Extremal behavior of finite EGARCH processes

Alexander M. Lindner Katharina M. M. Meyer *

Abstract

Extreme value theory for a class of EGARCH processes is developed. It is shown that the EGARCH process as well as the logarithm of its conditional variance lie in the domain of attraction of the Gumbel distribution. Norming constants are obtained and it is shown that the considered processes exhibit the same extremal behavior as their associated iid sequences. The results are then compared to related models, such as stochastic volatility models or Log–ACD models.

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*Center of Mathematical Sciences, Munich University of Technology, D-85747 Garching, Germany, email: {lindner,kmeyer}@mathematik.tu-muenchen.de, http://www.mathematik.tu-muenchen.de

1 Introduction

Nonlinear time series models such as ARCH and GARCH models introduced by Engle [En] and Bollerslev [Bo] are widely used to model financial time series, since they allow the conditional variance to depend on past information. As pointed out by Black [Bl], there is empirical evidence that stock returns are negatively correlated with changes in returns volatility, i.e. volatility tends to rise in response to "bad news" and tends to fall in response to "good news". In order to model this asymmetry, Nelson [Ne] introduced the *exponential GARCH model* $(\xi_t)_{t\in\mathbb{Z}}$, shortly EGARCH, defined as

$$\begin{aligned} \xi_t &= \sigma_t Z_t, \quad t \in \mathbb{Z}, \\ \log \sigma_t^2 &= \mu + \sum_{k=1}^{\infty} a_k g(Z_{t-k}), \quad t \in \mathbb{Z}. \end{aligned} \tag{1}$$

Here μ and a_k, $k \in \mathbb{N}$, are real numbers, $(Z_t)_{t\in\mathbb{Z}}$ is an iid (0,1) noise with symmetric distribution, and the function g is suitably chosen to model the asymmetry in the reaction to bad and good news. Nelson suggests to use

$$g(Z_t) = \theta Z_t + \gamma(|Z_t| - E|Z_t|), \quad t \in \mathbb{Z},$$

where θ and γ are fixed parameters such that $\theta^2 + \gamma^2 \neq 0$.

For the iid noise sequence $(Z_t)_{t\in\mathbb{Z}}$ he takes the *generalized error distribution* $GED(\nu)$ for $\nu > 0$, the density of which is given by

$$f_{GED}(x) = \frac{\nu \, \exp(-|x/\lambda|^{\nu}/2)}{\lambda \, 2^{1+1/\nu}\Gamma(1/\nu)}, \quad x \in \mathbb{R},$$

where

$$\lambda := \left(2^{-2/\nu}\Gamma(1/\nu)/\Gamma(3/\nu)\right)^{1/2}. \tag{2}$$

With these choices, the process $\xi_t = \sigma_t Z_t$, $t \in \mathbb{Z}$, as well as its conditional variance process $(\sigma_t^2)_{t\in\mathbb{Z}}$ are strictly and weakly stationary, provided $\sum_{k=1}^{\infty} |a_k| < \infty$ and $\nu > 1$. In fact they have finite moments of all orders under these conditions, see [Ne], [Me, Prop. 2.20]. Note that the choice $\nu = 2$ corresponds to the standard normal distribution. Also note that for arbitrary $\nu > 0$, the absolute first moment of Z_1 is given by $E|Z_1| = \Gamma(2/\nu)/(\Gamma(1/\nu)\Gamma(3/\nu))^{1/2}$.

EGARCH models are mainly used by economists to model stock returns volatility. He et al. [HTM], He [He] and Karanasos and Kim [KK] consider the moment structure of EGARCH models. In this paper we shall be concerned with the extremal behavior of the finite EGARCH process. We shall assume that $\nu > 1$. By "finite" we mean that the sum in (1) is a finite sum. For simplicity we also assume that $\mu = 0$. More precisely, we shall deal with the following model:

$$\begin{aligned}
\xi_t &= \sigma_t Z_t, \quad t \in \mathbb{Z}, && (3)\\
(Z_t)_{t\in\mathbb{Z}} &\ \text{iid}\ \ GED(\nu), \quad \nu > 1, && (4)\\
\log \sigma_t^2 &= \sum_{k=1}^{p} a_k g(Z_{t-k}), \quad t \in \mathbb{Z}, && (5)\\
g(z) &= \theta z + \gamma(|z| - E|Z_1|), \quad z \in \mathbb{R}, && (6)
\end{aligned}$$

where $p \in \mathbb{N}$, $a_1, \ldots, a_p$ are real coefficients, not all 0, and θ and γ are real parameters, not both 0. We shall determine the asymptotic tail and density behavior of all the considered processes $(\log \sigma_t^2)$, $(\log \xi_t^2)$, (ξ_t^2) and of (ξ_t) itself. In addition, we shall show that these processes lie in the domain of attraction of the Gumbel distribution and obtain the corresponding norming constants. It turns out that the processes behave like the corresponding associated iid sequences. In particular, exceedances of high tresholds do not occur in clusters. This is different from the behavior of ARCH(1) processes, as examined by de Haan et al. [HRRV]. A comparison will be given in Section 6.3.

We will only treat the $(\log \sigma_t^2)$ process for arbitrary $\nu > 1$. Although it seems likely that our methods also work for the processes $(\log \xi_t)$, (ξ_t^2) and (ξ_t) for arbitrary $\nu > 1$, we shall restrict to $\nu = 2$ for these processes, corresponding to a standard normally distributed noise sequence (Z_t). This makes calculations much easier. Furthermore, our results depend on the signs of $\theta + \gamma$ and of $\theta - \gamma$. For the cases that $\theta + \gamma$ and $\theta - \gamma$ have the same sign (referred to as cases 1 and 4 in Section 3), our results apply for any real coefficients $a_1, \ldots, a_p$. In the case of opposite signs, the results only apply if all of the coefficients are positive or all negative, depending on the particular case.

The paper is organized as follows: In the next section we will give some of the basic definitions and facts of extreme value theory which we shall need. In addition, in Theorem 2.1 we state a result of Balkema et al. [BKR] on the convolution of certain light tailed density functions. We shall make heavy use of this result in the sequel. In Sections 3 to 5 we will derive the tail and density behavior of the processes $(\log \sigma_t^2)$, $(\log \xi_t^2)$, (ξ_t^2) and (ξ_t) and show that they lie in the domain of attraction of the Gumbel distribution and large values do not cluster. In the last section, we compare our results to existing ones in the literature. In Section 6.1 we state the results of Settimi [S2] on the extremal behavior of the logvariance of an EGARCH(1,1) process. In Section 6.2, the similarity of EGARCH and stochastic volatility models is considered. Extreme value theory for stochastic volatility models has been developed by Breidt and Davis [BD] and been further extended by Diop and Guegan [DG]. We shall compare the results for stochastic volatility models to the results for EGARCH processes derived in this paper. In Section 6.3, the extremal results for ARCH and EGARCH processes will be compared. Finally, in Section 6.4, we consider the Log–ACD model of Bauwens and Giot [BG], a variant of the ACD model of Engle and Russell [ER]. The latter were introduced to model duration times between randomly occuring trading times, and the Log–ACD process has a structure similar to EGARCH processes. A few extreme value results for the Log–ACD model are indicated.

Many of the results of this paper are part of the second named author's diploma thesis [Me]. There also detailed calculations can be found.

