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ORIGINAL PAPER



Sustainability gap of public debt: importance of interest rates and a new decomposition with premia

Elmar Dönnebrink¹ · Nils Grevenbrock¹

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Abstract

Sustainability gaps (*S2* indicators) are frequently used in national and international reports to assess the sustainability of public finances. For instance, in the European Commission's Debt Sustainability Monitor (DSM) the indicators are analyzed in comparisons across (policy) scenarios, countries and time. The report's findings play a crucial role in the context of the Stability and Growth Pact and the European Semester. As a result, sustainability gaps have a significant indirect influence on policy decisions. In this paper, we analyze two non-transparent properties of these indicators. *First,* the response of these indicators to changes in the interest rate-growth (r-g) differential is not readily predictable in terms of both strength and direction. *Second,* in our examples for low values of r-g (in a range of 0.5%), highly uncertain projections for distant periods after 2070 explain about 80% of the indicators' values. To address these problems, we develop a new decomposition that takes into account the notion of premia (Reis 2021) and hence allows for a more transparent discussion of the sustainability of public debt.

Keywords Public debt · Interest rates · Fiscal sustainability · Public budget

JEL Classification $H6 \cdot E43 \cdot D52 \cdot J11$

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1 Introduction

The sustainability gap (S2 indicator) is frequently used in national and international debt sustainability reports to assess the long-term sustainability of public finances. The S2 indicator measures the permanent adjustment in the primary deficit that is necessary to satisfy the government's intertemporal budget constraint (IBC) which is defined for an infinite period. The S2 indicator is analyzed in detail in the European Commission's Debt Sustainability Monitor (DSM) in comparisons across (policy) scenarios, countries and time. The report's findings play a crucial role in the context of the Stability and Growth Pact and the European Semester. As a result, the S2 indicator has a significant indirect influence on policy decisions.

Arguably, sustainability indicators used for policy decision making should be transparent and simple to analyze. In this paper we look at two properties of the S2 indicator that do not satisfy these requirements. First, the response of the indicator to changes in the interest rate-growth (r-g) differential is not readily predictable in terms of both strength and direction. The reason is that a change in the interest rate-growth (r-g) differential acts through two channels, sometimes with opposite effects. These channels can be illustrated by representing the S2 indicator as the sum of two annuities-one annuity translating the baseline debt (d) level, S2(d), and a second annuity translating all primary deficits (pd) expected in the future, S2(pd). The first annuity (S2(d)) clearly increases (decreases) as the discount rate (r-g) increases (decreases). For the second annuity (S2(pd)), the response depends on the trajectory of primary deficits since the discount rate affects the weighting of primary deficits over time. When the discount rate is increasing (decreasing), primary deficits that are close (distant) in time are weighted more heavily. For instance, this means that when the discount rate is low, the indicator gives strong weight to projected primary deficits in the distant future. The sum of the two annuities (S2(d) + S2(pd)) responds to a change in interest rates in a clearly determinable direction only when deficits are projected to decline. When primary deficits are projected to increase over time, the direction of change in the S2 indicator is not easily determinable, since the interest rate responsiveness of S2(d) and S2(pd) partly cancel out each other. Second, for low values of r-g (in a range of 0.5%), the influence of highly uncertain projections for distant periods on the S2 indicators is very large. For example, projection values for periods after 2070 explain up to 80% of the S2 indicator in some scenarios.

To capture the unexpected properties of the *S2* indicator, we propose a new decomposition of the S2 indicator. The new decomposition exploits the observation that many sovereigns pay interest rates on their sovereign debt that are empirically lower than capital market interest rates on other asset classes, and hence, adopts the notion of premia (Reis 2021). Our decomposition allows for a more transparent discussion of the extent to which changes in the S2 indicators' values are due to direct fiscal policy actions or due to more indirect changes in the discount rate of the government's inter-temporal budget constraint. Additionally, the decomposition allows to disentangle parts of the indicator which are less dependent on very distant periods.

Related Literature. This paper relates to a broader literature that seeks to understand the sustainability of public debt based on indicators (Blanchard 1990; Escolano 2010; Debrun et al. 2019; Furman and Summers 2020; Gründler et al. 2022). Many of these indicators seek to gauge the fiscal adjustment that will bring the public debt level to a targeted debt level (in % of GDP) within a given number of periods (e.g. *S1* indicator). These indicators depend on a chosen debt target level and time horizon. Despite its practical appeal, the choice of target public debt levels and number of adjustment periods lacks theoretical rigor. Instead, the *S2* indicator's definition of long-term fiscal sustainability directly follows from the intertemporal budget constraint of the government and can be related to Blanchard (1990).

Several papers focus on discussing important properties of the S2 indicator. In a recent paper Werding (2021) describes how to calculate the S2 indicator without imposing special restrictions. Andersen (2020) and Werding et al. (2020) provide an intuition for unexpected responses of the S2 indicator with respect to changes in the interest rates. We complement their studies by showing analytically that an indepth decomposition analysis is indeed required to make the mechanisms transparent. Additionally, this paper contributes to a literature investigating the impact of the discount factor on the inter-temporal government budget constraint. In this regard, several papers provide explanations for why sovereigns pay interest rates on their debt that are empirically lower than the capital market interest rates on other assets (Reis 2021; Bayer et al. 2021; Krishnamurthy and Vissing-Jorgensen 2012). Some papers (e.g. Jiang et al. 2019) argue that the discount factor used for the government budget constraint should be the same as for pricing other risky assets. We use the findings of this literature for a new decomposition of the S2 indicator.

The remainder of the paper is organized as follows: Sect. 2 outlines the properties of the *S2* indicator and provides an in-depth decomposition of the response of the *S2* indicator to changes in the r-g differential. Additionally, we explore the reliance of the indicator on distant periods. In Sect. 3, we introduce our new decomposition of the *S2* indicator using the notion of premia. A concluding analysis is presented in Sect. 4.

2 S2 sustainability indicator

The S2 indicator is used to assess the sustainability of government debt by relating it to future primary deficits and current debt levels. Let D_t be the absolute level of government debt and $d_t = D_t/\text{GDP}_t$ be the relative debt as a percentage of GDP_t at the end of period t. Let the "growth-corrected real interest rate" be represented as $\gamma_t := \frac{1+r_t}{1+g_t} - 1 \approx r_t - g_t$ where r_t is the annual real interest rate on government debt and g_t is the annual growth rate of real GDP.¹ The primary deficit as a percentage of GDP at time t is pd_r . The debt level at the end of period t can then be described as $d_t = (1 + \gamma)d_{t-1} + pd_t$.

According to the standard government inter-temporal budget constraint (IBC), a fiscal policy is sustainable in the long run if the present value of all future primary surpluses $(-pd_{t+i}\forall i \ge 1)$ is equal to the current debt level (d_t) . Given actual projected primary deficits,

¹ In this paper we assume w.l.o.g. $\gamma_t = \gamma \ \forall t$.

