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Comparing External and Internal Instruments for Vector Autoregressions

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Abstract. In conventional proxy VAR analysis, the shocks of interest are identified by external instruments. This is typically accomplished by considering the covariance of the instruments and the reduced-form residuals. Alternatively, the instruments may be internalized by augmenting the VAR process by the instruments or proxies. These alternative identification methods are compared and it is shown that the resulting shocks obtained with the alternative approaches differ in general. Conditions are provided under which their impulse responses are nevertheless identical. If the conditions are satisfied, identification of the shocks is ensured without further assumptions. Empirical examples illustrate the results and the virtue of using the identification conditions derived in this study.

Key Words: Structural vector autoregression, proxy VAR, augmented VAR, fundamental shocks, invertible VAR

JEL classification: C32

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1 Introduction

Using external instruments or proxies to identify structural shocks has become an important tool in structural vector autoregressive (VAR) analysis. Nowadays some authors use several proxies to identify a set of shocks. In that case, the proxies generally identify only linear combinations of the shocks and typically additional assumptions are needed to identify the shocks individually (see, e.g., Mertens and Ravn (2013), Piffer and Podstawski (2017) or Jarociński and Karadi (2020)).

The dominant approaches for estimating the structural parameters and, hence, the shocks in proxy VAR analysis are based on the covariance of the instruments and the reduced-form residuals (see, e.g., Mertens and Ravn (2013)) or on augmenting the VAR model by the proxies and, hence, internalizing them (e.g., Kilian and Lütkepohl (2017, Sec. 15.2), Jarociński and Karadi (2020), Plagborg-Møller and Wolf (2021, 2022)). The distinction between external and internal instruments is also discussed by Stock and Watson (2018).

This study compares the two alternative approaches for using multiple proxies for identifying a set of shocks in structural VAR analysis and makes several contributions to the proxy VAR literature. (1) Conditions are derived under which the impulse response functions are identical in population for the external instruments and the augmented VAR approaches. (2) It is established that the shocks obtained with both approaches are different in population even if the conditions for identical impulse responses are satisfied. (3) It is shown that, if the conditions for identical impulse responses from both approaches are satisfied, the structural shocks are fully identified by the proxies such that additional restrictions for disentangling the shocks are unnecessary.

Specifically, we show that, if the proxies are mutually uncorrelated, each proxy is correlated with exactly one shock only and the proxies are not Granger-causal for the variables of interest, then the structural shocks are fully identified and the shocks can be scaled such that the structural impulse responses obtained from the external instruments and the augmented VAR approaches are identical in population. However, if the external instruments approach is used, the shocks will be linear transformations of the reduced-form residuals. In this setup, the proxies used for identification need not be direct measurements of the shocks of interest, but can contain some measurement error. If instead the augmented VAR approach is used, the resulting shocks will be linear transformations of the proxies, i.e., there is no built-in correction for measurement errors in the proxies.

We use two empirical examples to illustrate our theoretical results. The first one considers a model of the crude oil market based on a study by Känzig (2021). The second example is based on a study by Lunsford (2015) who explores the impact of total factor productivity (TFP) shocks on the U.S. economy. We will show how our theoretical findings can be used to identify the structural shocks even if some assumptions of earlier studies are relaxed.

The remainder of the paper is organized as follows. In the next section we present the model setup, compare the alternative identification and estimation methods formally, and present conditions for individually identified shocks when multiple proxies are used. In Section 3 we study the empirical examples and Section 4 concludes. Proofs are provided in the Appendix.

2 Model Setup and Identification

2.1 The Model

Our point of departure is a K-dimensional reduced-form VAR process,

$$y_t = \nu + A_1 y_{t-1} + \cdots + A_p y_{t-p} + u_t. \tag{1}$$

The error process, u_t, is zero-mean white noise with nonsingular covariance matrix Σ_u. In short, the u_t are serially uncorrelated and $u_t \sim (0, \Sigma_u)$. The structural shocks, denoted as $\mathbf{w}_t = (w_{1t}, \dots, w_{Kt})'$, are assumed to be linear combinations of the u_t, $\mathbf{w}_t = B^{-1}u_t$, such that $u_t = B\mathbf{w}_t$. In the following we will refer to shocks that are linear combinations of the reduced-form residuals u_t as fundamental.[2] The $(K \times K)$ transformation matrix B contains the impact effects of the structural shocks. They are assumed to have a diagonal covariance matrix $\Sigma_{\mathbf{w}}$ such that $B\Sigma_{\mathbf{w}}B' = \Sigma_u$. If the shocks are normalized to have unit variances and, hence, $\Sigma_{\mathbf{w}} = I_K$, the transformation matrix B has to be such that $BB' = \Sigma_u$.

We assume further that the first K_1 shocks, $\mathbf{w}_{1t} = (w_{1t}, \dots, w_{K_1 t})'$, are of primary interest and have to be properly identified as economic shocks, while the last $K - K_1$ shocks, $\mathbf{w}_{2t} = (w_{K_1+1,t}, \dots, w_{Kt})'$, are not of interest. Accordingly, we partition the vector of shocks as $\mathbf{w}_t' = (\mathbf{w}_{1t}', \mathbf{w}_{2t}')$. The matrix of impact effects, B, is partitioned correspondingly as $B = [B_1 : B_2]$, B_1 being a $(K \times K_1)$ matrix and B_2 being of dimensions $(K \times (K - K_1))$.

The matrix B contains the structural parameters of the model. The k-th column of B, say b_k, respresents the impact effects of the k-th shock on all the K variables. Thus, the columns of B_1 contain the impact effects of the shocks of interest, $\mathbf{w}_{1t}$. Having B_1, the latter shocks can be obtained from the reduced-form residuals as[3]

$$\mathbf{w}_{1t} = (B_1'\Sigma_u^{-1}B_1)^{-1}B_1'\Sigma_u^{-1}u_t. \tag{2}$$

The structural impulse responses of the shocks of interest for propagation horizon h are known to be $\Theta_{1,h} = \Phi_h B_1$, where the Φ_h are reduced-form quantities obtained recursively from the $A_1, \dots, A_p$ VAR slope coefficients as $\Phi_h = \sum_{j=1}^{h} \Phi_{h-j}A_j$, with $\Phi_0 = I_K$, for $h = 1, \dots,$ and $A_j = 0$ for $j > p$ (e.g., Lütkepohl (2005, Sec. 2.1.2)).

2.2 Identification via Proxy Variables

Identification of the structural parameters and, hence, the structural shocks is assumed to be based on a set of N instrumental variables (proxies) $z_t = (z_{1t}, \dots, z_{Nt})'$ satisfying

$$\mathbb{E}(\mathbf{w}_{1t}z_t') = \Sigma_{\mathbf{w}_1 z} \neq 0, \quad \Sigma_{\mathbf{w}_1 z}\ (K_1 \times N), \quad \mathrm{rk}(\Sigma_{\mathbf{w}_1 z}) = K_1 \quad \text{(relevance)}, \tag{3}$$

$$\mathbb{E}(\mathbf{w}_{2t}z_t') = 0 \quad \text{(exogeneity)}. \tag{4}$$

These conditions imply that

$$\mathbb{E}(u_t z_t') = B\mathbb{E}(\mathbf{w}_t z_t') = B_1\Sigma_{\mathbf{w}_1 z}. \tag{5}$$

[2] In some of the recent literature, this property is referred to as invertibility (see, e.g., Plagborg-Møller and Wolf (2021)).

[3] The relation follows from the fact that $w_{kt} = b_k'\Sigma_u^{-1}u_t / b_k'\Sigma_u^{-1}b_k$ (see, e.g., Stock and Watson (2018), Bruns and Lütkepohl (2022, Appendix A.1)) and $(B_1'\Sigma_u^{-1}B_1)^{-1} = \Sigma_{\mathbf{w}_1}$.

Obviously, there must be at least as many proxies as there are identified shocks such that $N \geq K_1$, to satisfy the rank condition for $\Sigma_{\mathbf{w}_1 z}$ which ensures that the N proxies contain identifying information for all shocks in $\mathbf{w}_{1t}$. As we can estimate $B_1\Sigma_{\mathbf{w}_1 z}$ by the usual covariance matrix estimator

$$\overline{\hat{u}z} = \frac{1}{T}\sum_{t=1}^{T} \hat{u}_t z_t', \tag{6}$$

where the $\hat{u}_t$ are reduced-form least squares (LS) residuals, the proxies contain identifying information for the first K_1 structural shocks collectively but the shocks are not necessarily individually identified. In the following, we will refer to this approach as the external proxy VAR approach to distinguish it from the approach based on augmenting the VAR model by the proxies to be discussed in Subsections 2.3 and 2.4. The shocks estimated with the external proxy VAR approach via equation (2) will be signified as $\hat{\mathbf{w}}_{1t}$(PVAR).

