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How to Detect Network Dependence in Latent Factor Models? A Bias-Corrected CD Test

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Abstract

In a recent paper Juodis and Reese (2022) (JR) show that the application of the CD test proposed by Pesaran (2004) to residuals from panels with latent factors results in over-rejection. They propose a randomized test statistic to correct for over-rejection, and add a screening component to achieve power. This paper considers the same problem but from a different perspective, and shows that the standard CD test remains valid if the latent factors are weak in the sense the strength is less than half. In the case where latent factors are strong, we propose a bias-corrected version, CD*, which is shown to be asymptotically standard normal under the null of error cross-sectional independence and have power against network type alternatives. This result is shown to hold for pure latent factor models as well as for panel regression models with latent factors. The case where the errors are serially correlated is also considered. Small sample properties of the CD* test are investigated by Monte Carlo experiments and are shown to have the correct size for strong and weak factors as well as for Gaussian and non-Gaussian errors. In contrast, it is found that JR.s test tends to over-reject in the case of panels with non-Gaussian errors, and has low power against spatial network alternatives. In an empirical application, using the CD* test, it is shown that there remains spatial error dependence in a panel data model for real house price changes across 377 Metropolitan Statistical Areas in the U.S., even after the effects of latent factors are filtered out.

JEL-Codes: C180, C230, C550.

Keywords: latent factor models, strong and weak factors, error cross-sectional independence, spatial and network alternatives, size and power.

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1 Introduction

It is now quite standard to use latent multi-factor models to characterize and explain cross-sectional dependence in panels when the cross section dimension (n) and the time series dimension (T) are both large. However, due to uncertainty regarding the nature of error cross-sectional dependence, it is arguable whether error cross-sectional dependence is fully accounted for by latent factors. Some of the factors could be semi-strong, and the errors might have spatial or network features that are not necessarily captured by common factors alone. Chudik et al. (2011) provide an early discussion of the different sources of cross-sectional dependence. Diagnostic tests of error cross-sectional independence in panels are required to safeguard against estimation bias and unreliable inference. See, for example, Bai (2003), Phillips and Sul (2003), Bai and Ng (2006), Pesaran (2006), Bai (2009), and Pesaran and Tosetti (2011). Such tests are also helpful to researchers interested in network or spatial dependence once the influence of common factors are filtered out, for forecasting and policy analyses. For example, how to detect spillover effects across regions or markets. See Bailey et al. (2016), Shi and Lee (2017), and Bai and Li (2021), amongst others.

One standard test for error cross-sectional independence is the CD test proposed by Pesaran (2004, 2021), which has been further developed later. For example, Hsiao et al. (2012) apply the CD test to panel data model with limited dependent variable, while Pesaran (2015) uses it to test weak error cross-sectional dependence in large panels. In a recent paper Juodis and Reese (2022) (JR) show that the application of the CD test to the residuals from panels with latent factors is invalid and can result in over-rejection of the null of error cross-sectional independence.¹ They propose a randomized CD test statistic as a solution. Their proposed test is constructed in two steps. First, they multiply the residuals from panel regressions with independent randomized weights to obtain their CD_W statistic, which will have a zero mean by construction. In this way they avoid the over-rejection problem of the CD test, but by the very nature of the randomization process they recognize that the CD_W test will lack power. To overcome the problem of lack of power, JR modify the CD_W test statistic by adding to it a screening component proposed by Fan et al. (2015) which is expected to tend to zero with probability approaching one under the null hypothesis, but to diverge at a reasonably fast rate under the alternative. This further modification of CD_W test is denoted by CD_{W+} . Accordingly, it is presumed that the CD_{W+} test can overcome both over-rejection and the low power problems. However, JR do not provide a formal proof establishing conditions under which the screening component tends to zero under the null and diverges sufficiently fast under alternatives, including spatial or network dependence type alternatives. Also, our Monte Carlo simulations show that the CD_{W+} test tends to over-reject when the errors are non-Gaussian and $n \gg T$, and seems to lack power under spatial network alternatives, which is likely to be particularly important in empirical applications.

In this paper we show that the standard CD test is in fact valid for testing error cross-sectional independence in panel data models with weak latent factors. However, when the latent factors are semi-strong or strong the use of CD test will result in over-rejection and will no longer be valid, extending JR's results to panels with semi-strong latent factors. In short, whilst

¹In the empirical finance literature Gagliardini et al. (2019) propose a diagnostic criterion to check if the errors from a (strong) factor model are weakly correlated or contain missing strong factor(s). These authors do not propose a test of cross-sectional error dependence but focus on the detection of potentially omitted strong factors in asset pricing models.

the CD_{W+} is a useful and welcome addition to testing for error cross-sectional independence, it would be interesting to develop a modified version of the test that simultaneously deals with the over-rejection problem and does not compromise power for a general class of alternatives. To that end, firstly we study testing for error cross-sectional independence in a pure latent factor model, and derive an explicit expression for the bias of the CD test statistic in terms of factor loadings and error variances. We then propose a bias-corrected version of the CD test statistic, denoted by CD^* , which is shown to have $N(0, 1)$ asymptotic distribution under the null hypothesis irrespective of whether the latent factors are weak or strong. When the latent factors are weak the correction tends to zero, CD and CD^* will be asymptotically equivalent. However, $CD - CD^*$ diverges if at least one of the underlying latent factors is (semi) strong. We show that under the null of cross-sectional independence, CD^* converges to a standard normal distribution when n and T tend to infinity so long as $n/T \rightarrow \kappa$, where $0 < \kappa < \infty$, and a test based on CD^* will have the correct size asymptotically. In addition, it is shown that the CD^* test has power against spatial and network type alternative. In particular, we are the first to give a formal derivation of power function for the CD tests against general spatial and network alternatives, which can be applied equally to CD test for panel data models without latent factors and therefore supplement earlier researches on CD test².

We then consider the application of the CD^* to test error cross-sectional independence in the case of panel regression models with latent factors, discussed in Pesaran (2006). It is shown that the asymptotic properties of CD^* in the case of pure latent factor models also carry over to panel regression models with latent factors. We also investigate the application of the CD^* to panel data models with serially correlated errors, and consider the method proposed by Baltagi et al. (2016) as well as using an autoregressive distributed lag (ARDL) representation which transforms the model with the serially correlated errors to one without error serial correlation.

The finite sample performance of the CD^* test is investigated by Monte Carlo simulations in the case of pure latent factor models, panel regression models with latent factors with and without error serial correlation. It is found that the CD^* test avoids the over-rejection problem under the null and diverges fast under spatial network alternatives, and has desirable small sample properties regardless of whether the errors are Gaussian or not, under different combinations of n and T . We also find that both adjustments for dealing with error serial correlation considered in the paper give desirable small sample properties. Finally, as compared to JR's CD_{W+} test, the proposed bias-corrected CD test is better in controlling the size of the test and has much better power properties against spatial (or network) alternatives.

The use of CD^* is illustrated by an empirical application in modeling real house price changes in the U.S. Because it is evident that real house price changes are driven by macroeconomic trends which can be modeled by latent factors, it is necessary to filter out these factors before testing for spillover effect. By applying CD^* to real house price changes in the U.S. we are able to show significant existence of weak cross-sectional dependence in addition to latent factors.

The rest of the paper is organized as follows. Section 2 sets out the latent factor model and its assumptions. Section 3 introduces the estimation of latent factors and the CD test. The bias corrected test, CD^* , is introduced in Section 4 and its asymptotic distributions are derived under the null and the alternative hypotheses. The extension to more general panel regression models with observed covariates as well as latent factors are discussed in Section 5.

²For instance, Pesaran (2004), which is the unpublished version of Pesaran (2021), also discusses the power of CD test against spatial dependence in its Section 8.2, but the power function of the test there is conjectured for a specific connection matrix and a sample with n fixed and $T \rightarrow \infty$.

Adjustments to the CD^* test for panels with serially correlated errors are discussed in Section 6. Using Monte Carlo techniques, the small sample properties of CD , CD^* and, CD_{W+} tests are discussed in Section 7. An empirical illustration is provided in Section 8. The proofs of the main theorems and the related lemmas are given in the online supplement.

2 The latent factor model

To simplify the exposition and to highlight the main issue of concern, namely the presence of *latent* (unobserved) factors in the panel regression model, initially we focus on the approximate factor model, due Chamberlain and Rothschild (1983), and assume that for each unit $i = 1, 2, \dots, n$,

$$y_{it} = \gamma_i' \mathbf{f}_t + u_{it}, \text{ for } t = 1, 2, \dots, T, \quad (1)$$

where

$$u_{it}/\sigma_i = \varepsilon_{it}(\lambda_T) = \varepsilon_{it} + \lambda_T \sum_{j=1}^n w_{ij} \varepsilon_{jt}, \quad (2)$$

$\sup_i \sigma_i^2 < C < \infty$ and $\inf_i \sigma_i^2 > c > 0$, $\{w_{ij} : j = 1, 2, \dots, n\}$ represent the strengths of connections of unit i with the rest of units, $\mathbf{f}_t = (f_{1t}, f_{2t}, \dots, f_{m_0t})'$ is an $m_0 \times 1$ vector of latent factors with m_0 fixed, and $\gamma_i = (\gamma_{i1}, \gamma_{i2}, \dots, \gamma_{im_0})'$ is the vector of associated factor loadings.

We make the following assumptions that are mostly standard in the analysis of latent factor models.

Assumption 1 (a) $\mathbf{f}_t$ is a covariance-stationary process with zero means and the covariance matrix, $E(\mathbf{f}_t \mathbf{f}_t') = \Sigma_{ff} > \mathbf{0}$. (b) $T^{-1} \sum_{t=1}^T [\|\mathbf{f}_t\|^j - E(\|\mathbf{f}_t\|^j)] \rightarrow_p 0$, for $j = 3, 4$, as $T \rightarrow \infty$. (c) There exists T_0 such that for all $T > T_0$, $T^{-1} \sum_{t=1}^T \mathbf{f}_t \mathbf{f}_t' = T^{-1} \mathbf{F}' \mathbf{F} = \Sigma_{T,ff} > \mathbf{0}$, and $\Sigma_{T,ff} \rightarrow_p E(T^{-1} \mathbf{F}' \mathbf{F}) = \Sigma_{ff} > \mathbf{0}$, where $\mathbf{F} = (\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_T)'$. (d) There exist constants r_1 , C_0 and $C_1 > 0$ such that

$$\sup_{j,t} \Pr(|f_{jt}| > a) \leq C_0 \exp(-C_1 a^{r_1}) \quad (3)$$

all $a > 0$.

Assumption 2 (a) $\varepsilon_{it} \sim IID(0, 1)$ for all i and t , and there exist constants r_2 , C_2 and $C_3 > 0$ such that

$$\sup_{i,t} \Pr(|\varepsilon_{it}| > a) \leq C_2 \exp(-C_3 a^{r_2}) \quad (4)$$

for all $a > 0$. (b) $\mu_{\max}(\mathbf{V}_{\varepsilon T}) = O_p(n/T)$, where $\mathbf{V}_{\varepsilon T} = T^{-1} \sum_{t=1}^T \boldsymbol{\varepsilon}_{ot} \boldsymbol{\varepsilon}_{ot}'$, and $\boldsymbol{\varepsilon}_{ot} = (\varepsilon_{1t}, \varepsilon_{2t}, \dots, \varepsilon_{nt})'$. (c) ε_{it} is distributed independently of $\mathbf{f}_{t'}$, for all i, t and t' , and there exists $v_0 > 0$ such that for all $v = T - m_0 > v_0$

$$\inf_i (v^{-1} \boldsymbol{\varepsilon}_{io}' \mathbf{M}_F \boldsymbol{\varepsilon}_{io}) > c > 0, \quad (5)$$

where $\boldsymbol{\varepsilon}_{io} = (\varepsilon_{i1}, \varepsilon_{i2}, \dots, \varepsilon_{iT})'$, and $\mathbf{M}_F = \mathbf{I}_T - \mathbf{F}(\mathbf{F}' \mathbf{F})^{-1} \mathbf{F}'$.

