{S}}(\varepsilon_n \psi^n + (1 - \varepsilon_n) \varphi^n) \geq \varepsilon_n V_T^{\tilde{S}}(\psi^n) + (1 - \varepsilon_n) V_T^{\tilde{S}}(\varphi^n) \rightarrow \infty,$$

which is a contradiction to $\varphi \in L$.

Step 2: Next, we show that for each nonnegative bounded function $\tilde{\psi}$,

$$\sup_{0 \leq \psi \leq \tilde{\psi}} V_T^{\tilde{S}}(\psi) \quad (4.7)$$

is attained by a maximiser ψ^* . To see this, let $(\psi^n)_{n \in \mathbb{N}}$ be a maximising sequence, i.e., $V_T^{\tilde{S}}(\psi^n) \rightarrow \sup_{0 \leq \psi \leq \tilde{\psi}} V_T^{\tilde{S}}(\psi)$. Since $\psi^n \bullet \tilde{S}_T \leq \sup_{t \in [0, T]} \tilde{\psi}_t \cdot \text{Var}_T(\tilde{S})$ for all $n \in \mathbb{N}$, the sequence of cost terms $(C_T^{\tilde{S}}(\psi^n))_{n \in \mathbb{N}}$ is bounded. In addition, the set $\{\bar{S} > \underline{S}\}$ can be written as a countable union of closed intervals on which we have either $\tilde{S} \geq \underline{S} + 1/3(\bar{S} - \underline{S})$ or $\tilde{S} \leq \underline{S} + 2/3(\bar{S} - \underline{S})$. In the first case, selling leads to essential costs on such an interval $[a, b]$. Consequently, one must have $\sup_{n \in \mathbb{N}} \text{Var}_a^b(\psi^n) < \infty$. Then by the same arguments as in Campi and Schachermayer [4, proof of Proposition 3.4], after passing to convex combinations, we obtain a pointwise limit $\lim_{n \rightarrow \infty} \psi^n =: \psi^*$ everywhere on $\{\bar{S} > \underline{S}\}$ and $\text{Var}(\tilde{S})$ -a.e. on $\{\bar{S} = \underline{S}\}$, which must be a maximiser by Theorem 3.19 (i).

Step 3: We now construct a sequence $(\hat{\varphi}^n)_{n \in \mathbb{N}}$ such that $\hat{\varphi}^n$ is a solution of (4.7) with $\tilde{\psi} = \varphi \wedge n$ for all $n \in \mathbb{N}$ and such that for $n < m$, the strategy $\hat{\varphi}^m$ has to “buy/sell” if $\hat{\varphi}^n$ “buys/sells”.

Starting with solutions $\hat{\eta}^k$ of (4.7) with $\tilde{\psi} = (\varphi - (k - 1))^+ \wedge 1$ for each $k \in \mathbb{N}$, we define the strategies $\eta^{n,k} := (\sum_{\ell=1}^n \hat{\eta}^\ell - (k - 1))^+ \wedge 1$ for $n \in \mathbb{N}$ and $k \leq n$. We have

$$\sum_{k=1}^n V_T^{\tilde{S}}(\eta^{n,k}) = V_T^{\tilde{S}}\left(\sum_{k=1}^n \eta^{n,k}\right) = V_T^{\tilde{S}}\left(\sum_{k=1}^n \hat{\eta}^k\right) \geq \sum_{k=1}^n V_T^{\tilde{S}}(\hat{\eta}^k).$$

Indeed, $V_T^{\tilde{S}}(\cdot)$ is superadditive, and additive for $\eta^{n,k}$, $k = 1, \dots, n$. The latter can be seen by the additivity of the cost term for approximating simple strategies. Together

with $V_T^{\tilde{S}}(\tilde{\eta}^k) \geq V_T^{\tilde{S}}(\eta^{n,k})$ for all $k \leq n$, this implies $V_T^{\tilde{S}}(\tilde{\eta}^k) = V_T^{\tilde{S}}(\eta^{n,k})$ for $k \leq n$ and $n \in \mathbb{N}$. Defining $\eta^k := \lim_{n \rightarrow \infty} \eta^{n,k} = (\sum_{\ell=1}^{\infty} \tilde{\eta}^{\ell} - (k-1))^+ \wedge 1$, $k \in \mathbb{N}$, we observe that $\eta^k = 0$ on $\{\eta^{k-1} < 1\}$ and $\eta^k \leq (\varphi - (k-1)) \wedge 1$. In addition, by Theorem 3.19 (i), we have $V_T^{\tilde{S}}(\eta^k) \geq \lim_{n \rightarrow \infty} V_T^{\tilde{S}}(\eta^{n,k}) = V_T^{\tilde{S}}(\tilde{\eta}^k)$, and thus η^k solves the problem (4.7) with $\tilde{\psi} = (\varphi - (k-1)) \wedge 1$. Finally, we set $\hat{\varphi}^n := \sum_{k=1}^n \eta^k$, $n \in \mathbb{N}$. Then for an arbitrary strategy ψ with $\psi \leq \varphi \wedge n$, the optimality of η^k yields

$$V_T^{\tilde{S}}(\psi) = \sum_{k=1}^n V_T^{\tilde{S}}((\psi - (k-1))^+ \wedge 1) \leq \sum_{k=1}^n V_T^{\tilde{S}}(\eta^k) = V_T^{\tilde{S}}(\hat{\varphi}^n),$$

i.e., $\hat{\varphi}^n$ solves (4.7) with $\tilde{\psi} = \varphi \wedge n$.

Step 4: Let $(\hat{\varphi}^n)_{n \in \mathbb{N}}$ be the sequence of maximisers from the previous step. Since short positions are forbidden, we can replace \underline{S}_T by \tilde{S}_T and assume that positions are sold at T . The goal is to construct a process S such that S is of finite variation, $V_T^{\tilde{S}}(\hat{\varphi}^n) = \hat{\varphi}^n \cdot S_T$, and $\psi \cdot S_T \leq \hat{\varphi}^n \cdot S_T$ for all strategies $0 \leq \psi \leq \varphi \wedge n$, i.e., S is a shadow price simultaneously for all problems (4.7) with $\tilde{\psi} = \varphi \wedge n$, $n \in \mathbb{N}$. Under Assumption 3.18 and by an exhaustion argument, it is possible to construct S in the following way. On the frictionless intervals, cf. Lemma 5.2, S is defined as $S = \bar{S} = \underline{S}$. Now let a be a “buying time” with $\bar{S}_a > \underline{S}_a$, i.e., there exists $n \in \mathbb{N}$ such that in any neighborhood of a , there are $t_1 < t_2$ with $\hat{\varphi}_{t_2}^n > \hat{\varphi}_{t_1}^n$. Let b be the next selling time (defined as infimum over $t \geq a$ such that for some $n \in \mathbb{N}$, in any neighborhood of t , $\hat{\varphi}^n$ is not nondecreasing), and d the next buying time after b . In addition, c is the last selling time before d . We have that $a < b \leq c \leq d$. The strict inequality is crucial for the exhaustion argument. It holds since by $\bar{S}_a > \underline{S}_a$ and the continuity of the bid–ask prices, any investment needs some time to amortise, and by Step 3, for any pair of buying and selling times, there is a joint strategy $\hat{\varphi}^n$ that realises this investment. Summing up, all $\hat{\varphi}^n$, $n \in \mathbb{N}$, are nondecreasing on (a, b) , nonincreasing on (b, c) , and constant on (c, d) .

For $t \in [a, b)$, we define

$$\tau_t := \inf \left\{ s \in [a, t] : \exists \varepsilon > 0 \text{ with } \inf_{u \in (s, t+\varepsilon)} \varphi_u > \inf_{u \in (t, b)} \varphi_u \right\} \wedge t$$

and

$$S_t := \inf_{u \in [\tau_t, b)} \bar{S}_u \wedge \underline{S}_b.$$

The process S can be interpreted as follows: The strategy has to be smaller than the function φ , and there is no selling before b . This induces a maximal number of shares that can be held at some time $t < b$. If this number is strictly smaller than φ_t , S is constant near t . Only at a “bottleneck”, S can increase, in the end up to the level \underline{S}_b . For $t \in [b, c)$, we define

$$\sigma_t := \sup \left\{ s \in [t, c) : \inf_{u \in (t+\varepsilon, s)} \varphi_u > \inf_{u \in [b, t]} \varphi_u, \forall \varepsilon > 0 \right\} \vee t$$

and

$$S_t := \sup_{u \in [b, \sigma_t]} \underline{S}_u. \quad (4.8)$$

For $[c, d)$, $c < d$, we make a case distinction. For $\widehat{\varphi}^1 = 0$ on (c, d) , we define S on $[c, d)$ as the Snell envelope of the process $L_t := \underline{S}_t \mathbb{1}_{\{t < d\}} + \bar{S}_d \mathbb{1}_{\{t = d\}}$, $t \in [c, d]$, i.e., $S_t := \sup_{u \in [t, d]} L_u$, $t \in [c, d)$. Otherwise, we define $S_t := \underline{S}_c \mathbb{1}_{\{t < \tilde{\tau}_d\}} + \bar{S}_d \mathbb{1}_{\{t \geq \tilde{\tau}_d\}}$, where $\tilde{\tau}_d := \inf\{s \in [c, d] : \inf_{u \in (s, d)} \varphi_u > \inf_{u \in (d, \tilde{b})} \varphi_u\} \wedge d$ with \tilde{b} being the next selling time after d . By using the maximality and the monotonicity of all $\widehat{\varphi}^n$, $n \in \mathbb{N}$, it is easy to check that S must lie in the bid–ask spread.

Now, any excursion of the spread away from zero, cf. Lemma 5.1, can be exhausted by intervals of the form $[a, b)$, $[b, c)$, and $[c, d)$. In the special case that there is no further buying time, (4.8) is applied to the closed interval from b to the end of the excursion of the spread away from zero or to T . The resulting process S is càdlàg and does not depend on the choice of the intervals. Note that $\bar{S}_a > \underline{S}_a$ is only needed to guarantee that $b > a$.

Step 5: Let us show that S is of finite variation and

$$\widehat{\varphi}^n \cdot S_T = V_T^S(\widehat{\varphi}^n) = V_T^{\tilde{S}}(\widehat{\varphi}^n), \quad n \in \mathbb{N}. \quad (4.9)$$

Let a be a buying time and \tilde{a} be the time $\inf\{t > a : \widehat{\varphi}_t^1 = 0\}$ truncated at the end of the excursion. We have that $S_a = \bar{S}_a \geq \tilde{S}_a$ and $S_{\tilde{a}} = \underline{S}_{\tilde{a}} \leq \tilde{S}_{\tilde{a}}$, and S is nondecreasing on $[a, \tilde{a}]$. From \tilde{a} up to (and including) the next buying time, S is nonincreasing. This yields $\text{Var}_T(S) \leq \text{Var}_T(\tilde{S}) < \infty$. Finally, by the construction of S , the cost terms $C^S(\widehat{\varphi}^n)$ vanish for all $n \in \mathbb{N}$ and thus (4.9) holds. For example, on $[a, b)$ the process $\widehat{\varphi}^n$ is nondecreasing and must be constant on $\{S < \bar{S}\}$ by optimality.

Step 6: Next, we show that

$$\psi \cdot S_T \leq \widehat{\varphi}^n \cdot S_T \quad \text{for all } n \in \mathbb{N} \text{ and all strategies } \psi \text{ with } 0 \leq \psi \leq \varphi \wedge n. \quad (4.10)$$

Of course, it is sufficient to show this assertion for excursions of the spread away from zero (cf. again Lemma 5.1).