Note that, if there is just one proxy that identifies a single shock ($K_1 = N = 1$), then an equivalent estimator of B_1 is obtained by including z_t as an additional regressor in the VAR model, i.e., by estimating the model

$$y_t = \nu + A_1 y_{t-1} + \cdots + A_p y_{t-p} + B_1 z_t + u_t^*$$

by LS. The LS estimator $\hat{B}_1$ is a multiple of $\overline{\hat{u}z}$ in this case (see Paul (2020, Online Appendix A.2, A.3)).

From now on we assume that $N = K_1$, i.e., there are as many proxies as there are shocks of interest. In practice, that assumption is not always satisfied (see, e.g., Hou (2024)) but it holds in most empirical studies. In that case, if each proxy is correlated with just one shock such that $\Sigma_{\mathbf{w}_1 z}$ is a diagonal square matrix, the shocks will be identified individually because the right-hand side of (5) will consist of multiples of the impact effects of the shocks that will provide multiples of the shocks via the relation (2). Thus, we can get shocks of the desired size by appropriately scaling the estimated shocks. In that case, we can equivalently estimate the impact effects of the shocks by using the proxies one-by-one. The estimates and shocks obtained in that way are obviously identical to those obtained by estimating the impact effects of all shocks at once as in (6).

One drawback of this approach is that, in practice, it may result in correlated estimated shocks, as shown, for example, by Gregory, McNeil and Smith (2024) and Bruns, Lütkepohl and McNeil (2024). If uncorrelatedness of the shocks is used as an additional restriction, the shocks of interest, $\mathbf{w}_{1t}$, are actually over-identified. Bruns et al. (2024) propose a GMM method that accounts for these additional restrictions. It not only provides uncorrelated shocks but it can also improve the estimation efficiency. In general, if $\Sigma_{\mathbf{w}_1 z}$ is not a diagonal matrix, the proxies identify only linear combinations of the shocks of interest. To identify them individually requires additional identifying assumptions, as mentioned earlier.

2.3 Population Results for VAR Models Augmented by the Proxies

In proxy VAR analysis, some authors augment the VAR model by the proxy variables (see, e.g., Angelini and Fanelli (2019), Jarociński and Karadi (2020), Plagborg-Møller and Wolf (2021)). We consider the augmented reduced-form VAR model

$$\begin{pmatrix} z_t \\ y_t \end{pmatrix} = \begin{pmatrix} \nu^z \\ \nu^y \end{pmatrix} + \begin{bmatrix} 0 & 0 \\ A_1^z & A_1^y \end{bmatrix} \begin{pmatrix} z_{t-1} \\ y_{t-1} \end{pmatrix} + \cdots + \begin{bmatrix} 0 & 0 \\ A_p^z & A_p^y \end{bmatrix} \begin{pmatrix} z_{t-p} \\ y_{t-p} \end{pmatrix} + \begin{pmatrix} u_t^z \\ u_t^y \end{pmatrix}, \quad (7)$$

where no lags appear in the z_t equations to simplify the exposition. This type of setup assumes that the proxy vector, z_t, is serially uncorrelated because there are no lags in the equations associated with the proxies. This assumption appears to be quite realistic for many proxies. Alternatively, one could also include lags of z_t and y_t in the z_t equations in the augmented VAR model (7).

An augmented model such as (7) is often used for Bayesian proxy VAR analysis because it allows to use standard Bayesian VAR methods that place priors on the reduced-form parameters. For example, one could use a Gaussian-inverse-Wishart prior that results in a convenient Gaussian-inverse-Wishart posterior for the reduced-form parameters of the augmented model. Thus, the following discussion is also relevant for Bayesian analysis (see also Caldara and Herbst (2019), Arias, Rubio-Ramírez and Waggoner (2021)).

The residual vector of the augmented VAR model (7) is

$$u_t^{aug} = \begin{bmatrix} u_t^z = z_t - \mathbb{E}(z_t) \\ u_t^y \end{bmatrix} \quad (8)$$

and its covariance matrix will be denoted by Σ_u^{aug}. In large VARs, identification of the shocks is often based on a recursiveness assumption. Hence, the shocks are identified based on a Cholesky decomposition of Σ_u^{aug} (e.g., Bańbura, Giannone and Reichlin (2010)). If that approach is applied here, $B^{aug} = \text{chol}(\Sigma_u^{aug})$ is used to obtain shocks

$$\mathbf{w}_t^{aug} = (B^{aug})^{-1} u_t^{aug} \quad (9)$$

(e.g., Martínez-Hernández (2020)). In that case, the first K_1 shocks are interpreted as the shocks identified by the K_1 proxies.

To understand the relationship of these shocks to the $\mathbf{w}_{1t}$ shocks obtained from the VAR model (1), it may be worth mentioning that the inverse of the lower-triangular B^{aug} is also a lower-triangular matrix and, hence, the first component of $\mathbf{w}_t^{aug}$ is just a multiple of the mean-adjusted z_{1t}, the second component is a linear combination of mean-adjusted z_{1t} and z_{2t}, and, more generally, the k-th component of $\mathbf{w}_t^{aug}$ is a linear combination of the mean-adjusted $z_{1t}, \dots, z_{kt}$. There are no linear combinations of the u_t^y involved in determining the first K_1 shocks of the augmented model. In other words, the shocks of interest from the augmented model are linear combinations of the proxies, while in the external proxy VAR approach the shocks are linear combinations of the reduced-form residuals of the VAR model (1) and are just correlated with the proxies such that the proxies are better thought of as shocks measured with error. In fact, the two sets of shocks can be quite different although they may result in identical impulse response functions, as we will see below. Empirical examples are provided in Section 3.

Given the way many proxies are constructed in practice, it may not be very appealing to view them directly as shocks. As an extreme case, consider for example the sign proxies proposed by Boer and Lütkepohl (2021) which are discrete variables with values $-1, 0$, and 1 only. A number of proxies used in the proxy VAR literature are explicitly constructed to be nonzero only for selected periods where shocks have occurred and set to zero for many other periods where no shock measurements are available (e.g., Gertler and Karadi (2015), Piffer and Podstawski (2017), Boer and Lütkepohl (2021)). It is easy to picture them as correlated with shocks of interest but perhaps less plausible to view them as the shocks of interest themselves. Thus, interpreting the proxies as shocks may not be natural in many situations.

It turns out, however, that, under suitable conditions, the impact effects of the shocks of interest, $\mathbf{w}_{1t}$, of the VAR model (1) are multiples of the impact effects of the first K_1 shocks in $\mathbf{w}_t^{aug}$, obtained from a Cholesky decomposition of Σ_u^{aug}. Thus, the shocks have equal impact effects if the scaling of the shocks is adjusted. In the following proposition, proven in Appendix A, we state conditions under which that equivalence of the impact effects of the shocks holds.

Proposition 1. Suppose the proxies, z_t, satisfy the relevance and exogeneity conditions (3) and (4) and, in addition, the following three conditions hold:

(a) In the augmented VAR model (7), $A_1^z = \cdots = A_p^z = 0$, i.e., no lags of the proxies appear in the y_t equations.

(b) The covariance matrix of z_t, Σ_z, is a diagonal matrix, i.e., the proxies are instantaneously uncorrelated.

(c) $\Sigma_{\mathbf{w}_1 z}$ is a diagonal matrix, i.e., each shock in $\mathbf{w}_{1t}$ is correlated with one proxy only.

Then the lower-left hand $(K \times K_1)$ block of $B^{aug} = \text{chol}(\Sigma_u^{aug})$ has columns that are scalar multiples of the columns of the impact effects matrix, B_1, of the shocks $\mathbf{w}_{1t}$. □

The proposition states that, if the proxies are contemporaneously uncorrelated and each of the proxies is correlated with one shock of interest only, then we can get scalar multiples of the impact effects of the first K_1 structural shocks, $\mathbf{w}_{1t}$, by considering $\text{chol}(\Sigma_u^{aug})$ and, using these multiples of the impact effects, we can get multiples of the shocks $\mathbf{w}_{1t}$ from the relation (2).

If there is just one shock of interest ($K_1 = 1$) and one proxy that satisfies the relevance and exogeneity conditions, the conditions (b) and (c) of Proposition 1 are automatically satisfied and, hence, if also $A_1^z = \cdots = A_p^z = 0$, the impact effects of the shock can be obtained directly from the relation $\mathbb{E}(u_t z_{1t}) = \sigma_1 b_1$, where σ_1 is a scalar, or, equivalently, by using the covariance matrix of the VAR residuals augmented by a single proxy. The latter fact follows from Corollary 1 in Appendix A which implies that, if there is just one shock identified by a single proxy ($N = K_1 = 1$), the last K elements of the first column of the Cholesky decomposition of the covariance matrix Σ_u^{aug} of the augmented VAR residual vector are a multiple of $\mathbb{E}(u_t z_t)$ and thus, upon standardization, are precisely the desired impact effects of the structural shock of interest. A closely related result was established earlier by Plagborg-Møller and Wolf (2021) (see in particular their Section 3.3).