Assumption 3 The $m_0 \times 1$ vector of factor loadings γ_i is bounded such that $\sup_i \|\gamma_i\| < C$, $n^{-1} \sum_{i=1}^n \gamma_i \gamma_i' = \Sigma_{n,\gamma\gamma} \rightarrow \Sigma_{\gamma\gamma} > \mathbf{0}$, and

$$1 - \theta_n > 0, \quad (6)$$

for all $n > n_0$ and as $n \rightarrow \infty$, where $\theta_n = 1 - n^{-1} \sum_{i=1}^n a_{i,n}^2$ with

$$a_{i,n} = 1 - \sigma_i \boldsymbol{\varphi}'_n \boldsymbol{\gamma}_i,$$

and $\boldsymbol{\varphi}_n = n^{-1} \sum_{i=1}^n \boldsymbol{\gamma}_i / \sigma_i$.

Remark 1 The above assumptions relate closely to those made in the literature on CD tests and high dimensional factor models. See, for example, the assumptions in Pesaran (2004, 2015, 2021), and assumptions in Bai (2003). Part (a) of Assumption 1 will be relaxed when we consider panel regression models with observed regressors. The sub-exponential type conditions (3) and (4) are needed for bounding the probabilities across all i , and are also adopted by Fan et al. (2011), Fan et al. (2013) and Chudik et al. (2018).

Remark 2 Condition (5) ensures that

$$E \left(\frac{\boldsymbol{\epsilon}'_{i0} \mathbf{M}_F \boldsymbol{\epsilon}_{i0}}{v} \right)^{-s} < \frac{1}{c^s} < C < \infty, \quad (7)$$

for all i and any fixed $s > 0$. It is unlikely to be violated in practice since

$$v^{-1} \boldsymbol{\epsilon}'_{i0} \mathbf{M}_F \boldsymbol{\epsilon}_{i0} = v^{-1} \sum_{t=1}^v \eta_{it}^2,$$

where $E(\eta_{it}^2) = 1$. This result follows since $\mathbf{M}_F$ is an idempotent matrix with rank v .

The focus of this paper is on testing the null hypothesis of error cross-sectional independence, which in the context of the model given by (1) and (2) and under Assumption 2 can be represented by

$$H_0 : \lambda_T = 0. \quad (8)$$

For the analysis of power, we consider local alternatives

$$H_{1T} : \lambda_T = c_\lambda T^{-1/2}, \quad (9)$$

with $c_\lambda \neq 0$. We also introduce the following assumption on $\mathbf{W} = (w_{ij})$.

Assumption 4 The connection matrix $\mathbf{W}$ has bounded maximum absolute column and row sum norms:

$$\|\mathbf{W}\|_1 = \sup_j \sum_{i=1}^n |w_{ij}| < C, \text{ and } \|\mathbf{W}\|_\infty = \sup_i \sum_{j=1}^n |w_{ij}| < C, \quad (10)$$

and $w_{ii} = 0$ for all i .

Remark 3 The connection matrix does not need to be symmetric. To see this, we can consider a more generalized setup of idiosyncratic errors,

$$\frac{u_{it}}{\sigma_i} = \varepsilon_{it} + \lambda_T \sum_{j=1}^n \frac{\rho_i \sigma_j \dot{w}_{ij}}{\sigma_i} \varepsilon_{jt}, \quad (11)$$

where $|\lambda_T| < C$ and $|\rho_i| < C$. It is clear that (11) can establish u_{it} in (2) by letting $w_{ij} = \rho_i \sigma_i^{-1} \sigma_j \dot{w}_{ij}$. In this way, $\mathbf{W}$ need not be symmetric even if the connection matrix $(\dot{w}_{ij})$ is

symmetric. Under local alternatives and Assumption 4, the specification of (2) allows for a wide range of spatial and network dependence characterized by the connection matrix. It is in accord with an early discussion in Chudik et al. (2011) that spatial dependence can be captured by a weak factor model, so long as the number of weak factors tends to infinity with n , which is ruled out in standard factor models where the number of latent factors is assumed to be fixed.

Remark 4 It is also easily seen that $\varepsilon_{it}(\lambda_T)$ defined by (2) is sub-exponential for any $|\lambda_T| < C$. This follows since by Assumption 2 $\{\varepsilon_{it}\}$ are independently and identically distributed sub-exponential processes, and by Assumption 4 $\sup_i \sum_{j=1}^n |w_{ij}| < C$. For a proof see part (b) of Theorem 5.5 in Goldie and Klüppelberg (1998) or Theorem 2.8.2 in Vershynin (2018).

3 Estimation of latent factors and the CD test

Following the literature we use principal component (PC) analysis to estimate the latent factors and their loadings. Let $\mathbf{Y} = (\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n)$ be the $T \times n$ matrix of observations on y_{it} , where $\mathbf{y}_i = (y_{i1}, y_{i2}, \dots, y_{iT})'$ and denote the first m_0 largest eigenvalues of $\mathbf{Y}'\mathbf{Y}$ by $(\hat{\rho}_1, \hat{\rho}_2, \dots, \hat{\rho}_{m_0})$, and its associated $n \times m_0$ matrix of orthonormal eigenvectors by $\hat{\mathbf{Q}}$. The PC estimators of factors $\mathbf{F} = (\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_T)'$ and their loadings $\mathbf{\Gamma} = (\gamma_1, \gamma_2, \dots, \gamma_n)'$ are then given by

$$\hat{\mathbf{F}} = (\hat{\mathbf{f}}_1, \hat{\mathbf{f}}_2, \dots, \hat{\mathbf{f}}_T)' = \frac{1}{\sqrt{n}} \mathbf{Y} \hat{\mathbf{Q}}, \text{ and } \hat{\mathbf{\Gamma}} = (\hat{\gamma}_1, \hat{\gamma}_2, \dots, \hat{\gamma}_n)' = \sqrt{n} \hat{\mathbf{Q}}. \quad (12)$$

By construction $n^{-1} \hat{\mathbf{\Gamma}}' \hat{\mathbf{\Gamma}} = \mathbf{I}_{m_0}$, and $T^{-1} \hat{\mathbf{F}}' \hat{\mathbf{F}} = \mathbf{D}_{nT}$, where $\mathbf{D}_{nT} = (nT)^{-1} \text{diag}(\hat{\rho}_1, \hat{\rho}_2, \dots, \hat{\rho}_{m_0})$. Under Assumptions 1-4 the asymptotic results derived by Bai (2003) for PCs continue to apply here, and u_{it} can be consistently estimated by

$$\hat{u}_{it} = y_{it} - \hat{\gamma}_i' \hat{\mathbf{f}}_t. \quad (13)$$

The CD test is based on the standardized residuals,

$$\tilde{\varepsilon}_{it,T} = \frac{\hat{u}_{it}}{\hat{\sigma}_{i,T}}, \quad (14)$$

where $\hat{\sigma}_{i,T} = \left(T^{-1} \sum_{t=1}^T \hat{u}_{it}^2 \right)^{1/2} = (T^{-1} \mathbf{y}_i' \mathbf{M}_{\hat{\mathbf{F}}} \mathbf{y}_i)^{1/2}$, and $\mathbf{M}_{\hat{\mathbf{F}}} = \mathbf{I}_T - \hat{\mathbf{F}} (\hat{\mathbf{F}}' \hat{\mathbf{F}})^{-1} \hat{\mathbf{F}}'$. Only units with non-zero $\hat{\sigma}_{i,T}^2$ are included in the construction of the CD test, namely

$$\inf_i \hat{\sigma}_{i,T}^2 > c > 0. \quad (15)$$

The standard CD test statistic based on the residuals, (13), is given by

$$CD = \sqrt{\frac{2T}{n(n-1)}} \left(\sum_{i=1}^{n-1} \sum_{j=i+1}^n \hat{\rho}_{ij,T} \right), \quad (16)$$

where $\hat{\rho}_{ij,T} = T^{-1} \sum_{t=1}^T \tilde{\varepsilon}_{it,T} \tilde{\varepsilon}_{jt,T}$.

Juodis and Reese (2022) apply the CD test to a panel regression model with latent factors, assuming that all the factors are strong. They show in that case $CD = O_p(\sqrt{T})$, and its use will lead to gross over-rejection of the null of error cross-sectional independence. To deal with the over-rejection problem, these authors propose a randomized CD test, CD_{W+} . However, as shown in Section S1.4 of the online supplement, the CD_{W+} test is likely to over-reject and tends to lack power against spatial/network alternatives. See also Section 7.3 for Monte Carlo evidence on the small sample performance of the CD_{W+} test.

4 The bias-corrected CD test

The main reason for the failure of the standard CD test in the case of latent factor models lies in the fact that both the factors and their loadings are unobserved and need to be estimated, and the differences between $\hat{\gamma}_i' \hat{\mathbf{f}}_t$ and $\gamma_i' \mathbf{f}_t$ do not tend to zero at a sufficiently fast rate for the CD test to be valid. Since the errors from estimation of $\gamma_i' \mathbf{f}_t$ are included in the residuals $\hat{u}_{it}$, the resultant CD statistic tends to over-state the degree of underlying error cross-sectional dependence. This problem also arises when latent factors are proxied by cross section averages, as is the case when panel data models are estimated using correlated common effect (CCE) estimators proposed by Pesaran (2006), which we shall address below in Section 5.

We propose a bias-corrected CD test statistic, which we denote by CD^* , that *directly* corrects the asymptotic bias of the CD test using the estimates of the factor loadings and error variances. To obtain the expression for the bias we first note under the null hypothesis of cross-sectional independence, $CD = z_{nT} + o_p(1)$ and

$$z_{nT} = \frac{1}{\sqrt{T}} \sum_{t=1}^T \left(\frac{\xi_{t,n}^2 - 1}{\sqrt{2}} \right) + o_p(1), \quad (17)$$

where

$$\xi_{t,n} = \frac{1}{\sqrt{n}} \sum_{i=1}^n a_{i,n} \varepsilon_{it}, \quad a_{i,n} = 1 - \sigma_i \boldsymbol{\varphi}_n' \boldsymbol{\gamma}_i, \quad (18)$$

$\boldsymbol{\varphi}_n = n^{-1} \sum_{i=1}^n \boldsymbol{\gamma}_i / \sigma_i$, which is established in proof of Theorem 1 in the online supplement. Since $a_{i,n}$ are given constants, then $E(\xi_{t,n}) = 0$,

$$E(\xi_{t,n}^2) \equiv \omega_n^2 = \frac{1}{n} \sum_{i=1}^n a_{i,n}^2 = n^{-1} \sum_{i=1}^n (1 - \sigma_i \boldsymbol{\varphi}_n' \boldsymbol{\gamma}_i)^2, \quad (19)$$

and

$$Var(\xi_{t,n}^2) = 2 \left(\frac{1}{n} \sum_{i=1}^n a_{i,n}^2 \right)^2 - \kappa_2 \left(\frac{1}{n^2} \sum_{i=1}^n a_{i,n}^4 \right), \quad (20)$$

where $\kappa_2 = E(\varepsilon_{it}^4) - 3$. Clearly, when the errors are Gaussian then $E(\varepsilon_{it}^4) = 3$, the second term of $Var(\xi_{t,n}^2)$ defined by (20) is exactly zero. But even for non-Gaussian errors the second term of $Var(\xi_{t,n}^2)$ is negligible when n is sufficiently large. To see this note that under Assumptions 2 and 3

$$\frac{1}{n^2} \sum_{i=1}^n a_{i,n}^4 = \frac{1}{n^2} \sum_{i=1}^n (1 - \sigma_i \boldsymbol{\varphi}_n' \boldsymbol{\gamma}_i)^4 \leq \frac{C}{n},$$

where C is a positive constant. Since ε_{it} (and henceforth $\xi_{t,n}$) are assumed to be serially independent, then we can also compute the mean and the variance of z_{nT} as

$$E(z_{nT}) = \frac{1}{\sqrt{T}} \sum_{t=1}^T \left(\frac{\omega_n^2 - 1}{\sqrt{2}} \right) = \sqrt{\frac{T}{2}} (\omega_n^2 - 1),$$

$$Var(z_{nT}) = \frac{1}{T} \sum_{t=1}^T Var\left(\frac{\xi_{t,n}^2}{\sqrt{2}}\right) = \frac{Var(\xi_{t,n}^2)}{2}.$$