From now on, we need the assumed lower semicontinuity, i.e.,

$$\varphi_t = \lim_{\varepsilon \rightarrow 0} \inf_{u \in [t-\varepsilon, t+\varepsilon]} \varphi_u \quad \text{for all } t \in (0, T) \text{ with } \bar{S}_t > \underline{S}_t. \quad (4.11)$$

We start with the buying period, i.e., the interval $[a, b)$ (cf. Step 4). For this, we define $\xi_t := \inf_{u \in [t, b)} \varphi_u$ and claim that

$$\int_{[a, b)} \psi_t dS_t \leq \int_{[a, b)} (\varphi_t \wedge n) dS_t \leq \int_{[a, b)} (\xi_t \wedge n) dS_t \leq \int_{[a, b)} \widehat{\varphi}_t^n dS_t \quad (4.12)$$

for every strategy ψ with $\psi \leq \varphi \wedge n$.

The first inequality is obvious as S is nondecreasing on $[a, b)$. We start by showing the second inequality in (4.12). It follows from (4.11) that $(\xi_t)_{t \in [a, b)}$ is left-continuous and the set $\{t \in [a, b) : \xi_t < \varphi_t\}$ is open. Hence we find a sequence of open intervals (u_1^k, u_2^k) , $u_1^k \leq u_2^k$, $k \in \mathbb{N}$, such that

$$\{t \in [a, b) : \xi_t < \varphi_t\} = \bigcup_{k \in \mathbb{N}} (u_1^k, u_2^k). \quad (4.13)$$

For all t_1, t_2 with $u_1^k < t_1 < t_2 < u_2^k$, we have $\inf_{t \in [t_1, t_2]} (\varphi_t - \xi_t) > 0$ and hence $S_{t_2} = S_{t_1}$. This yields $S_{u_2^k-} = S_{u_1^k}$ if $u_1^k < u_2^k$. So $\int_{[a,b)} (\varphi_t \wedge n) dS_t = \int_{[a,b)} (\xi_t \wedge n) dS_t$ follows due to (4.13).

Moving towards the last inequality in (4.12), we exclude the trivial case $\bar{S}_a = \underline{S}_b$. For a given $\varepsilon > 0$, there is a partition $a = t_0 < t_1 < \dots < t_m = b$ such that

$$\begin{aligned} \int_{[a,b)} (\xi_t \wedge n) dS_t &\leq \sum_{i=1}^{m-1} (\xi_{t_{i-1}} \wedge n) (S_{t_i} - S_{t_{i-1}}) \\ &\quad + (\xi_{t_{m-1}} \wedge n) (S_{b-} - S_{t_{m-1}}) + \varepsilon \end{aligned} \quad (4.14)$$

by Protter [37, Theorem II.21] and left-continuity of ξ . Next, we define a perturbation $\widehat{\varphi}^{n,p}$ of the optimal strategy $\widehat{\varphi}^n$ in the bid–ask model, which approximately realises the gains on the right-hand side of (4.14) on $[a, b)$. Let $s := \sup\{u > a : \bar{S}_u < \underline{S}_b\} \leq b$. We set $\widehat{\varphi}^{n,p} = \widehat{\varphi}^n$ on $[0, a) \cup [s, T]$ and construct $\widehat{\varphi}^{n,p}$ on $[a, s)$ by iteratively specifying possible purchases. At time $t_0 = a$, we buy until we reach $\widehat{\varphi}_a^{n,p} := \xi_{t_0} \wedge n \geq \widehat{\varphi}_a^n$, paying the price $\bar{S}_a = S_a$ per share (time t_0 has the special property that it is a “buying time” in the sense of Step 4). We proceed as follows. If $S_{t_1} < S_{t_2}$ (which is equivalent to $\inf_{u \in [\tau_{t_1}, \tau_{t_2})} \bar{S}_u < S_{t_2}$, and in this case, $S_{t_1} = \inf_{u \in [\tau_{t_1}, \tau_{t_2})} \bar{S}_u$), we buy until we reach $\xi_{t_1} \wedge n$ shares at time $t_1^* := \arg \min_{u \in [\tau_{t_1}, \tau_{t_2})} \bar{S}_u$. Then we have $\bar{S}_{t_1^*} < S_{t_2} \leq \underline{S}_b$, i.e., $t_1^* < s$, and since $t_1^* \geq \tau_{t_1}$, the constraint $\varphi \wedge n$ is also satisfied. This is repeated for the intervals $[\tau_{t_{i-1}}, \tau_{t_i})$ for $i = 3, \dots, m$. Since purchasing prices are strictly below \underline{S}_b , in the bid–ask market, purchases take place on $[a, s)$. We have $\widehat{\varphi}_s^{n,p} \leq \xi_s \wedge n = \widehat{\varphi}_s^n$ for $s < b$, where the equality follows from the optimality of $\widehat{\varphi}^n$ and (4.11). Finally, the missing position $\widehat{\varphi}_s^n - \widehat{\varphi}_s^{n,p} \geq 0$ is purchased at price $\bar{S}_s = \underline{S}_b$ if $s < b$. In the case $s = b$, we must have $\underline{S}_b = \bar{S}_b$ and need not care about the sign of the missing position. Hence the optimality of $\widehat{\varphi}^n$ together with $V_T^{\bar{S}}(\widehat{\varphi}^n) - V_T^{\bar{S}}(\widehat{\varphi}^{n,p}) = V_T^{\bar{S}}(\widehat{\varphi}^n) - V_T^{\bar{S}}(\widehat{\varphi}^{n,p})$ yields

$$0 \leq V_T^{\bar{S}}(\widehat{\varphi}^n) - V_T^{\bar{S}}(\widehat{\varphi}^{n,p}) \leq \int_{[a,b)} \widehat{\varphi}_t^n dS_t - \int_{[a,b)} (\xi_t \wedge n) dS_t + \varepsilon, \quad (4.15)$$

where the second inequality uses (4.14) and the fact that $\widehat{\varphi}^{n,p}$ does not produce any costs with respect to S . As $\varepsilon > 0$ is arbitrary, (4.15) implies the last inequality in (4.12).

It remains to show $\int_B \psi_t dS_t \leq \int_B \widehat{\varphi}_t^n dS_t$ for $B = [b, c)$ and $B = [c, d)$, i.e., on sets other than $[a, b)$ (cf. Step 4). After a time reversal, the proof for a selling interval $[b, c)$ is the same as that for a buying interval $[a, b)$. Namely, without loss of generality, we assume that $\underline{S}_c > \underline{S}_b$ and consider an approximation similar to (4.14) “backward in time” (the last point is $b-$ with $S_{b-} = \underline{S}_b$). Time s from above is replaced by $\tilde{s} := \inf\{u > b : \underline{S}_u > \underline{S}_b\} \leq c$. From the optimality of $\widehat{\varphi}^n$, the assumption that b is a selling time in the sense of Step 4, and (4.11), it follows that $\widehat{\varphi}_{b-}^n \geq \inf_{u \in [b, \tilde{s})} \varphi_u \wedge n$. We leave the details as an exercise for the reader. On intervals with $\widehat{\varphi}^1 = 0$, we use that the Snell envelope is nonincreasing.

Step 7: By $\varphi \in L$ and (4.3), we can find a sequence $(\varphi^n)_{n \in \mathbb{N}}$ of strategies with $\varphi^n \rightarrow \varphi$ and $0 \leq \varphi^n \leq \varphi \wedge n$ such that for all other strategies $(\tilde{\varphi}^n)_{n \in \mathbb{N}}$ with $\tilde{\varphi}^n \rightarrow \varphi$

and $0 \leq \tilde{\varphi}^n \leq \varphi \wedge n$, one has $(V_T^S(\tilde{\varphi}^n) - V_T^S(\varphi^n))^+ \rightarrow 0$. Let us show that $(\varphi^n \bullet S)_{n \in \mathbb{N}}$ must be Cauchy in $(\mathbb{S}, d_{\mathbb{S}})$. We first show that

$$\forall \varepsilon > 0 \exists K \in \mathbb{R}_+ \forall n \in \mathbb{N}, \forall B \in \mathcal{B}([0, T]) : (\mathbb{1}_{\{\varphi > K\} \cap B} \varphi^n) \bullet S_T \leq \varepsilon. \quad (4.16)$$

Indeed, by (4.10), (4.9) and (4.6), we have

$$(\mathbb{1}_{\{\varphi > K\} \cap B} \varphi^n) \bullet S_T \leq (\mathbb{1}_{\{\varphi > K\}} \hat{\varphi}^n) \bullet S_T \leq \sum_{k=1}^{\infty} ((\mathbb{1}_{\{\varphi > K\}} \eta^k) \bullet S_T) < \infty \quad (4.17)$$

for all $K \in \mathbb{R}_+$ and $B \in \mathcal{B}([0, T])$. By (4.17), $(\mathbb{1}_{\{\varphi > K\}} \eta^k) \bullet S_T \leq \eta^k \bullet S_T$ (which follows from (4.10)) and dominated convergence, we obtain (4.16). Let us show that

$$\forall \varepsilon > 0 \exists K \in \mathbb{R}_+ \exists N \in \mathbb{N} \forall n \geq N \forall B \in \mathcal{B}([0, T]) : (\mathbb{1}_{\{\varphi > K\} \cap B} \varphi^n) \bullet S_T \geq -\varepsilon. \quad (4.18)$$

Assume by contradiction that there exist $\varepsilon > 0$, a subsequence $(n_k)_{k \in \mathbb{N}}$ and a sequence $(B_k)_{k \in \mathbb{N}} \subseteq \mathcal{B}([0, T])$ such that $(\mathbb{1}_{\{\varphi > k\} \cap B_k} \varphi^{n_k}) \bullet S_T < -\varepsilon$ for all $k \in \mathbb{N}$. On the other hand, since $d_{\mathbb{S}}(\mathbb{1}_{\{\varphi > k\}} \bullet S, 0) \rightarrow 0$ for $k \rightarrow \infty$, there must exist a sequence $(\lambda_k)_{k \in \mathbb{N}} \subseteq \mathbb{R}_+$ with $\lambda_k \rightarrow \infty$ slowly enough such that $\mathbb{1}_{\{\varphi > k\} \cap B_k} (\varphi^{n_k} \wedge \lambda_k) \bullet S_T \rightarrow 0$ for $k \rightarrow \infty$. Thus we have $(\mathbb{1}_{\{\varphi > k\} \cap B_k} (\varphi^{n_k} - \lambda_k)^+) \bullet S_T < -\varepsilon/2$ for k large enough. As in (4.17), we can estimate

$$\begin{aligned} (\mathbb{1}_{[0, T] \setminus (\{\varphi > k\} \cap B_k)} (\varphi^{n_k} - \lambda_k)^+) \bullet S_T &= (\mathbb{1}_{\{\varphi > \lambda_k\} \setminus (\{\varphi > k\} \cap B_k)} (\varphi^{n_k} - \lambda_k)^+) \bullet S_T \\ &\leq \sum_{\ell=1}^{\infty} ((\mathbb{1}_{\{\varphi > \lambda_k\}} \eta^{\ell}) \bullet S_T). \end{aligned}$$

The right-hand side converges to 0 as $\lambda_k \rightarrow \infty$ for $k \rightarrow \infty$. This yields that for k large enough, we have $((\varphi^{n_k} - \lambda_k)^+) \bullet S_T < -\varepsilon/4$. Since the cost term of φ^{n_k} exceeds that of $\varphi^{n_k} \wedge \lambda_k$, we arrive at $V_T^S(\varphi^{n_k}) < V_T^S(\varphi^{n_k} \wedge \lambda_k) - \varepsilon/4$ for k large enough. This is a contradiction to the maximality of $(\varphi^n)_{n \in \mathbb{N}}$ stated at the beginning of this step. Thus (4.18) holds.

