Some aspects of Lévy copulas

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Abstract

Lévy processes and infinitely divisible distributions are increasingly defined in terms of their Lévy measure. In order to describe the dependence structure of a multivariate Lévy measure, Tankov (2003) introduced positive Lévy copulas. Together with the marginal Lévy measures they completely describe multivariate Lévy measures on $\mathbb{R}_+^m$. In this paper, we show that any such Lévy copula defines itself a Lévy measure with 1-stable margins, in a canonical way. A limit theorem is obtained, characterising convergence of Lévy measures with the aid of Lévy copulas. Homogeneous Lévy copulas are considered in detail. They correspond to Lévy processes which have a time-constant Lévy copula. Furthermore, we show how the Lévy copula concept can be used to construct multivariate distributions in the Bondesson class with prescribed margins in the Bondesson class. The construction depends on a mapping Υ, recently introduced by Barndorff-Nielsen and Thorbjørnsen (2004a,b) and Barndorff-Nielsen, Maejima and Sato (2004). Similar results are obtained for self-decomposable distributions and for distributions in the Thorin class.

1 Introduction

The concept of copulas for multivariate probability distributions has an analogue for multivariate Lévy measures, called *Lévy copulas*. The latter concept was introduced in a paper by Tankov [13] for Lévy measures on $\mathbb{R}_+^m$, and extended to Lévy measures on $\mathbb{R}^m$ by Kallsen and Tankov [10], see also the book by Cont and Tankov [8]. Similar to probabilistic copulas, a Lévy copula describes the dependence structure of a multivariate Lévy measure. The

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Lévy measure is then completely characterised by knowledge of the Lévy copula and the margins. Here and henceforth, by the *margins* of an m-dimensional Lévy measure ν (or distribution μ) we will always mean the m one-dimensional margins, which are obtained as projections of ν (or μ) onto the coordinate axes.

An advantage of modelling dependence via Lévy copulas is that the resulting probability law is automatically infinitely divisible. From the applied point of view, the usefulness of Lévy copulas hinges to a considerable extent on how feasible it is to obtain insight into relevant properties of the corresponding probability distributions. Much theoretical information in this regard can be gleaned from the book by Sato [16], while numerically there are now powerful methods that in many cases allow rather easy simulation of a probability law from its Lévy measure. In this latter respect, see Cont and Tankov [8] and references given there.

The present paper discusses some aspects of the Lévy copula concept. For simplicity, we consider only Lévy measures and Lévy copulas living on $\mathbb{R}_+^m$, where $\mathbb{R}_+ := [0,\infty)$. In the next section, we recall Tankov's definition of Lévy copulas and fix some notation used throughout the paper. Furthermore, we show that any (positive) Lévy copula defines itself a Lévy measure with 1-stable margins, when transformed under the mapping $Q_m : [0,\infty]^m \to [0,\infty]^m$, $(x_1,\ldots,x_m) \mapsto (x_1^{-1},\ldots,x_m^{-1})$. The latter transformation plays a natural role in the concept of Lévy copulas, and has also many relations to a mapping Υ_0 of Lévy measures recently introduced and studied by Barndorff-Nielsen and Thorbjørnsen [5, 6] and, in a multivariate version, by Barndorff-Nielsen, Maejima and Sato [2]. This will be discussed in Section 5.

Section 3 is concerned with a limit result for sequences of Lévy measures and Lévy copulas: we show that a sequence of Lévy measures converges vaguely to another Lévy measure if and only if the marginal Lévy measures converge vaguely, and the Lévy copulas converge pointwise on a suitable subset of $[0,\infty]^m$.

Section 4 discusses the special class of homogeneous Lévy copulas in more detail. They arise naturally as Lévy copulas which are constant in time for Lévy processes: if $(L^{(t)})_{t\geq 0}$ is a Lévy process with Lévy measure $\nu^{(t)}$ at time t and if the Lévy copula $C^{(1)}$ of $\nu^{(1)}$ is homogeneous, then $C^{(1)}$ is also a Lévy copula for $\nu^{(t)}$ for any $t > 0$. Furthermore, homogeneous Lévy copulas constitute the class of possible limits of Lévy copulas of Lévy processes as time approaches 0 or ∞. We additionally characterise homogeneous Lévy copulas as those for which the Lévy measure they define (via Q_m) is 1-stable.

In Section 5 we introduce the mapping Υ of Barndorff-Nielsen and Thorbjørnsen [5, 6] and Barndorff-Nielsen, Maejima and Sato [2], which will play a crucial role in Section 6 for the construction of Lévy measures with special properties. The mapping Υ maps the class of infinitely divisible distributions bijectively onto the Bondesson class. We discuss how Lévy copulas transform if the mapping Υ is applied, with particular emphasis on homogeneous Lévy copulas.

Section 6 is concerned with the construction of Lévy measures and distributions with special structures and prescribed margins. Suppose that $\mu_1, \ldots, \mu_m$ are one-dimensional infinitely divisible distributions, all of which are in the Bondesson class or Thorin class or are self-decomposable, respectively. Then using any Lévy copula gives an infinitely divisible distribution μ with margins $\mu_1, \ldots, \mu_m$. However, μ itself does not necessarily belong to the Bondesson class or to the Thorin class or to the class of self-decomposable laws, i.e. not every Lévy copula gives rise to such distributions. Here, we shall show how all distributions in the Bondesson class, Thorin class or the class of self-decomposable laws, with prescribed margins, can be obtained. For the Bondesson class, this is achieved in Section 6.1 with the help of the mapping Υ. Several examples are given, including stable, gamma and inverse Gaussian margins. In Section 6.2 a similar procedure is developed for self-decomposable distributions. Here, the role of the mapping Υ is replaced by a mapping Φ. The latter was shown by Sato and Yamazato [17] to map the class of all infinitely divisible distributions integrating $\max(1, \log|x|)$ bijectively onto the class of self-decomposable distributions. Finally, combining the mappings Φ and Υ, in Section 6.3 it is shown how to construct multivariate distributions in the Thorin class with prescribed margins in the Thorin class.

2 Lévy copulas and the derived Lévy measure

Recall that a *Lévy measure* is a measure ν on $\mathbb{R}^m$ which has no atom at zero and satisfies $\int_{\mathbb{R}^m}(|x|^2 \wedge 1)\nu(dx) < \infty$, where $|x|^2 = x_1^2 + \ldots + x_m^2$ denotes the Euclidean norm of $x = (x_1, \ldots, x_m)$. We call a Lévy measure *positive* if its support is contained in $\mathbb{R}_+^m = [0, \infty)^m$. For simplicity we shall restrict attention to the class $\mathcal{L}_+^m$ of positive Lévy measures.

Define the bijection

$$Q := Q_m : [0, \infty]^m \to [0, \infty]^m, \quad (x_1, \ldots, x_m) \mapsto (x_1^{-1}, \ldots, x_m^{-1}),$$

where $1/0$ has to be interpreted as ∞, and $1/\infty$ as 0. Then for $\nu \in \mathcal{L}_+^m$, define the measure χ as the image measure of ν under the mapping Q, i.e.

$$\chi(B) := (Q\nu)(B) = \nu(Q^{-1}(B))$$

for any Borel set B in $[0,\infty]^m$. Note that χ then does not have positive measure on hyperplanes of the form $\{(x_1,\dots,x_m) \in [0,\infty]^m : x_k = 0\}$ for fixed k, but can have positive measure on lines like $(0,\infty] \times \{\infty\} \times \ldots \times \{\infty\}$. However, since ν is a Lévy measure, χ is finite on any closed rectangle in $[0,\infty]^m$ not containing $(\infty,\dots,\infty)$. Then define the *volume function* $F = F_\nu : [0,\infty]^m \to [0,\infty]$ *of* ν as

$$F(x_1,\dots,x_m) := \begin{cases} \chi([0,x_1] \times \ldots \times [0,x_m]), & (x_1,\dots,x_m) \neq \{\infty,\dots,\infty\} \\ \infty, & (x_1,\dots,x_m) = (\infty,\dots,\infty). \end{cases}$$

Note that $F(\infty,\dots,\infty) = \chi([0,\infty]^m)$ if and only if ν is infinite. It is convenient when working with Lévy copulas to define $F(\infty,\dots,\infty) := \infty$ even for finite Lévy measures as above. This does not alter anything, since ν is completely described by knowledge of F on $[0,\infty]^m \setminus \{\infty,\dots,\infty\}$.

For $\nu \in \mathcal{L}_+^m$, denote the (one-dimensional) margins of ν by $\nu_1,\dots,\nu_m$. These margins are one-dimensional Lévy measures. In fact, $\nu_1,\dots,\nu_m$ are the Lévy measures of the one-dimensional margins of the probability measure corresponding to ν. To each of them we can associate the measure $\chi_k := Q_1\nu_k$ and thus define the volume function F_k of ν_k. Then $F_k(x_k) = F(\infty,\dots,\infty,x_k,\infty,\dots,\infty)$ for any $x_k \in [0,\infty]$ and we refer to F_k ($k = 1,\dots,m$) as the *marginal volume functions* of ν.

In analogy to probabilistic copulas, Tankov [13] defines a *(positive) Lévy copula* to be a function $C : [0,\infty]^m \to [0,\infty]$ such that $C(x_1,\dots,x_m) = 0$ if at least one of the x_i is zero (*groundedness*) and

$$C(\infty,\dots,\infty,x_k,\infty,\dots,\infty) = x_k \quad \forall\, x_k \in [0,\infty], \quad k = 1,\dots,m, \tag{2.1}$$

and such that C is an *m-increasing function*, i.e. $C(x_1,\dots,x_m) \neq \infty$ if $x_1,\dots,x_m$ are not all ∞, and for any set B of the form $B = (a_1,b_1] \times \ldots \times (a_m,b_m]$ with $0 \le a_k < b_k \le \infty$ it holds that $\sum \operatorname{sgn}(c)\, C(c) \ge 0$, where the sum is taken over all vertices $c = (c_1,\dots,c_m)$ of B, and sgn (c) is defined as

$$\operatorname{sgn}(c) = \begin{cases} 1, & \text{if } c_k = a_k \text{ for an even number of vertices,} \\ -1, & \text{if } c_k = a_k \text{ for an odd number of vertices.} \end{cases}$$

The most important feature of Lévy copulas is that they allow to separate the margins and the dependence structure of Lévy measures. This is made manifest in the following version of Sklar's theorem, proved by Tankov [13].

Theorem 2.1 *Let $\nu \in \mathcal{L}_+^m$ with volume function F and marginal volume functions $F_1,\dots,F_m$. Then there exists a (positive) Lévy copula C such that*

$$F(x_1,\dots,x_m) = C(F_1(x_1),\dots,F_m(x_m)) \quad \forall\ x_1,\dots,x_m \in [0,\infty]. \tag{2.2}$$

The Lévy copula C is uniquely determined on $Ran\, F_1 \times \ldots \times Ran\, F_m$. Conversely, if C is a positive Lévy copula and $F_1, \ldots, F_m$ are volume functions of one-dimensional positive Lévy measures $\nu_1, \ldots, \nu_m$, then (2.2) defines a Lévy measure $\nu \in \mathcal{L}_+^m$ with volume function F and marginal Lévy measures $\nu_1, \ldots, \nu_k$.

We shall refer to any Lévy copula C satisfying (2.2) as a Lévy copula *associated with* $\nu \in \mathcal{L}_+^m$.

