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ORIGINAL PAPER

Terminal value calculation with discontinuous financing and debt categories

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Abstract
Empirical analyses indicate that active and passive debt management have limited power to explain the financing behavior of firms. Therefore, discontinuous financing, as a combination of active and passive debt management, might be a more realistic financing strategy. However, the properties of this financing strategy for the steady state have not yet been sufficiently analyzed. For this reason, we investigate analytically terminal value calculation with discontinuous financing and derive adjustment formulas for the period-specific levered cost of equity. Since a single adjustment of the entire debt at the beginning of every planning phase might still not be close to the real financing behavior of firms, we modify discontinuous financing by introducing debt categories, which are adjusted successively and include the maturity of debt. For this new financing strategy, we derive valuation equations and an adjustment formula for the constant cost of equity. Finally, we discuss the relevance and applicability of discontinuous financing with debt categories and its impact on the market value of a firm.

Keywords Valuation · Terminal value calculation · Financing strategy · Discontinuous financing · Debt categories · Steady state

Mathematics Subject Classification G12 · G31 · G32

1 Introduction

Terminal value calculation is based on the assumption that a firm has reached a steady state after an explicit forecast phase (Koller et al. 2020, pp. 186–188; Brealey et al. 2020, pp. 95–99). To realistically depict the characteristics of a firm in a steady state, the choice of the financing strategy is a core issue. In corporate

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valuation practice, active or passive debt management are typically considered. However, empirical studies indicate that these financing strategies cannot model a firm's real financing policy with sufficient accuracy (see, e.g., Lewellen and Emery 1986; Barclay and Smith 2005; Grinblatt and Liu 2008). Consequently, discontinuous financing was developed as an alternative to active and passive debt management (Clubb and Doran 1995; Arnold et al. 2018, 2019; Dierkes and de Maeyer 2020). In particular, Arnold et al. (2018) derived valuation equations for the use of discontinuous financing in a steady state, but the effects of this financing strategy on the development of the market value of a firm, financial risk, and the cost of equity have not yet been investigated.

The aim of the paper is to analyze and further develop discontinuous financing as a more realistic financing policy for the steady state. First, we examine the properties of a steady state under discontinuous financing. We show that financial risk is inconstant under this financing policy, analyze the risk-free part of the tax shield, and derive an adjustment formula for the period-specific levered cost of equity. Furthermore, we outline that the sole adjustment of the entire debt level at the beginning of every planning phase could be improved by a consecutive adjustment of debt levels. Second, to model this consecutive adjustment, we develop a modified discontinuous financing policy, where every period a predetermined part of the overall debt level is adapted. Specifically, we introduce so-called debt categories, which include debt maturity and derive valuation equations with an adjustment formula for the levered cost of equity. Finally, we discuss its relevance, applicability, and impact on market value of a firm.

Findings on active and passive debt management are well known in the literature on corporate valuation (see, e.g., Kruschwitz and Löffler 2020, pp. 104–109). However, "neither purely active nor passive debt management assumptions are accurate reflections of corporate financial practice" (Clubb and Doran 1995, p. 690). In particular, empirical research showed that firms adjust their debt levels slowly (Fama and French 2002) and with a time lag (Leary and Roberts 2005; Huang and Ritter 2009). Therefore, Clubb and Doran (1995) introduced a lagged debt management policy that consists of consecutive planning phases in which passive debt management is used. In their first approach, debt levels are derived by multiplying the expected market value of a firm by the debt-to-market value ratio. After a predetermined number of periods, that is, after a planning phase, debt levels are adapted to the development of the firm and are redefined deterministically for the next planning phase, considering the updated expected market value of the firm. Despite the use of the debt ratio, debt levels are certain within a planning phase because they are linked to the expected and not the realized market value of the firm (Ashton and Atkins 1978). An extension of this mixed financing strategy was introduced by Arnold et al. (2018), who referred to it as discontinuous financing. Specifically, they pursued the second approach of Clubb and Doran (1995) that holds debt levels constant between rescheduling and analyzed the case of a perpetuity. Arnold et al. (2019) enhanced the valuation formula to a perpetuity with a constant growth rate, while Dierkes and de Maeyer (2020) examined the effects of discontinuous financing using a two-phase model.

In this study, we identify the differences between the approaches of Clubb and Doran (1995) and Arnold et al. (2018) and examine the properties of a steady state under discontinuous financing. It is apparent that financial risk cannot be deterministic given that passive debt management is used during a planning phase. However, the property of constant expected financial risk does not transfer from a steady state under passive debt management to a steady state under discontinuous financing. We highlight that financial risk varies depending on the remaining number of periods until the next planning phase. Furthermore, following the line of Inselbag and Kaufold (1997), we derive an adjustment formula for the period-specific cost of equity, which is also necessary for the unlevering and relevering of beta factors. We show that inconstant financial risk yields an inconstant growth of the market value of the firm. This effect was already briefly mentioned by Clubb and Doran (1995), who stated that "it does illustrate an interesting point, even if expectations [...] do not change [...] the value of the firm will still fluctuate" (Clubb and Doran 1995, p. 693). However, this effect has not been examined further thus far. Moreover, we outline that, compared to active and passive debt management, discontinuous financing might indeed be a more realistic depiction of a firm's financing behavior but the assumption that the entire debt level is adjusted according to the development only at the end of each planning phase might still not be practical. In particular, we find a partial adjustment in every period more realistic.

We implement this partial adjustment of debt by introducing debt categories, which constitutes the core contribution of our study. The resulting financing strategy is a modification of discontinuous financing that might come closer to a firm's real financing behavior. We assume that a firm adapts only a certain share of its overall debt in each period, requiring debt categories to be successively adjusted. That is, in each period, only one debt category is adjusted according to the development of the firm, and no other debt category is adapted to new information. In the subsequent period, another debt category is adjusted, and so on. Therefore, the planning phases overlap, and we, thus, consider debt maturity. At every point in time, a firm has various debt categories, characterized by different remaining maturities. First, we deduce a valuation equation for two debt categories and derive the adjustment formula for the levered cost of equity. Second, we extend the valuation and adjustment formulas to an arbitrary number of debt categories. Encouragingly, independent of the number of debt categories, the expected financial risk under this modified discontinuous financing policy is constant. Therefore, we obtain a constant levered cost of equity, making the application of the Gordon-Shapiro formula (Gordon and Shapiro 1956) possible. Although this successive adjustment of debt levels might be more realistic than a sole adjustment at the beginning of each planning phase, compared to discontinuous financing, its effect on the market value of the firm is small. Nevertheless, the analysis of this financing strategy is instructive for corporate valuation theory, and the valuation approach is also applicable in valuation practice. We discuss the application of both versions of discontinuous financing and outline that standard discontinuous financing can be used as an approximation for discontinuous financing with debt categories.

Overall, this study contributes to the literature on valuation by examining terminal value calculation with discontinuous financing and deriving a period-specific

adjustment formula for the levered cost of equity. Furthermore, the modification of standard discontinuous financing into discontinuous financing with debt categories yields a new financing policy with a constant levered cost of equity, which might be more suitable to describe the financing behavior of firms in a steady state.

The remainder of this study is organized as follows. In Sect. 2, discontinuous financing is analyzed, and the costs of equity under discontinuous financing are derived. The results are illustrated using an example. In Sect. 3, debt categories are introduced. First, the valuation formula for the special case of two debt categories is determined before generalizing it to an arbitrary number of debt categories. Second, the example from Sect. 2 is extended to debt categories to outline the impact of the new financing strategy on the market value of a firm and to compare it with other financing strategies. Third, it is outlined how discontinuous financing with debt categories can be applied in a two-phase model and how the risk of default can be included. Finally, we summarize and discuss the contributions of the study in Sect. 4.

2 Discontinuous financing

2.1 Fundamentals of discontinuous financing

Discontinuous financing is defined via planning phases in which debt levels are specified deterministically. The definition of debt levels within a planning phase is linked to the expected market value of a firm. After a planning phase, debt levels are adjusted according to the development of the firm and are again deterministically defined for the next planning phase. Figure 1 illustrates this structure for planning phases of length T.

There are different approaches on how to link debt levels to the expected market value of a firm. Initially, Clubb and Doran (1995) multiplied the debt-to-market value ratio by the expected market value of the firm and used a finite planning horizon for analysis. A second approach that Clubb and Doran (1995) discussed was to hold debt constant between rescheduling. This approach was also pursued by Arnold et al. (2018), who extended it to a steady state without growth. Arnold et al. (2019) examined this concept in a steady state with a constant growth rate g, and Dierkes and de Maeyer (2020) included this structure in a two-phase model. Table 1 summarizes the different debt level definitions, where θ is the debt-to-market value ratio,

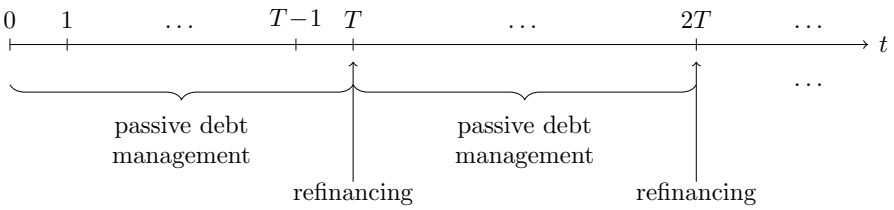

Fig. 1 Basic structure of discontinuous financing

Table 1 Overview of different debt level definitions in the case of discontinuous financing with $t \in \{0, \dots, T-1\}$

	1st planning phase $0, \dots, T-1$	2nd planning phase $T, \dots, 2T-1$
Clubb and Doran (1995) 1st appr	$D_t = \theta \cdot \mathbb{E}[\widetilde{V}_t^{\ell}]$	$D_{T+t} = \theta \cdot \mathbb{E}_T[\widetilde{V}_{T+t}^{\ell}]$
Clubb and Doran (1995) 2nd appr. and Arnold et al. (2018)	$D_t = \theta \cdot V_0^{\ell}$	$D_{T+t} = \theta \cdot V_T^{\ell}$
Arnold et al. (2019) and Dierkes and de Maeyer (2020)	$D_t = \theta \cdot V_0^{\ell} \cdot (1+g)^t$	$D_{T+t} = \theta \cdot V_T^{\ell} \cdot (1+g)^t$

and $\mathbb{E}_T[\cdot]$ is the expectation depending on the available information at time T. The tilde represents uncertain variables. Furthermore, V_t^{ℓ} and D_t denote the market value of the levered firm and the debt level at time t for $t \in \{0, \dots, T-1\}$, respectively. Superscript ℓ indicates the levered firm. Note that $T = 1$ constitutes active debt management according to Miles and Ezzell (ME), $T \to 0$ complies with active debt management according to Harris and Pringle (HP), and $T \to \infty$ represents passive debt management (Clubb and Doran 1995, pp. 687, 690; Arnold et al. 2018, p. 165; Arnold et al. 2019, pp. 352–353).

For all approaches in Table 1, the financial risk of a period depends on the remaining number of periods until the next refinancing date. If the firm is at a refinancing date, the tax shield is certain for the next T periods. However, if the firm will refinance at the end of the period, only the tax shield of the subsequent period is certain, which implies a higher financial risk. Therefore, in a steady state, in which the free cash flow (FCF) grows at a constant rate g, only the financial risks for periods in the same section of a planning phase coincide, being minimal at refinancing dates and increasing as the next refinancing date approaches. Such inconstant financial risk leads to a fluctuation in the market value of the firm even if cash flow expectations do not change. It follows that

$$\mathbb{E}[\widetilde{V}_{2T+t}^{\ell}] = \mathbb{E}[\widetilde{V}_{T+t}^{\ell}] \cdot (1+g)^T \qquad \text{but} \qquad \mathbb{E}[\widetilde{V}_{T+t}^{\ell}] \neq \mathbb{E}[\widetilde{V}_T^{\ell}] \cdot (1+g)^t, \tag{1}$$

for $t \neq n \cdot T$, $n \in \mathbb{N}$. Therefore, although the concepts in Table 1 may appear identical, there exists an important difference between the first and second approach of Clubb and Doran (1995) for $T \notin \{0, 1, \infty\}$. If the initial approach of Clubb and Doran (1995) was transferred to a steady state, debt levels would not grow at the constant growth rate g. For this reason, Arnold et al. (2018, 2019) and Dierkes and de Maeyer (2020) used the second approach of Clubb and Doran (1995) for their analysis of a steady state. While Arnold et al. (2018) examined a steady state without growth, Arnold et al. (2019) and Dierkes and de Maeyer (2020) included growth and allowed the debt levels to grow at the same constant growth rate as the FCFs within every planning phase. We build our analysis upon these results and use the second approach of Clubb and Doran (1995) with a constant growth of debt levels within planning phases.

To derive valuation equations for discontinuous financing on a clear theoretical basis, in a first step, we do not consider the costs of financial distress or the possibility of default. This is common in comparable basic analyses for other financing

strategies (see, e.g., Miles and Ezzell 1985). It follows that debt is risk-free and can be discounted at the risk-free interest rate r, which is constant over time. In a second step, consequences and relaxations of this assumption are discussed in Sect. 3.5. Moreover, in the presented valuation equations, we do not consider an explicit forecast phase, which means that the steady state phase starts at the valuation date. A detailed analysis on how to link standard discontinuous financing to an explicit forecast phase with passive debt management can be found in Dierkes and de Maeyer (2020).

