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Competition for Budget-Constrained Buyers: Exploring All-Pay Auctions

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Abstract. When faced with budget-constrained bidders, all-pay auctions revenue-dominate standard auctions (first and second-price), which, in a competitive market, gives an edge to the all-pay format. An equilibrium in which sellers compete with standard auctions fails to exist if the all-pay format is available. Assuming the budget is not severely limited, in the unique symmetric equilibrium sellers compete with all-pay auctions.

Keywords: All-pay Auctions, Budget Constraints, Directed Search, Competing Auctions

JEL: D4, D81, D83

1 Introduction

We consider a competitive market where transactions are settled with auctions and buyers differ in financial resources.¹ We then ask which auction format—first price, second price or all-pay—sellers should pick in their quest to attract customers. The choice of auction format can be crucial: should most sellers pick one format, a seller can strategically switch to another, and draw in a larger customer base and gain a competitive advantage.

We show that when faced with budget-constrained buyers, all-pay auctions revenue-dominate standard auctions (first and second-price). Under the all-pay rule the budget constraint does not bind as often, which leads to more aggressive bidding among all participants; thus, allowing the seller to collect a higher amount of revenue. Revenue dominance provides an edge in a competitive market and leads to two important results. First, an equilibrium in which sellers compete with standard auctions fails to exist if the all-pay format is available. Second, assuming the budget is not severely limited, in the unique symmetric equilibrium, sellers compete with all-pay auctions. What is more, despite the difference in budgets, buyers enjoy equal expected utility, which coincides with the outcome of homogeneous buyers. In other words, if buyers do not differ too much in terms of their ability to pay then the all-pay format avoids the budget constraint altogether.

¹Internet advertising serves as a prominent example. Many of these advertisements are auctioned off on platforms like Google, Facebook, and X (formerly Twitter). Advertisers allocate their marketing budgets to participate in these auctions, which are designed to maximize ad exposure within budget limits, thus introducing complexity to the process. Recently, there has been a transition towards first-price auctions over the previously prevalent second-price auctions. In 2019, Google shifted to first-price auctions for its ad exchange, followed by Twitter adopting first-price auctions for its mobile app ad slots in 2020 (Balseiro et al., 2023).

Real estate auction companies such as Auction.com, Williams & Williams, and Hubzu in the US, and Allsop, Auction House UK, and SDL Property Auctions in the UK serve as another example. These firms auction off thousands of residential and commercial properties, mostly through online platforms, and their typical method is a first-price auction with a reserve. The range of buyers at these auctions is diverse, spanning from deep-pocketed cash buyers, property flippers and developers to individual buyers with moderate budgets seeking affordable housing.

Auctions with budget-constraints are studied extensively; see Balseiro et al. (2023) or Kotowski (2020) for a recent review. However, this body of research overlooks competition: it typically assumes that sellers already have multiple bidders and does not examine how they initially attract the bidders. In our setup the selection of an auction rule affects not only the number of bidders but also the composition of high/low types among them. To account for competition, we rely on the directed search (competitive search) approach in the tradition of Burdett et al. (2001). Price mechanism selection within this class of models has been studied extensively; see for instance Severinov and Virag (2024) and the review therein. To the best of our knowledge, this note is the first attempt to integrate the concepts of all-pay auctions, budget constraints, and competition simultaneously.

2 Model

The economy consists of a large number of risk-neutral buyers and sellers, with buyer-seller ratio λ . Each seller has one unit of a good and aims to sell it at a price exceeding his reservation price, zero. Similarly, each buyer seeks to purchase one unit and is willing to pay up to his reservation price, one. While buyers share identical valuations of the good, their purchasing abilities vary: a fraction σ of buyers, “low types”, have constrained budgets and can pay only up to $b < 1$, while the remaining buyers, “high types”, can pay up to 1. The type of buyer is private information, but the parameters λ , σ , and b are common knowledge. The game proceeds over the course of two stages. In the first stage, sellers simultaneously and independently choose an auction format m and a reserve price r_m . The set of formats consists of first-price, second-price, and all-pay auctions. In the second stage, buyers observe sellers’ selections and pick one store to visit. If the customer is alone, then he pays the reserve price. If there are $n \geq 2$ buyers, then bidding ensues. If trade takes place at price r then the seller realizes payoff r and the buyer realizes $1 - r$.

Following the directed search literature, we focus on visiting strategies that are symmetric and anonymous on and off the equilibrium path, which, in a large market, imply that the distribution of demand at any store is Poisson (Burdett et al., 2001). Therefore, the probability that a seller with terms (m, r_m) meets n customers of type $i = h, l$ is given by $z_n(x_{i,m})$ where

$$z_n(x) = e^{-x} x^n / n!$$

Since high types and low types arrive at independent Poisson rates $x_{h,m}$ and $x_{l,m}$, the distribution of the *total demand* is also Poisson with $x_{h,m} + x_{l,m}$. Both $x_{h,m}$ and $x_{l,m}$ are endogenous and they depend on what the seller posts and how it compares with the rest of the market. This is analyzed in Section 4 but first we need to focus on auctions.

3 Auctions

Bidding ensues if $n \geq 2$, so consider a store with $n \geq 2$ customers. Low types arrive at Poisson rate x_l and high types arrive at rate x_h (for now drop the subscript m). The distribution of the number

of low types, therefore, is $\text{binomial}(n, \theta)$ where

$$\theta = x_l / (x_h + x_l).$$

Note that θ is the probability that a buyer is a low type and it is endogenous as it depends on the endogenous arrival rates x_h and x_l .

Remark 1 *With a first-price auction, in the unique symmetric equilibrium low types bid b while high types randomize in $[b, 1 - (1 - b)\theta^{n-1}]$. With a second price auction in the unique symmetric equilibrium low types bid b while high types bid their valuation, 1. Under both formats buyers earn $u_h(n) = \theta^{n-1}(1 - b)$ and $u_l(n) = \theta^{n-1}(1 - b)/n$ whereas the seller earns $\pi(n) = 1 - \theta^{n-1}(1 - b)(n - n\theta + \theta)$.*

The Remark is based on Selcuk (2017). In both auction formats, low types bid their budget b . High types exhibit less aggressive bidding in the first-price auction compared to the second-price auction. However, their expected earnings remain equal. Indeed, both formats yield identical expected payoffs for buyers as well as the seller. Given the payoff equivalence, sellers and buyers are indifferent to adopting or joining either auction format; thus we will treat first and second-price auctions equally, referring to them as *standard auctions*.