2 Preliminaries

Detailed introductions to extreme value theory can be found in Embrechts et al. [EKM], Leadbetter et al. [LLR] or Resnick [Re], for example. Let us recall some of the basic definitions and facts we shall use: Let $(Y_t)_{t\in\mathbb{Z}}$ be a strictly stationary sequence of random variables defined on a common probability space. The *associated iid sequence* $(\widetilde{Y}_t)_{t\in\mathbb{Z}}$ is defined to be an iid sequence with distribution Y_1. Let H be a distribution function. Then $(Y_t)_{t\in\mathbb{Z}}$ is *in the domain of attraction of* H, if there are positive constants $(c_n)_{n\in\mathbb{N}}$ and real constants $(d_n)_{n\in\mathbb{N}}$ such that $(\max(Y_1,\ldots,Y_n)-d_n)/c_n$ converges in distribution to H, as $n\to\infty$. The constants c_n and d_n are called *norming constants*. The distribution function H is unique up to affine linear transformations. If $(Y_t)_{t\in\mathbb{Z}}$ is an iid sequence, then H is (up to affine linear transformations) a Fréchet distribution with parameter $\alpha>0$, a Weibull distribution with parameter $\alpha>0$, or the Gumbel distribution, given by $\Lambda(x)=\exp(-e^{-x})$, $x\in\mathbb{R}$. For an arbitrary strictly stationary sequence the same holds if the condition $D(u_n)$ of Leadbetter, Lindgren and Rootzén given below is satisfied for sequences of the form $u_n=c_nx+d_n$, $x\in\mathbb{R}$, see [LLR, Th. 3.3.3]. If furthermore condition $D'(u_n)$ is satisfied, then $(Y_t)_{t\in\mathbb{Z}}$ exhibits the same extremal behavior as its associated iid sequence $(\widetilde{Y}_t)_{t\in\mathbb{Z}}$, with the same norming constants, see [LLR, Th. 3.5.2]. In that case, exceedances of $(Y_t)_{t\in\mathbb{Z}}$ of high tresholds cannot occur in clusters. This may be different if $D'(u_n)$ is not satisfied, as for example for GARCH processes. By definition, $(Y_t)_{t\in\mathbb{Z}}$ satisfies *condition* $D(u_n)$ for the sequence $(u_n)_{n\in\mathbb{N}}$, if for every $q_1,q_2,n\in\mathbb{N}$ and integers

$$1\le i_1<\ldots<i_{q_1}<j_1<\ldots<j_{q_2}\le n$$

such that $j_1-i_{q_1}\ge l$, it holds

$$\left|P\left(\max_{i\in A_1\cup A_2}Y_i\le u_n\right)-P\left(\max_{i\in A_1}Y_1\le u_n\right)P\left(\max_{i\in A_2}Y_i\le u_n\right)\right|\le\alpha_{n,l},$$

where $A_1 = \{i_1, \dots, i_{q_1}\}$, $A_2 = \{j_1, \dots, j_{q_2}\}$ and $\alpha_{n,l}$ tends to 0 as $n \to \infty$ for a sequence $l = l_n = o(n)$. $(Y_t)_{t \in \mathbb{Z}}$ satisfies *condition* $D'(u_n)$ if

$$\lim_{k \to \infty} \limsup_{n \to \infty} n \sum_{j=2}^{\lfloor n/k \rfloor} P(Y_1 > u_n, Y_j > u_n) = 0. \tag{7}$$

For the study of the extremal behavior of $(Y_t)_{t \in \mathbb{Z}}$, one often starts with the associated iid sequence $(\widetilde{Y}_t)_{t \in \mathbb{Z}}$ and then investigates conditions $D(u_n)$ and $D'(u_n)$. The extremal behavior of $(\widetilde{Y}_t)_{t \in \mathbb{Z}}$ is completely determined by the tail behavior of the distribution function of Y_1: Defining

$$\overline{F}_Y(x) := P(Y_1 > x), \quad x \in \mathbb{R},$$

one studies the behavior of $\overline{F}_Y(x)$ as x approaches the upper endpoint $x_\infty := \sup\{x \in \mathbb{R} : \overline{F}_Y(x) > 0\} \le \infty$ of the distribution of Y_1. Here it is desirable to find a "nice" function $h(x)$ such that $\overline{F}_Y(x) \sim h(x)$ as $x \uparrow x_\infty$, where by definition

$$f(x) \sim h(x) \text{ as } x \uparrow x_\infty \quad \text{if and only if} \quad \lim_{x \uparrow x_\infty} \frac{f(x)}{h(x)} = 1$$

for functions f and h.

A crucial ingredient in our proofs will be the following result by Balkema et al. [BKR], showing that the class of densities with Gaussian tails is closed under convolution, and giving an expression for the tail of the convolution.

Theorem 2.1. [BKR] *Let $Y_1, \dots, Y_d$ be independent random variables, such that each has a strictly positive density f_i in a neighborhood of its upper endpoint $x_{i\infty}$ satisfying the asymptotic equality*

$$f_i(x) \sim \gamma_i(x) e^{-\psi_i(x)}, \quad \text{as } x \uparrow x_{i\infty},$$

where the functions ψ_i are $\mathcal{C}^2$ and ψ_i'' is strictly positive. Suppose that with $\sigma_i := 1/\sqrt{\psi_i''}$ it holds

$$\frac{\sigma_i(x + y\sigma_i(x))}{\sigma_i(x)} \to 1 \quad \text{as well as} \quad \frac{\gamma_i(x + y\sigma_i(x))}{\gamma_i(x)} \to 1, \quad \text{as } x \uparrow x_{i\infty}, \tag{8}$$

uniformly on bounded y-intervals. Furthermore, suppose that $\sup_x \psi_i'(x) =: \tau_\infty \le \infty$ is independent of i. Then $Y_0 := Y_1 + \dots + Y_d$ has a strictly positive density f_0 in a left neighborhood of its upper endpoint $x_\infty = x_{1\infty} + \dots + x_{d\infty}$. It is of the same form

$$f_0(x) \sim \gamma_0(x) e^{-\psi_0(x)}, \quad \text{as } x \to x_\infty,$$

where ψ_0 is $\mathcal{C}^2$ with strictly positive second derivative. Explicit formulae for γ_0 and ψ_0 can be given in terms of the inverse functions $q_i(\tau) := (\psi_i')^{\leftarrow}(\tau)$ to the strictly increasing

derivatives ψ_i' as follows:
Write $x(\tau) = q_1(\tau) + \ldots + q_d(\tau)$. Then $x(\tau)$ is a strictly increasing function of τ and $x(\tau) \to x_\infty$ as $\tau \to \tau_\infty$. For τ in a left neighborhood of τ_∞ one can choose

$$\begin{aligned}
\psi_0(x(\tau)) &= \psi_1(q_1(\tau)) + \cdots + \psi_d(q_d(\tau)), \\
\sigma_0^2(x(\tau)) &= \sigma_1^2(q_1(\tau)) + \cdots + \sigma_d^2(q_d(\tau)), \\
\sqrt{2\pi}\,\sigma_0(x(\tau))\gamma_0(x(\tau)) &= \prod_{i=1}^{d} \sqrt{2\pi}\,\sigma_i(q_i(\tau))\gamma_i(q_i(\tau)).
\end{aligned}$$

Then $\sigma_0 = 1/\sqrt{\psi_0''}$, $\sup_x \psi_0'(x) = \tau_\infty$, and σ_0 and γ_0 satisfy (8).

The left hand limit in (8) means that σ_i is *self–neglecting*, and the right hand limit means that γ_i behaves roughly like a constant on intervals of length $\sigma(x)$.