In fact, the result even holds for the first shock if there are several proxies and shocks of interest. However, it does not hold for the other shocks in $\mathbf{w}_{1t}$ in general. Only if there are several proxies which are instantaneously uncorrelated, i.e., Σ_z is a diagonal matrix as in Proposition 1, then, by Corollary 2 given in Appendix A, we can get also the impact effects of all shocks from the external proxy VAR approach from the Cholesky decomposition of Σ_u^{aug}, provided $\Sigma_{\mathbf{w}_1 z}$ in (5) is a diagonal matrix, that is, if the i-th proxy is only correlated with the i-th shock and uncorrelated with all other shocks.

We emphasize that there is an important precondition specified in Proposition 1 for these results to hold, namely condition (a) which states that there are no lags of the proxies in the y_t equations. That condition implies that the proxies have to be Granger-noncausal for the y_t to ensure undistorted impulse responses. There are, in fact, good reasons for including lags of z_t in the y_t equations. If the proxies contain information on some of the structural shocks $\mathbf{w}_{1t}$, which after all is why we use them as proxies, they may well be Granger-causal for y_t. In that case, at least some of the A_i^z, $i = 1, \dots, p$, are nonzero. Including them leads to different impulse responses. Note the following relation between the reduced-form impulse responses of the augmented model (7), denoted by Φ_h^{aug}, and the original model (1), denoted by Φ_h:

$$\Phi_h^{aug} = \begin{bmatrix} 0 & 0 \\ * & \Phi_h \end{bmatrix}, \quad h = 1, 2, \dots, \tag{10}$$

where * stands for possibly nonzero elements. Thus, computing the structural impulse responses as $\Phi_h^{aug} B^{aug}$, the resulting impulse responses of y_t will in general not be identical to $\Phi_h B$ from the VAR model (1).

If the lags of the proxies enter the y_t equation in (7), one may even wonder whether the shocks of interest are fundamental and can be obtained as linear transformations of the reduced-form residuals u_t, as assumed in our model setup in Section 2, and in the external proxy VAR approach (see Plagborg-Møller and Wolf (2021) for discussion). Plagborg-Møller and Wolf (2021) discuss another point in favour of using an augmented VAR model. They point out that in general the impulse responses obtained from a VAR model will be distorted if the true shocks are nonfundamental and they mention that correct impulse responses in this case are obtained only if the assumed shock is a function of just the true shock and perhaps an error term which is not related to y_t. Plagborg-Møller and Wolf (2021) also mention that, in such a case, even if the shock is not fundamental, its impulse responses can be estimated properly by adding z_t as an additional variable to the VAR.

In summary, if $A_1^z = \cdots = A_p^z = 0$ and there is only one shock that is identified by a single proxy, then we can use a Cholesky decomposition of the residual covariance matrix of the augmented VAR model to compute the impulse responses. If there are several shocks identified by a set of proxies, then the impact effects of the first shock can be obtained from the first column of the Cholesky decomposition of the residual covariance matrix of the augmented VAR. The impact effects of the other shocks can be obtained from the Cholesky decomposition of the residual covariance matrix of the augmented VAR if both of the following conditions are satisfied: (b) the proxies are uncorrelated (the matrix Σ_z is a diagonal matrix) and (c) the i-th proxy is correlated with the i-th shock and not with any of the other shocks of interest (the matrix $\Sigma_{\mathbf{w}_1 z}$ is a diagonal matrix).

2.4 Estimation of Augmented VAR Models

The previous discussion refers to population quantities. It is important to emphasize that these results will carry over to estimated quantities in small samples as long as standard estimation methods are used. For example, if the VAR model (1) is estimated by LS and Σ_u^{aug} is estimated as

$$\widehat{\Sigma}_u^{aug} = \frac{1}{T}\sum_{t=1}^{T} \begin{bmatrix} z_t - \bar{z} \\ \hat{u}_t \end{bmatrix} [(z_t - \bar{z})', \hat{u}_t'], \tag{11}$$

where $\bar{z} = T^{-1}\sum_{t=1}^{T} z_t$ and the $\hat{u}_t$ are LS residuals, then the lower left-hand $(K \times K_1)$ block of $\text{chol}(\widehat{\Sigma}_u^{aug})$ can be used to estimate the impact effects B_1 of the structural shocks.

Alternatively, one may include lags of the proxies in the y_t equations of the augmented model (7) and consider the shocks and impulse responses obtained from the augmented model even if the true $A_1^z = \cdots = A_p^z = 0$. In that case, the estimated shocks, their interpretation and their impulse responses are different from those of the external proxy VAR approach. Of course, the additional parameters to be estimated in the augmented VAR may reduce the precision of the estimated impulse responses if that approach is used. In the following, to distinguish the shocks obtained from models with and without lags of z_t in the augmented VAR, we use the notation $\hat{\mathbf{w}}_{1t}(\text{augVAR})$ when the lags of z_t are included in the y_t equations and $\hat{\mathbf{w}}_{1t}(\text{augVAR}^0)$ when they are not included.

One issue that is often a concern in the estimation of the impact effects is that some proxies are only weakly related to the shocks of interest. In other words, the proxies may be weak instruments for which special estimation procedures are recommended (see Montiel Olea, Stock and Watson (2021)). Clearly, as the external proxy and augmented VAR estimates are identical up to a scalar multiple under the conditions of Proposition 1, both approaches are equally affected by the weakness of an instrument.

Of course, there are a number of other estimation methods for proxy VAR models. The method based on (11) does not account for the over-identifying restrictions from the diagonality of $\Sigma_{\mathbf{w}_1 z}$ and the uncorrelatedness of the structural shocks and is, hence, not efficient. To improve efficiency, one could use the GMM approach of Gregory et al. (2024), as mentioned earlier. There are also local projection (LP) and Bayesian estimation methods that could be applied. It may be useful to take the theoretical results of Proposition 1 into account in the estimation of proxy VAR models. In the present study we are interested in presenting and illustrating the results of Proposition 1 and therefore do not consider alternative estimation methods.

3 Empirical Examples

The following empirical examples illustrate the theoretical findings of the previous section. The first example considers the global market for crude oil and the second example studies the impact of TFP shocks on the U.S. economy.

3.1 Oil Market Shocks

Känzig (2021) considers a six-dimensional benchmark model to study the impact of oil market shocks on some key economic variables. In his benchmark analysis, he uses a single proxy to identify an oil supply news shock. To safeguard against distortions in his analysis, he also uses a setup with two proxies to identify two shocks, an oil supply

Table 1: Empirical Correlations of Proxies and Shocks for Oil Market Example with 95% Bootstrap Confidence Intervals

	z_t^{ops}	z_t^{news}	$\hat{w}_t^{ops}$(PVAR)	$\hat{w}_t^{news}$(PVAR)
z_t^{ops}	1	-0.065		
		$(-0.131, -0.008)$		
z_t^{news}		1		
$\hat{w}_t^{ops}$(PVAR)	0.173	-0.020	1	-0.088
	$(0.006, 0.319)$	$(-0.091, 0.050)$		$(-0.182, 0.005)$
$\hat{w}_t^{news}$(PVAR)	-0.015	0.226		1
	$(-0.130, 0.092)$	$(0.098, 0.346)$		

Note: The confidence intervals are obtained with a bootstrap suggested by Lunsford (2015) and presented in detail in the Appendix of Bruns et al. (2024).

news shock and an oil production shortfall shock. We will denote the corresponding proxies by z_t^{news} and z_t^{ops}, respectively.

Känzig uses a VAR(12) model with a constant term for the real price of oil (rp_t), world oil production ($prod_t$), world oil inventories (inv_t), world industrial production (ip_t^{World}), U.S. industrial production (ip_t^{US}), and the U.S. consumer price index (cpi_t^{US}) such that $y_t = (rp_t, prod_t, inv_t, ip_t^{World}, ip_t^{US}, cpi_t^{US})'$. All variables are in logs. Känzig uses monthly data from January 1974 to December 2017. Hence, his gross sample size is 528. Accounting for the presample values required for LS estimation of the VAR(12) model, we have a net sample size of $T = 516$. We use his sample period and data set to facilitate a comparison with his results although there is some evidence that the structural impulse responses may not be time-invariant across the full sample period (see Bruns and Lütkepohl (2023)). We also emphasize, that Känzig (2021) actually identifies the shocks with additional restrictions to ensure individually identified and uncorrelated shocks. Thus, his identification approach is different from ours.