Proposition 1 All-Pay Auctions. *If $b < \theta^{n-1}$ then a unique symmetric equilibrium exists where high types randomize in $[b, 1 - \theta^{n-1} + b]$ according to cdf $G_h(p)$, where*

$$G_h(p) = \frac{(p + \theta^{n-1} - b)^{\frac{1}{n-1} - \theta}}{1 - \theta}.$$

As for low types, if $b < \theta^{n-1}/n$ then they all bid b , but if $\theta^{n-1}/n \leq b < \theta^{n-1}$ then they employ a strategy $G_l(p)$ with support $[0, b]$ that has an atom at b , where

$$G_l(p) = \begin{cases} p^{\frac{1}{n-1}}/\theta & \text{if } 0 \leq p < (1 - \mu)^{n-1} \theta^{n-1} \\ 1 - \mu & \text{if } (1 - \mu)^{n-1} \theta^{n-1} \leq p < b \\ 1 & \text{if } p \geq b \end{cases}.$$

The size of the atom μ falls as b rises. Buyers' expected earnings are $u_h(n) = \theta^{n-1} - b$, while $u_l(n) = 0$ if $b \geq \theta^{n-1}/n$ and $u_l(n) = \theta^{n-1}/n - b$ otherwise.

If $\theta^{n-1} \leq b$ then in any symmetric equilibrium the seller extracts the entire surplus while all buyers earn zero i.e. $\pi(n) = 1$, and $u_h(n) = u_l(n) = 0$. The following strategies constitute such an outcome: low types randomize in $[0, \theta^{n-1}]$ whereas high types randomize in $[\theta^{n-1}, 1]$ according to

$$G_l(p) = \frac{p^{\frac{1}{n-1}}}{\theta} \quad \text{and} \quad G_h(p) = \frac{p^{\frac{1}{n-1} - \theta}}{1 - \theta}. \quad (1)$$

All proofs are in the online appendix. With all-pay auctions, bids are smaller. For instance, under standard auctions low types bid their entire budget b , but now they randomize within a lower range, especially if b is not too small. High types, too, reduce their bids. This is understandable

because, in the all-pay format, participants pay their bid regardless of winning or losing, whereas in standard formats, payment is due only in case of a win.

Lemma 1 *Fix $\theta \in (0, 1)$ and $n \geq 2$. All-pay auctions extract more revenue from buyers than standard auctions.*

Thanks to the smaller bids, the budget constraint does not bind as often as it does under standard auctions, which leads to more aggressive bidding among all participants.² Consequently, the seller collects a higher amount of revenue. Absent budget constraints, revenue equivalence among fixed-price, second-price and all-pay auctions is well known in the literature. We show that with financially constrained buyers, the all-pay format revenue-dominates the standard formats. Che and Gale (1996) prove a similar result when the budget distribution is continuous. We characterize the all-pay auction equilibrium under a discrete distribution and show that its revenue dominance remains valid.

The bids depend on θ , the fraction of low types among arrivals. For instance, if a seller attracts high types only, i.e. $\theta = 0$, then bidders randomize in the interval $[0, 1]$ according to $G = p^{\frac{1}{n-1}}$.³ If the seller attracts buyers with a different θ , then the shape of the cdf and the associated payoffs would, of course, change. With competition, θ depends on the seller's auction format, reserve price and how they compare to the rest of the market. In the next section we analyze this problem.

4 Competition

Let s represent standard (first or second price) auctions and ap represent all-pay auctions. A type i buyer's expected utility from visiting a store competing with the auction rule $m = s, ap$ is given by

$$U_{i,m}(r_m, x_{h,m}, x_{l,m}) = \sum_{n=0}^{\infty} z_n(x_{h,m} + x_{l,m}) u_{i,m}(n+1) \quad (2)$$

With probability $z_n(\cdot)$ the buyer finds $n = 0, 1, \dots$ other customers at the same store; so, in total there are $n+1$ customers (including himself) and the expected utility corresponding to this scenario is $u_{i,m}(n+1)$. The previous section pins down the expressions for $u_{i,m}(n)$ when $n \geq 2$. If $n = 1$, then $u_{i,m}(1) = 1 - r_m$, i.e. if the customer is alone, then he pays the reserve price. Now consider a store competing with rule m . The expected profit is given by

$$\Pi_m(r_m, x_{h,m}, x_{l,m}) = \sum_{n=1}^{\infty} z_n(x_{h,m} + x_{l,m}) \pi_m(n). \quad (3)$$

With probability $z_n(\cdot)$ the store gets n customers and the corresponding payoff is $\pi_m(n)$. As above, $\pi_m(1) = r$, i.e. if the store gets one customer then it charges r .

²As noted by Che and Gale (1996), with standard auctions, buyers bid for the right to obtain the good with probability 1, whereas with the all-pay format, they bid for the right to obtain the good with probability less than 1. The all-pay format divides the good into smaller probability units, inducing buyers to bid more aggressively for them.

³Baye et al. (1996) obtain the same outcome in a model with no budget constraints; see their Theorem 1.

Lemma 2 *The relationship between Π and U_h and U_l is given by*

$$\Pi_m = 1 - z_0(x_{h,m} + x_{l,m}) - x_{h,m}U_{h,m} - x_{l,m}U_{l,m}. \quad (4)$$

The expression $1 - z_0(\cdot)$ can be interpreted as the expected revenue. It is the value created by a sale (one), multiplied by the probability of trading ($1 - z_0$). The expression $x_{h,m}U_{h,m} + x_{l,m}U_{l,m}$ can be interpreted as the expected cost. The seller promises a payoff $U_{h,m}$ to each high type and $U_{l,m}$ to each low type customer. On average he gets $x_{h,m}$ high type and $x_{l,m}$ low type customers; so the total cost equals to $x_{h,m}U_{h,m} + x_{l,m}U_{l,m}$.