3 Extremal behavior of $\log \sigma_t^2$

In this section we derive the upper tail and density behavior of $\log \sigma_1^2$ and show that $(\log \sigma_t^2)_{t\in\mathbb{Z}}$ as well as its associated iid sequence belong to the domain of attraction of the Gumbel distribution. Consider the model (3) – (6) and set

$$X_t := \log \sigma_t^2 = \sum_{k=1}^{p} a_k g(Z_{t-k}), \quad t \in \mathbb{Z}. \tag{9}$$

To start with, we need the density $f_{g(Z)}$ of $g(Z_t)$. This can be easily calculated from the density f_Z of Z_t. Its general form depends on the parameters θ and γ. In the case that $\theta+\gamma > 0$ and $\theta - \gamma > 0$ (referred to as *case 1* in the following), we obtain

$$f_{g(Z)}(x) = \begin{cases} \dfrac{1}{\theta+\gamma} f_Z\left(\dfrac{x+\gamma E|Z_1|}{\theta+\gamma}\right), & x+\gamma E|Z_1| > 0 \\ \\ \dfrac{1}{\theta-\gamma} f_Z\left(\dfrac{x+\gamma E|Z_1|}{\theta-\gamma}\right), & x+\gamma E|Z_1| < 0, \end{cases} \tag{10}$$

and in the case $\theta+\gamma > 0$ and $\theta-\gamma < 0$ (referred to as *case 2*),

$$f_{g(Z)}(x) = \begin{cases} \dfrac{1}{\theta+\gamma} f_Z\left(\dfrac{x+\gamma E|Z_1|}{\theta+\gamma}\right) + \dfrac{1}{\gamma-\theta} f_Z\left(\dfrac{x+\gamma E|Z_1|}{\gamma-\theta}\right), & x+\gamma E|Z_1| > 0 \\ 0, & x+\gamma E|Z_1| < 0. \end{cases}$$

The case $\theta+\gamma < 0$ and $\theta-\gamma > 0$ (referred to as *case 3*) is similar to case 2, namely

$$f_{g(Z)}(x) = \begin{cases} 0, & x+\gamma E|Z_1| > 0 \\ -\dfrac{1}{\theta+\gamma} f_Z\left(\dfrac{x+\gamma E|Z_1|}{\theta+\gamma}\right) + \dfrac{1}{\theta-\gamma} f_Z\left(\dfrac{x+\gamma E|Z_1|}{\theta-\gamma}\right), & x+\gamma E|Z_1| < 0. \end{cases}$$

Case 4, namely when $\theta + \gamma < 0$ and $\theta - \gamma < 0$ is obtained from (10) in a similar manner, by replacing $\theta + \gamma$ by $\gamma - \theta$, and replacing $\theta - \gamma$ by $-(\theta + \gamma)$. The cases where $\theta + \gamma = 0$ or $\theta - \gamma = 0$ will not be considered here. In the following, we shall concentrate on case 1, and indicate what happens for the other cases.

Theorem 2.1 can be applied to obtain the tail behavior of $X_1 = \log \sigma_1^2$:

Theorem 3.1. *Let the model assumptions* (3) – (6) *be satisfied and* $(X_t)_{t\in\mathbb{Z}}$ *as in* (9)*. Suppose* $\theta + \gamma > 0$ *and* $\theta - \gamma > 0$*. Define*

$$a'_k := \begin{cases} (\gamma + \theta) a_k, & \text{if} \quad a_k \geq 0, \\ (\gamma - \theta) a_k, & \text{if} \quad a_k < 0, \end{cases} \tag{11}$$

and the constants

$$r := \frac{\nu}{\lambda 2^{1+1/\nu}\Gamma(1/\nu)} \qquad \text{and} \qquad s := \frac{1}{2\lambda^\nu}, \tag{12}$$

where λ *is the constant appearing in* (2)*. Denote by* p^* *the number of* k *in* $\{1, \ldots, p\}$ *such that* $a_k \neq 0$*. Then* X_1 *has a density satisfying*

$$f_X(x) \sim C_1\, x^{(p^*-1)(1-\nu/2)} \exp\left\{ -W \left(x + \gamma E|Z_1| \sum_{k=1}^p a_k \right)^\nu \right\}, \qquad \text{as } x \to \infty,$$

where

$$\begin{aligned} W &:= s \left(\sum_{k=1}^p (a'_k)^{\nu/(\nu-1)} \right)^{1-\nu}, & (13) \\ C_1 &:= \frac{(2\pi)^{(p^*-1)/2} r^{p^*}}{(s\nu(\nu-1))^{(p^*-1)/2}} \prod_{\substack{k=1 \\ a_k \neq 0}}^p (a'_k)^{(2-\nu)/(2(\nu-1))} \left(\sum_{k=1}^p (a'_k)^{\nu/(\nu-1)} \right)^{-(\nu-1)/2 - p^*(1-\nu/2)}. \end{aligned}$$

The tail $\overline{F}_X$ *of the distribution of* X_1 *satisfies*

$$\overline{F}_X(x) \sim C_2\, x^{p^* - (p^*+1)\nu/2} \exp\left\{ -W \left(x + \gamma E|Z_1| \sum_{k=1}^p a_k \right)^\nu \right\}, \qquad \text{as } x \to \infty,$$

where

$$C_2 := \frac{(2\pi)^{(p^*-1)/2} r^{p^*}}{(s\nu)^{(p^*+1)/2} (\nu-1)^{(p^*-1)/2}} \prod_{\substack{k=1 \\ a_k \neq 0}}^p (a'_k)^{(2-\nu)/(2(\nu-1))} \left(\sum_{k=1}^p (a'_k)^{\nu/(\nu-1)} \right)^{(\nu-1)/2 - p^*(1-\nu/2)}.$$

Proof. It is an easy matter to check that for large x the density of a summand $a_k g(Z_{t-k})$ for $a_k \neq 0$ is given by

$$f_k(x) = \gamma_k(x) e^{-\psi_k(x)},$$

where $\gamma_k(x) := r/a'_k$ and

$$\psi_k(x) = \begin{cases} s\left(\dfrac{\frac{x}{a_k} + \gamma E|Z_1|}{\gamma+\theta}\right)^\nu, & \text{if } a_k > 0 \\ \\ s\left(\dfrac{\frac{x}{a_k} + \gamma E|Z_1|}{\gamma-\theta}\right)^\nu, & \text{if } a_k < 0\,. \end{cases}$$

Then a direct application of Theorem 2.1 yields the asymptotic of the density f_X as given (see [Me, Th. 3.7] for detailed calculations). The assertion on the tail $\overline{F}_X$ then follows by l'Hospital's rule. □

The Theorem shows that the tail of the distribution of X_1 is Weibull–like. Hence, for the associated iid sequence we obtain immediately (cf. [EKM, Table 3.4.4]):

Corollary 3.2. *Let the assumptions and notations of Theorem 3.1 be satisfied and let $(\widetilde{X}_t)_{t\in\mathbb{Z}}$ be the associated iid sequence with $(X_t)_{t\in\mathbb{Z}}$. Then $(\widetilde{X}_t)_{t\in\mathbb{Z}}$ is in the domain of attraction of the Gumbel distribution. Possible choices for the norming constants are given by*

$$c_n^X = (W\nu)^{-1}(W^{-1}\log n)^{1/\nu-1}, \tag{14}$$

$$\begin{aligned} d_n^X &= (W^{-1}\log n)^{1/\nu} - \gamma E|Z_1|\sum_{k=1}^p a_k \\ &\quad + \frac{1}{\nu}(W^{-1}\log n)^{1/\nu-1}\left(\frac{p^* - (p^*+1)\nu/2}{W\nu}\log(W^{-1}\log n) + \frac{\log C_2}{W}\right). \end{aligned} \tag{15}$$