We proceed to show that Lévy copulas can be regarded as transformations of special Lévy measures: let $C : [0, \infty]^m \to [0, \infty]$ be a Lévy copula. That C is m-increasing means that C defines a measure χ_C on $(0, \infty]^m \backslash \{(\infty, \ldots, \infty)\}$ such that $\chi_C((a_1, b_1] \times \ldots \times (a_m, b_m]) = \sum \operatorname{sgn}(c)\, C(c)$ for $0 \le a_k < b_k \le \infty$, $k = 1, \ldots, m$, where the sum is taken over all vertices c as above. We extend this measure to $[0, \infty]^m$ by setting

$$\chi_C(\{(\infty, \ldots, \infty)\}) = \chi_C([0, \infty]^{k-1} \times \{0\} \times [0, \infty]^{m-k}) = 0, \quad 1 \le k \le m. \tag{2.3}$$

Then using the fact that C is grounded, we obtain

$$\chi_C([0, b_1] \times \ldots \times [0, b_m]) = \chi_C((0, b_1] \times \ldots \times (0, b_1]) = C(b_1, \ldots, b_m), \tag{2.4}$$

for $0 \le b_1, \ldots, b_m \le \infty$. Condition (2.1) means that χ_C has uniform margins, i.e.

$$\chi_C([0, \infty]^{k-1} \times [0, x_k] \times [0, \infty]^{m-k}) = x_k, \quad k = 1, \ldots, m. \tag{2.5}$$

Furthermore, it is easy to see that for any positive measure χ on $[0, \infty]^m$ satisfying (2.3) and having uniform margins, (2.4) defines a unique Lévy copula C such that $\chi_C = \chi$. Applying the map $Q_m^{-1} = Q_m$ to χ_C gives another measure ν_C. We summarize this in the following

Definition 2.2 For any (positive) Lévy copula C, the measure χ_C is defined to be the unique measure on $[0, \infty]^m$ satisfying (2.3) – (2.5). The measure ν_C is defined as

$$\nu_C := Q_m^{-1} \chi_C. \tag{2.6}$$

The following Theorem then shows that ν_C is a Lévy measure with 1-stable margins (i.e. there are constants $\delta_k \ge 0$, $k = 1, \ldots, m$, such that the marginal volume functions of ν_C are equal to $[0, \infty] \to [0, \infty]$, $x_k \mapsto \delta_k x_k$).

Theorem 2.3 *If C is an m-dimensional Lévy copula, then the measure ν_C is a Lévy measure with 1-stable margins. More precisely, the marginal volume functions of ν_C are equal to $[0, \infty] \to [0, \infty]$, $x_k \mapsto x_k$, and C is the volume function of ν_C. The Lévy measure ν_C is not of finite variation, i.e.*

$\int_{|x|<1} |x| \nu_C(dx) = \infty$. *Conversely, if* $\nu \in \mathcal{L}^m_+$ *is any Lévy measure with marginal volume functions* $[0,\infty] \to [0,\infty]$, $x_k \mapsto x_k$, *then there exists a unique Lévy copula* C *such that* $\nu_C = \nu$.

Proof. Let C be a Lévy copula. Since χ_C is finite outside neighbourhoods of $(\infty,\dots,\infty)$, ν_C is finite outside neighbourhoods of the origin. Denote by $(\chi_C)_k$ the k-th marginal measure of χ_C. Then

$$\begin{aligned}
&\int_{[0,1]^m} \sum_{k=1}^m x_k^2 \, d\nu_C(x_1,\dots,x_m) \\
&= \sum_{k=1}^m \int_{[1,\infty]^m} \frac{1}{y_k^2} \, d\chi_C(y_1,\dots,y_m) \\
&\le \sum_{k=1}^m \int_{[0,\infty]^{k-1}\times[1,\infty]\times[0,\infty]^{m-k}} \frac{1}{y_k^2} \, d\chi_C(y_1,\dots,y_m) \\
&= \sum_{k=1}^m \int_1^\infty \frac{1}{y_k^2} \, d(\chi_C)_k(y_k) \\
&= \sum_{k=1}^m \int_1^\infty \frac{1}{y_k^2} \, dy_k < \infty.
\end{aligned}$$

Hence, ν_C is a Lévy measure. Furthermore,

$$\int_{[0,1]^m} \sum_{k=1}^m x_k \, d\nu_C(x_1,\dots,x_m) \ge \int_{[1,\infty]^m} \frac{1}{y_1} \, d\chi_C(y_1,\dots,y_m) =$$

$$\int_{[1,\infty]\times[0,\infty]^{m-1}} \frac{1}{y_1} d\chi_C(y_1,\dots,y_m) - \int_{[1,\infty]\times([0,\infty]^{m-1}\setminus[1,\infty]^{m-1})} \frac{1}{y_i} d\chi_C(y_1,\dots,y_m).$$

But the first integral is equal to $\int_1^\infty \frac{1}{y_1} \, dy_1 = \infty$, while the second integral is finite since $\chi_C([1,\infty]\times([0,\infty]^{m-1}\setminus[1,\infty]^{m-1})) < \infty$. Thus, ν_C is not of finite variation. The remaining assertions are clear from the preceding discussion. ∎

3 Lévy copulas and convergence of Lévy measures

In this section we obtain a limit result for Lévy measures, characterising convergence of a sequence of Lévy measures by convergence of the margins and of the Lévy copulas. Let μ be an infinitely divisible distribution on $\mathbb{R}^m$

with characteristic triplet (A, ν, γ). Recall that ν is completely characterised by (A, ν, γ), and that the characteristic function $\widehat{\mu}$ of μ satisfies

$$\widehat{\mu}(z) = \exp\{-\frac{1}{2}\langle z, Az\rangle + i\langle \gamma, z\rangle + \int_{\mathbb{R}^m} (e^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle 1_{|x|\leq 1})\, d\nu(x)\}, \quad z \in \mathbb{R}^m.$$

Here, A is a symmetric nonnegative-definite $m \times m$-matrix, ν is the Lévy measure of μ, and $\gamma \in \mathbb{R}^m$ is a constant. $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product on $\mathbb{R}^m$.

Denote by $C_{\#}$ the class of bounded continuous functions from $\mathbb{R}^m$ to $\mathbb{R}$ vanishing in a neighbourhood of the origin. Let $(\mu^{(n)})_{n\in\mathbb{N}}$ be a sequence of infinitely divisible distributions on $\mathbb{R}^m$ with characteristic triplets $(A^{(n)}, \nu^{(n)}, \gamma^{(n)})$. For any $\varepsilon > 0$ define symmetric nonnegative-definite matrices $A^{(n),\varepsilon}$ by

$$\langle z, A^{(n),\varepsilon} z\rangle = \langle z, A^{(n)} z\rangle + \int_{|x|\leq\varepsilon} \langle z, x\rangle^2 \, d\nu^{(n)}(x), \quad z \in \mathbb{R}^m.$$

The following Theorem can be found in Sato [16], Theorem 8.7:

Theorem 3.1 *With the notations above, $\mu^{(n)}$ converges weakly to the infinitely divisible distribution μ as $n \to \infty$ if and only if*

$$\begin{aligned} \lim_{n\to\infty} \int_{\mathbb{R}^m} f(x)\, d\nu^{(n)}(x) &= \int_{\mathbb{R}^m} f(x)\, d\nu(x) \quad \forall\, f \in \mathcal{C}_{\#}, & (3.1)\\ \lim_{\varepsilon\to 0} \limsup_{n\to\infty} |\langle z, A^{(n),\varepsilon} z\rangle - \langle z, Az\rangle| &= 0 \quad \forall\, z \in \mathbb{R}^m, & (3.2)\\ \lim_{n\to\infty} \beta^{(n)} &= \beta, & (3.3) \end{aligned}$$

where

$$\beta := \gamma - \int_{|x|\leq 1} x|x|^2 \, d\nu(x)$$

and $\beta^{(n)}$ is defined similarly.

We see that the appropriate convergence concept for Lévy measures is described by relation (3.1). We shall write $\nu^{(n)} \xrightarrow{\#} \nu$ for this type of convergence of Lévy measures. In order to prove our main result of this section, we need to show the following lemma:

Lemma 3.2 *Let $(\nu^{(n)})_{n\in\mathbb{N}} \subset \mathcal{L}_+^m$, $\nu \in \mathcal{L}_+^m$, with volume functions $F^{(n)}$ and F, respectively. For any $i \in \{1, \ldots, m\}$, set*

$$\mathcal{A}_i := \{x_i \in [0, \infty) : F_i \text{ continuous in } x_i\} \cup \{\infty\},$$

where the F_i denote the marginal volume functions of ν. Then $\nu^{(n)} \stackrel{\#}{\to} \nu$ as $n \to \infty$ if and only if $F^{(n)}(x)$ converges pointwise to $F(x)$ at any $x \in \mathcal{A}_1 \times \ldots \times \mathcal{A}_m$. Moreover, this is equivalent to vague convergence of $\nu^{(n)}$ to ν on the set $E := [0,\infty]^m \setminus \{0,\ldots,0\}$.

Proof. For $i = 1,\ldots,m$, set $B_i := Q_1\mathcal{A}_i$, and $\widetilde{B}_i := B_i \setminus \{0,\infty\}$. Then

$$F\left(\frac{1}{y_1},\ldots,\frac{1}{y_m}\right) = \nu([y_1,\infty] \times \ldots \times [y_m,\infty]).$$

Let $\chi = Q_m\nu$ and $E := [0,\infty]^m \setminus \{(0,\ldots,0)\}$.

It is clear that $\nu^{(n)} \stackrel{\#}{\to} \nu$ as $n \to \infty$ implies vague convergence of $\nu^{(n)}$ to ν. So suppose that $\nu^{(n)}$ converges vaguely on E to ν. Let $k \in \{1,\ldots,m\}$ and let $y = (y_1,\ldots,y_m)$ such that $y_i \in \widetilde{B}_i$ for $0 \le i \le k$, and $y_i = 0$ for $i > k$. We have to show that $\nu^{(n)}([y_1,\infty] \times \ldots \times [y_m,\infty])$ converges to $\nu([y_1,\infty] \times \ldots \times [y_m,\infty])$ as $n \to \infty$. Let $g := 1_{[y_1,\infty]\times\ldots\times[y_m,\infty]}$. For any integer $t > (\min\{y_1,\ldots,y_k\})^{-1}$ and $1 \le i \le k$ set

$$y_i' := y_i - \frac{1}{t}, \quad y_i'' := y_i + \frac{1}{t}.$$

Furthermore, define

$$\begin{aligned} K_1 &:= [y_1,\infty] \times \ldots \times [y_m,\infty], \\ U_{1,t} &:= (y_1',\infty] \times \ldots \times (y_k',\infty] \times [0,\infty]^{m-k}, \\ K_{2,t} &:= [y_1'',\infty] \times \ldots \times [y_k'',\infty] \times [0,\infty]^{m-k}, \\ U_2 &:= (y_1,\infty] \times \ldots \times (y_k,\infty] \times [0,\infty]^{m-k}. \end{aligned}$$

Then K_1 and $K_{2,t}$ are compact in E and have (in the topology of E) open neighbourhoods $U_{1,t}$ and U_2, respectively. Urysohn's Lemma (e.g. Simmons [18], page 135) now implies the existence of continuous functions h_t and f_t on E such that $0 \le f_t \le g \le h_t \le 1$ and $h_t = 1$ on K_1, $f_t = 1$ on $K_{2,t}$, and the supports of h_t and f_t are contained in $U_{1,t}$ and U_2, respectively. Since $\nu^{(n)}$ converges vaguely to ν as $n \to \infty$, we conclude

$$\int_E f_t\, d\nu \le \liminf_{n\to\infty} \int_E g\, d\nu^{(n)} \le \limsup_{n\to\infty} \int_E g\, d\nu^{(n)} \le \int_E h_t\, d\nu.$$

Setting

$$D_t := \bigcup_{i=1}^{k} [0,\infty]^{i-1} \times [y_i', y_i''] \times [0,\infty]^{m-i}\big),$$

we have $\{h_t \neq f_t\} \subset D_t$. Since $y_i \in \widetilde{B}_i$ for $i = 1, \ldots, k$, $\nu(D_t) \to 0$ as $t \to \infty$. But this implies

$$\liminf_{n\to\infty} \int_E g\, d\nu^{(n)} = \limsup_{n\to\infty} \int_E g\, d\nu^{(n)} = \int_E g\, d\nu.$$

This shows that $F^{(n)}(x)$ converges pointwise to $F(x)$ at any $x \in \mathcal{A}_1 \times \ldots \times \mathcal{A}_m$.