The above expressions for $E(z_{nT})$ give the source of the asymptotic bias of CD as $E(z_{nT})$ rises with $\sqrt{T}$, unless

$$\lim_{n \rightarrow \infty} \omega_n^2 = \lim_{n \rightarrow \infty} n^{-1} \sum_{i=1}^n (1 - \sigma_i \boldsymbol{\varphi}'_n \boldsymbol{\gamma}_i)^2 = 1.$$

A bias-corrected version of CD can be defined by

$$CD^*(\theta_n) = \frac{CD + \sqrt{\frac{T}{2}}\theta_n}{1 - \theta_n}, \quad (21)$$

where

$$\theta_n = 1 - n^{-1} \mathbf{a}'_n \mathbf{a}_n, \quad \mathbf{a}_n = (a_{1,n}, a_{2,n}, \dots, a_{n,n})', \quad (22)$$

and $1 - \theta_n > 0$ by condition (6). Also upon using (18)

$$\theta_n = 2 \left(n^{-1} \sum_{i=1}^n \sigma_i \boldsymbol{\gamma}'_i \right) \boldsymbol{\varphi}_n - \boldsymbol{\varphi}'_n \left(\frac{1}{n} \sum_{i=1}^n \sigma_i^2 \boldsymbol{\gamma}_i \boldsymbol{\gamma}'_i \right) \boldsymbol{\varphi}_n. \quad (23)$$

The main difference between CD and $CD^*(\theta_n)$ depends on the magnitude of $\sqrt{T}\theta_n$, which in turn depends on the strengths of the factor loadings. Following Bailey et al. (2021), we measure the strength of factor j by α_j , defined by the rate at which the sum of absolute values of factor loadings rises with n , namely

$$\sum_{i=1}^n |\gamma_{ij}| = \Theta(n^{\alpha_j}), \quad \text{for } j = 1, 2, \dots, m_0, \quad (24)$$

where $0 \leq \alpha_j \leq 1$. Using (23) it is now easily established that $\theta_n = \Theta(n^{\alpha-1})$, where $\alpha = \max_{j=1,2,\dots,m_0}(\alpha_j)$, and θ_n does not tend to zero when there is at least one strong factor in the panel data model.³ Therefore, based on (21), the relationship between CD and $CD^*(\theta_n)$ is essentially controlled by the maximum factor strength α as $\sqrt{T}\theta_n = O(T^{1/2}n^{\alpha-1})$. Suppose now $T = \Theta(n^d)$ for some $d > 0$, then $\sqrt{T}\theta_n = \Theta(n^{d/2}n^{\alpha-1}) = \Theta(n^{\alpha+d/2-1})$, and the bias correction becomes negligible if $\alpha < 1 - d/2$. Under the required relative expansion rates of n and T entertained in this paper, we need to set $d = 1$, and for this choice the bias correction term, $\sqrt{T}\theta_n$, becomes negligible if $\alpha < 1/2$. As a result of this situation, CD and $CD^*(\theta_n)$ are asymptotically equivalent. In fact, the case of strong factors assumed in the PCA literature corresponds to $\alpha_j = 1$ for $j = 1, 2, \dots, m_0$, which is also fulfilled by Assumption 3 and used in our mathematical derivations.

The main theoretical result of the paper is summarized in the following theorem.

Theorem 1 *Suppose that observations on y_{it} , for $i = 1, 2, \dots, n$, and $t = 1, 2, \dots, T$ are generated from the pure latent factor model given by (1) and (2), where the number of factors, m_0 , is known. Consider the statistic $CD^*(\theta_n)$ defined by (21) and assume $(n, T) \rightarrow \infty$, such that $n/T \rightarrow \kappa$, and $0 < \kappa < \infty$.*

(a) Under the null hypothesis H_0 , defined by (8), and supposing that Assumptions 1 to 3 hold, then

$$CD^*(\theta_n) \rightarrow_d N(0, 1). \quad (25)$$

³For a proof see Section S1.3 of the online supplement.

(b) Under local alternatives H_{1T} , defined by (9), and supposing that Assumptions 1 to 4 hold, then

$$CD^*(\theta_n) \rightarrow_d N(\phi, 1), \quad (26)$$

where $\phi = \lim_{n \rightarrow \infty} \phi_n$ and

$$\phi_n = \frac{\sqrt{2}c_\lambda}{1 - \theta_n} n^{-1} \mathbf{a}'_n \mathbf{W} \mathbf{a}_n, \quad (27)$$

$\mathbf{W} = (w_{ij})$ is the connection matrix, $\mathbf{a}_n = (a_{1,n}, a_{2,n}, \dots, a_{n,n})'$, with $a_{i,n}$ and θ_n defined by (18) and (22), respectively.

For a proof see Section S1.1 of the online supplement.

The bias-corrected test statistic, $CD^*(\theta_n)$, depends on the unknown parameter, θ_n , which can be estimated by

$$\hat{\theta}_{nT} = 1 - \frac{1}{n} \sum_{i=1}^n \hat{a}_{i,nT}^2 \quad (28)$$

where

$$\hat{a}_{i,nT} = 1 - \hat{\sigma}_{i,T} (\hat{\boldsymbol{\varphi}}'_{nT} \hat{\boldsymbol{\gamma}}_i), \text{ and } \hat{\boldsymbol{\varphi}}_{nT} = \frac{1}{n} \sum_{i=1}^n \hat{\boldsymbol{\gamma}}_i / \hat{\sigma}_{i,T}. \quad (29)$$

The following proposition establishes the probability order of the difference between $\hat{\theta}_{nT}$ and θ_n .

Proposition 1 Suppose that observations on y_{it} , for $i = 1, 2, \dots, n$, and $t = 1, 2, \dots, T$ are generated from the pure latent factor model given by (1) and (2), where the number of factors, m_0 , is known, and $\lambda_T = c_\lambda T^{-1/2}$ with $|c_\lambda| < \infty$. Consider the term θ_n in the $CD^*(\theta_n)$ statistic given by (22) and its estimator $\hat{\theta}_{nT}$ given by (28). Let Assumptions 1 to 4 hold and $(n, T) \rightarrow \infty$, such that $n/T \rightarrow \kappa$, where $0 < \kappa < \infty$. Then

$$\sqrt{T} (\hat{\theta}_{nT} - \theta_n) = o_p(1). \quad (30)$$

For a proof see Section S1.1 of the online supplement.

Consider now the following feasible version of $CD^*(\theta_n)$,

$$CD^*(\hat{\theta}_{nT}) = \frac{CD + \sqrt{\frac{T}{2}} \hat{\theta}_{nT}}{1 - \hat{\theta}_{nT}}, \quad (31)$$

and note that in view of (21) and (30) we have

$$\begin{aligned} CD^*(\hat{\theta}_{nT}) &= \left(\frac{1 - \theta_n}{1 - \hat{\theta}_{nT}} \right) \frac{CD + \sqrt{\frac{T}{2}} \theta_n + \sqrt{\frac{T}{2}} (\hat{\theta}_{nT} - \theta_n)}{1 - \theta_n} \\ &= \left(\frac{1 - \theta_n}{1 - \hat{\theta}_{nT}} \right) [CD^*(\theta_n) + o_p(1)]. \end{aligned}$$

Also,

$$\frac{1 - \theta_n}{1 - \hat{\theta}_{nT}} = 1 + \frac{\sqrt{T} (\hat{\theta}_{nT} - \theta_n)}{\sqrt{T} (1 - \theta_n) - \sqrt{T} (\hat{\theta}_{nT} - \theta_n)} = 1 + o_p(1),$$

and hence $CD^* (\hat{\theta}_{nT}) = CD^* (\theta_n) + o_p(1)$. We refer to $CD^* (\hat{\theta}_{nT})$ simply as CD^* and the test based on it as the CD^* test. The main result of the paper for pure latent factor models is summarized in the following theorem.

Theorem 2 *Suppose that observations on y_{it} , for $i = 1, 2, \dots, n$, and $t = 1, 2, \dots, T$ are generated from the pure latent factor model given by (1) and (2), where the number of factors, m_0 , is known. Consider the statistic CD^* defined by (31), and assume $(n, T) \rightarrow \infty$, such that $n/T \rightarrow \kappa$, and $0 < \kappa < \infty$.*

(a) *Under the null hypothesis H_0 , defined by (8), and supposing that Assumptions 1 to 3 hold, then*

$$CD^* \rightarrow_d N(0, 1).$$

(b) *Under local alternatives H_{1T} , defined by (9), and supposing that Assumptions 1 to 4 hold, then*

$$CD^* \rightarrow_d N(\phi, 1),$$

where $\phi = \lim_{n \rightarrow \infty} \phi_n$, and ϕ_n is defined by (27).

For a proof see Section S1.1 of the online supplement.

This theorem establishes the conditions under which the proposed CD^* test has the correct size asymptotically. It also shows that the CD^* has power against network alternatives if the limit of ϕ_n defined by (27) is nonzero, namely so long as $\lim_{n \rightarrow \infty} n^{-1} \mathbf{a}'_n \mathbf{W} \mathbf{a}_n \neq 0$. This condition is likely to be satisfied if the connection matrix, $\mathbf{W}$, is not too sparse, although it must be sufficiently sparse so that Assumption 4 is met. In the case where there are no latent factors, $\mathbf{a}_n = (1, 1, \dots, 1)'$, and it is sufficient that $n^{-1} \sum_{i=1}^n \sum_{j=1}^n w_{ij} \neq 0$.

To our knowledge, we are the first to give a formal derivation of power function of the CD tests against spatial network alternatives, which applies equally to CD test for panel data models without latent factors. Hence, our derivation of power function can be used to supplement earlier researches on CD test.

As we shall see from the Monte Carlo results reported below, the CD^* test performs well even if some of the latent factors happen to be weak with $\alpha_j \in (0, 1/2]$ or semi-strong with $\alpha_j \in (1/2, 1)$. This is because when a factor is weak, it does not matter if its estimation by PCA is not consistent at the standard rate of $\delta_{nT} = \min(n^{1/2}, T^{1/2})$, and its inclusion or exclusion from the analysis has no material impact on the CD^* statistics for n and T sufficiently large. In view of this result, in the mathematical derivations it is sufficient to consider the case of strong factors, and let the weak factors to be absorbed in the error term.

However, it should be acknowledged that our derivations do not take account of the case when one or more of the factors are semi-strong. Such an extension is beyond the scope of the present paper, although recent studies by Bai and Ng (2023) and Jiang et al. (2023) show that PCA estimation is asymptotically valid for factor models so long as factor strengths are all above $1/2$. It is therefore reasonable to conjecture that CD^* test applied to PCA residuals will be asymptotically valid even if some of the factors are semi-strong, namely if $1/2 < \alpha_j < 1$.

In practice, the true number of factors, m_0 , is unknown. In cases where the estimated number of factors, $\hat{m}$, is underestimated ($\hat{m} < m_0$), the CD^* test has power against the missing strong factors. However, the rejection of the null hypothesis by the CD^* test does not necessarily mean there are missing factors, since the rejection could be due to network error dependence. It is, therefore, important for the investigator to decide on the number of strong latent factors before the implementation of the proposed CD^* test. To that end, we refer the reader to the information criterion approach advanced by Bai and Ng (2002) and the eigenvalue ratio test of Ahn and Horenstein (2013), for example.