Thus far, to investigate the market value of a firm under discontinuous financing, the adjusted present value (APV) approach has been theoretically analyzed. As for all financing strategies, it is instructive to also analyze other discounted cash flow (DCF) methods (for DCF approaches, see, e.g., Kruschwitz and Löffler 2020). We start by citing existing results, which we then use to analyze the adjustment formula for the cost of equity in Sect. 2.2, which is required for the FCF and the Flow to Equity (FtE) approach. Dierkes and de Maeyer (2020) used a recursive procedure to derive the valuation equation for a steady state and derived for the terminal value (Dierkes and de Maeyer 2020, p. 1327)

$$V_0^{\ell} = \sum_{t=1}^{T} \frac{\mathbb{E}[\widetilde{FCF}_1] \cdot (1+g)^{t-1}}{(1+\rho^u)^t} + \sum_{t=1}^{T} \frac{\tau \cdot r \cdot \theta \cdot V_0^{\ell} \cdot (1+g)^{t-1}}{(1+r)^t} + \frac{\mathbb{E}[\widetilde{V}_T^{\ell}]}{(1+\rho^u)^T}\,, \quad (2)$$

where τ denotes the corporate tax rate and ρ^u the cost of equity of an unlevered firm. Superscript u indicates the unlevered firm. The market value of the firm in the first planning phase is computed in the first and second terms by calculating the values of the unlevered firm and of the tax shields separately. As tax shields are certain over the planning phase, they can be discounted using the risk-free interest rate. The value of the levered firm at the beginning of the second planning phase is added in the third term. Dierkes and de Maeyer (2020, pp. 1327–1328) used the relation $\mathbb{E}[\widetilde{V}_T^{\ell}] = V_0^{\ell} \cdot (1+g)^T$, solved the circularity problem, and deduced the expression

$$V_0^{\ell} = \frac{\mathbb{E}[\widetilde{FCF}_1] \cdot \mathrm{PVA}(\rho^u, g, T)}{1 - \tau \cdot r \cdot \theta \cdot \mathrm{PVA}(r, g, T) - \frac{(1+g)^T}{(1+\rho^u)^T}}\,, \quad (3)$$

where

$$\mathrm{PVA}(k, g, T) = \frac{1}{k-g} \cdot \left(1 - \frac{(1+g)^T}{(1+k)^T}\right), \quad (4)$$

for $k \in \{\rho^u, r\}$, determines the present value of a growing annuity. Therefore, there exists a valuation equation for discontinuous financing that can be used to calculate the terminal value at the valuation date without circularity problems.

2.2 Cost of equity derivation

A thorough analysis of a financing strategy does not only consist of a valuation formula based on the APV approach but includes also an analysis of the levered

cost of equity. In particular, an adjustment formula for the levered cost of equity is necessary to apply other DCF approaches. Therefore, in this section, we first extend existing research by calculating the market value of a firm within a planning phase. Second, we use this result to derive an adjustment formula for period-specific levered costs of equity. Moreover, this analysis functions as a theoretical foundation for the extension of discontinuous financing in Sect. 3.

We identified that financial risk is inconstant under discontinuous financing, which yields inconstant levered costs of equity. Furthermore, since the debt-to-market value ratios vary with the development of the firm, they generally are random variables. Consequently, to calculate the levered cost of equity, we need to determine the market value of the firm between refinancing dates. As in Eq. (2), we compute the market values of the firm in the first planning phase by partitioning it into the planning phase value and the value of the firm at the end of the planning phase. For $t \in \{0, \dots, T-1\}$, the market value of the firm is given by

$$\begin{aligned}\mathbb{E}[\widetilde{V}_t^{\ell}] &= \sum_{k=1}^{T-t} \frac{\mathbb{E}[\widetilde{FCF}_1] \cdot (1+g)^{t+k-1}}{(1+\rho^u)^k} + \sum_{k=1}^{T-t} \frac{\tau \cdot r \cdot \theta \cdot V_0^{\ell} \cdot (1+g)^{t+k-1}}{(1+r)^k} \\ &\quad + \frac{\mathbb{E}[\widetilde{V}_T^{\ell}]}{(1+\rho^u)^{T-t}} \\ &= \mathbb{E}[\widetilde{FCF}_1] \cdot (1+g)^t \cdot \mathrm{PVA}(\rho^u, g, T-t) + \tau \cdot r \cdot \theta \cdot V_0^{\ell} \\ &\quad \cdot (1+g)^t \cdot \mathrm{PVA}(r, g, T-t) + V_0^{\ell} \cdot \frac{(1+g)^T}{(1+\rho^u)^{T-t}}.\end{aligned} \tag{5}$$

The market value of the firm of the current planning phase can be divided into the value of the unlevered firm and that of the tax shields. The planning phases consist of T periods, which implies that the firm will refinance in $T-t$ periods. The computation of the tax shield includes the market value of the firm at the valuation date. This value is known in period t so that we can use the risk-free interest rate as discount rate. The market value of the firm at the end of the first planning phase is added in the third term. The equity value can be derived by subtracting the debt level of period t, that is,

$$\mathbb{E}[\widetilde{E}_t^{\ell}] = \mathbb{E}[\widetilde{V}_t^{\ell}] - \theta \cdot V_0^{\ell} \cdot (1+g)^t. \tag{6}$$

Note that $t = 0$ results in valuation Eq. (2) for the levered firm at the valuation date. By the following proposition, we derive an adjustment formula for the levered cost of equity and the weighted average cost of capital (WACC).

Proposition 1 *Under the assumption of discontinuous financing, the levered cost of equity is given by*

$$\rho_t^{\ell} = \rho^u + (\rho^u - r) \cdot (1 - \tau \cdot r \cdot \mathrm{PVA}(r, g, T-t)) \cdot L_t\,, \tag{7}$$

where

$$L_t = \frac{D_t}{\mathbb{E}[\widetilde{E}_t^{\ell}]} \tag{8}$$

is the leverage in period t for $t \in \{0, \ldots, T-1\}$. *Furthermore, the WACC is given by*

$$k_t^{\tau} = (1 - \tau \cdot r \cdot \theta_t \cdot \mathrm{PVA}(r, g, T-t)) \cdot \rho^u - \tau \cdot r \cdot \theta_t \cdot (1 - r \cdot \mathrm{PVA}(r, g, T-t)) \,, \tag{9}$$

where $\theta_t = \frac{D_t}{\mathbb{E}[\widetilde{V}_t^{\ell}]}$ *is the debt-to-market value ratio in period t.*

Note that the leverage is defined as a quotient of a deterministic quantity and an expectation of a random variable. Thereby, we use a similar definition of the leverage as for passive debt management. However, due to Jensen's inequality, the leverage is generally not equal to the expression $\mathbb{E}\left[\frac{D_t}{\widetilde{E}_t^{\ell}}\right]$ since the equity value is a random variable. The same argumentation holds for the debt-to-market value ratio.

In the proof of Proposition 1, we derive the WACC by using the definition of the cost of capital. We define the cost of capital as the amount by which the expected value of the sum of the FCF and the market value of the firm at the end of the period is to be discounted to obtain the expected market value of the firm at the beginning of the period:

$$k_t^{\tau} = \frac{\mathbb{E}[\widetilde{FCF}_1] \cdot (1+g)^t + \mathbb{E}[\widetilde{V}_{t+1}^{\ell}]}{\mathbb{E}[\widetilde{V}_t^{\ell}]} - 1. \tag{10}$$

Note that the expectations are not conditioned on the level of information, that is, we consider the cost of capital at the valuation date. Subsequently, deducing the cost of equity is straightforward. A detailed computation is provided in the Appendix.

Proposition 1 shows that the formula for the derivation of the costs of equity under discontinuous financing has a similar structure as that for the levered cost of equity under other financing strategies (for the adjustment formulas under active or passive debt management, see, e.g., Miles and Ezzell 1985; Inselbag and Kaufold 1997; Kruschwitz and Löffler 2020). The first term is the unlevered cost of equity and depicts operational risk. To consider the financial risk that results from debt financing, a risk premium is added, which depends on the difference between the unlevered cost of equity and the risk-free interest rate. The term $\tau \cdot r \cdot \mathrm{PVA}(r, g, T-t)$ considers that the tax shields are certain until the end of the planning phase, that is, for $T-t$ periods. The risk premium increases linearly with leverage. For $T = 1$, the formula simplifies to the adjustment formula for active debt management of Miles and Ezzel [see Inselbag and Kaufold 1997, Eq. (10)] and $T \to \infty$ results in the adjustment formula for passive debt management [see Inselbag and Kaufold 1997, Eq. (7)]

$$\rho_t^{\ell} = \rho^u + (\rho^u - r) \cdot \frac{D_t - VTS_t}{\mathbb{E}[\widetilde{E}_t]}. \tag{11}$$

Following the line of Inselbag and Kaufold (1997) and adapting the adjustment formula of passive debt management is another possibility to derive the adjustment formula of Proposition 1. In a steady state with passive debt management, all future tax shields are certain such that the value of the tax shield is calculated as $VTS_t = \frac{\tau \cdot r \cdot D_t}{r-g}$. For discontinuous financing, the tax shield is only certain for the next $T-t$ periods. The value of the risk-free part of the tax shields VTS_t^R, that is, the part that can be discounted at the risk-free interest rate, amounts to

$$\begin{aligned} VTS_t^R &= \tau \cdot r \cdot \theta \cdot V_0^{\ell} \cdot (1+g)^t \cdot \mathrm{PVA}(r, g, T-t) \\ &= \tau \cdot r \cdot D_t \cdot \mathrm{PVA}(r, g, T-t)\,, \end{aligned} \tag{12}$$

see Eq. (5). The remaining part of the value of the tax shield is uncertain and, therefore, discounted at the unlevered cost of equity (Miles and Ezzell 1985). The value of the risk-free part of the tax shield lowers the risk premium that is added to the business risk in the formula for the levered cost of equity. We conclude

$$\rho_t^{\ell} = \rho^u + (\rho^u - r) \cdot \frac{D_t - VTS_t^R}{\mathbb{E}[\widetilde{E}_t]}. \tag{13}$$

Plugging VTS_t^R, see Eq. (12), into Eq. (13) yields the adjustment formula from Eq. (7). It follows that the basic structure of the adjustment formula does not change, but the share of the risk-free tax shield is adjusted (for similar observations, see Inselbag and Kaufold 1997). The factor that displays the risk-free part of the tax shield can be expressed in percentage of V_0^{ℓ}.

Similar observations can be conducted for the formula of the WACC. In the case of passive debt management, the WACC can be computed as [Inselbag and Kaufold 1997, Eq. (8)]

$$k_t^{\tau} = \rho^u \cdot \left(1 - \frac{VTS_t}{V_t^{\ell}}\right) + r \cdot \frac{VTS_t - \tau \cdot D_t}{V_t^{\ell}}. \tag{14}$$

To derive the WACC for discontinuous financing we can again replace the value of the tax shield VTS_t by the risk-free part of the value of the tax shield VTS_t^R, see Eq. (12), to obtain the WACC from Eq. (9).

At $t = T-1$, the firm is in the last period of the first planning phase and will refinance in the next period. Therefore, the tax shield is only certain for the subsequent period, which complies with an active debt management according to ME. Note that Eq. (7) and (9) indeed simplify to the cost of equity and WACC under active debt management with leverage L_{T-1} for $t = T-1$, respectively (Miles and Ezzell 1985). At the valuation date, that is, $t = 0$, the leverage L_0 complies with the specified ratio $L = \frac{\theta}{1-\theta}$. It follows that the cost of equity is exceptionally a deterministic variable during this period. Furthermore, for $t = 0$, Eq. (7) is in line with the formula for unlevering β in Arnold et al. (2018). However, in their analysis, it remains unclear how to adjust β or the cost of equity in the other periods of a planning phase. Insofar, we extend the study of Arnold et al. (2018) by analyzing periodic-specific costs of equity and by deriving the corresponding adjustment formulas.

As outlined above, the debt-to-market value ratio in period t differs from the specified ratio θ, and the adjusted leverage L_t needs to be used. This ratio depends on the equity value at time t. It follows that the computation of both the levered cost of equity and the WACC contains a circularity problem. They cannot be computed without determining first the debt level and expected market value of the firm at time t. However, solving circularity problems is a common and recurring procedure in corporate valuation practice. It is usually addressed using a spreadsheet software.

Since we derived a closed-form solution for the levered costs of equity and corresponding WACCs within the first planning phase, we can compute the levered costs of equity for all periods. We have shown that financial risk coincides for periods in the same section of a planning phase, which implies $\rho^{\ell}_{n\cdot T+t} = \rho^{\ell}_{t}$ for $n \in \mathbb{N}$.

2.3 Example

To illustrate the findings of the previous subsections, we present an example. We assume a steady state in which the input parameters grow at a constant rate of $g = 1.5\%$. The FCF of period one is 1000, and the debt-to-market value ratio is $\theta = 60\%$.[1] The risk-free interest rate, unlevered cost of equity, and tax rate amount to $r = 4\%$, $\rho^{u} = 10\%$, and $\tau = 30\%$, respectively. We use Eq. (3) to compute the market value of the firm V^{ℓ}_0 and deduce

$$V^{\ell}_0 = \frac{1000 \cdot 3.90}{1 - 30\% \cdot 4\% \cdot 60\% \cdot 4.58 - \frac{(1+1.5\%)^5}{(1+10\%)^5}} = 13,066.70.$$

By multiplying this value by the debt-to-market value ratio, we obtain the debt level $D_0 = 7840.02$, and by subtracting this from the market value of the firm, we obtain the equity value $E^{\ell}_0 = 5226.68$ at the valuation date. Given that the firm is situated in a steady state, we assume that the debt level grows at the constant growth rate of $g = 1.5\%$. To calculate the market values of the firm within the first planning phase, we use Eq. (5). In period five, refinancing is carried out so that the financial risk for this period coincides with the financial risk at the valuation date. Therefore, we obtain

$$\mathbb{E}[\widetilde{E}^{\ell}_5] = E^{\ell}_0 \cdot (1+g)^5 = 5226.68 \cdot (1+1.5\%)^5 = 5630.62.$$

Furthermore, by applying Eq. (8), we obtain the debt-to-market value ratios of periods 1 to 4, and, by using Eq. (7), we then derive the levered costs of equity.

Although the differences in the costs of equity are small, this example illustrates that financial risk differs depending on the remaining number of periods until the next refinancing date. At a refinancing date, the tax shield is certain for the next five

[1] The calculations can be easily adjusted if the debt level, rather than the debt-to-market value ratio, is defined deterministically. Then the debt-to-market value ratio is obtained by dividing the debt level by the market value of the firm. This ratio is used in the subsequent periods, see Dierkes and de Maeyer (2020) for further explanations.