Putting (4.16), (4.18), and $\varphi^n \rightarrow \varphi$ with $\varphi^n \leq \varphi$ for all $n \in \mathbb{N}$ together, we obtain that $(\varphi^n \bullet S)_{n \in \mathbb{N}}$ is Cauchy in $(\mathbb{S}, d_{\mathbb{S}})$. This implies that $\varphi \in L(S)$ (cf. Note 4.4). \square

Remark 4.6 The proof demonstrates how the maximality condition in the definition of L works. For $\varphi \in L$, the maximisation problem (4.6) must be finite, but its solution $\hat{\varphi} := \lim_{n \rightarrow \infty} \hat{\varphi}^n$ can be different from $\varphi = \lim_{n \rightarrow \infty} \varphi^n$. Also, in the frictionless shadow price market, $\hat{\varphi}^n$ dominates all other strategies that are bounded by $\varphi \wedge n$. This upper bound is key to show that $\varphi^n \bullet S$ is Cauchy with respect to the semimartingale topology.

It is an open problem whether the theorem also holds in the general stochastic case. The construction of the shadow price S is essentially based on the assumptions that the model is deterministic and φ is lower semicontinuous. The latter is needed since on the intervals with friction, S has its upward movements at the “bottlenecks” of the constraint $\varphi \wedge n$.

Nevertheless, we think that the proof already provides the basic intuition for the relation between L and $L(S)$ in the general stochastic case. In addition, the sequence of strategies constructed in Step 3 and the ideas from Step 7 should also be of general use to solve related problems in a stochastic model. By contrast, the other assumptions are less essential. They are made to focus on the main ideas and to avoid further case distinctions and technicalities.

5 Proof of Theorem 3.19

We start with two lemmas that prepare the proof of Theorem 3.19. In the following, we set $X := \bar{S} - \underline{S}$ with the convention that $X_{0-} := 0$. Let M be the set of starting points of excursions of the spread away from zero, i.e.,

$$M := (\{X = 0\} \cup \{X_- = 0\}) \\ \cap \left\{ (\omega, t) \in \Omega \times [0, T) : \exists \varepsilon > 0 \forall s \in (t, (t + \varepsilon) \wedge T) : X_s(\omega) > 0 \right\}.$$

Here, we follow the convention that an excursion also ends (and thus a new excursion can start) if only the left limit of the spread process is zero. Under the usual conditions and Assumption 3.18, the process $Y := \mathbb{1}_{\{(\omega, t) \in \Omega \times [0, T) : \exists \varepsilon > 0 \forall s \in (t, (t + \varepsilon) \wedge T) : X_s(\omega) > 0\}}$ is right-continuous on $\Omega \times [0, T)$ and adapted. The latter uses that for all $t \in [0, T)$ and $\tilde{\varepsilon} \in (0, T - t)$, one has

$$\begin{aligned} & \{\omega \in \Omega : \exists \varepsilon > 0 \forall s \in (t, (t + \varepsilon) \wedge T) : X_s(\omega) > 0\} \\ &= \Omega \setminus \{\omega \in \Omega : \exists \varepsilon \in (0, \tilde{\varepsilon}) \cap \mathbb{Q} \forall s \in (t, t + \varepsilon) \cap \mathbb{Q} : X_s(\omega) = 0\}. \end{aligned}$$

Thus Y is a progressive process (see e.g. He et al. [23, Theorem 3.11]), which implies that M is a progressive set. Consequently, $\{\omega \in \Omega : \tau(\omega) < \infty, (\omega, \tau(\omega)) \notin M\} \in \mathcal{F}$ if τ is a stopping time.

For a stopping time τ , we define the associated stopping time $\Gamma(\tau)$ by

$$\Gamma(\tau) := \inf\{t > \tau : X_t = 0 \text{ or } X_{t-} = 0\}.$$

Lemma 5.1 *There exists a sequence $(\tau_1^n)_{n \in \mathbb{N}}$ of stopping times with*

$$\mathbb{P}[\{\omega \in \Omega : \tau_1^n(\omega) < \infty, (\omega, \tau_1^n(\omega)) \notin M\}] = 0$$

for all $n \in \mathbb{N}$, $\mathbb{P}[\tau_1^{n_1} = \tau_1^{n_2} < \infty] = 0$ for all $n_1 \neq n_2$ and

$$\{X_- > 0\} \subseteq \bigcup_{n \in \mathbb{N}} \llbracket \tau_1^n, \Gamma(\tau_1^n) \rrbracket \quad \text{up to evanescence.}$$

Proof We define a finite measure μ on the predictable σ -algebra by

$$\mu(A) := \sum_{k=1}^{\infty} 2^{-k} \mathbb{P}[\{\omega \in \Omega : (\omega, q_k) \in A\}], \quad A \in \mathcal{P},$$

where $(q_k)_{k \in \mathbb{N}}$ is an enumeration of the rational numbers. Let \mathcal{M} be the set of predictable processes of the form $\mathbb{1}_{\tau, \Gamma(\tau)}$, where τ runs through all stopping times satisfying $\mathbb{P}[\{\omega \in \Omega : \tau(\omega) < \infty, (\omega, \tau(\omega)) \notin M\}] = 0$. The essential supremum of \mathcal{M} with respect to μ can be written as

$$\text{esssup } \mathcal{M} = \sup_{n \in \mathbb{N}} \mathbb{1}_{\tau_1^n, \tau_2^n} = \mathbb{1}_{\bigcup_{n \in \mathbb{N}} \tau_1^n, \tau_2^n} \quad \mu\text{-a.e.},$$

where $\tau_2^n := \Gamma(\tau_1^n)$. Obviously, we can and do choose the sequence $(\tau_1^n)_{n \in \mathbb{N}}$ such that $\mathbb{P}[\tau_1^{n_1} = \tau_1^{n_2} < \infty] = 0$ holds for all $n_1 \neq n_2$. Then by the definition of M and Γ , one has that $\mathbb{1}_{\tau_1^{n_1}, \tau_2^{n_1}} \cap \mathbb{1}_{\tau_1^{n_2}, \tau_2^{n_2}} = \emptyset$ up to evanescence for all $n_1 \neq n_2$.

Now consider the random time

$$\sigma := \inf \left\{ t \in (0, T] : X_{t-} > 0 \text{ and } t \notin \bigcup_{n \in \mathbb{N}} (\tau_1^n, \tau_2^n) \right\}.$$

Then σ is a stopping time since it can be written as the debut $\inf\{t \in (0, T] : Z_t > 0\}$, where $Z := X_- (1 - \sum_{n=1}^{\infty} \mathbb{1}_{\tau_1^n, \tau_2^n})$ is a finite-valued predictable process (see e.g. Cohen and Elliott [6, Theorem 7.3.4]). By the definition of the infimum and Γ , we must have $X_\sigma = 0$ or $X_{\sigma-} = 0$ on the set $\{\sigma < \infty\}$. Together with Assumption 3.18, this means that an excursion starts in σ , and $\mathbb{1}_{\sigma, \Gamma(\sigma)} \cap (\bigcup_{n \in \mathbb{N}} \tau_1^n, \tau_2^n) = \emptyset$. By the definition of the essential supremum, one has $\mu(\mathbb{1}_{\sigma, \Gamma(\sigma)}) = 0$. Since $\Gamma(\sigma) > \sigma$ on $\{\sigma < \infty\}$, this is only possible if $\mathbb{P}[\sigma < \infty] = 0$, and thus,

$$\mathbb{P} \left[\left\{ \omega \in \Omega : \exists t \in (0, T] : X_{t-}(\omega) > 0 \text{ and } t \notin \bigcup_{n \in \mathbb{N}} (\tau_1^n(\omega), \tau_2^n(\omega)) \right\} \right] = 0. \quad \square$$

Next, we analyse the time the spread spends at zero. Define

$$M_1 := \{(\omega, t) \in \Omega \times [0, T] : t = 0 \text{ or } \forall \varepsilon > 0, \exists s \in ((t - \varepsilon) \vee 0, t) : X_s(\omega) > 0\} \\ \cap \{X_- = 0\}$$

and

$$M_2 := \{X_- > 0\} \cap \{X = 0\}.$$

The optional set $M_1 \cup M_2$ consists of the ending points of an excursion and of their accumulation points. For a stopping time τ , we define the starting point of the next excursion after τ by $(\Lambda(\tau))(\omega) := \inf\{t \geq \tau(\omega) : (\omega, t) \in M\}$ for $\omega \in \Omega$. Then $\Lambda(\tau)$ is the debut of a progressive set and thus a stopping time by [6, Theorem 7.3.4]. Recall the notation $\tau_A := \mathbb{1}_A + \infty \mathbb{1}_{A^c}$.

Lemma 5.2 *There exists a sequence of stopping times $(\sigma_1^n)_{n \in \mathbb{N}}$ with*

$$\mathbb{P}[\{\omega \in \Omega : \sigma_1^n(\omega) < \infty, (\omega, \sigma_1^n(\omega)) \notin M_1 \cup M_2\}] = 0$$

such that $(\sigma_1^n)_{\{X_{\sigma_1^n-}=0\}}$ is predictable for all $n \in \mathbb{N}$, $\mathbb{P}[\sigma_1^{n_1} = \sigma_1^{n_2} < \infty] = 0$ for all $n_1 \neq n_2$ and

$$\{X_- = 0\} \subseteq \bigcup_{n \in \mathbb{N}} (\llbracket (\sigma_1^n)_{\{X_{\sigma_1^n-}=0\}} \rrbracket \cup \llbracket \sigma_1^n, \Lambda(\sigma_1^n) \rrbracket) \quad \text{up to evanescence} \quad (5.1)$$

for Λ defined above.

Equation (5.1) can be interpreted as follows. If the spread approaches zero continuously at some time t , the investment between $t-$ and t already falls into the “frictionless regime”. On the other hand, if the spread jumps to zero at time t , the frictionless regime only starts immediately after t (if at all).

Proof of Lemma 5.2 We take the starting points τ_1^n of the excursions from Lemma 5.1 and define the measure

$$\mu(A) := \sum_{n=1}^{\infty} 2^{-n} \mathbb{P}[\{\omega \in \Omega : (\omega, \tau_1^n(\omega)) \in A\}] + \mathbb{P}[\{\omega \in \Omega : (\omega, T) \in A\}]$$

for all $A \in \mathcal{P}$. Consider the essential supremum with respect to μ of the set of predictable processes $\llbracket \llbracket \sigma_{\{X_{\sigma-}=0\}} \rrbracket \cup \llbracket \sigma, \Lambda(\sigma) \rrbracket$, where σ runs through the set of stopping times satisfying $\mathbb{P}[\{\omega \in \Omega : \sigma(\omega) < \infty, (\omega, \sigma(\omega)) \notin M_1 \cup M_2\}] = 0$ with the further constraint that $\sigma_{\{X_{\sigma-}=0\}}$ is a predictable stopping time. Again, the supremum can be written as

$$\mathbb{1}_{\bigcup_{n \in \mathbb{N}} (\llbracket (\sigma_1^n)_{\{X_{\sigma_1^n-}=0\}} \rrbracket \cup \llbracket \sigma_1^n, \Lambda(\sigma_1^n) \rrbracket)} \quad \mu\text{-a.e.}$$

Consider the random time

$$\sigma := \inf \left\{ t \geq 0 : X_{t-} = 0 \text{ and } t \notin \bigcup_{n \in \mathbb{N}} (\llbracket (\sigma_1^n)_{\{X_{\sigma_1^n-}=0\}} \rrbracket \cup \llbracket \sigma_1^n, \sigma_2^n \rrbracket) \right\}, \quad (5.2)$$

where $\sigma_2^n := \Lambda(\sigma_1^n)$. Since $\sigma = \inf\{t \geq 0 : Z_t = 0\}$, where

$$Z := X_- + \sum_{n=1}^{\infty} \mathbb{1}_{\llbracket (\sigma_1^n)_{\{X_{\sigma_1^n-}=0\}} \rrbracket \cup \llbracket \sigma_1^n, \sigma_2^n \rrbracket}$$

is predictable, σ is a stopping time (see e.g. [6, Theorem 7.3.4]). In addition, one has

$$\begin{aligned} \llbracket \sigma_{\{X_{\sigma-}=0\}} \rrbracket &= \llbracket \sigma \rrbracket \cap \{X_- = 0\} \\ &= \left(\llbracket 0, \sigma \rrbracket \setminus \bigcup_{n \in \mathbb{N}} (\llbracket (\sigma_1^n)_{\{X_{\sigma_1^n-}=0\}} \rrbracket \cup \llbracket \sigma_1^n, \sigma_2^n \rrbracket) \right) \cap \{X_- = 0\} \in \mathcal{P}, \end{aligned}$$

where we use that the infimum in (5.2) must be attained if $X_{\sigma-} = 0$. Thus $\sigma_{\{X_{\sigma-}=0\}}$ is a predictable stopping time. Finally, we have that

$$\mathbb{P}[\{\omega \in \Omega : \sigma(\omega) < \infty, (\omega, \sigma(\omega)) \notin M_1 \cup M_2\}] = 0.$$

By the maximality of the supremum, one has $\mu(\llbracket \sigma_{\{X_{\sigma-}=0\}} \rrbracket \cup \llbracket \sigma, \Lambda(\sigma) \rrbracket) = 0$. As the intervals overlap T or some $\tau_1^n(\omega)$ if they are nonempty, we arrive at $\mathbb{P}[\sigma < \infty] = 0$, and thus (5.1) holds. \square

Note 5.3 For any $\varphi \in \mathbf{bP}$ and any σ -finite measure μ on \mathcal{P} with $\mu^S \ll \mu$, there exists a uniformly bounded sequence of simple strategies $(\varphi^n)_{n \in \mathbb{N}}$ with $\varphi^n \rightarrow \varphi$ μ -a.e., and for any such sequence $(\varphi^n)_{n \in \mathbb{N}}$, one has $\varphi^n \bullet S \rightarrow \varphi \bullet S$ uniformly in probability.