Example 3.3. The formulae for the tail and the norming constants become considerably easier for $\nu = 2$, i.e. when the $(Z_t)_{t\in\mathbb{Z}}$ are standard normally distributed. In that case we obtain asymptotically as $x \to \infty$,

$$\begin{aligned} f_X(x) &\sim \left(2\pi\sum_{k=1}^p (a'_k)^2\right)^{-1/2} \exp\left\{-\left(x + \gamma\sqrt{2/\pi}\sum_{k=1}^p a_k\right)^2 \Big/ \left(2\sum_{k=1}^p (a'_k)^2\right)\right\}, \\ \overline{F}_X(x) &\sim \left(\frac{\sum_{k=1}^p (a'_k)^2}{2\pi}\right)^{1/2} x^{-1}\exp\left\{-\left(x + \gamma\sqrt{2/\pi}\sum_{k=1}^p a_k\right)^2 \Big/ \left(2\sum_{k=1}^p (a'_k)^2\right)\right\}. \end{aligned}$$

For the norming constants we obtain

$$
\begin{aligned}
c_n^X &= \left(\sum_{k=1}^{p}(a_k')^2\right)^{1/2}(2\log n)^{-1/2},\\
d_n^X &= \left(\sum_{k=1}^{p}(a_k')^2\right)^{1/2}(2\log n)^{1/2}-\gamma\sqrt{2/\pi}\sum_{k=1}^{p}a_k\\
&\quad -\frac{1}{2}\left(\sum_{k=1}^{p}(a_k')^2\right)^{1/2}(2\log n)^{-1/2}(\log(4\pi)+\log\log n).
\end{aligned}
$$

We now show that Corollary 3.2 holds for the process $(X_t)_{t\in\mathbb{Z}}$, too. In particular, large values of $(X_t)_{t\in\mathbb{Z}}$ do not occur in clusters.

Theorem 3.4. *Let the assumptions of Theorem 3.1 be satisfied. Then $(X_t)_{t\in\mathbb{Z}}$ lies in the domain of attraction of the Gumbel distribution with the same norming constants as its associated iid sequence. These constants are given by* (14) *and* (15).

Proof. Since the convergence of $P(\max(\widetilde{X_1},\dots,\widetilde{X_n})\le c_n^X x+d_n^X)$ to $\Lambda(x)$ is equivalent to $\lim_{n\to\infty} n\overline{F}_X(c_n^X x+d_n^X)=e^{-x}$, it suffices to show conditions $D(u_n)$ and $D'(u_n)$ for sequences u_n such that $n\overline{F}_X(u_n)$ converges to some $\tau\in(0,\infty)$, as $n\to\infty$. The condition $D(u_n)$ here is trivially satisfied since X_t and X_{t+h} are independent for $h\ge p$. It remains to verify condition $D'(u_n)$, i.e. (7). Observe that

$$
\begin{aligned}
& n\sum_{j=2}^{[n/k]}P(X_1>u_n,X_j>u_n)\\
&= nP(X_1>u_n,X_2>u_n)+\cdots+nP(X_1>u_n,X_p>u_n)+n(\lfloor n/k\rfloor-p)(P(X_1>u_n))^2\\
&= nP(X_1>u_n,X_2>u_n)+\cdots+nP(X_1>u_n,X_p>u_n)+\tau^2/k+o(1),\quad \text{as } n\to\infty.
\end{aligned}
$$

Thus, it remains to show that for $2\le j\le p$,

$$nP(X_1>u_n,X_j>u_n)\to 0\quad \text{as } n\to\infty.$$

Since $nP(X_1>u_n,X_j>u_n)\le nP(X_1+X_j>2u_n)$ and since $nP(X_1>u_n)\to\tau$, it suffices to show that

$$\frac{P(X_1+X_j>2u_n)}{P(X_1>u_n)}\to 0\quad \text{as } n\to\infty \tag{16}$$

for $2\le j\le p$. Set

$$G:=X_1+X_j=\sum_{k=1}^{p+j-1}\widetilde{a}_k g(Z_{j-k}),$$

where the coefficients $\widetilde{a}_k$ are defined by

$$\widetilde{a}_k := \begin{cases} a_k, & \text{for} \quad 1 \le k \le j-1\,, \\ a_{k-j+1} + a_k, & \text{for} \quad j \le k \le p\,, \\ a_{k-j+1}, & \text{for} \quad p+1 \le k \le p+j-1. \end{cases}$$

Set

$$\widetilde{a}'_k := \begin{cases} (\gamma+\theta)\widetilde{a}_k, & \text{if} \quad \widetilde{a}_k \ge 0, \\ (\gamma-\theta)\widetilde{a}_k, & \text{if} \quad \widetilde{a}_k < 0. \end{cases}$$

From Theorem 3.1 follows that G has distribution tail

$$\overline{F}_G(2x) \sim \widetilde{C}_2 (2x)^q \exp\left\{ -\widetilde{W} \left(2x + \gamma E|Z_1| \sum_{k=1}^{p+j-1} \widetilde{a}_k \right)^{\nu} \right\}, \quad \text{as } x \to \infty,$$

with some constants $\widetilde{C}_2 > 0$ and $q \in \mathbb{R}$ and

$$\widetilde{W} := s \left(\sum_{k=1}^{p+j-1} (\widetilde{a}'_k)^{\nu/(\nu-1)} \right)^{1-\nu},$$

where s is as in (12). Then (16) will follow if $2^\nu \widetilde{W} > W$ can be shown, with W as in (13). With $\mu := \nu/(\nu-1) > 1$, this is equivalent to

$$\sum_{k=1}^{p+j-1} (\widetilde{a}'_k)^\mu < \sum_{k=1}^{p} (2a'_k)^\mu. \tag{17}$$

To show (17), observe that $(\widetilde{a}'_k)^\mu = (a'_k)^\mu$ for $k \in \{1, \dots, j-1\}$, and $(\widetilde{a}'_k)^\mu = (a'_{k-j+1})^\mu$ for $k \in \{p+1, \dots, p+j-1\}$. For $k \in \{j, \dots, p\}$, it holds

$$(\widetilde{a}'_k)^\mu \le 2^{\mu-1}((a'_k)^\mu + (a'_{k-j+1})^\mu) \tag{18}$$

with equality if and only if $a_k = a_{k-j+1}$. If a_k and a_{k-j+1} have different signs, this follows from the fact that $|a_k + a_{k-j+1}| \le \max(|a_k|, |a_{k-j+1}|)$. If, for example, both are positive, then (18) follows from the fact that for fixed $c > 0$ the function $[0, c] \to \mathbb{R}$, $a \mapsto c^\mu/(a^\mu + (c-a)^\mu)$ attains its maximum value $2^{\mu-1}$ if and only if $a = c/2$. Thus it follows

$$\sum_{k=1}^{p+j-1} (\widetilde{a}'_k)^\mu \le \sum_{k=1}^{j-1} (a'_k)^\mu + \sum_{k=j}^{p} 2^{\mu-1}((a'_k)^\mu + (a'_{k-j+1})^\mu) + \sum_{k=p+1}^{p+j-1} (a'_{k-j+1})^\mu \le \sum_{k=1}^{p} (2a'_k)^\mu,$$

and at least one of these inequalities is strict if not all of the a_k are 0 (which is a trivial case and we have excluded). This is (17). $\square$

Remark 3.5. We have done all the calculations for the case $\theta+\gamma>0$ and $\theta-\gamma>0$ (case 1). In case 4, the same results hold by replacing $\theta+\gamma$ by $\gamma-\theta$ and replacing $\theta-\gamma$ by $-(\theta+\gamma)$. In the second case, if all coefficients are nonnegative, then if $\theta>0$ we obtain exactly the same results as in case 1, and if $\theta<0$ the same results as in case 4. If some of the coefficients are negative, we cannot apply Theorem 2.1 and we do not get results. In case 3, similar statements to case 2 hold. However, here we only get results if all of the a_k are nonpositive.