Känzig (2021) constructs one proxy (z_t^{news}) based on OPEC announcements about their production plans. It is used to identify an 'oil supply news shock' (w_t^{news}), while the other proxy (z_t^{ops}) is based on work by Kilian (2008) and Bastianin and Manera (2018) and captures the shortfall of OPEC oil production caused by exogenous political events such as wars or civil disturbances and, hence, may be related to the first proxy. Känzig considers the two proxies to exclude possible distortions due to omitting effects related to his oil supply news shock.

Based on the shocks obtained by using the proxies one-by-one in an external proxy VAR approach, we get the correlations between the proxies and shocks of interest presented in Table 1. Although the empirical correlation between the proxies is small, there is evidence that the proxies are correlated as zero is not in the 95% bootstrap confidence interval. However, the estimated correlation between the resulting structural shocks is small and not significantly different from zero. Moreover, the estimated correlation matrix corresponding to $\Sigma_{\mathbf{w}_1 z}$ has off-diagonal elements not significantly different from zero. Diagonality of the latter matrix already ensures identification of the shocks, while the conditions (a) and (b) of Proposition 1 ensure that both

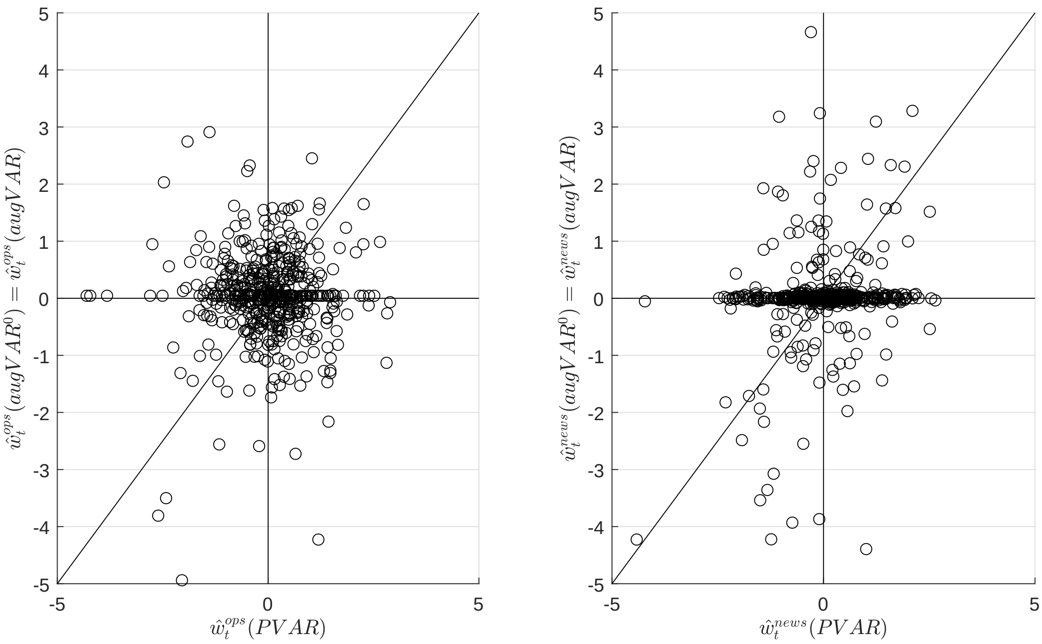

Figure 1: Scatter plots of shocks for oil market example.

approaches provide identical impulse responses. Given that the conditions of Proposition 1 for using the augmented model without additional identifying assumptions are roughly satisfied if we ignore that there may be some small correlation between the proxies, the impulse responses may be expected to be very similar.

In Figure 1 we show scatter plots of the shocks obtained from the external proxy VAR ($\hat{\mathbf{w}}_{1t}$(PVAR)) and the two versions of the augmented VAR approaches with and without lagged proxies in the y_t equations ($\hat{\mathbf{w}}_{1t}$(augVAR) and $\hat{\mathbf{w}}_{1t}$(augVAR^0)). Table 2 shows the corresponding correlations. Note that we have computed the augVAR shocks using the relation (9). Obviously, the augmented VAR shocks are identical ($\hat{\mathbf{w}}_{1t}$(augVAR^0) $=$ $\hat{\mathbf{w}}_{1t}$(augVAR)) because they are equal to the same linear combinations of the proxies, and corr($\hat{w}_t^{news}$(augVAR^0), $\hat{w}_t^{ops}$(augVAR^0)) $=$ corr($\hat{w}_t^{news}$(augVAR), $\hat{w}_t^{ops}$(augVAR)) $= 0$ by construction. Note that z_t^{news} contains many zero elements, leading to $\hat{w}_t^{news}$(augVAR^0) and $\hat{w}_t^{news}$(augVAR) to take on a constant value for much of the sample. In contrast, $\hat{w}_t^{news}$(PVAR) varies across the full sample period. This outcome illustrates the external proxy VAR's ability to account for measurement errors in the proxies rather than intending to measure the shock directly.

Pairs of corresponding shocks differ quite substantially between the external PVAR and the augmented VAR approaches, although the conditions of Proposition 1 are approximately satisfied. Given that these conditions are roughly satisfied, we would expect similar estimates for the impact effects, B_1, for the external PVAR and the augmented VAR approaches. The reason for the difference in estimated shocks is that for the external proxy VAR approach, shocks are recovered as linear combinations of VAR residuals according to equation (2). For the augmented VAR approach instead, the estimated shocks are linear combinations of the proxies themselves, as apparent from equations (8) and (9). Note that, as a consequence, the $\hat{w}_t^{news}$(augVAR) are constant across parts of the sample while the $\hat{w}_t^{news}$(PVAR) vary. It is therefore not

Table 2: Empirical Correlations of Shocks Obtained with Alternative Estimation Procedures for Oil Market Example with 95% Bootstrap Confidence Intervals

	$\hat{w}_t^{ops}(\text{augVAR}^0) = \hat{w}_t^{ops}(\text{augVAR})$	$\hat{w}_t^{news}(\text{augVAR}^0) = \hat{w}_t^{news}(\text{augVAR})$
$\hat{w}_t^{ops}(\text{PVAR})$	0.172	-0.009
	$(0.006, 0.320)$	$(-0.080, 0.062)$
$\hat{w}_t^{news}(\text{PVAR})$	-0.015	0.225
	$(-0.130, 0.092)$	$(0.098, 0.346)$

surprising that the correlation of $\hat{w}_t^{news}(\text{PVAR})$ and $\hat{w}_t^{news}(\text{augVAR})$ is low (see Table 2) despite the conditions of Proposition 1 being approximately satisfied and therefore a similarity in impact effects of the shocks might be expected.

The impulse responses to the frist shock, the ops shock, obtained by the external proxy VAR approach and the augmented VAR approach without lags of z_t are identical, as discussed in Section 2.3. Therefore we focus on the impulse responses of the news shock and compare them for all three approaches in Figure 2. We follow Känzig (2021) and consider shocks that increase the real oil price by 10% on impact. As the conditions of Proposition 1 are (almost) satisfied, one would expect that the responses to $\hat{w}_t^{news}(\text{PVAR})$ and $\hat{w}_t^{news}(\text{augVAR}^0)$ shocks are similar if restricting the A_i^z to zero in model (7) is not a severe restriction. In other words, one would expect the impulse responses to be similar if the shocks are fundamental. As can be seen in Figure 2, the responses to $\hat{w}_t^{news}(\text{PVAR})$ and $\hat{w}_t^{news}(\text{augVAR}^0)$ are indeed very similar. We have also performed a standard Wald test of $\mathbb{H}_0 : A_1^z = \cdots A_p^z = 0$. It returns a value of 116.9 which corresponds to a p-value of 0.95 of a χ^2 limiting distribution with 144 degrees of freedom. Thus, the null hypothesis is clearly not rejected.[4] Although this suggests that the shock may indeed be fundamental, it is clear that nonrejection of a null hypothesis just means that the test may not have enough power to reject. The null hypothesis may still be false. Actually, Plagborg-Møller and Wolf (2022) find that the Känzig shocks may not be fundamental. For illustrative purposes we nevertheless treat the shock as fundamental in the following.