Following the directed search literature, let \bar{U}_i denote the maximum expected utility ("market utility") a type i customer can obtain in the market. For now we treat \bar{U}_i as given, subsequently it will be determined endogenously. Consider a seller who advertises (m, r_m) and suppose that buyers respond to this with arrivals $x_{h,m} \geq 0$ and $x_{l,m} \geq 0$. These rates satisfy

$$x_{i,m} > 0 \text{ if } U_{i,m}(r_m, x_{h,m}, x_{l,m}) = \bar{U}_i \text{ and } 0 \text{ otherwise.} \quad (5)$$

The tuple $(r_m, x_{h,m}, x_{l,m})$ must generate an expected utility of \bar{U}_h for high types, else they will stay away, and \bar{U}_l for low types, else they will stay away (Note that $U_{i,m}$ cannot exceed \bar{U}_i by definition.) Each seller picks some m and r_m but realizes that $x_{h,m}$ and $x_{l,m}$ are determined via (5).

Lemma 3 *Fix (r, x_h, x_l) . The all-pay format generates strictly lower payoffs for buyers, i.e. $U_{h,ap} < U_{h,s}$ and $U_{l,ap} < U_{l,s}$. Equivalently, it generates strictly higher profits for the seller, i.e. $\Pi_{ap} > \Pi_s$.*

This result echoes Lemma 1. There, the the number of buyers at a store, n , was assumed to be known, so the claim was based on ex-post payoffs $u_h(n)$ and $u_l(n)$. Here the claim is about the ex-ante payoffs U_h and U_l where n is uncertain; however if one fixes the arrival rates and the reserve price, then the claim still goes through.

Proposition 2 *If the all-pay format is available, then a standard auction equilibrium fails to exist.*

The proof is by contradiction. We conjecture a scenario in which sellers compete with standard auctions. We then show that if a particular seller switches to the all-pay format, he can offer the buyers the same amount of payoffs that the other stores offer, however the composition of his buyers changes: the all-pay auction store attracts a higher percentage of low types and a lower percentage of high types. This allows the seller to earn more; rendering our initial conjecture a non-equilibrium. The revenue-dominance of all-pay auctions (Lemma 3) plays a key role in obtaining this result.

Proposition 3 *If $b \geq \sigma$ then in the unique symmetric equilibrium sellers compete with all-pay auctions. Each store, on average, receives $\sigma\lambda$ low type and $(1 - \sigma)\lambda$ high type buyers. Stores set a*

reserve price of zero, which is charged if $n = 1$. If $n \geq 2$ then auctions take place, where low types randomize in $[0, \sigma^{n-1}]$ and high types randomize in $[\sigma^{n-1}, 1]$ according to

$$G_l(p) = \frac{p^{\frac{1}{n-1}}}{\sigma} \quad \text{and} \quad G_h(p) = \frac{p^{\frac{1}{n-1}} - \sigma}{1 - \sigma}.$$

In this equilibrium buyers earn $U_{h,ap} = U_{l,ap} = z_0(\lambda)$, while sellers earn $\Pi_{ap} = 1 - z_0(\lambda) - z_1(\lambda)$.

The payoffs here are the same as in a model with homogenous buyers.⁴ Indeed, the equilibrium holds when $b = 1$ (no budget constraint), or $b = \sigma$ or anywhere in between. The all-pay format manages to maintain the homogenous outcome and afford both types of buyers the same expected utility, despite the difference in their ability to pay; thus avoiding the budget constraint. This remains the case so long as the budget is not too low, or the percentage of low types in the market is not too large, or both ($b \geq \sigma$).

If $b < \sigma$ then, per Proposition 1, low types and high types can no longer have the same expected utility in a symmetric equilibrium (high types are better off when b falls below σ); thus the budget constraint starts to bite.

5 Conclusion

In our model buyers have identical valuations, which somewhat restricts the scope of our findings. Pai and Vohra (2014) study the optimal auction design involving financially constrained buyers with differing valuations and conclude that a modified all-pay auction format is the most effective approach. This suggests that in a competitive market with differing valuations the all-pay format should still maintain its edge over alternative formats.

With widening income inequality, budget limitations are now more critical than ever. Although the all-pay format may not be as widely utilized as the standard auction formats in current business practices, our results as well as the preceding literature on all-pay auctions suggests that its potential to address budget issues should not be underestimated.

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⁴In a setup with homogenous buyers Eeckhout and Kircher (2010) show that there exists a continuum of equilibria in which sellers compete with “payoff complete” mechanisms. In any such equilibrium, the expected demand at a store is λ , sellers earn $1 - z_0(\lambda) - z_1(\lambda)$ no matter which rule they compete with whereas buyers earn $z_0(\lambda)$ no matter which seller’s rule they join in. See also Kultti (1999), and Julien et al. (2000). We show that all-pay auctions achieve the same outcome even with budget-constrained buyers (provided $b \geq \sigma$).

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6 Online Appendix - Not intended for publication

Proof of the Proposition 1. We focus on symmetric mixed strategies where bidders of the same type pick the same cumulative distribution function (cdf) $G_i(p) : [\underline{s}_i, \bar{s}_i] \rightarrow [0, 1]$. A point p is an *increasing point* of G_i if G_i is not constant in an ε neighborhood of p_i , i.e. if for each $\varepsilon > 0$ the probability of having a value in $(p - \varepsilon, p + \varepsilon)$ is positive. Conversely, p is a *constant point* if G_i is constant in an ε neighborhood of p . If there is an atom at p then p is an increasing point. If the pair (G_h, G_l) corresponds to an equilibrium, then type i buyers earn their equilibrium payoff u_i at each increasing point p of G_i . Similarly they earn an expected payoff that is less than or equal to u_i at each constant point p of G_i (Hillman and Samet, 1987; Baye et al., 1996).