Table 2 Illustration of discontinuous financing

	$t=0$	$t=1$	$t=2$	$t=3$	$t=4$	$t=5$
$\mathbb{E}[\widetilde{FCF}_t]$		1000.00	1015.00	1030.23	1045.68	1061.36
$\mathbb{E}[\widetilde{V}_t^{\ell}]$	13, 066.70	13, 253.43	13, 447.02	13, 648.20	13, 857.75	14, 076.55
D_t	7840.02	7957.62	8076.99	8198.14	8321.11	8445.93
$\mathbb{E}[\widetilde{E}_t^{\ell}]$	5226.68	5295.81	5370.04	5450.06	5536.64	5630.62
$\mathbb{E}[\widetilde{TS}_t]$		94.08	95.49	96.92	98.38	99.85
$\mathbb{E}[\widetilde{TCF}_t]$		1094.08	1110.49	1127.15	1144.06	1161.22
$\mathbb{E}[\widetilde{VTS}_t]$	1302.00	1312.25	1326.73	1346.10	1371.12	1402.62
VTS_t^R	431.08	354.24	272.92	186.91	96.01	464.40
θ_t	60.00%	60.04%	60.07%	60.07%	60.05%	60.00%
ρ_t^{ℓ}	18.51%	18.61%	18.72%	18.82%	18.91%	18.51%
g_t		1.32%	1.40%	1.49%	1.59%	1.70%

periods so that the cost of equity is lower than that in the subsequent period in which the tax shield is only certain for the next four periods. In the fourth period, financial risk is maximal, since only the tax shield of the subsequent period is certain. Furthermore, note that

$$E_0^{\ell} \cdot (1+g) = 5,226.68 \cdot (1+1.5\%) = 5,305.08 \neq 5,295.81 = \mathbb{E}[\widetilde{E}_1^{\ell}].$$

It follows that the equity value, and therefore the market value of the firm, grows at a smaller rate than $g = 1.5\%$ from period 0 to period 1, since financial risk increases. In terms of the equity value, this growth rate amounts to $g_1 = 1.32\%$. Conversely, for example, from period 4 to period 5, financial risk decreases so that the equity value grows at a higher rate, $g_5 = 1.70\%$. However, from period 0 to period 5, from period 1 to period 6, and so on, the growth rate is exactly $g = 1.5\%$ for both the market value of the firm and equity value. Since the growth rate is inconstant, the debt-to-market value ratio varies. On the one hand, at the beginning of a planning phase, the debt levels grow faster than the market value of the firm, which yields an increase in the debt-to-market value ratio. On the other hand, at the end of a planning phase, the market value of the firm grows faster than the debt levels, resulting in a decreasing debt-to-market value ratio. In period 5, the second planning phase starts, and the structure is repeated.

In Table 2, we also included the expected tax shield $\mathbb{E}[\widetilde{TS}_t]$, expected total cash flow $\mathbb{E}[\widetilde{TCF}_t]$ (as the sum of FCF and tax shield), expected value of the tax shield $\mathbb{E}[\widetilde{VTS}_t]$, and risk-free part of the value of the tax shield VTS_t^R. As outlined in the previous subsection, the latter can alternatively be used to compute the levered cost of equity.

This example illustrates that, in a steady state under discontinuous financing, financing risk is inconstant, which yields inconstant costs of equity and market value fluctuations. In summary, the operating, investing, and financing activities yield a constant growth of every relevant quantity, but the financing activities still do not

result in a constant financial risk. This setting is the result of the sole and, therefore, big refinancing every T periods. It follows that discontinuous financing provides an opportunity to depict a broad range of financing behaviors of firms, but might still not come close to the real financing behavior. It might be more practical and more realistic to adjust a certain part of the debt level in every period. After every T periods, the entire debt level has still been adjusted, but the refinancing is partitioned into several periods. This motivates the analysis of debt categories.

3 Discontinuous financing with debt categories

3.1 Derivation of a valuation formula for two debt categories

In this section, we introduce a modification of discontinuous financing as follows. We consider a firm that has various debt categories, which are adjusted successively. In each period, some debt category is adjusted by multiplying the debt-to-market value ratio by the market value of the firm, and no other category is adjusted. Since we examine this financing strategy in a steady state, these other debt categories grow at the same constant growth rate g as the FCF. In the subsequent period, another category is adjusted according to the updated market value of the firm, and so on. Consequently, at each point in time, the debt categories reflect shares of the overall debt that have different remaining maturities. We obtain an overlapping sequence of bonds with an identical time to maturity, each of which is prolonged at an adjusted level every year. Therefore, instead of consecutive planning phases, this financing strategy incorporates overlapping planning phases and includes the maturity of debt. In particular, active and passive debt management is mixed in every period.[2] The successive adjustment of proportions of the overall debt in each period seems more practical than the sole adjustment of the entire debt level at the valuation date or at the beginning of a planning phase. It follows that the assumption of discontinuous financing with debt categories might come closer to a firm's financing behavior as opposed to the assumption of standard discontinuous financing.

In this subsection, we examine a steady state of a firm that has two debt categories. Thus, compared to Miles and Ezzell (1980), we introduce one additional layer of debt. If θ is the pursued debt-to-market value ratio, we define $\theta^{(2)} := \frac{1}{2}\theta$. Superscript (2) refers to the number of debt categories. Furthermore, let D_t^j be the amount of debt in category $j \in \{0, 1\}$ over period t. Category 0 is adjusted in period $0, 2, 4, \ldots$, and category 1 is adjusted in period $1, 3, 5, \ldots$. Figure 2 illustrates the concept of two debt categories.

[2] A mix of active and passive debt management in every period can also be found in Dierkes and Schäfer (2017). In their study, the mix of active and passive debt management can be arbitrarily determined in each period. Furthermore, they define a deterministic part of the overall debt for all future periods at the valuation date, which is different from our study. We adjust the deterministic part successively according to the debt-to-market value ratio.

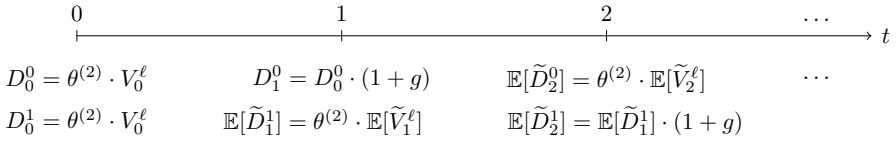

Fig. 2 Discontinuous financing with two debt categories

At the valuation date, the entire debt level has to be specified, but the part D_0^1 should have been defined in the previous period. If the assumption of debt categories is used in a two-phase model and an explicit forecast phase is planned before the steady state, the debt level that results from an explicit planning could be used. Since we want to concentrate on the valuation formulas for the steady state phase, we exclude this detailed planning in our derivations and discuss possible links in Sect. 3.4. It follows that, at the valuation date, the firm exceptionally adjusts both debt categories, that is 0 and 1, according to $\theta^{(2)}$, which yields $D_0 = \theta \cdot V_0^\ell$. The expected total amount of debt, $\mathbb{E}[\widetilde{D}_t]$, at some time $t \neq 0$ can be derived by adding the two debt categories:

$$\mathbb{E}[\widetilde{D}_t] = \theta^{(2)} \cdot \mathbb{E}[\widetilde{V}_{t-1}^\ell] \cdot (1+g) + \theta^{(2)} \cdot \mathbb{E}[\widetilde{V}_t^\ell]\,. \tag{15}$$

We use a backward inductive approach to derive the valuation formula, which is similar to the approaches of Miles and Ezzell (1980), Inselbag and Kaufold (1997) and Dierkes and Schäfer (2017). First, we assume a finite time horizon of $T < \infty$ periods to derive valuation equations. Afterwards, we extend the formulas for $T \to \infty$. We start with the valuation formula in period $T-1$. To calculate the value of the levered firm, we use the concept of value-additivity and compute the values of the unlevered firm and of the tax shield separately. We derive the former by discounting the firm's expected FCF at time T at the unlevered cost of equity. The value of the tax shield can be computed by discounting the tax savings due to both debt categories. In period $T-1$, the firm alters one of its debt categories. The other has already been adjusted in period $T-2$ and grows at the constant growth rate g. Therefore, the value of the levered firm in period $T-1$ depends on the market value of the firm in period $T-2$. Since the market value of the firm in periods $T-2$ and $T-1$ is certain in period $T-1$, the amount of debt for both categories is certain, and it can be discounted at the risk-free interest rate r. Again, we begin with a theoretical framework and assume risk-free debt to concentrate on the derivation of the valuation equations and adjustment formulas. For a discussion of the integration of the risk of default, we refer to Sect. 3.5. We obtain

$$V_{T-1}^\ell = \frac{\mathbb{E}_{T-1}[\widetilde{FCF}_T]}{1+\rho^u} + \frac{\tau \cdot r \cdot \theta^{(2)} \cdot V_{T-2}^\ell \cdot (1+g)}{1+r} + \frac{\tau \cdot r \cdot \theta^{(2)} \cdot V_{T-1}^\ell}{1+r}\,. \tag{16}$$

Solving the circularity (i.e., solving for the value of the levered firm in period $T-1$) yields

$$V_{T-1}^{\ell} = \frac{\mathbb{E}_{T-1}[\widetilde{FCF}_T]}{(1+\rho^u)\cdot\eta_1^{(2)}} + \frac{\tau\cdot r\cdot\theta^{(2)}\cdot V_{T-2}^{\ell}\cdot(1+g)}{(1+r)\cdot\eta_1^{(2)}}, \tag{17}$$

where

$$\eta_1^{(2)} := 1 - \frac{\tau\cdot r\cdot\theta^{(2)}}{1+r}. \tag{18}$$

To derive the value of the levered firm in period $T-2$, we consider that one debt category has been adjusted in period $T-3$. Furthermore, we have to include the value of the levered firm in period $T-1$. Therefore, the market value of the firm in period $T-2$ is

$$\begin{aligned} V_{T-2}^{\ell} = {} & \frac{\mathbb{E}_{T-2}[\widetilde{FCF}_{T-1}]}{1+\rho^u} + \frac{\tau\cdot r\cdot\theta^{(2)}\cdot V_{T-3}^{\ell}\cdot(1+g)}{1+r} + \frac{\tau\cdot r\cdot\theta^{(2)}\cdot V_{T-2}^{\ell}}{1+r} \\ & + \frac{\mathbb{E}_{T-2}[\widetilde{V}_{T-1}^{\ell}]}{1+d}. \end{aligned} \tag{19}$$

The amount of debt for both debt categories in period $T-2$ is again certain, and it can be discounted at the risk-free interest rate r. Since the appropriate discount rate, d, of the value of the levered firm in period $T-1$ is not apparent, we can again apply the value-additivity principle and divide the last term into its components to obtain

$$\begin{aligned} V_{T-2}^{\ell} = {} & \frac{\mathbb{E}_{T-2}[\widetilde{FCF}_{T-1}]}{1+\rho^u} + \frac{\tau\cdot r\cdot\theta^{(2)}\cdot V_{T-3}^{\ell}\cdot(1+g)}{1+r} + \frac{\tau\cdot r\cdot\theta^{(2)}\cdot V_{T-2}^{\ell}}{1+r} \\ & + \frac{\mathbb{E}_{T-2}[\widetilde{FCF}_T]}{(1+\rho^u)^2\cdot\eta_1^{(2)}} + \frac{\tau\cdot r\cdot\theta^{(2)}\cdot V_{T-2}^{\ell}\cdot(1+g)}{(1+r)^2\cdot\eta_1^{(2)}}. \end{aligned} \tag{20}$$

The FCF reflects the cash flow of an unlevered firm and can, thus, be discounted using the unlevered cost of equity. Since the market value of the firm in period $T-2$ is certain, we discount it at the risk-free interest rate r. Solving for the market value of the firm in period $T-2$ yields

$$V_{T-2}^{\ell} = \frac{\mathbb{E}_{T-2}[\widetilde{FCF}_{T-1}]}{(1+\rho^u)\cdot\eta_2^{(2)}} + \frac{\mathbb{E}_{T-2}[\widetilde{FCF}_T]}{(1+\rho^u)^2\cdot\eta_2^{(2)}\cdot\eta_1^{(2)}} + \frac{\tau\cdot r\cdot\theta^{(2)}\cdot V_{T-3}^{\ell}\cdot(1+g)}{(1+r)\cdot\eta_2^{(2)}}, \tag{21}$$

where

$$\eta_2^{(2)} := 1 - \frac{\tau\cdot r\cdot\theta^{(2)}}{1+r} - \frac{\tau\cdot r\cdot\theta^{(2)}\cdot(1+g)}{(1+r)^2\cdot\eta_1^{(2)}}. \tag{22}$$

With these calculations, a general formula can be deduced for the sequence $(\eta_k^{(2)})_{k\in\mathbb{N}}$. For $k>1$, we define

$$\eta_k^{(2)} := 1 - \frac{\tau \cdot r \cdot \theta^{(2)}}{1+r} - \frac{\tau \cdot r \cdot \theta^{(2)} \cdot (1+g)}{(1+r)^2 \cdot \eta_{k-1}^{(2)}}. \tag{23}$$

This sequence considers that the tax shield of a specific period depends not only on the debt level of this period but also on thatof the previous period. To derive a valuation equation for a perpetuity, we need to show that this sequence converges to a limit $\eta^{(2)}$ for $k \to \infty$, which we do in Corollary 1, see the Appendix.

From these results, we can derive a valuation formula for the levered firm at the valuation date. Since we exclude an explicit planning of debt levels, we use $D_0^1 = \theta^{(2)} \cdot V_0^{\ell}$ and deduce a valuation formula for the market value of the firm for a perpetuity.

Proposition 2 *If $g < k^*$, the market value of the firm for two debt categories is given by*

$$V_0^{\ell} = \frac{E[\widetilde{FCF}_1]}{k^* - g} + \frac{\tau \cdot r \cdot D_0^1}{1+r^*} = \frac{E[\widetilde{FCF}_1]}{k^* - g} \cdot \left(1 - \frac{\tau \cdot r \cdot \theta^{(2)}}{1+r^*}\right)^{-1}, \tag{24}$$

where $k^* := (1+\rho^u) \cdot \eta^{(2)} - 1$ and $r^* := (1+r) \cdot \eta^{(2)} - 1$.

Proposition 2 exemplifies that the FCFs are discounted at the adjusted cost of capital, k^*, which we deduced by solving the emerging circularity problems. We add the tax shield that results from the amount of debt D_0^1 of the category that is again adapted in period 1. By using the relation $D_0^1 = \theta^{(2)} \cdot V_0^{\ell}$, we can rewrite the expression and deduce a circularity-free valuation formula for two debt categories. Multiplying the market value of the firm by the debt-to-market value ratio yields the equity value. For a detailed derivation of Eq. (24), see the Proof of Proposition 2 in the Appendix.

As mentioned in Sect. 2, in corporate valuation practice, it is common to use the FCF or the FtE approach. To do so, we need to derive an adjustment formula for the levered cost of equity and deduce the WACC to be able to apply a valuation formula of the form

$$V_0^{\ell} = \frac{\mathbb{E}[\widetilde{FCF}_1]}{k^{\tau} - g}, \tag{25}$$

with $k^{\tau} = (1-\theta) \cdot \rho^{\ell} + r \cdot (1-\tau) \cdot \theta$. The adjustment formula and the expression for the WACC are captured in the following proposition.