5 CD^* test for panel regression model with interactive effects

Consider now the factor model (1) augmented with observed regressors

$$y_{it} = \boldsymbol{\alpha}'_i \mathbf{d}_t + \boldsymbol{\beta}'_i \mathbf{x}_{it} + \boldsymbol{\gamma}'_i \mathbf{f}_t + u_{it}, \quad (32)$$

where $\mathbf{d}_t$ is a $k_d \times 1$ vector of observed common factors, $\mathbf{x}_{it}$ is a $k_x \times 1$ vector of unit-specific observed covariates, $\boldsymbol{\alpha}_i = (\alpha_{i1}, \alpha_{i2}, \dots, \alpha_{ik_d})'$, and $\boldsymbol{\beta}_i = (\beta_{i1}, \beta_{i2}, \dots, \beta_{ik_x})'$ are their associated unknown coefficients. To highlight the relevance of the CD test for this set up, model (32) can be written alternatively as

$$y_{it} = \boldsymbol{\alpha}'_i \mathbf{d}_t + \boldsymbol{\beta}'_i \mathbf{x}_{it} + v_{it}, \quad (33)$$

where the errors, v_{it} , follow the factor structure

$$v_{it} = \boldsymbol{\gamma}'_i \mathbf{f}_t + u_{it}. \quad (34)$$

The CD test is applicable, without any modifications, to test the null hypothesis that the errors of the panel regression model, v_{it} , are cross-sectionally independent, so long as the regressors, $\mathbf{d}_t$ and $\mathbf{x}_{it}$, are strictly exogenous with respect to v_{it} . When the regressors are correlated with the errors, the least squares estimates of v_{it} become inconsistent and the standard CD test will fail. One important example of endogeneity arises when both y_{it} and $\mathbf{x}_{it}$ are driven by the same latent factor(s). Pesaran (2006) formalizes this form of endogeneity by assuming that

$$\mathbf{x}_{it} = \mathbf{A}'_i \mathbf{d}_t + \boldsymbol{\Gamma}'_i \mathbf{f}_t + \boldsymbol{\varepsilon}_{xit}, \quad (35)$$

where $\mathbf{A}_i$ and $\boldsymbol{\Gamma}_i$ are $k_d \times k_x$ and $m_0 \times k_x$ factor loading matrices and $\boldsymbol{\varepsilon}_{xit}$ are distributed independently of $\mathbf{f}_t$. The system of equations (33), (34) and (35) fully specify the dependence of $\mathbf{x}_{it}$ and v_{it} , and allows consistent estimation of v_{it} which can then be used to test the hypothesis that u_{it} are cross-sectionally independent in the pure latent factor model (34). We now show that the CD^* test applied to these residuals will be valid. To this end we make the following additional standard assumptions.

Assumption 5 (a) The $k_d \times 1$ vector $\mathbf{d}_t$ is a covariance stationary process, with absolute summable autocovariance and $\mathbf{d}_t$ is distributed independently of $\mathbf{f}_{t'}$, for all t and t' , such that $T^{-1} \mathbf{D}' \mathbf{F} = O_p(T^{-1/2})$, where $\mathbf{D} = (\mathbf{d}_1, \mathbf{d}_2, \dots, \mathbf{d}_T)'$ and $\mathbf{F} = (\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_T)'$ are matrices of observations on $\mathbf{d}_t$ and $\mathbf{f}_t$. (b) $(\mathbf{d}_t, \mathbf{f}_t)$ is distributed independently of u_{is} and $\boldsymbol{\varepsilon}_{xis}$ for all i, t, s .

Assumption 6 The unobserved factor loadings $\boldsymbol{\Gamma}_i$ are bounded, i.e. $\|\boldsymbol{\Gamma}_i\|_2 < C$ for all i .

Assumption 7 The individual-specific errors ε_{it} and $\varepsilon_{xi,t'}$ are distributed independently for all i, j, t and t' , and ε_{xit} follows the linear stationary process $\varepsilon_{xit} = \sum_{l=0}^{\infty} \mathbf{S}_{il} \boldsymbol{\eta}_{xi,t-l}$, where for each i , $\boldsymbol{\eta}_{xit}$ is a $k_x \times 1$ vector of serially uncorrelated random variables with mean zero, the variance matrix $\mathbf{I}_{k_x}$, and finite fourth-order cumulants. For each i , the coefficient matrices $\mathbf{S}_{il}$ satisfy the condition

$$\text{Var}(\varepsilon_{xit}) = \sum_{l=0}^{\infty} \mathbf{S}_{il} \mathbf{S}_{il}' = \boldsymbol{\Sigma}_{xi},$$

where $\boldsymbol{\Sigma}_{xi}$ is a positive definite matrix, such that $\sup_i \|\boldsymbol{\Sigma}_{xi}\|_2 < C$.

Assumption 8 Let $\tilde{\Gamma} = E(\boldsymbol{\gamma}_i, \boldsymbol{\Gamma}_i)$. We assume that $\text{Rank}(\tilde{\Gamma}) = m_0$.

Assumption 9 Consider the cross-sectional averages of the individual-specific variables, $\mathbf{z}_{it} = (y_{it}, \mathbf{x}_{it}')'$ defined by $\bar{\mathbf{z}}_t = n^{-1} \sum_{i=1}^n \mathbf{z}_{it}$, and let $\bar{\mathbf{M}} = \mathbf{I}_T - \bar{\mathbf{H}}(\bar{\mathbf{H}}'\bar{\mathbf{H}})^{-1}\bar{\mathbf{H}}'$, and $\mathbf{M}_g = \mathbf{I}_T - \mathbf{G}(\mathbf{G}'\mathbf{G})^{-1}\mathbf{G}'$, where $\bar{\mathbf{H}} = (\mathbf{D}, \bar{\mathbf{Z}})$, $\mathbf{G} = (\mathbf{D}, \mathbf{F})$, and $\bar{\mathbf{Z}} = (\bar{\mathbf{z}}_1, \bar{\mathbf{z}}_2, \dots, \bar{\mathbf{z}}_T)'$ is the $T \times (k_x + 1)$ matrix of observations on the cross-sectional averages. Let $\mathbf{X}_i = (\mathbf{x}_{i1}, \mathbf{x}_{i2}, \dots, \mathbf{x}_{iT})'$, then the $k \times k$ matrices $\hat{\Psi}_{i,T} = T^{-1} \mathbf{X}_i' \bar{\mathbf{M}} \mathbf{X}_i$ and $\Psi_{ig} = T^{-1} \mathbf{X}_i' \mathbf{M}_g \mathbf{X}_i$ are non-singular, and $\hat{\Psi}_{i,T}^{-1}$ and Ψ_{ig}^{-1} have finite second-order moments for all i .

Remark 5 The above assumptions are standard in the panel data models with multi-factor error structure. See, for example, Pesaran (2006). But in our setup under Assumption 2 we require the error term, ε_{it} , to be serially independent, since our focus is on testing ε_{it} for cross-sectional independence, and this assumption is needed for asymptotic normality of the bias-corrected CD test. Later in Section 6, we will consider models with serially correlated errors and show that the bias-corrected CD test remains valid. Nevertheless, we allow ε_{xit} , the errors in the $\mathbf{x}_{it}$ equations to be serially correlated. Assumption 5 separates the observed and the latent factors, as in Assumption 11 of Pesaran and Tosetti (2011). This assumption is required to obtain the probability order of estimated residuals needed for computation of CD^* statistic. A necessary condition for the rank condition in Assumption 8 to hold is $k_x \geq m_0 - 1$.

To estimate v_{it} we first filter out the effects of observed covariates using the CCE estimators proposed in Pesaran (2006), namely for each i we estimate $\boldsymbol{\beta}_i$ by

$$\hat{\boldsymbol{\beta}}_{CCE,i} = (\mathbf{X}_i' \bar{\mathbf{M}} \mathbf{X}_i)^{-1} (\mathbf{X}_i' \bar{\mathbf{M}} \mathbf{y}_i), \quad (36)$$

and following Pesaran and Tosetti (2011), estimate $\boldsymbol{\alpha}_i$ by

$$\hat{\boldsymbol{\alpha}}_{CCE,i} = (\mathbf{D}' \mathbf{D})^{-1} \mathbf{D}' (\mathbf{y}_i - \mathbf{X}_i \hat{\boldsymbol{\beta}}_{CCE,i}). \quad (37)$$

Then we have the following estimator of v_{it}

$$\hat{v}_{it} = y_{it} - \hat{\boldsymbol{\alpha}}_{CCE,i}' \mathbf{d}_t - \hat{\boldsymbol{\beta}}_{CCE,i}' \mathbf{x}_{it}. \quad (38)$$

Using results in Pesaran and Tosetti (2011) (p. 189) it follows that under Assumptions 1-9

$$\hat{v}_{it} = v_{it} + O_p\left(\frac{1}{n}\right) + O_p\left(\frac{1}{\sqrt{T}}\right) + O_p\left(\frac{1}{\sqrt{nT}}\right). \quad (39)$$

Note when $\alpha_i = \mathbf{0}$ and $\beta_i = \mathbf{0}$, (33) reduces to the pure latent factor model, (1), where PCA can be applied to v_{it} directly. In the case of panel regressions $\hat{v}_{it}$ can be used instead of v_{it} to compute the bias-corrected CD statistic given by (31). The errors involved will become asymptotically negligible in view of the fast rate of convergence of $\hat{v}_{it}$ to v_{it} , uniformly for each i and t . Specifically, as in the case of the pure latent factor model, we first compute m_0 PCs of $\{\hat{v}_{it}; i = 1, \dots, n; \text{ and } t = 1, \dots, T\}$ and the associated factor loadings, $(\hat{\gamma}_i, \hat{\mathbf{f}}_t)$, subject to the normalization $n^{-1} \sum_{i=1}^n \hat{\gamma}_i \hat{\gamma}_i' = \mathbf{I}_{m_0}$. The residuals

$$\hat{u}_{it} = \hat{v}_{it} - \hat{\gamma}_i' \hat{\mathbf{f}}_t, \text{ for } i = 1, \dots, n; \text{ and } t = 1, \dots, T, \quad (40)$$

can then be used to compute the standard CD statistic, (16), and its bias-corrected version, CD^* , using (31).

Remark 6 *It is important to bear in mind that $\hat{u}_{it}$ is not the same as the CCE residuals that result from running the panel regressions of y_{it} on $(\mathbf{d}_t, \mathbf{x}_{it}, \bar{\mathbf{z}}_t)$. As shown by Juodis and Reese (2022), the standard CD test applied to the CCE residuals will result in over-rejection and is not recommended. In our approach, we filter out the latent factors from $\hat{v}_{it}$ and use the filtered residuals, $\hat{u}_{it}$, to compute the CD statistic and correct it, as in CD^* , to allow for errors associated with estimation of factors and their loadings.*

The following theorem extends the result of Theorem 2.

Theorem 3 *Suppose that observations on y_{it} , for $i = 1, 2, \dots, n$, and $t = 1, 2, \dots, T$ are generated from the panel regression model defined by (33), (34) and (35), where the number of latent factors in (34), m_0 , is known. Consider the statistic CD^* given by (31) using the filtered residuals defined by (40). Suppose that $(n, T) \rightarrow \infty$, such that $n/T \rightarrow \kappa$, and $0 < \kappa < \infty$.*

(a) *Under the null hypothesis H_0 , defined by (8), and supposing that Assumptions 1 to 3 and Assumptions 5 to 9 hold, then*

$$CD^* \rightarrow_d N(0, 1).$$

(b) *Under local alternatives H_{1T} , defined by (9), and supposing that Assumptions 1 to 9 hold, then*

$$CD^* \rightarrow_d N(\phi, 1),$$

where $\phi = \lim_{n \rightarrow \infty} \phi_n$ and ϕ_n defined by (27).