 $\widehat{pd}_{t+i} \forall i \ge 1$, the IBC is not necessarily satisfied and a residual Z arises. If Z > 0, then fiscal adjustment is required to comply with the IBC.²

$$d_t = -\left(\sum_{i=1}^{\infty} \frac{1}{(1+\gamma)^i} \widehat{pd}_{t+i}\right) + Z \tag{1}$$

Sustainability indicators are a measure of the level of fiscal adjustment needs. In case of the *S2* indicator, the fiscal adjustment requirement *Z* is translated into an annuity.³ Each primary deficit \widehat{pd}_t is thus adjusted in each period by an annuity amount *k* (constant in % of the respective GDP_t) so large that the IBC is exactly fulfilled (e.g. Sustainability Report of the German Federal Ministry of Finance (BMF) 2020; European Commission 2006, 2018, 2020). It follows:

$$0 \equiv d_t + \sum_{i=1}^{\infty} \frac{1}{(1+\gamma)^i} \left(\widehat{pd}_{t+i} - \underbrace{k}_{:=S2} \right)$$
(2)

The S2 indicator at a time t corresponds exactly to the level of k. We next discuss important properties of this indicator.

2.1 Two sub-indicators

From Eq. (2) we directly get

$$S2_t := k = \underbrace{d_t \gamma}_{:=S2(d)} + \underbrace{\sum_{i=1}^{\infty} w_{t+i}(\gamma) \widehat{pd}_{t+i}}_{:=S2(pd)}$$
(3)

The $S2_t$ indicator⁴ at time *t* can be divided into two sub-indicators: one that maps the importance of already explicit debt (*S2(d)*) and one that maps the importance of future primary deficits (*S2(pd)*) for the *S2* indicator. Note that *S2(d)* interacts directly with γ and *S2(pd)* with the weights $w_{t+i}(\gamma) = \frac{\gamma}{(1+\gamma)^i}$, which depend on *i* and γ . The *S2* indicator is the exact sum of the two sub-indicators.⁵

² Since for $\gamma > 0$ there is always a risk that the debt ratio will "explode", the Ponzi condition, which prevents an explosion qua assumption, is imposed (see, inter alia, Escolano 2010). The Ponzi condition is not necessary for $\gamma < 0$, since the evolution of debt dynamics is stable in this case. For example, the debt equation converges to the value $d_{\infty} = -\frac{pd}{\gamma}$ for constant pd and $\gamma < 0$. Note that the assumption $\gamma_{t+i} < 0 \forall i$ is very strong as it ignores the fact that γ_{t+i}' might increase above 0 for a large i = I, possibly in response to a change in $pd_{t+i'}$.

³ In some publications (e.g. Peters et al. 2019), the level Z is shown as an indicator. This total debt adjustment is rarely used as an fiscal policy indicator in a decision making context.

⁴ We only make use of subscripts when necessary.

⁵ Werding et al. (2020) describe the possibility of splitting the *S2* indicator, as does the Debt Sustainability Monitor (DSM) of the European Union (European Commission, 2018, 2020). In the DSM, this is referred to as Initial Budgetary Positions (IBP) and Cost of Ageing (CoA), with primary deficits reflecting ageing costs.



Fig. 1 Periodic a, b, and cumulative weights c over time for various values of γ . Source: Own illustration

2.1.1 Explicit debt, S2(d) sub-indicator

The sub-indicator is described by $S2(d) = d_t \gamma$. The S2(d) sub-indicator shows which annuity payment k would have to be chosen to satisfy the IBC if total debt consisted of explicit debt only. To keep government debt constant at the level of year t, a S2(d) sub-indicator would always cover (growth-corrected) interest expenditures from current primary surpluses. With $d_t = 90\%$ and $pd_{t+i} = 0\%\forall i > 0$, and a growth-corrected real interest rate of $\gamma = 2\%$, according to S2(d) the IBC would be satisfied with an annual primary surplus equal to S2(d) := k = 0.02 * 0.9 = 1.8% of GDP. The following general statements regarding the S2(d) sub-indicator can be made:

- (a) The higher the initial debt level d_t , the higher the value of S2(d).
- (b) The higher the growth-corrected real interest rate γ , the higher the value of S2(d).

2.1.2 Future primary deficits, S2(pd) sub-indicator

To understand the *S2(pd)* sub-indicator, it is instructive to look first at the isolated effect of the weighting of time and then at its interaction with the projected trajectories of primary deficits.

Step I: Weighting of time

Figure 1 shows periodic and cumulative weights for the case where a *S2* indicator is calculated for year t = 2019, i.e. $S2_{t=2019}$. As in the usual present value calculation, Eq. (2) weights a single future period t + i by a factor of $\frac{1}{(1+\gamma)^i}$. Figure 1a plots this for alternative values of γ . The usual result holds: when the growth-corrected real interest rate is $\gamma > 0$, more distant time points are weighted less. This effect is stronger the larger γ is.

Alternatively, the "importance" of a period can be put in relation to all other considered future periods $i \ge 1$ $(\sum_{i=1}^{\infty} \frac{1}{(1+\gamma)^i} = \frac{1}{\gamma})$. The weighting of a period in this case



Fig.2 *S2*(*pd*) indicator with different trajectories for \widehat{pd}_{t+i} -with $\gamma = 0.5\%$ and $\gamma = 2.5\%$. Source: Own illustration

is $w_{t+i} = \gamma \frac{1}{(1+\gamma)^t}$, cf. Fig. 1b.⁶ Figuratively, this causes a vertical shift of the graphs shown in Fig. 1a by a factor γ . Figure 1c further shows the cumulative weights from t=2020 to t=2080 (i.e., $1 \le i \le 61$). When calculating the S2 indicator for the year t=2019, all future periods (≥ 1 , i. e. ≥ 2020) together always yield exactly a cumulative weight of 100%, i.e. $\sum_{i=1}^{\infty} w_{t+i}(\gamma) = 1$. In this context, distant periods have a larger weight for small γ . For example, with $\gamma = 0.5\%$ the period for t > T = 2070 receives a weight of $\sum_{i=52}^{\infty} w_{t+i}(\gamma = 0.5\%) = 1 - \sum_{i=1}^{51} w_{t+i}(0.5\%) = 77.5\%$.

Step II: Weighting of time in interaction with trajectories of primary deficits.

Now, we consider the entire S2(pd) sub-indicator. The S2(pd) sub-indicator shows which annuity payment would have to be chosen to meet the IBC if total debt consisted solely of future primary deficits. The sub-indicator is described by $S2(pd) := k = \sum_{i=1}^{\infty} w_{t+i}(\gamma) \widehat{pd}_{t+i}$

For purposes of illustration, Fig. 2 shows three examples for trajectories of primary deficits: (+) strongly monotonically increasing primary deficits, (o) comparatively moderately monotonically increasing primary deficits, and (–) monotonically decreasing primary deficits. By assumption, the values of the primary deficits for $t \in [2071, \infty)$ remain at the value as in T = 2070. The lower the value of γ , the more weight is put on distant time points.