Proposition 3 *Under the assumption of two debt categories, the levered cost of equity can be obtained by*

$$\rho^{\ell} = \rho^u + (\rho^u - r) \cdot \left(1 - \frac{\tau \cdot r}{1+r} - \frac{1}{2} \cdot \frac{\tau \cdot r \cdot (1+g)}{(1+r)^2 \cdot \eta^{(2)}}\right) \cdot L, \tag{26}$$

where $L = \frac{\theta}{1-\theta}$ is the leverage. Furthermore, the WACC is given by

$$k^{\tau} = \rho^{u} - \tau \cdot r \cdot \theta \cdot \frac{(1+\rho^{u})}{1+r} - (\rho^{u} - r) \cdot \frac{1}{2} \cdot \frac{\tau \cdot r \cdot \theta \cdot (1+g)}{(1+r)^2 \cdot \eta^{(2)}}. \tag{27}$$

A proof is provided in the Appendix. Proposition 3 clarifies that the derivation of the levered cost of equity can again be divided into the operational risk ρ^{u} and a risk premium that depends on the difference of the unlevered cost of equity and the risk-free interest rate. This difference is multiplied by a factor that incorporates the financial risk due to both debt categories. Compared to the adjustment formula for active debt management according to HP, where all tax shields are uncertain and this factor equals 1, we subtract two terms to depict the smaller financial risk. Both debt categories are certain in the subsequent period, which corresponds to active debt management of ME and yields the subtraction of $\frac{\tau \cdot r}{1+r}$. Additionally, half of the debt level is certain in the period after next; that is, it can be discounted at the risk-free interest rate for two periods. The effect of this additional certainty of the tax shield is reflected in the term $\frac{1}{2} \frac{\tau \cdot r \cdot (1+g)}{(1+r)^2 \cdot \eta^{(2)}}$. The WACC is also similar to the WACC for active debt management (for the WACC of active debt management, see e.g., Miles and Ezzell 1980, Eq. (20); Kruschwitz and Löffler 2020, p. 105). In addition to the one half of the debt that is adjusted in the next period, we must consider the effects of the other debt category.

For a comparison to passive debt management, we compute the value of the risk-free part of the tax shields at the valuation date, as we have done for standard discontinuous financing. It can be derived by similar recursive steps as above, which yields

$$VTS_0^{R} = \frac{\tau \cdot r \cdot \theta \cdot V_0^{\ell}}{1+r} + \frac{1}{2} \cdot \frac{\tau \cdot r \cdot \theta \cdot V_0^{\ell} \cdot (1+g)}{(1+r)^2 \cdot \eta^{(2)}} \tag{28}$$

It displays that the entire debt level is certain for one period and half of the debt level is certain for two periods. Inserting this into the adjustment formula derived by Inselbag and Kaufold (1997) see Eq. (13), yields the adjustment formula from Eq. (26). This alternative derivation of the adjustment formula for the cost of equity highlights that the value of the tax shield compared to passive debt management is again decreased by a factor that depends on V_0^{ℓ}.

The same holds for the WACC compared to the WACC under passive debt management, see Eq. (14). It becomes clearer, if we rearrange Eq. (27) to

$$\begin{aligned} k^{\tau} = \rho^{u} \cdot \left(1 - \frac{\tau \cdot r \cdot \theta}{1+r} - \frac{1}{2} \cdot \frac{\tau \cdot r \cdot \theta \cdot (1+g)}{(1+r)^2 \cdot \eta^{(2)}}\right) \\ + r \cdot \left(\frac{\tau \cdot r \cdot \theta}{1+r} + \frac{1}{2} \cdot \frac{\tau \cdot r \cdot \theta \cdot (1+g)}{(1+r)^2 \cdot \eta^{(2)}} - \tau \cdot \theta\right). \end{aligned} \tag{29}$$

The value of the risk-free part of the tax shields, see Eq. (28) can again be inserted into the formula for the WACC, see Eq. (14), to derive the expression from Eq. (29).

Note that the levered cost of equity is, generally, a random variable. In each period, the debt-to-market value ratio of the debt category that is not adjusted depends on the development of the firm. An exception displays the debt-to-market

value ratio at the valuation date since, in this period, both debt categories are adjusted. However, the expected financial risk is constant for all periods. In every period, half of the tax shield is certain for two periods and the other half is certain for one period. Consequently, the construction of debt categories is similar to that of standard discontinuous financing. The difference is that by partially adjusting the debt level, we obtain a financial risk that does not vary depending on the remaining number of periods in the planning phase. It follows that this financing policy could be more suitable to model the financing behavior and financial risk of a firm. Since we were able to derive a closed-form solution for the levered cost of equity, the market value of the firm for two debt categories can be easily calculated using the Gordon–Shapiro formula (Gordon and Shapiro 1956).

3.2 Derivation of a valuation formula for an arbitrary number of debt categories

While the previous analysis was based on a firm with two debt categories, it is now of interest to derive a valuation formula for a firm with various debt categories. Their number can be company specific. First, we consider three debt categories. In some period t, one category is adjusted according to the debt-to-market value ratio $\theta^{(3)} := \frac{1}{3}\theta$, and the amount of debt for the other two debt categories depends on the market value of the firm in periods $t-1$ and $t-2$, respectively. At the valuation date, we again assume that categories 1 and 2 are exceptionally adjusted according to the specified ratio $\theta^{(3)}$.[3] The expected total amount of debt in some period $t \geq 2$ is

$$\mathbb{E}[\widetilde{D}_t] = \theta^{(3)} \cdot \mathbb{E}[\widetilde{V}^{\ell}_{t-2}] \cdot (1+g)^2 + \theta^{(3)} \cdot \mathbb{E}[\widetilde{V}^{\ell}_{t-1}] \cdot (1+g) + \theta^{(3)} \cdot \mathbb{E}[\widetilde{V}^{\ell}_{t}]\,. \quad (30)$$

It follows that, to derive a valuation formula for this setting, the sequence $(\eta_k^{(2)})_{k\in\mathbb{N}}$ needs to be adjusted. By repeating the above backward iteration for three debt categories, we obtain

$$\begin{aligned}
\eta_1^{(3)} &:= 1 - \frac{\tau \cdot r \cdot \theta^{(3)}}{1+r}\,,\\
\eta_2^{(3)} &:= 1 - \frac{\tau \cdot r \cdot \theta^{(3)}}{1+r} - \frac{\tau \cdot r \cdot \theta^{(3)} \cdot (1+g)}{(1+r)^2 \cdot \eta_1^{(3)}} \quad \text{and}\,,\\
\eta_k^{(3)} &:= 1 - \frac{\tau \cdot r \cdot \theta^{(3)}}{1+r} - \frac{\tau \cdot r \cdot \theta^{(3)} \cdot (1+g)}{(1+r)^2 \cdot \eta_{k-1}^{(3)}} - \frac{\tau \cdot r \cdot \theta^{(3)} \cdot (1+g)^2}{(1+r)^3 \cdot \eta_{k-1} \cdot \eta_{k-2}^{(3)}}\,,
\end{aligned} \quad (31)$$

for $k > 2$. This sequence considers that one third of the overall debt is certain only in the subsequent period, one part is certain for two periods, and one part is certain for three periods. The latter is displayed in the last term of $\eta_k^{(3)}$. Since deriving an analytic solution for the limit $\eta^{(3)}$ of the sequence $(\eta_k^{(3)})_{k\in\mathbb{N}}$ is difficult, a spreadsheet software or other programs should be used to calculate the limit numerically.

[3] In accordance with our analysis in the previous subsection, in a first step, we only consider the steady state phase and refer to Sect. 3.4 for possible links to the explicit forecast phase.

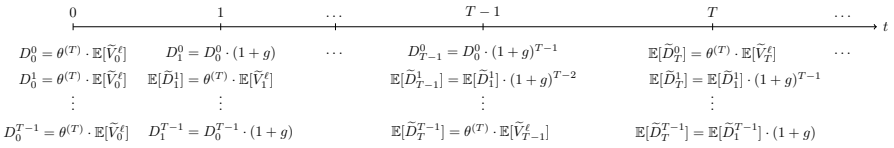

Fig. 3 Discontinuous financing with T debt categories

With these observations, we can now deduce a valuation formula for a firm with T debt categories. Let $\theta^{(T)} := \frac{1}{T}\theta$. We exclude a detailed planning of debt levels and adjust all debt levels according to $\theta^{(T)}$ at the valuation date. Thereafter, in period one, category one is adjusted and so on. For an illustration of T debt categories, see Fig. 3. It follows that the expected total amount of debt in some period $t \geq T-1$ is

$$\mathbb{E}[\widetilde{D}_t] = \sum_{s=0}^{T-1} \theta^{(T)} \cdot \mathbb{E}[\widetilde{V}_{t-s}^\ell] \cdot (1+g)^s. \tag{32}$$

In this setting, the total amount of debt in period t depends on the market values of the firm in periods $t-(T-1)$ to t. Accordingly, the sequence $(\eta_k^{(T)})_{k\in\mathbb{N}}$ must be derived. Following the structure for two and three debt categories, respectively, we define $\eta_1^{(T)} := 1 - \frac{\tau \cdot r \cdot \theta^{(T)}}{1+r}$. For $k > 1$, we derive

$$\eta_k^{(T)} := 1 - \frac{\tau \cdot r \cdot \theta^{(T)}}{1+r} - \sum_{t=1}^{\min\{k,T\}-1} \frac{\tau \cdot r \cdot \theta^{(T)} \cdot (1+g)^t}{(1+r)^{t+1} \cdot \prod_{s=1}^{t} \eta_{k-s}^{(T)}}. \tag{33}$$

The sequence $(\eta_k^{(T)})_{k\in\mathbb{N}}$ considers the dependencies of the debt levels on the market value of the firm of the previous $k-(T-1)$ periods.

As in the case of two debt categories, we can show that the sequence $(\eta_k^{(T)})_{k\in\mathbb{N}}$ converges to $\eta^{(T)} := \lim_{k\to\infty} \eta_k^{(T)}$ (see Lemma 3). For the limit holds

$$\eta^{(T)} := 1 - \tau \cdot r \cdot \theta^{(T)} \cdot \sum_{t=0}^{T-1} \frac{(1+g)^t}{(1+r)^{t+1} \cdot (\eta^{(T)})^t}. \tag{34}$$

We cannot derive a closed-form solution for this limit, but it can be computed using a spreadsheet software or other programs.

The general valuation formula for a perpetuity has the same structure as Eq. (24), where we assumed a perpetual annuity for two debt categories. If $g < k^*$, we can apply Lemma 2 and derive for an arbitrary number of debt categories

$$V_0^\ell = \frac{E[\widetilde{FCF}_1]}{k^* - g} + \sum_{s=1}^{T-1} \sum_{j=1}^{s} \frac{\tau \cdot r \cdot D_0^{T-j} \cdot (1+g)^{T-s-1}}{(1+r^*)^{T-s}}, \tag{35}$$

where $k^* := (1+\rho^u) \cdot \eta^{(T)} - 1$ and $r^* := (1+r) \cdot \eta^{(T)} - 1$. The expression D_0^{T-j}, $j \in \{1, \ldots, T-1\}$, represents the amount of debt of category $T-j$ that should not be adjusted in period 0, but in period $T-j$. These debt levels have to be exceptionally adjusted simultaneously at the valuation date. They are risk-free and can be

discounted at the risk-free interest rate. The value of the tax shield of these debt categories is computed in the second term. In period zero, these are $T-1$ categories; in period one, these are $T-2$ categories; until in period $T-1$, it is only one debt category. Thereafter, starting in period T, every debt category has been adjusted once according to the ratio $\theta^{(T)}$, see Fig. 3.

Since we exclude the link to an explicit planning of the debt levels, we adjust all debt categories according to the market value of the firm at the valuation date, that is, $D_0^j = \theta^{(T)} \cdot V_0^\ell$ for $j \in \{0, \ldots, T-1\}$. Plugging this in, we can simplify valuation Eq. (35) as we do in the following proposition.

Proposition 4 *If $g < k^*$, the market value of the firm for T debt categories can be computed by*

$$V_0^\ell = \frac{\mathbb{E}[\widetilde{FCF}_1]}{k^* - g} \cdot \left(1 - \frac{\tau \cdot r \cdot \theta^{(T)}}{1+r} \cdot \frac{(1+x)^t - Tx - 1}{x^2}\right)^{-1}, \tag{36}$$

where

$$x = \frac{1+g}{(1+r) \cdot \eta^{(T)}} - 1. \tag{37}$$

As in the case of two debt categories, the FCFs are discounted at the adjusted cost of capital k^*. Compared to Eq. (35), we inserted the relation $D_0^j = \theta^{(T)} \cdot V_0^\ell$ and solved the circularity problems to deduce a circularity-free valuation formula. Multiplying the market value of the firm by the debt-to-market value ratio yields the equity value. For detailed calculations, see the Proof of Proposition 4 in the Appendix.

To be able to apply the FCF approach in conjunction with the Gordon–Shapiro formula, see Eq. (25), we need to determine the levered cost of equity and the WACC. The formulas are captured in the following proposition.

Proposition 5 *Under the assumption of T debt categories, the levered cost of equity can be obtained by*

$$\rho^\ell = \rho^u + (\rho^u - r) \cdot \left(1 - \tau \cdot r \cdot \sum_{s=0}^{T-1} \left(1 - \frac{s}{T}\right) \cdot \frac{(1+g)^s}{(1+r)^{s+1} \cdot (\eta^{(T)})^s}\right) \cdot L \tag{38}$$

$$= \rho^u + (\rho^u - r) \cdot \left(1 - \frac{\tau \cdot r}{1+r} \cdot \frac{(1+x)^{T+1} - (T+1) \cdot x - 1}{T \cdot x^2}\right) \cdot L. \tag{39}$$

Furthermore, the WACC is given by

$$k^\tau = \rho^u - (\rho^u - r) \cdot \frac{\tau \cdot r}{1+r} \cdot \frac{(1+x)^{T+1} - (T+1) \cdot x - 1}{T \cdot x^2} \cdot \theta - r \cdot \tau \cdot \theta. \tag{40}$$

See the Appendix for a proof. While Eq. (39) can be used for computation, we can obtain a better interpretation from Eq. (38): We can again divide the derivation of the levered cost of equity into the operational risk ρ^u and a risk premium that depends on the difference of the unlevered cost of equity and the risk-free interest rate. The factor that incorporates the financial risk due to all debt categories can be interpreted as follows. Compared to active debt management according to HP, we have a reduced risk. The first term of the sum in Eq. (39) is $\frac{\tau \cdot r}{1+r}$ and reflects that the entire debt is certain in the subsequent period. The second term is $\left(1 - \frac{1}{T}\right) \cdot \frac{1+g}{(1+r)^2 \cdot \eta^{(T)}}$ and represents that the entire debt, except the part that is adjusted in the next period (i.e., a share of $1 - \frac{1}{T}$ of the overall debt), is certain for two periods. This continues until, in the last term, only $\frac{1}{T}$ of the overall debt is considered, which is the debt category that is defined in the current period, and is, therefore, certain for T periods.