The equilibrium G_h cannot have a mass point on its support $[\underline{s}_h, \bar{s}_h]$. A mass point means tying with other bidders in which case the surplus is divided via random rationing. A financially unconstrained bidder can always beat the tie and improve his payoff by placing a bid that is slightly above the mass point, which is inconsistent with G_h being an equilibrium distribution. The argument applies to the upper bound \bar{s}_h as well: If $\bar{s}_h < 1$ then there is room to beat a potential tie at \bar{s}_h . If $\bar{s}_h = 1$ then a mass at 1 would result in a negative payoff.

The equilibrium cdf G_l cannot have an atom anywhere below b for the same reason, however, it may have an atom at b . There are three scenarios for G_l :

- L1 — The entire mass is at b .
- L2 — There is a partial mass at b .
- L3 — There is no mass at b .

Assuming that p is not a mass point, the expected payoff associated with bidding p is given by

$$EU(p) = \sum_{i=0}^{n-1} \binom{n-1}{i} [\theta G_l(p)]^i [(1-\theta)G_h(p)]^{n-1-i} - p = [\theta G_l(p) + (1-\theta)G_h(p)]^{n-1} - p. \quad (6)$$

This is true for both types of buyers. Indeed buyers have identical valuations for the item, so both types earn the same expected payoff $EU(p)$, assuming, of course, $p \leq b$. If $p > b$ then low types are sure to be outbid; the expected payoff for high types can be obtained by substituting $G_l(p) = 1$ into (6).

- **Region 1:** Suppose that $b < \frac{\theta^{n-1}}{n}$

1A. The minimum expected payoff for high types is $\theta^{n-1} - b$ whereas for low types is $\frac{\theta^{n-1}}{n} - b$, both of which are positive. Low types cannot bid more than b , so if a high type bids $b + \varepsilon$, then even if he loses against all other high types, he can still win the item with probability θ^{n-1} (everyone else is a low type) and would obtain a minimum expected payoff $\theta^{n-1} - b$. Similarly if a low type bids b , then in the worst case scenario he loses against all high types and ties with every other low type, so his payoff is at least $\frac{\theta^{n-1}}{n} - b$. It follows that $u_h \geq u_l > 0$.

1B. We rule out scenarios L2 and L3, which leaves L1 as the only possible scenario for G_l . In scenarios L2 and L3 the cdf G_l is assumed to be continuous over some interval $[\underline{s}_l, \bar{s}_l]$ where $0 \leq \underline{s}_l < \bar{s}_l \leq b$. Recall that G_h is also continuous over $[\underline{s}_h, \bar{s}_h]$, so there are three possibilities: either

$\underline{s}_h > \underline{s}_l$ or $\underline{s}_l > \underline{s}_h$ or $\underline{s}_l = \underline{s}_h$. Consider the first one, i.e. suppose $\underline{s}_h > \underline{s}_l$. This implies $u_l = -\underline{s}_l$. To see why, note that \underline{s}_l is an increasing point of G_l and in a mixed strategy equilibrium any increasing point, including \underline{s}_l , must yield the equilibrium payoff to the bidder. If the buyer bids $p = \underline{s}_l$ then he is sure to lose the item as everyone else is sure to bid more than \underline{s}_l (recall that $\underline{s}_l < \underline{s}_h$). His payoff, therefore, is $-\underline{s}_l \leq 0$ because all participants forfeit their bids (if the lower bound \underline{s}_l is set to be zero then the resulting payoff is zero, but if \underline{s}_l is positive then the payoff is negative). This, of course, contradicts the fact that $u_l > 0$. Now suppose $\underline{s}_l > \underline{s}_h$. This implies $u_h = -\bar{s}_h \leq 0$, which contradicts $u_h > 0$. Finally if $\underline{s}_l = \underline{s}_h$ then $u_h = u_l = -\underline{s}_l \leq 0$, which again is a contradiction. In words, if both cdfs have continuous bits then the one with the lower bound on the far left is bound to yield at most a zero payoff. It follows that G_l cannot have a continuous part over some interval below b ; the entire mass must be at point b .

1C. Given that the entire mass of G_l is placed at b the lower bound of G_h cannot be below b , i.e. we must have $\underline{s}_h \geq b$. Indeed if $\underline{s}_h < b$ then $u_h = -\underline{s}_h \leq 0$, a contradiction. Since all low types bid b and high types are sure to outbid the low types ($\underline{s}_h \geq b$) the equilibrium payoff of a low type equals to $u_l = \frac{\theta^{n-1}}{n} - b$.

1D. As discussed earlier, G_h cannot have an atom, i.e. there are no jumps. We now argue that it cannot have intermittent flat spots either. By contradiction, suppose G_h is constant at some interval $(a_1, a_2) \subset [\underline{s}_h, \bar{s}_h]$. Both a_1 and a_2 are increasing points of the distribution function G_h hence they both must deliver the same payoff u_h . Since G_h is flat in this interval the probability of winning the auction is the same at both points (notice also $G_l = 1$ at both points). This, however, means that the player gets a lower payoff at a_2 than a_1 since $a_2 > a_1$; so, he cannot be indifferent; a contradiction.

1E. We can now characterize G_h . We established that G_h is monotonically increasing on its support $[\underline{s}_h, \bar{s}_h]$. The expected payoff associated with bidding any $p \in [\underline{s}_h, \bar{s}_h]$ is given by

$$EU(p) = [\theta + (1 - \theta) G_h(p)]^{n-1} - p,$$

which is obtained by substituting $G_l(p) = 1$ into (6) (since $\underline{s}_h \geq b$ we have $G_l(p) = 1$ for all $p \geq \underline{s}_h$). High types must earn their equilibrium payoff u_h at any increasing point of G_h . Since G_h is monotonically increasing we must have

$$EU(p) = u_h \text{ for all } p \in [\underline{s}_h, \bar{s}_h].$$

Substituting for $EU(p)$ and solving for G_h we have

$$G_h(p) = \frac{(p + u_h)^{\frac{1}{n-1}} - \theta}{1 - \theta}.$$

We know $G_h(\underline{s}_h) = 0$ and $G_h(\bar{s}_h) = 1$, hence $u_h = \theta^{n-1} - \underline{s}_h$ and $u_h = 1 - \bar{s}_h$. Recall that $u_h \geq \theta^{n-1} - b$ and that $\underline{s}_h \geq b$. This means $\underline{s}_h = b$ and $u_h = \theta^{n-1} - b$ and therefore $\bar{s}_h = 1 - \theta^{n-1} + b$. Substituting for u_h yields the expression of G_h in the body of the proposition.