Proof The existence of such a sequence with $\varphi^n \rightarrow \varphi$ μ -a.e. follows from the approximation theorem for measures (see e.g. Klenke [33, Theorem 1.65 (ii)]). Then the convergence of the integrals follows for the martingale parts by Jacod and Shiryaev [26, (3) after Theorem 4.40], and for the finite variation parts by dominated convergence. \square

Proof of Theorem 3.19 Obviously, it is sufficient to show the theorem under an equivalent measure $\mathbb{Q} \approx \mathbb{P}$. Hence we assume without loss of generality that $\mathbb{P} = \mathbb{Q}$, where \mathbb{Q} is the measure introduced above (3.10).

(i) Let $(\varphi^n)_{n \in \mathbb{N}} \subseteq \mathbf{bP}$ satisfy $\varphi^n \rightarrow \varphi$ pointwise on $\{\bar{S}_- > \underline{S}_-, A = 1\}$. For any $J \in \mathcal{I}$ from (3.1), Proposition 3.11 yields that

$$\liminf_{n \rightarrow \infty} C(\varphi^n, J \cap [0, t])(\omega) \geq C(\varphi, J \cap [0, t])(\omega) \quad \text{for all } (\omega, t) \in \{A = 1\}.$$

It follows that

$$\liminf_{n \rightarrow \infty} C_t(\varphi^n)(\omega) \geq \sup_{J \in \mathcal{I}} C(\varphi, J \cap [0, t])(\omega) = C_t(\varphi)(\omega) \quad \text{for all } (\omega, t) \in \{A = 1\}.$$

If in addition $(\varphi^n)_{n \in \mathbb{N}}$ is uniformly bounded and

$$\varphi^n \longrightarrow \varphi \quad \mu^S\text{-a.e. on } \{\bar{S}_- = \underline{S}_-, A = 1\},$$

we have that

$$(\varphi^n \mathbb{1}_{\{A=1\}}) \bullet S \longrightarrow (\varphi \mathbb{1}_{\{A=1\}}) \bullet S \quad \text{uniformly in probability} \quad (5.3)$$

(see Note 5.3). Since $\{A = 1\}$ is a predictable set of interval type, there is an increasing sequence $(T^m)_{m \in \mathbb{N}}$ of stopping times such that

$$\{A = 1\} \cup (\Omega \times \{0\}) = \bigcup_{m \in \mathbb{N}} [0, T^m]$$

(see e.g. He et al. [23, Theorem 8.18]). For each $m \in \mathbb{N}$, we obviously have

$$((\mathbb{1}_{[0, T^m]} \varphi) \bullet S) \mathbb{1}_{[0, T^m]} = (\varphi \bullet S)^{T^m} \mathbb{1}_{[0, T^m]} = (\varphi \bullet S) \mathbb{1}_{[0, T^m]}.$$

Letting $m \rightarrow \infty$, this yields by Note 5.3 that

$$(\varphi \mathbb{1}_{\{A=1\}} \bullet S) \mathbb{1}_{\{A=1\}} = (\varphi \bullet S) \mathbb{1}_{\{A=1\}}$$

up to evanescence, and analogously $((\varphi^n \mathbb{1}_{\{A=1\}}) \bullet S) \mathbb{1}_{\{A=1\}} = (\varphi^n \bullet S) \mathbb{1}_{\{A=1\}}$ up to evanescence for $n \in \mathbb{N}$. Thus together with (5.3), we have

$$\liminf_{n \rightarrow \infty} (\varphi^n \bullet S - \varphi \bullet S)^+ \mathbb{1}_{\{A=1\}} = 0 \quad \text{up to evanescence.}$$

Putting the cost terms and the trading gains with respect to S together, we arrive at (i).

(ii) The following analysis is based on the stopping times $(\tau_1^n)_{n \in \mathbb{N}}$ and $(\sigma_1^n)_{n \in \mathbb{N}}$ from Lemmas 5.1 and 5.2, respectively. We can and do choose $(\sigma_1^n)_{n \in \mathbb{N}}$ such that

$$\mathbb{P}[\sigma_1^n = \tau_1^m < \infty, X_{\sigma_1^n -} > 0] = 0, \quad \forall n, m \in \mathbb{N}. \quad (5.4)$$

This means that if the spread X only touches zero at a single point and its left limit is non-zero, the next excursion directly starts without a one-point frictionless regime in between. For the rest of the proof, we write $\{X_{\tau-} \in B\}$ for the set

$$\{\omega \in \Omega : \exists t \in [0, T] : \tau(\omega) = t, X_{t-}(\omega) \in B\},$$

where τ is a $[0, T] \cup \{\infty\}$ -valued stopping time and $B \subseteq \mathbb{R}$. Let

$$A^n := \llbracket (\tau_1^n)_{\{X_{\tau_1^n -} > 0\}}, \Gamma(\tau_1^n) \llbracket \cup \llbracket (\Gamma(\tau_1^n))_{\{X_{\tau_1^n -} > 0\} \cap \{X_{\Gamma(\tau_1^n) -} > 0\}} \rrbracket \in \mathcal{P}, \quad n \in \mathbb{N},$$

$$B^n := \llbracket (\sigma_1^n)_{\{X_{\sigma_1^n -} = 0\}} \rrbracket \cup \llbracket \sigma_1^n, \Lambda(\sigma_1^n) \rrbracket \in \mathcal{P}, \quad n \in \mathbb{N},$$

$$\tilde{B}^n := \llbracket \Lambda(\sigma_1^n), \Gamma(\Lambda(\sigma_1^n)) \llbracket \cup \llbracket (\Gamma(\Lambda(\sigma_1^n)))_{X_{\{\Gamma(\Lambda(\sigma_1^n)) -} > 0\}} \rrbracket \rrbracket \in \mathcal{P}, \quad n \in \mathbb{N},$$

and

$$\varphi^N := \varphi \mathbb{1}_{\cup_{n=1, \dots, N} (A^n \cup B^n \cup \tilde{B}^n)}, \quad N \in \mathbb{N}.$$

Excursions away from zero are either included in some A^n or in some \tilde{B}^m with the frictionless forerunner B^m . In the first case, the spread cannot jump away from zero since $X_{\tau_1^n} = 0$ on $\{X_{\tau_1^n -} > 0\}$. In the latter case, the frictionless forerunner avoids that φ^N produces costs when the spread jumps away from zero, which do not occur with the strategy φ . Namely, at a time the spread jumps away from zero, φ^N either remains zero or it already coincides with φ . Note that the frictionless forerunner may consist of a single point only. For example, this is the case if the jump time is an accumulation point of starting/ending points of excursions shortly before.

First, we approximate φ by the strategies φ^N .

Step 1: Let $E \in \mathcal{F}_T$ be a set with $\mathbb{P}[E] = 1$ and such that the properties from Lemmas 5.1 and 5.2 hold for all $\omega \in E$. Let us show that $\varphi_t^N(\omega) \rightarrow \varphi_t(\omega)$ for all $t \in [0, T]$ and $\omega \in E$. By construction of φ^N , we only have to show that for $n \in \mathbb{N}$, the excursion starting in $\tau_1^n(\omega)$ is overlapped by $A_\omega^n := \{t \in [0, T] : (\omega, t) \in A^n\}$,

the ω -intersection of A^n , or by some \tilde{B}_ω^m , $m \in \mathbb{N}$. In the case that $X_{\tau_1^n(\omega)-}(\omega) > 0$, the excursion is overlapped by A_ω^n . In the case that $X_{\tau_1^n(\omega)-}(\omega) = 0$, we have by Lemma 5.2 that $\tau_1^n(\omega) \in [\sigma_1^m(\omega), \Lambda(\sigma_1^m(\omega))]$ for some $m \in \mathbb{N}$, and thus the excursion starting in $\tau_1^n(\omega)$ is overlapped by \tilde{B}_ω^m . By Note 5.3, it follows that $\varphi^N \bullet S \rightarrow \varphi \bullet S$ uniformly in probability for $N \rightarrow \infty$.

Step 2: Without loss of generality, we assume that the bounded process φ takes values in $[-1/2, 1/2]$ to get rid of a further constant. Let us show that

$$\sup_{t \in [0, T]} |C_t(\varphi^N) - C_t(\varphi)| \mathbb{1}_{\{C_t(\varphi) \leq K\}} \longrightarrow 0 \quad \text{as } N \rightarrow \infty, \quad (5.5)$$

pointwise on E , for all $K \in \mathbb{N}$.

From $X_{\tau_1^n} = 0$ on $\{X_{\tau_1^n-} > 0\}$ and $X_{\sigma_1^n} = 0$ on $\{X_{\sigma_1^n-} > 0\}$, we conclude that for fixed $\omega \in E$ and $a \leq b$ with $\inf_{u \in [a, b]} X_u(\omega) > 0$, we either have $\varphi_u^N(\omega) = \varphi_u(\omega)$ for all $u \in [a, b]$ or $\varphi_u^N(\omega) = 0$ for all $u \in [a, b]$. By the definition of the cost term in (3.2), this yields $C(\varphi^N, I \cap [0, t]) \leq C(\varphi, I \cap [0, t])$ for all $I \in \mathcal{I}$, $(\omega, t) \in E \times [0, T]$, and thus $C_t(\varphi^N) \leq C_t(\varphi)$ for all $(\omega, t) \in E \times [0, T]$. We define

$$\theta^m := \inf\{t \geq 0 : C_t(\varphi) > m\} \wedge T \quad \text{for } m \in \mathbb{N}. \quad (5.6)$$

Due to $\Delta^- C_{\theta^m}(\varphi) \leq \sup_{u \in [0, T]} X_u$, the *paths* of the stopped process $C^{\theta^m}(\varphi)$ are bounded. Fix $\omega \in E$ and $\varepsilon > 0$. For $K \in \mathbb{N}$, we set $u := \theta^K$. Using Proposition 3.7, we get that $C(\varphi, I \cap [0, u]) = C(\varphi, I \cap [0, t]) + C(\varphi, I \cap [t, u])$ for all $I \in \mathcal{I}$ and $t \leq u$. Therefore, together with Proposition 3.9 (i), there exists $I \in \mathcal{I}$ such that

$$\sup_{t \in [0, T]} (C_t(\varphi) - C(\varphi, I \cap [0, t])) \mathbb{1}_{\{C_t(\varphi) \leq K\}} \leq \varepsilon.$$

The set I is overlapped by finitely many ω -intersections of A^n and $B^n \cup \tilde{B}^n$. Thus one has $I \subseteq \bigcup_{n \leq N} (A^n \cup B^n \cup \tilde{B}^n)_\omega$, i.e., $C(\varphi^N, I \cap [0, t]) = C(\varphi, I \cap [0, t])$, for N large enough, and consequently

$$\begin{aligned} & (C_t(\varphi) - C_t(\varphi^N)) \mathbb{1}_{\{C_t(\varphi) \leq K\}} \\ & \leq (C(\varphi, I \cap [0, t]) - C(\varphi^N, I \cap [0, t])) \mathbb{1}_{\{C_t(\varphi) \leq K\}} + \varepsilon = \varepsilon \end{aligned}$$

for all $t \in [0, T]$. This implies (5.5). Together with Step 1, we have that

$$\begin{aligned} & \varphi^N \longrightarrow \varphi \text{ pointwise up to evanescence and} \\ & \sup_{t \in [0, T]} |V_t(\varphi^N) - V_t(\varphi)| \mathbb{1}_{\{C_t(\varphi) \leq K\}} \longrightarrow 0 \text{ in probability} \end{aligned} \quad (5.7)$$

for $N \rightarrow \infty$ and every $K \in \mathbb{N}$.

