Consistency and asymptotic normality of the maximum likelihood estimator in a zero-inflated generalized Poisson regression

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ABSTRACT. Poisson regression models for count variables have been utilized in many applications. However, in many problems overdispersion and zero-inflation occur. We study in this paper regression models based on the generalized Poisson distribution (Consul (1989)). These regression models which have been used for about 15 years do not belong to the class of generalized linear models considered by McCullagh and Nelder (1989) for which an established asymptotic theory is available. Therefore we prove consistency and asymptotic normality of a solution to the maximum likelihood equations for zero-inflated generalized Poisson regression models. Further the accuracy of the asymptotic normality approximation is investigated through a simulation study. This allows to construct asymptotic confidence intervals and likelihood ratio tests.

Key words: central limit theorem, likelihood, maximum likelihood estimator, overdispersion, zero-inflated generalized Poisson regression.

1 Introduction

Poisson regression models are often used to analyze count data. However count regression data often exhibit substantial overdispersion which is present when the data has higher variability as is allowed by the model. In particular, equality of mean and variance for count data analyzed under a Poisson assumption is often violated. Various reasons such as missing covariates and correlation among the measurements make counts overdispersed. Consequently, a number of different regression models in the literature have been proposed, which commonly handle overdispersion in two general approaches:

1) inclusion of random effects;

2) extension of the parametric model by extra parameters to allow for a more general variance structure.

Excellent surveys on this topic can be found in Cameron and Trivedi (1998) as well as in Winkelmann (2003).

One way to extend a parametric model is to consider a distribution with a more flexible variance function. A negative binomial (NB) and a generalized Poisson (GP) distributions are standard count distributions used in analyzing overdispersed data. Lawless (1987) first systematically studied the NB regression model and showed asymptotic normality of its maximum likelihood (ML) estimator. Consul and Famoye (1992) introduced the GP regression model and applied it to several data sets. However they did not examined the asymptotic properties of the ML estimator in the GP regression and this has remained an open problem.

It became popular over the past decade to model overdispersed data with a large frequency of zeros using a mixture of a count distribution with a degenerate distribution supported at zero. This is another way to treat overdispersion by introducing an additional parameter in regression model. Zero-inflated Poisson (ZIP) regression is one of frequently used models for count data with large proportion of zeros. Lambert (1992) investigated the asymptotic properties of the ML estimator in ZIP regression models. However she stated asymptotic results without giving rigorous proofs and exact assumptions required. Jansakul and Hinde (2002) derived score tests for ZIP models and investigated their power via a simulation study. Famoye and Singh (2003) have recently proposed score tests for a k-inflated GP regression model where k is a fixed nonnegative integer. Gupta, Gupta, and Tripathi (2004) independently studied score tests for a zero-inflated generalized Poisson (ZIGP) regression model. Asymptotic properties of the ML estimator in the ZIGP regression model have also not been investigated.

The objective of this paper is to derive the appropriate asymptotic theory for ZIGP regression models and to examine the accuracy of the normal approximation for the ML estimator. It should be noted that our results remain valid for GP and ZIP regression models. The paper is organized as follows. In Section 2 we introduce zero-inflated and GP distributions. Their basic properties will be also briefly discussed. The ZIGP regression model will be defined in Section 3. Section 4 gives the asymptotic existence, the consistency and the asymptotic normality of the ML estimators in ZIGP regression model. Results of a simulation study are reported in Section 5. Computation of the Fisher information matrix and the proof of Theorem 1 are given in Appendix.

2 Zero-inflated count distributions and the generalized Poisson (GP) distribution

Suppose that we observe realizations of a count random variable Y and we believe that Y has a specified discrete count distribution. Further suppose that the observed data exhibits an excess of zeros which can not be modelled by the assumed model. This means that we cannot rely anymore on our hypothesis. But an assumption, that zeros arise from a mixture of a Bernoulli distribution and the conjectured distribution, makes it possible for us to investigate our conjecture. More precisely, we assume that the probability mass function of the observed response Y is given by

$$P(Y=y)=\begin{cases}\omega+(1-\omega)P(\tilde{Y}=0) & y=0,\\ (1-\omega)P(\tilde{Y}=y) & y=1,2,\ldots, \qquad 0\le\omega\le 1,\end{cases}$$

where $\tilde{Y}$ is distributed according to the conjectured distribution with finite second moment. Simple calculations show that mean and variance of the zero-inflated random variable Y are given by

$$E(Y)=(1-\omega)E(\tilde{Y}) \tag{1}$$

and

$$Var(Y)=(1-\omega)Var(\tilde{Y})+\omega(1-\omega)\left(E(\tilde{Y})\right)^2. \tag{2}$$

Throughout this paper, we assume that the conjectured distribution of the response variable Y, i.e. the distribution of $\tilde{Y}$, is a generalized Poisson (GP) distribution with two parameters μ and φ denoted by $GP(\mu,\varphi)$. This distribution was first introduced by Consul and Jain (1970) and subsequently studied in detail by Consul (1989). The probability mass function of the GP distribution is given by

$$P_{\mu,\varphi}(y):=\begin{cases}\mu(\mu+y(\varphi-1))^{y-1}\varphi^{-y}e^{-(\mu+y(\varphi-1))/\varphi}/y! & \text{for } y=0,1,\ldots\\ 0 & \text{for } y>m, \quad \text{when } \varphi<1\end{cases} \tag{3}$$

and its real-valued parameters μ and φ satisfy the following constraints:

- $\mu>0$;
- $\varphi\ge\max\{1/2,1-\mu/m\}$, where m $(m\ge 4)$ is a largest natural number such that $\mu+m(\varphi-1)>0$ when $\varphi<1$.

If $\varphi<1$ then (3) does not correspond to a probability distribution. The lower limit, imposed on φ in this case, guarantees us that the total error of truncation is less than 0.5% (see Consul and Shoukri (1985)). Since all discrete distributions are truncated under sampling procedures this is a quite reasonable condition.

One particular property of the GP distribution is that the variance of this distribution is greater than, equal to or less than the mean according to whether the second parameter φ is greater than, equal to or less than 1. More precisely (for details see Consul (1989), page 12), if $Y\sim GP(\mu,\varphi)$ then mean and variance of Y are given by

$$E(Y) \quad = \quad \mu \tag{4}$$

and

$$Var(Y) \quad = \quad \varphi^2\mu. \tag{5}$$

This implies that a regression model associated with the GP distribution can be used to fit count regression data which has overdispersion or underdispersion or as well as equidispersion. In the sequel this regression model will be called a GP regression. It should be noted that the GP regression does not belong to well-studied generalized linear models (GLM) (see for example McCullagh and Nelder (1989)) and consequently, there is no asymptotic theory available at the moment.

3 Zero-inflated generalized Poisson regression

A random variable Y is said to be distributed according to the zero-inflated generalized Poisson (ZIGP) distribution with parameters μ, φ and ω, which we further denote by $ZIGP(\mu, \varphi, \omega)$, if its probability mass function is given by

$$\begin{aligned} P_{\mu,\varphi,\omega}(y) &:= P(Y = y) \\ &= \begin{cases} \omega + (1-\omega)P_{\mu,\varphi}(0), & \text{if } \ y = 0 \\ (1-\omega)P_{\mu,\varphi}(y), & \text{if } \ y = 1, 2, \ldots, \\ 0 & \text{for } y > m \quad \text{when } \varphi < 1, \end{cases} \end{aligned} \tag{6}$$

and zero otherwise, where $0 \leq \omega \leq 1$, $\mu > 0$, $\varphi \geq \max\{1/2, 1-\mu/m\}$ and m $(m \geq 4)$ is a largest natural number for which $\mu + m(\varphi - 1) > 0$ when $\varphi < 1$. Thus, the ZIGP distribution is a mixture of a Bernoulli distribution with parameter $1-\omega$ and the GP distribution with parameters μ and φ. Equations (1), (2), (4) and (5) imply that mean and variance of the ZIGP distribution are connected with its parameters as follows

$$E(Y) = (1-\omega)\mu \tag{7}$$

and

$$Var(Y) = E(Y)\left(\varphi^2 + \mu\omega\right). \tag{8}$$

One of the main benefits of considering a regression model based on the ZIGP distribution is that it gives a large class of regression models for count response data. In particular, it reduces to Poisson regression when $\varphi = 1$ and $\omega = 0$, to GP regression when $\omega = 0$ and to the zero-inflated Poisson regression when $\varphi = 1$. Moreover, by virtue of (7) and (8) this regression can be used to fit count regression data exhibiting overdispersion or underdispersion.

Analogously to GLM, we now introduce a regression model with response Y_i and (known) explanatory variables $\mathbf{x}_i = (x_{i0}, x_{i1}, \ldots, x_{ip})^t$ with $x_{i0} = 1$ for $i = 1, \ldots, n$:

1. *Random components:*
   $\{Y_i,\ 1 \leq i \leq n\}$ are independent where $Y_i \sim ZIGP(\mu_i, \varphi, \omega)$.

2. *Systematic component:*
   The linear predictors $\eta_i(\boldsymbol{\beta}) = \mathbf{x}_i^t\boldsymbol{\beta}$ for $i = 1, \ldots, n$ influence the response Y_i. Here $\boldsymbol{\beta} = (\beta_0, \beta_1, \ldots, \beta_p)^t$ are unknown regression parameters. The matrix $\mathbf{X} = (\mathbf{x}_1, \ldots, \mathbf{x}_n)^t$ is called the design matrix.

3. *Parametric link component:*
   The linear predictors $\eta_i(\boldsymbol{\beta})$ are related to the parameter μ_i of Y_i by $\mu_i = \exp(\eta_i(\boldsymbol{\beta}))$ for $i = 1, \ldots, n$.

Here and in the sequel, $\mathbf{A}^t$ and $\mathbf{a}^t$ denote the transpose of the matrix $\mathbf{A}$ and the vector $\mathbf{a}$, respectively. To stress the fact that the distribution of the responses Y_i's does not belong to the exponential family, this regression will be called the ZIGP regression model. Further, we denote the joint vector of the regression parameters $\boldsymbol{\beta}$ and the parameters φ and ω of the ZIGP distribution by $\boldsymbol{\delta}$, i.e. $\boldsymbol{\delta} := (\boldsymbol{\beta}^t, \varphi, \omega)^t$.

The following abbreviations for $i = 1, \ldots, n$ will be used throughout in the paper:

$$\begin{aligned} \mu_i(\boldsymbol{\beta}) &:= \exp\left(\mathbf{x}_i^t \boldsymbol{\beta}\right) \\ f_i(\boldsymbol{\beta}, \varphi) &:= \exp\left(-\mu_i(\boldsymbol{\beta})/\varphi\right) \\ g_i(\boldsymbol{\delta}) &:= \omega + (1-\omega) f_i(\boldsymbol{\beta}, \varphi) = P_{\mu_i(\beta),\varphi,\omega}(0). \end{aligned}$$

For observations $y_1, \ldots, y_n$, the log-likelihood $l(\boldsymbol{\delta})$ derived from the ZIGP regression can be written as

$$\begin{aligned} l_n(\boldsymbol{\delta}) &= \sum_{i=1}^n \mathbb{1}_{\{y_i=0\}} \log\left(g_i(\boldsymbol{\delta})\right) \\ &+ \sum_{i=1}^n \mathbb{1}_{\{y_i>0\}} \Big(\log(1-\omega) + \mathbf{x}_i^t \boldsymbol{\beta} - \frac{1}{\varphi}\mu_i(\boldsymbol{\beta}) + (y_i - 1)\log\left[\mu_i(\boldsymbol{\beta}) + y_i(\varphi - 1)\right] \\ &\quad - y_i \log\varphi - y_i \frac{1}{\varphi}(\varphi - 1) - \log(y_i!) \Big). \end{aligned}$$

The maximum likelihood (ML) equations for estimating the true $\boldsymbol{\delta}_0 = (\boldsymbol{\beta}_0^t, \varphi_0, \omega_0)^t$ are obtained by equating to zero the score vector which has the following representation:

$$\mathbf{s}_n(\boldsymbol{\delta}) = \left(s_0(\boldsymbol{\delta}), \ldots, s_p(\boldsymbol{\delta}), s_{p+1}(\boldsymbol{\delta}), s_{p+2}(\boldsymbol{\delta})\right)^t, \tag{9}$$

where

$$s_r(\boldsymbol{\delta}) := \frac{\partial l_n(\boldsymbol{\delta})}{\partial \beta_r} = \sum_{i=1}^n s_{r,i}(\boldsymbol{\delta})$$

with

$$\begin{aligned} s_{r,i}(\boldsymbol{\delta}) &:= -x_{ir} \mathbb{1}_{\{y_i=0\}} \frac{(1-\omega) f_i(\boldsymbol{\beta}, \varphi)\mu_i(\boldsymbol{\beta})}{\varphi g_i(\boldsymbol{\delta})} \\ &+ x_{ir} \mathbb{1}_{\{y_i>0\}} \left(1 + \frac{\mu_i(\boldsymbol{\beta})(y_i - 1)}{\mu_i(\boldsymbol{\beta}) + (\varphi - 1) y_i} - \frac{\mu_i(\boldsymbol{\beta})}{\varphi}\right) \end{aligned} \tag{10}$$

for $r = 0, \ldots, p$,

$$s_{p+1}(\boldsymbol{\delta}) := \frac{\partial l_n(\boldsymbol{\delta})}{\partial \varphi} = \sum_{i=1}^n s_{p+1,i}(\boldsymbol{\delta})$$

with

$$\begin{aligned} s_{p+1,i}(\boldsymbol{\delta}) &:= \mathbb{1}_{\{y_i=0\}} \frac{(1-\omega) f_i(\boldsymbol{\beta}, \varphi)\mu_i(\boldsymbol{\beta})}{\varphi^2 g_i(\boldsymbol{\delta})} \\ &+ \mathbb{1}_{\{y_i>0\}} \left(\frac{y_i(y_i - 1)}{\mu_i(\boldsymbol{\beta}) + (\varphi - 1) y_i} - \frac{y_i}{\varphi} + \frac{\mu_i(\boldsymbol{\beta}) - y_i}{\varphi^2}\right), \end{aligned} \tag{11}$$

$$s_{p+2}(\boldsymbol{\delta}) := \frac{\partial l_n(\boldsymbol{\delta})}{\partial \omega} = \sum_{i=1}^n s_{p+2,i}(\boldsymbol{\delta})$$

with

$$s_{p+2,i}(\boldsymbol{\delta}) \quad := \quad \mathbb{1}_{\{y_i=0\}} \frac{1-f_i(\boldsymbol{\beta},\varphi)}{g_i(\boldsymbol{\delta})} - \mathbb{1}_{\{y_i>0\}} \frac{1}{1-\omega}, \tag{12}$$

for $i = 1, \ldots, n$.