Now suppose that $F^{(n)}$ converges pointwise to F on $\mathcal{A}_1 \times \ldots \times \mathcal{A}_m$. To show vague convergence of $\nu^{(n)}$ to ν on E, let f be a continuous function on E with compact support in E. Choose $y_i \in \widetilde{B}_i$ $(i = 1, \ldots, m)$ such that the support of f is contained in the compact space $E' := E \setminus ([0, y_1) \times \ldots \times [0, y_m))$. Denote by $\mathcal{R}$ the class of all rectangles of the form $R = R_1 \times \ldots \times R_m$, where $R_i = [a_i, b_i)$ or $R_i = [a_i, \infty]$ with $a_i \in B_i \setminus \{\infty\}$, $b_i \in B_i$, $a_i < b_i$ and not all $a_i = 0$. Note that if $R_i = [a_i, \infty]$ for some i, then the Lévy measure of R is the same as if R_i is replaced by $[a_i, \infty)$. Then for each R, we have

$$\nu(R) = \chi\left(\left(\frac{1}{b_1}, \frac{1}{a_1}\right] \times \ldots \times \left(\frac{1}{b_m}, \frac{1}{a_m}\right]\right) = \sum \operatorname{sgn}(c) F(c),$$

where the sum is taken over all vertices c of $(\frac{1}{b_1}, \frac{1}{a_1}] \times \ldots \times (\frac{1}{b_m}, \frac{1}{a_m}]$. By assumption, this then implies that $\nu^{(n)}(R)$ converges to $\nu(R)$ as $n \to \infty$ for any $R \in \mathcal{R}$. It then follows that $\nu^{(n)}(S)$ converges to $\nu(S)$ for any S which is a finite union of elements of $\mathcal{R}$. In particular, $\lim_{n\to\infty} \nu^{(n)}(E') = \nu(E')$. Since B_i is dense in $[0, \infty]$ and $0, \infty \in B_i$, every (in the topology of E') open set G in E' is a countable union of elements of $\mathcal{R}$. Then if $G \subset E'$ is open in E' and if $S \subset G$ is a finite union of elements of $\mathcal{R}$, it follows that

$$\nu(S) = \lim_{n\to\infty} \nu^{(n)}(S) \leq \liminf_{n\to\infty} \nu^{(n)}(G),$$

and hence $\nu(G) \leq \liminf_{n\to\infty} \nu^{(n)}(G)$. If $K \subset E'$ is compact, then

$$\begin{aligned}\limsup_{n\to\infty} \nu^{(n)}(K) &= \lim_{n\to\infty} \nu^{(n)}(E') - \liminf_{n\to\infty} \nu^{(n)}(E' \setminus K) \\ &\leq \nu(E') - \nu(E' \setminus K) = \nu(K).\end{aligned}$$

But this means that $\nu^{(n)}_{|E'}$ converges vaguely to $\nu_{|E'}$ in E' (e.g. Resnick [12], Proposition 3.12). In particular, $\int_E f\, d\nu^{(n)} \to \int_E f\, d\nu$, since the support of f is contained in E'. Since f was arbitrary, this implies vague convergence of $\nu^{(n)}$ to ν on E.

Now let f be a continuous function on $[0, \infty)^m$, bounded by a constant M and vanishing in a neighbourhood of the origin. Let $\varepsilon > 0$ and choose $y_i \in \widetilde{B}_i$ such that $\nu([0, \infty)^m \setminus ([0, y_1) \times \ldots \times [0, y_m))) < \frac{\varepsilon}{2M}$. Let g_ε be a continuous function with compact support in E such that $f = g_\varepsilon$ on $[0, y_1] \times \ldots \times [0, y_m]$ and g_ε is bounded by M. Since $y_i \in \widetilde{B}_i$, for sufficiently large n follows, by assumption, that $\nu^{(n)}([0,\infty)^m \setminus ([0, y_1) \times \ldots \times [0, y_m))) < \frac{\varepsilon}{M}$. This implies

$$\left| \int_E g_\varepsilon \, d\nu^{(n)} - \int_E f \, d\nu^{(n)} \right| \le \varepsilon, \quad \left| \int_E g_\varepsilon \, d\nu - \int_E f \, d\nu \right| \le \varepsilon.$$

Since $\lim_{n\to\infty} \int_E g_\varepsilon \, d\nu^{(n)} = \int_E g_\varepsilon \, d\nu$ by vague convergence, as already shown, it follows that $\lim_{n\to\infty} \int_E f \, d\nu^{(n)} = \int_E f \, d\nu$, i.e. $\nu^{(n)} \stackrel{\#}{\to} \nu$ as $n \to \infty$. ■

We can now show that a sequence of Lévy measures converges to a Lévy measure if and only if the margins converge and the Lévy copulas converge pointwise on a suitable subset. This is an analogue of a result of Lindner and Szimayer [11] for probabilistic copulas.

Theorem 3.3 *Let $(\nu^{(n)})_{n\in\mathbb{N}} \subset \mathcal{L}_+^m$, $\nu \in \mathcal{L}_+^m$, with margins $\nu_i^{(n)}$ and ν_i ($i = 1, \ldots, m$), and associated Lévy copulas $C^{(n)}$ and C, respectively. Then $\nu^{(n)} \stackrel{\#}{\to} \nu$ as $n \to \infty$ if and only if $\nu_i^{(n)} \stackrel{\#}{\to} \nu_i$ as $n \to \infty$ for $i = 1, \ldots, m$, and $C^{(n)}$ converges pointwise to C on $\mathrm{Ran}\, F_1 \times \ldots \times \mathrm{Ran}\, F_m$ as $n \to \infty$, where the F_i denote the marginal volume functions of ν. In that case, the convergence of $C^{(n)}$ to C is uniform on any set of the form $(\overline{\mathrm{Ran}\, F_1} \times \ldots \times \overline{\mathrm{Ran}\, F_m}) \cap (K_1 \times \ldots \times K_m)$, where K_i is a compact subset of $[0,\infty)$, or $K_i = \{\infty\}$.*

Proof. Since the proof is similar to the proof of Theorem 2.1 in Lindner and Szimayer [11], we only give a sketch of it. The main difference to the proof in [11] is that the Lipschitz continuity property has to be modified in the following sense: if D is any Lévy copula, if $1 \le k \le m$ and if $u_1, \ldots, u_k, v_1, \ldots, v_k \in [0, \infty)$, then

$$|D(u_1, \ldots, u_k, \infty, \ldots, \infty) - D(v_1, \ldots, v_k, \infty, \ldots, \infty)| \le \sum_{i=1}^k |u_i - v_i|. \quad (3.4)$$

This follows readily from the fact that any Lévy copula defines a measure with uniform margins. Let $\mathcal{M}_i := \{F_i(x_{u,i}) : x_{u,i} \in \mathcal{A}_i \setminus \{\infty\}\}$, where F_i and $\mathcal{A}_i$ are as in Lemma 3.2.

Suppose that $\nu^{(n)} \stackrel{\#}{\to} \nu$ as $n \to \infty$. Then $\nu_i^{(n)} \stackrel{\#}{\to} \nu_i$ by Lemma 3.2. Furthermore, for any $(u_1 \ldots, u_k) \in \mathcal{M}_1 \times \ldots \times \mathcal{M}_k$ such that $u_i = F_i(x_{u,i})$, $x_{u,i} \in \mathcal{A}_i$, we obtain using (3.4), similarly to the proof in [11],

$$\begin{aligned} & |C^{(n)}(u_1, \ldots, u_k, \infty, \ldots, \infty) - C(u_1, \ldots, u_k, \infty, \ldots, \infty)| \\ \le \; & \sum_{i=1}^k |F_i(x_{u,i}) - F_i^{(n)}(x_{u,i})| \\ & + |F^{(n)}(x_{u,1}, \ldots, x_{u,k}, \infty, \ldots, \infty) - F(x_{u,1}, \ldots, x_{u,k}, \infty, \ldots, \infty)|. \end{aligned}$$

Lemma 3.2 then implies convergence of $C^{(n)}$ to C at $(u_1, \ldots, u_k, \infty, \ldots, \infty)$. Convergence on $\operatorname{Ran} F_1 \times \ldots \times \operatorname{Ran} F_m$ and the assertion on the uniform convergence follows as in [11].

For the converse, suppose that $\nu_i^{(n)} \xrightarrow{\#} \nu_i$ as $n \to \infty$, and that $C^{(n)}$ converges pointwise on $\mathcal{M}_1 \times \ldots \times \mathcal{M}_k \times \{\infty\} \times \ldots \times \{\infty\}$. Then for $x = (x_1, \ldots, x_k, \infty, \ldots, \infty)$ with $x_i \in \mathcal{A}_i \setminus \{\infty\}$, it follows as in [11] that

$$
\begin{aligned}
&|F^{(n)}(x) - F(x)| \leq \\
&\quad \sum_{i=1}^{k} |F_i^{(n)}(x_i) - F_i(x_i)| + \\
&\quad |C^{(n)}(F_1(x_1), \ldots, F_k(x_k), \infty, \ldots, \infty) - C(F_1(x_1), \ldots, F_k(x_k), \infty, \ldots, \infty)|.
\end{aligned}
$$

Lemma 3.2 then gives the claim. ■

Since any Lévy copula C defines itself a measure ν_C via Definition 2.2, it is natural to ask whether the pointwise convergence condition of $C^{(n)}$ can be replaced by vague convergence of $\nu_{C^{(n)}}$. Since the limit copula C is not necessarily unique if $\operatorname{Ran} F_i \neq [0, \infty]$ for some i, vague convergence is not to be expected in general. However, if $\operatorname{Ran} F_i = [0, \infty]$ for all $i = 1, \ldots, m$, then the statement on the pointwise convergence of $C^{(n)}$ in Theorem 3.3 can be replaced by vague convergence of $\nu_{C^{(n)}}$. This follows from the following corollary to Lemma 3.2.

Corollary 3.4 *Let $(C^{(n)})_{n \in \mathbb{N}}$ and C be Lévy copulas. Then $C^{(n)}$ converges pointwise on $[0, \infty]^m$ to C if and only if $\nu_{C^{(n)}} \xrightarrow{\#} \nu_C$ as $n \to \infty$.*

Recalling that weak convergence of infinitely divisible distributions can be described by convergence of the characteristic triplets as in Theorem 3.1, we obtain the following corollary to Theorem 3.3:

Corollary 3.5 *Let $(\mu^{(n)})_{n \in \mathbb{N}}$ and μ be infinitely divisible distributions with characteristic triplets $(A^{(n)}, \nu^{(n)}, \gamma^{(n)})$ and (A, ν, γ), such that ν and $\nu^{(n)}$ are in $\mathcal{L}_+^m$. Let $\mu^{(n)} = (\mu_1^{(n)}, \ldots, \mu_m^{(n)})$ and $\mu = (\mu_1, \ldots, \mu_m)$. Suppose that $A^{(n)}$ converges pointwise to A as $n \to \infty$. Then $\mu^{(n)}$ converges weakly to μ as $n \to \infty$ if and only if all the margins $\mu_i^{(n)}$ converge weakly to μ_i as $n \to \infty$, $i = 1, \ldots, m$, and the Lévy copula of ν_n converges pointwise to the Lévy copula of ν on $\operatorname{Ran} F_1 \times \ldots \times \operatorname{Ran} F_m$ as $n \to \infty$, where the F_i denote the marginal volume functions of ν.*

It should be noted that the assumption $\lim_{n \to \infty} A^{(n)} = A$ is somehow restrictive. It implies that in the limit the Lévy measures do not contribute to an extra Gaussian part. This then makes an easy description by the Lévy copula convergence feasible.