In Figure 2 we also show the impulse responses corresponding to $\hat{w}_t^{news}(\text{augVAR})$, where the A_i^z are not restricted to zero. Although the A_i^z are not significally different from zero as a group, they have a substantial impact on the estimated impulse responses. First of all, the confidence intervals (in green color in Figure 2) are much wider than the confidence intervals of the corresponding impulse responses estimated with the other two approaches (in blue and red in Figure 2) which may be due to the larger number of parameters in the model. The impulse responses also have partly very different shapes and, hence, are likely to lead to very different interpretations of the dynamics in the system. For example, the cpi^{US} index is much less persistent than the response estimated with the other two methods. Thus, augmenting the model with many additional insignificant parameters, may not be a good idea if the lagged proxies are actually not needed in the y_t equations. On the other hand, the differences in the impulse responses may be indicative for the lagged z_t to be important in

[4]Based on a model $y_t = [\nu, A_1^z, \dots, A_p^z, A_1, \dots, A_p]X_t + u_t = DX_t + u_t$ or, for $t = 1, \dots, T$, $Y = DX + U$, we use the Wald statistic $W = \text{vec}(\hat{D})'R'[R((XX')^{-1} \otimes \hat{\Sigma}_u)R']^{-1}R\text{vec}(\hat{D})$ for testing $\mathbb{H}_0 : R\text{vec}(D) = 0$. Here $\hat{D} = YX'(XX')^{-1}$ and $R = [0_{pKK_1 \times K}, I_{pKK_1}, 0_{pKK_1 \times pK^2}]$.

the y_t equations in which case the PVAR and augVAR^0 impulse responses would be distorted.

We have also reversed the order of the proxies and hence the shocks such that $\hat{w}_t^{news}$ is first and $\hat{w}_t^{ops}$ becomes the second shock. In Figure B.1 in Appendix B we show the corresponding impulse responses of the oil production shortfall shock. The results are qualitatively similar to those in Figure 2 in that the responses to $\hat{w}_t^{ops}$(PVAR) and $\hat{w}_t^{ops}$(augVAR^0) shocks are almost identical, while the responses to $\hat{w}_t^{ops}$(augVAR) have wider confidence intervals and are partly quite different from the corresponding other two impulse responses.

As mentioned earlier, Känzig (2021) identifies the shocks with additional restrictions to ensure individually identified and uncorrelated shocks. Assuming that $\Sigma_{\mathbf{w}_1 z}$ is a diagonal matrix, as suggested by the results in Table 1, such additional restrictions would be unnecessary in this case for fully identifying the shocks. Of course, additional information can always be used to improve the related inference procedures. For example, impulse responses may be estimated more precisely if additional restrictions for the structural parameters are available. Because our objective is to illustrate the theoretical results of Section 2, we do not consider alternative estimation methods here.

3.2 U.S. TFP Shocks

Lunsford (2015) considers the dynamic effects of two types of TFP shocks on the U.S. economy. He uses data from 1948Q2 to 2015Q2 which implies a gross sample size of 269 observations. His model contains $K = 5$ variables: GDP growth, employment growth, inflation, consumption growth, and investment growth. There are two proxies to identify $K_1 = 2$ TPF shocks and the VAR model has $p = 4$ lags[5] and a constant.[6] The model was also considered by Arias et al. (2021).

The two proxies are meant to be related to shocks to consumption and investment TFP. They are based on two utilization-adjusted TFP measures constructed by Fernald (2014), one for the consumption sector excluding durable goods (consumption TFP) and the second one for durable goods and equipment investment (investment TFP). Consumption TFP and investment TFP are regressed on four lags of y_t and a constant and the resulting residuals are used as proxies z_t^{cons} and z_t^{inv}, respectively. The corresponding shocks will be denoted by w_t^{cons} and w_t^{inv}, respectively.

In Table 3 we present the estimated correlations between the proxies and the shocks of interest together with 95% bootstrap confidence intervals. The correlation between the two proxies is 0.295 which is significantly different from zero. The resulting shocks $\hat{w}_t^{cons}$(PVAR) and $\hat{w}_t^{inv}$(PVAR), obtained by the external proxy VAR approach, are well correlated with their respective proxy and have no significant correlation with the other proxy. For instance, $\hat{w}_t^{cons}$(PVAR) has correlation 0.582 with the proxy z_t^{cons} and correlation 0.011 with z_t^{inv}. The latter correlation is not significantly different from zero in that zero is within its 95% bootstrap confidence interval. The situation is similar for $\hat{w}_t^{inv}$(PVAR). Thus, we conclude that each proxy is correlated with one shock only ($\Sigma_{\mathbf{w}_1 z}$ is diagonal). Hence, the shocks are identified without imposing further restrictions. Moreover, the estimated structural shocks $\hat{w}_t^{cons}$(PVAR)

[5] Lunsford (2015) states that he uses 3 lags. However, in a later revised version of the paper, $p = 4$ is claimed. Thus, we use the latter lag length.

[6] The dataset is available at https://sites.google.com/site/kurtglunsford/research.

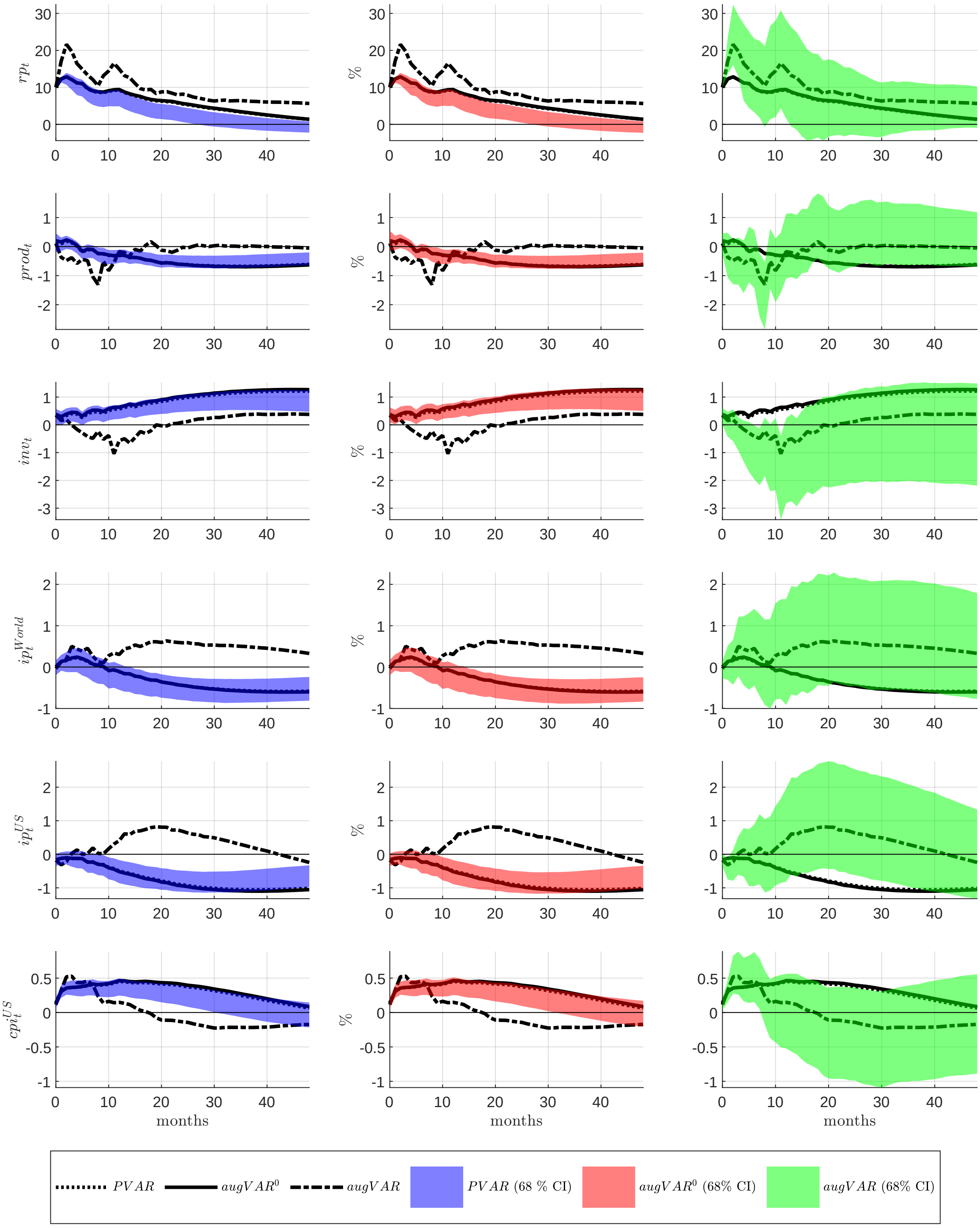

Figure 2: Oil market example: Comparison of impulse responses of a w_t^{news} shock estimated with a single proxy, z_t^{news}, and by the augmented model with two proxies. The shock is normalized to increase oil prices by 10 percent on impact. The confidence intervals around the impulse responses are based on 5000 bootstrap samples.