For a proof see Section S1.1 of the online supplement.

6 CD^* test for models with serially correlated errors

As shown by Baltagi et al. (2016), when the errors ε_{it} in (32) are serially correlated the variance of standard CD test statistic is not unity (even asymptotically) and the test is no longer valid. The same also applies to the CD^* test. To deal with this problem, we propose two solutions which involve different ways of adjusting the CD^* test so that it will become applicable to panels with serially correlated errors. The first method closely follows the variance adjustment proposed by Baltagi et al. (2016), in which CD^* is scaled by ϖ where

$$\varpi^2 = \frac{2T}{n(n-1)} \sum_{i=2}^n \sum_{j=1}^{i-1} \tilde{\varepsilon}_{i,T}' (\tilde{\varepsilon}_{j,T} - \tilde{\varepsilon}_{(ij),T}) \tilde{\varepsilon}_{j,T}' (\tilde{\varepsilon}_{i,T} - \tilde{\varepsilon}_{(ij),T}), \quad (41)$$

with $\tilde{\boldsymbol{\varepsilon}}_{i,T} = (\tilde{\varepsilon}_{i1,T}, \tilde{\varepsilon}_{i2,T}, \dots, \tilde{\varepsilon}_{iT,T})'$, $\tilde{\varepsilon}_{it,T}$ defined in (14) and

$$\tilde{\boldsymbol{\varepsilon}}_{(ij),T} = \frac{1}{n-2} \sum_{1 \leq \tau \neq i, j \leq n} \tilde{\boldsymbol{\varepsilon}}_{\tau,T}.$$

The expression in (41) is the equivalent to that provided in Theorem 3 of Baltagi et al. (2016) but the factor of 2 in (41) is missing in their paper. The same adjustment is also applied to the CD_{W+} test to allow for serially correlated errors.

Alternatively, following Pesaran (2004), we first transform the panel regression models to eliminate the error serial correlation and then apply the CD^* test to the residuals of the transformed model. This is possible so long as the error serial correlation can be approximated by a finite order stationary autoregressive process. As a simple illustration consider the pure latent factor model $y_{it} = \gamma_i f_t + u_{it}$, in which factor f_t and loading γ_i are both latent, and the errors u_{it} are generated as $AR(1)$ processes, $u_{it} = \rho_i u_{it-1} + \epsilon_{it}$, where ρ_i is the autoregression coefficient and ϵ_{it} is serially independent, as well as being distributed independently of $f_{t'}$ for all i and $t, t' = 1, 2, \dots, T$. Testing the cross-sectional independence of u_{it} is equivalent to testing the cross-sectional independence of ϵ_{it} in the following autoregressive distributed lag (ARDL) representation of y_{it}

$$y_{it} = \rho_i y_{i,t-1} + \gamma_i f_t - \rho_i \gamma_i f_{t-1} + \epsilon_{it},$$

which can be written equivalently as a multi-factor AR panel regression model

$$y_{it} = \rho_i y_{i,t-1} + \tilde{\boldsymbol{\gamma}}_i' \tilde{\mathbf{f}}_t + \epsilon_{it}, \quad (42)$$

where $\tilde{\mathbf{f}}_t = (f_t, f_{t-1})'$, and $\tilde{\boldsymbol{\gamma}}_i = (\gamma_i, -\rho_i \gamma_i)'$. Since $y_{i,t-1}$ is weakly exogenous, the transformed model satisfies the setup of panel regression model (33) with $\tilde{\mathbf{f}}_t$ viewed as a vector of latent variables with the associated factor loadings, $\tilde{\boldsymbol{\gamma}}_i$. It therefore follows that the CD^* test can now be applied to test the cross-sectional independence of ϵ_{it} in (42). We refer to this test as *ARDL* adjusted CD^* test.

The same approach can also be used for panels with observed covariates. In general, testing cross-sectional independence of ε_{it} in model (32) is equivalent to testing the cross-sectional independence of ϵ_{it} in

$$y_{it} = \sum_{s=0}^S \boldsymbol{\alpha}_{i,s}' \mathbf{d}_{t-s} + \sum_{s=1}^S \rho_{i,s} y_{it-s} + \sum_{s=0}^S \boldsymbol{\beta}_{i,s}' \mathbf{x}_{it-s} + \mathbf{g}_i' \mathbf{h}_t + \epsilon_{it}, \quad (43)$$

where $\mathbf{h}_t$ is an extended set of latent factors (that encompass $\mathbf{f}_t$), and $\mathbf{g}_i$ are the associated factor loadings. The number of lags S is determined by the order of the AR specification assumed for ε_{it} in (32).

The variance adjustment is simpler to implement but it requires theoretical justification in the context of panel data models with latent factors. The ARDL adjustment is theoretically justified so long as the underlying errors follow finite order AR processes. As we shall see both approaches work well in dealing with serially correlated errors, at least in the context of the limited MC designs that we are considering. Clearly, further theoretical and Monte Carlo investigations are needed for a better understanding of the relative merits of the two approaches.

7 Small sample properties of CD^* and CD_{W+} tests

7.1 Data generating process

We consider the following data generating process

$$y_{it} = a_i + \sigma_i \left[\beta_{i1}d_t + \beta_{i2}x_{it} + m_0^{-1/2} \boldsymbol{\gamma}'_i \mathbf{f}_t + \varepsilon_{it}(\lambda) \right], \quad i = 1, 2, \dots, n; t = 1, 2, \dots, T, \quad (44)$$

where $\varepsilon_{it}(\lambda)$ follows the first order spatial autoregressive process, $SAR(1)$, such that

$$\varepsilon_{it}(\lambda) = \lambda \sum_{j=1}^n w_{ij} \varepsilon_{jt}(\lambda) + c \varepsilon_{it}. \quad (45)$$

a_i is a unit-specific effect, d_t is the observed common factor, x_{it} is the observed regressor that varies across i and t , $\mathbf{f}_t$ is the $m_0 \times 1$ vector of unobserved factors, $\boldsymbol{\gamma}_i$ is the vector of associated factor loadings. The scalar constants, $\sigma_i > 0$, are generated as $\sigma_i^2 = 0.5 + \frac{1}{2}(s_i^2 - 1)$, with $s_i^2 \sim IID\chi^2(2)$, which ensures that $E(\sigma_i^2) = 1$.

7.1.1 DGP under the null hypothesis

Under the null hypothesis, we set $\lambda = 0$ and $c = 1$, and consider both serially independent errors and serially correlated errors, which are generated by both Gaussian and non-Gaussian distributions:

- Serially independent errors: Gaussian errors, $\varepsilon_{it} \sim IIDN(0, 1)$; chi-squared distributed errors, $\varepsilon_{it} \sim IID\left(\frac{\chi^2(2)-2}{2}\right)$.
- Serially correlated errors: $\varepsilon_{it} = \rho_\varepsilon \varepsilon_{it-1} + \sqrt{1 - \rho_\varepsilon^2} e_{\varepsilon it}$, for $i = 1, 2, \dots, n$ and $t = 1, 2, \dots, T$, where $\rho_\varepsilon = 0.5$ and $e_{\varepsilon it}$ are generated as Gaussian errors, $e_{\varepsilon it} \sim IIDN(0, 1)$, or chi-squared distributed errors, $e_{\varepsilon it} \sim IID\left(\frac{\chi^2(2)-2}{2}\right)$.

The focus of the experiments is on testing the null hypothesis that ε_{it} are cross-sectional independent, whilst allowing for the presence of m_0 unobserved factors, $\mathbf{f}_t = (f_{1t}, f_{2t}, \dots, f_{m_0 t})'$. We consider $m_0 = 1$ and $m_0 = 2$, and generate the factor loadings $\boldsymbol{\gamma}_i = (\gamma_{i1}, \gamma_{i2})'$ as:

$$\begin{aligned} \gamma_{i1} &\sim IIDN(0.5, 0.5) \text{ for } i = 1, 2, \dots, [n^{\alpha_1}], \\ \gamma_{i2} &\sim IIDN(1, 1) \text{ for } i = 1, 2, \dots, [n^{\alpha_2}], \\ \gamma_{ij} &= 0 \text{ for } i = [n^{\alpha_j}] + 1, [n^{\alpha_j}] + 2, \dots, n, \text{ and } j = 1, 2. \end{aligned}$$

In the one-factor case ($m_0 = 1$), we only include f_{1t} as the latent factor and denote its factor strength by α . Three values of α are considered, namely $\alpha = 1, 2/3, 1/2$, respectively representing strong, semi-strong and weak factors. Similarly, in the two-factor case ($m_0 = 2$), we include both f_{1t} and f_{2t} as the latent factors and consider the following combinations of factor strengths: $(\alpha_1, \alpha_2) = [(1, 1), (1, 2/3), (2/3, 1/2)]$. The intercepts a_i are generated as $IIDN(1, 2)$ and fixed thereafter. The observed common factor is generated as $d_t = \rho_d d_{t-1} + \sqrt{1 - \rho_d^2} v_{dt}$,

with $\rho_d = 0.8$, and $v_{dt} \sim IIDN(0, 1)$, thus ensuring that $E(d_t) = 0$ and $Var(d_t) = 1$. The observed unit-specific regressors, x_{it} , for $i = 1, 2, \dots, n$ are generated to have non-zero correlations with the unobserved factors:

$$x_{it} = \gamma_{xi1}f_{1t} + \gamma_{xi2}f_{2t} + e_{xit}, \quad (46)$$

where $f_{jt} = r_j f_{j,t-1} + \sqrt{1 - r_j^2} v_{jt}$, with $r_j = 0.9$ and $v_{jt} \sim IID\left(\frac{\chi^2(2)-2}{2}\right)$, for $j = 1, 2$. The factor loadings in (46) are generated as $\gamma_{xi1} \sim IIDU(0.25, 0.75)$ and $\gamma_{xi2} \sim IIDU(0.1, 0.5)$. The error term of (46) is generated as $e_{xit} = \rho_i e_{xi,t-1} + \sqrt{1 - \rho_i^2} v_{xit}$, where $\rho_i \sim IIDU(0, 0.95)$ and $v_{xit} \sim IIDN(0, 1)$.

We will examine the small sample properties of the CD and the bias-corrected CD tests for both the pure latent factor model and for the panel regression model which also includes observed covariates.

- In the case of the pure latent factor model we set $\beta_{i1} = \beta_{i2} = 0$.
- In the case of the panel regression model with latent factors, we allow for heterogeneous slopes and generate the slopes of observed covariates, d_t and x_{it} , as $\beta_{i1} \sim IIDN(\mu_{\beta 1}, \sigma_{\beta 1}^2)$, and $\beta_{i2} \sim IIDN(\mu_{\beta 2}, \sigma_{\beta 2}^2)$ where $\mu_{\beta 1} = \mu_{\beta 2} = 0.5$ and $\sigma_{\beta 1}^2 = \sigma_{\beta 2}^2 = 0.25$, respectively.