Proof. The "only-if"-direction is clear from Theorems 3.1 and 3.3, so we only have to show the converse. This is done by checking the conditions in Theorem 3.1. Here, (3.1) holds by Theorem 3.3. The characteristic triplet of μ_i is $(A_{ii}, \nu_i, \widetilde{\gamma}_i)$, where A_{ii} denotes the i'th diagonal element of A, $\widetilde{\gamma}_i = \gamma_i + \int_{\mathbb{R}^m_+} x_i(1_{|x_i|\le 1} - 1_{|x|\le 1})\, d\nu(x)$, and γ_i denotes the i'th coordinate of γ, see Sato [16], Proposition 11.10. Let $\widetilde{\beta}_i := \widetilde{\gamma}_i - \int_{|x_i|\le 1} x_i|x_i|^2\, d\nu_i(x_i)$. To show (3.2) and (3.3), note that convergence of $A^{(n)}$ to A implies convergence of $A^{(n)}_{ii}$ to A_{ii}. Since $\mu_i^{(n)}$ converges weakly to μ_i, Theorem 3.1 implies that

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\left|\int_{|x_i|\le\varepsilon} x_i^2\, d\nu_i^{(n)}(x_i)\right| = 0, \tag{3.5}$$

and that $\widetilde{\beta}_i^{(n)}$ converges to $\widetilde{\beta}_i$ as $n\to\infty$. Again, by convergence of A_n to A and (3.5) it then follows, for any $z\in\mathbb{R}^m$, that

$$\begin{aligned}
&\limsup_{\varepsilon\to 0}\limsup_{n\to\infty}|\langle z, A^{(n),\varepsilon}z\rangle - \langle z, Az\rangle| \\
&\le\ |z|^2 \limsup_{\varepsilon\to 0}\limsup_{n\to\infty}\left|\int_{|x|\le\varepsilon} |x|^2\, d\nu^{(n)}(x)\right| \\
&\le\ |z|^2 \limsup_{\varepsilon\to 0}\limsup_{n\to\infty}\sum_{i=1}^m\left|\int_{|x_i|\le\varepsilon} x_i^2\, d\nu_i^{(n)}(x_i)\right| = 0.
\end{aligned}\tag{3.6}$$

This shows (3.2). For (3.3), note that

$$\beta_i - \widetilde{\beta}_i = \int_{|x_i|\le 1} x_i^3\, d\nu_i(x_i) - \int_{|x|\le 1} x_i|x|^2\, d\nu(x) - \int_{\mathbb{R}^m_+} x_i(1_{|x_i|\le 1} - 1_{|x|\le 1})\, d\nu(x),$$

where β_i denotes the i'th coordinate of β (as appearing in Theorem 3.1). From $\nu^{(n)} \overset{\#}{\to} \nu$, (3.5) and (3.6) one can show that $\beta_i^{(n)} - \widetilde{\beta}_i^{(n)}$ converges to $\beta_i - \widetilde{\beta}_i$ as $n\to\infty$. Since $\widetilde{\beta}_i^{(n)}$ converges to $\widetilde{\beta}_i$, this proves that $\beta_i^{(n)}$ converges to β_i as $n\to\infty$, verifying (3.3). This finishes the proof. ∎

4 Homogeneous Lévy copulas

In this section we discuss the special class of homogeneous Lévy copulas. A Lévy copula C is called *homogeneous* (of order 1), if

$$C(u_1,\dots,u_m) = t\,C(u_1/t,\dots,u_m/t) \quad \forall\, u_1,\dots,u_m\in[0,\infty] \quad \forall\, t>0.$$

These Lévy copulas appear naturally, because they correspond to Lévy processes with time constant Lévy copulas. We study this in subsection 4.1. In 4.2 we investigate further properties of homogeneous Lévy copulas, which are not in the dynamical context of Lévy processes.

Examples of homogeneous Lévy copulas are the Lévy copula of *complete dependence*

$$C(u_1, \ldots, u_m) = \min\{u_1, \ldots, u_m\},$$

the copula of *independence*

$$C(u_1, \ldots, u_m) = \sum_{i=1}^{m} u_i \, 1_{u_1 = \ldots = u_{i-1} = u_{i+1} = \ldots = u_m = \infty},$$

and the family of *Clayton Lévy copulas*, defined for $\theta > 0$ by

$$C(u_1, \ldots, u_m) = \left(\sum_{i=1}^{m} u_i^{-\theta} \right)^{-1/\theta},$$

$u_1, \ldots, u_m \in [0, \infty]$, see Cont and Tankov [8], Chapter 5. An example of a non-homogeneous Lévy copula is given in Example 4.2.

Euler showed that a continuously differentiably function $f : (0, \infty)^k \to \mathbb{R}$ is homogeneous (of order 1) if and only if Euler's formula

$$\sum_{i=1}^{k} u_i \frac{\partial f(u_1, \ldots, u_k)}{\partial u_i} = f(u_1, \ldots, u_k) \tag{4.1}$$

holds on $(0, \infty)^k$. We will not use this characterisation in the sequel. However, note that for Lévy copulas (4.1) not only has to be checked for C on $(0, \infty)^m$, but also on hyperplanes where one or more of the u_i are ∞. For example, on $\{u_{m-1} = u_m = \infty\}$, (4.1) has to be checked for the function

$$(0, \infty)^{m-2} \to \mathbb{R}, \quad (u_1, \ldots, u_{m-2}) \mapsto C(u_1, \ldots, u_{m-2}, \infty, \infty)$$

(provided it is continuously differentiable).

4.1 Time-wise properties of Lévy copulas of Lévy processes

Let $(L^{(t)})_{t \geq 0}$ be a Lévy process in $\mathbb{R}^m$. Then at any time t, $L^{(t)}$ has an infinitely divisible distribution. If $\nu^{(t)}$ denotes the Lévy measure of $L^{(t)}$ then

$\nu^{(t)} = t\nu^{(1)}$. Now suppose that $\nu^{(1)} \in \mathcal{L}_+^m$ with associated Lévy copula $C^{(1)}$. Then it follows readily that

$$C^{(t)}(u_1, \ldots, u_m) := t\, C^{(1)}(u_1/t, \ldots, u_m/t), \quad \forall\, u_1, \ldots, u_m \in [0, \infty], \quad (4.2)$$

gives a Lévy copula associated with $\nu^{(t)}$. In particular, if $C^{(1)}$ is homogeneous, then the Lévy process is described by the same Lévy copula $C^{(1)}$ at any time t. On the other hand, if C is a Lévy copula and $(L^{(t)})_{t\geq 0}$ is a Lévy process such that C is associated with $\nu^{(t)}$ for every t, and if there is some $\varepsilon > 0$ such that $\mathrm{Ran}\, F_i^{(1)} \supset [0, \varepsilon]$ for all $i = 1, \ldots, m$ (where $F_i^{(1)}$ denote the marginal volume functions of $\nu^{(1)}$), then C must be homogeneous. This follows from the fact that the Lévy copula of $\nu^{(t)}$ is unique on $t(\mathrm{Ran}\, F_1^{(1)} \times \ldots \times \mathrm{Ran}\, F_m^{(1)})$ for any $t > 0$ by Theorem 2.1, hence on $[0, \infty]^m$. Thus, the Lévy copulas at times t and 1 satisfy (4.2), showing that C is homogeneous.

We now turn to convergence of Lévy copulas of Lévy processes as time goes to infinity and to zero. Again, the homogeneous Lévy copulas appear naturally as possible limit copulas.

Theorem 4.1 *Let $(L^{(t)})_{t\geq 0}$ be a Lévy process with positive Lévy measure and with Lévy copula $C^{(t)}$ at time t given by (4.2). Then:*
(a) $C^{(t)}$ converges pointwise to a finite function D on $[0, \infty]^m \setminus \{(\infty, \ldots, \infty)\}$ as $t \to \infty$ if and only if all for all directions $(u_1, \ldots, u_m) \in \mathbb{R}_+^m$ the directional derivative of $C^{(1)}$ exists at the origin. In that case, the function D is a homogeneous Lèvy copula. The convergence is uniform on $[0, \infty]^m$ if and only if $C^{(1)}$ is homogeneous.
(b) If $C^{(t)}$ converges pointwise to a finite function D on $[0, \infty]^m \setminus \{(\infty, \ldots, \infty)\}$ as $t \to 0$, then the function D is a homogeneous Lévy copula. $C^{(t)}$ converges uniformly on $[0, \infty]^m$ to D as $t \to 0$ if and only if $\|C^{(1)} - D\|_\infty < \infty$, where $\|\cdot\|_\infty$ denotes the supremum norm on $[0, \infty]^m \setminus \{\infty, \ldots, \infty\}$.

Proof. From (4.2) follows readily that if $C^{(t)}$ converges pointwise to a finite function D on $[0, \infty]^m \setminus \{\infty, \ldots, \infty\}$ as $t \to \infty$ or $t \to 0$, then D must be a homogeneous Lévy copula. Further, noting that for $u = (u_1, \ldots, u_m)$ and $t > 0$ we have

$$t\, C^{(1)}(u/t) = \frac{C^{(1)}(u/t) - C^{(1)}(0)}{1/t},$$

it follows that $\lim_{t\to\infty} C^{(t)}(u)$ exists if and only if if the directional derivative of $C^{(1)}$ in direction u exists at the origin. If $C^{(1)}$ is homogeneous, then uniform convergence of $C^{(t)}$ as $t \to \infty$ is clear. For the converse, suppose uniform convergence, but that $C^{(1)}$ is not homogeneous. Then there is $u \in [0, \infty]^m$ and $t_0 > 0$ such that $|C^{(1)}(t_0 u) - t_0 C^{(1)}(u)| =: \varepsilon > 0$. From the uniform convergence follows the existence of $t_1 > 0$ such that $|t\, C^{(1)}(v/t) - D(v)| \leq \varepsilon$ for any $v \in [0,\infty]^m \setminus \{\infty,\ldots,\infty\}$ and any $t > t_1$. Using the homogeneity of D we conclude for $t > t_1$

$$\begin{aligned} |tt_0 C^{(1)}(u) - tt_0 D(u)| &= \left|t_0 t C^{(1)}(tu/t) - t_0 D(tu)\right| \leq t_0\varepsilon, \\ |tC^{(1)}(t_0 u) - tt_0 D(u)| &= \left|tC^{(1)}(tt_0 u/t) - D(tt_0 u)\right| \leq \varepsilon. \end{aligned}$$

This implies

$$t\varepsilon = t|t_0 C^{(1)}(u) - C^{(1)}(t_0 u)| \leq (1+t_0)\varepsilon \quad \forall\, t \geq t_1,$$

which clearly is a contradiction. This proves (a).

For the proof of (b), note that, by homogeneity of D, $C^{(t)}$ converges uniformly to D as $t \to 0$ if and only if $t|C^{(1)}(v/t) - D(v/t)|$ converges uniformly in v to 0. But this is equivalent to $\|C^{(1)} - D\|_\infty < \infty$. ■

We give a few examples which are concerned with the convergence of Lévy copulas of Lévy processes.

Example 4.2 Consider a Lévy process such that the Lévy copula at time 1 is given by

$$C^{(1)}(u_1,\ldots,u_m) := \log\left(\left(\sum_{i=1}^m \frac{e^{-u_i}}{1-e^{-u_i}}\right)^{-1} + 1\right).$$

This Lévy copula was introduced in Tankov [13], see Cont and Tankov [8], page 150. Let $D_\infty(u_1,\ldots,u_m) := \left(\sum_{i=1}^m (1/u_i)\right)^{-1}$ and $D_0(u_1,\ldots,u_m) := \min\{u_1,\ldots,u_m\}$. Then easy calculations show that $C^{(t)}$ converges pointwise to D_∞ as $t \to \infty$. The convergence is not uniform, since $C^{(1)}$ is not homogeneous. On the other hand, it is easy to show that $\|C^{(1)} - D_0\|_\infty < \infty$, so that $C^{(t)}$ converges uniformly to D_0 as $t \to 0$.