The following general statements can be made regarding the S2(pd) sub-indicator: Ceteris paribus,

- (a) The higher the value of \widehat{pd}_{t+i} for any $i \ge 1$, the higher the value of $S2_t(pd)$.
- (b) In the case of monotonically rising primary deficits, cf. Fig. 2a,b:
- in earlier periods $(i < i^*)$, $k > \widehat{pd}_{t+i}$; in later periods $(i > i^*)$, $k < \widehat{pd}_{t+i}$.
- the higher the growth-corrected real interest rate γ , the more weight is put on earlier periods and the *lower* the value of $S2_t(pd)$.
- In earlier periods, $k < pd_{t+i}$; in later periods, $k > pd_{t+i}$.

⁶ A similar representation of the weights as in Fig. 1b can be found in Andersen (2020).

⁷ It can be seen that due to $k = k \sum_{i=1}^{\infty} w_{t+i}$ the condition $\sum_{i=1}^{\infty} w_{t+i} \left(k - p \partial_{t+i}\right) = 0$ holds. Accordingly, the annuity payment is described by the level of k that ensures that the w_{t+i} -weighted differences of annuity payments and primary deficits add up to zero.

- The higher the growth-corrected real interest rate γ , the more weight is put on earlier periods and the *higher* the value of $S2_r(pd)$.
- (c) In case of monotonically declining primary deficits, cf. Fig. 2c:
- in earlier periods $(i < i^*)$, $k > \widehat{pd}_{t+i}$; in later periods $(i > i^*)$, $k < \widehat{pd}_{t+i}$.
- the higher the growth-corrected real interest rate γ , the more weight is put on earlier periods and the *lower* the value of $S2_t(pd)$.
- (d) The more the primary deficits increase/decrease in level, i.e., the more extreme the difference between near-time and far-time primary deficits, the more $S2_t(pd)$ responds to a change in γ .
- (e) Long-term projections are more uncertain than short-term projections. Since with lower γ , more distant, and thus, more uncertain projections receive more weight, the *S*2(*pd*) indicator itself becomes more uncertain.

2.2 Two drawbacks of the S2 indicator

The *S2* indicator is associated with two drawbacks. First, sensitivity to changes in the growth-corrected interest rate is not immediately transparent. Second, projection periods in the very distant future significantly drive the indicator's value.

2.2.1 Non-unique dependence on effective interest rate γ

The S2 indicator's sensitivity in response to a change in the growth-corrected real interest rate can be illustrated through numerical examples: Fig. 3 shows S2(d), S2(pd) and S2 for the primary deficit trajectories shown in Fig. 2, for an assumed initial debt level of $d_{1=2019} = 90\%$, and for various values of γ (horizontal axis).

The sub-indicator of explicit debt S2(d) is clearly positively dependent on γ . The indicator increases from 0.5% * 90%=0.45% for γ =0.5% to 2.5% * 90%=2.25% for γ = 2.5%, i.e. the lower the effective interest rate, the lower the sustainability gap according to the S2(d) sub-indicator. This is independent of the trajectory of primary deficits and thus holds for all three cases described above. The dependence of the S2(pd) sub-indicator on γ , on the other hand, depends on the paths of primary deficits. In the case of increasing primary deficits, S2(pd) decreases when γ increases, cf. (a) and (b), i.e., the lower the effective interest rate, the higher the sustainability gap according to the S2(pd) sub-indicator. In the case of declining primary deficits, S2(pd) increases when γ increases, cf. (c), i.e., the lower the effective interest rate, the lower the sustainability gap according to the S2(pd) sub-indicator.

The overall interest rate sensitivity of the S2 indicator depends on how the interest rate sensitivities of the two sub-indicators relate to each other quantitatively. When primary deficits are projected to increase, the interest rate sensitivity of the S2 indicator is relatively small



Fig. 3 Interest rate response of the S2 indicator. S2, S2(d), S2(pd): Various trajectories for pd_{t+i} and initial debt level of $d_t = 90\%$. Source: Own illustration

because the sensitivities of S2(d) and S2(pd) with respect γ run in opposite directions and partially offset each other. The direction of change of the S2 indicator is not even easily predictable a priori in this case. In example (a), the S2 indicator follows a U-shaped path. In example (b), the indicator increases continuously because the change in S2(d) outweighs the change in S2(pd) for all γ considered. Since in the case of falling primary deficits in example (c) both S2(d) and S2(pd) increase when γ increases, the response of the S2 indicator is clear and interest rate sensitivity is high.⁸

2.2.2 Dominant importance of distant forecast periods

When the effective interest rate γ decreases, the influence of the $S2(d) = d_t \gamma$ sub-indicator decreases. At the same time, the S2(pd) sub-indicator becomes comparatively important, cf. Figure 3. Decomposing the S2(pd) sub-indicator into subperiods also provides an understanding of the contribution of the periods $2020 \le t \le 2070 = T$ and t > 2070, cf. Fig. 4. To show this, we split S2(pd) for the year 2019 into two projection sub-periods, i.e.

$$S2_{2019}(pd) = \underbrace{\sum_{i=1}^{5^{-1}} \mathbf{w}_{t+i}(\gamma)\widehat{pd}_{t+i}}_{:=S2(pd)_{\leq T}} + \underbrace{\sum_{i=52}^{\infty} \mathbf{w}_{t+i}(\gamma)\widehat{pd}_{t+i}}_{:=S2(pd)_{>T}}$$

period after 2070 receives a lot of weight.

As in Figs. 3 and 4 shows the plots of the *S*2 indicator and the *S*2(*pd*) sub-indicator for all three cases described above. In addition, the staggered bar graphs indicate how quantitatively important the sub-periods T>2070 and T ≤ 2070 are for the level of the *S*2(*pd*) sub-indicator. For instance, for $\gamma \approx r - g = 0.5\%$, the *S*2 indicator in example (a) shows a value of 3.31%. The value is composed of *S*2(*pd*)=2.86% and *S*2(*d*)=0.45%. The period after T=2070 is very important with *S*2(pd)_{>T} = $\sum_{i=52}^{\infty} w_{t+i}(\gamma) p d_{t+i} =$ 2.71%. Thus, in total, the forecast of primary deficits for the period after 2070 explains

⁸ Trajectories of declining primary deficits are not uncommon in forecasts, especially in countries with projected declining cost of ageing cf. the European Commission's 2021 Ageing Report. Please be referred to "Appendix 1" for a more detailed example of how the *S2* indicator is used in practical applications.



Fig. 4 Stronger weighting of distant periods at lower γ . Contribution to the *S2(pd)* indicator of periods before and after 2070 and *S2* as a function of γ . Source: Own illustration

 $\frac{2.71}{2.86} = 94.7\%$ of the *S2(pd)* sub-indicator and $\frac{2.71}{3.31} = 81.9\%$ of the total *S2* indicator. These and other values can be found in tabular form in "Appendix 2".