As our theoretical results show the null distribution of the CD and bias-corrected CD tests do not depend on α_i , β_{i1} and β_{i2} , it is therefore innocuous what values are chosen for these parameters. Moreover, the average fit of the panel is controlled in terms of the limiting value of the pooled R-squared defined by

$$PR_{nT}^2 = 1 - \frac{(nT)^{-1} \sum_{i=1}^n \sum_{t=1}^T \sigma_i^2 E(\varepsilon_{it}^2)}{(nT)^{-1} \sum_{i=1}^n \sum_{t=1}^T Var(y_{it})}. \quad (47)$$

Since the underlying processes, (44) and (46), are stationary and $E(\varepsilon_{it}^2) = 1$, we have

$$\lim_{T \rightarrow \infty} PR_{nT}^2 = PR_n^2 = \frac{n^{-1} \sum_{i=1}^n \sigma_i^2 [\beta_{i1}^2 + \beta_{i2}^2 Var(x_{it}) + m_0^{-1} \gamma_i' \gamma_i + 2Cov(x_{it}, \gamma_i' \mathbf{f}_t)]}{n^{-1} \sum_{i=1}^n Var(y_{it})},$$

where $\gamma_i = (\gamma_{i1}, \gamma_{i2})'$, $Var(x_{it}) = \gamma_{xi}' \gamma_{xi} + 1$, $Cov(x_{it}, \gamma_i' \mathbf{f}_t) = \gamma_{xi}' \gamma_i$, $\gamma_{xi} = (\gamma_{xi1}, \gamma_{xi2})'$, and

$$Var(y_{it}) = \sigma_i^2 \left[\beta_{i1}^2 + \beta_{i2}^2 Var(x_{it}) + m_0^{-1} \gamma_i' \gamma_i + 2m_0^{-1/2} Cov(x_{it}, \gamma_i' \mathbf{f}_t) + 1 \right].$$

Also since σ_i^2 and β_{ij} are independently distributed and $E(\sigma_i^2) = 1$, it then readily follows that $\lim_{n \rightarrow \infty} PR_n^2 = \eta^2 / (1 + \eta^2)$, where

$$\eta^2 = \mu_{\beta 1}^2 + \sigma_{\beta 1}^2 + (\mu_{\beta 2}^2 + \sigma_{\beta 2}^2) \left[1 + E(\gamma_{xi}' \gamma_{xi}) \right] + \frac{2\mu_{\beta 2} E(\gamma_{xi}' \gamma_i)}{\sqrt{m_0}} + \frac{E(\gamma_i' \gamma_i)}{m_0}.$$

By controlling the value of η^2 across the experiments we ensure that the pooled R^2 in large samples is the same for all values of σ_i^2 . In particular, in the case of the pure latent model we have $\eta^2 = m_0^{-1} E(\gamma_i' \gamma_i) = O(n^{\alpha-1})$, where $\alpha = \max(\alpha_1, \alpha_2)$.

7.1.2 DGP under alternative hypotheses

Under alternative hypotheses, using (45), we consider a spatial network alternative defined by

$$\boldsymbol{\varepsilon}_{ot}(\lambda) = c(\mathbf{I}_n - \lambda \mathbf{W})^{-1} \boldsymbol{\varepsilon}_{ot},$$

where $\boldsymbol{\varepsilon}_{ot}(\lambda) = (\varepsilon_{1t}(\lambda), \varepsilon_{2t}(\lambda), \dots, \varepsilon_{nt}(\lambda))'$, $\mathbf{W} = (w_{ij})$, and $\boldsymbol{\varepsilon}_{ot} = (\varepsilon_{1t}, \varepsilon_{2t}, \dots, \varepsilon_{nt})'$. The errors ε_{it} are generated as described above. For the spatial weights w_{ij} , we first set $w_{ij}^0 = 1$ if $j = i - 2, i - 1, i + 1, i + 2$, and zero otherwise. We then row normalize the weights such that $w_{ij} = \left(\sum_{j=1}^n w_{ij}^0\right)^{-1} w_{ij}^0$. We also set $c^2 = n/\text{tr}[(\mathbf{I}_n - \lambda \mathbf{W})^{-1}(\mathbf{I}_n - \lambda \mathbf{W})'^{-1}]$, which ensures that $n^{-1} \sum_{i=1}^n \text{Var}(\varepsilon_{it}(\lambda)) = 1$, for all values of λ . In practice, only positive values of λ are of interest, and power function need not be symmetric for all positive and negative values of λ .

7.2 CD, CD* and CD_{W+} tests

All experiments are carried out for $n = 100, 200, 500, 1000$ and $T = 100, 200, 500$ and the number of replications is set to 2,000. Firstly we consider the DGPs with serially independent errors. For the pure latent factor models, we compute the filtered residuals as $\hat{v}_{it} = y_{it} - \hat{a}_i$, where $\hat{a}_i = T^{-1} \sum_{t=1}^T y_{it}$. For the panel regressions with latent factors, the filtered residuals are computed as

$$\hat{v}_{it} = y_{it} - \hat{a}_{CCE,i} - \hat{\beta}_{CCE,i1} d_t - \hat{\beta}_{CCE,i2} x_{it}, \quad (48)$$

where $(\hat{a}_{CCE,i}, \hat{\beta}_{CCE,i1}, \hat{\beta}_{CCE,i2})$ is the CCE estimator of a_i , β_{i1} and β_{i2} , as set out in Pesaran (2006). The residuals $\{\hat{v}_{it}; i = 1, 2, \dots, n; \text{ and } t = 1, 2, \dots, T\}$, together with their first m PCs and the associated factor loadings, $(\hat{\gamma}_i, \hat{\mathbf{f}}_t)$, are then used to compute the filtered residuals, $\hat{u}_{it} = \hat{v}_{it} - \hat{\gamma}_i' \hat{\mathbf{f}}_t$, to compute the CD test statistics, CD and CD^* , given by (16) and (31), respectively. For comparison, we also consider the power enhanced version of randomized CD statistic proposed by JR given by

$$CD_{W+} = CD_W + \Delta_{nT}, \quad (49)$$

where

$$CD_W = \sqrt{\frac{2}{Tn(n-1)}} \sum_{t=1}^T \sum_{i=2}^n \sum_{j=1}^{i-1} (w_i \hat{u}_{it}) (w_j \hat{u}_{jt}). \quad (50)$$

The weights w_i , for $i = 1, 2, \dots, n$ are independent drawn from a Rademacher distribution and

$$\Delta_{nT} = \sum_{i=2}^n \sum_{j=1}^{i-1} |\hat{\rho}_{ij,T}| \mathbf{1} \left(|\hat{\rho}_{ij,T}| > 2\sqrt{\frac{\ln(n)}{T}} \right), \quad (51)$$

where $\hat{\rho}_{ij,T} = T^{-1} \sum_{t=1}^T \tilde{\varepsilon}_{it,T} \tilde{\varepsilon}_{jt,T}$, and $\tilde{\varepsilon}_{it,T}$ is defined by (14). As shown by JR, CD_W has a zero mean by construction to avoid the over-rejection problem of the CD test, but it also lacks power by the very nature of the randomization process. JR further suggest CD_{W+} by adding a screening component Δ_{nT} proposed by Fan et al. (2015), which enhances the power of test since Δ_{nT} converges to zero as n and $T \rightarrow \infty$ under the null hypotheses, but can diverge under alternatives with a sufficient number of (i, j) pairs with non-zero correlations, ρ_{ij} .

As discussed in Section 6, CD^* is not valid when the errors are serially correlated. In the simulations, we apply the variance and ARDL adjustments to CD , CD_{W+} , and CD^* .

The variance adjusted versions are computed by scaling the original statistics by the standard deviation of the CD statistics using the expression in (41) with $\hat{u}_{it} = \hat{v}_{it} - \hat{\gamma}'_i \hat{\mathbf{f}}_t$. The ARDL adjusted versions of CD , CD^* , and CD_{W+} , are computed using the residuals from the following dynamic panel data model with latent factors,

$$y_{it} = a_i + \sum_{s=1}^S \rho_{is} y_{i,t-s} + \sum_{s=0}^S \beta_{i1,s} d_{t-s} + \sum_{s=0}^S \beta_{i2,s} x_{i,t-s} + \mathbf{g}'_i \mathbf{h}_t + \epsilon_{it}. \quad (52)$$

In the simulations we set $S = 1$, but higher order values can also be considered. The number of latent factors in $\mathbf{h}_t$ depends on S and is given by $m_h = (S + 1)m_0$. Accordingly, the number of PCs, m , should satisfy $m \geq (S + 1)m_0$. In the simulations if $S = 0$, we consider $m = 1$ and 2 if $m_0 = 1$, and $m = 2$ and 4 if $m_0 = 2$. But if $S = 1$ we consider $m = 2$ and 4 if $m_0 = 1$, and $m = 4$ and 6 if $m_0 = 2$. Seen from this perspective, the variance adjustment approach to dealing with error serial correlation seems preferable since it does not require specifying the lag order S .

7.3 Simulation results

We first report the simulation results for the DGPs with normally distributed errors, followed by the results based on DGPs with chi-squared distributed errors. Next, we report simulation results for the DGPs with serially correlated errors, using the variance and ARDL adjusted CD tests discussed in Section 6. Finally, to investigate the power of the CD^* test we consider the spatial $SAR(1)$ alternative with $\lambda = 0.25$. As to be expected power rises very quickly as λ deviates from 0.⁴

7.3.1 Serially independent errors: normally distributed errors

The simulation results for the DGPs with the errors following Gaussian distribution are shown in Tables 1-4. Table 1 reports the test results for the latent factor model with one factor. The top panel gives the results for the case where the number of selected PCs, denoted by m , is the same as the true number of factors, m_0 , while the bottom panel reports the results when $m = 2$. As to be expected the standard CD test over-rejects when the factor is strong, namely when $a = 1$. By comparison, the rejection frequencies of both CD^* and CD_{W+} tests under null ($\lambda = 0$) are generally around the nominal size of 5 per cent. Under the alternative (when $\lambda = 0.25$), the CD^* has satisfactory power properties with significantly high rejection frequencies even when the sample size is small. But CD_{W+} test performs quite poorly under spatial alternatives, especially when T is small.

Table 2 summarizes the size and power results for the latent factor model with $m_0 = 2$, and reports the results when m (the selected number PCs) is set to 2 (the top panel) and 4 (the bottom panel). The results are qualitatively similar to the ones reported for the one factor model. The CD test over-rejects if at least one of the factors is strong, and the empirical sizes of CD^* and CD_{W+} tests are close to their nominal value of 5 per cent, although we now observe some mild over-rejection when $n = 100$ and the selected number of PCs is 4. In terms of power, the CD^* performs well, although there is some loss of power as the number of factors

⁴Simulated power functions are provided in Section S2 of the online supplement for $\lambda = \pm 0.05, \pm 0.1, \pm 0.2, \pm 0.3, \pm 0.4, \pm 0.5, \pm 0.6, \pm 0.7, \pm 0.8, \pm 0.9, \pm 0.95$.

and selected PCs rise. Similarly, the power of the CD_{W+} test is now even lower and quite close to 5 per cent when $T < 500$ even if the number of PCs is set to $m_0 = 2$.

Turning to panel regression models with latent factors estimated by CCE, the associated simulation results are summarized in Tables 3 and 4. As can be seen, the results are very close to the ones reported in Tables 1 and 2 for the latent factor model, and are in line with the asymptotic result in (39) that underlie the use of CCE approach to filter out the effects of observed covariates, as well as latent factors.

7.3.2 Serially independent errors: chi-squared distributed errors

To save space, the simulation results for the DGPs with chi-squared errors are provided in Tables S.1 to S.4 in the online supplement. For standard CD test and its biased-corrected version, CD^* , as shown in Tables S.1 and S.2, the results are very similar to the ones with Gaussian errors, suggesting that CD^* test is likely to be robust to departures from Gaussianity. As with the experiments with Gaussian errors, the standard CD test continues to over-reject unless $\alpha < 2/3$, and CD^* has the correct size for all n and T combinations, except when the number of selected PCs is large relative to m_0 , and $T = 100$. The main difference between the results with and without Gaussian errors is the tendency for the CD_{W+} test to over-reject when $n > T$, which seems to be a universal feature of this test and holds for *all* choices of m_0 and the number of selected PCs, irrespective of whether the factors are strong or weak. This could be due to the screening component of CD_{W+} not tending to zero sufficiently fast with n and T . Furthermore, the CD^* test continues to have satisfactory power, but CD_{W+} clearly lacks power against spatial or network alternatives that are of primary interest.

Similar results are obtained for panel regression models with latent factors, summarized in Tables S.3 and S.4 in the online supplement.