Example 4.3 Let the probabilistic copulas H_1 and H_2 on $[0,1]^2$ be given by $H_1(u,v) := uv$ and $H_2(u,v) := \min\{u,v\}$. For any integer $n \in \mathbb{Z}$ and $u,v \in [2^n, 2^{n+1}]$ let

$$C^{(1)}(u,v) := 2^n + 2^n H_i\left(\frac{u-2^n}{2^n}, \frac{v-2^n}{2^n}\right),$$

where $i = 1$ if n is odd and $i = 2$ if n is even. If $u \in [2^n, 2^{n+1}]$ for some n and $v > 2^{n+1}$, set $C^{(1)}(u,v) := C^{(1)}(u, 2^{n+1})$, and if $u > v$ set $C^{(1)}(u,v) = C^{(1)}(v,u)$. It can be easily checked that $C^{(1)}$ defines a Lévy copula. Let $u_n :=$

$2^n + 2^{n-1}$. Then $C^{(1)}(u_n, u_n) = u_n$ if n is even, and $C^{(1)}(u_n, u_n) = 2^n + 2^{n-2}$ if n is odd. In particular,

$$\frac{C^{(1)}(u_n, u_n)}{u_n} = \begin{cases} 5/6, & n \text{ odd}, \\ 1, & n \text{ even}. \end{cases}$$

This shows that for a Lévy process with Lévy copula $C^{(t)}$ at time $t > 0$, $C^{(t)}(1,1)$ does neither converge as $t \to 0$ nor as $t \to \infty$.

There remains the question whether there are Lévy processes such that the Lévy copula $C^{(t)}$ converges pointwise but not uniformly as $t \to 0$. By now we have not been able to decide this question.

4.2 Further properties of homogeneous Lévy copulas

In this subsection we investigate further properties of homogeneous Lévy copulas. The following proposition shows that they are rarely associated with finite Lévy measures:

Proposition 4.4 *Let ν be a finite Lévy measure, concentrated on $(0,\infty)^m$, and suppose that the Lévy copula C associated with ν is homogeneous. Then C must be the Lévy copula of complete dependence, i.e.*

$$C(u_1, \ldots, u_m) = \min\{u_1, \ldots, u_m\} \quad \forall\, u_1, \ldots, u_m \in [0, \infty].$$

Proof. Denote by M the total mass of ν and its (marginal) volume functions by F_i and F. Then $\lim_{x_i \to \infty} F_i(x_i) = M$ for $i \in \{1, \ldots, m\}$, and $\lim_{x\to\infty} F(x, \ldots, x) = M$. Therefore $C(F_1(x), \ldots, F_m(x))$ converges to M as $x \to \infty$, by (2.2). From the continuity property (3.4) then follows $C(M, \ldots, M) = M$. Since C was assumed to be homogeneous, we conclude $C(u, \ldots, u) = u$ for any $u > 0$. Now let $u_1, \ldots, u_m \in [0, \infty]$ and suppose w.l.o.g. that their minimum is at u_1. Then

$$u_1 = C(u_1, \ldots, u_1) \le C(u_1, u_2, \ldots, u_m) \le C(u_1, \infty, \ldots, \infty) = u_1,$$

showing the claim. ■

The following theorem provides a stepping stone to Corollary 4.6 below, which characterises homogeneous Lévy copulas C in terms of the Lévy measure ν_C they define. We say that a Lévy measure is stable or self-decomposable, if it is the Lévy measure of a stable or self-decomposable infinitely divisible distribution, respectively. For the definitions and properties of such distributions, we refer to Sato [16], Chapters 13-15.

Theorem 4.5 *Let $\alpha \in (0,2)$ and ν be a Lévy measure with non-degenerate α-stable margins and associated Lévy copula C. Then*
(a) ν is stable if and only if ν_C is 1-stable.
(b) ν is selfdecomposable if and only if ν_C is selfdecomposable.

Proof. We first prove part (b). Let $F_i(x_i) = k_i x_i^\alpha$ ($k_i > 0$, $i = 1, \ldots, m$) be the marginal volume functions of ν. By Sato [16], Theorem 15.8, ν is selfdecomposable if and only if $\nu(t^{-1}B) \geq \nu(B)$ for all Borel sets B in $[0,\infty)^m$ and all $t \geq 1$, or what is the same if

$$\chi(t^{-1}B) \leq \chi(B) \tag{4.3}$$

for all Borel sets B in $(0,\infty]^m$ and all $t \geq 1$; here $\chi = Q\nu$. It is enough to check (4.3) for all Borel sets of the form $B := (a_1, b_1] \times \ldots \times (a_m, b_m]$. With the aid of the volume function of ν we can write

$$\begin{aligned} \chi(B) &= \sum \operatorname{sgn}(c)\, F(c) \\ &= \sum \operatorname{sgn}(c)\, C(F_1(c_1), \ldots, F_m(c_m)) \\ &= \sum \operatorname{sgn}(c)\, C(k_1\, x_1^\alpha, \ldots, k_m\, x_m^\alpha), \end{aligned}$$

where the sum is taken over all vertices $c = (c_1, \ldots, c_m)$ of B. Thus, ν is selfdecomposable if and only if

$$\sum \operatorname{sgn}(c)\, C(t^{-\alpha} k_1 c_1^\alpha, \ldots, t^{-\alpha} k_m c_m^\alpha) \leq \sum \operatorname{sgn}(c)\, C(k_1 c_1^\alpha, \ldots, k_m c_m^\alpha)$$

for all $t \geq 1$ and all sets $(a_1, b_1] \times \ldots \times (a_m, b_m]$. Substituting $u_i = k_i a_i^\alpha$, $v_i = k_i b_i^\alpha$, this is the same as

$$\sum \operatorname{sgn}(d)\, C(t^{-\alpha} d) \leq \sum \operatorname{sgn}(d)\, C(d),$$

where the sum ranges over all vertices d of $(u_1, v_1], \ldots, (u_m, v_m]$. The latter is the condition for the Lévy measure with volume function C, i.e. ν_C, to be selfdecomposable.

The proof of (a) is similar, using Sato [16], Theorem 14.3. ■

Tankov [13] showed that if $\alpha \in (0,2)$ and if a positive Lévy measure ν has non-degenerate α-stable margins, then ν is α-stable if and only if the associated Lévy copula is homogeneous. Now we immediately obtain:

Corollary 4.6 *A Lévy copula C is homogeneous if and only if ν_C is a 1-stable Lévy measure.*

5 The mapping Υ and Lévy copulas

In this section we shall recall the definition of the mapping $\Upsilon^{(m)}$ and investigate its action on copulas. This mapping will play a crucial role in the next section, when we construct arbitrary Lévy measures in the Bondesson or Thorin class with prescribed margins. For self-decomposable distributions a similar construction, using another mapping Φ, will be given.

The mapping $\Upsilon^{(m)}$ was introduced by Barndorff-Nielsen and Thorbjørnsen [5, 6] for the one-dimensional case $m = 1$ and extended by Barndorff-Nielsen, Maejima and Sato [2] to the multivariate setting. It maps infinitely divisible distributions to infinitely divisible distributions. More precisely, if μ is an infinitely divisible distribution on $\mathbb{R}^m$ with characteristic triplet (A, ν, γ), then $\widetilde{\mu} := \Upsilon(\mu) := \Upsilon^{(m)}(\mu)$ is the infinitely divisible distribution with characteristic triplet $(\widetilde{A}, \widetilde{\nu}, \widetilde{\gamma})$, where

$$\begin{aligned}
\widetilde{A} &:= 2A, \\
\widetilde{\nu}(B) &:= \int_0^\infty e^{-s}\,\nu(s^{-1}B)\,ds \quad \forall\ B \text{ Borel set in } \mathbb{R}^m, \\
\widetilde{\gamma} &= \gamma + \int_0^\infty e^{-s}\,s \int_{\mathbb{R}^m} x \left(\frac{1}{1+|x|^2 s^2} - \frac{1}{1+|x|^2} \right) \nu(dx)\,ds.
\end{aligned}$$

It can be shown that this is well defined, in particular $\widetilde{\nu}$ is a Lévy measure. Furthermore, extending results for dimension 1 due to Barndorff-Nielsen and Thorbjørnsen [5], Barndorff-Nielsen, Maejima and Sato [2] prove that $\Upsilon(\mu)$ is the law of the stochastic integral $\int_0^1 (-\log t)\,dX_\mu^{(t)}$, where $(X_\mu^{(t)})_{t\geq 0}$ is a Lévy process with distribution μ at time 1. They show moreover that Υ is a bijection from the class of infinitely divisible distributions on $\mathbb{R}^m$ to the Bondesson class $B(\mathbb{R}^m)$. We shall give the definition of $B(\mathbb{R}^m)$ in Section 6.1. For the moment, we mention only that $B(\mathbb{R}^m)$ contains all the stable distributions and is a proper subclass of the class of infinitely divisible distributions on $\mathbb{R}^m$. Furthermore, any element in $B(\mathbb{R}^1)$ has a Lévy density.

The transformation of ν to $\widetilde{\nu}$ when applying $\Upsilon^{(m)}$ is the most interesting part. We can restrict Υ to a mapping of Lévy measures, sending ν to $\widetilde{\nu}$; this mapping will be denoted by $\Upsilon_0^{(m)}$. If the dimension m is clear from the context, we will occasionally skip m from the notation. So we have for any Borel set $B \subset \mathbb{R}^m$,

$$\Upsilon_0(\nu)(B) := \Upsilon_0^{(m)}(\nu)(B) := \widetilde{\nu}(B) = \int_0^\infty e^{-s}\,\nu(s^{-1}B)\,ds. \qquad (5.1)$$

In particular, $\Upsilon_0^{(m)}$ is a bijection from $\mathcal{L}_+^m$ to the class of Lévy measures in $\mathcal{L}_+^m$ which correspond to infinitely divisible distributions in the Bondesson class. Furthermore, $\Upsilon_0^{(1)}$ can be viewed as a regularizer, since $\Upsilon_0^{(1)}(\nu)$ will have a Lévy density.

One important feature of Υ_0, which will be used in Section 6.1, is that it commutes with projection onto the axes. More precisely, if $\nu \in \mathcal{L}_+^m$ and $\Pi_i : \mathbb{R}_+^m \to \mathbb{R}_+$ denotes the projection onto the i'th axis ($i = 1, \ldots, m$), then

$$\widetilde{\nu}_i := \Pi_i(\Upsilon_0^{(m)}(\nu)) = \Upsilon_0^{(1)}(\Pi_i \nu). \tag{5.2}$$

This can be seen easily from the definition of Υ_0. Next, we show how the copulas transform when Υ_0 is applied to positive Lévy measures.

Lemma 5.1 *Let $\nu \in \mathcal{L}_+^m$ with marginal volume functions $F_1, \ldots, F_m$ and Lévy copula C. Let $M_i := \lim_{x_i \to \infty} F_i(x_i)$, the total mass of the marginal Lévy measure ν_i. Let $\widetilde{\nu} := \Upsilon_0^m(\nu)$ with marginal volume functions $\widetilde{F}_i$, $i = 1, \ldots, m$ and Lévy copula $\widetilde{C}$. Then, for any $x_1, \ldots, x_m \in [0, \infty]$,*

$$\widetilde{C}(\widetilde{F}_1(x_1), \ldots, \widetilde{F}_m(x_m)) = \int_0^\infty e^{-s}\, C\left(F_1(sx_1), \ldots, F_m(sx_m)\right) ds. \tag{5.3}$$

The Lévy copula $\widetilde{C}$ is uniquely determined on $([0, M_1] \cup \{\infty\}) \times \ldots \times ([0, M_m] \cup \{\infty\})$. If the marginal Lévy measures ν_i are non-degenerate κ_i-stable with $\kappa_i \in (0, 2)$, then $M_i = \infty$, and for any $u_1, \ldots, u_m \in [0, \infty]$,

$$\widetilde{C}(u_1, \ldots, u_m) = \int_0^\infty e^{-s}\, C\left(s^{\kappa_1} \frac{u_1}{\Gamma(\kappa_1 + 1)}, \ldots, s^{\kappa_m} \frac{u_m}{\Gamma(\kappa_m + 1)}\right) ds. \tag{5.4}$$

Proof. Denote by $\widetilde{F}$ the volume function of $\widetilde{\nu}$. Let $x_1, \ldots, x_m \in [0, \infty]$. Taking $B := [1/x_1, \infty) \times \ldots \times [1/x_m, \infty)$ in (5.1) gives for the volume functions F and $\widetilde{F}$,

$$\widetilde{F}(x_1, \ldots, x_m) = \int_0^\infty e^{-s}\, F(sx_1, \ldots, sx_m)\, ds, \tag{5.5}$$

which is equivalent to (5.3) by Theorem 2.1. Furthermore, taking $B_i := [0, \infty)^{i-1} \times (0, \infty) \times [0, \infty)^{m-i}$, (5.1) and (5.2) imply $\widetilde{\nu}_i((0, \infty)) = \nu_i((0, \infty)) = M_i$. Since $\widetilde{\nu}_i$ has a Lévy density, $\operatorname{Ran} \widetilde{F}_i \supset [0, M_i) \cup \{\infty\}$, and the uniqueness-assertion follows from Theorem 2.1 and the continuity property (3.4).