A solution of the ML equations $\mathbf{s}_n(\boldsymbol{\delta}) = \mathbf{0}$ will be denoted by $\hat{\boldsymbol{\delta}}$ if it exists. If there are more than one local maxima of the log-likelihood then we take any of them as the ML estimator $\hat{\boldsymbol{\delta}}$. Otherwise set $\hat{\boldsymbol{\delta}}$ as an arbitrary constant in the interior of the set K_δ (for a definition of K_δ see Assumption (A3) of the next section). Further, the expected Fisher information matrix will be denoted by $\mathbf{F}_n(\boldsymbol{\delta})$, i.e. $\mathbf{F}_n(\boldsymbol{\delta}) = E_\delta\left(\mathbf{s}_n(\boldsymbol{\delta})(\mathbf{s}_n(\boldsymbol{\delta}))^t\right)$ and computations of its entries are given in Appendix 1.

4 Asymptotic theory for the maximum likelihood estimator in ZIGP regression

Fahrmeir and Kaufmann (1985) proved consistency and asymptotic normality of the ML estimator in GLM for canonical as well as noncanonical link functions under mild assumptions. Their method can be adapted for proving similar results for the ZIGP regression.

Analogously to Fahrmeir and Kaufmann (1985), we use the Cholesky square root matrix for normalizing the ML estimator. The left Cholesky square root matrix $\mathbf{A}^{1/2}$ of a positive definite matrix $\mathbf{A}$ is the unique lower triangular matrix with positive diagonal elements such that $\mathbf{A}^{1/2}\left(\mathbf{A}^{1/2}\right)^t = \mathbf{A}$ (see Stewart (1998), p. 188). For convenience, set $\mathbf{A}^{t/2} := \left(\mathbf{A}^{1/2}\right)^t$, $\mathbf{A}^{-1/2} := \left(\mathbf{A}^{1/2}\right)^{-1}$ and $\mathbf{A}^{-t/2} := \left(\mathbf{A}^{t/2}\right)^{-1}$. In this paper we deal only with the spectral norm of square matrices denoted by $\|\cdot\|$. The spectral norm of a real-valued matrix $\mathbf{A}$ is given by

$$\|\mathbf{A}\| = \left(\text{maximum eigenvalue of } \mathbf{A}^t\mathbf{A}\right)^{1/2} = \sup_{\|\mathbf{u}\|_2=1} \|\mathbf{A}\mathbf{u}\|_2\,,$$

where $\|\cdot\|_2$ denotes the L^2– norm of vectors. We drop subindex 2 in $\|\cdot\|_2$ since the spectral norm is generated by the L^2–norm of vectors and arguments of considered norms are always clearly defined. The minimal (maximal) eigenvalue of a square matrix $\mathbf{A}$ will be further denoted by $\lambda_{min}(\mathbf{A})$ ($\lambda_{max}(\mathbf{A})$).

Now denote by

$$N_n(\varepsilon) = \{\boldsymbol{\delta} : \|\mathbf{F}_n^{t/2}(\boldsymbol{\delta}_0)(\boldsymbol{\delta}-\boldsymbol{\delta}_0)\| \leq \varepsilon\} \tag{13}$$

a neighborhood of the unknown true parameter $\boldsymbol{\delta}_0$ for $\varepsilon > 0$.

For convenience, we drop the arguments $\boldsymbol{\delta}_0$, $\boldsymbol{\beta}_0$ and φ_0 as well as the subindex $\boldsymbol{\delta}_0$ in $\mu_i(\boldsymbol{\beta}_0)$, $f_i(\boldsymbol{\beta}_0,\varphi_0)$, $g_i(\boldsymbol{\delta}_0)$, P_{δ_0}, E_{δ_0} etc. and write μ_i, f_i, g_i, P, E etc. Constants will be further denoted by C and c, with subindexes or without them. They may depend on $\boldsymbol{\delta}_0$ but not on n. The same C's and c's in different places denote different constants. Finally, the k-dimensional unit matrix will be denoted by $\mathbf{I}_k$ and an admissible set for a regression parameter $\boldsymbol{\beta}$ will be denoted by B.

In the paper we make the following assumptions.

(A1)

$$\frac{n}{\lambda_{min}(\mathbf{F}_n)} \leq C_1 \quad \forall\, n \geq 1,$$

where C_1 is a positive constant.

(A2) $\{\mathbf{x}_n,\ n \geq 1\} \subset K_x$, where $K_x \subset \mathbb{R}^{p+3}$ is a compact set.

(A3) Assume that $B \subset \mathbb{R}^{p+1}$ is an open set and $\boldsymbol{\delta}_0$ is an interior point of the set $K_\delta := B \times \Phi \times \Omega$, where $\Phi := [1, \infty)$ and $\Omega := [0, 1]$.

Now we state our main theorem which is the analogue to Theorem 4 of Fahrmeir and Kaufmann (1985).

Theorem 1. *Under the assumptions* (A1)–(A3)*, there exists a sequence of random variables* $\hat{\boldsymbol{\delta}}_n$*, such that*

(i) $P(\mathbf{s}_n(\hat{\boldsymbol{\delta}}_n) = 0) \rightarrow 1$ *as* $n \rightarrow \infty$ *(asymptotic existence),*

(ii) $\hat{\boldsymbol{\delta}}_n \xrightarrow{P} \boldsymbol{\delta}_0$ *as* $n \rightarrow \infty$ *(weak consistency),*

(iii) $\mathbf{F}_n^{t/2}(\hat{\boldsymbol{\delta}}_n - \boldsymbol{\delta}_0) \stackrel{\mathcal{D}}{\Longrightarrow} N_p(\mathbf{0}, \mathbf{I}_{p+3})$ *as* $n \rightarrow \infty$ *(asymptotic normality).*

The proof is given in Appendix 2.

Remarks

(i) Assumption (A1) is more restrictive than the corresponding condition (D) of Fahrmeir and Kaufmann (1985).

(ii) Assumption (A2) simply means that we deal with compact regressors.

(iii) If $\boldsymbol{\delta}_0$ lies on the boundary of parameter space K_δ, i.e. (A3) is violated, then statements of Theorem 1 do not hold anymore. However, one may investigate asymptotic properties of the ML estimator $\hat{\boldsymbol{\delta}}$ using results of Self and Liang (1987) and Moran (1971).

(iv) Theorem 1 allows to construct confidence intervals for φ_0 and ω_0 and tests based on them. However, we cannot test the adequacy of the Poisson regression, the GP regression or the ZIP regression models, i.e. $\varphi_0 = 1$ and/or $\omega_0 = 0$, using Theorem 1.

(v) It is not difficult to see that the asymptotic results of Theorem 1 remain valid in GP or ZIP regression models subject to appropriate changes are performed in the log-likelihood, the ML equations and the Fisher information matrix as well as in Assumption (A3).

5 Simulation study

We investigated the accuracy of the normal approximation based on Theorem 1 by performing a small simulation study in S-PLUS for samples of size $n = 50, 100$ and 200. We use a similar simulation setup as Stekeler (2004). It should be noted here that the maximization routine have been written by Stekeler (2004) and can be downloaded (*www-m4.ma.tum.de/Diplarb/*). A simple model with intercept and single covariate x was considered for the linear predictors $\eta_i(\boldsymbol{\beta})$'s, i.e. $\eta_i(\boldsymbol{\beta}) = \beta_0 + \beta_1 x_i$ for $i = 1, \dots, n$. The values of the covariate x were chosen equally spaced between -1 and 1. Further we examined two choices for β_1 and set $\beta_0 = -1$. In the first case we put $\beta_1 = 2$ while $\beta_1 = 3$ was set in the second case, which will be in the sequel called Setting-1 and Setting-2, respectively. This allows us to compare models with a small (Setting-1) and large (Setting-2) range of the parameter μ of the $ZIGP$ distribution. Since we are mostly interested in the case when Poisson regression does not satisfactorily fit the count regression data, the following values of ω and φ were considered: $\omega = 0.1, 0.25$ and $\varphi = 1.25, 3$. For each combination of sample size n, setting, ω and φ we simulated 100 samples of responses Y_i's, i.e. $Y_i \sim ZIGP(\exp(\beta_0 + \beta_1 x_i), \varphi, \omega)$ for $i = 1, \dots, n$.

We computed the average estimate and the estimated mean squared error (MSE) of the ML estimators $\hat{\beta}_0$, $\hat{\beta}_1$, $\hat{\varphi}$ and $\hat{\omega}$ in 100 replications for each considered case. Simulation results for Setting-1 are a bit more accurate than for Setting-2 but they demonstrate similar patterns. This is natural to expect since μ has an influence on the range the of data. Here we present the results only for Setting-2 given in Table 1. Standard errors of the average estimate and estimated MSE are given in parentheses. From Table 1 we see as expected that the bias and MSE always decrease as the sample size n increases. An opposite pattern is observed with respect to φ. If φ increases, while n and ω remain fixed, the accuracy of the estimates becomes worse. This is explained by allowing for more dispersed data for larger φ. A similar pattern holds for ω. If ω increases, while n and φ remain fixed, then the accuracy becomes worse. Since a larger ω increases the overdispersion in the data this is to be expected. Note that in our simulation study φ has a larger influence on the accuracy of ML estimators than ω. This can be seen from the estimated MSE's. For instance, the estimated MSE of $\hat{\beta}_1$ is equal to 0.09 when $\varphi = 1.25$, $\omega = 0.1$ and $n = 200$. Now if ω is increased by 2.5 times then the estimated MSE approximately increases 20% while if φ is increased by 2.4 times then the estimated MSE approximately increases 110%.

To draw a normal quantile-quantile (QQ) plot for the empirical distribution of each component of the random vector $\mathbf{F}_n^{t/2}(\hat{\boldsymbol{\delta}}_n - \boldsymbol{\delta}_0)$ considered in Theorem 1 and the standard normal distribution, the ML estimators $\hat{\beta}_0$, $\hat{\beta}_1$, $\hat{\varphi}$ and $\hat{\omega}$ were centered by the corresponding true value and normalized by the corresponding square root of diagonal element of the inverse of the Fisher information matrix evaluated at the true values of parameters $(\varphi, \omega, \beta_0, \beta_1)$. The normalized and centered ML estimators are further denoted by $\hat{\beta}_0^{st}$, $\hat{\beta}_1^{st}$, $\hat{\varphi}^{st}$ and $\hat{\omega}^{st}$. Figures 1 ($\omega = 0.1$) and 2 ($\omega = 0.25$) display the QQ–plots for Setting-2. For a better visualization we connected points of QQ–plots with different type of lines. The solid, dotted and dashed broken lines correspond to sample sizes $n = 50$, $n = 100$ and $n = 200$, respectively. The straight line corresponds to 45^0 degree line and indicates where the points of a standard normal distribution in a normal QQ–plot would fall.

From these plots we see that the normal approximation for $\hat{\beta}_0^{st}$ and $\hat{\beta}_1^{st}$ is quite satisfactory. This is only partially true for $\hat{\varphi}^{st}$ and $\hat{\omega}^{st}$ since we observe horizontal segments in the left bottom corner of the corresponding QQ-plot. A reason of the above anomaly is the closeness of the true values of φ and ω to their left boundary values 1 and 0, respectively.