To compare these equations to passive debt management, we compute the risk-free part of the value of the tax shield. It is

$$\begin{aligned} VTS_0^R &= \tau \cdot r \cdot \theta \cdot V_0^\ell \cdot \sum_{s=0}^{T-1} \left(1 - \frac{s}{T}\right) \cdot \frac{(1+g)^s}{(1+r)^{s+1} \cdot (\eta^{(T)})^s} \\ &= \tau \cdot r \cdot \theta \cdot V_0^\ell \cdot \frac{(1+x)^{T+1} - (T+1) \cdot x - 1}{T \cdot x^2 \cdot (1+r)}. \end{aligned} \tag{41}$$

Compared to passive debt management, the risk-free part of the tax shield is reduced. As explained above, only parts of the overall debt level are deterministic, which is expressed in the sum. Inserting Eq. (41) into Eqs. (13) and (14) is an alternative way to derive the adjustment formula for the cost of equity and the WACC, respectively.

With these findings, it becomes clear that the expected financial risk is constant. In each period, a proportion of $\frac{1}{T}$ of the overall debt is certain for T periods, another proportion is certain for $T-1$ periods, and so on. It follows that it is possible to use Eq. (25) to calculate the market value of the firm. Therefore, we have determined a formula for the levered cost of equity and the WACC, which is easy to apply to calculate the market value of a firm with an arbitrary number of debt levels with the Gordon–Shapiro formula. Note that, for $T = 2$, the formula simplifies to the above derived formulas for two debt categories (see Proposition 3). Furthermore, $T = 1$ complies with active debt management according to ME since the firm has only one debt category that is adjusted every period.

Compared to standard discontinuous financing, we considered the maturity of debt and constructed debt categories in a way that yields a partial adjustment of the debt level. Thus, discontinuous financing with T debt categories might be more suitable to model the financing behavior of a firm. For more than two debt categories, the limit η has to be computed numerically. However, with the help of a spreadsheet software or other programs, this does not pose a problem. Overall, the results allow for a deep theoretical understanding not only of this new financing strategy but also of standard discontinuous financing. The following subsection illustrates the approach of discontinuous financing with debt categories using an example.

Table 3 Market values of the firm under different financing strategies

Financing strategy	$t=0$	$t=1$	$t=2$	$t=3$	$t=4$	$t=5$
Unlevered firm	11,764.71	11,941.18	12,120.29	12,302.10	12,486.63	12,673.93
Debt categories	13,057.81	13,253.68	13,452.49	13,654.27	13,859.09	14,066.97
Discontinuous financing	13,066.70	13,253.43	13,447.02	13,648.20	13,857.75	14,076.55
Active debt management	12,922.47	13,166.30	13,313.05	13,512.74	13,715.43	13,921.17
Passive debt management	16,523.46	16,771.32	17,022.88	17,278.23	17,537.40	17,800.46

3.3 Illustration and comparison with other financing policies

We use the above example, see Sect. 2.3, to illustrate discontinuous financing with debt categories and to compare it with other financing strategies. We consider a firm with five debt categories and the same input parameters as above. To compute the market value of the firm under this assumption of five debt categories, we use the definition of $\eta^{(T)}$ from Eq. (34) and numerically obtain $\eta^{(5)} \approx 0.993$. Furthermore, we assume that all debt categories D_0^j, $j \in \{1,\dots,4\}$, are adjusted at the valuation date by multiplying the debt ratio $\theta^{(5)}$ by the market value of the firm, that is, $D_0^j = \theta^{(T)} \cdot V_0^\ell = 12\% \cdot V_0^\ell$. Therefore, we can apply the FCF approach and compute the WACC according to Eq. (40). To do so, we calculate $x = -0.017$, see Eq. (37), and obtain for the WACC

$$\begin{aligned} k^\tau &= 0.1 - (0.1-0.04) \cdot \frac{0.3 \cdot 0.04}{1+0.04} \cdot \frac{(1-0.017)^6 - 6 \cdot 0.017 - 1}{5 \cdot 0.017^2} \cdot 0.6 \\ &\quad - 0.04 \cdot 0.3 \cdot 0.6 \\ &= 9.16\%. \end{aligned} \tag{42}$$

For the value of the levered firm, we can use the Gordon–Shapiro formula, see Eq. (25), and obtain

$$V_0^{\ell,\mathrm{DC}} = \frac{1000}{9.16\% - 1.5\%} = 13{,}057.81\,, \tag{43}$$

with a total amount of debt of

$$D_0^{\mathrm{DC}} = \theta \cdot V_0^{\ell,\mathrm{DC}} = 60\% \cdot 13{,}057.81 = 7834.69\,,$$

where DC denotes the assumption of debt categories. For the equity value follows

$$E_0^{\ell,\mathrm{DC}} = V_0^{\ell,\mathrm{DC}} \cdot (1-\theta) = 13{,}057.81 \cdot 40\% = 5{,}223.13.$$

Since the firm is in a steady state, the expected market value of the firm, equity value, and debt levels grow at the constant growth rate g (see Tables 3, 4, and 5, respectively). Moreover, we can derive the levered cost of equity according to Eq. (39), which amounts to $\rho^\ell = 18.70\%$ and is constant in every period (see Table 6).

Table 4 Equity values under different financing strategies

Financing strategy	$t = 0$	$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$
Unlevered firm	11,764.71	11,941.18	12,120.29	12,302.10	12,486.63	12,673.93
Debt categories	5223.13	5301.47	5380.99	5461.71	5543.64	5626.79
Discontinuous financing	5226.68	5295.81	5370.04	5450.06	5536.64	5630.62
Active debt management	5168.99	5246.52	5325.22	5405.10	5486.17	5568.47
Passive debt management	6609.39	6708.53	6809.15	6911.29	7014.96	7120.19

Table 5 Debt levels under different financing strategies

Financing strategy	$t = 0$	$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$
Unlevered firm	0.00	0.00	0.00	0.00	0.00	0.00
Debt categories	7834.69	7952.21	8071.49	8192.56	8315.45	8440.18
Discontinuous financing	7840.02	7957.62	8076.99	8198.14	8321.11	8445.93
Active debt management	7753.48	7869.78	7987.83	8107.65	8229.26	8352.70
Passive debt management	9914.08	10,062.79	10,213.73	10,366.94	10,522.44	10,680.28

Table 6 Levered costs of equity under different financing strategies

Financing strategy	$t = 0$	$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$
Unlevered firm	10.00%	10.00%	10.00%	10.00%	10.00%	10.00%
Debt categories	18.70%	18.70%	18.70%	18.70%	18.70%	18.70%
Discontinuous financing	18.51%	18.61%	18.72%	18.82%	18.91%	18.51%
Active debt management	18.90%	18.90%	18.90%	18.90%	18.90%	18.90%
Passive debt management	14.68%	14.68%	14.68%	14.68%	14.68%	14.68%

Table 7 Value of the risk-free part of the tax shield under different financing strategies

Financing strategy	$t = 0$	$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$
Unlevered firm	0.00	0.00	0.00	0.00	0.00	0.00
Debt categories	264.97	268.94	272.97	277.07	281.23	285.44
Discontinuous financing	431.08	354.24	272.92	186.91	96.01	464.40
Active debt managements	89.46	90.81	92.17	93.55	94.95	96.38
Passive debt managements	4758.76	4830.14	4902.59	4976.13	5050.77	5126.53

The cost of equity can alternatively be derived by computing the value of the risk-free part of the tax shields according to Eq. (41) and applying Eq. (13). We included the value of the risk-free part of the tax shields in Table 7.

In the following, we compare the equity value in the case of discontinuous financing with debt categories to that under the assumption of other financing strategies. In Sect. 2.3, we have already calculated the equity value in the case of standard discontinuous financing and obtained

$$E_0^{\ell,\mathrm{DF}} = 5226.68,$$

where DF denotes discontinuous financing. This value is slightly higher than the market value of the firm in the case of debt categories. We derive a negligible deviation of

$$\frac{E_0^{\ell,\mathrm{DF}} - E_0^{\ell,\mathrm{DC}}}{E_0^{\ell,\mathrm{DC}}} = \frac{5226.68 - 5223.13}{5223.13} = 0.07\%.$$

The deviation occurs because, under standard discontinuous financing, at the valuation date, the tax shield is certain for five periods. Under debt categories, at the valuation date, only one fifth of the tax shield is certain for five periods. However, under discontinuous financing, in periods 1, 2, 3, and 4, the tax shield is only certain for 4, 3, 2, and 1 periods, respectively, which yields a value advantage of debt categories (see Table 3). This characteristic can also be observed for the costs of equity. Due to the lower financial risk, the cost of equity under standard discontinuous financing is lower than the cost of equity under debt categories at the valuation date, but this relation changes in periods 2, 3, and 4. Looking at a complete planning phase, the differences almost cancel each other out.

To compare the equity value under debt categories with that under the assumption of pure financing strategies, we first use the FCF approach to calculate the market value of the firm in the case of active debt management according to ME, which yields a levered cost of equity of $\rho^{\ell} = 18.90\%$. We deduce $k^{\tau} = 9.24\%$ and calculate thereby

$$E_0^{\ell,\mathrm{ADM}} = \frac{\mathbb{E}[\widetilde{FCF}_1]}{k^{\tau} - g} \cdot (1 - \theta) = \frac{1000}{9.24\% - 1.5\%} \cdot 40\% = 5.168{,}99, \tag{44}$$

where ADM denotes active debt management. The equity value is lower than in the case of discontinuous financing and debt categories, since the tax shields are only certain in the period of their occurrence and uncertain in all preceding periods. This can also be observed in the higher levered cost of equity, which implies that financial risk is higher. We deduce a deviation of the equity value of 1.05% and 1.12% for debt categories and standard discontinuous financing, respectively. For a higher T, the differences will become larger. In period 4, the levered cost of equity under standard discontinuous financing is nearly the same as under active debt management. In this period, the tax shield under discontinuous financing is also only certain for one period. However, the value remains slightly higher for discontinuous financing since the expected debt-to-market value ratio is 60.05% (see Table 2), which is higher than the debt-to-market value ratio of 60% under active debt management.

Table 8 Value of the tax shield under different financing strategies

Financing strategy	$t = 0$	$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$
Unlevered firm	0.00	0.00	0.00	0.00	0.00	0.00
Debt categories	1293.11	1312.51	1332.19	1352.18	1372.46	1393.04
Discontinuous financing	1302.00	1312.25	1326.73	1346.10	1371.12	1402.62
Active debt managements	1157.76	1175.13	1192.75	1210.64	1228.80	1247.24
Passive debt managements	4758.76	4830.14	4902.59	4976.13	5050.77	5126.53

For passive debt management we also assume a debt-to-market value ratio of 60%, which yields a WACC of 7.55%. For the equity value, we obtain

$$E_0^{\ell,\mathrm{PDM}} = \frac{E[\widetilde{FCF}_1]}{k^\tau - g} \cdot (1-\theta) = \frac{1,000}{7.55\% - 1.5\%} \cdot 40\% = 6609.39, \tag{45}$$

where PDM denotes passive debt management. Unsurprisingly, the equity value under passive debt management is considerably higher than the other equity values. We derive deviations to discontinuous financing and debt categories of more than 25% because, for passive debt management, all tax shields are certain and can be discounted at the risk-free interest rate r, rather than the unlevered cost of equity ρ^u. Hence, the levered cost of equity is also lower. For a higher T, the deviations decrease since the planning phases, and, therefore, the maturity of debt becomes longer.

For every financing strategy, a comparison of Tables 7 and 8 illustrates that for active debt management, discontinuous financing, and debt categories, only a small portion of the value of the tax shield comes from risk-free debt. For passive debt management, all future debt levels are risk-free such that the value of the risk-free part of the tax shields coincides with the value of the tax shield.

From period 5 onward, the structure is repeated. For discontinuous financing, the next planning phase starts, which implies a debt-to-market value ratio of 60% in period 5. For every other financing strategy, the equity value, market value of the firm, debt levels, value of the tax shield, and value of the risk-free part of the tax shields continue to grow at the growth rate g.

In this example, we always defined the debt-to-market value ratio deterministically since we excluded an explicit planning of debt levels. Thereby, we followed the line of the example in Clubb and Doran (1995, pp. 690–692). One could also define coinciding debt levels for every financing strategy and include an explicit forecast phase. Consequently, the debt-to-market value ratios would vary, but interpretations would be similar.

We conclude that the financing behavior of a firm should be carefully analyzed for choosing the most suitable financing strategy. The terminal value calculation with standard discontinuous financing shows shortcomings that can be rectified by debt categories. However, in the presented example, the value differences between standard discontinuous financing and discontinuous financing with

debt categories are negligible. For other input parameters, similar results can be expected. An advantage of standard discontinuous financing is that the market value of the firm and, therefore, the equity value at the valuation date can be calculated without circularity problems. For debt categories, this is only possible for the special case of two debt categories. For more debt categories, the limit η has to be computed numerically. However, to calculate market values within a planning phase, the application of standard discontinuous financing also involves circularity problems. Since a spreadsheet software is usually used for valuation, the application of both financing strategies can be conducted easily. Overall, the assumption of debt categories seems more realistic since the shares of the overall debt are adjusted successively, instead of all at once every T periods. If standard discontinuous financing is used, it can be interpreted as an approximation of debt categories.