If the margins ν_i are non-degenerate κ_i-stable, then $F_i(x_i) = b_i\, x_i^{\kappa_i}$ for some $b_i > 0$, and an easy calculation shows $\widetilde{F}_i(x_i) = b_i \Gamma(\kappa_i + 1)\, x_i^{\kappa_i}$. Inserting this in (5.3) gives (5.4). ■

In the following Theorem we consider the effect of the mapping Υ_0 to measures with homogeneous Lévy copulas:

Theorem 5.2 *Let $\nu \in \mathcal{L}_+^m$ have stable non-degenerate margins with indices $\kappa_1, \ldots, \kappa_m \in (0,2)$. Then the Lévy copula associated with ν is homogeneous if and only if the Lévy copula associated with $\Upsilon_0(\nu)$ is homogeneous.*

Proof. Let ν_i, $i = 1, \ldots, m$, be the non-degenerate κ_i stable margins of ν. From (5.4) we see immediately that if C is homogeneous then so is $\widetilde{C}$. For the converse, suppose that $\widetilde{C}$ is homogeneous. It then follows from (5.4) that for any $t > 0$,

$$t^{-1}\widetilde{C}(t^{-\kappa_1}u_1, \ldots, t^{-\kappa_m}u_m) = \int_0^\infty e^{-rt}\, C\left(\frac{r^{\kappa_1}u_1}{\Gamma(\kappa_1+1)}, \ldots, \frac{r^{\kappa_m}u_m}{\Gamma(\kappa_m+1)}\right) dr.$$

For fixed $u = (u_1, \ldots, u_m) \in [0,\infty]^m \setminus \{\infty, \ldots, \infty\}$, define

$$\begin{aligned} f_u &: (0,\infty) \to \mathbb{R}, \qquad r \mapsto C\left(r^{\kappa_1}\frac{u_1}{\Gamma(\kappa_1+1)}, \ldots, r^{\kappa_m}\frac{u_m}{\Gamma(\kappa_m+1)}\right), \\ g_u &: (0,\infty) \to \mathbb{R}, \qquad t \mapsto t^{-1}\widetilde{C}\left(t^{-\kappa_1}u_1, \ldots, t^{-\kappa_m}u_m\right). \end{aligned}$$

Then g_u is the Laplace transform of f_u, $g_u = Lap(f_u)$. Further, for fixed $s > 0$, $\frac{1}{s}g_{su} = Lap(\frac{1}{s}f_{su})$. Now if $\widetilde{C}$ is homogeneous, then $g_u = \frac{1}{s}g_{su}$. From the uniqueness theorem for Laplace transforms then follows that $\frac{1}{s}f_{su}(r) = f_u(r)$ almost everywhere in r, and even everywhere in r since both functions are continuous by (3.4). In particular, $\frac{1}{s}f_{su}(1) = f_u(1)$, showing that C is homogeneous. ■

One might wonder if both ν and $\Upsilon_0^{(m)}(\nu)$ having homogeneous Lévy copulas implies stability of the margins. This, however, is not the case:

Example 5.3 Let $\nu \in \mathcal{L}_+^m$ with marginal volume functions $F_1(x) \leq F_2(x) \leq \ldots \leq F_m(x)$ $\forall\ x \in [0,\infty]$ and associated Lévy copula $C(u_1, \ldots, u_m) = \min\{u_1, \ldots, u_m\}$. Then (5.3) shows that $\widetilde{C} = C$ is associated with $\Upsilon_0^{(m)}(\nu)$. In particular, C and $\widetilde{C}$ are both homogeneous, although the margins of ν are not necessarily stable.

The following example shows that without assumptions on the margins, homogeneity of C does not imply homogeneity of $\widetilde{C}$.

Example 5.4 Let $\nu \in \mathcal{L}_+^2$ with marginal volume functions

$$F_1(x_1) = \begin{cases} 2x_1, & x_1 \leq 2, \\ 3 + x_1/2, & x_1 > 2, \end{cases}$$

$F_2(x_2) = x_2$ and the homogeneous Lévy copula $C(u_1, u_2) = \min\{u_1, u_2\}$. Then, evaluating the integrals in (5.5), it follows that the (marginal) volume functions of $\widetilde{\nu}$ satisfy $\widetilde{F}_1(x_1) = 2x_1 - \frac{3}{2}x_1 \exp(-2/x_1)$, $\widetilde{F}_2(x_2) = x_2$, and

$$\widetilde{F}(x, \widetilde{F}_1(x)) = \widetilde{F}_1(x) - \left(\widetilde{F}_1(x) - \frac{x}{2}\right) \exp\left(-\frac{3}{\widetilde{F}_1(x) - x/2}\right) \quad \forall\, x > 0.$$

From this it can be easily seen that $\widetilde{F}(x, \widetilde{F}_1(x))/\widetilde{F}_1(x)$ is not constant in $x > 0$, from which it follows that the copula associated with $\widetilde{\nu}$ cannot be homogeneous.

Barndorff-Nielsen, Maejima and Sato [2] have shown that $\Upsilon^{(m)}$ maps the class of stable random variables bijectively onto itself. For the subclass of stable random variables with Lévy measure in $\mathcal{L}_+^m$, another proof of this now follows easily by combining Theorem 5.2 and Tankov's characterisation of homogeneous Lévy copulas.

Finally, in this section we define a new mapping Υ_0^{cop}, acting directly on Lévy copulas. Recall from Definition 2.2 that any Lévy copula C defines a Lévy measure ν_C with marginal volume functions $F_i : x_i \mapsto x_i$. Then $\widetilde{F}_i(x_i) = x_i$, so by (2.2) the volume function of $\Upsilon_0(\nu_C)$ is identical to its Lévy copula. Thus, we can define $\Upsilon_0^{cop}(C)$ to be the unique Lévy copula C' such that $\nu_{C'} = \Upsilon_0(\nu_C)$. By (5.4), this is equivalent to the following

Definition 5.5 For any Lévy copula C, the transformed Lévy copula $\Upsilon_0^{cop}(C)$ is defined by

$$\Upsilon_0^{cop}(C)(u_1, \ldots, u_m) = \int_0^\infty e^{-s}\, C(su_1, \ldots, su_m)\, ds \quad \forall\, u_1, \ldots, u_m \in [0, \infty].$$

Note that $\Upsilon_0^{cop}(C)$ can be defined for any Lévy copula C, while $\widetilde{C}$ as appearing in Lemma 5.1 depends on the margins of a Lévy measure, as shown in Example 5.4. Furthermore, if C is homogeneous, then $\Upsilon_0^{cop}(C) = C$.

6 Constructing special Lévy measures with prescribed margins

Let $\nu_1, \ldots, \nu_m$ be prescribed one-dimensional positive Lévy measures which are in the Bondesson class. Then an easy description of the Lévy copulas which give rise to multivariate Lévy measures ν in the Bondesson class with these margins does not seem to be available. However, in Section 6.1 we shall show how Lévy measures ν in the m-dimensional Bondesson class with margins $\nu_1, \ldots, \nu_m$ can be constructed, using the mapping Υ. Then, in Sections 6.2 and 6.3 we shall obtain similar results for self-decomposable Lévy measures and for Lévy measures in the Thorin class. Again, easy descriptions of the relevant Lévy copulas do not seem to be at hand.

6.1 Lévy measures in the Bondesson class

Bondesson [7] considered the smallest class of probability distributions on $[0, \infty)$ which is closed under weak convergence and convolution and contains all mixtures of exponential distributions. This class was extended to distributions on the real line, and we shall refer to that as the Bondesson class $B(\mathbb{R})$. Barndorff-Nielsen, Maejima and Sato [2] generalised this further to distributions on $\mathbb{R}^m$: by definition, the *multivariate Bondesson class* $B(\mathbb{R}^m)$ consists of all infinitely divisible distributions μ whose Lévy measure ν can be expressed as

$$\nu(B) = \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi)\, l_\xi(r)\, dr \quad \forall\, B \text{ Borel set in } \mathbb{R}^m \setminus \{0\}. \tag{6.1}$$

Here, λ is a positive measure on $S = \{\xi \in \mathbb{R}^m : |\xi| = 1\}$ and $(l_\xi)_{\xi \in S}$ is a family of functions on $(0, \infty)$ such that $l_\xi(r)$ is completely monotone in r for λ-a.e. ξ, and $l_\xi(r)$ is measurable in ξ for each $r > 0$. A characterisation of $B(\mathbb{R}^m)$ as the smallest class closed under weak convergence and convolution and containing all "elementary mixtures" of signed exponential random variables in $\mathbb{R}^m$ was also obtained in [2]; we shall not make use of this characterisation in the sequel.

We shall be interested in the subclass $B(\mathbb{R}_+^m)$, consisting of all elements of $B(\mathbb{R}^m)$ whose *Lévy measure* is concentrated on $\mathbb{R}_+^m$. For notational convenience, since for any infinitely divisible distribution μ the property of belonging to $B(\mathbb{R}_+^m)$ is completely determined by its Lévy measure ν, we shall also say that ν belongs to $B(\mathbb{R}_+^m)$.

In one dimension, $B(\mathbb{R}_+)$ consists of all infinitely divisible distributions whose Lévy measure is concentrated on $(0, \infty)$ and has a completely monotone Lévy density there. Recall that a function on $(0, \infty)$ is *completely monotone* if it is C^∞ and if $(-1)^n (d^n/dx^n) f(x) \geq 0$ on $(0, \infty)$ for all $n \in \mathbb{N}_0$. By Bernstein's theorem a function f on $(0, \infty)$ is completely monotone if and only if it is the Laplace transform

$$f(x) = \int_{(0,\infty)} e^{-xy}\, d\psi(y), \quad x > 0$$

of some positive measure ψ for which the integral is finite. Such an f is a Lévy density if and only if

$$\int_{(0,\infty)} \left(y^{-3} \int_0^y r^2 e^{-r}\, dr \; + y^{-1} e^{-y} \right) d\psi(y) < \infty,$$

see [2].

Barndorff-Nielsen, Maejima and Sato [2] showed that the mapping $\Upsilon^{(m)}$ maps the class of infinitely divisible distributions one-to-one onto $B(\mathbb{R}^m)$. From this follows easily that the class of infinitely divisible distributions whose Lévy measure is concentrated on $\mathbb{R}_+^m$ is mapped bijectively onto $B(\mathbb{R}_+^m)$. This will be the key property for us when constructing multivariate distributions in the Bondesson class.

The following example shows that not every Lévy measure whose one-dimensional margins are in $B(\mathbb{R}_+)$ belongs to $B(\mathbb{R}_+^m)$.