7.3.3 Serially correlated errors

To save space, the results for the DGPs with serially correlated errors are summarized in Tables S.5-S.20 in the online supplement. Tables S.5 to S.12 give the simulation results for the variance adjusted CD tests, whilst Tables S.13 to S.20 provide the results for the ARDL adjusted tests. Overall, the results corroborate our earlier findings obtained for DGPs with serially independent errors. Both adjustments for serial error correlation work well, with size and power of the adjusted CD^* tests being quite close to the results already reported for DGPs with serially independent errors. It is also clear that without adjustments for latent factors and error serial correlation, the standard CD test will lead to large size distortions when the latent factors are strong. But in line with our theoretical results, the standard CD test, when adjusted for error serial correlation if needed, tends to have the correct size when the latent factors are weak.

Comparing the two types of adjustments for error serial correlations (for pure latent factor models as well as for panel regression models with latent factors), the variance adjusted CD^* test works particularly well, and only shows mild over-rejection in the case where $T = 100$ and $n > T$. In contrast, the CD_{W+} with variance adjustment over-rejects for all combinations of n and T .

The ARDL adjusted version of the CD^* test also works well when the number PCs are not too large, and tends to have the correct size for all (n, T) combinations and only shows slight over-rejection when $n = 100$. The CD_{W+} test using ARDL adjustment does better in

Table 1: Size and power of tests of error cross-sectional dependence for the latent factor model with one factor ($m_0 = 1$) and serially independent Gaussian errors

$m = 1$		Size ($H_0 : \lambda = 0$)												Power ($H_1 : \lambda = 0.25$)												
		$\alpha = 1$			$\alpha = 2/3$			$\alpha = 1/2$			$\alpha = 1$			$\alpha = 2/3$			$\alpha = 1/2$									
Tests	$n \setminus T$	100	200	500	100	200	500	100	200	500	100	200	500	100	200	500	100	200	500							
	100	64.7	88.1	97.5	5.8	9.7	22.5	5.3	5.9	9.4	23.8	37.5	55.7	68.9	86.7	97.7	81.0	93.8	99.4							
	200	67.7	92.3	99.4	5.3	7.1	14.2	5.9	5.4	7.0	16.0	31.0	50.6	75.4	93.1	99.8	84.9	97.6	100.0							
	500	71.0	95.2	100.0	5.1	4.2	8.6	6.2	4.9	4.3	10.9	22.9	46.3	82.3	97.5	100.0	89.6	98.9	100.0							
	1000	69.1	95.2	100.0	5.1	4.5	5.7	6.1	5.4	4.5	9.4	20.0	44.6	84.0	98.1	100.0	89.7	99.0	100.0							
CD^*	100	5.7	3.9	4.4	4.8	5.2	5.8	5.9	5.9	5.5	58.0	82.0	98.4	86.1	98.8	100.0	88.6	98.7	100.0							
	200	5.5	4.9	5.3	5.5	5.1	5.2	5.9	5.2	5.1	59.3	81.1	98.9	84.8	98.3	100.0	88.8	98.8	100.0							
	500	5.3	5.2	4.4	5.7	5.0	4.8	6.3	5.1	4.6	57.9	83.4	99.4	87.2	98.7	100.0	90.4	99.1	100.0							
	1000	4.4	5.3	5.1	5.5	4.8	4.6	6.1	5.5	4.9	60.1	84.0	99.4	86.7	99.2	100.0	90.1	99.2	100.0							
	100	5.8	5.1	5.0	5.5	5.5	5.9	5.8	5.4	7.6	6.9	7.8	49.5	6.4	9.7	59.8	6.9	8.5	64.5							
CD_{W+}	200	6.1	5.7	5.5	4.7	5.4	4.5	5.9	7.1	5.9	6.5	7.3	51.9	5.7	7.8	61.0	6.9	8.0	60.3							
	500	5.4	5.4	5.2	5.7	5.6	5.4	5.1	5.5	5.3	5.5	5.9	52.9	5.9	6.7	56.9	5.8	6.6	57.4							
	1000	5.1	4.6	4.8	4.6	5.9	5.3	4.7	5.9	6.0	4.9	5.6	51.7	4.6	6.2	54.2	5.6	5.6	52.8							
	$m = 2$		Size ($H_0 : \lambda = 0$)												Power ($H_1 : \lambda = 0.25$)											
			$\alpha = 1$			$\alpha = 2/3$			$\alpha = 1/2$			$\alpha = 1$			$\alpha = 2/3$			$\alpha = 1/2$								
Tests	$n \setminus T$	100	200	500	100	200	500	100	200	500	100	200	500	100	200	500	100	200	500							
	100	65.2	87.9	97.6	5.9	10.2	21.8	5.6	6.2	8.8	26.6	44.2	62.7	59.4	73.8	85.8	70.5	83.8	92.2							
	200	68.3	91.7	99.5	5.1	7.6	14.6	5.8	5.1	7.1	17.3	34.5	56.9	70.5	89.3	98.3	81.1	95.2	99.4							
	500	70.7	94.9	100.0	5.0	4.5	8.2	6.3	5.0	5.0	12.0	25.2	49.7	80.2	96.8	100.0	88.2	98.3	100.0							
	1000	68.9	95.1	100.0	4.8	4.8	5.3	5.9	5.7	4.9	9.3	20.7	45.7	83.0	97.4	100.0	88.7	98.9	100.0							
CD^*	100	5.7	4.7	5.8	5.5	6.3	6.2	5.8	6.5	6.3	57.8	81.8	98.8	83.9	98.4	100.0	85.8	98.3	100.0							
	200	5.1	5.0	5.9	6.1	5.3	5.7	5.8	5.3	5.1	57.9	81.0	98.8	84.2	98.1	100.0	87.8	98.6	100.0							
	500	5.3	5.3	4.3	5.6	5.1	4.7	6.5	5.0	4.3	57.4	82.5	99.4	86.8	98.6	100.0	89.3	98.9	100.0							
	1000	4.3	5.1	5.0	5.8	5.1	5.0	6.3	5.8	4.6	59.7	83.6	99.4	86.2	99.1	100.0	89.4	99.2	100.0							
	100	5.1	5.4	6.5	3.8	5.4	6.1	4.8	5.4	8.4	5.4	8.1	33.8	5.1	8.2	36.7	5.9	8.2	44.1							
CD_{W+}	200	5.5	5.3	6.1	4.9	5.8	5.3	6.2	5.3	5.5	5.5	6.6	38.8	5.7	7.1	41.7	6.2	7.0	43.6							
	500	4.7	4.5	4.3	5.3	5.1	4.7	5.3	5.7	4.1	4.9	6.1	45.2	5.5	5.9	48.2	6.2	6.2	47.0							
	1000	3.9	4.6	4.9	4.9	5.7	4.6	5.9	4.6	4.8	3.7	5.2	43.9	5.3	6.6	48.3	6.0	6.1	47.3							

Note: The DGP is given by (44) with $\beta_{11} = \beta_{12} = 0$ and contains a single latent factor with factor strength $\alpha = 1, 2/3, 1/2$. λ is the spatial autocorrelation coefficient of the error term. m_0 is the true number of factors and m is the number of selected PCs used to estimate factors. CD denotes the standard CD test statistic while CD^* denotes the bias-corrected CD test statistic. CD_{W+} denotes the power-enhanced randomized CD test statistic.

Table 2: Size and power of tests of error cross-sectional dependence for the latent factor model with two factors ($m_0 = 2$) and serially independent Gaussian errors