4 Extremal behavior of $\log\xi_t^2$

We now consider the process

$$L_t := \log\xi_t^2 = \log\sigma_t^2 + \log Z_t^2, \quad t\in\mathbb{Z}.$$

We derive the asymptotic tail and density of L_1 and show that $(L_t)_{t\in\mathbb{Z}}$ as well as the associated iid sequence show the same extremal behavior and lie in the domain of attraction of the Gumbel distribution. Norming constants will be given. As already noted in the introduction we shall restrict ourselves to the case where $\nu=2$, i.e. the $(Z_t)_{t\in\mathbb{Z}}$ are iid standard normally distributed. We then obtain:

Theorem 4.1. *Let the assumptions of Theorem 3.1 be satisfied with $\nu=2$. Define a_k' as in (11) and set*

$$v := -\gamma\sqrt{2/\pi}\sum_{k=1}^{p} a_k \quad \textit{and} \quad w := \left(\sum_{k=1}^{p}(a_k')^2\right)^{1/2}. \tag{19}$$

For $x>0$, define

$$\begin{aligned} f_{v,w}(x) &:= \frac{1}{w\sqrt{\pi}}\exp\left\{-\frac{(x-\log x)^2}{2w^2}+\frac{(\log 2-\log w^2-1+v)x}{w^2}\right.\\ &\qquad -\frac{(\log 2-\log w^2+v)\log x}{w^2}-\frac{\log^2 w^2}{2w^2}\\ &\qquad \left.+\frac{\log 2\log w^2}{w^2}-\frac{(v+\log 2)^2}{2w^2}+\frac{v\log w^2}{w^2}\right\}, &(20)\\ \overline{F}_{v,w}(x) &:= \frac{w^2}{x}f_{v,w}(x). &(21)\end{aligned}$$

Then the density f_L and the tail $\overline{F}_L$ of $L_1=\log\xi_1^2$ behave asymptotically like

$$f_L(x)\sim f_{v,w}(x), \quad \overline{F}_L(x)\sim\overline{F}_{v,w}(x), \quad \textit{as } x\to\infty.$$

Proof. Let f_1 be the density of $\log Z_1^2$ and f_2 be the density of $\log \sigma_1^2$. Define

$$\begin{aligned}
\gamma_1(x) &:= \frac{1}{\sqrt{2\pi}}, \qquad \psi_1(x) := \frac{1}{2}\left(e^x - x\right), \qquad \sigma_1^2(x) := 1/\psi_1''(x) = 2e^{-x}, \\
\gamma_2(x) &:= \frac{1}{\sqrt{2\pi w^2}}, \quad \psi_2(x) := \frac{1}{2w^2}(x-v)^2 \quad \text{and} \quad \sigma_2^2(x) := 1/\psi_2''(x) = w^2.
\end{aligned}$$

Then a straightforward calculation and an application of Theorem 3.1, respectively, show that as $x \to \infty$,

$$f_1(x) \sim \gamma_1(x) e^{-\psi_1(x)} \quad \text{and} \quad f_2(x) \sim \gamma_2(x) e^{-\psi_2(x)}.$$

To obtain the density of f_L we can apply Theorem 2.1, and obtain

$$f_L(x) \sim \gamma_L(x) e^{-\psi_L(x)}, \quad \text{as} \quad x \to \infty,$$

where the functions γ_L and ψ_L can be calculated as follows: Set

$$\begin{aligned}
q_1(\tau) &:= (\psi_1')^{\leftarrow}(\tau) = \log(2\tau + 1), \\
q_2(\tau) &:= (\psi_2')^{\leftarrow}(\tau) = w^2\tau + v, \\
x(\tau) &:= q_1(\tau) + q_2(\tau) = \log(2\tau + 1) + w^2\tau + v. \qquad (22)
\end{aligned}$$

Then

$$\begin{aligned}
\psi_L(x) &= \psi_1(q_1(\tau)) + \psi_2(q_2(\tau)) = \frac{1}{2}\left(2\tau + 1 - \log(2\tau+1) + w^2\tau^2\right), \qquad (23) \\
\sigma_L^2(x) &= \sigma_1^2(q_1(\tau)) + \sigma_2^2(q_2(\tau)) = \frac{2}{2\tau+1} + w^2, \\
\gamma_L(x) &= \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma_L(x)} \sqrt{2\pi}\, \sigma_1(q_1(\tau))\gamma_1(q_1(\tau)) \sqrt{2\pi}\, \sigma_2(q_2(\tau))\gamma_2(q_2(\tau)) \\
&\sim \frac{1}{\sqrt{2\pi}} x^{-1/2}. \qquad (24)
\end{aligned}$$

Approximative inversion of equation (22) results in

$$\tau(x) = \frac{x}{w^2} - \frac{1}{w^2}\log\left(\frac{2x}{w^2}\right) - \frac{v}{w^2} + \frac{1}{w^2 x}\log\left(\frac{2x}{w^2}\right) - \frac{1}{2x} + \frac{v}{w^2 x} + o\left(\frac{1}{x}\right).$$

Inserting this in (23) and using (24) then shows $f_L(x) \sim f_{v,w}(x)$ as $x \to \infty$. Detailed calculations are given in [Me, Th. 3.22]. The assertion on $\overline{F}_L(x)$ follows from l'Hospital's rule. □

Corollary 4.2. *Let the assumptions of Theorem 4.1 be satisfied. Let $(\widetilde{L}_t)_{t\in\mathbb{Z}}$ be the associated iid sequence with $(L_t)_{t\in\mathbb{Z}}$. Then $(\widetilde{L}_t)_{t\in\mathbb{Z}}$ is in the domain of attraction of the Gumbel distribution with norming constants*

$$\begin{aligned}
c_n^L &= w(2\log n)^{-1/2} \qquad (25) \\
d_n^L &= e_1(\log n)^{1/2} + e_2 \log(\log n)^{1/2} + e_3 + e_4 \frac{\log(\log n)^{1/2}}{(\log n)^{1/2}} + e_5 \frac{1}{(\log n)^{1/2}}, \qquad (26)
\end{aligned}$$

where

$$e_1 = \sqrt{2}\, w, \quad e_2 = 1, \quad e_3 = \frac{3}{2}\log 2 - \frac{1}{2}\log w^2 - 1 + v, \quad e_4 = -\frac{w}{\sqrt{2}},$$

$$e_5 = -\frac{1 + w^2 \log(2\pi)}{2\sqrt{2}\, w},$$

and v and w are as in (19).

Proof. Since the derivative $f'_{v,w}(x)$ of $f_{v,w}$ is negative for all x in a left neighborhood of ∞ and since

$$\lim_{x\to\infty} \frac{f'_{v,w}(x)\overline{F}_L(x)}{(f_L(x))^2} = -1,$$

it follows that $(\widetilde{L}_t)_{t\in\mathbb{Z}}$ is in the domain of attraction of the Gumbel distribution, see e.g. Resnick [Re, Prop. 1.18]. Furthermore, pairs of normalizing constants c_n and d_n are given by the equations

$$\log \overline{F}_{v,w}(d_n) + \log n = 0 \quad \text{and} \quad c_n = \frac{\overline{F}_{v,w}(d_n)}{f_{v,w}(d_n)}.$$

From this and (21) follows

$$d_n = \sqrt{2}w(\log n)^{1/2} + o((\log n)^{1/2}), \quad \text{as} \quad n \to \infty,$$

and hence

$$c_n^L \sim c_n, \quad \text{as } n \to \infty.$$

Inserting (26) into (21) shows

$$\log \overline{F}_{v,w}(d_n^L) = \log \overline{F}_{v,w}(d_n) + o(1), \quad n \to \infty.$$

From this it can be deduced that $(d_n^L - d_n)/c_n = o(1)$. This shows that $(c_n^L)_{n\in\mathbb{N}}$ and $(d_n^L)_{n\in\mathbb{N}}$ are pairs of norming constants. □

Again, the maxima of the process $(L_t)_{t\in\mathbb{Z}}$ behave in the same way as those of the associated iid sequence, in particular, no clustering occurs.