Therefore the log-likelihood reaches its maximum at $\varphi = 1 + 10^{-99}$ and $\omega = 10^{-99}$ which are the lower bound for φ and ω in the maximization routine. The standard normal QQ-plots for $\hat{\varphi}^{st}$ in Figure 1 and $\hat{\omega}^{st}$ in Figure 2 justify this fact. Note that the normal approximation for $\hat{\omega}^{st}$ in Figure 2 is worse for $\varphi = 3$ than for $\varphi = 1.25$. This occurs since data becomes more dispersed for large φ. The above anomaly is resolved when a higher sample size is used in these cases. This can be seen by comparing the first column of QQ-plots in Figure 1 with the corresponding QQ-plots in Figure 3 for $n = 500$.

Since Theorem 1 covers such important regression models as ZIP and GP regressions, we were also interested in investigating the accuracy of the normal approximation in these special models. In the case of a ZIP regression we simulated 100 samples of responses Y_i's from $ZIP(\exp(\beta_0 + \beta_1 x_i), \omega)$ for $i = 1, \dots, n$, where sample size n, the parameters ω, β_0, β_1 and the covariate x were defined as in the case of the $ZIGP$ regression simulation. Table 2 displays the average estimate and estimated MSE of the ML estimators $\hat{\beta}_0$, $\hat{\beta}_1$ and $\hat{\omega}$ in 100 replications for the ZIP regression simulation study. Their standard errors are also given in parentheses. Figure 4 displays the corresponding QQ-plots of empirical distributions of normalized and centered $\hat{\omega}^{st}$, $\hat{\beta}_0^{st}$, $\hat{\beta}_1^{st}$ and the standard normal distribution. From estimated MSE's in Table 2 we see that $\hat{\omega}$ is more accurate estimated in Setting 2 while $\hat{\beta}_0$ and $\hat{\beta}_1$ is more accurate estimated in Setting 1. This is natural to expect since if range of μ gets larger then data becomes more overdispersed and zeros from Poisson distribution occur rarer. In general, patterns discussed for the case of the ZIGP regression model are also valid here.

In the case of a GP regression we simulated 100 samples of responses Y_i's from $GP(\exp(\beta_0 + \beta_1 x_i), \varphi)$ for $i = 1, \dots, n$, where sample size n, the parameters φ, β_0, β_1 and the covariate x were set as in the case of the $ZIGP$ regression. Table 3 displays the average estimate and the estimated MSE of the ML estimators $\hat{\beta}_0$, $\hat{\beta}_1$ and $\hat{\varphi}$ in 100 replications for the GP regression simulation study. Figure 5 displays the corresponding QQ-plots of empirical distributions of normalized and centered $\hat{\varphi}^{st}$, $\hat{\beta}_0^{st}$, $\hat{\beta}_1^{st}$ and the standard normal distribution. From Table 3 we see that the accuracy of the ML estimators becomes better as n gets larger and becomes worse as φ increases. Since μ does not bear an influence on overdispersion (see (4) and (5)), there is not much difference in the accuracy of the ML estimators for Setting-1 and Setting-2.

We also investigated the coverage of the true values of the parameters φ, ω, β_0 and β_1 of asymptotic confidence intervals based on Theorem 1 for sample size $n = 500$. Results of simulations show good agreement with true values of φ, ω, β_0 and β_1. More detailed research on this topic will be carried out in the future.

Using a negative binomial regression model Czado and Sikora (2002) analyzed data on patents US high-tech firms in 1976 from Wang, Cockburn, and Puterman (1998). Czado and Sikora (2002) rejected Poisson model in favor of negative binomial model using a p–value approach. We applied a $ZIGP$ regression model to their model setup for the patent data. Since $[2.098, 3.263]$ and $[-0.03, 0.06]$ were obtained as asymptotic 95% confidence intervals for φ and ω, respectively, we reject the Poisson regression model in favor of a GP regression model. Zero-inflation is not present in this data set. An application of a GP regression to the patent data produces $[2.152, 3.342]$ as an asymptotic 95% confidence interval. Thus, the GP regression is preferred over the Poisson regression model.

Table 1: Average estimate and estimated MSE of $\hat{\varphi}$, $\hat{\omega}$, $\hat{\beta}_0$, $\hat{\beta}_1$ in Setting-2 for a *ZIGP* regression model on the basis of 100 replications replications (estimated standard errors are given in parentheses).

Parameter	**True value**	**n**	**Estimate**		**MSE**		**Parameter**	**True value**	**n**	**Estimate**		**MSE**	
φ	1.25	50	1.222	(0.214)	0.047	(0.009)	φ	1.25	50	1.208	(0.213)	0.047	(0.007)
		100	1.219	(0.148)	0.023	(0.003)			100	1.190	(0.151)	0.026	(0.004)
		200	1.238	(0.122)	0.015	(0.002)			200	1.207	(0.126)	0.018	(0.002)
ω	0.1	50	0.112	(0.110)	0.012	(0.002)	ω	0.25	50	0.259	(0.147)	0.022	(0.003)
		100	0.104	(0.081)	0.007	(0.001)			100	0.264	(0.095)	0.009	(0.001)
		200	0.096	(0.056)	0.003	$(4 \cdot 10^{-4})$			200	0.250	(0.072)	0.005	(0.001)
β_0	-1	50	-1.018	(0.438)	0.192	(0.029)	β_0	-1	50	-1.206	(0.591)	0.391	(0.058)
		100	-1.044	(0.310)	0.098	(0.011)			100	-1.095	(0.357)	0.136	(0.023)
		200	-1.064	(0.237)	0.060	(0.010)			200	-1.027	(0.264)	0.070	(0.013)
β_1	3	50	2.976	(0.512)	0.263	(0.034)	β_1	3	50	3.228	(0.689)	0.526	(0.077)
		100	3.083	(0.406)	0.172	(0.019)			100	3.122	(0.464)	0.230	(0.037)
		200	3.065	(0.293)	0.090	(0.013)			200	3.022	(0.327)	0.107	(0.019)

Parameter	**True value**	**n**	**Estimate**		**MSE**		**Parameter**	**True value**	**n**	**Estimate**		**MSE**	
φ	3	50	2.672	(1.404)	2.079	(0.324)	φ	3	50	2.563	(1.255)	1.765	(0.214)
		100	2.865	(0.914)	0.853	(0.143)			100	2.936	(0.957)	0.921	(0.113)
		200	2.915	(0.514)	0.272	(0.037)			200	2.933	(0.648)	0.424	(0.061)
ω	0.1	50	0.165	(0.184)	0.038	(0.006)	ω	0.25	50	0.258	(0.220)	0.048	(0.005)
		100	0.117	(0.137)	0.019	(0.004)			100	0.227	(0.181)	0.033	(0.003)
		200	0.090	(0.095)	0.009	(0.001)			200	0.251	(0.129)	0.017	(0.002)
β_0	-1	50	1.382	(1.107)	1.372	(0.372)	β_0	-1	50	-1.423	(1.110)	1.412	(0.319)
		100	1.137	(0.606)	0.386	(0.075)			100	-1.241	(0.674)	0.512	(0.082)
		200	1.127	(0.388)	0.166	(0.029)			200	-1.095	(0.440)	0.203	(0.046)
β_1	3	50	3.430	(1.256)	1.762	(0.534)	β_1	3	50	3.399	(1.275)	1.784	(0.430)
		100	3.207	(0.720)	0.561	(0.116)			100	3.259	(0.748)	0.626	(0.103)
		200	3.120	(0.414)	0.186	(0.031)			200	3.141	(0.537)	0.308	(0.086)

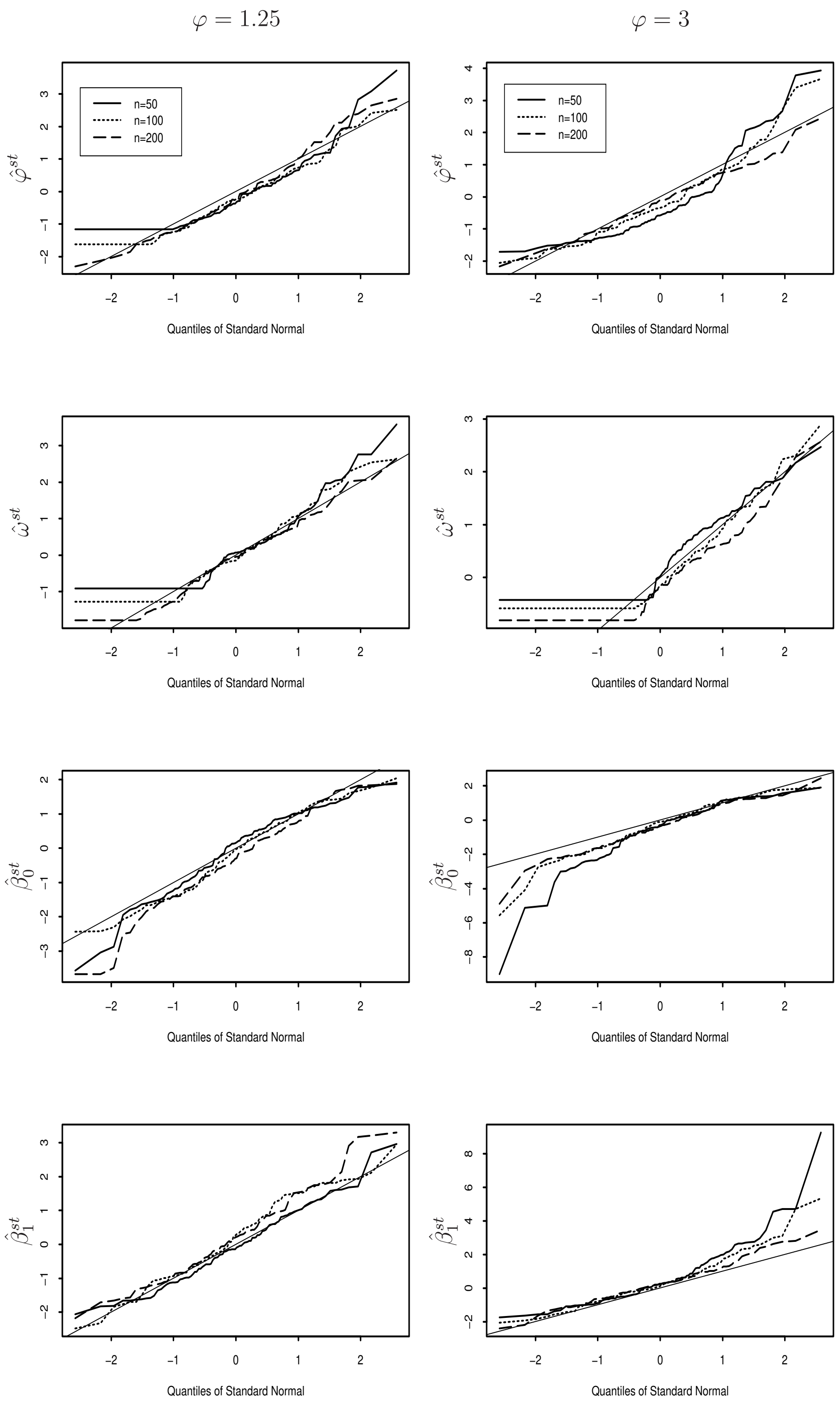

Figure 1: Normal QQ-plots of centered and normalized ML estimators in Setting-2 for a *ZIGP* regression model with $\omega = 0.1$ based on 100 replications