Example 6.1 Let ν_1 and ν_2 be one-dimensional Lévy measures with volume functions $F_1(x_1) = x_1^\alpha$ and $F_2(x_2) = x_2^\beta$, where $0 < \alpha, \beta < 2$ and $\alpha \neq \beta$. Define the bivariate Lévy measure ν using the Lévy copula $C(x_1, x_2) = \min(x_1, x_2)$. Then the volume function of ν is given by $F(x_1, x_2) = \min(x_1^\alpha, x_2^\beta)$. But this implies that the Lévy measure ν is concentrated on the curve $x_2 = x_1^{\alpha/\beta}$. In particular, its radial component cannot have a Lebesgue density, so $\nu \notin B(\mathbb{R}_+^2)$. However, the marginals ν_1 and ν_2 of ν are α- and β-stable, respectively, and hence in $B(\mathbb{R}_+)$.

So we have seen that not every Lévy copula can be used on margins in the Bondesson class to obtain a Lévy measure in $B(\mathbb{R}_+^m)$. The following Theorem gives a complete description of all possibilities to construct such measures:

Theorem 6.2 *Let $\widetilde{\nu}_1, \dots, \widetilde{\nu}_m \in B(\mathbb{R}_+)$ be prescribed marginal Lévy measures. Set*

$$\nu_i := (\Upsilon_0^{(1)})^{-1} \widetilde{\nu}_i, \quad i = 1, \dots, m. \tag{6.2}$$

Let C be any m-dimensional Lévy copula and define the Lévy measure ν with margins $\nu_1, \dots, \nu_m$ by (2.2). Then

$$\widetilde{\nu} := \Upsilon_0^{(m)}(\nu)$$

defines a Lévy measure in the Bondesson class $B(\mathbb{R}_+^m)$ with margins $\widetilde{\nu}_1, \dots, \widetilde{\nu}_m$. Furthermore, all Lévy measures in $B(\mathbb{R}_+^m)$ with these margins are obtained in this way.

Proof. It is clear that ν is a Lévy measure with margins $\nu_1, \ldots, \nu_m$, and from (5.2) then follows that $\widetilde{\nu}$ has margins $\widetilde{\nu}_1, \ldots, \widetilde{\nu}_m$. Since the range of $\Upsilon^{(m)}$ is the Bondesson class, $\widetilde{\nu} \in B(\mathbb{R}^m_+)$ follows. The fact that all such measures are obtained this way follows since $\Upsilon_0^{(m)}$ is a bijection and from Theorem 2.1. ■

In the following we give some examples applying Theorem 6.2.

Example 6.3 Let $0 < \alpha < \beta < 2$. Let $\widetilde{\nu}_1$ and $\widetilde{\nu}_2$ have α- and β-stable margins, respectively. In Example 6.1 we have seen that we cannot use any Lévy copula together with $\widetilde{\nu}_1$, $\widetilde{\nu}_2$ to insure that the resulting bivariate Lévy measure is in the Bondesson class. Now let $\nu_i := (\Upsilon_0^{(1)})^{-1}(\widetilde{\nu}_i)$, $i = 1, 2$, and define ν with margins ν_1 and ν_2 using the Lévy copula $C(u_1, u_2) = \min(u_1, u_2)$. It then follows from (5.4) that

$$\widetilde{C}(u_1, u_2) = \int_0^\infty e^{-s} \min\left(s^\alpha \frac{u_1}{\Gamma(\alpha+1)}, s^\beta \frac{u_2}{\Gamma(\beta+1)}\right) ds$$

is the Lévy copula of $\widetilde{\nu} = \Upsilon_0^{(2)}(\nu)$. Setting $z = z(u_1, u_2) := \left(\frac{u_1}{u_2}\frac{\Gamma(\beta+1)}{\Gamma(\alpha+1)}\right)^{\frac{1}{\beta-\alpha}}$, it follows that

$$\begin{aligned}
&\widetilde{C}(u_1, u_2) \\
&= \int_0^z e^{-s} s^\beta \frac{u_2}{\Gamma(\beta+1)} ds + \int_z^\infty e^{-s} s^\alpha \frac{u_1}{\Gamma(\alpha+1)} ds \\
&= u_2 P(\beta+1, z(u_1, u_2)) + u_1 (1 - P(\alpha+1, z(u_1, u_2))), \qquad (6.3)
\end{aligned}$$

where

$$P(a, x) := \frac{1}{\Gamma(a)} \int_0^x e^{-s} s^{a-1} ds, \quad a > 0, \; x > 0,$$

denotes the incomplete Γ-function (see e.g. Abramowitz and Stegun [1], formula 6.5.1). So, using the (homogeneous) copula (6.3) on the margins $\widetilde{\nu}_1$ and $\widetilde{\nu}_2$, we obtain $\widetilde{\nu} \in B(\mathbb{R}^2_+)$ with α- and β-stable margins. Tables of $P(a, x)$ can be found in [1], and many software packages have routines implemented to compute it.

Example 6.4 Let $c_i, \alpha_i > 0$ $(i = 1, \ldots, m)$ be parameters. Let $\widetilde{\nu}_1, \ldots, \widetilde{\nu_m}$ be Lévy measures of Γ_{c_i,α_i} distributions. Then $\widetilde{\nu}_i$ has Lévy density $\widetilde{f}_i(x) = c_i\, x^{-1} e^{-\alpha_i x} 1_{(0,\infty)}(x)$. We aim to construct a Lévy measure $\widetilde{\nu}$ in the Bondesson class with margins $\widetilde{\nu}_1, \ldots, \widetilde{\nu_m}$. Setting

$$h_i(s) := \frac{c_i}{s} 1_{(\alpha_i,\infty)}(s),$$

we recognize $\widetilde{f}_i$ as the Laplace transform of $s \mapsto s\, h_i(s)$. From Barndorff-Nielsen and Thorbjørnsen [5] then follows that h_i is the Lebesgue density of the measure $\chi_i := Q_1^{-1}(\nu_i)$, where $\nu_i = (\Upsilon_0^{(1)})^{-1}(\widetilde{\nu}_i)$. In order to construct a Lévy measure ν with margins ν_i using Lévy copulas, we need the marginal volume functions

$$F_i(x) = \int_0^x h_i(s)\,ds = \begin{cases} 0, & x \leq \alpha_i, \\ c_i \log \frac{x}{\alpha_i}, & x > \alpha_i. \end{cases}$$

Then if C is any Lévy copula, (2.2) defines a Lévy measure ν, and $\widetilde{\nu} = \Upsilon_0^m(\nu)$ is then a Lévy measure in the Bondesson class with gamma margins $\widetilde{\nu}_1, \ldots, \widetilde{\nu}_m$.

Example 6.5 In Example 6.4, specialise to $m = 2$ and $c_1 = c_2 =: c$. Let the copula C be given by

$$C(u_1, u_2) := \log\left(\left(\sum_{i=1}^{2} \frac{e^{-u_i}}{1 - e^{-u_i}}\right)^{-1} + 1\right)$$

as in Example 4.2. Inserting the marginal volume functions into C we obtain a Lévy measure ν with volume function F such that $F(x_1, x_2) = 0$ if $x_i \leq \alpha_i$ for some i, and else

$$F(x_1, x_2) = \log(x_1^c x_2^c - \alpha_1^c \alpha_2^c) - \log(\alpha_2^c x_1^c + \alpha_1^c x_2^c - 2\alpha_1^c \alpha_2^c).$$

From (5.5) then follows that

$$\widetilde{F}(x_1, x_2) = \int\limits_{\max(\frac{\alpha_1}{x_1}, \frac{\alpha_2}{x_2})}^{\infty} e^{-s} \left\{\log(s^{2c} x_1^c x_2^c - \alpha_1^c \alpha_2^c) - \log(s^c(\alpha_2^c x_1^c + \alpha_1^c x_2^c) - 2\alpha_1^c \alpha_2^c)\right\} ds.$$

To simplify further, we suppose that $c = 1$ and $\alpha_1 = \alpha_2 = 1$, so that $\widetilde{\nu}_i$ is the Lévy measure of an exponential distribution with parameter 1. Then, substituting and using partial integration, it can be shown that

$$\begin{aligned} \widetilde{F}(x_1, x_2) &= e^{1/\sqrt{x_1 x_2}}\, E_1\left(\frac{1}{x_1} + \frac{1}{\sqrt{x_1 x_2}}\right) + e^{-1/\sqrt{x_1 x_2}}\, E_1\left(\frac{1}{x_1} - \frac{1}{\sqrt{x_1 x_2}}\right) \\ &\quad - e^{-2/(x_1 + x_2)}\, E_1\left(\frac{x_2}{x_1(x_1 + x_2)} - \frac{1}{x_1 + x_2}\right) \end{aligned}$$

for $x_1 < x_2$. For $x_1 > x_2$ we have $\widetilde{F}(x_1, x_2) = \widetilde{F}(x_2, x_1)$, and for $x_1 = x_2$ it holds that $\widetilde{F}(x_1, x_1) = e^{1/x_1}\, E_1(2/x_1)$. Here

$$E_1(x) := \int_x^\infty s^{-1}\, e^{-s}\, ds, \quad x > 0$$

denotes the exponential integral (see e.g. Abramowitz and Stegun [1], formula 5.1.1).

Example 6.6 Now we aim to construct bivariate distributions in the Bondesson class with tempered stable margins. For this, it is necessary to get the inverse under $\Upsilon_0^{(1)}$ of these distributions. Let $\widetilde{\nu}_1, \dots, \widetilde{\nu}_m$ be Lévy measures of tempered stable distributions $TS(\kappa_i, \delta_i, \gamma_i)$, where $\kappa_i \in (0,1)$ and $\delta_i, \gamma_i > 0$, $i = 1, \dots, m$. Denote by $\widetilde{f}_i$ the Lévy density of $\widetilde{\nu}_i$, which is given by

$$\widetilde{f}_i(x) = \delta_i 2^{\kappa_i} \frac{\kappa_i}{\Gamma(1-\kappa_i)} x^{-1-\kappa_i} \exp\{-\frac{1}{2}\gamma_i^{1/\kappa_i} x\} 1_{(0,\infty)}(x),$$

see Barndorff-Nielsen and Shephard [3]. Let $\nu_i := (\Upsilon_0^{(1)})^{-1}(\widetilde{\nu}_i)$ and set $\chi_i := Q_1^{-1}(\nu_i)$. Define $h_i(s)$ by

$$s\, h_i(s) := \delta_i 2^{\kappa_i} \frac{\kappa_i}{\Gamma(1-\kappa_i)} \frac{(s - \gamma_i^{1/\kappa_i}/2)^{\kappa_i}}{\Gamma(1+\kappa_i)} 1_{[\gamma_i^{1/\kappa_i}/2,\infty)}(s).$$

Then $\widetilde{f}_i$ is the Laplace transform of $s \mapsto s\, h_i(s)$, see e.g. [1], formula 29.3.63. Again, from [5] follows that h_i is the Lebesgue density of the measure χ_i. Simple calculations using the properties of the Γ-function show that

$$h_i(s) = \delta_i\, 2^{\kappa_i} \frac{\sin(\pi\kappa_i)}{\pi} \frac{(s - \gamma_i^{1/\kappa_i}/2)^{\kappa_i}}{s} 1_{[\gamma_i^{1/\kappa_i}/2,\infty)}(s).$$

In order to construct a Lévy measure ν with margins ν_i using Lévy copulas, we need the marginal volume functions $F_i(x) = \int_0^x h_i(s)\, ds$. To calculate these in an explicit form, we specialise to $\kappa_i = 1/2$, the case where $\widetilde{\nu}_i$ is the Lévy measure of an inverse Gaussian law. Then $F_i(x) = 0$ for $x \le \gamma_i^2/2$, and for $x \ge \gamma_i^2/2$ we obtain

$$F_i(x) = \frac{\delta_i\sqrt{2}}{\pi} \int_{\gamma_i^2/2}^{x} \frac{(s-\gamma_i^2/2)^{1/2}}{s}\, ds = \frac{\delta_i\sqrt{2}}{\pi} \int_0^{x-\gamma_i^2/2} \frac{s^{1/2}}{s+\gamma_i^2/2}\, ds.$$

The last integral can be calculated explicitly, see e.g. Dwight [9], formula 185.11., and we obtain

$$F_i(x) = \begin{cases} 0, & x \le \gamma_i^2/2, \\ \frac{\delta_i\sqrt{2}}{\pi} \left\{ 2(x-\gamma_i^2/2)^{1/2} - \sqrt{2}\gamma_i \arctan \frac{\sqrt{2}(x-\gamma_i^2/2)^{1/2}}{\gamma_i} \right\}, & x \ge \gamma_i^2/2. \end{cases}$$

Then if C is any Lévy copula, (2.2) defines a Lévy measure ν, and $\widetilde{\nu} = \Upsilon_0^m(\nu)$ is then a Lévy measure in $B(\mathbb{R}_+^m)$ with inverse Gaussian margins.