3 S2 indicator: a new decomposition

It has been shown that the S2 indicator does not respond in an easily determinable way to a change in γ . Moreover, the indicator is highly dependent on assumptions regarding the very distant future. In order to interpret the S2 indicator in a meaningful way against the background of non-transparent interest rate reactions, we amend the model which allows for a new decomposition of the S2 indicator. The decomposition is presented below.

3.1 Debt decomposition with premia

The starting point is the observation that many sovereigns pay interest rates r on their sovereign debt that are empirically lower than capital market interest rates m on other asset classes, i.e. r < m. Investors are therefore willing to forego part of the yield achievable on the capital market in order to hold government bonds and thus pay a premium (convenience yield, seignorage). This premium has been justified, among other things, by government bonds having a liquidity and safety feature (Reis 2021; Bayer et al. 2021; Krishnamurthy and Vissing-Jorgensen 2012). In this regard investors value that government bonds carry minimal credit risk and are highly liquid. Other complementary explanations emphasize the role of market power. The argumentations follow the line that with an increase in mark ups, profits increase, and hence drive a wedge between the average return to capital and the interest rate on government bonds (Fahri and Gourio 2018; Egg-ertsson et al. 2021; Ball and Mankiw 2021).

If the government had to pay the standard capital market interest rate *m*, or in growthcorrected form $\delta := \frac{(1+m)}{(1+g)} - 1$, for its government debt, the adjustment requirement to comply with the IBC would be different from the adjustment requirement *Z* shown in Eq. (1). The difference in the adjustment requirement due to the difference between δ and γ (*premium*) represents a *premium revenue*. Equation (1) can be rewritten in light of the above consideration into a part that captures the value of government debt as the sum of a fundamental value (v_t), the premium revenue, and the residual adjustment requirement Z from above:

$$d_{t} = -\left(\sum_{i=1}^{\infty} \frac{1}{(1+\gamma)^{i}} \widehat{pd}_{t+i}\right) + Z$$

$$= \underbrace{-\left(\sum_{i=1}^{\infty} \frac{1}{(1+\delta)^{i}} \widehat{pd}_{t+i}\right)}_{:=v_{t}}$$

$$+ \underbrace{(\delta-\gamma)}_{:=premium} \underbrace{\left(\sum_{i=0}^{\infty} \frac{1}{(1+\delta)^{i+1}} d_{t+i}\right)}_{:=PV(d)} + Z$$

$$\underbrace{(4)}_{:=premium}$$

The fundamental value v_t does not depend on γ . In contrast, the premium revenue, $(\delta - \gamma) PV(d)$, depends on γ^9 : Imagine an initial situation with positive government debt, in which the government pays positive interest equal to the capital market rate of return $(m = r \rightarrow \delta = \gamma)$, there is a positive amount of debt throughout $(d_{t+i} > 0 \forall i)$, and hence $PV(d) > 0 \forall t$ also holds. If, ceteris paribus, the interest rate *r* is lowered from this starting point, this has an effect on the premium revenue via two channels:

- On the one hand, the interest rate differential (*m*-*r*) increases and so does (δ γ). This is favorable for the government. For a given PV (d) > 0 this results in a positive premium revenue the government additionally generates an implicit surplus.
- On the other hand, the present value of all future government debt, PV(d), decreases because it grows more slowly due to the lower interest rate $(d_{t+1} = d_t(1 + \gamma) + pd_{t+1})$. In other words, the amount of government debt at which the interest rate differential can be exploited decreases.

The decomposition of government debt presented here can be used for the S2 indicator.

3.2 S2 indicator decomposition with premia

Based on Eq. (4) and in analogy to the derivation of the S2 indicator in Sect. 2, the S2 indicator can now be decomposed into three parts.

⁹ As long as the economy is dynamically efficient (g < m and therefore $\delta > 0$), Eq. (4) is clearly defined. If the interest rate differential in a dynamically efficient economy is sufficiently large so that not only r < m holds but even r < g < m holds, then a deficit can be financed permanently due to the premium and debt converges to a constant level.

$$S2 := k = \underbrace{d_t \delta}_{:=F2(d)} + \underbrace{\sum_{i=1}^{\infty} w_{t+i}(\delta) \widehat{pd}_{t+i}}_{:=F2(nd)} - \underbrace{\delta((\delta - \gamma) PV(d; S2))}_{:=F2(S)}$$
(5)

Here PV(d;S2) describes the present value of debt given an adjustment according to the S2 indicator. The new decomposition of the S2 indicator is similar to the original decomposition from Eq. (3). F2(d) and F2(pd) correspond to S2(d) and S2(pd) in Eq. (3), with the difference that δ is now used instead of γ , thus replacing the government bond yield r with the capital market yield m.¹⁰ The sum of F2(pd) and F2(d) gives a variant of the S2 indicator when valued at the growth-corrected capital market rate δ and is independent of γ . By analogy with the interpretation introduced in Sect. 2, the sum of F2(d) and F2(pd) describes by how much the primary deficit would have to be permanently adjusted to satisfy the IBC if the government debt were to earn interest at the capital market rate m.

This is not the case, however. In fact, the interest rate on government debt is not *m*, but *r*. The difference is reflected in *F*2(*S*). For example, if m - r > 0 (hence $\delta - \gamma > 0$) and PV(d;S2) > 0, then F2(S) > 0, i.e. the *S*2 indicator decreases due to the implicit surplus being favorable to the government. Note that PV(d;S2) depends on the path of $d_{t+i} \forall i \ge 1$ which inter alia depends on *S*2. From Sect. 2 we know that the response of *S*2 with respect to changes in γ crucially depends on the trajectory of primary deficits. As a result, the response of PV(d;S2) with respect to changes in γ hinges on the trajectory of the primary deficits as well. In other words, *F*2(*S*) being a function of PV(d;S2) inherits the opaqueness of *S*2, and therefore is less straightforward to interpret than PV(d)=PV(d;S2=0). Furthermore, note that the *S*2 indicator can increase even with a favorable interest rate differential m - r > 0 if PV(d;S2) < 0.¹¹

In Fig. 5, the new decomposition of the S2 indicator is performed using the examples of primary deficit trajectories introduced in Sect. 2. F2(d) and F2(pd) are independent of γ in all examples. Only F2(S) is dependent on γ . If $\gamma = \delta$ holds, then F2(S)=0 holds in all examples. From the non-unique plots of F(S) versus γ , it can be seen that both $\delta - \gamma$ and PV(d;S2) act on F(S). "Appendix 4" additionally presents the values in tabular form. The tables include the differential $(\delta - \gamma)$, and the present value of debt (PV(d; S2)).

3.3 Applying the new decomposition

Overall, the new decomposition can be used to make the comparison of the S2 indicator across (policy) scenarios, time, and countries more transparent. Consider the example of a comparison between country A and B, $S2_{j=A}$ and $S2_{j=B}$: In Fig. 6 below we assume the primary deficits for country A to follow the path of the (+) scenario and for country B to follow the path of the (-) scenario of the examples introduced in Sect. 2. In country A the growth adjusted real interest rate γ_A is assumed to be 1.0%, and of country B it is assumed to be $\gamma_B = 0.5\%$. Like above, both countries are assumed to start with a debt level of 90%. Furthermore, we assume $\delta_A = \delta_B = \delta = 2.5\%$.