- **Region 2:** Suppose that $\frac{\theta^{n-1}}{n} < b < \theta^{n-1}$

2A. Per the discussion in 1A, high types are still guaranteed to receive a positive payoff, $\theta^{n-1} - b$; however, low types are no longer guaranteed to receive a positive payoff i.e. $u_h > 0$, $u_l \geq 0$.

2B. There are three scenarios for G_l which are outlined in L1, L2 and L3. Suppose L1 is valid, i.e. low types bid b for sure. Then $\underline{s}_h \geq b$ and the equilibrium payoff of a low type equals to $u_l = \frac{\theta^{n-1}}{n} - b$ (see 1C). This, however, is negative because $\frac{\theta^{n-1}}{n} < b$; a contradiction. Since L1 is ruled out, we have either L2 or L3. In either case G_l is continuous over some interval $[\underline{s}_l, \bar{s}_l]$ where $0 \leq \underline{s}_l < \bar{s}_l \leq b$. Recall that G_h is continuous on its support $[\underline{s}_h, \bar{s}_h]$, so there are three possibilities: either $\underline{s}_h > \underline{s}_l$ or $\underline{s}_l > \underline{s}_h$ or $\underline{s}_l = \underline{s}_h$. Per the discussion in 1B, we cannot have $\underline{s}_l > \underline{s}_h$ or $\underline{s}_l = \underline{s}_h$ because in either case $u_h \leq 0$, a contradiction. So we must have $\underline{s}_h > \underline{s}_l$. This implies $u_l = -\underline{s}_l \leq 0$. Clearly if $\underline{s}_l > 0$ then u_l is negative; hence we must have $\underline{s}_l = 0$ and therefore $u_l = 0$.

2C. We will show that $\underline{s}_h \geq \bar{s}_l$, i.e. the supports of G_h and G_l cannot overlap. Suppose they do, i.e. suppose that $\underline{s}_h < \bar{s}_l$ so that $G_h(\bar{s}_l) > 0$. Pick some point $p \in [\underline{s}_h, \bar{s}_l]$ at which G_h is increasing (since $G_h(\bar{s}_l) > 0$ such a point must exist). Recall that the expected payoff associated with bidding p is equal to $EU(p)$, which is given by (6), and it is the same for both types of buyers. Since p is an increasing point of G_h we have $EU(p) = u_h$. Now p is either an increasing point of G_l or it is a constant point of G_l . The first case implies $EU(p) = u_l$ whereas the second one implies $EU(p) \leq u_l$. In either case we have a contradiction since $u_h > 0$ and $u_l = 0$. It follows that the supports of G_h and G_l cannot overlap, so we must have $\underline{s}_h \geq \bar{s}_l$.

2D. Now we will rule out scenario L3. Again, by contradiction, suppose L3 is valid, i.e. G_l is atomless and does not jump anywhere on its support $[0, \bar{s}_l]$. In 2C we proved that $\underline{s}_h \geq \bar{s}_l$, hence $G_h(p) = 0$ for all $p \in [0, \bar{s}_l]$. Furthermore, per the discussion in 1D, G_l cannot have intermittent flat spots either; hence G_l is monotonically increasing on its support. The expected payoff associated with bidding any $p \in [0, \bar{s}_l]$ is given by

$$EU(p) = [\theta G_l(p)]^{n-1} - p,$$

which is obtained by substituting $G_h(p) = 0$ into (6). Low types must earn their equilibrium payoff $u_l = 0$ at any increasing point of G_l . Since G_l is monotonically increasing we must have $EU(p) = u_h$ for all $p \in [\underline{s}_h, \bar{s}_h]$. Substituting for $EU(p)$ and solving for G_l we have

$$G_l(p) = \frac{p^{\frac{1}{n-1}}}{\theta}. \quad (7)$$

Note that $G_l(\bar{s}_l)$ must be equal to 1; however this is impossible because $\bar{s}_l \leq b$ and $G_l(b) = \frac{b^{\frac{1}{n-1}}}{\theta} < 1$ since $b < \theta^{n-1}$. Without a mass point at b , the function G_l cannot be a valid cdf.

2E. Since L3 is ruled out, the only possible scenario is L2 where G_l has some partial mass μ at point b while the remaining mass is spread over some interval starting at the lower bound $\underline{s}_l = 0$. Per the discussion in 2D, at any point of increase in the region where G_l is atomless low types must earn $u_l = 0$, which implies $G_l(p) = \frac{p^{\frac{1}{n-1}}}{\theta}$. Buyers should get the same payoff $u_l = 0$ at the mass point b

as well. The expected payoff associated with bidding b is given by

$$\theta^{n-1} \sum_{i=0}^{n-1} \binom{n-1}{i} \frac{\mu^i}{i+1} (1-\mu)^{n-i-1} - b = \frac{\theta^{n-1} [1 - (1-\mu)^n]}{\mu n} - b.$$

For this to be equal to zero, μ must solve

$$\frac{bn}{\theta^{n-1}} - \frac{1 - (1-\mu)^n}{\mu} = 0$$

It is straightforward to show that so as long as $\frac{\theta^{n-1}}{n} < b < \theta^{n-1}b$ there exists a unique $\mu \in (0, 1)$ satisfying above. The upper bound of the atomless portion of G_l , call it \bar{p} , satisfies $G_l(\bar{p}) = 1 - \mu$ hence $\bar{p} = \theta^{n-1} (1 - \mu)^{n-1}$. An argument similar to the one in 1D reveals that G_l cannot have flat spots in the region $(0, \bar{p})$. So G_l monotonically rises in $[0, \bar{p}]$, has mass point at b and it is flat in between.