3.4 Application of debt categories in a two-phase model

In the previous subsections, we excluded an explicit forecast phase and specified the entire debt level according to the debt-to-market value ratio at the valuation date. In this section, we analyze the possibilities of a link to a detailed planning of debt levels. This explicit planning is typically based on deterministic debt levels such that we assume passive debt management in the explicit forecast phase. Thereafter, we mix active and passive debt management by applying discontinuous financing with T debt categories in the steady state phase. We assume that the explicit forecast phase consists of S periods. As for every other combination of financing strategies in a two-phase model, it can be distinguished between an abrupt and a successive transition from the explicit forecast phase to the steady state (Koller et al. 2020, pp. 259–260).

In case of an abrupt change of financing strategies, all debt categories are adjusted according to $\theta^{(T)}$ at the beginning of the steady state phase; that is, the debt level of period S is computed as $\widetilde{D}_S = \theta \cdot \widetilde{V}_S^{\ell}$. Thus, at the beginning of the steady state phase, exceptionally, all debt categories are adjusted in the same period according to $\theta^{(T)}$, see Fig. 4. This approach does not directly consider the debt level D_{S-1} that results from the last period of the explicit forecast phase, but yields a refinancing. However, the debt ratio and the associated refinancing is planned together with the explicit debt levels and, therefore, does not constitute a problem. To calculate the market value of the firm at the beginning of the steady state phase, the adjustment formulas from Proposition 5 can be used. This approach of an abrupt change of financing strategies is very similar to the assumption of a steady state with active debt management, which is a common approach in corporate valuation practice. When active debt management is used for a steady state after an explicit forecast phase with passive debt management, the firm has to refinance according to the specified debt ratio (see also studies on hybrid financing, e.g., Kruschwitz et al. 2007; Dierkes and Gröger 2010; Dierkes and de Maeyer 2020).

Fig. 4 Two-phase model with an abrupt change of financing strategies

It is also possible to adjust the debt-to-market value ratio θ such that the expected debt level at the end of the explicit forecast phase, $\mathbb{E}[\widetilde{D}_S] = \theta \cdot \mathbb{E}[\widetilde{V}_S^{\ell}]$, coincides with an explicitly planned debt level D_S. This can, for example, be conducted by using a spreadsheet software. The debt levels are then adjusted exactly as in the first approach: In the first period of the steady state, all debt categories are adjusted according to the specified debt ratio. Whether D_S or $\mathbb{E}[\widetilde{D}_S]$ is used has no effect on the market value of the firm at the beginning of the steady state phase (for a more detailed analysis, see, Dierkes and de Maeyer 2020) such that it can again be computed with the help of Proposition 5.

Alternatively, a successive transition from passive debt management to debt categories can be assumed by including a convergence phase. To apply such an approach, the debt level D_S^* of period S must be explicitly planned. However, the resulting debt level $\widetilde{D}_S$ of this period S does not coincide with D_S^* since debt category 0 is adjusted according to the debt-to-market value ratio. All other debt categories are defined as a fraction of $\frac{1}{T}$ of the fixed level D_S^*, see Fig. 5. In period $S+1$, category 1 is adjusted according to the defined debt-to-market value ratio $\theta^{(T)}$. All other categories grow at the specified growth rate. In the second period after the end of the explicit planning, category 2 is adjusted according to this ratio and so on. In period $S+T-1$, all categories have been adjusted once according to $\theta^{(T)}$ such that this corresponds to the first period of the steady state phase. It follows that the convergence phase consists of T periods, after which the steady state with debt categories begins. To calculate the market value of the firm $\widetilde{V}_S^{\ell}$ at the of the explicit forecast phase, Eq. (35) can be used. For the debt levels D_S^{T-j}, the debt levels that result from an explicit planning have to be inserted. The example from Sect. 3.3 can be adjusted accordingly.

3.5 Risk of default for discontinuous financing and debt categories

In the above analysis, we concentrate on the derivation of the valuation equations and adjustment formulas, as well as their consequences for a steady state. We abstained from the integration of the risk of default to keep this focus. This is a common procedure when analyzing new financing strategies. However, the risk of default and the potential losses in value due to costs of financial distress should be taken into account (see, e.g., Almeida and Philippon 2007; Korteweg 2007; Lahmann et al. 2018). These costs consist of, for example, legal fees, costs due to customer losses and qualified employees leaving the firm in a crisis (Korteweg 2007, footnote 3; Lahmann et al. 2018, pp. 80–81). The consideration of the risk of default in corporate valuation has already been extensively analyzed, but is still intensively discussed (see, e.g., Sick 1990; Kruschwitz et al. 2005; Friedrich 2016; Lahmann et al. 2018). The most pragmatic solution, which is often applied, is to use a risk-adjusted cost of capital, rather than the risk-free interest rate for calculating the tax shields and discounting the risk-free part of the tax shields. This was, for example, performed in the analysis of active and passive debt management in Inselbag and Kaufold (1997), and the implementation of discontinuous financing of Clubb and Doran (1995) and Arnold et al. (2018). Accordingly, our analysis can be easily

Fig. 5 Two-phase model with a successive change of financing strategies

adapted by inserting the cost of debt r_D for the risk-free interest rate r. The discount rate r_D depends on assumptions regarding the tax treatment (for different possibilities of considering taxes in the case of default and the cost of debt, see, e.g., Sick 1990; Kruschwitz et al. 2005; Rapp 2006; Krause and Lahmann 2016; Baule 2019).

For a more explicit analysis of the risk of default in the case of discontinuous financing, we refer to Lahmann et al. (2018) and Arnold et al. (2019). They analyze the connection between the probability of default and the length of the planning phases T (Arnold et al. 2019, pp. 356–358). In particular, they consider a continuous time model with a geometric Brownian motion with drift-rate μ and volatility σ for the changes in the value of the unlevered firm and analyze the probability of default at refinancing dates. Lahmann et al. (2018) distinguish between an endogenous and an exogenous insolvency trigger (Lahmann et al. 2018, pp. 87–88). They outline, for example, that the longer the planning phases (the higher T) and the higher the debt-to-market value ratio, the higher the probability of default (Lahmann et al. 2018, pp.109–110). Furthermore, Arnold et al. (2019) discuss other possibilities of considering the risk of default in the case of discontinuous financing. Overall, they develop a framework that displays well the consequences of the integration of the risk of default for discontinuous financing.

The argumentation of Lahmann et al. (2018) and Arnold et al. (2019) can be transferred to debt categories. Since only parts of the overall debt level are adjusted according to a debt-to-market value ratio, the geometric Brownian Motion may drive down the value of the unlevered firm $\widetilde{V}_t^u$ at some period t so low that it is less than the debt level $\widetilde{D}_t$ in this period. The corresponding probability might be very low, but this should trigger a default. Furthermore, the fact that the probability of default increases for a higher T and a higher debt-to-market value ratio, as is the case with standard discontinuous financing, is also true for discontinuous financing with debt categories.

These fundamental considerations form an overview of the integration of the risk of default in the case of debt categories. However, analyses on how exactly the risk of default can be integrated into the valuation equations are beyond the scope of this study. We laid the groundwork for an investigation of financing with debt categories by obtaining valuation equations under the assumption of risk-free debt and leave additional analyses for further research. For the application of this financing strategy in corporate valuation practice, we recommend the common approach to use of the cost of debt, instead of the risk-free interest rate, which can be easily implemented. After all, the sound integration of the risk of default is not a specific problem of discontinuous financing with debt categories but applies to all financing strategies.

4 Conclusions

The choice of a financing strategy is a central issue for terminal value calculation. Since the terminal value accounts for a large part of the equity value, the financing strategy should accurately reflect the real financing behavior of a firm. We addressed this problem and introduced debt categories as a suitable financing strategy in a steady state. Under this assumption, different layers of debt were adjusted

successively. We obtained valuation equations and an adjustment formula for the cost of equity.

The foundation for this new financing strategy is standard discontinuous financing. As a mix of active and passive debt management, it provides the opportunity to depict a broad range of firm financing strategies with a slow adjustment of debt levels toward a fixed debt ratio. In this study, we clarified the differences between the approaches of discontinuous financing of Clubb and Doran (1995), Arnold et al. (2018, 2019), and Dierkes and de Maeyer (2020). We followed the approach of a perpetuity with growth of Arnold et al. (2019) and Dierkes and de Maeyer (2020), and showed that it results in an inconstant financial risk and, thus, an inconstant levered cost of equity. Moreover, we derived an adjustment formula for the period-specific levered cost of equity. The adjustment formula has a similar form to those for active and passive debt management. The knowledge of the adjustment formula offers the possibility to unlever and relever beta factors, as well as to calculate the market value of the firm using the FCF or FtE approach with period-specific costs of capital. Since the sole adjustment of the entire debt level after a planning phase and the associated consequences for financial risk might still not be close to the real financing behavior of firms, we introduced debt categories.

For discontinuous financing with debt categories, we assumed that a specified share of the overall debt is adapted in every period. The resulting debt categories were successively adjusted while considering the maturity of debt. First, we derived a valuation formula for two debt categories and an adjustment formula for the levered cost of equity. We showed that discontinuous financing with debt categories results in a constant expected financial risk and a constant levered cost of equity. Second, we extended the approach to an arbitrary number of debt categories. Independent of the number of debt categories, we obtain a financing policy for the steady state with the property of constant financial risk. Consequently, the Gordon-Shapiro formula can be applied.

Additionally, we presented an example to illustrate the theoretical findings and analyzed the value effects of the different financing strategies. When comparing standard discontinuous financing to active debt management, we obtained a small deviation. The difference is much larger when standard discontinuous financing is compared to passive debt management. The same results hold for discontinuous financing with debt categories. Moreover, we found that the deviation between the market value of the firm under standard discontinuous financing and discontinuous financing with debt categories is small. Consequently, despite the advantage of discontinuous financing with debt categories of depicting a broader range of real financing strategies of firms, valuation with standard discontinuous financing can still be applied. The latter can be interpreted as an approximation for the assumption of discontinuous financing with debt categories.

Our analysis focused on the consequences for a steady state. In particular, in the main part, we excluded an explicit forecast phase. Thereafter, we discussed possibilities of linking the assumption of a steady state with debt categories to a detailed planning of debt levels. The derived valuation equations can be easily adjusted to one of these models. This enables the application of debt categories in a two-phase model. Furthermore, we excluded the risk of default and the costs of financial

distress in our basic analysis in order to derive the valuation equations und adjustment formulas on a clear theoretical basis for our new financing policy, as it is common in comparable analyses for other financing policies. Nevertheless, it is important to analyze the additional incorporation of risk of default, such that we discussed limitations and possible solutions of this assumption afterwards. We pointed out that the application of the cost of debt can be easily implemented and laid out fundamental characteristics of an explicit analysis, but left a more detailed analysis to further research. Overall, by introducing discontinuous financing with debt categories, we presented a new possibility to depict the financing behavior of firms in a steady state and contribute to the ongoing discussion on terminal value calculation.

Appendix

Proof of Proposition 1 First, we deduce the formula for the WACC. To do so, we introduce additional notation to simplify the calculations. Let

$$G = 1 + g, \qquad R = 1 + r, \qquad K = 1 + \rho^u \, .$$

Since the function PVA does always depend on g in our setting, we denote it as a function of C and s, where $C \in \{R, K\}$ and $s \in \mathbb{N}$. It is

$$f(C, s) = \text{PVA}(C - 1, G - 1, s) = \frac{1}{C - G} \cdot \left(1 - \frac{G^s}{C^s}\right).$$

We use this new defined notation to rewrite the definition of the cost of capital, see Eq. (10), and the claim

$$\begin{aligned} 1 + k_t^\tau &:= \frac{\mathbb{E}[\widetilde{FCF}_1] \cdot G^t + \mathbb{E}[\widetilde{V}_{t+1}]}{\mathbb{E}[\widetilde{V}_t^\ell]} \\ &\overset{!}{=} (1 - \tau \cdot r \cdot \theta_t \cdot f(R, T - t)) \cdot K - \tau \cdot r \cdot \theta_t \cdot (1 - R \cdot f(R, T - t)). \end{aligned} \tag{46}$$

Furthermore, we can rewrite the computation of the market value of the firm at time t, see Eq. (5), as

$$\mathbb{E}[\widetilde{V}_t^\ell] = \mathbb{E}[\widetilde{FCF}_1] \cdot G^t \cdot f(K, T - t) + \tau \cdot r \cdot \theta \cdot V_0 \cdot G^t \cdot f(R, T - t) + V_0 \cdot \frac{G^T}{K^{T-t}}. \tag{47}$$

To prove the claim, we note that

$$f(C, s - 1) = f(C, s) - \frac{G^{s-1}}{C^s} \tag{48}$$

and

$$\left(1 - \frac{G^s}{C^s}\right) = f(C, s) \cdot (C - G). \tag{49}$$

By using these relations, we can rearrange the numerator of Eq. (46) to obtain

$$\begin{aligned}
\mathbb{E}[\widetilde{FCF}_1] \cdot G^t + \mathbb{E}[\widetilde{V}^{\ell}_{t+1}] &\overset{(47)}{=} \mathbb{E}[\widetilde{FCF}_1] \cdot G^t \cdot (1 + G \cdot f(K, T-t-1)) \\
&\quad + \tau \cdot r \cdot \theta \cdot V_0 \cdot G^t \cdot G \cdot f(R, T-t-1) \\
&\quad + K \cdot V_0 \cdot \frac{G^T}{K^{T-t}} \\
&\overset{(48)}{=} \mathbb{E}[\widetilde{FCF}_1] \cdot G^t \cdot \left(1 + G \cdot f(K, T-t) - \frac{G^{T-t}}{K^{T-t}}\right) \\
&\quad + \tau \cdot r \cdot \theta \cdot V_0 \cdot G^t \cdot \left(G \cdot f(R, T-t) - \frac{G^{T-t}}{R^{T-t}}\right) \\
&\quad + K \cdot V_0 \cdot \frac{G^T}{K^{T-t}} \\
&\overset{(49)}{=} \mathbb{E}[\widetilde{FCF}_1] \cdot G^t \cdot K \cdot f(K, T-t) \\
&\quad + \tau \cdot r \cdot \theta \cdot V_0 \cdot G^t \cdot (R \cdot f(R, T-t) - 1) \\
&\quad + K \cdot V_0 \cdot \frac{G^T}{K^{T-t}}.
\end{aligned}$$

We add and subtract $K \cdot \tau \cdot r \cdot \theta \cdot V_0 \cdot G^t \cdot f(R, T-t)$ and use again Eq. (47), which yields

$$\mathbb{E}[\widetilde{FCF}_1] \cdot G^t + \mathbb{E}[\widetilde{V}^{\ell}_{t+1}] = K \cdot \mathbb{E}[\widetilde{V}_t] + \tau \cdot r \cdot \theta \cdot V_0 \cdot G^t \cdot ((R-K) \cdot f(R, T-t) - 1). \tag{50}$$

Since the debt-to-market value ratio is inconstant, we need to calculate the debt-to-market value ratio of period t. It is

$$\frac{\theta \cdot V_0 \cdot G^t}{\mathbb{E}[\widetilde{V}_t]} = \frac{D_0 \cdot G^t}{\mathbb{E}[\widetilde{V}_t]} = \frac{D_t}{\mathbb{E}[\widetilde{V}_t]} = \theta_t.$$

We can use this relation and the modified expression of the numerator, see Eq. (50), to compute

$$\frac{\mathbb{E}[\widetilde{FCF}_1] \cdot G^t + \mathbb{E}[\widetilde{V}_{t+1}]}{\mathbb{E}[\widetilde{V}^{\ell}_t]} = K + \tau \cdot r \cdot \theta_t \cdot ((R-K) \cdot f(R, T-t) - 1).$$

Rearranging and inserting the definition of k^{τ}_t, see Eq. (46), yields

$$1 + k^{\tau}_t = K \cdot (1 - \tau \cdot r \cdot \theta_t \cdot f(R, T-t)) - \tau \cdot r \cdot \theta_t \cdot (1 - R \cdot f(R, T-t)).$$

By inserting the original notation, we obtain

$$\begin{aligned}
1 + k^{\tau}_t = \rho^u \cdot (1 - \tau \cdot r \cdot \theta_t \cdot \mathrm{PVA}(r, g, T-t)) - \tau \cdot r \cdot \theta_t \cdot (1 - r \cdot \mathrm{PVA}(r, g, T-t)) \\
+ 1 - \tau \cdot r \cdot \theta_t \cdot \mathrm{PVA}(r, g, T-t) + \tau \cdot r \cdot \theta_t \cdot \cdot \mathrm{PVA}(r, g, T-t)\,,
\end{aligned}$$

which yields Eq. (9).