Step 3: It remains to approximate the strategies φ^N , $N \in \mathbb{N}$, by almost simple strategies. Since the pointwise convergence that we need on $\{X_- > 0\} \cap \{C(\varphi) < \infty\}$ is not metrisable, it is *not* sufficient to approximate each φ^N separately by a sequence of almost simple strategies. Recall μ^S from (3.10) and let μ be a σ -finite measure on \mathcal{P} with $\mu^S \ll \mu$. We fix some $N \in \mathbb{N}$ and let $\varepsilon := 2^{-N}$. In the following, we

construct an almost simple strategy step by step on disjoint stochastic intervals. The main idea is to approximate the cost term on subintervals of excursions where the spread is bounded away from zero while controlling the error at the beginning and the end of the excursions. We start with the construction of an almost simple strategy on A^n with $n \leq N$. We recall that $\tau_2^n := \Gamma(\tau_1^n)$. There exists a stopping time $\tau_1^{n,N}$ with $\theta^N \wedge \tau_2^n \geq \tau_1^{n,N} > \tau_1^n$ on $\{\tau_1^n < \theta^N\} \cap \{X_{\tau_1^n-} > 0\}$, $\tau_1^{n,N} = \theta^N$ on $\{\theta^N \leq \tau_1^n\}$ and, for notational convenience, $\tau_1^{n,N} = \tau_1^n$ elsewhere such that

$$\mathbb{P}[\tau_1^n \wedge \theta^N \leq \tau_1^{n,N} \leq \tau_1^n + \varepsilon] = 1, \quad (5.8)$$

as well as

$$\begin{aligned} \mathbb{P}[(\varphi^N \mathbb{1}_{\llbracket \tau_1^n, \tau_1^{n,N} \rrbracket} \bullet S)^* > \varepsilon] &\leq \varepsilon, \mathbb{P}[\tau_1^n < \infty, |X_{\tau_1^{n,N}} - X_{\tau_1^n \wedge \theta^N}| > \varepsilon] \\ &\leq \varepsilon, \\ \mathbb{P}[\tau_1^n < \infty, C_{\tau_1^{n,N}}(\varphi^N) - C_{\tau_1^n \wedge \theta^N}(\varphi^N) > \varepsilon] &\leq \varepsilon, \end{aligned}$$

where θ^N is defined in (5.6). This follows from the right-continuity of the processes $(\varphi^N \mathbb{1}_{\llbracket \tau_1^n, T \rrbracket}) \bullet S$, X and from the definition of the cost process together with $X_{\tau_1^n} = 0$ on $\{X_{\tau_1^n-} > 0\}$. In addition, since $\llbracket (\tau_2^n)_{\{X_{\tau_2^n-}=0\}} \rrbracket = \llbracket \tau_1^n, \tau_2^n \rrbracket \cap \{X_- = 0\} \in \mathcal{P}$, the stopping time $(\tau_2^n)_{\{X_{\tau_2^n-}=0\}}$ is predictable. Thus by the existence of an announcing sequence (see e.g. He et al. [23, Theorem 4.34]), there is a stopping time $\tau_2^{n,N}$ which satisfies $\tau_1^{n,N} \leq \tau_2^{n,N} \leq \tau_2^n \wedge \theta^N$,

$$\tau_2^{n,N} < \tau_2^n \quad \text{on } \{X_{\tau_2^n-} = 0, \tau_1^{n,N} < \tau_2^n\},$$

$$\mathbb{P}[\tau_2^{n,N} < \tau_2^n \wedge \theta^N - \varepsilon] \leq \varepsilon, \mathbb{P}[X_{\tau_2^n-} > 0, \tau_2^{n,N} < \tau_2^n \wedge \theta^N] \leq \varepsilon,$$

$$\mathbb{P}\left[\left((\varphi^N \mathbb{1}_{\llbracket \tau_2^{n,N}, \tau_2^n \wedge \theta^N \rrbracket} \llbracket \cup \llbracket (\tau_2^n)_{\{X_{\tau_2^n-}=0\}} \rrbracket \right) \bullet S\right)^* > \varepsilon\right] \leq \varepsilon, \quad (5.9)$$

$$\mathbb{P}[X_{\tau_2^{n,N}} > \varepsilon, \tau_2^{n,N} < \tau_2^n \wedge \theta^N] \leq \varepsilon \text{ and}$$

$$\mathbb{P}[\tau_2^n < \infty, C_{\tau_2^n \wedge \theta^N}(\varphi^N) - C_{\tau_2^{n,N}}(\varphi^N) > \varepsilon] \leq \varepsilon.$$

By Proposition 3.17 applied to the stopping times $\tau_1^{n,N} \leq \tau_2^{n,N}$, there exists an almost simple strategy $\tilde{\psi}^N$ with $\tilde{\psi}_{\tau_1^{n,N}}^N = \varphi_{\tau_1^{n,N}}^N$,

$$\sup_{t \in [\tau_1^{n,N}, \tau_2^{n,N}]} |\tilde{\psi}_t^N - \varphi_t^N| \leq \varepsilon, \quad (5.10)$$

$$\mathbb{P}\left[\sup_{t \in [\tau_1^{n,N}, \tau_2^{n,N}]} |C_t(\tilde{\psi}^N) - C_{\tau_1^{n,N}}(\tilde{\psi}^N) - (C_t(\varphi^N) - C_{\tau_1^{n,N}}(\varphi^N))| > \varepsilon\right] \leq \varepsilon,$$

and $\mathbb{P}[\mathbb{P}(((\tilde{\psi}^N - \varphi^N) \mathbb{1}_{\llbracket \tau_1^{n,N}, \tau_2^{n,N} \rrbracket}) \bullet S)^* > \varepsilon] \leq \varepsilon$ (the latter also uses Note 5.3). We define an almost simple strategy by

$$\psi_t^N := \tilde{\psi}_t^N \mathbb{1}_{(\tau_1^{n,N} < t \leq \tau_2^{n,N})} \quad \text{on } A^n. \quad (5.11)$$

Since ψ^N can be updated for free at the left endpoint of A^n , we get for the increments of the process $V(\psi^N) - V(\varphi^N) = (\psi^N - \varphi^N) \bullet S - (C(\psi^N) - C(\varphi^N))$ the estimate

$$\begin{aligned} & \mathbb{P} \left[\sup_{t \in (\tau_1^n, \tau_2^n) \cup [(\tau_2^n)_{\{X_{\tau_2^n-} > 0\}}]} |V_t(\psi^N) - V_{\tau_1^n}(\psi^N) - (V_t(\varphi^N) - V_{\tau_1^n}(\varphi^N))| \right. \\ & \quad \times \mathbb{1}_{\{C_t(\varphi) \leq K\}} > 8\varepsilon, \\ & \left. \tau_1^n < \infty, X_{\tau_1^n-} > 0 \right] \leq 8\varepsilon \quad \text{for all } n = 1, \dots, N, K \leq N, \end{aligned} \quad (5.12)$$

regardless of how ψ^N is defined outside of A^n , especially at time τ_1^n . Indeed, in the worst case, there are 2 error terms on $(\tau_1^n, \tau_1^{n,N}]$, 3 error terms on $(\tau_1^{n,N}, \tau_2^{n,N}]$ and 3 error terms between $(\tau_2^{n,N}, \tau_2^n) \cup [(\tau_2^n)_{\{X_{\tau_2^n-} > 0\}}]$.

We proceed with the construction of the almost simple strategy on $B^n \cup \tilde{B}^n$ with $n \leq N$. A strategy with support B^n has zero costs, and by Note 5.3, we find an (almost) simple strategy $\hat{\psi}^N$ with

$$\mu(|\hat{\psi}^N - \varphi^N| \mathbb{1}_{B^n} > \varepsilon) \leq \varepsilon, \quad (5.13)$$

$$\mathbb{P}[\Lambda(\sigma_1^n) < \infty, |\hat{\psi}_{\Lambda(\sigma_1^n)}^N - \varphi_{\Lambda(\sigma_1^n)}^N| X_{\Lambda(\sigma_1^n)} > \varepsilon] \leq \varepsilon, \quad (5.14)$$

and $\mathbb{P}[\mathbb{P}(((\hat{\psi}^N - \varphi^N) \mathbb{1}_{B^n}) \bullet S)^* > \varepsilon] \leq \varepsilon$. After $\Lambda(\sigma_1^n)$, we proceed similarly to (5.11). Setting $\tilde{\tau}_2^n := \Gamma(\Lambda(\sigma_1^n))$, there exists a stopping time $\tilde{\tau}_1^{n,N}$ with

$$\tilde{\tau}_1^{n,N} = \theta^N \quad \text{on } \{\theta^N \leq \Lambda(\sigma_1^n)\},$$

$\tilde{\tau}_1^{n,N} = \Lambda(\sigma_1^n)$ on $\{\Lambda(\sigma_1^n) < \theta^N, X_{\Lambda(\sigma_1^n)} > 0\}$ and $\theta^N \wedge \tilde{\tau}_2^n \geq \tilde{\tau}_1^{n,N} > \Lambda(\sigma_1^n)$ on the set $\{\Lambda(\sigma_1^n) < \theta^N, X_{\Lambda(\sigma_1^n)} = 0\}$ such that $\mathbb{P}[\Lambda(\sigma_1^n) \wedge \theta^N \leq \tilde{\tau}_1^{n,N} \leq \Lambda(\sigma_1^n) + \varepsilon] = 1$, $\mathbb{P}[\mathbb{P}(((\varphi^N - \varphi_{\Lambda(\sigma_1^n)}^N) \mathbb{1}_{\llbracket \Lambda(\sigma_1^n), \tilde{\tau}_1^{n,N} \rrbracket}) \bullet S)^* > \varepsilon] \leq \varepsilon$ and

$$\mathbb{P}[\Lambda(\sigma_1^n) < \infty, |X_{\tilde{\tau}_1^{n,N}} - X_{\Lambda(\sigma_1^n) \wedge \theta^N}| > \varepsilon] \leq \varepsilon,$$

$$\mathbb{P}[\Lambda(\sigma_1^n) < \infty, C_{\tilde{\tau}_1^{n,N}}(\varphi^N) - C_{\Lambda(\sigma_1^n) \wedge \theta^N}(\varphi^N) > \varepsilon] \leq \varepsilon.$$