2F. We now characterize G_h . We know $\underline{s}_h \geq \bar{s}_l = b$, so for any $p \geq \underline{s}_h$ we have $G_l(p) = 1$. At any point of increase in the support of G_h we must have $EU(p) = u_h$ where

$$EU(p) = [\theta + (1-\theta)G_h(p)]^{n-1} - p,$$

which is obtained by substituting $G_l(p) = 1$ into (6). It follows that

$$G_h(p) = \frac{(p + u_h)^{\frac{1}{n-1}} - \theta}{1 - \theta}.$$

Since $G_h(\underline{s}_h) = 0$ we have $\underline{s}_h = \theta^{n-1} - u_h$. Since $u_h \geq \theta^{n-1} - b$ and $\underline{s}_h \geq b$ we have $\underline{s}_h = b$ and $u_h = \theta^{n-1} - b$ and therefore $\bar{s}_h = 1 - \theta^{n-1} + b$. Per the discussion in 1D, G_h cannot have intermittent flat spots anywhere in its support.

• **Region 3:** Suppose $\theta^{n-1} \leq b$.

3A. Since $\theta^{n-1} \leq b$, per 1A, buyers are no longer guaranteed to receive a positive payoff. In regions 1 and 2 at least one of the equilibrium payoffs was positive, and this feature played a key role in establishing the uniqueness of the equilibrium. Without this information proving uniqueness becomes a challenge; so, instead of attempting to characterize the equilibrium, we focus on expected payoffs and prove that in any symmetric equilibrium, the seller extracts the entire surplus, i.e. $u_l = u_h = 0$. We then verify that the strategy profile in (1) constitutes such an equilibrium.

3B. Consider a symmetric mixed strategy equilibrium with cdfs G_h, G_l and associated equilibrium payoffs u_h, u_l . WLOG let $u_h \geq u_l$, so there are three possibilities: (i) $u_h \geq u_l > 0$ or (ii) $u_h > u_l = 0$ or (iii) $u_h = u_l = 0$. We will rule out (i) and (ii). To start, suppose $u_h \geq u_l > 0$. Recall that G_h is atomless whereas there are three scenarios for G_l . Per 1B, if both cdfs have continuous bits then the one with the lower bound on the far left will yield at most a zero payoff. So, if u_l is positive then G_l cannot have a continuous part over some interval below b ; the entire mass must be at point b . This

scenario is analyzed in 1C and the equilibrium payoff of a low type equals to $u_l = \frac{\theta^{n-1}}{n} - b$, which is negative since $\theta^{n-1} \leq b$; a contradiction.

3C. Suppose $u_h > u_l = 0$. The cdf G_l cannot have the entire mass at b (3B). So either it has a partial mass at b or it is atomless everywhere. Suppose it has a partial mass at b . Then, per 2C, $\underline{s}_h \geq \bar{s}_l = b$ (cdfs cannot overlap). For a high type the expected payoff associated with bidding \underline{s}_h equals to $EU(\underline{s}_h) = \theta^{n-1} - b$, which is less than or equal to zero since $\theta^{n-1} \leq b$. The lower bound \underline{s}_h is an increasing point of G_h , so we must have $EU(\underline{s}_h) = u_h$. This, however, is a contradiction because $u_h > 0$ but $EU(\underline{s}_h) \leq 0$. The final scenario for G_l is where it is atomless everywhere; so, suppose this is the case. Per 2B, 2C and 2D, G_l is given by (7) and it must be monotonically increasing on its support $[0, \bar{s}_l]$ with no flat spots. Solving $G_l(\bar{s}_l) = 1$ yields $\bar{s}_l = \theta^{n-1}$. Recall that $\underline{s}_h \geq \bar{s}_l$, so $\underline{s}_h \geq \theta^{n-1}$. It follows that $EU(\underline{s}_h) \leq 0$, which, again is a contradiction since we must have $EU(\underline{s}_h) = u_h > 0$.

3D. Arguments in 3B and 3C imply that we must have $u_l = u_h = 0$. What remains to be done is to characterize equilibrium strategies G_l and G_h delivering these payoffs. For the purpose of the paper the fact that any symmetric equilibrium yields zero payoffs is sufficient, so we refrain from attempting to characterize all possible combinations of G_l and G_h ; instead we will verify that the specific forms of G_l and G_h , given by (1), indeed correspond to an equilibrium and they yield zero payoffs. So suppose all players adopt the cdfs in (1) and consider a potential deviation by a low type buyer who picks a different cdf, say, $\tilde{G} : [0, b] \rightarrow [0, 1]$. His expected payoff \tilde{u} is given by

$$\tilde{u} = \int_0^b \{[\theta G_l(p) + (1 - \theta) G_h(p)]^{n-1} - p\} d\tilde{G}(p).$$

Observe that $G_l(p) = 1$ for $p \geq \theta^{n-1}$ and $G_h(p) = 0$ for $p \leq \theta^{n-1}$; therefore

$$\tilde{u} = \int_0^{\theta^{n-1}} \{\theta^{n-1} G_l^{n-1}(p) - p\} d\tilde{G}(p) + \int_{\theta^{n-1}}^b \{[\theta + (1 - \theta) G_h(p)]^{n-1} - p\} d\tilde{G}(p).$$

After substituting for $G_l(p)$ and $G_h(p)$, which are given by (1), the expressions inside the curly brackets vanish, thus $\tilde{u} = 0$ irrespective of \tilde{G} , i.e. if everyone else sticks to (1) then a low type cannot earn anything but zero irrespective of the cdf he picks. Using the same technique, one can show that high types do not have a profitable deviation either. ■

Proof of Lemma 1. Consider an auction $m = s, ap$ involving $n \geq 2$ buyers and the seller. The total surplus obtained by buyers and the seller adds up to the total surplus 1; thus

$$\pi_m(n) + \sum_{i=0}^n \binom{n}{i} \theta^i (1 - \theta)^{n-i} [i u_{l,m}(n) + (n - i) u_{h,m}(n)] = 1.$$