6.2 Self-decomposable Lévy measures

The Lévy measure constructed in Example 6.1 not only is not in the Bondesson class, but also is not self-decomposable, although it has self-decomposable margins. In the following we shall show how to construct all self-decomposable distributions with given margins. Much the same way as we constructed Lévy measures in the Bondesson class using the mapping Υ_0, we can construct self-decomposable Lévy measures using the mapping Φ_0 defined below. Recall that an infinitely divisible distribution μ is self-decomposable if and only if its Lévy measure ν has representation (6.1), where l_ξ does not need to be completely monotone, but $r \mapsto r l_\xi(r)$ has to be decreasing on $(0, \infty)$, see Sato [16], Theorem 15.10. By abuse of language, we shall also say that the Lévy measure ν is self-decomposable.

Denote by $L(\mathbb{R}_+^m)$ the class of self-decomposable distributions with Lévy measure in $\mathcal{L}_+^m$, and by $ID_{\log}(\mathbb{R}_+^m)$ the class of infinitely divisible distributions μ with Lévy measure in $\mathcal{L}_+^m$ and which satisfy

$$\int_{|x|>1} \log |x| \, d\mu(x) < \infty.$$

If μ is infinitely divisible with Lévy measure $\nu \in \mathcal{L}_+^m$, then $\mu \in ID_{\log}(\mathbb{R}_+^m)$ if and only if

$$\int_{|x|>1} \log |x| \, d\nu(x) < \infty,$$

see Sato [16], Theorem 25.3; we shall also write $\nu \in L(\mathbb{R}_+^m)$.

Let $\mu \in ID_{\log}(\mathbb{R}_+^m)$ and $(X_\mu^{(t)})_{t \geq 0}$ be a Lévy process with distribution μ at time 1. Then $\Phi^{(m)}(\mu) := \Phi(\mu) :\stackrel{d}{=} \int_0^\infty e^{-t} \, dX_\mu^{(t)}$ exists. Sato and Yamazato [17], Section 4, have shown that Φ defines a bijection from $ID_{\log}(\mathbb{R}_+^m)$ onto $L(\mathbb{R}_+^m)$. (Similar results hold without the restriction that the Lévy measure be concentrated on $\mathbb{R}_+^m$.) The action of Φ on μ can also be defined in terms of the characteristic triplets, cf. Sato [16], Theorem 17.5. In particular, if ν is the Lévy measure of μ and $\breve{\nu}$ denotes the Lévy measure of $\breve{\mu} := \Phi(\mu)$, then

$$\Phi_0^{(m)}(\nu)(B) := \Phi_0(\nu)(B) := \breve{\nu}(B) = \int_0^\infty \nu(e^s B) \, ds \quad \forall\, B \text{ Borel set in } \mathbb{R}_m^+.$$

Again, Φ_0 defines a bijection between the class of Lévy measures in $ID_{\log}(\mathbb{R}_+^m)$ and the Lévy measures in $L(\mathbb{R}_+^m)$. From the definition of the bijection $\Phi^{(m)}$ we see in particular that if $\breve{\nu}$ is the Lévy measure of a self-decomposable distribution $\breve{\mu}$, then $\breve{\mu}$ can be represented as $\int_0^\infty e^{-t} \, dX_\mu^{(t)}$, where $(X_\mu^{(t)})_{t \geq 0}$ is the background driving Lévy process with distribution μ and Lévy measure $\nu = (\Phi_0^{(m)})^{-1}(\breve{\nu})$ at time 1.

Let ν be a Lévy measure in $ID_{\log}(\mathbb{R}_+^m)$, and let F, F_i, C and $\breve{F}$, $\breve{F}_i$ and $\breve{C}$ be the (marginal) volume functions and copulas of ν and $\breve{\nu}$, respectively. Then for any $x = (x_1, \dots, x_m) \in [0, \infty]^m$,

$$\begin{aligned} \breve{F}(x) &= \int_0^\infty F(e^{-s}x)\, ds, \\ \breve{C}(\breve{F}_1(x_1), \dots, \breve{F}_m(x_m)) &= \int_0^\infty C(F_1(e^{-s}x_1), \dots, F_m(e^{-s}x_m))\, ds. \end{aligned} \tag{6.4}$$

Before we can use the mapping Φ_0 to construct self-decomposable Lévy measures, we need the following lemma:

Lemma 6.7 *Let $\nu \in \mathcal{L}_+^m$ with margins $\nu_1, \dots, \nu_m$. Then $\nu \in ID_{\log}(\mathbb{R}_+^m)$ if and only if $\nu_i \in ID_{\log}(\mathbb{R}_+)$ for all $i = 1, \dots, m$.*

Proof. If $\nu \in ID_{\log}(\mathbb{R}_+^m)$, then for any $i \in \{1, \dots, m\}$, writing $x = (x_1, \dots, x_m)$,

$$\int_{|x_i|>1} \log x_i^2\, d\nu_i(x_i) \le \int_{|x_i|>1} \log |x|^2\, d\nu(x) \le \int_{|x|>1} \log |x|^2\, d\nu(x) < \infty.$$

On the other hand, if $\nu_i \in ID_{\log}(\mathbb{R}_+)$ for $i = 1, \dots, m$, then

$$\begin{aligned} &\int_{|x_1|>1} \cdots \int_{|x_m|>1} \log |x|^2\, d\nu(x) \\ &\quad \le \int_{|x_1|>1} \cdots \int_{|x_m|>1} \left(\log m + \sum_{i=1}^m \log |x_i|^2 \right) d\nu(x) < \infty. \end{aligned}$$

■

Hence, we can combine any marginal Lévy measures $\nu_1, \dots, \nu_m$ in $ID_{\log}(\mathbb{R}_+)$ with any Lévy copula, and obtain a Lévy measure ν in $ID_{\log}(\mathbb{R}_+^m)$. From the definition of Φ_0 follows readily that it commutes with the projection Π_i on the i-th axis, more precisely:

$$\breve{\nu}_i := \Pi_i(\Phi_0^{(m)})(\nu)) = \Phi_0^{(1)}(\Pi_i \nu).$$

In analogy to Theorem 6.2, with the same line of proof, we now obtain:

Theorem 6.8 *Let $\breve{\nu}_1, \dots, \breve{\nu}_m$ be prescribed marginal Lévy measures in $L(\mathbb{R}_+^m)$. Set*

$$\nu_i := (\Phi_0^{(1)})^{-1} \breve{\nu}_i, \quad i = 1, \dots, m.$$

Let C be any m-dimensional Lévy copula and define the Lévy measure ν with margins $\nu_1, \ldots, \nu_m$ using (2.2)*. Then*

$$\breve{\nu} := \Phi_0^{(m)}(\nu)$$

defines a selfdecomposable Lévy measure on $\mathbb{R}_+^m$ with margins $\breve{\nu}_1, \ldots, \breve{\nu}_m$. Furthermore, all Lévy measures in $L(\mathbb{R}_+^m)$ with these margins are obtained in this way.

It is quite easy to obtain $(\Phi_0^{(1)})^{-1}\breve{\nu}_i$, i.e. the Lévy measure of the background driving Lévy process, from the Lévy density of $\breve{\nu}_i$: if this Lévy density is denoted by $\breve{f}_i$, then the marginal volume function F_i of $(\Phi_0^{(1)})^{-1}\breve{\nu}_i$ satisfies

$$F_i(x_i) = x_i^{-1}\, \breve{f}_i(x_i^{-1}),$$

see Barndorff-Nielsen and Shephard [4], Equation (4.17). For example, if $\breve{\nu}_i$ is a tempered stable distribution $TS(\kappa_i, \delta_i, \gamma_i)$, i.e. if

$$\breve{f}_i(x_i) = \delta_i 2^{\kappa_i} \frac{\kappa_i}{\Gamma(1-\kappa_i)}\, x_i^{-1-\kappa_i}\, \exp\{-\frac{1}{2}\gamma_i^{1/\kappa_i} x_i\}\, 1_{(0,\infty)}(x_i),$$

then

$$F_i(x) = \delta_i 2^{\kappa_i} \frac{\kappa_i}{\Gamma(1-\kappa_i)} x_i^{\kappa_i} \exp\{\gamma_i^{1/\kappa_i} x_i^{-1}/2\}.$$

Then using any Lévy copula C in (6.4) leads to a multivariate selfdecomposable Lévy measure $\breve{\nu}$ with tempered stable margins $\breve{\nu}_1, \ldots, \breve{\nu}_m$. In particular, when $\kappa_i = 1/2$ the $\breve{f}_i$ correspond to the Lévy densities of inverse Gaussian distributions.

6.3 Lévy measures in the Thorin class

In Barndorff-Nielsen, Maejima and Sato [2], the m-dimensional Thorin class $T(\mathbb{R}^m)$ is defined to be the class of all infinitely divisible distributions μ whose Lévy measure ν has representation (6.1), where $r \mapsto r l_\xi(r)$ has to be completely monotone on $(0, \infty)$. This is a generalisation of the one-dimensional Thorin class $T(\mathbb{R})$ introduced by Thorin [14]. It can be shown that $T(\mathbb{R}^m)$ is a proper subclass of $B(\mathbb{R}^m) \cap L(\mathbb{R}^m)$. A probabilistic interpretation as for the Bondesson class is given in [2]. There, it is also shown that $T(\mathbb{R}^m)$ is the image of $L(\mathbb{R}^m)$ under $\Upsilon^{(m)}$ (for $m = 1$ this was proved in [5]), and also the image of $B(\mathbb{R}^m) \cap ID_{\log}(\mathbb{R}^m)$ under $\Phi^{(m)}$. Furthermore, $\Phi^{(m)}$ and $\Upsilon^{(m)}$ commute, i.e. $\Phi^{(m)}\Upsilon^{(m)}(\mu) = \Upsilon^{(m)}\Phi^{(m)}(\mu)$ for $\mu \in ID_{\log}(\mathbb{R}^m)$. Denote by $T(\mathbb{R}_+^m)$ the class of all infinitely divisible distributions in the Thorin class whose Lévy measure is in $\mathcal{L}_+^m$. Then the results of Sections 6.1 and 6.2

can be used to construct all distributions in $T(\mathbb{R}_+^m)$ with any prescribed marginal Lévy measures $\widetilde{\nu}_i$ in $T(\mathbb{R}_+)$: take the inverses ν_i of the marginal Lévy measures $\widetilde{\nu}_i$ under $\Upsilon_0^{(1)}$, construct a Lévy measure $\nu \in L(\mathbb{R}_+^m)$ with margins ν_i as in Section 6.2, and set $\widetilde{\nu} := \Phi_0^{(m)}(\nu)$. Alternatively, one can set $\breve{\nu}_i := (\Phi_0^{(1)})^{-1}(\widetilde{\nu}_i) \in B(\mathbb{R}_+) \cap ID_{\log}(\mathbb{R}_+)$, construct $\breve{\nu} \in B(\mathbb{R}_+^m) \cap ID_{\log}(\mathbb{R}_+^m)$ as in Section 6.1, and set $\widetilde{\nu} := \Phi(\breve{\nu})$.

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