The stopping time $\tilde{\tau}_2^{n,N}$ is defined completely analogously to $\tau_2^{n,N}$ from above. We set

$$\psi_t^N := \hat{\psi}_t^N \mathbb{1}_{\{t \leq \Lambda(\sigma_1^n) \wedge \theta^N\}} + \overline{\psi}_t^N \mathbb{1}_{\{\tilde{\tau}_1^{n,N} < t \leq \tilde{\tau}_2^{n,N}\}} \quad \text{on } B^n \cup \tilde{B}^n \quad (5.15)$$

for some almost simple strategy $\bar{\psi}^N$ with $\bar{\psi}_{\tau_1^{n,N}}^N = \varphi_{\tau_1^{n,N}}^N$ and

$$\sup_{t \in [\tau_1^{n,N}, \tau_2^{n,N}]} |\bar{\psi}_t^N - \varphi_t^N| \leq \varepsilon.$$

As in (5.12), but with the additional error terms on B^n and (5.14) for the case that the spread jumps away from zero, we get that

$$\begin{aligned} & \mathbb{P} \left[\sup \left\{ |V_t(\psi^N) - V^1 - (V_t(\varphi^N) - V^2)| \right. \right. \\ & \quad \times \mathbb{1}_{\{C_t(\varphi) \leq K\}} : t \in [(\sigma_1^n)_{\{X_{\sigma_1^n-}=0\}} \\ & \quad \cup (\sigma_1^n, \Gamma(\Lambda(\sigma_1^n))) \\ & \quad \left. \left. \cup \left[(\Gamma(\Lambda(\sigma_1^n)))_{\{X_{\Gamma(\Lambda(\sigma_1^n))-}>0\}} \right] \right\} > 10\varepsilon \right] \\ & \leq 10\varepsilon \quad \text{for all } n = 1, \dots, N, K \leq N, \end{aligned} \quad (5.16)$$

where we define $V^1 := V_{\sigma_1^n-}(\psi^N)$, $V^2 := V_{\sigma_1^n-}(\varphi^N)$ on the set $\{X_{\sigma_1^n-} = 0\}$ and $V^1 := V_{\sigma_1^n}(\psi^N)$, $V^2 := V_{\sigma_1^n}(\varphi^N)$ on $\{X_{\sigma_1^n-} > 0\}$. By (5.4), A^n and $B^n \cup \tilde{B}^n$ are disjoint. Thus (5.11) and (5.15) can be used to define an almost simple strategy on $\Omega \times [0, T]$:

For $n \leq N$, define ψ^N on $\bigcup_{n \leq N} (A^n \cup B^n \cup \tilde{B}^n)$ as above and set $\psi^N := 0$ on $(\Omega \times [0, T]) \setminus \bigcup_{n \leq N} (A^n \cup B^n \cup \tilde{B}^n)$. Due to $V_0(\psi^N) = V_0(\varphi^N) = 0$ and the construction of A^n and $B^n \cup \tilde{B}^n$, for each (ω, t) , $(V_t(\psi_t^N)(\omega) - V_t(\varphi^N)(\omega)) \mathbb{1}_{\{C_t(\varphi) \leq K\}}(\omega)$ can be written as a finite sum of increments from (5.12) and (5.16). For this, we again use that at the right endpoint of A^n and \tilde{B}^n , the position can be liquidated without any costs. Summing up the error terms and recalling that $\varepsilon = 2^{-N}$, this yields $\mathbb{P}[\sup_{t \in [0, T]} |V_t(\psi^N) - V_t(\varphi^N)| \mathbb{1}_{\{C_t(\varphi) \leq K\}} > 18N2^{-N}] \leq 18N2^{-N}$ for all $N \geq K$. Together with (5.7), we obtain $\sup_{t \in [0, T]} |V_t(\psi^N) - V_t(\varphi)| \mathbb{1}_{\{C_t(\varphi) \leq K\}} \rightarrow 0$ in probability for $N \rightarrow \infty$ and all $K \in \mathbb{N}$.

The sequence $(\psi^N)_{N \in \mathbb{N}}$ converges to φ μ -a.e. on $\{X_- = 0\} \cap \{C(\varphi) < \infty\}$ by (5.13) and (5.15). It remains to show that $(\psi^N)_{N \in \mathbb{N}}$ converges pointwise up to evanescence to φ on the set $\{X_- > 0\} \cap \{C(\varphi) < \infty\}$. Fix $(\omega, t) \in \Omega \times [0, T]$ with $X_{t-}(\omega) > 0$ and $C_t(\varphi)(\omega) < \infty$. There exists an $n \in \mathbb{N}$ with $(\omega, t) \in A^n \cup \tilde{B}^n$ by the arguments in Step 1. Without loss of generality, we can assume $(\omega, t) \in A^n$. One has $\tau_1^{n,N}(\omega) \leq \tau_1^n(\omega) + 2^{-N} < t$ for N large enough by (5.8), and as the costs at t are finite, $\theta^N(\omega) \geq t$ for N large enough.

Case 1: $t < \tau_2^n(\omega)$. By (5.9) and the Borel–Cantelli lemma, we have $\mathbb{P}[E^n] = 0$, where $E^n := \bigcap_{\tilde{N} \in \mathbb{N}} \bigcup_{N \geq \tilde{N}} \{\tau_2^{n,N} < \tau_2^n - 2^{-N}\}$. If $\omega \notin E^n$, this implies that $t < \tau_2^n(\omega) - 2^{-N} \leq \tau_2^{n,N}(\omega)$ for N large enough and thus $|\psi_t^N(\omega) - \varphi_t^N(\omega)| \leq 2^{-N}$ for N large enough by (5.10).

Case 2: $t = \tau_2^n(\omega)$ and thus $X_{\tau_2^n(\omega)-} > 0$. By (5.9) and the Borel–Cantelli lemma, we have $\mathbb{P}[\tilde{E}^n] = 0$, where $\tilde{E}^n := \bigcap_{\tilde{N} \in \mathbb{N}} \bigcup_{N \geq \tilde{N}} \{X_{\tau_2^n-} > 0, \tau_2^{n,N} < \tau_2^n\}$. If $\omega \notin \tilde{E}^n$, this implies that $t = \tau_2^{n,N}(\omega)$ for N large enough. Thus by (5.10) and for N large enough, we have $|\psi_t^N(\omega) - \varphi_t^N(\omega)| \leq 2^{-N}$.

Since $\varphi_t^N(\omega) = \varphi_t(\omega)$ for all $N \geq n$, we conclude that the sequence $(\psi^N)_{N \in \mathbb{N}}$ converges pointwise up to evanescence to φ on the set $\{X_- > 0\} \cap \{C(\varphi) < \infty\}$. \square

Appendix A: Construction of the cost term: technical proofs

Proof of Propositions 3.3 and 3.10 As the two propositions are interrelated, we give their proofs together. Recall that the arguments below are path by path, i.e., $\omega \in \Omega$ is fixed. The uniqueness of the cost term in Definition 3.2 is straightforward, and its proof is omitted. We turn towards existence.

Step 1: Let us show that there exists a sequence of partitions satisfying the assumptions (i) and (ii) of Proposition 3.10. For this, let $(\delta_n)_{n \in \mathbb{N}}, (\eta_n)_{n \in \mathbb{N}} \subseteq (0, \infty)$ be sequences with $\delta_n \downarrow 0$ and $\eta_n \downarrow 0$. It follows from a minor adjustment of Mikosch and Norvaiša [35, Lemma 2.1] that for each $n \in \mathbb{N}$, there is a partition $P_n = \{t_0^n, \dots, t_{k_n}^n\}$ of the considered interval I such that

$$\text{osc}(\overline{S} - S, [t_{i-1}^n, t_i^n]) < \delta_n \quad \text{and} \quad \text{osc}(S - \underline{S}, [t_{i-1}^n, t_i^n]) < \delta_n \quad (\text{A.1})$$

for $i = 1, \dots, k_n$. By the definition of the oscillation of a function, (A.1) also holds for every refinement of P_n . Hence P_n can be chosen such that we also have

$$\begin{cases} \sum_{i=1}^{k_n} |\varphi_{t_i^n} - \varphi_{t_{i-1}^n}| + \eta_n \geq \text{Var}_a^b(\varphi), & \text{if } \text{Var}_a^b(\varphi) < \infty, \\ \sum_{i=1}^{k_n} |\varphi_{t_i^n} - \varphi_{t_{i-1}^n}| > 1/\eta_n, & \text{if } \text{Var}_a^b(\varphi) = \infty, \end{cases} \quad \text{for all } n \in \mathbb{N}. \quad (\text{A.2})$$

In addition, we can obviously choose the sequence $(P_n)_{n \in \mathbb{N}}$ such that it is refining. This shows that there exists a refining sequence of partitions satisfying assertions (i) and (ii) of Proposition 3.10.

Step 2: Next, let $(P_n)_{n \in \mathbb{N}}$ be a refining sequence of partitions from Step 1, i.e., $P_n = \{t_0^n, \dots, t_{k_n}^n\}$ satisfies (A.1) and (A.2).

Case 1: Let us first assume $\text{Var}_a^b(\varphi) < \infty$. Let $M := \sup_{t \in I} (\overline{S}_t - \underline{S}_t)$. We claim that for all subdivisions $\lambda = \{s_1, \dots, s_{k_n}\}$ of P_n , all refinements $P' = \{t'_0, \dots, t'_m\}$ of P_n and all subdivisions $\lambda' = \{s'_1, \dots, s'_m\}$ of P' , we have

$$|R(\varphi, P', \lambda') - R(\varphi, P_n, \lambda)| \leq \eta_n M + \delta_n \text{Var}_a^b(\varphi). \quad (\text{A.3})$$

The key estimate to derive (A.3) is

$$\begin{aligned}
& \left| (\bar{S}_{S_i} - S_{S_i})(\varphi_{t_i^n} - \varphi_{t_{i-1}^n})^+ - \sum_{k=1}^{n_i} (\bar{S}_{S'_{i_k}} - S_{S'_{i_k}})(\varphi_{t'_{i_k}} - \varphi_{t'_{i_{k-1}}})^+ \right| \\
& \leq \left| (\bar{S}_{S_i} - S_{S_i}) \left((\varphi_{t_i^n} - \varphi_{t_{i-1}^n})^+ - \sum_{k=1}^{n_i} (\varphi_{t'_{i_k}} - \varphi_{t'_{i_{k-1}}})^+ \right) \right| \\
& \quad + \left| \sum_{k=1}^{n_i} ((\bar{S}_{S'_{i_k}} - S_{S'_{i_k}}) - (\bar{S}_{S_i} - S_{S_i}))(\varphi_{t'_{i_k}} - \varphi_{t'_{i_{k-1}}})^+ \right| \\
& \leq M \left(\sum_{k=1}^{n_i} (\varphi_{t'_{i_k}} - \varphi_{t'_{i_{k-1}}})^+ - (\varphi_{t_i^n} - \varphi_{t_{i-1}^n})^+ \right) + \delta_n \sum_{k=1}^{n_i} (\varphi_{t'_{i_k}} - \varphi_{t'_{i_{k-1}}})^+,
\end{aligned}$$

where $i \in \{1, \dots, k_n\}$ and the points $t'_{i_1}, \dots, t'_{i_{n_i}}$ denote the elements of the partition P' with $t'_{i-1} = t'_{i_1} < \dots < t'_{i_{n_i}} = t_i^n$.