After rearranging

$$\pi_m(n) + n \theta u_{l,m}(n) + n (1 - \theta) u_{h,m}(n) = 1. \quad (8)$$

Under standard auctions we have $u_{h,s}(n) = \theta^{n-1} (1 - b)$ and $u_{l,s}(n) = \theta^{n-1} (1 - b) / n$, whereas with

all-pay auctions we have $u_{h,ap}(n) = \max(\theta^{n-1} - b, 0)$ and $u_{l,ap}(n) = \max(\theta^{n-1}/n - b, 0)$. Clearly $u_{h,ap}(n) < u_{h,s}(n)$ and $u_{l,ap}(n) < u_{l,s}(n)$. The fact that $\pi_{ap}(n) > \pi_s(n)$ follows from (8). ■

Proof of Lemma 2 and Lemma 3. Fix x_l and x_h and let $x = x_l + x_h$ and $\theta = x_l/x$. Equation (8) pins down the relationship between $\pi_m(n)$, $u_{h,m}(n)$ and $u_{l,m}(n)$ when $n \geq 2$. If, however, $n = 1$ then $u_{h,m}(1) = u_{l,m}(1) = 1 - r$, while $\pi_m(1) = r$; thus the equation still holds. Dropping the subscript m when understood, we have

$$\begin{aligned} \Pi &= \sum_{n=1}^{\infty} z_n(x) \pi(n) = \sum_{n=1}^{\infty} z_n(x) [1 - n\theta u_l(n) - n(1-\theta)u_h(n)] \\ &= 1 - z_0(x) - x\theta \sum_{n=0}^{\infty} z_n(x) u_l(n+1) - x(1-\theta) \sum_{n=0}^{\infty} z_n(x) u_h(n+1) \\ &= 1 - z_0(x) - x_l U_l - x_h U_h \end{aligned}$$

In the first line, we substitute (8) into (3). In the last line we use $x_l = \theta x$ and $x_h = (1-\theta)x$ as well as the definitions of U_h and U_l from (2). This completes the proof of Lemma 2. Now turn to Lemma 3. Equation (2) implies

$$U_{i,m} = z_0(x) (1-r) + \sum_{n=1}^{\infty} z_n(x) u_{i,m}(n+1). \quad (9)$$

Recall that $u_{i,ap}(n) < u_{i,s}(n)$ for all $i = h, l$ and $n \geq 2$. Since r and $z_n(x)$ are controlled for, we have $U_{i,ap} < U_{i,s}$, for $i = h, l$, i.e. buyers earn less under the all-pay format. Substituting these inequalities into (4) establishes that $\Pi_{ap} > \Pi_s$, i.e. sellers earn more under all-pay auctions. ■

Proof of Proposition 2. Conjecture a symmetric equilibrium in which all sellers adopt standard auctions. Selcuk (2017) characterizes this equilibrium in detail, and shows that if sellers are confined to use standard auctions, then there exists a unique symmetric equilibrium where all sellers set the same reserve price r_s and each seller receives λ customers with a composition of $\sigma\lambda$ low types and $(1-\sigma)\lambda$ high types. Along this outcome sellers earn

$$\Pi_s(r_s, \lambda, \sigma) = 1 - z_0(\lambda) - \lambda(1-\sigma)U_{h,s}(r_s, \lambda, \sigma) - \sigma\lambda U_{l,s}(r_s, \lambda, \sigma),$$

while buyers earn $U_{h,s}$ and $U_{l,s}$. Crucially, we have $U_{h,s} > U_{l,s}$, i.e. high types obtain a higher payoff than low types. To see why, recall that $u_{h,s}(n) > u_{l,s}(n)$ for all $n \geq 2$ (Remark 1). Substituting these into (9) yields the inequality.

In our model sellers can pick all-pay auctions, too, so consider a seller who switches to all-pay auctions. We will show that this seller can earn more while still providing his customers with payoffs $U_{h,s}$ and $U_{l,s}$. After such a switch, there are three key parameters: the reserve price r , the total demand, x , and the composition of demand, θ . The fact that they are potentially different from (r_s, λ, σ) makes the comparison difficult. To get around this issue, fix the total demand λ and note that Lemma 3 implies that $U_{h,ap}(r, \lambda, \theta) < U_{h,s}(r_s, \lambda, \sigma)$ and $U_{l,ap}(r, \lambda, \theta) < U_{l,s}(r_s, \lambda, \sigma)$ when

$r = r_s$ and $\theta = \sigma$. Furthermore, both $U_{h,ap}$ and $U_{l,ap}$ fall in r and rise in θ . The first claim is immediate from (9) whereas the second one follows from the fact that $u_{h,ap}(n)$ and $u_{l,ap}(n)$ both rise in θ . Thus, there exists some $\hat{\theta} > \sigma$ and $\hat{r} < r_s$ satisfying

$$U_{h,ap}(\hat{r}, \lambda, \hat{\theta}) = U_{h,s}(r_s, \lambda, \sigma) \quad \text{and} \quad U_{l,ap}(\hat{r}, \lambda, \hat{\theta}) = U_{l,s}(r_s, \lambda, \sigma), \quad (10)$$

which means that if the all-pay store sets a reserve \hat{r} then he attracts λ buyers in total, but with a composition $\hat{\theta}$ instead of σ . The store earns

$$\begin{aligned} \Pi_{ap}(\hat{r}, \lambda, \hat{\theta}) &= 1 - z_0(\lambda) - \lambda(1 - \hat{\theta})U_{h,ap}(\hat{r}, \lambda, \hat{\theta}) - \hat{\theta}\lambda U_{l,ap}(\hat{r}, \lambda, \hat{\theta}) \\ &= 1 - z_0(\lambda) - \lambda(1 - \hat{\theta})U_{h,s}(r_s, \lambda, \sigma) - \hat{\theta}\lambda U_{l,s}(r_s, \lambda, \sigma). \end{aligned}$$