¹⁰ If $\delta = \gamma$, then it holds that S2(d) = F2(d), S2(pd) = F2(pd) and F2(S) = 0.

¹¹ see "Appendix 3" for a more detailed discussion of PV(d;S2).



Fig. 5 New decomposition of S2 with premium. S2 decomposition with various trajectories for pd_{t+t} with $\delta = 2.5\%$ and $d_t = 90\%$. Source: Own illustration

Country	Α	в	Country	Α	в
Primary deficits	(+)	(-)	Primary deficits	(+)	(-)
$\gamma_j \approx r_j \ - g_j$	1.0%	0.5%	$\gamma_j\approx r_j\ -g_j$	1.0%	0.5%
$\delta_j \approx m_j \ - g_j$	n.a.	n.a.	$\delta_j \approx m_j \ - g_j$	2.5%	2.5%
S2 _j (d)	0.90%	0.45%	F2 _j (d)	2.25%	2.25%
S2 _j (pd)	2.31%	1.12%	F2 _j (pd)	1.12%	2.07%
$S2_j(pd)_{\leq T=2070}$	0.20%	0.52%	F2 _j (pd) _{≤T=2070}	0.13%	1.85%
$S2_{j}(pd)_{>T}=2070$	2.11%	0.60%	$F2_{j}(pd)_{>T}=2070$	0.99%	0.22%
			F2 _j = F2 _j (d) + F2 _j (pd)	3.37%	4.32%
			F2j(S)	0.16%	2.75%
			$\delta - \gamma$	1.5%	2.0%
			PV(d; S2)	422%	5496%
$S2_j = S2_j(d) + S2_j(pd)$	3.21%	1.57%	S2j	3.21%	1.57%

Fig. 6 *S2*, *S2(pd)*, *S2(d)*, *F(d)*, *F(pd)*, and *F(S)* for two countries starting with debt level 90%. Source: Own illustration

On the left-hand table one can see that $S2_A = S2_A(d) + S2_A(pd) > S2_B$. On the righthand side table we observe that $F2_A(d) + F2_A(pd) < F2_B(d) + F2_B(pd)$. Discounting with δ instead of γ leads to $F2_A(d) = F2_B(d)$ because both countries start with the same debt level. Additionally, the comparison of the sub-indicators reflecting the effect of the primary deficits is revealing. It holds that $S2_A(pd) > S2_B(pd)$ but $F2_A(pd) < F2_B(pd)$. The reason is that distant periods are discounted more heavily given the assumption of $\delta > \gamma$. This leads to F2(pd) < (>) S2(pd) if primary deficits are projected to be increasing (decreasing). Hence, with $\delta > \gamma$ the decomposition allows to shift more weight of the analysis to the part of the sustainability gap that is less dependent on the distant future. However, as discussed in sub-Sect. 3.2 the part capturing PV(d;S2) still hinges fundamentally on the assumptions regarding the very distant future.

Moreover, the effect of the implicit return earned by the government due to the premia is very different comparing both countries, $F2_A(S) < F2_B(S)$. The reason is a combination of different $(\delta - \gamma)$ differentials and present values of debt PV(d;S2). In other words, the sum $F2_j(d) + F2_j(pd)$ allows to compare sustainability gaps as if there were no additional effects through premia in a ceteris paribus framework.

3.4 Discussion of the new decomposition

Comparisons across countries, time, and/or (policy) scenarios hinge on choosing appropriate values for δ . How should δ be chosen? We motivate the choice of δ by the fact that m > r (Reis 2021). As a result, for a given country j and time i it holds that $\delta_{j,i} \approx m_{j,i} - g_{j,i} > r_{j,i} - g_{j,i} \approx \gamma_{j,i}$. However, in comparisons across time for a given country j (and similarly across countries given a point in time i) it might hold that $\delta_{j,i} \neq \delta_{j,i+1}$ even if $m_{j,i} = m_j \forall i$, simply because $g_{j,i} \neq g_{j,i+1}$.¹²

If one wants to separate the effect of different (or changing) growth-adjusted real interest rates $\gamma_{j,i} = r_{j,i} - g_{j,i}$ across countries (or over time), $\delta_{i,j}$ should be chosen to be the same. In fact, our decomposition approach does not rely on a very specific value for δ other than being larger than γ . To put it differently, the observed relationship m > r rather motivates the assumption $\delta > \gamma$ than it does necessarily dictate a specific value for δ . Instead, in analyses the level of δ can be chosen taking additional considerations, such as uncertainty regarding the future or risk aversion, into account.

Differences in sustainability gaps across countries and time may be due to fiscal policy decisions, differences in projection assumptions, the economic environment and/or different discount rates. Our new decomposition facilitates explaining differences in sustainability gaps caused by different discount rates. In this type of analysis is imposed a ceteris paribus assumption as it abstracts from the fact that primary deficits and the level of γ (or δ) may interact in a direct way.

Relatedly, a question to ask is: What drives premia, and to what extent should these premia be considered as exogenously given or endogenous? More specifically, to what extent do government spending behavior and monetary policy decisions affect m and r? Reis (2021) explains premia as arising from liquidity and safety, and studies how fiscal (redistributive) policies, inflation and financial repression have an influence on premia. Additionally, governments may face a trade-off in exploiting the premium by increasing public debt and sustaining the premium by not increasing public debt levels too much. Bayer et al. (2021) have made some progress in that direction of research. Future research should expand the analysis to shed light on the effect of fiscal and monetary policies on premia within a monetary union. For instance, under which circumstances does a centralized monetary authority affect premia of member states differently. Similarly, do fiscal policies of a country have spill-over effects on premia of other countries? Answering these questions remains for future research.

We have shown that the S2 indicator can be decomposed in various ways (e.g. projection intervals, premia) to enrich the analysis of the long-term sustainability of public debt. The broader result is that the decomposition suggested in our paper can be extended in many ways. For instance, one could decompose the present value of debt, $PV(d;S2) = \sum_{i=0}^{\infty} \frac{1}{(1+\delta)^{i+1}} d_{t+i}$, into a part that is not affected by changes in S2, PV(d;S2 = 0), and a residual part where PV(d;R) = PV(d;S2) - PV(d;S2 = 0). This could help to disentangle further the opaque impact of the S2 indicator on the premium

¹² In fact, $\delta_{j,i}$ might even fluctuate more than $\gamma_{j,i}$. To see this take a given country *j*: Assume $m_i = m$ for each point in time *i*, however r_i fluctuates over time. For instance, if then g_i co-moves with r_i in such a way that $\gamma_i = \frac{1+r_i}{1+\epsilon_i} - 1 = \gamma \forall i$ holds, δ_i displays a higher volatility than γ_i .

revenue. Additional decomposition along projection intervals are possible. The exact choice of the decompositions should be guided by the interest of the researcher or analyst.