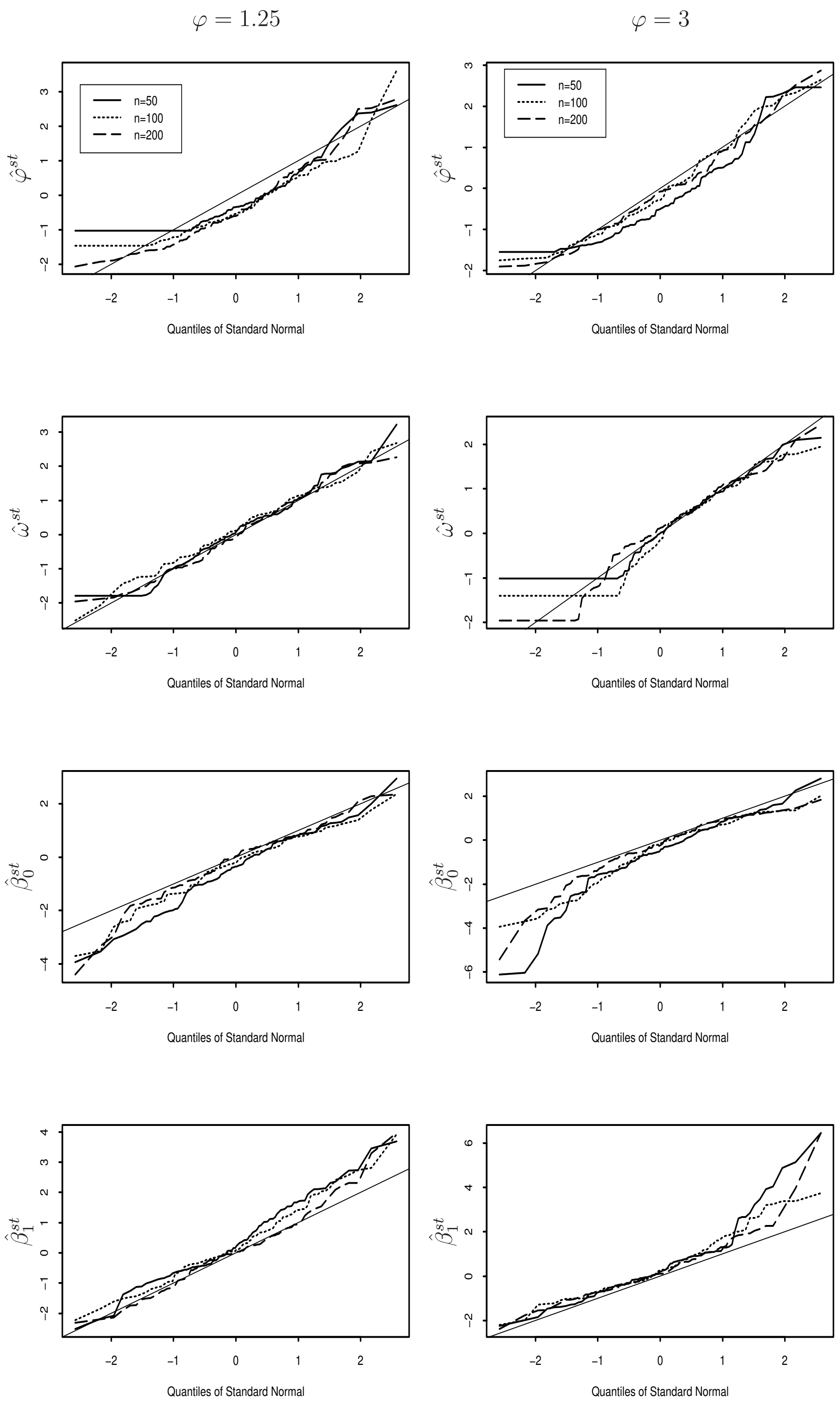

Figure 2: Normal QQ-plots of centered and normalized ML estimators in Setting-2 for a *ZIGP* regression model with $\omega = 0.25$ based on 100 replications

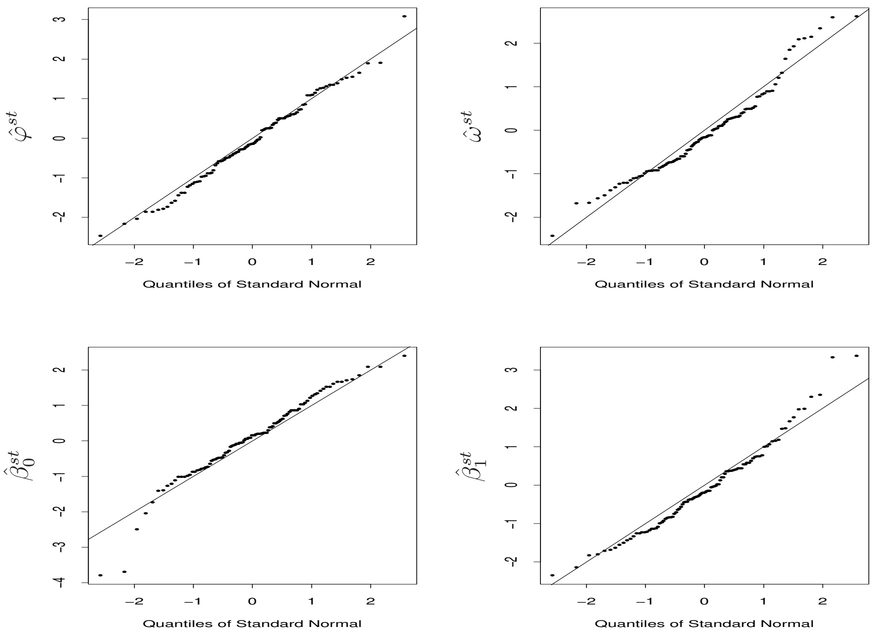

Figure 3: Normal QQ-plots of centered and normalized ML estimators in Setting-2 for a $ZIGP$ regression model with $n = 500$, $\omega = 0.1$ and $\varphi = 1.25$ based on 100 replications

Table 2: Average estimate and estimated MSE of $\hat{\omega}, \hat{\beta}_0, \hat{\beta}_1$ for a ZIP regression model on the basis of 100 replications (estimated standard errors are given in parentheses)

Para-meter	**True value**	**n**	**Setting-1**				**Para-meter**	**True value**	**n**	**Setting-2**			
			Estimate		**MSE**					**Estimate**		**MSE**	
ω	0.1	50	0.121	(0.123)	0.016	$(7 \cdot 10^{-6})$	ω	0.1	50	0.105	(0.095)	0.009	$(2 \cdot 10^{-6})$
		100	0.092	(0.092)	0.009	(10^{-6})			100	0.104	(0.067)	0.005	$(3 \cdot 10^{-7})$
		200	0.083	(0.066)	0.005	$(2 \cdot 10^{-7})$			200	0.101	(0.055)	0.003	$(2 \cdot 10^{-7})$
β_0	-1	50	-1.017	(0.388)	0.151	(0.001)	β_0	-1	50	-1.09	(0.386)	0.157	(0.001)
		100	-1.049	(0.214)	0.048	$(5 \cdot 10^{-5})$			100	-1.009	(0.267)	0.071	$(9 \cdot 10^{-5})$
		200	-1.061	(0.183)	0.037	$(3 \cdot 10^{-5})$			200	-0.99	(0.183)	0.033	$(2 \cdot 10^{-5})$
β_1	2	50	1.992	(0.491)	0.241	(0.001)	β_1	3	50	3.093	(0.501)	0.260	(0.001)
		100	2.072	(0.285)	0.086	(10^{-4})			100	3.022	(0.353)	0.125	$(3 \cdot 10^{-4})$
		200	2.051	(0.231)	0.056	$(5 \cdot 10^{-5})$			200	2.987	(0.23)	0.053	$(4 \cdot 10^{-5})$
Para-meter	**True value**	**n**	**Setting-1**				**Para-meter**	**True value**	**n**	**Setting-2**			
			Estimate		**MSE**					**Estimate**		**MSE**	
ω	0.25	50	0.213	(0.164)	0.028	(10^{-5})	ω	0.25	50	0.228	(0.132)	0.018	$(4 \cdot 10^{-6})$
		100	0.234	(0.125)	0.016	$(3 \cdot 10^{-6})$			100	0.264	(0.085)	0.007	$(9 \cdot 10^{-7})$
		200	0.249	(0.093)	0.009	$(2 \cdot 10^{-6})$			200	0.243	(0.058)	0.003	$(3 \cdot 10^{-7})$
β_0	-1	50	-1.006	(0.37)	0.137	$(4 \cdot 10^{-4})$	β_0	-1	50	-1.085	(0.385)	0.156	$(5 \cdot 10^{-4})$
		100	-1.052	(0.285)	0.084	(10^{-4})			100	-1.007	(0.27)	0.073	$(2 \cdot 10^{-4})$
		200	-1.014	(0.202)	0.041	$(4 \cdot 10^{-5})$			200	-1.038	(0.201)	0.042	$(3 \cdot 10^{-5})$
β_1	2	50	1.939	(0.434)	0.192	(0.001)	β_1	3	50	3.062	(0.476)	0.231	(0.001)
		100	1.992	(0.398)	0.159	$(5 \cdot 10^{-4})$			100	2.993	(0.371)	0.138	$(4 \cdot 10^{-4})$
		200	2.031	(0.252)	0.064	$(7 \cdot 10^{-5})$			200	3.043	(0.242)	0.060	$(8 \cdot 10^{-5})$

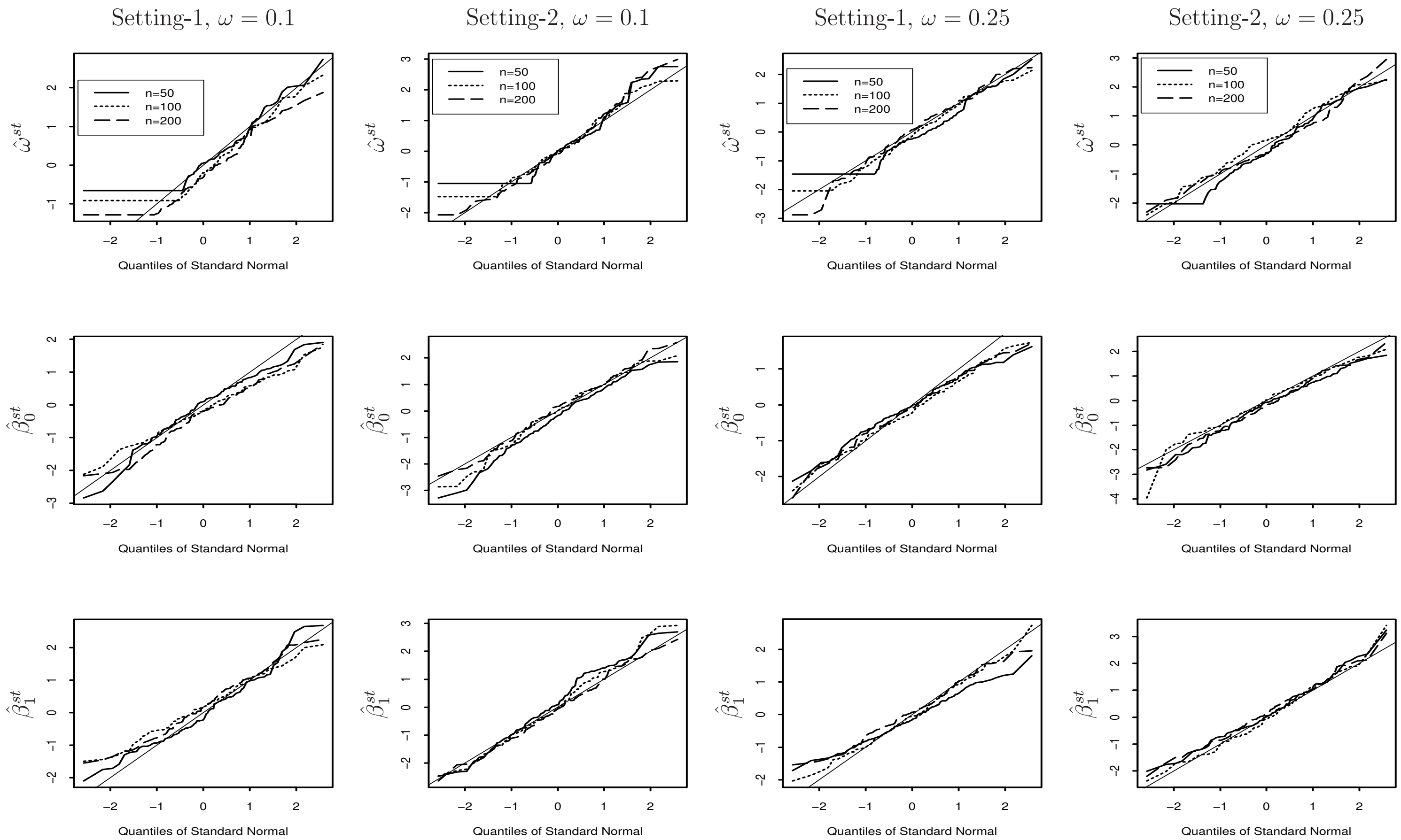

Figure 4: Normal QQ-plots of centered and normalized ML estimators for a ZIP regression model based on 100 replications

Table 3: Average estimate and estimated MSE of $\hat{\varphi}$, $\hat{\beta}_0$, $\hat{\beta}_1$ for a *GP* regression model on the basis of 100 replications (estimated standard errors are given in parentheses).