The first line follows from (4) and the second one from (10). The inequalities $U_{h,s} > U_{l,s}$ and $\hat{\theta} > \sigma$ imply that $\Pi_{ap} > \Pi_s$, rendering our initial conjecture a non-equilibrium. In words, the deviating seller offers the buyers the same amount of payoffs that the other stores offer ($U_{l,ap} = U_{l,s}$ and $U_{h,ap} = U_{h,s}$), however he attracts a higher percentage of low types and a lower percentage of high types ($\hat{\theta} > \sigma$); thus, the deviation is profitable. ■

Proof of Proposition 3. First, we will characterize the symmetric all-pay auction equilibrium. Then we will show that no seller has a profitable deviation by selecting first or second-price auctions. To start, conjecture an outcome where all sellers adopt all-pay auctions. Symmetry in buyers' visiting strategies implies that each seller receives λ customers with a composition of $\sigma\lambda$ low types and $(1 - \sigma)\lambda$ high types. The condition $b > \sigma$ ensures that $b > \sigma^{n-1}$ for all $n \geq 2$; thus per Proposition 1 in an auction with n customers, low types randomize in $[0, \sigma^{n-1}]$ whereas high types randomize in $[\sigma^{n-1}, 1]$ yielding expected payoffs $u_h(n) = u_l(n) = 0$. Substituting these into (2) yields $U_{h,ap} = U_{l,ap} = z_0(\lambda)(1 - r)$. Each seller solves

$$\max_{\lambda} \Pi_{ap} = \max_{\lambda} 1 - z_0(\lambda) - \lambda_l U_{l,ap} - \lambda_h U_{h,ap} \quad \text{s.t.} \quad U_{l,ap} = \bar{U}_l \text{ and } U_{h,ap} = \bar{U}_h.$$

Since $U_{h,ap} = U_{l,ap}$ the market utilities satisfy $\bar{U}_l = \bar{U}_h = \bar{U}$. Thus, the problem becomes $\max_{\lambda} 1 - z_0(\lambda) - \lambda\bar{U}$. The objective function is concave and the first-order condition yields $z_0(\lambda) = \bar{U}$. It follows that $z_0(\lambda)(1 - r) = z_0(\lambda)$ implying that the equilibrium reserve price is $r^* = 0$. Each seller, thus, earns

$$\Pi_{ap} = 1 - z_0(\lambda) - z_1(\lambda).$$

We now show that one cannot profitably switch to a standard format. At all-pay stores we have $U_{h,ap} = U_{l,ap} = \bar{U}$, whereas at the deviating store we have $U'_{h,s} > U'_{l,s}$, i.e. at a standard auction store, a low type obtains a strictly lower expected utility than a high type (Selcuk (2017), see Section 2.4). There are three scenarios, therefore:

- The store attracts low types, while high types stay away. This requires $U'_{l,s} = \bar{U} > U'_{h,s}$, but contradicts $U'_{h,s} > U'_{l,s}$.

- The store attracts both types. This requires $U'_{l,s} = \bar{U}$ and $U'_{h,s} = \bar{U}$, but contradicts $U'_{h,s} > U'_{l,s}$.
- The store attracts high types, while low types stay away, i.e. $U'_{h,s} = \bar{U} > U'_{l,s}$. This is feasible, so we focus on this scenario.

The seller solves $\max_x \Pi'_s = 1 - z_0(x) - xU'_{h,s}$ s.t. $U'_{h,s} = \bar{U}$. The first order condition implies $z_0(x) = \bar{U}$. In a scenario where a standard auction store attracts high types only we have $\theta = 0$, thus $u_h(n) = 0$ for all $n \geq 2$ (Remark 1). Substituting this into (2) yields $U'_{h,s} = z_0(x)(1-r)$. Combining this relationship with the first order condition $z_0(x)(1-r) = z_0(x)$ implies that $r = 0$. Furthermore since $U'_{h,s} = \bar{U} = z_0(\lambda)$ implies $x = \lambda$. Consequently, the seller earns as much as Π_{ap} , which does not present an incentive to deviate; hence the all-pay equilibrium remains. Uniqueness follows from Proposition 2. ■

The fact that the deviating seller can earn Π_{ap} hints at the presence of other, non-symmetric equilibria. Indeed, in addition to this symmetric equilibrium, there exists a continuum of other non-symmetric equilibria where a fraction $\alpha \in [\sigma, 1]$ of sellers adopt all pay auctions and attract both types of customers and the remaining $(1-\alpha)$ of sellers adopt standard auctions and attract high types only. The composition of low types among the all-pay buyers is equal to $\theta = \frac{\sigma}{\alpha}$, thus the minimum budget required for the existence of such equilibria is $b > \frac{\sigma}{\alpha}$ which becomes more stringent as α falls.

Such equilibria are possible only because of the availability of all-pay format.

Such equilibria require a lot coordination among buyers, and sellers as to hwo is going to adopt what and who is going where. The symmetric equilibrium requires no such coordination.

Details: Both types of stores have a total of λ customers and provide their buyers with $z_0(\lambda)$.

$$\begin{aligned} \alpha x_{l,ap} + (1-\alpha) \underbrace{x_{l,s}}_{=0} &= \sigma \lambda \\ \alpha x_{h,ap} + (1-\alpha) \underbrace{x_{h,s}}_{=\lambda} &= (1-\sigma) \lambda \end{aligned}$$

Thus

$$x_{l,ap} = \frac{\sigma}{\alpha} \lambda \text{ and } x_{h,ap} = \frac{\alpha - \sigma}{\alpha} \lambda \quad \mapsto \quad \text{total demand} = \frac{\sigma}{\alpha} \lambda + \frac{\alpha - \sigma}{\alpha} \lambda = \lambda$$

The implied composition

$$\theta = \frac{\frac{\sigma}{\alpha} \lambda}{\lambda} = \frac{\sigma}{\alpha}$$

So we need $b > \frac{\sigma}{\alpha}$ to avoid the all-pay budget issue. Furthermore note that $\alpha \in [\sigma, 1]$ because if α falls below σ then $x_{l,ap} = \frac{\sigma}{\alpha} \lambda > \lambda$, which cannot be. The case $\alpha = 1$ is the symmetric all pay equilibrium and it requires $b \geq \sigma$. If α goes up, then the budget constraint gets more stringent.