Table 3: Empirical Correlations of Proxies and Shocks for U.S. TFP Example with 95% Bootstrap Confidence Intervals

	z_t^{cons}	z_t^{inv}	$\hat{w}_t^{cons}$(PVAR)	$\hat{w}_t^{inv}$(PVAR)
z_t^{cons}	1	0.295		
		$(0.175, 0.412)$		
z_t^{inv}		1		
$\hat{w}_t^{cons}$(PVAR)	0.582	0.011	1	0.025
	$(0.493, 0.663)$	$(-0.113, 0.140)$		$(-0.125, 0.176)$
$\hat{w}_t^{inv}$(PVAR)	0.015	0.445		1
	$(-0.131, 0.158)$	$(0.344, 0.539)$		

Table 4: Empirical Correlations of Shocks Obtained with Alternative Estimation Procedures for U.S. TFP Example with 95% Bootstrap Confidence Intervals

	$\hat{w}_t^{cons}(\text{augVAR}^0) = \hat{w}_t^{cons}(\text{augVAR})$	$\hat{w}_t^{inv}(\text{augVAR}^0) = \hat{w}_t^{inv}(\text{augVAR})$
$\hat{w}_t^{cons}$(PVAR)	0.582	-0.168
	$(0.493, 0.663)$	$(-0.286, -0.036)$
$\hat{w}_t^{inv}$(PVAR)	0.015	0.461
	$(-0.131, 0.158)$	$(0.355, 0.556)$

and $\hat{w}_t^{inv}$(PVAR) are not significantly correlated. The estimated correlation of 0.025 has zero in the 95% confidence interval. Thus, in this example, one could justify the assumption of uncorrelated shocks that are each correlated with one proxy only.

Because the proxies are correlated, the second shock, $\hat{w}_t^{inv}(\text{augVAR}^0)$, and its impulse responses obtained from the augmented VAR approach in this case are expected to be different from $\hat{w}_t^{inv}$(PVAR) as condition (b) of Proposition 1 is not satisfied (and because the shocks are recovered from VAR residuals for the PVAR case and from proxies for the augmented VAR case). The shocks are plotted against the external proxy VAR shocks in Figure 3, where we have again used the augmented VAR model with and without lags of the proxies in the y_t equations for computing the $\hat{\mathbf{w}}_t$(augVAR) and $\hat{\mathbf{w}}_t(\text{augVAR}^0)$ shocks, respectively. As expected, $\hat{\mathbf{w}}_t$(PVAR) are correlated with but not identical to the corresponding shocks obtained from the augmented VAR approach. Again, by construction, the two augmented VAR approaches yield identical shocks.

The numerical correlations between the shocks estimated with the external proxy VAR and augmented VAR approaches are presented in Table 4. The corresponding shocks estimated with the different methods are correlated, but the PVAR and the augmented VAR approach yield notably different shocks, as one would expect since the proxies are correlated. The $\hat{w}_t^{cons}(\text{augVAR}^0)$ and $\hat{w}_t^{inv}(\text{augVAR}^0)$ are uncorrelated by construction. The same is, of course, true for $\hat{w}_t^{cons}$(augVAR) and $\hat{w}_t^{inv}$(augVAR).

The impulse responses of the investment TFP shocks estimated by all three approaches are presented in Figure 4. Unlike in the previous example, the impulse responses of the $\hat{w}_t^{inv}$(PVAR) (dotted line) and $\hat{w}_t^{inv}(\text{augVAR}^0)$ shocks (solid line) are

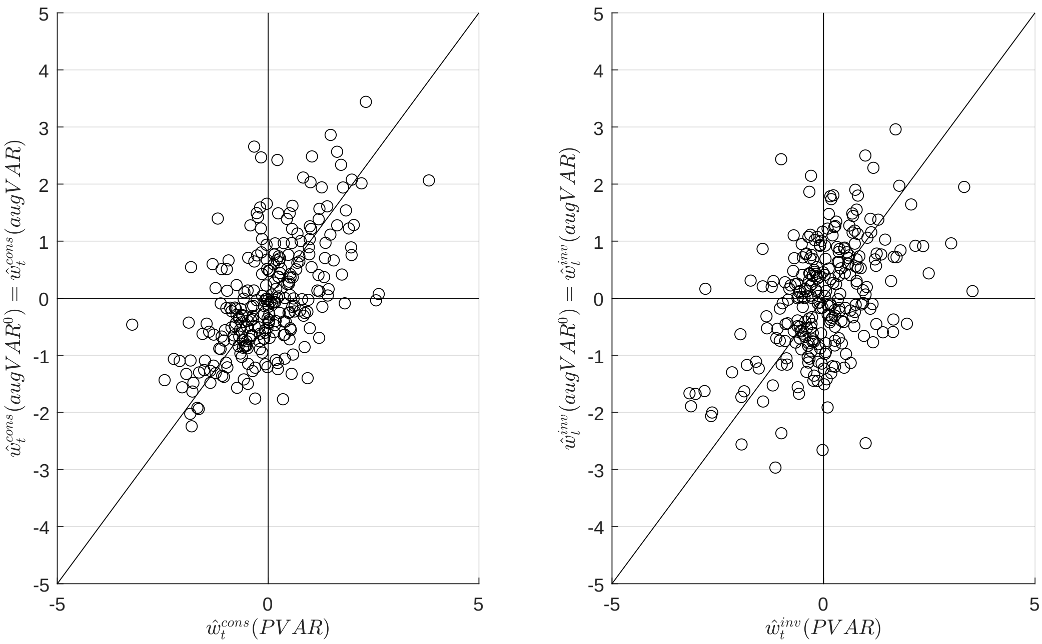

Figure 3: Scatter plots of TFP shocks

now quite different, as expected. In some cases, impulse responses of the $\hat{w}_t^{inv}$(PVAR) are not even covered by the confidence intervals around the impulse response estimates obtained for $\hat{w}_t^{inv}$(augVAR^0) and vice versa. Given that the proxies are correlated and, hence, the conditions of Proposition 1 are not satisfied, this outcome is not surprising. Again the impulse responses of $\hat{w}_t^{inv}$(augVAR) shocks (dashed line) are partly quite different from those of the other shocks. Note in particular the response of inflation which is even qualitatively different when the lags of proxies are included in the model.

We have also reversed the order of the proxies and shocks and performed an impulse response analysis with the consumption TFP shock being the second shock. The impulse responses are presented in Figure B.2 in Appendix B. The general picture in Figure B.2 is similar to that in Figure 4 in that the impulse responses of the external proxy VAR approach partly differ from those obtained by the augmented VAR approach without lagged proxies in the y_t equations.

To investigate, whether this may also be driven by having a model with nonfundamental shocks, we have again applied a standard Wald test for Granger-causality of the proxies. The test value is 81.1 which corresponds to a p-value of 0.0001 of the limiting χ^2 null distribution with 40 degrees of freedom. Thus, Granger-noncausality is strongly rejected and, hence, the shocks of interest may not be linear transformations of the u_t which may contribute to the clear differences in some responses to $\hat{w}_t^{inv}$(PVAR) and $\hat{w}_t^{inv}$(augVAR^0) shocks in Figures 4 and B.2. Thus, this example illustrates that the conditions of Proposition 1, in this case notably the fundamentalness condition (a), are crucial for getting properly identified impulse responses by the external proxy VAR methodology without further assumptions.