Parameter	**True value**	**n**	**Setting-1**				**Parameter**	**True value**	**n**	**Setting-2**			
			Estimate		**MSE**					**Estimate**		**MSE**	
φ	1.25	50	1.221	(0.192)	0.038	$(5 \cdot 10^{-5})$	φ	1.25	50	1.204	(0.176)	0.033	$(2 \cdot 10^{-5})$
		100	1.222	(0.123)	0.016	$(4 \cdot 10^{-6})$			100	1.224	(0.151)	0.023	(10^{-5})
		200	1.239	(0.094)	0.009	(10^{-6})			200	1.245	(0.093)	0.009	(10^{-6})
β_0	-1	50	-1.041	(0.389)	0.153	(0.001)	β_0	-1	50	-1.112	(0.373)	0.151	(0.001)
		100	-1.053	(0.245)	0.063	$(9 \cdot 10^{-5})$			100	-1.037	(0.289)	0.085	$(3 \cdot 10^{-4})$
		200	-1.015	(0.187)	0.035	$(2 \cdot 10^{-5})$			200	-1.016	(0.162)	0.027	$(2 \cdot 10^{-5})$
β_1	2	50	1.990	(0.540)	0.291	(0.002)	β_1	3	50	3.120	(0.468)	0.233	(0.001)
		100	2.026	(0.392)	0.154	(0.001)			100	3.046	(0.373)	0.141	$(5 \cdot 10^{-4})$
		200	2.006	(0.250)	0.063	$(6 \cdot 10^{-5})$			200	3.007	(0.234)	0.055	$(6 \cdot 10^{-5})$

Parameter	**True value**	**n**	**Setting-1**				**Parameter**	**True value**	**n**	**Setting-2**			
			Estimate		**MSE**					**Estimate**		**MSE**	
φ	3	50	2.720	(1.182)	1.477	(0.037)	φ	3	50	2.699	(0.921)	0.939	(0.025)
		100	3.114	(0.93)	0.878	(0.024)			100	2.941	(0.824)	0.683	(0.010)
		200	3.025	(0.591)	0.349	(0.003)			200	2.931	(0.504)	0.258	(0.001)
β_0	-1	50	-1.323	(0.841)	0.812	(0.035)	β_0	-1	50	-1.195	(0.718)	0.554	(0.021)
		100	-1.101	(0.496)	0.256	(0.001)			100	-1.060	(0.471)	0.226	(0.001)
		200	-1.052	(0.308)	0.098	$(2 \cdot 10^{-4})$			200	-1.031	(0.319)	0.103	$(5 \cdot 10^{-4})$
β_1	2	50	2.116	(0.982)	0.978	(0.04)	β_1	3	50	3.080	(0.856)	0.740	(0.024)
		100	2.085	(0.619)	0.390	(0.002)			100	3.033	(0.599)	0.360	(0.003)
		200	2.094	(0.410)	0.177	(0.001)			200	3.001	(0.389)	0.151	(0.001)

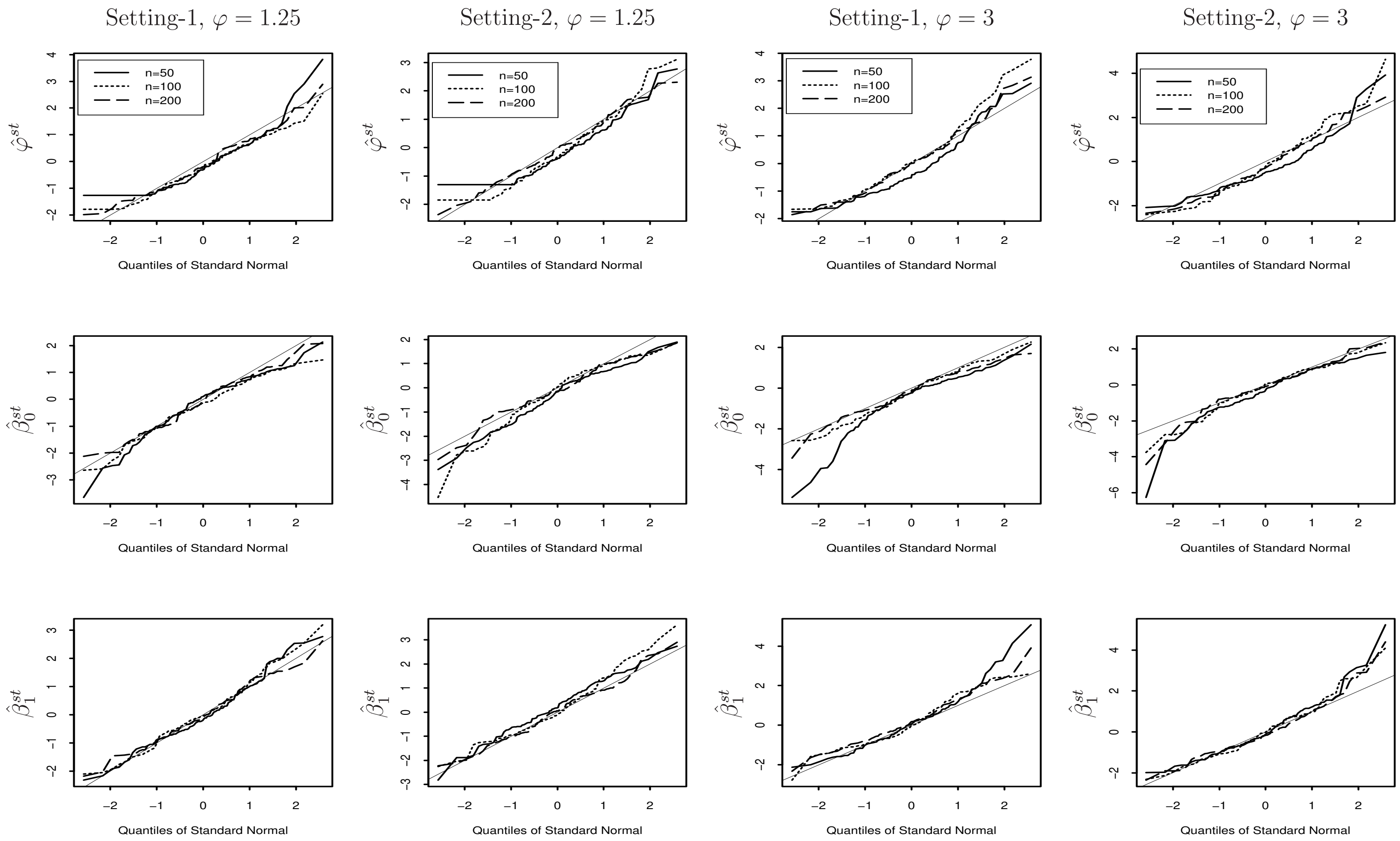

Figure 5: Normal QQ-plots of centered and normalized ML estimators for a GP regression model based on 100 replications

6 Conclusions

This paper shows that solutions to the ML equations in ZIGP (GP, ZIP) regression models possess analogous asymptotic properties as they do in GLM. General results of Fahrmeir and Kaufmann (1985) for noncanonical links in GLM have been adopted for this purpose. Simulation study illustrates that the normal approximation is satisfactory for moderate and large sample sizes. In particular for moderate overdispersion ($\varphi = 1.25$) and moderate zero-inflation ($\omega = 0.25$) sample sizes of $n = 200$ are sufficient.

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References

Cameron, A. and P. Trivedi (1998). *Regression analysis of count data*. Cambridge University Press.

Consul, P. C. (1989). *Generalized Poisson distributions*, Volume 99 of *Statistics: Textbooks and Monographs*. New York: Marcel Dekker Inc. Properties and applications.

Consul, P. C. and F. Famoye (1992). Generalized Poisson regression model. *Comm. Statist. Theory Methods 21*(1), 89–109.

Consul, P. C. and G. C. Jain (1970). On the generalization of Poisson distribution. *Ann. Math. Statist. 41*, 1387.

Consul, P. C. and L. R. Shenton (1974). On the probabilistic structure and propertis of discrete lagrangian distribution. In *A modern course on statistical distribution in scientific work*, Volume 1. Boston.

Consul, P. C. and M. M. Shoukri (1985). The generalized Poisson distribution when the sample mean is large than the sample variance. *Comm. Statist. Theory Methods 14*(3), 667–681.

Czado, C. and I. Sikora (2002). Quantifying overdispersion effects in count regression data. *Discussion paper 289 of SFB 386* (*http://www.stat.uni-muenchen.de/sfb386/*).

Fahrmeir, L. and H. Kaufmann (1985). Consistency and asymptotic normality of the maximum likelihood estimator in generalized linear models. *Ann. Statist. 13*(1), 342–368.

Famoye, F. and K. P. Singh (2003). On inflated generalized Poisson regression models. *Adv. Appl. Stat. 3*(2), 145–158.

Gupta, P. L., R. C. Gupta, and R. C. Tripathi (2004). Score test for zero inflated generalized Poisson regression model. *Comm. Statist. Theory Methods 33*(1), 47–64.

Hoffmann-Jørgensen, J. (1994). *Probability with a view toward statistics. Vol. I.* Chapman & Hall Probability Series. New York: Chapman & Hall.

Jansakul, N. and J. P. Hinde (2002). Score tests for zero-inflated Poisson models. *Comput. Statist. Data Anal. 40*(1), 75–96.

Lambert, D. (1992). Zero-inflated poisson regression, with an application to defects in manufacturing. *Technometrics 34*(1), 1–14.

Lawless, J. (1987). Negative binomial and mixed poisson regression. *The Canadian Journal of Statistics 15*, 209–225.

Mardia, K. V., J. T. Kent, and J. M. Bibby (1979). *Multivariate analysis*. London: Academic Press [Harcourt Brace Jovanovich Publishers]. Probability and Mathematical Statistics: A Series of Monographs and Textbooks.

McCullagh, P. and J. Nelder (1989). *Generalized linear models*. Chapman & Hall.

Moran, P. A. P. (1971). Maximum-likelihood estimation in non-standard conditions. *Proc. Cambridge Philos. Soc. 70*, 441–450.

Petrov, V. V. (1995). *Limit theorems of probability theory sequences of independent variables*. Oxford studeis in probability. Oxford: Clarendon Press.

Self, S. G. and K.-Y. Liang (1987). Asymptotic properties of maximum likelihood estimators and likelihood ratio tests under nonstandard conditions. *J. Amer. Statist. Assoc. 82*(398), 605–610.

Stekeler, D. (2004). Verallgemeinerte Poissonregression und daraus abgeleitete Zero-Inflated und Zero-Hurdle Regressionsmodelle. Master's thesis, Technische Universität München (*www-m4.ma.tum.de/Diplarb/*).

Stewart, G. W. (1998). *Matrix algorithms. Vol. I*. Philadelphia, PA: Society for Industrial and Applied Mathematics. Basic decompositions.

Wang, P., I. Cockburn, and M. Puterman (1998). Analysis of patent data – a mixed Poisson regression model approach. *J. Bus. Econom. Statist. 16*, 27–41.

Winkelmann, R. (2003). *Econometric analysis of count data* (fourth ed.). Berlin: Springer-Verlag.

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Appendix 1: Derivation of the Fisher information matrix for ZIGP regression

In order to compute the Fisher information matrix we need the following lemma derived in Stekeler (2004), pp. 80–81. For the reader's convenience we provide its proof here.

Lemma 1. *If* $\tilde{Y} \sim GP(\mu, \varphi)$ *then*

$$E\left(\frac{\tilde{Y}(\tilde{Y}-1)}{[\mu+(\varphi-1)\tilde{Y}]^2}\right) = \frac{\mu}{\varphi^2(\mu-2+2\varphi)}$$

and

$$E\left(\frac{\tilde{Y}^2(\tilde{Y}-1)}{[\mu+(\varphi-1)\tilde{Y}]^2}\right)=\frac{\mu(\mu+2\varphi)}{\varphi^2(\mu-2+2\varphi)}.$$

Proof. We give a proof for the second assertion of Lemma 1. The first one can be proven in the same way. Using (3), (4) and making change of variable $y=\tilde{y}+2$, we find

$$\begin{aligned}
E\left(\frac{\tilde{Y}^2(\tilde{Y}-1)}{[\mu+(\varphi-1)\tilde{Y}]^2}\right) &= \sum_{y=2}^{\infty}\frac{y^2(y-1)}{[\mu+(\varphi-1)y]^2}P_{\mu,\varphi}(y)\\
&= \frac{\mu}{\varphi^2(\mu+2(\varphi-1))}\sum_{\tilde{y}=0}^{\infty}(\tilde{y}+2)P_{\mu+2(\varphi-1),\varphi}(\tilde{y})\\
&= \frac{\mu}{\varphi^2(\mu+2(\varphi-1))}(\mu+2(\varphi-1)+2)\\
&= \frac{\mu(\mu+2\varphi)}{\varphi^2(\mu-2+2\varphi)}.
\end{aligned}$$

□

The following lemma will be directly used for computing entries of the Fisher information matrix.