It remains to deduce the formula for the cost of equity. The computation is straightforward. By using the definition of the WACC, $k_t^\tau = \rho^\ell \cdot (1-\theta_t) + r \cdot (1-\tau) \cdot \theta_t$, and equating it to Eq. (9), we obtain

$$\begin{aligned}\rho^\ell \cdot (1-\theta_t) &= k_t^\tau - r \cdot (1-\tau) \cdot \theta_t \\ &= \rho^u - (\rho^u - r) \cdot \tau \cdot r \cdot \theta_t \cdot \mathrm{PVA}(r, g, T-t) - \tau \cdot r \cdot \theta_t \\ &\quad - r \cdot (1-\tau) \cdot \theta_t \\ &= \rho^u \cdot (1-\theta_t) + (\rho^u - r) \cdot (1 - \tau \cdot r \cdot \mathrm{PVA}(r, g, T-t)) \cdot \theta_t.\end{aligned}$$

Dividing both sides by $(1-\theta_t)$ and defining $L_t := \frac{\theta_t}{1-\theta_t}$ yields the levered cost of equity. □

Proof of Proposition 2 From the results in Sect. 3.1, we deduce

$$V_0^\ell = \lim_{T\to\infty} \sum_{t=1}^{T} \frac{E[\widetilde{FCF}_t]}{\prod_{s=1}^{t}(1+\rho^u) \cdot \eta_{T-s+1}^{(2)}} + \frac{\tau \cdot r \cdot D_0^1}{(1+r) \cdot \eta_T^{(2)}}. \tag{51}$$

To show that the first sum converges, we want to apply Lemma 2. To do so, we define

$$a_k := \eta_k \cdot \frac{1+\rho^u}{1+g}.$$

For $k \to \infty$, follows $a_k \searrow a$ with

$$a := \eta \cdot \frac{1+\rho^u}{1+g},$$

since $\eta_k \searrow \eta$, see the proof of Lemma 1. By assumption holds $a > 1$. It follows that, by applying Lemma 2, Eq. (51) simplifies to

$$V_0^\ell = \frac{\mathbb{E}[\widetilde{FCF}_1]}{k^* - g} + \frac{\tau \cdot r \cdot D_0^1}{1+r^*} = \frac{\mathbb{E}[\widetilde{FCF}_1]}{k^* - g} + \frac{\tau \cdot r \cdot \theta^{(2)} \cdot V_0^\ell}{1+r^*}.$$

Solving the circularity problem, that is, solving for the market value of the firm, yields the claim. □

Proof of Proposition 3 In this proof, we forgo the labeling of the case of two debt categories in the exponent of the adjustment sequence and write η instead of $\eta^{(2)}$. The formula for the computation of the value of the levered firm in Eq. (24) must be equal to Eq. (25). Solving for the WACC yields

$$k^\tau = k^* - (k^* - g) \cdot \frac{\tau \cdot r \cdot \theta^{(2)}}{1+r^*}. \tag{52}$$

Applying the relation $k^\tau = \rho^\ell \cdot (1-\theta) + r \cdot (1-\tau) \cdot \theta$, equating Eq. (52) and this expression, and solving for $\rho^\ell \cdot (1-\theta)$ yields

$$\rho^{\ell} \cdot (1-\theta) = k^* - (k^* - g) \cdot \frac{\tau \cdot r \cdot \theta^{(2)}}{1+r^*} - r \cdot \theta + \tau \cdot r \cdot \theta.$$

By plugging in the definition of k^* and r^*, we obtain

$$\begin{aligned}\rho^{\ell} \cdot (1-\theta) &= (1+\rho^u) \cdot \eta - 1 - \tau \cdot r \cdot \theta^{(2)} \cdot \frac{(1+\rho^u) \cdot \eta - 1 - g}{(1+r) \cdot \eta} - r \cdot \theta + \tau \cdot r \cdot \theta \\ &= (1+\rho^u) \cdot \eta - 1 - \tau \cdot r \cdot \theta^{(2)} \cdot \left(\frac{1+\rho^u}{1+r} - \frac{1+g}{(1+r) \cdot \eta} \right) - r \cdot \theta \\ &\quad + \tau \cdot r \cdot \theta.\end{aligned}$$

By using that η is a fixed point of the sequence $(\eta_k)_{k \in \mathbb{N}}$, see Eq. (23), we obtain

$$\begin{aligned}\rho^{\ell} \cdot (1-\theta) &= (1+\rho^u) \cdot \left(1 - \frac{\tau \cdot r \cdot \theta^{(2)}}{1+r} - \frac{\tau \cdot r \cdot \theta^{(2)} \cdot (1+g)}{(1+r)^2 \cdot \eta} \right) - 1 \\ &\quad - \tau \cdot r \cdot \theta^{(2)} \cdot \left(\frac{1+\rho^u}{1+r} - \frac{1+g}{(1+r) \cdot \eta} \right) - r \cdot \theta + \tau \cdot r \cdot \theta \\ &= (1+\rho^u) \cdot \left(1 - 2 \cdot \frac{\tau \cdot r \cdot \theta^{(2)}}{1+r} - \frac{\tau \cdot r \cdot \theta^{(2)} \cdot (1+g)}{(1+r)^2 \cdot \eta} \right) - 1 \\ &\quad + (1+r) \cdot \left(\frac{\tau \cdot r \cdot \theta}{1+r} + \frac{\tau \cdot r \cdot \theta^{(2)} \cdot (1+g)}{(1+r)^2 \cdot \eta} \right) - r \cdot \theta.\end{aligned}$$

Adding and subtracting $\rho^u \cdot \theta$ results in

$$\rho^{\ell} \cdot (1-\theta) = (\rho^u - r) \cdot \left(\theta - \frac{\tau \cdot r \cdot \theta}{1+r} - \frac{1}{2} \cdot \frac{\tau \cdot r \cdot \theta \cdot (1+g)}{(1+r)^2 \cdot \eta} \right) + \rho^u \cdot (1-\theta).$$

Dividing both sides by $(1-\theta)$ and defining $L := \frac{\theta}{1-\theta}$ as the leverage yields the cost of equity.

The expression of k^{τ} can be derived by applying the definition of the WACC and inserting ρ^{ℓ}. We obtain

$$\begin{aligned}k^{\tau} &= (1-\theta) \cdot \rho^{\ell} + r \cdot (1-\tau) \cdot \theta \\ &= (1-\theta) \cdot \rho^u + (\rho^u - r) \cdot \left(1 - \frac{\tau \cdot r}{1+r} - \frac{1}{2} \cdot \frac{\tau \cdot r \cdot (1+g)}{(1+r)^2 \cdot \eta} \right) \cdot \theta + r \cdot \theta \\ &\quad - r \cdot \tau \cdot \theta \\ &= \rho^u - (\rho^u - r) \cdot \left(\frac{\tau \cdot r}{1+r} - \frac{1}{2} \cdot \frac{\tau \cdot r \cdot (1+g)}{(1+r)^2 \cdot \eta} \right) \cdot \theta - (1+r) \cdot \frac{r \cdot \tau \cdot \theta}{1+r} \\ &= \rho^u - \tau \cdot r \cdot \theta \cdot \frac{(1+\rho^u)}{1+r} - (\rho^u - r) \cdot \frac{1}{2} \cdot \frac{\tau \cdot r \cdot \theta \cdot (1+g)}{(1+r)^2 \cdot \eta},\end{aligned}$$

which proves the claim. □

Proof of Proposition 4 Note that

$$\sum_{s=1}^{T-1}\sum_{t=1}^{s}\frac{(1+g)^{T-s-1}}{(1+r^*)^{T-s}}=\sum_{s=1}^{T-1}\frac{(T-s)\cdot(1+g)^{s-1}}{(1+r^*)^s}.$$

Using this result, inserting $D_0^t=\theta^{(T)}\cdot V_0^{\ell}$, and solving the circularity problem in Eq. (35) yields

$$V_0^{\ell}=\frac{\mathbb{E}[\widetilde{FCF}_1]}{k^*-g}\cdot\left(1-\tau\cdot r\cdot\theta^{(T)}\cdot\sum_{s=1}^{T-1}\frac{(T-s)\cdot(1+g)^{s-1}}{(1+r^*)^s}\right)^{-1}. \tag{53}$$

We can simplify this expression using Lemma 4 to

$$V_0^{\ell}=\frac{\mathbb{E}[\widetilde{FCF}_1]}{k^*-g}\cdot\left(1-\frac{\tau\cdot r\cdot\theta^{(T)}}{1+r}\cdot\frac{(1+x)^T-Tx-1}{x^2}\right)^{-1},$$

where x is defined as in Eq. (37), which shows the claim. □

Proof of Proposition 5 The proof of the adjustment formula has the same structure as the proof of Proposition 3. The formula for the computation of the value of the levered firm, see Eq. (53) must be equal to Eq. (25). Solving for the WACC yields

$$\begin{aligned}k^{\tau}&=(k^*-g)\cdot\left(1-\tau\cdot r\cdot\theta^{(T)}\cdot\sum_{s=1}^{T-1}\frac{(T-s)\cdot(1+g)^{s-1}}{(1+r^*)^s}\right)+g\\&=k^*-(k^*-g)\cdot\tau\cdot r\cdot\theta^{(T)}\cdot\sum_{s=1}^{T-1}\frac{(T-s)\cdot(1+g)^{s-1}}{(1+r^*)^s}.\end{aligned}$$

Note that we again forgo the labeling of the case of T debt categories and write η instead of $\eta^{(T)}$ in the following. Applying the relation $k^{\tau}=\rho^{\ell}\cdot(1-\theta)+r\cdot(1-\tau)\cdot\theta$, equating it to the above equation, and solving for $\rho^{\ell}\cdot(1-\theta)$ yields

$$\begin{aligned}\rho^{\ell}\cdot(1-\theta)&=k^*-(k^*-g)\cdot\tau\cdot r\cdot\theta^{(T)}\cdot\sum_{s=1}^{T-1}\frac{(T-s)\cdot(1+g)^{s-1}}{(1+r^*)^s}-r\cdot(1-\tau)\cdot\theta\\&=(1+\rho^u)\cdot\eta-1-((1+\rho^u)\cdot\eta-(1+g))\cdot\tau\cdot r\cdot\theta^{(T)}\\&\quad\cdot\sum_{s=1}^{T-1}\frac{(T-s)\cdot(1+g)^{s-1}}{(1+r^*)^s}-r\cdot(1-\tau)\cdot\theta.\end{aligned}$$

Using that η is a fixed point of the sequence $(\eta_k)_{k\in\mathbb{N}}$, see Eq. (33), results in

$$\begin{aligned}\rho^{\ell} \cdot (1-\theta) = (1+\rho^u) \cdot \left(1 - \frac{\tau \cdot r \cdot \theta^{(T)}}{1+r} - \sum_{t=1}^{T-1} \frac{\tau \cdot r \cdot \theta^{(T)} \cdot (1+g)^t}{(1+r)^{t+1} \cdot \eta^t}\right) - 1 \\ - (1+\rho^u) \cdot \tau \cdot r \cdot \theta^{(T)} \cdot \sum_{s=1}^{T-1} \frac{(T-s) \cdot (1+g)^{s-1}}{(1+r)^s \cdot \eta^{s-1}} \\ + (1+r) \cdot \tau \cdot r \cdot \theta^{(T)} \cdot \sum_{s=1}^{T-1} \frac{(T-s) \cdot (1+g)^s}{(1+r)^{s+1} \cdot \eta^s} - r \cdot \theta \\ + (1+r) \cdot \frac{\tau \cdot r \cdot \theta^{(T)} \cdot T}{1+r},\end{aligned}$$

which can be written as

$$\begin{aligned}\rho^{\ell} \cdot (1-\theta) = (1+\rho^u) \cdot \left(1 - \tau \cdot r \cdot \theta^{(T)} \cdot \left(\sum_{t=0}^{T-1} \frac{(1+g)^t}{(1+r)^{t+1} \cdot \eta^t} - 1\right.\right. \\ \left.\left. + \sum_{s=0}^{T-1} \frac{(T-s-1) \cdot (1+g)^s}{(1+r)^{s+1} \cdot \eta^s}\right)\right) \\ + (1+r) \cdot \tau \cdot r \cdot \theta^{(T)} \cdot \sum_{s=0}^{T-1} \frac{(T-s) \cdot (1+g)^s}{(1+r)^{s+1} \cdot \eta^s} \\ - r \cdot \theta + \rho^u \cdot \theta - \rho^u \cdot \theta \\ = (\rho^u - r) \cdot \left(\theta - \tau \cdot r \cdot \theta^{(T)} \cdot \sum_{s=0}^{T-1} \frac{(T-s) \cdot (1+g)^s}{(1+r)^{s+1} \cdot \eta^s}\right) + \rho^u \cdot (1-\theta).\end{aligned}$$