4 Conclusion

The S2 indicator (sustainability gap) is an established measure for assessing the *long-term* sustainability of public finances and shows the level of fiscal policy action required, taking into account trajectories of future primary deficits. The factors influencing the S2 indicator are diverse, and in some cases may be irritating for non-specialists.

First, a change in the discount rate γ acts through two channels, sometimes with opposite effects. These can be illustrated by representing the S2 indicator as the sum of two annuities—one annuity translating the baseline debt level (S2(d)) and a second annuity translating all primary deficits expected in the future (S2(pd)). Clearly, the first annuity (S2(d)) increases (decreases) as the discount rate γ increases (decreases). For the second annuity (S2(pd)) the response depends on the trajectory of primary deficits since the discount rate γ affects the weighting of primary deficits over time. When the discount rate is increasing (decreasing), primary deficits that are close (distant) in time are weighted more heavily. For instance, this means that when the discount rate is low, the indicator gives strong weight to projected primary deficits in the distant future. The sum of the two annuities (S2(d)+S2(pd)) responds to a change in interest rates in a clearly determinable direction only when deficits are projected to decline. When primary deficits are projected to increase over time, the direction of change in the S2 indicator is not easily determinable, since the interest rate responsiveness of S2(d) and S2(pd) partly cancel each other out.

Second, for low values of r-g, the influence of highly uncertain projections for distant periods on the S2 indicators is very large. This feature may be desirable as long-term analyses are meant to uncover long-term risks. Yet, long-term projections are more uncertain than short-term projections. Hence, the S2 indicator itself becomes more uncertain. For example, projection values for periods after 2070 explain up to 80% of the S2 indicator in some scenarios with r-g in a range of 0.5%. Importantly, since projected primary deficits are to a large extent driven by projected future policies, one important driver of uncertainty is the credibility of the implementation of these policies.

We propose a new decomposition of the *S2* indicator to capture these two properties. The decomposition makes use of the observation that many sovereigns pay interest rates on their sovereign debt that are empirically lower than capital market interest rates on other asset classes. In other words, our decomposition explicitly takes into account the notion of premia (Reis 2021). These premia may differ across countries, (policy) scenarios, and over time. Therefore, explicitly introducing premia makes a comparison of sustainability gaps more transparent. Additionally, our new decomposition mitigates the reliance of highly uncertain projections for the distant future. By imposing the assumption that $\delta > \gamma$, the decomposition allows to perform the sustainability analysis by attaching more weight to less distant periods, and thus, is less prone to projection uncertainty.

Sustainability gaps indicate a need for action but do not constitute a recommendation for a particular fiscal policy. The assumed constant adjustment level of the primary deficit S2 := k is the technical result of the assumed annuity translation under assumptions and does not specify the adjustment path of the primary deficit that should be targeted with respect to welfare maximization. This is evident, among other things, from the property of non-uniqueness in changes to the indicator, which can show the same indicator value for different combinations of initial debt, trajectories of discount rates and primary deficits (see Sect. 2). For instance, in most cases, the policy recommendations for a particular value of the *S2* indicator would be different if primary deficits were trending upwards than if primary deficits were trending downwards.

Overall, the *S2* indicator remains an important measure for estimating the need for action to ensure fiscal sustainability. However, in contrast to purely present-oriented indicators (e.g. debt-to-GDP ratio), the notion that there can be only one indicator value for forward-looking indicators needs to be overcome. The *S2* indicator (and its decompositions) is an indicator alongside debt-to-GDP ratios, primary deficits and refinancing costs. In the case of low discount rates, analysing *S1* indicators that refrain from including infinite time periods, may be a complementary tool to *S2* indicators.¹³ However, the orientation on *S1* indicators relinquishes the IBC for a target value for debt levels and adjustment periods which are rather arbitrary. Under no circumstances should *S1/S2* indicators be the sole and binding benchmark for political action.

Appendix 1

Sustainability gaps in reports analysing ageing-related costs

The sustainability gap is used in standard sustainability analyses primarily to assess demographic risks to the long-term development of public finances. In sustainability analyses, it is common practice to assume that revenues and non-demographic expenditures increase at the same rate as GDP growth. For demography-dependent expenditures, on the other hand, projections are made regarding their expected rate of growth. Together with the debt level and the fiscal balance of the baseline situation, demography-dependent expenditures shape the fiscal action required in the sustainability analyses.

In practical application, however, no meaningful projections can be made for annual \widehat{pd}_{t+i} in very distant periods. In practice, therefore, it is assumed that a new steady state is reached starting from a chosen period *T* (here T = 2070), i.e. after T=t+I periods, and that the variables no longer change, i.e., the primary deficit as a % of GDP and the growth-corrected real interest rate remain unchanged for $t \ge T$. Therefore, the practical formula of the *S2* indicator is as follows:

¹³ In "Appendix 5" we briefly discuss the *S1* indicator in more detail.

$$S2_{t} := k = \underbrace{d_{t}\gamma}_{:=S2(d)} + \underbrace{\sum_{i=1}^{l} w_{t+i}(\gamma)\widehat{pd}_{t+i}}_{:=S2(pd) \leq T} + \underbrace{\sum_{i=l+1}^{\infty} w_{t+i}\widehat{pd}_{T}}_{:=S2(pd) > T}$$
$$= \underbrace{d_{t}\gamma}_{:=S2(d)} + \underbrace{\left(\sum_{i=1}^{l} w_{t+i}(\gamma)\widehat{pd}_{t+i}\right)}_{:=S2(pd)} + \frac{\widehat{pd}_{t+l}}{(1+\gamma)^{l}}$$

Appendix 2

Decomposition of the S2 indicator

See Fig. 7.

Case: Primary deficits (+)					
$\gamma \approx r - g$	0.5%	1.0%	1.5%	2.0%	2.5%
S2(d)	0.45%	0.90%	1.35%	1.80%	2.25%
S2(pd)	2.86%	2.31%	1.85%	1.46%	1.12%
S2(pd) _{≤T=2070}	0.14%	0.20%	0.21%	0.18%	0.13%
$S2(pd)_{>T=2070}$	2.71%	2.11%	1.64%	1.27%	0.99%
\$2	3.31%	3.21%	3.20%	3.26%	3.37%
Case: Primary deficits (o)					
$\gamma \approx r - g$	0.5%	1.0%	1.5%	2.0%	2.5%
S2(d)	0.45%	0.90%	1.35%	1.80%	2.25%
S2(pd)	1.02%	0.71%	0.45%	0.23%	0.03%
S2(pd) _{≤T=2070}	-0.05%	-0.11%	-0.19%	-0.27%	-0.36%
S2(pd) _{>T=2070}	1.06%	0.82%	0.64%	0.50%	0.39%
S2	1.47%	1.61%	1.80%	2.03%	2.28%
Case: Primary deficits (_)					
$v \approx r - q$	0.5%	1.0%	1 5%	2.0%	2 5%
60(J)	0.0 /6	0.00%	4.250/	4.000/	2.0 /0
52(a)	0.45%	0.90%	1.35%	1.80%	2.25%
S2(pd)	1.12%	1.42%	1.67%	1.89%	2.08%