Lemma 2. *If* $Y\sim ZIGP(\mu,\varphi,\omega)$ *and* $\tilde{Y}\sim GP(\mu,\varphi)$ *then for any measurable function* u

$$E_\delta\left(\mathbb{1}_{\{Y>0\}}u(Y)\right)=(1-\omega)E_\delta\left(u(\tilde{Y})\right)-u(0)(1-\omega)\exp\left(-\frac{\mu}{\varphi}\right)$$

and

$$E_\delta\left(\mathbb{1}_{\{Y=0\}}u(Y)\right)=u(0)\left(\omega+(1-\omega)e^{-\mu/\varphi}\right).$$

If furthermore the function u *is non-negative on* $[0,\infty)$ *then*

$$E_\delta\left(\mathbb{1}_{\{Y>0\}}u(Y)\right)\le(1-\omega)E_\delta\left(u(\tilde{Y})\right).$$

Proof. Let us prove the first statement of Lemma 2.

$$\begin{aligned}
E_\delta\left(\mathbb{1}_{\{Y>0\}}u(Y)\right) &= \sum_{y=1}^{\infty}u(y)P_{\mu,\varphi}(y)\\
&= (1-\omega)E\left(u(\tilde{Y})\right)-u(0)(1-\omega)P_{\mu,\varphi}(0)\\
&= (1-\omega)E\left(u(\tilde{Y})\right)-u(0)(1-\omega)\exp\left(-\frac{\mu}{\varphi}\right).
\end{aligned}$$

The second statement of Lemma 2 is obvious. The third statement follows from the first one. □

The Fisher information matrix will be computed via the Hessian matrix. Simple calculations show that the Hessian matrix is given by

$$\mathcal{H}_n(\boldsymbol{\delta})=\begin{pmatrix}
h_{0,0}(\boldsymbol{\delta}) & h_{0,1}(\boldsymbol{\delta}) & \dots & h_{0,p}(\boldsymbol{\delta}) & h_{0,p+1}(\boldsymbol{\delta}) & h_{0,p+2}(\boldsymbol{\delta})\\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
h_{p,0}(\boldsymbol{\delta}) & h_{p,1}(\boldsymbol{\delta}) & \dots & h_{p,p}(\boldsymbol{\delta}) & h_{p,p+1}(\boldsymbol{\delta}) & h_{p,p+2}(\boldsymbol{\delta})\\
h_{p+1,0}(\boldsymbol{\delta}) & h_{p+1,1}(\boldsymbol{\delta}) & \dots & h_{p+1,p}(\boldsymbol{\delta}) & h_{p+1,p+1}(\boldsymbol{\delta}) & h_{p+1,p+2}(\boldsymbol{\delta})\\
h_{p+2,0}(\boldsymbol{\delta}) & h_{p+2,1}(\boldsymbol{\delta}) & \dots & h_{p+2,p}(\boldsymbol{\delta}) & h_{p+2,p+1}(\boldsymbol{\delta}) & h_{p+2,p+2}(\boldsymbol{\delta})
\end{pmatrix}$$

with entries:

$$
\begin{aligned}
h_{r,s}(\boldsymbol{\delta}) &:= \frac{\partial^2 l_n(\delta)}{\partial\beta_r\partial\beta_s} \\
&= -\sum_{i=1}^{n} \mathbb{1}_{\{y_i=0\}} x_{ir}x_{is}(1-\omega)\mu_i(\boldsymbol{\beta}) \\
&\times \frac{\left[1-f_i(\boldsymbol{\beta},\varphi)/\varphi\right]g_i(\boldsymbol{\delta})+(1-\omega)\left[f_i(\boldsymbol{\beta},\varphi)\right]^2\mu_i(\boldsymbol{\beta})/\varphi}{\varphi\left[g_i(\boldsymbol{\delta})\right]^2} \\
&- \sum_{i=1}^{n} \mathbb{1}_{\{y_i>0\}} x_{ir}x_{is}\mu_i(\boldsymbol{\beta})\left(\frac{1}{\varphi}-\frac{y_i(y_i-1)(\varphi-1)}{\left[\mu_i(\boldsymbol{\beta})+(\varphi-1)y_i\right]^2}\right)
\end{aligned}
$$

for $r,s=0,\ldots,p$;

$$
\begin{aligned}
h_{p+1,r}(\boldsymbol{\delta}) &= h_{r,p+1}(\boldsymbol{\delta}) := \frac{\partial^2 l_n(\delta)}{\partial\beta_r\partial\varphi} \\
&= -\sum_{i=1}^{n} \mathbb{1}_{\{y_i=0\}} x_{ir}(1-\omega)f_i(\boldsymbol{\beta},\varphi)\mu_i(\boldsymbol{\beta}) \\
&\times \frac{g_i(\boldsymbol{\delta})\left[\mu_i(\boldsymbol{\beta})/\varphi-1\right]-(1-\omega)f_i(\boldsymbol{\beta},\varphi)\mu_i(\boldsymbol{\beta})/\varphi}{\varphi^2\left[g_i(\boldsymbol{\delta})\right]^2} \\
&+ \sum_{i=1}^{n} \mathbb{1}_{\{y_i>0\}} x_{ir}\mu_i(\boldsymbol{\beta})\left(\frac{1}{\varphi^2}-\frac{y_i(y_i-1)}{\left[\mu_i(\boldsymbol{\beta})+(\varphi-1)y_i\right]^2}\right)
\end{aligned}
$$

for $r=0,\ldots,p$;

$$
\begin{aligned}
h_{p+2,r}(\boldsymbol{\delta}) &= h_{r,p+2}(\boldsymbol{\delta}) := \frac{\partial^2 l_n(\delta)}{\partial\beta_r\partial\omega} \\
&= \sum_{i=1}^{n} \mathbb{1}_{\{y_i=0\}} \frac{x_{ir}f_i(\boldsymbol{\beta},\varphi)\mu_i(\boldsymbol{\beta})}{\varphi\left[g_i(\boldsymbol{\delta})\right]^2}
\end{aligned}
$$

for $r=0,\ldots,p$;

$$
\begin{aligned}
h_{p+1,p+1}(\boldsymbol{\delta}) &:= \frac{\partial^2 l_n(\delta)}{\partial\varphi\partial\varphi} \\
&= \sum_{i=1}^{n} \mathbb{1}_{\{y_i=0\}}(1-\omega)f_i(\boldsymbol{\beta},\varphi)\mu_i(\boldsymbol{\beta})\frac{g_i(\boldsymbol{\delta})\left(\mu_i(\boldsymbol{\beta})-2\varphi\right)-(1-\omega)f_i(\boldsymbol{\beta},\varphi)\mu_i(\boldsymbol{\beta})}{\varphi^4\left[g_i(\boldsymbol{\delta})\right]^2} \\
&- \sum_{i=1}^{n} \mathbb{1}_{\{y_i>0\}}\left(\frac{y_i^2(y_i-1)}{\left[\mu_i(\boldsymbol{\beta})+(\varphi-1)y_i\right]^2}+2\frac{\mu_i(\boldsymbol{\beta})-y_i}{\varphi^3}-\frac{y_i}{\varphi^2}\right);
\end{aligned}
$$

$$
\begin{aligned}
h_{p+2,p+1}(\boldsymbol{\delta}) &= h_{p+1,p+2}(\boldsymbol{\delta}) := \frac{\partial^2 l_n(\delta)}{\partial\varphi\partial\omega} \\
&= -\sum_{i=1}^{n} \mathbb{1}_{\{y_i=0\}} \frac{f_i(\boldsymbol{\beta},\varphi)\mu_i(\boldsymbol{\beta})}{\varphi^2\left[g_i(\boldsymbol{\delta})\right]^2};
\end{aligned}
$$

$$
\begin{aligned}
h_{p+2,p+2}(\boldsymbol{\delta}) &:= \frac{\partial^2 l_n(\delta)}{\partial\omega\partial\omega} \\
&= -\sum_{i=1}^{n}\left(\mathbb{1}_{\{y_i=0\}}\frac{[1-f_i(\boldsymbol{\beta},\varphi)]^2}{[g_i(\boldsymbol{\delta})]^2}+\mathbb{1}_{\{y_i>0\}}\frac{1}{(1-\omega)^2}\right).
\end{aligned}
$$

Now set

$$
\mathbf{H}_n(\boldsymbol{\delta}) = -\mathcal{H}_n(\boldsymbol{\delta}).
$$

It is well known (see for example Mardia, Kent, and Bibby (1979), p.98) that under mild general regularity assumptions $\mathbf{F}_n(\boldsymbol{\delta}) = E_\delta \mathbf{H}_n(\boldsymbol{\delta})$. Thus, using Lemmas 1 and 2, the Fisher information matrix can be straightforward compute and is given by

$$
\mathbf{F}_n(\boldsymbol{\delta}) = \begin{pmatrix}
f_{0,0}(\boldsymbol{\delta}) & f_{0,1}(\boldsymbol{\delta}) & \dots & f_{0,p}(\boldsymbol{\delta}) & f_{0,p+1}(\boldsymbol{\delta}) & f_{0,p+2}(\boldsymbol{\delta}) \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
f_{p,0}(\boldsymbol{\delta}) & f_{p,1}(\boldsymbol{\delta}) & \dots & f_{p,p}(\boldsymbol{\delta}) & f_{p,p+1}(\boldsymbol{\delta}) & f_{p,p+2}(\boldsymbol{\delta}) \\
f_{p+1,0}(\boldsymbol{\delta}) & f_{p+1,1}(\boldsymbol{\delta}) & \dots & f_{p+1,p}(\boldsymbol{\delta}) & f_{p+1,p+1}(\boldsymbol{\delta}) & f_{p+1,p+2}(\boldsymbol{\delta}) \\
f_{p+2,0}(\boldsymbol{\delta}) & f_{p+2,1}(\boldsymbol{\delta}) & \dots & f_{p+2,p}(\boldsymbol{\delta}) & f_{p+2,p+1}(\boldsymbol{\delta}) & f_{p+2,p+2}(\boldsymbol{\delta})
\end{pmatrix}
$$

with entries:

$$
\begin{aligned}
f_{r,s}(\boldsymbol{\delta}) &= f_{s,r}(\boldsymbol{\delta}) = \sum_{i=1}^{n} x_{ir}x_{is}(1-\omega)\mu_i(\boldsymbol{\beta}) \\
&\times \frac{[1-f_i(\boldsymbol{\beta},\varphi)/\varphi]\,g_i(\boldsymbol{\delta})+(1-\omega)\,[f_i(\boldsymbol{\beta},\varphi)]^2\,\mu_i(\boldsymbol{\beta})/\varphi}{\varphi g_i(\boldsymbol{\delta})} \\
&+ \sum_{i=1}^{n}(1-\omega)x_{ir}x_{is}\mu_i(\boldsymbol{\beta})\left(\frac{\mu_i(\boldsymbol{\beta})-2\varphi+2\varphi^2}{\varphi^2(\mu_i(\boldsymbol{\beta})-2+2\varphi)}-\frac{1}{\varphi}f_i(\boldsymbol{\beta},\varphi)\right)
\end{aligned}
$$

for $r, s = 0, \dots, p$;

$$
\begin{aligned}
f_{p+1,r}(\boldsymbol{\delta}) &= f_{r,p+1}(\boldsymbol{\delta}) = \sum_{i=1}^{n} x_{ir}(1-\omega)f_i(\boldsymbol{\beta},\varphi)\mu_i(\boldsymbol{\beta}) \\
&\times \frac{g_i(\boldsymbol{\delta})\,[\mu_i(\boldsymbol{\beta})/\varphi-1]-(1-\omega)f_i(\boldsymbol{\beta},\varphi)\mu_i(\boldsymbol{\beta})/\varphi}{\varphi^2 g_i(\boldsymbol{\delta})} \\
&- \sum_{i=1}^{n}(1-\omega)x_{ir}\mu_i(\boldsymbol{\beta})\left(\frac{2(\varphi-1)}{\varphi^2(\mu_i(\boldsymbol{\beta})-2+2\varphi)}-\frac{f_i(\boldsymbol{\beta},\varphi)}{\varphi^2}\right)
\end{aligned}
$$

for $r = 0, \dots, p$;

$$
f_{p+2,r}(\boldsymbol{\delta}) = f_{r,p+2}(\boldsymbol{\delta}) = -\sum_{i=1}^{n}\frac{x_{ir}f_i(\boldsymbol{\beta},\varphi)\mu_i(\boldsymbol{\beta})}{\varphi g_i(\boldsymbol{\delta})}
$$

for $r = 0, \dots, p$;

$$
\begin{aligned}
f_{p+1,p+1}(\boldsymbol{\delta}) &= -\sum_{i=1}^{n}(1-\omega)f_i(\boldsymbol{\beta},\varphi)\mu_i(\boldsymbol{\beta})\frac{g_i(\boldsymbol{\delta})\,(\mu_i(\boldsymbol{\beta})-2\varphi)-(1-\omega)f_i(\boldsymbol{\beta},\varphi)\mu_i(\boldsymbol{\beta})}{\varphi^4 g_i(\boldsymbol{\delta})} \\
&+ \sum_{i=1}^{n}2(1-\omega)\mu_i(\boldsymbol{\beta})\left(\frac{1}{\varphi^2(\mu_i(\boldsymbol{\beta})-2+2\varphi)}-\frac{f_i(\boldsymbol{\beta},\varphi)}{\varphi^3}\right);
\end{aligned}
$$

$$f_{p+2,p+1}(\boldsymbol{\delta}) = f_{p+1,p+2}(\boldsymbol{\delta}) = \sum_{i=1}^{n} \frac{f_i(\boldsymbol{\beta},\varphi)\mu_i(\boldsymbol{\beta})}{\varphi^2 g_i(\boldsymbol{\delta})};$$

$$f_{p+2,p+2}(\boldsymbol{\delta}) = \sum_{i=1}^{n} \left(\frac{[1-f_i(\boldsymbol{\beta},\varphi)]^2}{g_i(\boldsymbol{\delta})} + \frac{1-f_i(\boldsymbol{\beta},\varphi)}{1-\omega} \right).$$

Appendix 2: Proof of Theorem 1

The proof of Theorem 1 follows the proof of Theorem 4 given in Fahrmeir and Kaufmann (1985). In particular, we have to prove asymptotic normality of the normalized score vectors $\mathbf{F}_n^{t/2}\mathbf{s}_n$ (Lemma 5) and show (Lemma 6) that

$$\max_{\delta \in N_n(\varepsilon)} \|\mathbf{V}_n(\boldsymbol{\delta}) - \mathbf{I}_{p+3}\| \xrightarrow{P} 0 \qquad \text{for all} \quad \epsilon > 0,$$

where $\mathbf{V}_n(\boldsymbol{\delta}) := \mathbf{F}_n^{-1/2}\mathbf{H}_n(\boldsymbol{\delta})\mathbf{F}_n^{-t/2}$ for $n = 1, 2, \ldots$

We start the appendix with two preliminary lemmas. Recall that we drop the dependency on $\boldsymbol{\delta}_0$,$\boldsymbol{\beta}_0$, φ_0 and use μ_i, $\mathbf{F}_n$, etc.