Dividing both sides by $(1-\theta)$ yields

$$\begin{aligned}\rho^u &= \rho^u + (\rho^u - r) \cdot \left(1 - \tau \cdot r \cdot \frac{1}{T} \cdot \sum_{s=0}^{T-1} \frac{(T-s) \cdot (1+g)^s}{(1+r)^{s+1} \cdot \eta^s}\right) \cdot \frac{\theta}{1-\theta} \\ &= \rho^u + (\rho^u - r) \cdot \left(1 - \tau \cdot r \cdot \sum_{s=0}^{T-1} \left(1 - \frac{s}{T}\right) \cdot \frac{(1+g)^s}{(1+r)^{s+1} \cdot \eta^s}\right) \cdot \frac{\theta}{1-\theta}.\end{aligned}$$

Inserting the leverage $L = \frac{\theta}{1-\theta}$ yields Eq. (38). As in the Proof of Proposition 4, we can simplify the sum using Lemma 4, which results in

$$\rho^{\ell} = \rho^u + (\rho^u - r) \cdot \left(1 - \frac{\tau \cdot r}{1+r} \cdot \frac{(1+x)^{T+1} - (T+1) \cdot x - 1}{Tx^2}\right) \cdot \frac{\theta}{1-\theta},$$

where x is defined as in Eq. (37). This proves Eq. (39)

Inserting this expression for ρ^{ℓ} in the definition of the WACC yields

$$k^{\tau} = \rho^{u} \cdot (1-\theta) + (\rho^{u} - r) \cdot \left(1 - \frac{\tau \cdot r}{1+r} \cdot \frac{(1+x)^{T+1} - (T+1) \cdot x - 1}{Tx^2}\right) \cdot \theta$$
$$+ r \cdot (1-\tau) \cdot \theta$$
$$= \rho^{u} - (\rho^{u} - r) \cdot \frac{\tau \cdot r}{1+r} \cdot \frac{(1+x)^{T+1} - (T+1) \cdot x - 1}{Tx^2} \cdot \theta - r \cdot \tau \cdot \theta \,,$$

which shows the claim. □

Lemma 1 *Let $a > 0$, $b > 0$, and $a^2 > b$. We define $(\gamma_k)_{k\in\mathbb{N}}$ as a recursive sequence, with $\gamma_1 := 2a$ and*

$$\gamma_k := 2a - \frac{b}{\gamma_{k-1}}$$

for $k > 1$. Then, this sequence converges to

$$\gamma = a + \sqrt{a^2 - b}.$$

Proof We show that $(\gamma_k)_{k\in\mathbb{N}}$ is monotonously decreasing and bounded from below. We can show the former by induction. Since $a > 0$ and $b > 0$, it follows that $\gamma_2 < \gamma_1$. For $k \in \mathbb{N}$, we assume that $\gamma_k < \gamma_{k-1}$ and conclude

$$\gamma_{k+1} = 2a - \frac{b}{\gamma_k} < 2a - \frac{b}{\gamma_{k-1}} = \gamma_k.$$

Next, we show that it is bounded from below by

$$\gamma := a + \sqrt{a^2 - b}.$$

If $\gamma_k = \gamma + \varepsilon$ for $\varepsilon > 0$, that is, γ_k is greater than γ, we show that the next term is also greater than γ: Since $\gamma > 0$, we have

$$\gamma_{k+1} = 2a - \frac{b}{\gamma_k} = 2a - \frac{b}{\gamma+\varepsilon} = 2a - \frac{b}{\gamma} + \frac{b}{\gamma} - \frac{b}{\gamma+\varepsilon}$$
$$= \gamma + \left(\frac{b}{\gamma} - \frac{b}{\gamma+\varepsilon}\right) > \gamma.$$

Additionally, since $a > 0$ and $b > 0$, it follows that $\gamma_1 = 2a > \gamma$. Hence, the sequence $(\gamma_k)_{k\in\mathbb{N}}$ is monotonously decreasing and bounded from below by γ which implies that $(\gamma_k)_{k\in\mathbb{N}}$ converges. The limit follows from

$$\lim_{k\to\infty} \gamma_{k+1} = 2a - \frac{b}{\lim_{k\to\infty} \gamma_k} \,,$$

which yields

$$\lim_{k\to\infty} \gamma_k = a + \sqrt{a^2 - b} = \gamma.$$

□

Corollary 1 *Let* $\tau, r, \theta \in [0,1], \theta^{(2)} := \frac{1}{2}\theta$, *and*

$$-1 < g < \frac{1}{4} \cdot \frac{(1+r-\tau \cdot r \cdot \theta^{(2)})^2}{\tau \cdot r \cdot \theta^{(2)}} - 1.$$

In the case $T = 2$, *that is, for two debt categories, the sequence* $(\eta_k^{(2)})_{k\in\mathbb{N}}$ *with* $\eta_1^{(2)} := 1 - \frac{\tau \cdot r \cdot \theta^{(2)}}{1+r}$ *and*

$$\eta_k^{(2)} = 1 - \frac{\tau \cdot r \cdot \theta^{(2)}}{1+r} - \frac{\tau \cdot r \cdot \theta^{(2)} \cdot (1+g)}{(1+r)^2 \cdot \eta}$$

converges to $\eta^{(2)}$ *for* $k \to \infty$, *where*

$$\eta^{(2)} = \frac{1+r-\tau \cdot r \cdot \theta^{(2)}}{2 \cdot (1+r)} + \frac{1}{2 \cdot (1+r)} \cdot \sqrt{(1+r)^2 - 2 \cdot (1+r) \cdot \tau \cdot r \cdot \theta^{(2)} + (\tau \cdot r \cdot \theta^{(2)} - 4 \cdot (1+g)) \cdot \tau \cdot r \cdot \theta^{(2)}}.$$

Proof We want to apply Lemma 1. To do so, we define

$$a := \frac{1}{2} \cdot \left(1 - \frac{\tau \cdot r \cdot \theta^{(2)}}{1+r}\right) \text{ and } b := \frac{\tau \cdot r \cdot \theta^{(2)} \cdot (1+g)}{(1+r)^2}.$$

It remains to show that the assumptions of Proposition 1 are valid. Since $\tau \cdot r \cdot \theta^{(2)} < 1$ and $1 + r > 1$, it follows $a > 0$. Moreover, $b > 0$ holds by assumption. To check $a^2 > b$, note that

$$\begin{aligned} & g < \frac{1}{4} \cdot \frac{(1+r-\tau \cdot r \cdot \theta^{(2)})^2}{\tau \cdot r \cdot \theta^{(2)}} - 1 \\ \Leftrightarrow \quad & \frac{1}{4} \cdot \left(1 - \frac{\tau \cdot r \cdot \theta^{(2)}}{1+r}\right)^2 > \frac{\tau \cdot r \cdot \theta^{(2)} \cdot (1+g)}{(1+r)^2}. \end{aligned}$$

□

Lemma 2 *Let* $(a_n)_{n\in\mathbb{N}}$ *be a sequence that converges from above to* $a > 1$ *for* $n \to \infty$. *It follows*

$$\lim_{T\to\infty} \sum_{t=1}^{T} \frac{1}{\prod_{s=1}^{t} a_{T-s+1}} = \sum_{t=1}^{\infty} \frac{1}{a^t} = \frac{1}{a-1}.$$

Proof We define

$$S_T := \sum_{t=1}^{T} \frac{1}{\prod_{s=1}^{t} a_{T-s+1}}.$$

By assumption holds $a_t \geq a > 1$, which implies $S_T \leq \frac{1}{a-1}$. Furthermore, by definition, we have

$$S_{T+1} = \frac{1}{a_{T+1}} \cdot (1 + S_T).$$

It follows $S_{T+1} \geq S_T$. Thus, the sequence $(S_T)_{T\in\mathbb{N}}$ is monotonously increasing and bounded from above. Hence, the sequence converges. For the limit holds

$$\lim_{T\to\infty} S_{T+1} = \frac{1}{\lim_{T\to\infty} a_{T+1}} \cdot \Big(1 + \lim_{T\to\infty} S_T\Big).$$

Since $\lim_{T\to\infty} a_t = a$, the claim follows. □

Lemma 3 *Let $\tau, r, \theta \in [0,1]$, and $\theta^{(T)} := \frac{1}{T}\theta$. For an arbitrary number of T debt categories, the sequence $(\eta_k^{(T)})_{k\in\mathbb{N}}$ with $\eta_1^{(T)} := 1 - \frac{\tau\cdot r\cdot\theta^{(T)}}{1+r}$ and*

$$\eta_k^{(T)} := 1 - \frac{\tau\cdot r\cdot\theta^{(T)}}{1+r} - \sum_{t=1}^{\min\{k,T\}-1} \frac{\tau\cdot r\cdot\theta^{(T)}\cdot(1+g)^t}{(1+r)^{t+1}\cdot\prod_{s=1}^{t}\eta_{k-s}^{(T)}}$$

converges to $\eta^{(T)}$ for $k\to\infty$ if T is even,

$$-1 \leq g \leq (1+r)\cdot\Big(\frac{1}{T^2}\cdot\frac{1+r}{\tau\cdot r\cdot\theta^{(T)}}\Big)^{\frac{1}{n-1}}\cdot\frac{T-1}{T} - 1\,,$$

and

$$\frac{\tau\cdot r\cdot\theta}{1+r} \leq \frac{1}{T}\,,$$

or if $g > -1$ and T is uneven.

Proof The proof is similarly structured to the proof of Lemma 1. First, we need to show that the function

$$f(x) = x^T - x^{T-1} + \frac{\tau\cdot r\cdot\theta^{(T)}}{1+r}\cdot\sum_{j=0}^{T-1}\frac{(1+g)^j}{(1+r)^j}\cdot x^{T-1-j}$$

has at least one real root. For T odd, this is a well-known result (see e.g., Kriz and Pultr 2013, p. 9). For T even, note that $f(x) > 0$ for $x \in \{0,1\}$. If we can show that there exists an $x \in (0,1)$ with $f(x) < 0$, it follows that f has a root $x \in (0,1)$. We want to show

$$f\Big(\frac{T-1}{T}\Big) < 0. \tag{54}$$

From the assumptions follows

$$\frac{\tau \cdot r \cdot \theta^{(T)}}{1+r} \cdot T \cdot \left(\max\left\{ \frac{1+g}{1+r}, x \right\} \right)^{T-1} \leq x^{T-1} \cdot (1-x) \,,$$

for $x = \frac{T-1}{T}$. We obtain

$$\frac{\tau \cdot r \cdot \theta^{(T)}}{1+r} \cdot \sum_{j=0}^{T-1} \frac{(1+g)^j}{(1+r)^j} \cdot x^{T-1-j} \leq x^{T-1} \cdot (1-x) \,,$$

which implies that Eq. (54) is true. We conclude that $\eta_k^{(T)}$ has at least one fixed point $\eta \in \mathbb{R}$.

Let η^* be the largest real fixed point. We want to show that the sequence converges to η^*. To do so, we show that $(\eta_k^{(T)})_{k\in\mathbb{N}}$ is monotonously decreasing and bounded from below. The former assumption is proven by induction. Note that $\eta_1^{(T)} \geq \eta_2^{(T)} \geq \cdots \geq \eta_T^{(T)}$ holds. We now assume that $\eta_{k-T+1}^{(T)} \geq \eta_{k-T+2}^{(T)} \geq \cdots \geq \eta_k^{(T)}$ for $k \geq T$ and conclude

$$\begin{aligned}
\eta_{k+1}^{(T)} &= 1 - \frac{\tau \cdot r \cdot \theta^{(T)}}{1+r} - \sum_{t=1}^{T-1} \frac{\tau \cdot r \cdot \theta^{(T)} \cdot (1+g)^t}{(1+r)^{t+1} \cdot \prod_{s=1}^{t} \eta_{k+1-s}^{(T)}} \\
&\leq 1 - \frac{\tau \cdot r \cdot \theta^{(T)}}{1+r} - \sum_{t=1}^{T-1} \frac{\tau \cdot r \cdot \theta^{(T)} \cdot (1+g)^t}{(1+r)^{t+1} \cdot \prod_{s=1}^{t} \eta_{k-s}^{(T)}} \\
&= \eta_k^{(T)} .
\end{aligned}$$

It remains to show that $(\eta_k^{(T)})_{k\in\mathbb{N}}$ is bounded from below by η^*. We assume that $\eta_k^{(T)} \geq \eta^*$ and show $\eta_{k+1}^{(T)} \geq \eta^*$. The inequality $\eta_k^{(T)} \geq \eta^*$ implies $\eta_{k-1}^{(T)}, \ldots, \eta_{k+2-T}^{(T)} \geq \eta^*$, which yields

$$\begin{aligned}
\eta_{k+1}^{(T)} &= 1 - \frac{\tau \cdot r \cdot \theta^{(T)}}{1+r} - \sum_{t=1}^{T-1} \frac{\tau \cdot r \cdot \theta^{(T)} \cdot (1+g)^t}{(1+r)^{t+1} \cdot \prod_{s=1}^{t} \eta_{k+1-s}^{(T)}} \\
&\geq 1 - \frac{\tau \cdot r \cdot \theta^{(T)}}{1+r} - \sum_{t=1}^{T-1} \frac{\tau \cdot r \cdot \theta^{(T)} \cdot (1+g)^t}{(1+r)^{t+1} \cdot (\eta^*)^t} \\
&= \eta^* \,.
\end{aligned}$$

We conclude that $(\eta_t^{(T)})_{k\in\mathbb{N}}$ converges to the largest real fixed point η^*. By defining $\eta^{(T)} := \eta^*$, the claim follows. □

Lemma 4 *For $T \in \mathbb{N}$ and $x > 0$ holds*

$$\sum_{s=0}^{T-1} (T-s) \cdot (1+x)^s = \frac{(1+x)^{T+1} - (T+1) \cdot x - 1}{x^2} .$$

Proof For a proof see Gradshteyn and Ryzhik (2007, Eq. 0.113). □

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