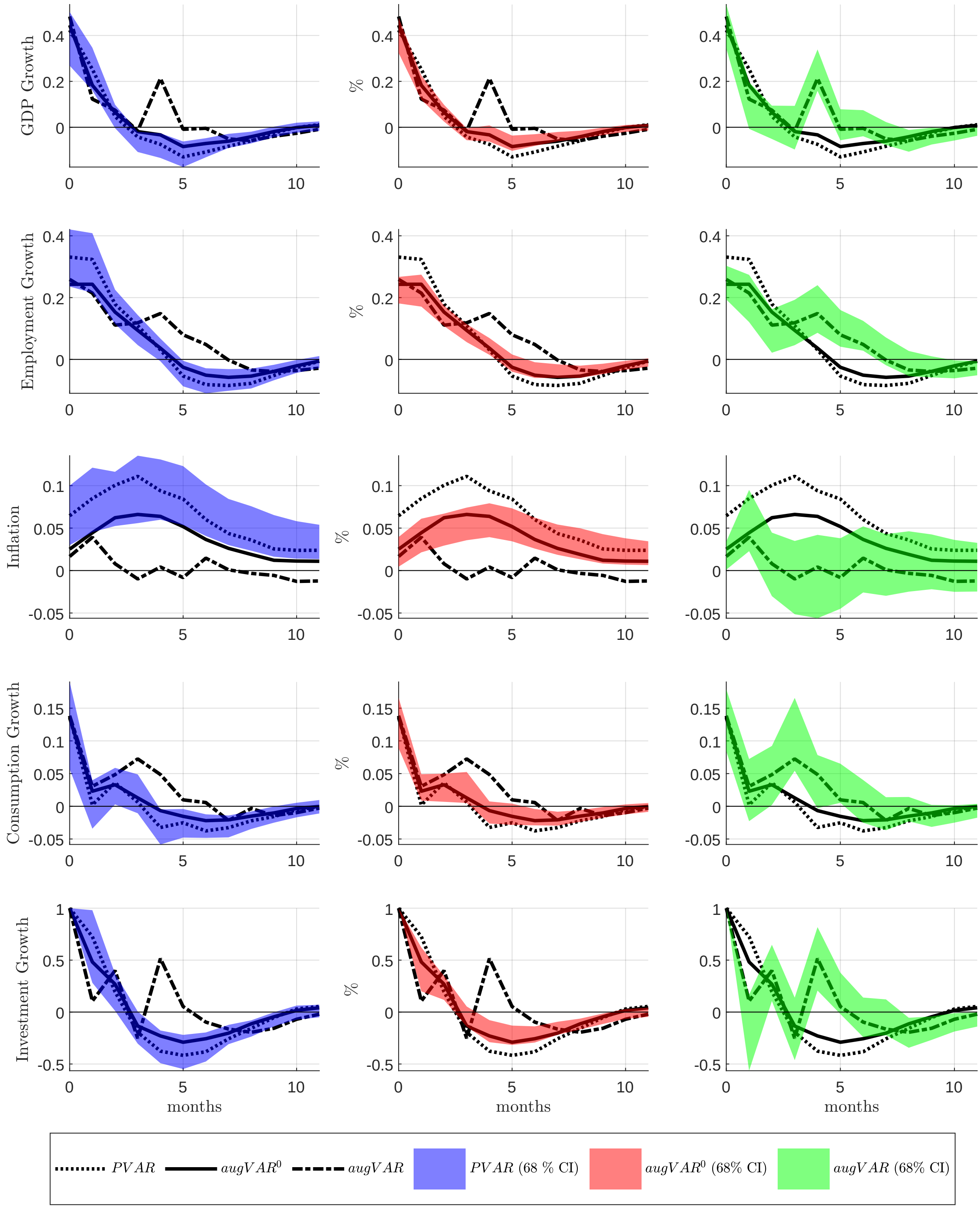

Figure 4: TFP example: Comparison of impulse responses of a w_t^{inv} shock estimated with different methods. The investment TFP shock is normalized to increase investment TFP by 1 percent on impact. The confidence intervals around the impulse responses are based on 5000 bootstrap samples.

4 Conclusions

When instruments are used in a proxy VAR study to identify a single shock or a set of shocks, a couple of alternative approaches for estimating the structural parameters are in common use. The first one is based on the covariance of the proxies and the reduced-form residuals and another one augments the VAR by the proxies. We have pointed out important differences and similarities between these approaches for the case of multiple proxies. Thereby, we not only provide some new insights for researchers using multiple proxies to identify a set of shocks but also generalize results that were previously known for the case of identifying a single shock only.

In general, both approaches for identifying multiple shocks by a set of proxies have to be complemented with additional identifying assumptions in order to properly identify the shocks of interest individually. However, if there are exactly as many proxies as there are shocks to be identified and if the proxies are mutually uncorrelated, and each of them is correlated with a single shock only, then no additional information is needed to fully identify the shocks of interest individually. If the proxies are not Granger-causal for the variables of the VAR model, the impulse responses obtained with the two alternative estimation approaches are identical in population and may be very similar if in the augmented VAR no lags of the proxies are included. If the lagged proxies are nevertheless included in the model, the increased estimation uncertainty due to the additional parameters in the model may distort the impulse responses.

We consider two examples to illustrate these theoretical results. The first one studies the impact of oil market shocks on the economy and the second one considers two different types of TFP shocks and their impact on the U.S. economy. In the first example, the conditions for identified impulse responses are nearly satisfied and the shocks estimated by the external proxy and the augmented VAR approaches are very similar, if no lagged proxies are included in the equations for the variables of interest. Dropping the lagged proxies in this example is supported by the nonsignificant outcome of a Granger-causality test. If nevertheless lags of the proxies are included on the right-hand side of the VAR model, the additional estimation uncertainty due to model augmentation is reflected in much wider confidence intervals around the impulse responses and quite different impulse response estimates that may lead the researcher to draw different conclusions regarding the dynamics of the model. Thus, the example illustrates that it may not be a good idea to include unnecessarily many variables in the VAR model.

In the second example, the proxies are correlated and are also found to be Granger-causal for the variables of the original VAR model. Therefore the estimates obtained from the external proxy VAR approach are different from those of the augmented VAR approach and may be distorted. In contrast, the augmented VAR impulse responses may still be valid.

Our results imply the following strategies for applied work. If the shocks can be thought of as linear combinations of the proxies, then the augmented VAR approach may be useful. If, however, the proxies are better thought of as shocks measured with error, then the external proxy VAR approach is perhaps more suitable. If the conditions of Proposition 1 are satisfied and the researcher is interested only in the impulse responses, then s/he is free to choose between the external proxy VAR and the augmented VAR approaches because they imply identical impulse responses under the conditions of Proposition 1. Finally, if the proxies are Granger-causal for the variables of the model, then using the augmented VAR approach with lags of the proxies in the model is called for and the two approaches will provide different shocks and impulse responses.

In practice, one may also want to consider alternative estimation methods which may be more efficient or account for weak instruments. Also Bayesian methods can be considered instead of the frequentist methods mentioned in this study. We have not discussed those methods here because the objective of this study is to raise awareness of the theoretical relations between external proxies and internalizing them. Considering the implications for the various possible estimation methods for proxy VAR analysis may be an interesting topic for future research.

A Proof of Proposition 1

Proposition 1 follows from the following matrix result.

Lemma 1. Let Σ_{11} be a symmetric positive definite $(N \times N)$ matrix, Σ_{22} be a symmetric positive definite $(K \times K)$ matrix, and Σ_{21} a $(K \times N)$ matrix such that the $((N+K) \times (N+K))$ matrix

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{21}' \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$$

is positive definite. Then the lower-triangular Cholesky decomposition of Σ is

$$\operatorname{chol}(\Sigma) = \begin{bmatrix} \operatorname{chol}(\Sigma_{11}) & 0 \\ \Sigma_{21}\operatorname{chol}(\Sigma_{11})^{-1\prime} & G \end{bmatrix}, \tag{A.1}$$

where $G = \operatorname{chol}(\Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{21}')$, i.e., $GG' = \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{21}'$. □

Proof of Lemma 1. The lemma follows by multiplying the right-hand side of equation (A.1) by its transpose and noting that $\operatorname{chol}(\Sigma_{11})^{-1\prime}\operatorname{chol}(\Sigma_{11})^{-1} = \Sigma_{11}^{-1}$. □

Using this Lemma, it is easy to prove Proposition 1.

Proof of Proposition 1. Setting $\Sigma_{11} = \Sigma_z$ and $\Sigma_{21} = \mathbb{E}(u_t z_t') = \mathbb{E}\left(u_t(z_t - \mathbb{E}(z_t))'\right)$ and using that Σ_z is a diagonal matrix and, hence, $\operatorname{chol}(\Sigma_z)^{-1} = \operatorname{chol}(\Sigma_z)^{-1\prime}$, Lemma 1 implies that the lower-left hand $(K \times K_1)$ block of $\operatorname{chol}(\Sigma_u^{aug})$ is $\mathbb{E}(u_t z_t')\operatorname{chol}(\Sigma_z)^{-1}$. The latter matrix is equal to $B_1\Sigma_{\mathbf{w}_1 z}\operatorname{chol}(\Sigma_z)^{-1}$ according to the relevance and exogeneity conditions (3) and (4) (see also expression (5)). Thus, Proposition 1 follows by noting that $\operatorname{chol}(\Sigma_z)^{-1}$ is a diagonal matrix. □

We also state the following straightforward implications of Lemma 1 for future reference.