Now let $(\lambda_n)_{n \in \mathbb{N}}$ be arbitrary modified intermediate subdivisions of $(P_n)_{n \in \mathbb{N}}$. Then, as the sequence $(P_n)_{n \in \mathbb{N}}$ is refining, (A.3) yields

$$\sup_{m \geq n} |R(\varphi, P_m, \lambda_m) - R(\varphi, P_n, \lambda_n)| \leq \eta_n M + \delta_n \text{Var}_a^b(\varphi).$$

Thus the sequence $(R(\varphi, P_n, \lambda_n))_{n \in \mathbb{N}}$ is Cauchy in \mathbb{R}_+ and $C := \lim_{n \rightarrow \infty} R(\varphi, P_n, \lambda_n)$ exists. It remains to show that C satisfies Definition 3.2 (i). Therefore, let $\varepsilon > 0$ and choose $n \in \mathbb{N}$ such that $\eta_n M + \delta_n \text{Var}_a^b(\varphi) < \varepsilon/2$ and $|C - R(\varphi, P_n, \lambda_n)| < \varepsilon/2$. Together with (A.3), this implies that for all refinements P' of P_n and subdivisions λ' of P' , we have

$$|C - R(\varphi, P', \lambda')| \leq |C - R(\varphi, P_n, \lambda_n)| + |R(\varphi, P_n, \lambda_n) - R(\varphi, P', \lambda')| < \varepsilon.$$

Thus C satisfies Definition 3.2 (i).

Case 2: We now treat the case $\text{Var}_a^b(\varphi) = \infty$. We have to show that the cost term exists and $C(\varphi, I) = \infty$. Recall that we assume $\delta := \inf_{t \in [a, b]} (\bar{S}_t - \underline{S}_t) > 0$. We define a sequence $(\sigma_k)_{k \geq 0}$ by $\sigma_0 = a$ and

$$\sigma_k := \begin{cases} \inf\{t \geq \sigma_{k-1} : S_t \leq \underline{S}_t + \delta/3\} \wedge b, & k \text{ odd,} \\ \inf\{t \geq \sigma_{k-1} : S_t \geq \bar{S}_t - \delta/3\} \wedge b, & k \text{ even.} \end{cases}$$

As \underline{S} , S and \bar{S} are càdlàg, we have $\sigma_k = b$ for k large enough. Hence, let $K \in \mathbb{N}$ denote the smallest number such that $\sigma_K = b$. In addition, note that we also have $\sigma_0 \leq \sigma_1 < \sigma_2 < \dots < \sigma_K = b$, and by construction,

$$\inf_{t \in [\sigma_{2k}, \sigma_{2k+1})} (S_t - \underline{S}_t) \geq \delta/3 \quad \text{and} \quad \inf_{t \in [\sigma_{2k+1}, \sigma_{2(k+1)})} (\bar{S}_t - S_t) \geq \delta/3, \quad k \in \mathbb{N}_0. \quad (\text{A.4})$$

Recall that $\text{Var}_a^b(\varphi) = \infty$ implies that $\sum_{i=1}^{k_n} |\varphi_{t_i^n} - \varphi_{t_{i-1}^n}| \rightarrow \infty$ as $n \rightarrow \infty$ by (A.2). Since $K < \infty$ and φ is bounded, this implies that for at least one $k \in \{0, 1, \dots, K-1\}$,

we have

$$\sum_{\substack{t_{i-1}^n, t_i^n \in P_n \\ t_{i-1}^n, t_i^n \in [\sigma_k, \sigma_{k+1}]}} |\varphi_{t_i^n} - \varphi_{t_{i-1}^n}| \longrightarrow \infty \quad \text{as } n \rightarrow \infty,$$

which again by the boundedness of φ implies that

$$\begin{aligned} \sum_{\substack{t_{i-1}^n, t_i^n \in P_n \\ t_{i-1}^n, t_i^n \in [\sigma_k, \sigma_{k+1}]}} (\varphi_{t_i^n} - \varphi_{t_{i-1}^n})^+ &\longrightarrow \infty \quad \text{as } n \rightarrow \infty, \\ \sum_{\substack{t_{i-1}^n, t_i^n \in P_n \\ t_{i-1}^n, t_i^n \in [\sigma_k, \sigma_{k+1}]}} (\varphi_{t_i^n} - \varphi_{t_{i-1}^n})^- &\longrightarrow \infty \quad \text{as } n \rightarrow \infty. \end{aligned} \quad (\text{A.5})$$

By (A.4), this implies that $R(\varphi, P_n, \lambda_n) \rightarrow \infty$ as $n \rightarrow \infty$ for arbitrary subdivisions λ_n of P_n . Since the sums in (A.5) get even bigger if the P_n are replaced by refining partitions, the cost term $C(\varphi, I)$ exists and is infinity. \square

We now turn to the proof of Lemma 3.13. It relies on the following concept and result of Doob [15].

Definition A.1 Let φ be a stochastic process. A sequence $(T_n)_{n \in \mathbb{N}}$ of predictable stopping times is called a *predictable separability set* for φ if for each $\omega \in \Omega$, the set $\{T_n(\omega) : n \in \mathbb{N}\}$ contains 0 and is dense in $[0, T]$ and

$$\{(t, \varphi_t(\omega)) : t \in [0, T]\} = \overline{\{(T_n(\omega), \varphi_{T_n(\omega)}(\omega)) : n \in \mathbb{N}\}}, \quad (\text{A.6})$$

i.e., the graph of the sample function $t \mapsto \varphi_t(\omega)$ is the closure of the graph restricted to the set $\{T_n(\omega) : n \in \mathbb{N}\}$. A stochastic process φ having a predictable separability set is called *predictably separable*.

We recall Doob [15, Theorem 5.2]:

Theorem A.2 A predictable process coincides with some predictably separable predictable process up to evanescence.

Proof of Lemma 3.13 By Theorem A.2, we have to show that for a predictably separable predictable process φ , the process $C(\varphi, [\sigma \wedge \cdot, \tau \wedge \cdot])$ is predictable.

By (A.6), we can find a sequence of finite sequences of (not necessarily predictable) stopping times $\sigma = T_0^n \leq T_1^n \leq \dots \leq T_{m_n}^n = \tau$ such that

$$\sum_{i=1}^{m_n} |\varphi_{T_i^n \wedge t} - \varphi_{T_{i-1}^n \wedge t}| \longrightarrow \text{Var}_{\sigma \wedge t}^{\tau \wedge t}(\varphi) \quad \text{pointwise for } n \rightarrow \infty, t \in [0, T].$$

Next, we define for each $n \in \mathbb{N}$ and $i \in \{1, \dots, m_n\}$ a sequence $(V_\ell^{n,i})_{\ell \in \mathbb{N}_0}$ of stopping times by $V_0^{n,i} := T_{i-1}^n$ and recursively

$$V_\ell^{n,i} := \inf \left\{ t > V_{\ell-1}^{n,i} : |\bar{S}_t - S_t - (\bar{S}_{V_{\ell-1}^{n,i}} - S_{V_{\ell-1}^{n,i}})| > \frac{1}{2n} \text{ or } \right. \\ \left. |S_t - \underline{S}_t - (S_{V_{\ell-1}^{n,i}} - \underline{S}_{V_{\ell-1}^{n,i}})| > \frac{1}{2n} \right\} \wedge T_i^n.$$

This leads to the sequence of random partitions $\bar{P}_n := \bigcup_{k \leq n} \bigcup_{i=1, \dots, m_k} \bigcup_{\ell \in \mathbb{N}_0} \{V_\ell^{k,i}\}$, $n \in \mathbb{N}$, which is for each ω refining. Note that for ω and n fixed, P_n is finite. Rearranging the resulting stopping times in increasing order yields a refining sequence of increasing stopping times $(v_k^n)_{k \in \mathbb{N}_0}$, $n \in \mathbb{N}$, such that $\#\{k : v_k^n(\omega) < \infty\} < \infty$ for all $n \in \mathbb{N}$, $\text{Var}_{\sigma \wedge t}^{\tau \wedge t}(\varphi) = \lim_{n \rightarrow \infty} \sum_{k=1}^\infty |\varphi_{v_k^n \wedge t} - \varphi_{v_{k-1}^n \wedge t}|$ for all $t \in [0, T]$, and such that $\max(\text{osc}(\bar{S} - S, [v_k^n, v_{k+1}^n]), \text{osc}(S - \underline{S}, [v_k^n, v_{k+1}^n])) \leq 1/n$ for all $k \in \mathbb{N}_0$ and $n \in \mathbb{N}$. In particular, this means that for each $\omega \in \{\sigma < \tau\}$ and $t \in [0, T]$, the sequence of partitions $(P_n(\omega))_{n \in \mathbb{N}}$ defined by $P_n(\omega) := \{v_k^n(\omega) \wedge t : k \in \mathbb{N}\}$ satisfies the assumptions of Proposition 3.10. Hence Proposition 3.10 together with $C(\varphi, [\sigma \wedge \cdot, \tau \wedge \cdot]) = 0$ on $\{\sigma = \tau\}$ implies that the sequence of predictable processes

$$\sum_{k=1}^\infty (\bar{S}_{v_{k-1}^n} - S_{v_{k-1}^n})(\varphi_{v_k^n \wedge \cdot} - \varphi_{v_{k-1}^n \wedge \cdot})^+ + \sum_{k=1}^\infty (S_{v_{k-1}^n} - \underline{S}_{v_{k-1}^n})(\varphi_{v_k^n \wedge \cdot} - \varphi_{v_{k-1}^n \wedge \cdot})^-,$$

$n \in \mathbb{N}$, converges pointwise to $C(\varphi, [\sigma \wedge \cdot, \tau \wedge \cdot])$, which yields the assertion. \square

Proof of Proposition 3.17 In the following, we can and do assume with no loss of generality that σ and τ are $[0, T]$ -valued stopping times. In addition, by Proposition 3.3, we have $\text{Var}_\sigma^\tau(\varphi) < \infty$ a.s. and thus without loss of generality also for all paths. This implies that the paths of φ are $\text{l\grave{a}g\text{l}\grave{a}d}$ on $[\![\sigma, \tau]\!]$.

Step 1. We start by constructing an approximating sequence of almost simple strategies $(\varphi^n)_{n \in \mathbb{N}}$ as in Guasoni et al. [20]. For this, we define

$$T_0^n := \sigma, \quad T_k^n := \inf\{t \in (T_{k-1}^n, \tau] : |\varphi_t - \varphi_{T_{k-1}^n}| \geq 1/n\}, \quad k \in \mathbb{N},$$

which are obviously stopping times. In addition, we have $T_{k-1}^n < T_k^n$ on $\{T_{k-1}^n < \infty\}$ and $\#\{k : T_k^n(\omega) \leq \tau(\omega)\} < \infty$ for all $\omega \in \Omega$ as $\text{Var}_\sigma^\tau(\varphi) < \infty$. One has to distinguish between a portfolio adjustment at T_k^n and at T_k^n+ . For this, we introduce further stopping times, namely

$$\pi_0^n := \sigma, \quad \pi_k^n := (T_k^n)_{\{|\varphi_{T_k^n} - \varphi_{T_{k-1}^n}| \geq 1/n\}}, \quad k \in \mathbb{N}$$

and note that π_k^n is a predictable stopping time for all $k \in \mathbb{N}$. Indeed, for $k \geq 1$ we have

$$\llbracket \pi_k^n \rrbracket = \llbracket 0, T_k^n \rrbracket \cap \{(\omega, t) : Y_t(\omega) \geq 1/n\} \in \mathcal{P}$$

since the process $Y_t := |\varphi_t - \varphi_{T_{k-1}^n + \lfloor \mathbb{1} \rfloor T_{k-1}^n, \tau}|$ is predictable. We define $(\varphi^n)_{n \in \mathbb{N}}$ by

$$\varphi^n := \sum_{k=0}^{\infty} (\varphi_{\pi_k^n} \mathbb{1}_{\llbracket \pi_k^n \rrbracket} + \varphi_{T_k^n + \mathbb{1}_{\llbracket T_k^n, T_{k+1}^n \rrbracket \setminus \llbracket \pi_{k+1}^n \rrbracket}}),$$

which satisfies $\varphi_\sigma^n = \varphi_\sigma$ and $\varphi^n \mathbb{1}_{\llbracket \sigma, \tau \rrbracket}$ is predictable and consequently almost simple. In addition, the construction ensures that $|\varphi^n - \varphi| \leq 1/n$ on $\llbracket \sigma, \tau \rrbracket$.