Lemma 3. *Let $\tilde{Y}_i \sim GP(\mu_i, \varphi_0)$ for $i = 1, \ldots, n$ be a sequence of random variables. Then under assumptions* (A2) *and* (A3),

$$\max_{i=1,\ldots,n} E\left(\frac{1}{(\mu_i + (\varphi_0 - 1)\tilde{Y}_i)^k}\right) \leq C_1,$$

$$\max_{i=1,\ldots,n} E(\tilde{Y}_i^k) \leq C_2$$

for any finite integer $k > 0$, where C_1 and C_2 are positive constants depending only on k and $\boldsymbol{\delta}_0$.

Proof. Let us show the first inequality of the Lemma. It is evident using (A3) that

$$E\left(\frac{1}{(\mu_i + (\varphi_0 - 1)\tilde{Y}_i)^k}\right) \leq \frac{1}{\mu_i^k}.$$

Now it follows

$$\begin{aligned}
\max_{i=1,\ldots,n} \frac{1}{\mu_i^k} &= \max_{i=1,\ldots,n} \frac{1}{\exp(k\mathbf{x}_i^t\boldsymbol{\beta}_0)} \\
&\leq \max_{\mathbf{x}\in K_x} \frac{1}{\exp(k\mathbf{x}^t\boldsymbol{\beta}_0)} \\
&\leq C_1(\boldsymbol{\beta}_0, k),
\end{aligned}$$

since K_x is a compact and $\exp(k\mathbf{x}^t\boldsymbol{\beta}_0)$ is a continuous function of $\mathbf{x}$. It should be noted that $C_1(\boldsymbol{\beta}_0, k)$ is continuous with respect to $\boldsymbol{\beta}_0$ and well defined for all $\boldsymbol{\beta}_0 \in B$.

Now we show the second inequality of the lemma. First, we reparameterize the GP distribution by introducing new parameters $\theta_i := \mu_i/\varphi_0$ and $\lambda_0 := (\varphi_0 - 1)/\varphi_0$, $i = 1, \ldots, n$.

Consul and Shenton (1974) gave the following recurrence formula for the noncentral moments of the $GP(\theta_i, \lambda_0)$ distribution:

$$(1-\lambda_0)m_{i,k+1} = \theta_i m_{i,k} + \theta_i \frac{\partial m_{i,k}}{\partial \theta_i} + \lambda_0 \frac{\partial m_{i,k}}{\partial \lambda_0}, \quad k = 0, 1, 2, \ldots,$$

where $m_{i,k} := E(\tilde{Y}_i^k)$.

Solving this recursion for fixed k shows that $m_{i,k}$ is a polynomial in θ_i, λ_0 and $1/(1-\lambda_0)$. Thus, $m_{i,k}$ is a continuous function with respect to (θ_i, λ_0) and consequently, it is also continuous with respect to (μ_i, φ_0). It follows now that

$$\begin{aligned}
\max_{i=1,\ldots,n} E(\tilde{Y}_i^k) &= \max_{i=1,\ldots,n} m_{i,k}\,(\theta_i, \lambda_0) \\
&= \max_{i=1,\ldots,n} m_{i,k}\,(\mu_i/\varphi_0, \mu_i(\varphi_0 - 1)/\varphi_0) \\
&\leq \max_{\mathbf{x}\in K_x} m_k \left(e^{\mathbf{x}^t\beta_0}/\varphi_0, e^{\mathbf{x}^t\beta_0}(\varphi_0 - 1)/\varphi_0\right) \\
&\leq C_2(\boldsymbol{\delta}_0),
\end{aligned}$$

where $m_k := E(\tilde{Y}^k)$ and $\tilde{Y} \sim GP(\exp(\mathbf{x}^t\boldsymbol{\beta}_0), \varphi_0)$. It is not difficult to see that $C_2(\boldsymbol{\delta}_0)$ is continuous with respect to $\boldsymbol{\delta}_0$ and well defined for all $\boldsymbol{\delta}_0 \in K_\delta$. □

Lemma 4. *Let $Q_k(y)$ be a polynomial of a finite order k ($k \in \mathbb{N}$) whose coefficients are positive continuous functions of $\mathbf{x}$, $\boldsymbol{\delta}$ and $\boldsymbol{\delta}_0$. Further, let $Y_i \sim ZIGP(\exp(\mathbf{x}_i^t\boldsymbol{\beta}_0), \varphi_0, \omega_0)$ for $i = 1, \ldots, n$. If* (A1)–(A3) *hold then*

$$\max_{\delta\in N_n(\varepsilon)} \max_{i=1,\ldots,n} E\left(\mathbb{1}_{\{Y_i>0\}} Q_k(Y_i)\right) < C,$$

where C is a positive constant depending on k and $\boldsymbol{\delta}_0$.

Proof. Note that under (A1) the neighborhood $N_n(\varepsilon)$ is a compact for any $n \in \mathbb{N}$ and shrinks to $\boldsymbol{\delta}_0$ for any $\varepsilon > 0$ as $n \to \infty$. Using Lemmas 2 and 3 and the continuity of the coefficients of Q_k, it follows now that

$$\begin{aligned}
\max_{\delta\in N_n(\varepsilon)} \max_{i=1,\ldots,n} E\left(\mathbb{1}_{\{Y_i>0\}} Q_k(Y_i)\right) &\leq \max_{\delta\in N_n(\varepsilon)} \max_{i=1,\ldots,n} (1-\omega_0) E\left(Q_k(\tilde{Y}_i)\right) \\
&\leq \max_{\delta\in N_1(\varepsilon)} \max_{\mathbf{x}\in K_x} (1-\omega_0) E\left(Q_k(\tilde{Y})\right) \\
&\leq C,
\end{aligned}$$

where $\tilde{Y}_i \sim GP(\exp(\mathbf{x}_i^t\boldsymbol{\beta}_0), \varphi_0)$ and $\tilde{Y} \sim GP(\exp(\mathbf{x}^t\boldsymbol{\beta}_0), \varphi_0)$. □

Lemma 5. *Under assumptions* (A1)–(A3), $\mathbf{F}_n^{-1/2}\mathbf{s}_n \overset{\mathcal{D}}{\Rightarrow} N_{p+3}(\mathbf{0}, \mathbf{I}_{p+3})$ *as $n \to \infty$, where $N_{p+3}(\mathbf{0}, \mathbf{I}_{p+3})$ is a $(p+3)$-dimensional normal distribution with mean vector $\mathbf{0}$ and covariance matrix $\mathbf{I}_{p+3}$.*

Proof. According to the Cramer-Wald device, it is sufficient to show that a linear combination $\mathbf{a}^t\mathbf{F}_n^{-1/2}\mathbf{s}_n$ converges in distribution to $N(0, \mathbf{a}^t\mathbf{a})$ for any vector $\mathbf{a} \in \mathbb{R}^{p+3}$ ($\mathbf{a} \neq \mathbf{0}$). Without loss of generality, we set $\|\mathbf{a}\| = 1$.

Now observe that $\mathbf{s}_n$ can be written as a sum of independent random vectors, namely

$$\mathbf{s}_n = \sum_{i=1}^{n} \mathbf{s}_{ni},$$

where $\mathbf{s}_{ni} = (s_{0,i}, \ldots, s_{p,i}, s_{p+1,i}, s_{p+2,i})^t$ with $s_{k,i} := s_{k,i}(\boldsymbol{\delta}_0)$ defined in (10), (11) and (12) for $k = 0, \ldots, p+2$ and $i = 1, \ldots, n$, respectively. Further, define independent random variables ξ_{in} by $\xi_{in} := \mathbf{a}^t \mathbf{F}_n^{-1/2} \mathbf{s}_{ni}$. Since $E(\xi_{in}) = 0$ and $Var\left(\sum_{i=1}^n \xi_{in}\right) = 1$, it is enough to show that the Lyapunov condition is satisfied, i.e.

$$L_s := \sum_{i=1}^n E|\xi_{in}|^s \overset{n\to\infty}{\longrightarrow} 0, \quad \text{for some } s > 2,$$

say $s = 3$ (see for example Hoffmann-Jørgensen (1994), p. 393). Noticing that $\|\mathbf{F}_n^{-1/2}\|^2 = 1/\lambda_{min}(\mathbf{F}_n)$, it follows from $(A1)$ that

$$\begin{aligned} L_3 &\leq \sum_{i=1}^n E\left(\left\|\mathbf{a}^t\right\|^3 \left\|\mathbf{F}_n^{-1/2}\right\|^3 \|\mathbf{s}_{ni}\|^3\right) \\ &\leq \frac{C}{n^{3/2}} \sum_{i=1}^n E\left\|\mathbf{s}_{ni}\right\|^3 \\ &\leq \frac{C}{\sqrt{n}} \max_{i=1,\ldots,n} E\left\|\mathbf{s}_{ni}\right\|^3 . \end{aligned}$$

Using an extension of the c_r-inequality given by

$$E\left|\sum_{i=1}^m \zeta_i\right|^k \leq m^{k-1} \sum_{i=1}^m E|\zeta_i|^k \qquad (\ k > 1,\ k \in \mathbb{R}), \tag{14}$$

to m arbitrary random variables $\zeta_1, \ldots, \zeta_m$ (see, for example, Petrov (1995), p.58) yields that

$$E\left\|\mathbf{s}_{ni}\right\|^3 \leq C\left(E\left|s_{0,i}\right|^3 + \ldots + E\left|s_{p,i}\right|^3 + E\left|s_{p+1,i}\right|^3 + E\left|s_{p+2,i}\right|^3\right).$$

Thus, it remains to establish that $\max_{i=1,\ldots,n} E\left|s_{r,i}\right|^3$ is uniformly bounded in n for $r = 0, \ldots, p+2$. This will be shown for case $r = 0, \ldots, p$. The remaining cases can be treated similarly. Without loss of generality, set $r = p$. Using now (14) with $m = 2$, we have

$$\begin{aligned} \max_{i=1,\ldots,n} E\left|s_{p,i}\right|^3 &\leq 2^2 \max_{i=1,\ldots,n} E\left|x_{ip}\mathbb{1}_{\{y_i=0\}} \frac{(1-\omega_0)f_i\mu_i}{\varphi_0 g_i}\right|^3 \\ &+ 2^2 \max_{i=1,\ldots,n} E\left(\left|x_{ip}\mathbb{1}_{\{y_i>0\}}\left(1 + \frac{\mu_i(y_i-1)}{\mu_i + (\varphi_0-1)y_i} - \frac{\mu_i}{\varphi_0}\right)\right|^3\right) \\ &=: 4A_p(\boldsymbol{\delta}_0) + 4B_p(\boldsymbol{\delta}_0). \end{aligned}$$

The last step in the proof is now to show that

$$A_p(\boldsymbol{\delta}_0) < C_1 \qquad \text{and} \qquad B_p(\boldsymbol{\delta}_0) < C_3, \tag{15}$$

where C_1 and C_3 are some constants depending on $\boldsymbol{\delta}_0$.