Corollary 1. The first column of $\operatorname{chol}(\Sigma)$ is a multiple of the first column of Σ. More precisely, denoting the upper left-hand element of Σ by σ_{11}, the first column of $\operatorname{chol}(\Sigma)$ is $1/\sqrt{\sigma_{11}}$ times the first column of Σ. □

Corollary 2. If Σ_{11} in Proposition 1 is a diagonal matrix, then the first N columns of chol(Σ) are multiples of the corresponding columns of Σ. More precisely, denoting the i-th diagonal element of Σ_{11} by σ_{ii}, the i-th column of chol(Σ) is $1/\sqrt{\sigma_{ii}}$ times the i-th column of Σ for $i = 1, \ldots, N$. $\square$

Proof of Corollary 2. The corollary follows by noting that, for a diagonal matrix $\Sigma_{11} = \mathrm{diag}(\sigma_{11}, \ldots, \sigma_{NN})$, $\sigma_{ii} > 0$, the Cholesky decomposition is $\mathrm{chol}(\Sigma_{11}) = \mathrm{diag}(\sqrt{\sigma_{11}}, \ldots, \sqrt{\sigma_{NN}})$. $\square$

B Additional Impulse Responses

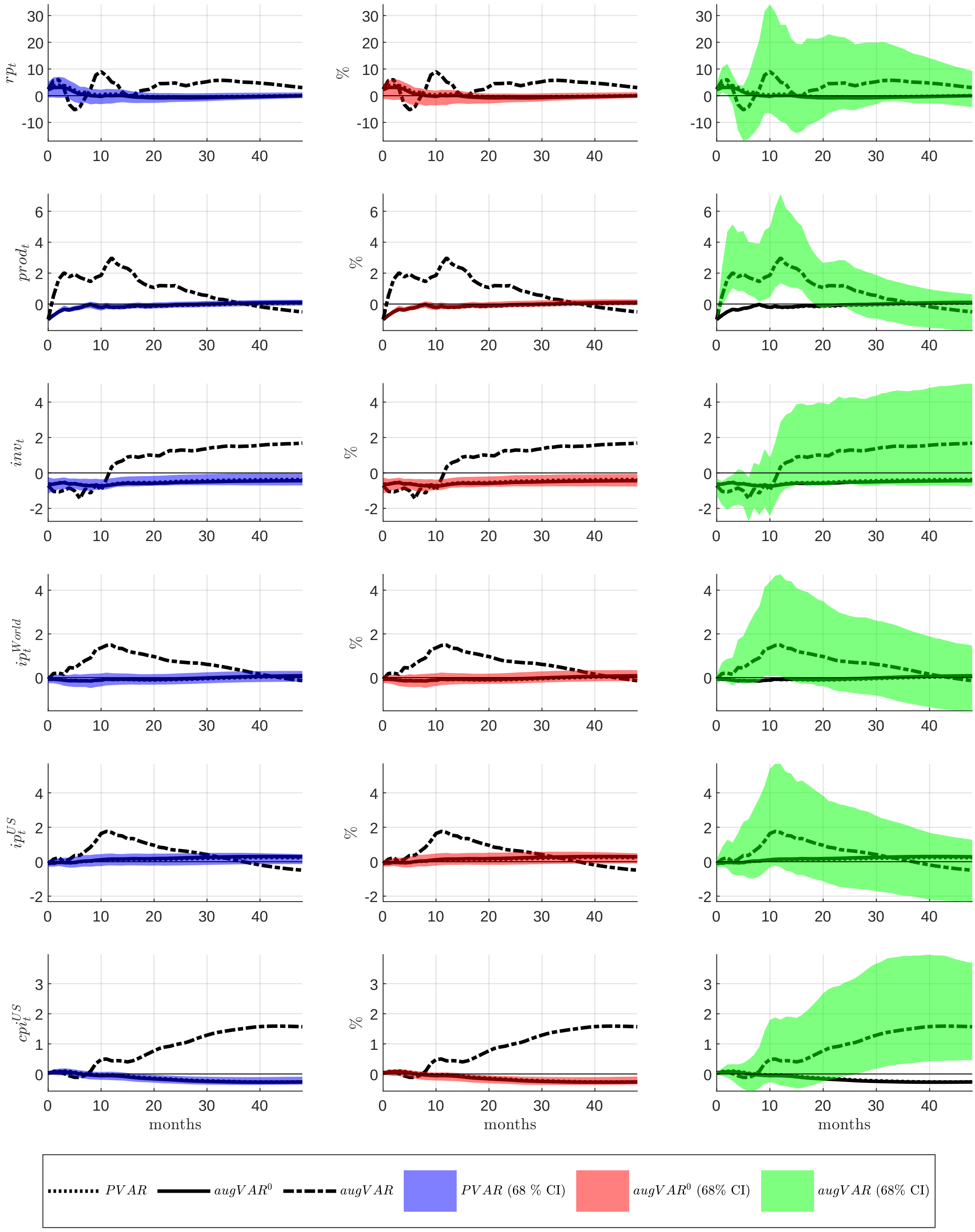

Figure B.1: Oil market example: Comparison of impulse responses of a w_t^{ops} shock estimated with a single proxy, z_t^{ops}, and by the augmented model with two proxies. The oil production shortfall shock is normalized to decrease oil production by 1 percent on impact.The confidence intervals around the impulse responses are based on 5000 bootstrap samples.

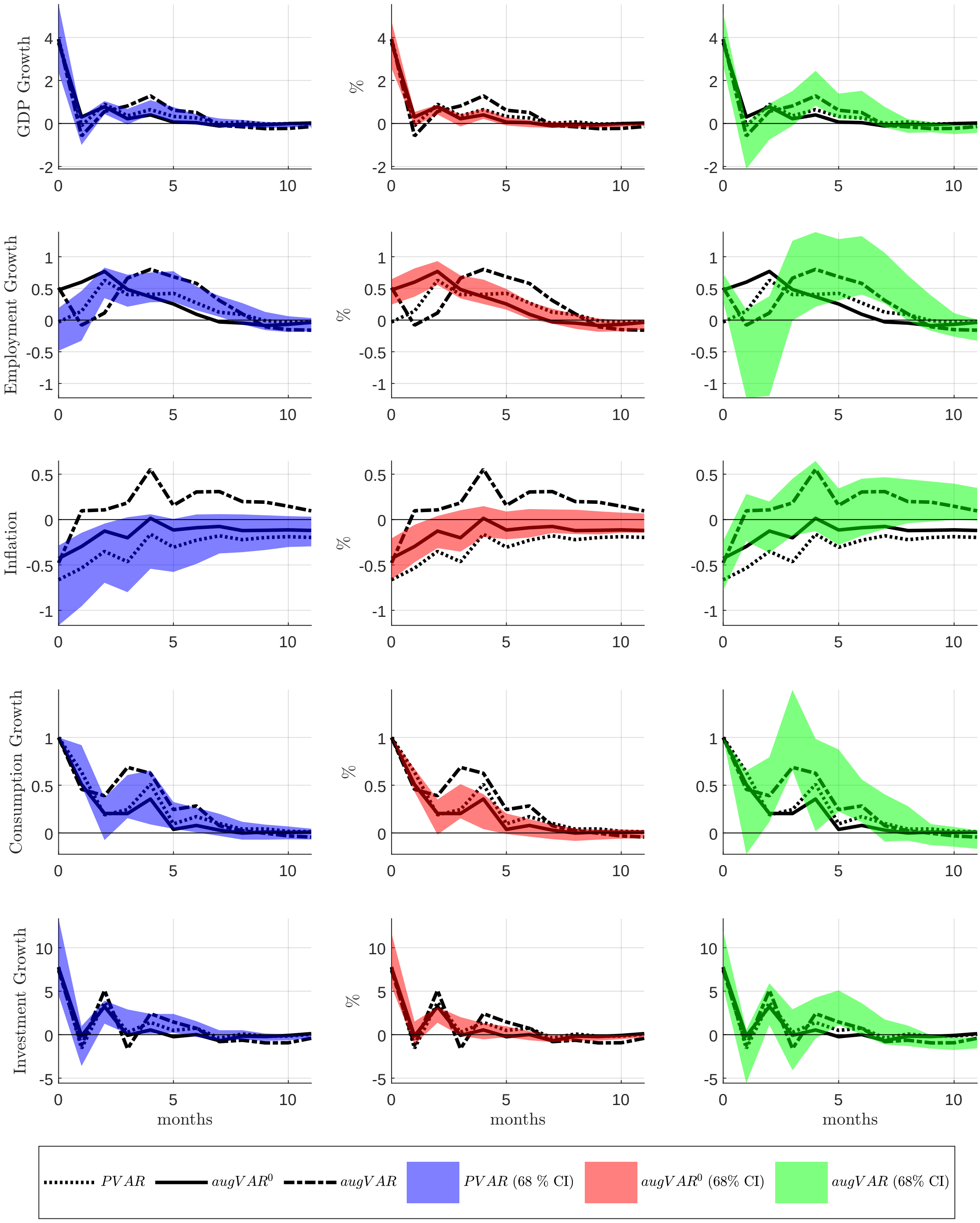

Figure B.2: TFP example: Comparison of impulse responses of a w_t^{cons} shock estimated with different methods. The consumption TFP shock is normalized to increase consumption TFP by 1 percent on impact. The confidence intervals around the impulse responses are based on 5000 bootstrap samples.

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