Theorem 4.3. *Let the assumptions of Theorem 4.1 be satisfied. Then $(L_t)_{t\in\mathbb{Z}}$ lies in the domain of attraction of the Gumbel distribution with the same norming constants as its associated iid sequence, which are given by* (25) *and* (26).

Proof. The idea of the proof is similar to the proof of Theorem 3.4, by showing that for sequences u_n such that $\lim_{n\to\infty} n\overline{F}_L(u_n) = \tau \in (0,\infty)$ conditions $D(u_n)$ and $D'(u_n)$ hold. Here however, the verification of $D'(u_n)$ turns out to be more elaborate.

Again, $D(u_n)$ is satisfied by independence of L_t and L_{t+h} for $h \geq p+1$, and $D'(u_n)$ follows if

$$\frac{P(L_1 + L_j > 2u_n)}{P(L_1 > u_n)} \to 0 \quad \text{as} \quad n \to \infty \tag{27}$$

for $2 \leq j \leq p+1$. To show (27), we need the tail behavior of

$$H := L_1 + L_j := \log Z_j^2 + \big(\log Z_1^2 + a_{j-1} g(Z_1)\big) + \sum_{k=1}^{p+j-1} \widetilde{a}_k g(Z_{j-k}),$$

where the coefficients $\widetilde{a}_k$ are defined by

$$\widetilde{a}_k := \begin{cases} a_k, & \text{if} \quad 1 \leq k \leq j-2, \\ 0, & \text{if} \quad k = j-1, \\ a_{k-j+1} + a_k, & \text{if} \quad j \leq k \leq p, \\ a_{k-j+1}, & \text{if} \quad p+1 \leq k \leq p+j-1. \end{cases}$$

Let f_1 be the density of $\log Z_j^2$, f_3 the density of $\sum_{k=1}^{p+j-1} \widetilde{a}_k g(Z_{t-k})$, and f_4 the density of $\log Z_1^2 + a_{j-1} g(Z_1)$. Then

$$f_1(x) \sim \frac{1}{\sqrt{2\pi}} \exp(-(e^x - x)/2), \quad \text{as } x \to \infty,$$

and by Example 3.3,

$$f_3(x) \sim \frac{1}{\sqrt{2\pi w_3^2}} \exp\left(-\frac{1}{2}\left(\frac{x - v_3}{w_3}\right)^2\right), \quad \text{as } x \to \infty,$$

where

$$v_3 := -\gamma\sqrt{2/\pi} \sum_{k=1}^{p+j-1} \widetilde{a}_k \quad \text{and} \quad w_3 := \left(\sum_{k=1}^{p+j-1} (\widetilde{a}_k')^2\right)^{1/2},$$

with

$$\widetilde{a}_k' := \begin{cases} (\gamma+\theta)\widetilde{a}_k, & \text{if} \quad \widetilde{a}_k \geq 0, \\ (\gamma-\theta)\widetilde{a}_k, & \text{if} \quad \widetilde{a}_k < 0. \end{cases}$$

To obtain the density f_4, define for $z > 0$,

$$\delta(z) := \begin{cases} \log z^2 + a_{j-1} g(z) = \log z^2 + a_{j-1}(\theta+\gamma) z - \mathit{const.}, & \text{if} \quad a_{j-1} > 0, \\ \log z^2 + a_{j-1} g(-z) = \log z^2 + a_{j-1}(\gamma-\theta) z - \mathit{const.}, & \text{if} \quad a_{j-1} < 0. \end{cases}$$

We shall assume that $a_{j-1} \neq 0$. For $a_{j-1} = 0$ similar proofs hold with slight modifications. For large x, the density $f_4(x)$ is given by

$$f_4(x) = \frac{1}{\frac{2}{\delta^{\leftarrow}(x)} + a'_{j-1}} \frac{1}{\sqrt{2\pi}} e^{-(\delta^{\leftarrow}(x))^2/2} \sim \frac{1}{\sqrt{2\pi}\, a'_{j-1}} e^{-(\delta^{\leftarrow}(x))^2/2}, \quad \text{as } x \to \infty.$$

Here, a'_{j-1} is defined as in (11). Now we can again apply Theorem 2.1 to obtain the asymptotic behavior of the density f_H as $f_H(x) \sim \gamma_H(x) e^{-\psi_H(x)}$, as $x \to \infty$. We have

$$f_1(x) \sim \gamma_1(x) e^{-\psi_1(x)}, \quad f_3(x) \sim \gamma_3(x) e^{-\psi_3(x)}, \quad f_4(x) \sim \gamma_4(x) e^{-\psi_4(x)}, \quad \text{as } x \to \infty,$$

where

$$\begin{aligned}
\gamma_1(x) &= 1/\sqrt{2\pi}, & \psi_1(x) &= (e^x - x)/2, & q_1(\tau) &= \psi_1'^{\leftarrow}(\tau) = \log(2\tau + 1), \\
\gamma_3(x) &= 1/\sqrt{2\pi w_3^2}, & \psi_3(x) &= ((x - v_3)/w_3)^2/2, & q_3(\tau) &= \psi_3'^{\leftarrow}(\tau) = w_3^2 \tau + v_3, \\
\gamma_4(x) &= 1/(\sqrt{2\pi} a'_{j-1}), & \psi_4(x) &= (\delta^{\leftarrow}(x))^2/2, & q_4(\tau) &= \psi_4'^{\leftarrow}(\tau), \\
x &= q_1(\tau) + q_3(\tau) + q_4(\tau), \\
\psi_H(x) &= \psi_1(q_1(\tau)) + \psi_3(q_3(\tau)) + \psi_4(q_4(\tau)) = \frac{1}{2}\left[(\delta^{\leftarrow}(q_4(\tau)))^2 + w_3^2 \tau^2 + o(\tau^2)\right].
\end{aligned}$$

Since, as $x \to \infty$ and $\tau \to \infty$,

$$\delta^{\leftarrow}(x) = \frac{1}{a'_{j-1}} x + o(x) \quad \text{and} \quad q_1(\tau) = (a'_{j-1})^2 \tau + o(\tau),$$

it follows that

$$\psi_H(x) = \frac{1}{2} \frac{1}{(a'_{j-1})^2 + w_3^2} x^2 + o(x^2).$$

Similar calculations show $\gamma_H(x) \sim \frac{1}{\sqrt{2\pi}} x^{-1/2}$ as $x \to \infty$. Having now the asymptotic density $f_H(x) \sim \gamma_H(x) e^{-\psi_H(x)}$ and the density of L_1 given by Theorem 4.1, (27) will follow if

$$(a'_{j-1})^2 + \sum_{k=1}^{p+j-1} (\widetilde{a}'_k)^2 < \sum_{k=1}^{p} (2a'_k)^2$$

for $2 \leq j \leq p+1$. This however has already been shown to be true in (17) for $j \leq 2 \leq p$ (note that there $\widetilde{a}'_{j-1}$ was defined in a different manner), and for $j = p+1$ it is obvious. This shows that (27) and thus $D'(u_n)$ holds. □

Remark 4.4. As in Remark 3.5, one obtains similar results for the other cases 2, 3 and 4, where exactly the same changes have to be made as in Remark 3.5.