Step 2: Let us show that $\sup_{t \in [\sigma, \tau]} |\text{Var}_\sigma^t(\varphi^n) - \text{Var}_\sigma^t(\varphi)| \rightarrow 0$ pointwise. We fix $\omega \in \Omega$ and $\varepsilon > 0$ and take a partition $P = \{t_0, \dots, t_m\}$ such that

$$\text{Var}_\sigma^\tau(\varphi)(\omega) \leq \sum_{i=1}^m |\varphi_{t_i}(\omega) - \varphi_{t_{i-1}}(\omega)| + \varepsilon.$$

This yields

$$\text{Var}_\sigma^t(\varphi)(\omega) \leq \sum_{i=1}^m |\varphi_{t_i \wedge t}(\omega) - \varphi_{t_{i-1} \wedge t}(\omega)| + \varepsilon, \quad \forall t \in [\sigma(\omega), \tau(\omega)].$$

Recall from Step 1 that $\varphi^n(\omega) \rightarrow \varphi(\omega)$ uniformly in time on $[\sigma(\omega), \tau(\omega)]$. Thus we may choose $N \in \mathbb{N}$ large enough such that we have $|\varphi_t^n(\omega) - \varphi_t(\omega)| \leq \varepsilon/(2m)$ for all $t \in [\sigma(\omega), \tau(\omega)]$ and $n \geq N$. Therefore, we get

$$\begin{aligned} \text{Var}_\sigma^t(\varphi)(\omega) - \text{Var}_\sigma^t(\varphi^n)(\omega) &\leq \sum_{i=1}^m |\varphi_{t_i \wedge t}(\omega) - \varphi_{t_{i-1} \wedge t}(\omega)| + \varepsilon - \text{Var}_\sigma^t(\varphi^n)(\omega) \\ &\leq \sum_{i=1}^m |\varphi_{t_i \wedge t}^n(\omega) - \varphi_{t_{i-1} \wedge t}^n(\omega)| + 2\varepsilon - \text{Var}_\sigma^t(\varphi^n)(\omega) \\ &\leq 2\varepsilon \end{aligned}$$

for all $t \in [\sigma(\omega), \tau(\omega)]$. Together with $\text{Var}_\sigma^t(\varphi)(\omega) \geq \text{Var}_\sigma^t(\varphi^n)(\omega)$, we have proved the claim.

Step 3: Let us show that (3.8) holds. We again argue path by path, i.e., $\omega \in \Omega$ is fixed without explicitly mentioning it. The jumps of the cost term on $[\sigma, \tau]$ are given by

$$\begin{aligned} \Delta C_t(\varphi) &= \lim_{s \uparrow t} C(\varphi, [s, t]) \\ &= (\bar{S}_{t-} - S_{t-})(\Delta \varphi_t)^+ + (S_{t-} - \underline{S}_{t-})(\Delta \varphi_t)^-, \quad t \in (\sigma, \tau], \\ \Delta^+ C_t(\varphi) &= \lim_{s \downarrow t} C(\varphi, [t, s]) \\ &= (\bar{S}_t - S_t)(\Delta^+ \varphi_t)^+ + (S_t - \underline{S}_t)(\Delta^+ \varphi_t)^-, \quad t \in [\sigma, \tau). \end{aligned}$$

For $k \in \mathbb{N}$, we use the notation $C(\varphi, (T_{k-1}^n, T_k^n)) := C(\varphi, [T_{k-1}^n, T_k^n]) - \Delta^+ C_{T_{k-1}^n}(\varphi)$ and $C(\varphi, (T_{k-1}^n, T_k^n)) := C(\varphi, (T_{k-1}^n, T_k^n]) - \Delta C_{T_k^n}(\varphi)$, where it is tacitly assumed

that $T_k^n \leq \tau$. In particular, this means that for φ^n , we have

$$C(\varphi^n, (T_{k-1}^n, T_k^n]) = (\bar{S}_{T_k^n-} - S_{T_k^n-})(\varphi_{T_k^n}^n - \varphi_{T_k^n-}^n)^+ + (S_{T_k^n-} - \underline{S}_{T_k^n-})(\varphi_{T_k^n}^n - \varphi_{T_k^n-}^n)^-$$

as $C(\varphi^n, (T_{k-1}^n, T_k^n)) = 0$ according to Proposition 3.16. We now want to get an estimate for

$$\left| C(\varphi^n, (T_{k-1}^n, T_k^n]) + \Delta^+ C_{T_k^n}(\varphi^n) - \left(C(\varphi, (T_{k-1}^n, T_k^n]) + \Delta^+ C_{T_k^n}(\varphi) \right) \right| \quad (\text{A.7})$$

(this means that we move forward from T_{k-1}^n+ to T_k^n+ and tacitly assume $T_k^n < \tau$).

We start by establishing a strong bound on the difference (A.7) that only holds if the prices do not vary too much between T_{k-1}^n and T_k^n . To formalise this, we take $\delta > 0$, which will be specified later, and define $(\rho_m)_{m \geq 0}$ by $\rho_0 := \sigma$ and

$$\rho_m := \inf\{t \in (\rho_{m-1}, \tau] : |\bar{S}_t - S_t - (\bar{S}_{\rho_{m-1}} - S_{\rho_{m-1}})| > \delta \text{ or} \\ |S_t - \underline{S}_t - (S_{\rho_{m-1}} - \underline{S}_{\rho_{m-1}})| > \delta\}.$$

We claim that on $\{\rho_{m-1} \leq T_{k-1}^n < T_k^n < \rho_m\}$ for some $m \geq 1$, we have for $k \geq 1$ that

$$\left| C(\varphi^n, (T_{k-1}^n, T_k^n]) + \Delta^+ C_{T_k^n}(\varphi^n) - \left(C(\varphi, (T_{k-1}^n, T_k^n]) + \Delta^+ C_{T_k^n}(\varphi) \right) \right| \\ \leq (\text{Var}_{T_{k-1}^n+}^{T_k^n+}(\varphi) - \text{Var}_{T_{k-1}^n+}^{T_k^n+}(\varphi^n)) \sup_{t \in [0, T]} (\bar{S}_t - \underline{S}_t) + 2\delta \text{Var}_{T_{k-1}^n+}^{T_k^n+}(\varphi). \quad (\text{A.8})$$

Indeed, in a related model with artificial times for T_{k-1}^n+ and T_k^n+ (that we do not write down), the cost term of φ^n is a modified Riemann–Stieltjes sum from Definition 3.1(v), and the cost term of φ is the limit in \mathbb{R} of such sums with refined partitions. Since the oscillation of the processes $\bar{S} - S$ and $S - \underline{S}$ on $[\rho_{m-1}, \rho_m]$ is bounded by 2δ , (A.8) is just an application of (A.3) with $\delta_n = 2\delta$.

We still need a bound for (A.7) for the general case when prices can vary by more than 2δ between T_{k-1}^n and T_k^n . Fortunately, a weaker bound is sufficient here, namely

$$\left| C(\varphi^n, (T_{k-1}^n, T_k^n]) + \Delta^+ C_{T_k^n}(\varphi^n) - \left(C(\varphi, (T_{k-1}^n, T_k^n]) + \Delta^+ C_{T_k^n}(\varphi) \right) \right| \\ \leq \left(\text{Var}_{T_{k-1}^n+}^{T_k^n+}(\varphi) - \text{Var}_{T_{k-1}^n+}^{T_k^n+}(\varphi^n) + \frac{1}{n} \right) \sup_{t \in [0, T]} (\bar{S}_t - \underline{S}_t), \quad k \geq 1. \quad (\text{A.9})$$

It is sufficient to show (A.9) for the very special case that $\varphi_{T_k^n}^n = \varphi_{T_{k-1}^n+}^n + 1/n$ and $\varphi_{T_k^n+}^n = \varphi_{T_k^n}^n$. In this case, one has $\text{Var}_{T_{k-1}^n+}^{T_k^n+}(\varphi^n) = 1/n$ and the estimation is obvious (cf. Proposition 3.9 (iii)). If (A.9) holds for this case, it also holds in general since by the construction of φ^n , overshoots of $\varphi_{T_{k-1}^n+}^n + 1/n$ and jumps between T_{k-1}^n and T_k^n+ affect both the two cost processes and the two variation processes in the same way. In addition, by (A.9) and $\text{Var}_{T_{k-1}^n+}^{T_k^n-}(\varphi^n) = 0$, we obtain the estimate, for $t \in (T_{k-1}^n, T_k^n]$,

$$\begin{aligned}
& |C(\varphi^n, (T_{k-1}^n, t]) - C(\varphi, (T_{k-1}^n, t])| \\
& \leq \left(\text{Var}_{T_{k-1}^n+}^t(\varphi) - \text{Var}_{T_{k-1}^n+}^t(\varphi^n) + \frac{1}{n} \right) \sup_{u \in [0, T]} (\bar{S}_u - \underline{S}_u). \quad (\text{A.10})
\end{aligned}$$

Step 4: We complete the proof by putting the different estimates together. Let $a(\delta) := \#\{m : \rho_m \leq \tau\}$ and note that $a(\delta) < \infty$ (recall that $\omega \in \Omega$ is fixed). Next, note that we have $\Delta^+ C_\sigma(\varphi^n) = \Delta^+ C_\sigma(\varphi)$ by construction of φ^n . For $t \in [\sigma, \tau]$, let $K_n := \#\{k : T_k^n < t\}$. By Proposition 3.9 (ii), we have

$$\begin{aligned}
|C(\varphi^n, [\sigma, t]) - C(\varphi, [\sigma, t])| & \leq \sum_{k=1}^{K_n} |C(\varphi^n, (T_{k-1}^n, T_k^n]) - C(\varphi, (T_{k-1}^n, T_k^n])| \\
& \quad + (\Delta^+ C_{T_k^n}(\varphi^n) - \Delta^+ C_{T_k^n}(\varphi)) \mathbb{1}_{\{T_k^n < t\}} \\
& \quad + |C(\varphi^n, (T_{K_n}^n, t]) - C(\varphi, (T_{K_n}^n, t])|. \quad (\text{A.11})
\end{aligned}$$

To the last summand, we apply the estimate (A.10). The estimate (A.9) is applied to all $k = 1, \dots, K_n$ such that there is at least one $m = 1, \dots, a(\delta)$ with $T_{k-1}^n < \rho_m \leq T_k^n$, and for all other k , we use the stronger estimate (A.8). Plugging this into (A.11) and adding up the variation terms along time, we arrive at

$$\begin{aligned}
& |C(\varphi^n, [\sigma, t]) - C(\varphi, [\sigma, t])| \\
& \leq \left(\text{Var}_\sigma^t(\varphi) - \text{Var}_\sigma^t(\varphi^n) + \frac{a(\delta)}{n} \right) \sup_{u \in [0, T]} (\bar{S}_u - \underline{S}_u) + 2\delta \text{Var}_\sigma^t(\varphi) \\
& \leq \left(\sup_{u \in [\sigma, \tau]} (\text{Var}_\sigma^u(\varphi) - \text{Var}_\sigma^u(\varphi^n)) + \frac{a(\delta)}{n} \right) \sup_{u \in [0, T]} (\bar{S}_u - \underline{S}_u) + 2\delta \text{Var}_\sigma^\tau(\varphi) \quad (\text{A.12})
\end{aligned}$$

for all $t \in [\sigma, \tau]$. Given $\varepsilon > 0$, we first choose $\delta < \varepsilon/(4\text{Var}_\sigma^\tau(\varphi))$. Then by $a(\delta) < \infty$ and Step 2, we find $N \in \mathbb{N}$ such that for $n \geq N$, the first summand in the last line of (A.12) is smaller than $\varepsilon/2$. This yields $\sup_{t \in [\sigma, \tau]} |C(\varphi^n, [\sigma, t]) - C(\varphi, [\sigma, t])| < \varepsilon$ for $n \geq N$, and we are done. \square

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