0.47%

2.32%

0.37%

3.02%

0.29%

3.69%

0.23%

4.33%

S2

Fig. 7 S2, S2(pd), S2(d). Source: Own illustration

S2(pd)_{>T=2070}

0.60%

1.57%

Appendix 3

PV(d; S2)

Here we show that the response of the present value of debt, PV(d; S2), with respect to a change in the growth adjusted real interest rate γ hinges on the path of primary deficits. More specifically, PV(d; S2) inherits some of the opaque properties of the S2 indicator. Additionally, we show that PV(d; S2) can be negative. To illustrate this. we proceed in several steps: As a first step, we discuss the evolution of debt levels independent of an adjustment according to the S2 indicator. As a second step we, show how debt levels evolve taking into account an adjustment according to the S2 indicator. As a third, step we illustrate the responsiveness of the long-term debt level (in % of GDP), d_{∞} , with respect to γ . As a final step, we discuss how these findings affect the properties of PV(d;S2).

Evolution of debt levels

From the equation describing the government debt dynamics without imposing the no-Ponzi game condition follows:

$$\lim_{i \to \infty} \frac{d_{t+i}}{(1+\gamma)^i} = d_t + \sum_{i=1}^{\infty} \frac{1}{(1+\gamma)^i} p d_{t+i}$$

The no-Ponzi game condition (and the transversality condition, TVC) result in:

$$\lim_{i \to \infty} \frac{d_{t+i}}{\left(1+\gamma\right)^i} = 0$$

This condition states that the debt ratio must not grow permanently at a rate exceeding the growth-corrected real interest rate. This would be the case if both debt and interest were permanently financed by new debt. The government would borrow more and more to finance both its primary deficits and the interest on existing debt. This "Ponzi scheme" would ultimately not work out. The no-Ponzi condition does not say that the debt ratio must converge to zero, so $\lim_{i\to\infty} d_{t+i} = 0$. Instead, according to the no-Ponzi condition, the debt level d_{t+i} can converge to a finite d_{∞} .

To see this, take the following example: The evolution of debt is described by $d_{t+1} = (1 + \gamma)d_t + \widehat{pd}_{t+1}$. From a point t + I, let the primary deficits be constant, i.e., $\widehat{pd}_{t+i} = \widehat{pd}_{t+i} \forall i \ge I$. For $\gamma > 0$, the system $d_{t+i+1} = (1 + \gamma)d_{t+i} + \widehat{pd}_{t+i+1}$ is then known to be latently unstable. The solution of this equation is

$$d_{t+i} = \left(d_{t+1} - \frac{-\widehat{pd}_{t+1}}{\gamma}\right)(1+\gamma)^{i} - \frac{-\widehat{pd}_{t+1}}{\gamma}$$

There are three possibilities for the evolution of d_{t+i} in the long-run:

(a)
$$d_{t+I} > \frac{-\widehat{pd}_{t+I}}{\gamma}$$
 : $\lim_{i \to \infty} d_{t+i} = \infty$

(b)
$$d_{t+I} < \frac{-\widehat{pd}_{t+I}}{\gamma}$$
 : $\lim_{i \to \infty} d_{t+i} = -\infty$

(c)
$$d_{t+I} = \frac{-pd_{t+I}}{\gamma}$$
 : $d_{t+i} = d_{t+I} \forall i \ge I$

Thus, even with primary surpluses, $-pd_{t+I} > 0$, an "explosion" of the debt ratio can occur. For sustainable government finances, the primary surpluses necessary for debt sustainability depend on the level of the debt ratio at time t + I. In case (a), the debt explodes, i.e. a classic Ponzi game is played. In case (b), savings are even infinite in the long run, which cannot be optimal in welfare terms, since consumption opportunities would be foregone (transversality condition, TVC). In case (c) the debt ratio converges to a constant level and only in this case the Ponzi condition and the TVC is fulfilled.

Adjusting debt levels according to the S2 indicator

The S2 indicator is now to be chosen exactly such that case (c) occurs, i.e. such that $d_{t+1}(S2) = \frac{-(\widehat{pd}_{t+I}-S2)}{\gamma}$ holds. Here it is helpful to observe that $\frac{\partial d_{t+1}(S2)}{\partial S2} < 0$ and $\frac{\partial [-(\widehat{pd}_{t+I}-S2)]}{\partial S2} > 0$. Therefore, for relevant values of \widehat{pd}_{t+i} and γ and applying the intermediate value theorem, we obtain exactly one S2 for which $d_T(S2) = \frac{-(\widehat{pd}_{t+I}-S2)}{\gamma}$ is satisfied. Hence, $S2 = \gamma d_{t+1} + \widehat{pd}_{t+I}$. Assuming constancy of the parameters from time t + I on, this also means that after calculating the S2 indicator, we can say exactly to which debt ratio, d_{∞} , the series converges. It is instructive to consider the evolution of debt levels for the examples of primary deficits introduced in the main text, and an initial debt level of $d_t = 90\%$, c.f. Fig. 8.

One sees that in examples (a) and (b) it holds that $d_{\infty} < d_t$ and that the derivative w.r.t. to γ is larger than zero, $\frac{\partial d_{\infty}}{\partial \gamma} > 0$. In example (c) it holds that $d_{\infty} > d_t$ and that $\frac{\partial d_{\infty}}{\partial \gamma} < 0$. This has an immediate effect on PV(d; S2) as well. To scrutinise these properties analytically, we provide w.l.o.g. an example with I = 1.

Properties of d_{∞} (for l = 1)

From $d_{t+1} = (1+\gamma)d_t + (\widehat{pd}_{t+1} - S2) = \frac{-(\widehat{pd}_{t+2} - S2)}{\gamma}$ we get $S2 = \gamma d_t + \frac{\gamma}{1+\gamma}\widehat{pd}_{t+1} + \frac{1}{1+\gamma}\widehat{pd}_{t+2}$. Plugging this S2 into $d_{\infty} = \frac{-(\widehat{pd}_{t+2} - S2)}{\gamma}$ yields:

$$d_{\infty} = d_{t+1} = d_t + \frac{1}{1+\gamma} \left(\widehat{pd}_{t+1} - \widehat{pd}_{t+2} \right)$$

It follows that $d_{\infty} < d_t$ if $(\widehat{pd}_{t+1} - \widehat{pd}_{t+2}) < 0$, i.e. the long-term debt ratio is smaller than the initial debt ratio for increasing primary deficits. The response of d_{∞} with respect to a change in γ can be described as its derivative:



Fig. 8 Long-term debt levels for various values of γ , with $d_{t=2019} = 90\%$. Source: Own illustration

$$\frac{\partial d_{\infty}}{\partial \gamma} = -\frac{1}{\left(1+\gamma\right)^2} \left(\widehat{pd}_{t+1} - \widehat{pd}_{t+2} \right)$$

For example, with increasing γ , long-term debt levels (in % of GDP) increase when primary deficits are increasing, i.e. $\frac{\partial d_{\infty}}{\partial \gamma} > 0$ if $(\widehat{pd}_{t+1} - \widehat{pd}_{t+2}) < 0$.