5 Extremal behavior of ξ_t^2 and ξ_t

In this section we shall show how the results of the previous section imply similar results on the extremal behavior of $(\xi_t^2)_{t\in\mathbb{Z}}$ and $(\xi_t)_{t\in\mathbb{Z}}$. In particular, $(\xi_t^2)_{t\in\mathbb{Z}}$ and $(\xi_t)_{t\in\mathbb{Z}}$ lie in the domain of attraction of the Gumbel distribution and exceedances of high tresholds do not occur in clusters. More precisely we have the following result:

Theorem 5.1. *Let the assumptions of Theorem 4.1 be satisfied. Then with the notations of Theorem 4.1, the densities f_{ξ^2} and f_ξ as well as the tails $\overline{F}_{\xi^2}$ and $\overline{F}_\xi$ of ξ_1^2 and ξ_1 satisfy as $x\to\infty$ the asymptotic relations*

$$f_{\xi^2}(x) \sim \frac{1}{x} f_{v,w}(\log x), \qquad \overline{F}_{\xi^2}(x) \sim \frac{w^2}{\log x} f_{v,w}(\log x),$$
$$f_\xi(x) \sim \frac{1}{x} f_{v,w}(2\log x), \qquad \overline{F}_\xi(x) \sim \frac{w^2}{4\log x} f_{v,w}(2\log x).$$

The sequences $(\xi_t^2)_{t\in\mathbb{Z}}$ and $(\xi_t)_{t\in\mathbb{Z}}$ as well as their associated iid sequences lie in the domain of attraction of the Gumbel distribution. Norming constants $c_n^{(2)}$ and $d_n^{(2)}$ for $(\xi_t^2)_{t\in\mathbb{Z}}$ and its associated iid sequence are given by

$$c_n^{(2)} := c_n^L \exp(d_n^L) \quad \textit{and} \quad d_n^{(2)} := \exp(d_n^L),$$

where c_n^L and d_n^L are given in (25) and (26). Norming constants $c_n^{(1)}$ and $d_n^{(1)}$ for $(\xi_t)_{t\in\mathbb{Z}}$ and its associated iid sequence are given by

$$c_n^{(1)} := \frac{c_n^{(2)}}{2\sqrt{d_n^{(2)}}} \quad \textit{and} \quad d_n^{(1)} := \sqrt{d_n^{(2)}} - \frac{c_n^{(2)}\log 2}{2\sqrt{d_n^{(2)}}}.$$

Proof. The assertions on the asymptotic behavior of f_{ξ^2} and $\overline{F}_{\xi^2}$ follow directly from Theorem 4.1. For $\overline{F}_\xi$ note that for positive x

$$\overline{F}_\xi(x) = P(\xi_1 > x) = P(\sigma_1^2 Z_1^2 > x^2, Z_1 > 0) = \frac{1}{2} P(\sigma_1^2 Z_1^2 > x^2),$$

since Z_1 is symmetric and independent of σ_1. This then immediately implies the assertions on f_ξ and $\overline{F}_\xi$. That the iid sequences associated with $(\xi_t^2)_{t\in\mathbb{Z}}$ and $(\xi_t)_{t\in\mathbb{Z}}$ are in the domain of attraction of the Gumbel distribution follows as in Corollary 4.2 by application of Proposition 1.18 in [Re]. That $c_n^{(2)}$ and $d_n^{(2)}$ are norming constants for the associated iid sequence with $(\xi_t^2)_{t\in\mathbb{Z}}$ can be shown in a similar manner to the derivation of norming constants for the lognormal distribution, see e.g. [EKM, Ex. 3.3.31]. Thus, for any $x\in\mathbb{R}$, $n\,\overline{F}_{\xi^2}(c_n^{(2)}x + d_n^{(2)})$ converges to $\exp(-x)$ as $n\to\infty$, implying that $n\,\overline{F}_\xi(\sqrt{c_n^{(2)}x + d_n^{(2)}})$

converges to $(\exp(-x))/2 = \exp(-x - \log 2)$. Using

$$\sqrt{c_n^{(2)} x + d_n^{(2)}} = \sqrt{d_n^{(2)}} + \frac{c_n^{(2)}}{2\sqrt{d_n^{(2)}}} x + o\left(\frac{c_n^{(2)}}{\sqrt{d_n^{(2)}}}\right), \quad \text{as} \quad n \to \infty,$$

it follows that $c_n^{(1)}$ and $d_n^{(1)}$ as defined are norming constants for the iid sequence associated with $(\xi_t)_{t\in\mathbb{Z}}$. The proof that the maxima of $(\xi_t^2)_{t\in\mathbb{Z}}$ and $(\xi_t)_{t\in\mathbb{Z}}$ behave as the maxima of the corresponding associated iid sequences follows from the proof of Theorem 4.3. In particular, for the verification of $D'(u_n)$ for the ξ^2 process, suppose that $n\overline{F}_{\xi^2}(u_n) \to \tau \in (0,\infty)$ as $n \to \infty$. Then

$$nP(\log \xi_1^2 > \log u_n) = nP(\xi_1^2 > u_n) \to \tau \quad \text{as} \quad n \to \infty.$$

But then it follows

$$\frac{P(\xi_1^2 > u_n, \xi_j^2 > u_n)}{P(\xi_1^2 > u_n)} = \frac{P(\log \xi_1^2 > \log u_n, \log \xi_j^2 > \log u_n)}{P(\log \xi_1^2 > \log u_n)} \to 0 \quad \text{as} \quad n \to \infty,$$

as was shown in the proof of Theorem 4.3. That $D'(v_n)$ holds for the process $(\xi_t)_{t\in\mathbb{Z}}$ for sequences $(v_n)_{n\in\mathbb{N}}$ such that $n\overline{F}_{\xi}(v_n) \to \tau \in (0,\infty)$ as $n \to \infty$ then follows from the fact that

$$\frac{P(\xi_1 > v_n, \xi_j > v_n)}{P(\xi_1 > v_n)} \leq \frac{P(\xi_1^2 > v_n^2, \xi_j^2 > v_n^2)}{P(\xi_1^2 > v_n^2)/2} \to 0 \quad \text{as} \quad n \to \infty.$$

□

As in the previous section, the same remarks as in Remark 3.5 hold for the other cases.

6 Connections to other results

In this section we discuss connections between the results of this paper and other results in the literature.

6.1 The logvariance of the EGARCH(1,1) process

In [S1, S2], Settimi derives the extremal behavior of the logvariance of an EGARCH(1,1) process. An EGARCH(1,1) process is given by (3), (4), (6), and

$$\log \sigma_t^2 = \alpha_0 + \alpha_1 g(Z_{t-1}) + \beta_1 \log \sigma_{t-1}^2, \quad t \in \mathbb{Z}, \tag{28}$$

where $|\beta_1| < 1$. Settimi considers the case where $\alpha_1 > 0$, $0 < \beta_1 < 1$, $\nu = 2$ in (4) and $\gamma = 1$ and $\theta \in (-1, 0)$ in (6). This corresponds to case 2 in our classification given in Section 3. Rewriting (28) as an infinite moving average process as in (1), with positive coefficients, she obtains that the logvariance process $(\log \sigma_t^2)_{t\in\mathbb{Z}}$ is in the domain of attraction of the Gumbel distribution, and she gives explicit norming constants. While in [S2] she mainly states her results and ideas, proofs are given in [S1]. Her proofs are based on a result of Rootzén on the extremal behavior of infinite moving average processes [Ro1] and its companion paper [Ro2]. It should be noted that an application of the results in [Ro2] would have given the tail behavior of $\overline{F}_X$ in Theorem 3.1, too.