For proving (15) note that Lemma 2 implies

$$\begin{aligned} A_p(\boldsymbol{\delta}_0) &\leq \max_{\mathbf{x}\in K_x} \|\mathbf{x}\|^3 \left|\frac{(1-\omega_0)f_i\mu_i}{\varphi_0 g_i}\right|^3 g_i \\ &\leq C_1. \end{aligned}$$

Let us now consider $B_p(\boldsymbol{\delta}_0)$. Using Lemma 2, Inequality (14), Cauchy-Schwarz inequality and Lemma 3, respectively, gives

$$\begin{aligned}
B_p(\boldsymbol{\delta}_0) &\leq \max_{i=1,\ldots,n} E\left((1-\omega_0)\,|x_{ir}|^3 \cdot \left|1+\frac{\mu_i(\tilde{Y}_i-1)}{\mu_i+(\varphi_0-1)\tilde{Y}_i}-\frac{\mu_i}{\varphi_0}\right|^3\right) \\
&\leq C\max_{\mathbf{x}\in K_x}(1-\omega_0)\|\mathbf{x}\|^3\left(1^3+E\left|\frac{\mu_i(\tilde{Y}-1)}{\mu_i+(\varphi_0-1)\tilde{Y}}\right|^3+\left(\frac{\mu_i}{\varphi_0}\right)^3\right) \\
&\leq C_1(\boldsymbol{\delta}_0)+C_2(\boldsymbol{\delta}_0)\max_{\mathbf{x}\in K_x} E\left|\tilde{Y}-1\right|^3 \\
&\leq C_1(\boldsymbol{\delta}_0)+C_2(\boldsymbol{\delta}_0)\max_{\mathbf{x}\in K_x}\sqrt{E\left(\tilde{Y}-1\right)^6} \\
&\leq C_3(\boldsymbol{\delta}_0),
\end{aligned}$$

where $\tilde{Y}_i \sim GP\left(\mu_i,\varphi_0\right)$ for $i=1,\ldots,n$ and $\tilde{Y}\sim GP\left(\exp(\mathbf{x}^t\boldsymbol{\beta}_0),\varphi_0\right)$. ☐

Lemma 6. *Under the assumptions* (A1)–(A3),

$$\max_{\delta\in N_n(\varepsilon)}\|\mathbf{V}_n(\boldsymbol{\delta})-\mathbf{I}_{p+3}\| \xrightarrow{P} 0 \qquad \textit{for all} \quad \epsilon>0. \tag{16}$$

Proof. We have a.s.

$$\begin{aligned}
\|\mathbf{V}_n(\boldsymbol{\delta})-\mathbf{I}_{p+3}\| &= \left\|\mathbf{F}_n^{-1/2}\left[\mathbf{H}_n(\boldsymbol{\delta})-\mathbf{F}_n\right]\mathbf{F}_n^{-t/2}\right\| \\
&\leq \frac{1}{\lambda_{min}(\mathbf{F}_n)}\|\mathbf{H}_n(\boldsymbol{\delta})-\mathbf{F}_n\| \\
&\leq \frac{C}{n}\|\mathbf{H}_n(\boldsymbol{\delta})-\mathbf{F}_n\| \\
&\leq C\left\|\frac{1}{n}\left(\mathbf{H}_n(\boldsymbol{\delta})-E\mathbf{H}_n(\boldsymbol{\delta})\right)\right\|+C\left\|\frac{1}{n}\left(E\mathbf{H}_n(\boldsymbol{\delta})-\mathbf{F}_n\right)\right\|.
\end{aligned}$$

Thus, conditions

$$\max_{\delta\in N_n(\varepsilon)}\left\|\frac{1}{n}\left(\mathbf{H}_n(\boldsymbol{\delta})-E\mathbf{H}_n(\boldsymbol{\delta})\right)\right\| \xrightarrow{P} 0 \tag{17}$$

and

$$\max_{\delta\in N_n(\varepsilon)}\left\|\frac{1}{n}\left(E\mathbf{H}_n(\boldsymbol{\delta})-\mathbf{F}_n\right)\right\| \longrightarrow 0 \tag{18}$$

imply (16).

In order to show (17) it is enough to establish that the maximum over $\boldsymbol{\delta}\in N_n(\varepsilon)$ of the absolute value of the (r,s)-element of the random matrix $[\mathbf{H}_n(\boldsymbol{\delta})-E\mathbf{H}_n(\boldsymbol{\delta})]/n$ converges to zero in probability, i.e.

$$\max_{\delta\in N_n(\varepsilon)}\frac{|h_{rs}(\boldsymbol{\delta})-Eh_{rs}(\boldsymbol{\delta})|}{n} \xrightarrow{P} 0.$$

Recall that the Hessian matrix has 6 different types of entries. We shall illustrate the above convergence in the case of $0\leq r,s\leq p$. The remaining cases can be treated similarly. Without loss of generality, we only show

$$\max_{\delta\in N_n(\varepsilon)}\left|\frac{1}{n}\left(h_{p,p}(\boldsymbol{\delta})-Eh_{p,p}(\boldsymbol{\delta})\right)\right| \xrightarrow{P} 0. \tag{19}$$

Let $Z_i := \mathbb{1}_{\{Y_i>0\}} Y_i(Y_i-1)$, $U_i(\boldsymbol{\beta},\varphi) := \mu_i(\boldsymbol{\beta}) + (\varphi-1)Y_i$, $q_{i,p}(\boldsymbol{\delta}) := x_{ip}^2\mu_i(\boldsymbol{\beta})(\varphi-1)$ and

$$v_{i,p}(\boldsymbol{\delta}) := x_{ip}^2(1-\omega)f_i(\boldsymbol{\beta},\varphi)\mu_i(\boldsymbol{\beta})\frac{[1-\mu_i(\boldsymbol{\beta})/\varphi]\, g_i(\boldsymbol{\delta}) + (1-\omega)f_i(\boldsymbol{\beta},\varphi)\mu_i(\boldsymbol{\beta})/\varphi}{\varphi\,[g_i(\boldsymbol{\delta})]^2}.$$

It easy to see that (19) will follow from the next three conditions:

$$\begin{aligned}
&\max_{\delta\in N_n(\varepsilon)}\left|\frac{1}{n}\sum_{i=1}^n v_{i,p}(\boldsymbol{\delta})\left(\mathbb{1}_{\{Y_i=0\}} - E(\mathbb{1}_{\{Y_i=0\}})\right)\right| \xrightarrow{P} 0,\\
&\max_{\delta\in N_n(\varepsilon)}\left|\frac{1}{n}\sum_{i=1}^n \frac{q_{i,p}(\boldsymbol{\delta})}{\varphi}\left(\mathbb{1}_{\{Y_i>0\}} - E(\mathbb{1}_{\{Y_i>0\}})\right)\right| \xrightarrow{P} 0\\
&\max_{\delta\in N_n(\varepsilon)}\left|\frac{1}{n}\sum_{i=1}^n q_{i,p}(\boldsymbol{\delta})\left[\frac{Z_i}{[U_i(\boldsymbol{\beta},\varphi)]^2} - E\left(\frac{Z_i}{[U_i(\boldsymbol{\beta},\varphi)]^2}\right)\right]\right| \xrightarrow{P} 0.
\end{aligned}\tag{20}$$

Since they have a similar structure we only establish the validity of the last relation. It is worth to recall that the dependency on $\boldsymbol{\delta}_0$, $\boldsymbol{\beta}_0$ and φ_0 is always dropped.

Observe that the right hand side of (20) may be bounded by a sum of

$$\begin{aligned}
A_n &= \max_{\delta\in N_n(\varepsilon)}\left|\frac{1}{n}\sum_{i=1}^n q_{i,p}(\boldsymbol{\delta})\left(\frac{Z_i}{[U_i(\boldsymbol{\beta},\varphi)]^2} - \frac{Z_i}{U_i^2}\right)\right|,\\
B_n &= \max_{\delta\in N_n(\varepsilon)}\left|\frac{1}{n}\sum_{i=1}^n q_{i,p}(\boldsymbol{\delta})\left[E\frac{Z_i}{[U_i(\boldsymbol{\beta},\varphi)]^2} - E\left(\frac{Z_i}{U_i^2}\right)\right]\right|,\\
D_n &= \max_{\delta\in N_n(\varepsilon)}\left|\frac{1}{n}\sum_{i=1}^n q_{i,p}(\boldsymbol{\delta})\left[\frac{Z_i}{U_i^2} - E\left(\frac{Z_i}{U_i^2}\right)\right]\right|.
\end{aligned}$$

For A_n we have the following bounds:

$$\begin{aligned}
A_n &\le \max_{\delta\in N_n(\varepsilon)}\frac{1}{n}\sum_{i=1}^n\frac{|q_{i,p}(\boldsymbol{\delta})Z_i|}{\mu_i^2(\boldsymbol{\beta})\mu_i^2}\cdot|U_i(\boldsymbol{\beta},\varphi)+U_i|\,|\mu_i(\boldsymbol{\beta})-\mu_i+(\varphi-\varphi_0)Y_i|\\
&\le \max_{\delta\in N_n(\varepsilon)}\frac{1}{n}\sum_{i=1}^n\frac{|q_{i,p}(\boldsymbol{\delta})Z_i|}{\mu_i^2(\boldsymbol{\beta})\mu_i^2}\cdot|(Y_i+1)(\mu_i(\boldsymbol{\beta})+\mu_i+\varphi+\varphi_0-2)|\\
&\quad\times|\mu_i(\boldsymbol{\beta})-\mu_i+(\varphi-\varphi_0)Y_i|\\
&\le \frac{C_1}{n}\left(\sum_{i=1}^n Z_i(Y_i+1)\right)\max_{\delta\in N_n(\varepsilon)}\max_{\mathbf{x}\in K_x}\left|\exp(\mathbf{x}^t\boldsymbol{\beta})-\exp(\mathbf{x}^t\boldsymbol{\beta}_0)\right|\\
&+ \frac{C_1}{n}\left(\sum_{i=1}^n Z_iY_i(Y_i+1)\right)\max_{\delta\in N_n(\varepsilon)}|\varphi-\varphi_0|\\
&=: AB_n + AC_n.
\end{aligned}\tag{21}$$

It is not difficult to see that

$$\frac{1}{n}\sum_{i=1}^n Z_i(Y_i+1)$$

converges in probability as $n\to\infty$ to

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n E\left(Z_i(Y_i+1)\right)$$

which is finite by Lemma 4.

These facts and the continuity in $\boldsymbol{\beta}$ of the function $\max_{\mathbf{x}\in K_x} |\exp(\mathbf{x}^t\boldsymbol{\beta}) - \exp(\mathbf{x}^t\boldsymbol{\beta}_0)|$ with value zero at $\boldsymbol{\beta} = \boldsymbol{\beta}_0$ yield that AB_n converges to 0 in probability as $n \to \infty$. Convergence of AC_n to 0 in probability may be proven in the same way.

Using similar arguments as above one can show that B_n converges to 0 in probability. To prove $D_n \to 0$ in probability, observe that the function $\max_{i=1,\dots,n} |q_{i,p}(\boldsymbol{\delta}) - q_{i,p}(\boldsymbol{\delta}_0)|$ can be bounded from above by the following continuous function of $\boldsymbol{\delta}$

$$C \max_{\mathbf{x}\in K_x} \left|\exp(\mathbf{x}^t\boldsymbol{\beta})(\varphi - 1) - \exp(\mathbf{x}^t\boldsymbol{\beta}_0)(\varphi_0 - 1)\right|$$

with zero at $\boldsymbol{\delta} = \boldsymbol{\delta}_0$. The desired result now follows from the law of large numbers and standard arguments.

It remains to show (18). We will show

$$\max_{\delta\in N_n(\varepsilon)} \left|\frac{[E\mathbf{H}_n(\boldsymbol{\delta}) - \mathbf{F}_n]_{rs}}{n}\right| \to 0 \tag{22}$$

and again restrict ourself to the case $r = s = p$. It easy to see that condition (22) will follow from the next three conditions :

$$\max_{\delta\in N_n(\varepsilon)} \left|\frac{1}{n}\sum_{i=1}^{n} (v_{i,p}(\boldsymbol{\delta}) - v_{i,p})\, E(\mathbb{1}_{\{Y_i=0\}})\right| \to 0, \tag{23}$$

$$\max_{\delta\in N_n(\varepsilon)} \left|\frac{1}{n}\sum_{i=1}^{n} \left(\frac{q_{i,p}(\boldsymbol{\delta})}{\varphi} - \frac{q_{i,p}}{\varphi_0}\right) E(\mathbb{1}_{\{Y_i>0\}})\right| \to 0, \tag{24}$$

$$\max_{\delta\in N_n(\varepsilon)} \left|\frac{1}{n}\sum_{i=1}^{n} \left(q_{i,p}(\boldsymbol{\delta})E\left(\frac{Z_i}{[U_i(\boldsymbol{\beta},\varphi)]^2}\right) - q_{i,p}E\left(\frac{Z_i}{U_i^2}\right)\right)\right| \to 0. \tag{25}$$

Now we see that the same technique used for deriving (20) can be employed to establish the convergence results (23)–(25). □