Properties of PV(d;S2) (for I = 1)

The present value of debt can be described as follows:

$$PV(d;S2) = \frac{1}{1+\delta} \left(d_t + \frac{d_{\infty}}{\delta} \right)$$

PV(d; S2) is not contrained to be positive, e.g. when pd_{t+2} is sufficiently large. Furthermore, we describe the response of the present value of debt with respect to a change in the growth-adjusted interest rate as follows:

$$\frac{\partial PV(d;S2)}{\partial \gamma} = \left(\frac{1}{1+\delta}\right) \frac{1}{\delta} \frac{\partial d_{\infty}}{\partial \gamma}$$

For instance, for increasing primary deficits $(\widehat{pd}_{t+1} - \widehat{pd}_{t+2} < 0)$ the present value of debt PV(d; S2) is increasing if γ is increasing. In conclusion, the sign of $\frac{\partial PV(d;S2)}{\partial \gamma}$ depends on the path of primary deficits. Note that if primary deficits are monotonically increasing or decreasing with at least one $\widehat{pd}_{t+i} \neq \widehat{pd}_{t+i+1}$, the results hold true for I > 2 as well. Hence, PV(d; S2) inherits the opaque properties of the S2-indicator.

Appendix 4

Decomposition with premium

See Fig. 9.

2.5%

2 254

2.0%

2 250

rz(u)	2.23%	2.23%	2.25%	2.25%	2.25%
F2(pd)	1.12%	1.12%	1.12%	1.12%	1.12%
F2(pd) _{≤T=2070}	0.13%	0.13%	0.13%	0.13%	0.13%
F2(pd) _{>T=2070}	0.99%	0.99%	0.99%	0.99%	0.99%
F2(S)	0.06%	0.16%	0.17%	0.11%	0.00%
$\delta - \gamma$	2.0%	1.5%	1.0%	0.5%	0.0%
PV(d; S2)	128%	422%	680%	906%	1106%
S2	3.31%	3.21%	3.20%	3.26%	3.37%
Case: Primary deficits (o)				
$\gamma \approx r - g$	0.5%	1.0%	1.5%	2.0%	2.5%
F2(d)	2.25%	2.25%	2.25%	2.25%	2.25%
F2(pd)	0.03%	0.03%	0.03%	0.03%	0.03%
F2(pd) _{≤T=2070}	-0.36%	-0.36%	-0.36%	-0.36%	-0.36%
F2(pd) _{>T=2070}	0.39%	0.39%	0.39%	0.39%	0.39%
F2(S)	0.82%	0.67%	0.48%	0.26%	0.00%
$\delta - \gamma$	2.0%	1.5%	1.0%	0.5%	0.0%
PV(d; S2)	1633%	1786%	1922%	2042%	2148%
S2	1.47%	1.61%	1.80%	2.03%	2.28%
Case: Primary deficits (-)				
$\gamma \approx r - g$	0.5%	1.0%	1.5%	2.0%	2.5%
F2(d)	2.25%	2.25%	2.25%	2.25%	2.25%
F2(pd)	2.07%	2.07%	2.07%	2.07%	2.07%
F2(pd) _{≤T=2070}	1.85%	1.85%	1.85%	1.85%	1.85%
F2(pd) _{>T=2070}	0.22%	0.22%	0.22%	0.22%	0.22%
F2(S)	2.75%	2.00%	1.30%	0.63%	0.00%

1.0%

2 250

1.5%

2 254

Case: Primary deficits (+)

0.5%

2.0%

5496%

1.57%

2 254

Fig. 9	S2,	F2(pd),	F2(d),	F(S).	Source:	Own	illustration;	Note:	In	caclulations	for	the	values	when
$\gamma = 2.5$	5%, v	ve chose	a value	for δ t	hat was s	slightly	y larger							

1.5%

5339%

2.32%

1.0%

5198%

3.02%

0.5%

5059%

3.69%

0.0%

4862%

4.32%

Appendix 5

 $\delta - \gamma$

S2

PV(d; S2)

S1 indicator

In this appendix we discuss some properties of the *S1* indicator that are different from the *S2* indicator. Additionally, we show that our new decomposition can be applied to the *S1* as well.

The S1 indicator is different from the S2 indicator: it does not have to satisfy the IBC defined for $T \rightarrow \infty$. Instead, the S1 indicator indicates the size of the fiscal adjustment required (via annuity adjustments) to achieve a target debt level in I years, \overline{d}_{t+I} , i.e. by a point in time t + I. The consequence of this simplification is that S1 indicators relinquish the IBC for a target value

 $\gamma \approx r - g$

E2(-1)

for debt levels and adjustment periods which are rather arbitrary. Mechanically however, the *S1* indicator shares many properties of the *S2* indicator. Formally, the *S1* indicator is described as

$$S1 := k = \underbrace{(d_t - \frac{\overline{d}_{t+I}}{(1+\gamma)^I})\overline{\gamma}}_{:=S1(d)} + \underbrace{\sum_{i=1}^{I} \overline{w}_{t+i}(\gamma)\widehat{pd}_{t+i}}_{:=S1(pd)}$$

where $\overline{\gamma}$ is defined as $\overline{\gamma} = \frac{1}{\sum_{i=1}^{l} \frac{1}{(1+\gamma)^{i}}}$ and $\overline{w}_{t+i}(\gamma) = \overline{\gamma} \frac{1}{(1+\gamma)^{i}}$. It should be noted that, unlike the *S2* indicator, no special restrictions need to be placed on the interest rate. However, analogously to the *S2* indicator the development of *S1(pd)* does not have an unambiguous response to changes in the interest rate-growth differential. The dependence on distant projection periods can be gauged by the distance of the target

Amending the *S1* indicator to allow for a decomposition analogous to the decomposition for the *S2* indicator yields the following result:

$$S1 := \underbrace{\left[d_t - \frac{d_{t+I}}{(1+\delta)^I}\right]}_{:=F1(d)} \bar{\delta} + \underbrace{\sum_{i=1}^I \bar{w}_{t+i}(\delta) p d_{t+i}}_{:=F1(pd)} - \underbrace{\bar{\delta}[(\delta - \gamma) PV(d;S1)]}_{:=F1(S)}$$

where $\overline{\delta} = \frac{1}{\sum_{i=1}^{l} \frac{1}{(1+\delta)^{i}}}$, $\overline{w}_{t+i}(\delta) = \overline{\delta} \frac{1}{(1+\delta)^{i}}$, and $PV(d;S1) = \sum_{i=1}^{l} \frac{1}{(1+\delta)^{i}} d_{t+i-1}$.

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Declarations

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