6.2 Stochastic volatility models

Consider a process given by

$$\xi_t = \sigma_t Y_t, \quad \log \sigma_t^2 = \sum_{k=0}^{\infty} a_k Z_{t-k}, \quad t \in \mathbb{Z}, \tag{29}$$

where $(Y_t)_{t\in\mathbb{Z}}$ is iid $N(0,1)$, $(Z_t)_{t\in\mathbb{Z}}$ is iid $N(0,1)$ independent of $(Y_t)_{t\in\mathbb{Z}}$, and $(a_k)_{k\in\mathbb{N}_0}$ is a sequence of real parameters such that $\sum_{k=0}^{\infty} a_k^2 < \infty$. Then this defines a certain stochastic volatility (SV) model. The extreme value behavior of the process given in (29) has been studied by Breidt and Davis [BD]. Some of their results have been generalized by Diop and Guegan [DG] to allow the sequence $(Z_t)_{t\in\mathbb{Z}}$ to be iid $GED(\nu)$ with some $\nu > 0$, however still assuming an iid $N(0,1)$ sequence $(Y_t)_{t\in\mathbb{Z}}$ independent of $(Z_t)_{t\in\mathbb{Z}}$. The SV model (29) as studied by Breidt and Davis has certain similarities with the EGARCH model considered in this article. For the SV model, the logarithm of the conditional variance σ_t^2 is expressed as an infinite sum of independent Gaussian random variables, while in (5) it is expressed as a finite sum of independent random variables $g(Z_{t-k})$. Here, $g(Z_{t-k})$ is Gaussian only if $\gamma = 0$. In the general case however, the tail of $g(Z_{t-k})$ behaves like a Gaussian tail, as noticed in the beginning of Section 3. The next difference is that in the SV model the process $(Y_t)_{t\in\mathbb{Z}}$ is assumed to be independent of the whole process $(Z_t)_{t\in\mathbb{Z}}$ and hence independent of $(\sigma_t)_{t\in\mathbb{Z}}$. In the EGARCH model (3) – (6), where $\xi_t = \sigma_t Z_t$, Z_t is only independent of $\sigma_t, \sigma_{t-1}, \ldots$, but not of $\sigma_{t+1}, \ldots, \sigma_{t+p}$. For the study of the tailbehavior of ξ_1 and its transforms such as $\log \xi_1^2$ this makes no difference. However, the dependence structure of the sequence $(\xi_t)_{t\in\mathbb{Z}}$ is different for the SV and the EGARCH model.

Breidt and Davis derive the upper tail behavior for $\log \xi_1^2$ in model (29), and show that $(\log \xi_t^2)_{t\in\mathbb{Z}}$ is in the domain of attraction of the Gumbel distribution, provided the correlation function of $(\xi_t)_{t\in\mathbb{Z}}$ decays faster than $1/(\log h)$ as $h \to \infty$. They also give norming constants. Via a point process argument they show that the untransformed $(\xi_t)_{t\in\mathbb{Z}}$

lies in the domain of attraction of the Gumbel distribution, too, and they give norming constants.

As already pointed out, if $(a_k)_{k\in\mathbb{N}_0}$ is a finite sequence in (29) with $a_0 = 0$, and if $\gamma = 0$ and $\theta = 1$ in the EGARCH model (3) – (6), then the distributions of the ξ_1 coincide. In particular, the tails of $\log \xi_1^2$ in both models must be the same. Furthermore the norming constants for the associated iid sequences must be the same. A comparison of the results of Breidt and Davis and the results of this article shows that this is in fact the case. Furthermore, for neither of the processes $(\xi_t)_{t\in\mathbb{Z}}$ large values occur in clusters. This latter similarity is not obvious, since both processes have different dependence structures.

6.3 ARCH-processes

It is natural to compare the extremal behavior of EGARCH processes with the extremal behavior of GARCH processes. In [HRRV], de Haan et al. studied the extremal behavior of an ARCH(1) process, given by

$$\xi_t = \sigma_t Z_t, \quad \sigma_t^2 = \omega + a_1 \sigma_{t-1}^2 Z_{t-1}^2, \quad t \in \mathbb{Z},$$

where $\omega > 0$ and $a_1 \in (0, 2\exp(\gamma))$, where γ is Euler's constant. They obtained that the tail of the ARCH(1) process behaves asymptotically like $cx^{-2\kappa}/2$ with positive constants κ and c. In particular, the associated iid sequence with $(\xi_t)_{t\in\mathbb{Z}}$ lies in the domain of attraction of the Fréchet distribution $\Phi_{2\kappa}$ with parameter 2κ and norming constants given by $c_n = (cn)^{1/(2\kappa)}$, $d_n = 0$. The ARCH process $(\xi_t)_{t\in\mathbb{Z}}$ however does not satisfy the $D'(u_n)$ condition, and maxima scaled with c_n do not converge to the same limit as for the associated iid sequence. Rather do they converge to $\Phi_{2\kappa}^{\theta}$, where $\theta \in (0,1)$ is the extremal index. This means that exceedances of certain high tresholds of the ARCH(1) process occur in clusters with an average cluster length of $1/\theta$. All this is different from the results obtained for the finite EGARCH process $(\xi_t)_{t\in\mathbb{Z}}$ as defined in (3) – (6). Here, the tail behaves like $\exp(-2(\log x)^2/w^2 + o(\log x)^2)$, as $x \to \infty$, which is heavier than exponential, but lighter than polynomial. As seen, the EGARCH process $(\xi_t)_{t\in\mathbb{Z}}$ as well as its associated iid sequence lie in the domain of attraction of the Gumbel distribution, and subsequent maxima of $(\xi_t)_{t\in\mathbb{Z}}$ do not cluster.

6.4 Log–ACD models

Engle and Russell [ER] have introduced the class of ACD models to model durations x_t between the $(t-1)$'th and t'th event (such as trades) that occur randomly during the market hours of stock exchanges. A variant of these models is the *logarithmic ACD model* as given by Bauwens and Giot in [BG] or by Bauwens et al. [BGG] in the Log–ACD(p,q)-form:

$$\begin{aligned} x_t &= \sigma_t Z_t, \quad t \in \mathbb{Z}, \\ \log \sigma_t^2 &= \omega + \sum_{k=1}^{p} a_k g(Z_{t-k}) + \sum_{k=1}^{q} b_k \log \sigma_{t-k}^2, \quad t \in \mathbb{Z}, \end{aligned}$$

with constants ω, a_k and b_k, a deterministic function g and an iid sequence $(Z_t)_{t\in\mathbb{Z}}$ with support of distribution on the positive axis and with finite variance. A priori there are not made any further restrictions on the distribution of Z_t. In [BG] however, it is assumed that the Z_t follow a Weibull distribution with parameter $\alpha > 0$. The Log–ACD model is very similar to the EGARCH(p,q)–model, the latter which is a straightforward generalisation of the EGARCH(1,1)–model given in (28). The main difference between the two models is in the assumption of the distribution of the Z_t, which are symmetric in the EGARCH case and concentrated on the positive axis in the Log–ACD model. In case that $g = id$ in the Log–ACD model (referred to as *Log–ACD*$_2$*–model* in [BG, BGG]), that $q = 0$, that the Z_t are Weibull distributed with parameter $\alpha > 1$, and that $a_1, \ldots, a_p$ are all nonnegative, the same methods as in Section 3 show that the $(\log \sigma_t^2)$ process lies in the domain of attraction of the Gumbel distribution. Its extremal behavior is the same as the one of the associated iid sequence, implying that large values do not occur in clusters. It seems likely that, under these restrictions on the distribution and the model parameters, the extremal behavior of the duration process x_t itself can be treated with methods similar to the ones applied in Section 4.

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