$m = 2$		Size ($H_0 : \lambda = 0$)																		Power ($H_1 : \lambda = 0.25$)																	
		$\alpha_1 = 1, \alpha_2 = 1$						$\alpha_1 = 1, \alpha_2 = 2/3$						$\alpha_1 = 2/3, \alpha_2 = 1/2$						$\alpha_1 = 1, \alpha_2 = 1$						$\alpha_1 = 1, \alpha_2 = 2/3$						$\alpha_1 = 2/3, \alpha_2 = 1/2$					
		100	200	500	1000	100	200	500	1000	100	200	500	1000	100	200	500	1000	100	200	500	1000	100	200	500	1000	100	200	500	1000	100	200	500	1000				
Tests	$n \setminus T$	100	99.9	100.0	100.0	100.0	98.3	99.9	100.0	100.0	8.3	15.4	40.7	99.4	100.0	100.0	100.0	99.4	100.0	100.0	100.0	99.4	100.0	100.0	100.0	93.1	99.2	100.0	71.5	84.3	97.8	83.1	95.4	100.0	85.5	97.7	100.0
	CD	200	100.0	100.0	100.0	100.0	99.4	100.0	100.0	100.0	6.6	8.6	24.1	99.8	100.0	100.0	100.0	99.9	100.0	100.0	100.0	99.9	100.0	100.0	100.0	93.4	99.8	100.0	83.1	95.4	100.0	85.5	97.7	100.0	85.5	97.7	100.0
	500	100.0	100.0	100.0	100.0	100.0	99.7	100.0	100.0	100.0	5.7	5.8	13.0	99.9	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	94.4	100.0	100.0	83.1	95.4	100.0	85.5	97.7	100.0	85.5	97.7	100.0
	1000	100.0	100.0	100.0	100.0	100.0	99.9	100.0	100.0	100.0	7.6	4.9	7.8	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	94.4	100.0	100.0	83.1	95.4	100.0	85.5	97.7	100.0	85.5	97.7	100.0
CD^*	100	5.5	4.7	3.8	5.6	5.3	4.9	8.8	7.2	6.8	23.8	35.2	60.0	31.3	49.3	78.9	83.7	97.8	100.0	23.8	35.2	60.0	31.3	49.3	78.9	83.7	97.8	100.0	71.5	84.3	97.8	83.1	95.4	100.0	85.5	97.7	100.0
	200	5.4	5.0	4.9	6.3	4.6	4.6	8.3	5.5	5.3	22.4	33.1	60.6	32.1	49.4	81.8	86.4	98.2	100.0	22.4	33.1	60.6	32.1	49.4	81.8	86.4	98.2	100.0	71.5	84.3	97.8	83.1	95.4	100.0	85.5	97.7	100.0
	500	4.8	4.7	6.0	5.8	4.7	4.1	7.4	5.9	5.1	21.1	35.1	64.6	33.5	51.3	83.4	88.4	98.6	100.0	21.1	35.1	64.6	33.5	51.3	83.4	88.4	98.6	100.0	71.5	84.3	97.8	83.1	95.4	100.0	85.5	97.7	100.0
	1000	5.1	4.5	4.8	5.4	5.4	4.7	8.3	6.2	5.2	23.5	34.2	63.1	36.2	52.8	85.2	88.4	98.7	100.0	23.5	34.2	63.1	36.2	52.8	85.2	88.4	98.7	100.0	71.5	84.3	97.8	83.1	95.4	100.0	85.5	97.7	100.0
CD_{W+}	100	4.4	6.8	5.9	5.5	5.9	7.8	5.8	5.3	9.4	5.4	9.4	38.1	6.4	9.3	48.1	7.1	10.1	63.8	5.4	9.4	38.1	6.4	9.3	48.1	7.1	10.1	63.8	7.1	10.1	63.8	7.1	10.1	63.8	7.1	10.1	63.8
	200	5.5	5.4	5.3	6.4	5.6	6.3	5.9	5.9	5.6	6.0	6.1	44.7	6.4	6.9	46.9	6.9	7.3	56.7	6.0	6.1	44.7	6.4	6.9	46.9	6.9	7.3	56.7	6.9	7.3	56.7	6.9	7.3	56.7	6.9	7.3	56.7
	500	5.3	5.6	5.1	5.5	5.1	4.8	5.4	4.7	6.4	5.9	6.6	49.5	5.9	6.6	49.5	5.7	5.8	50.1	5.9	6.6	49.5	5.7	5.8	50.1	5.7	5.8	50.1	6.3	5.0	57.3	6.3	5.0	57.3	6.3	5.0	57.3
	1000	4.9	4.6	4.6	5.1	4.3	5.4	6.6	5.4	5.1	5.0	5.5	50.1	5.0	5.5	50.1	4.9	5.2	50.1	5.0	5.5	50.1	4.9	5.2	50.1	4.9	5.2	50.1	6.7	6.3	53.4	6.7	6.3	53.4	6.7	6.3	53.4
$m = 4$		Size ($H_0 : \lambda = 0$)																		Power ($H_1 : \lambda = 0.25$)																	
Tests	$n \setminus T$	$\alpha_1 = 1, \alpha_2 = 1$						$\alpha_1 = 1, \alpha_2 = 2/3$						$\alpha_1 = 2/3, \alpha_2 = 1/2$						$\alpha_1 = 1, \alpha_2 = 1$						$\alpha_1 = 1, \alpha_2 = 2/3$						$\alpha_1 = 2/3, \alpha_2 = 1/2$					
		100	200	500	1000	100	200	500	1000	100	200	500	1000	100	200	500	1000	100	200	500	1000	100	200	500	1000	100	200	500	1000	100	200	500	1000				
CD	100	99.8	100.0	100.0	100.0	98.8	100.0	100.0	100.0	7.4	15.9	40.8	99.4	100.0	100.0	100.0	99.4	100.0	100.0	100.0	99.4	100.0	100.0	100.0	93.6	98.8	99.8	39.9	43.0	55.4	39.9	43.0	55.4				
	200	100.0	100.0	100.0	100.0	99.6	100.0	100.0	100.0	5.7	8.9	24.3	99.8	100.0	100.0	100.0	99.8	100.0	100.0	100.0	99.8	100.0	100.0	100.0	95.4	99.5	100.0	61.7	73.3	87.8	61.7	73.3	87.8				
	500	100.0	100.0	100.0	100.0	99.6	100.0	100.0	100.0	5.9	5.7	12.8	99.9	100.0	100.0	100.0	99.9	100.0	100.0	100.0	99.9	100.0	100.0	100.0	94.6	99.9	100.0	79.2	93.4	99.6	79.2	93.4	99.6				
	1000	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	7.4	4.7	8.0	99.9	100.0	100.0	100.0	99.9	100.0	100.0	100.0	99.9	100.0	100.0	100.0	94.2	100.0	100.0	83.3	96.7	100.0	83.3	96.7	100.0				
CD^*	100	7.6	7.9	15.4	7.0	6.8	13.2	9.5	8.4	10.1	26.4	39.2	67.5	26.4	39.2	67.5	26.4	39.2	67.5	26.4	39.2	67.5	26.4	39.2	67.5	33.7	56.4	86.4	79.8	96.6	100.0	79.8	96.6	100.0			
	200	5.5	6.0	6.6	6.8	6.3	7.0	7.6	6.3	6.5	22.8	36.8	66.0	22.8	36.8	66.0	22.8	36.8	66.0	22.8	36.8	66.0	22.8	36.8	66.0	33.3	51.7	85.1	84.4	98.0	100.0	84.4	98.0	100.0			
	500	5.1	4.7	6.3	6.2	5.1	4.8	7.8	6.8	5.1	21.0	35.2	66.1	21.0	35.2	66.1	21.0	35.2	66.1	21.0	35.2	66.1	21.0	35.2	66.1	34.3	51.9	84.1	87.2	98.1	100.0	87.2	98.1	100.0			
	1000	5.8	5.0	4.7	5.4	5.0	4.8	8.6	5.7	5.7	23.4	34.5	63.4	23.4	34.5	63.4	23.4	34.5	63.4	23.4	34.5	63.4	23.4	34.5	63.4	35.9	52.4	84.9	87.3	98.7	100.0	87.3	98.7	100.0			
CD_{W+}	100	5.5	6.8	26.2	5.9	6.3	15.7	6.2	6.1	11.8	6.2	8.8	39.5	6.2	8.8	39.5	6.2	8.8	39.5	6.2	8.8	39.5	6.2	8.8	39.5	6.2	8.7	40.5	7.3	7.9	34.9	7.3	7.9	34.9			
	200	4.5	5.7	6.4	5.2	4.7	5.3	5.5	5.4	6.7	5.4	7.2	37.4	5.4	7.2	37.4	5.4	7.2	37.4	5.4	7.2	37.4	5.4	7.2	37.4	5.6	6.5	29.6	6.3	6.2	31.2	6.3	6.2	31.2			
	500	5.6	5.0	5.3	5.9	5.1	5.7	5.9	5.3	4.8	6.3	6.1	44.2	6.3	6.1	44.2	6.3	6.1	44.2	6.3	6.1	44.2	6.3	6.1	44.2	6.7	6.4	38.8	5.9	5.9	39.0	5.9	5.9	39.0			
	1000	4.6	5.8	4.9	5.3	4.9	4.6	5.9	5.5	5.9	5.0	6.6	45.6	5.0	6.6	45.6	5.0	6.6	45.6	5.0	6.6	45.6	5.0	6.6	45.6	4.8	5.0	42.4	6.1	5.7	43.7	6.1	5.7	43.7			

Note: The DGP is given by (44) with $\beta_{11} = \beta_{12} = 0$, and contains two latent factors with factor strengths $(\alpha_1, \alpha_2) = [(1, 1), (1, 2/3), (2/3, 1/2)]$. λ is the spatial autocorrelation coefficient of the error term. m_0 is the true number of factors and m is the number of selected PCs used to estimate factors. CD denotes the standard CD test statistic while CD^* denotes the bias-corrected CD test statistic. CD_{W+} denotes the power-enhanced randomized CD test statistic.

Table 3: Size and power of tests of error cross-sectional dependence for panel regression model with one latent factor ($m_0 = 1$) and serially independent Gaussian errors

$m = 1$		Size ($H_0 : \lambda = 0$)												Power ($H_1 : \lambda = 0.25$)											
Tests	$n \setminus T$	$\alpha = 1$			$\alpha = 2/3$						$\alpha = 1/2$			$\alpha = 1$			$\alpha = 2/3$			$\alpha = 1/2$			$\alpha = 1/2$		
		100	200	500	100	200	500	100	200	500	100	200	500	100	200	500	100	200	500	100	200	500	100	200	500
CD	100	67.9	88.6	98.5	6.7	9.5	20.5	6.9	6.3	9.0	25.1	37.6	55.6	71.1	86.6	97.7	81.4	93.9	99.3						
	200	68.9	92.5	99.7	5.5	6.7	13.4	6.8	5.3	5.9	17.2	29.7	49.7	77.2	94.0	99.7	86.5	98.1	100.0						
	500	67.4	94.7	100.0	4.9	6.1	8.5	5.3	5.0	5.9	11.8	22.1	45.7	83.2	97.0	100.0	89.5	99.0	100.0						
	1000	69.0	95.1	100.0	5.9	4.3	6.6	6.8	5.8	5.6	10.1	19.7	44.8	85.8	98.3	100.0	89.8	99.0	100.0						
CD^*	100	5.1	5.6	4.6	7.4	5.5	6.1	7.8	6.1	6.2	57.5	81.3	98.2	85.9	98.0	100.0	88.9	98.8	100.0						
	200	5.5	6.0	4.2	6.2	5.8	4.6	6.8	5.7	4.9	58.9	83.0	99.1	85.8	98.3	100.0	89.2	99.2	100.0						
	500	5.0	5.0	4.5	5.5	5.7	5.5	5.5	5.3	5.5	59.0	83.7	99.3	87.4	98.6	100.0	90.6	99.4	100.0						
	1000	4.5	4.6	5.4	6.3	5.0	5.6	7.0	5.9	5.9	60.0	83.3	99.5	88.3	99.0	100.0	90.1	99.2	100.0						
CD_{W+}	100	5.4	5.3	6.4	5.3	5.7	6.2	4.5	5.9	6.4	5.9	8.6	46.4	5.7	8.9	60.1	6.2	8.8	64.6						
	200	5.8	5.1	5.2	5.3	4.9	5.3	6.1	6.4	4.8	6.0	6.6	51.8	6.0	6.3	60.2	6.2	7.6	59.3						
	500	5.1	5.4	5.9	5.6	4.5	5.1	5.6	6.4	5.7	5.4	6.7	55.0	5.9	6.1	57.4	6.2	7.0	58.0						
	1000	4.9	5.5	4.5	6.0	4.6	4.7	5.1	6.0	4.8	4.6	5.6	48.9	5.4	5.3	52.8	4.8	5.9	51.9						

$m = 2$		Size ($H_0 : \lambda = 0$)												Power ($H_1 : \lambda = 0.25$)											
Tests	$n \setminus T$	$\alpha = 1$			$\alpha = 2/3$						$\alpha = 1/2$			$\alpha = 1$			$\alpha = 2/3$			$\alpha = 1/2$			$\alpha = 1/2$		
		100	200	500	100	200	500	100	200	500	100	200	500	100	200	500	100	200	500	100	200	500	100	200	500
CD	100	67.5	88.9	98.6	7.7	9.3	21.8	6.8	6.8	9.4	28.3	43.6	63.0	59.0	75.0	86.3	69.5	84.5	90.9						
	200	69.0	92.2	99.5	5.6	6.7	13.3	6.8	5.4	5.8	18.7	32.3	55.7	71.5	89.4	98.3	80.8	96.1	99.5						
	500	68.0	94.9	100.0	5.0	5.7	8.0	5.1	5.1	5.5	12.4	23.7	47.6	80.9	96.0	100.0	87.6	98.6	100.0						
	1000	69.9	95.0	100.0	5.9	4.2	6.7	6.7	5.7	5.7	11.0	20.4	46.2	84.1	97.9	100.0	88.4	98.9	100.0						
CD^*	100	5.6	6.5	6.4	7.9	5.8	7.1	8.0	6.6	6.7	57.0	81.0	98.4	83.3	97.8	100.0	86.6	98.6	100.0						
	200	5.8	5.7	4.9	6.2	6.9	5.7	7.0	5.6	4.7	58.2	82.6	98.9	84.4	98.1	100.0	87.0	99.0	100.0						
	500	5.5	5.0	4.4	5.6	5.3	5.3	5.6	5.3	5.5	58.0	83.4	99.4	86.5	98.7	100.0	89.1	99.3	100.0						
	1000	4.6	4.7	5.3	6.3	5.1	5.6	7.0	6.2	6.1	58.5	83.0	99.4	87.1	98.9	100.0	89.4	99.2	100.0						
CD_{W+}	100	6.1	5.8	6.6	6.9	5.9	6.4	5.5	6.5	7.4	7.6	7.9	32.6	8.0	8.0	36.4	7.2	7.8	41.4						
	200	5.9	4.8	5.6	5.2	6.1	5.6	5.3	5.4	4.4	5.8	6.1	39.4	6.1	6.2	43.4	5.8	6.8	42.8						
	500	5.2	4.9	5.3	5.1	5.1	5.7	5.7	5.1	5.3	5.1	5.6	48.6	5.4	6.5	47.9	5.9	6.3	47.6						
	1000	5.1	5.1	4.4	5.1	4.7	5.6	6.2	4.8	4.9	5.1	5.7	47.7	5.9	4.6	49.9	5.6	5.3	48.8						

Note: The DGP is given by (44) with β_{11} and β_{12} both generated from normal distribution, and contains a single latent factor with factor strength $\alpha = 1, 2/3, 1/2$. λ is the spatial autocorrelation coefficient of the error term. m_0 is the true number of factors and m is the number of selected PCs used to estimate factors. CD denotes the standard CD test statistic while CD^* denotes the bias-corrected CD test statistic. CD_{W+} denotes the power-enhanced randomized CD test statistic.

Table 4: Size and power of tests of error cross-sectional dependence for panel regression model with two latent factors ($m_0 = 2$) and serially independent Gaussian errors

$m = 2$		Size ($H_0 : \lambda = 0$)												Power ($H_1 : \lambda = 0.25$)																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																
		$\alpha_1 = 1, \alpha_2 = 1$						$\alpha_1 = 1, \alpha_2 = 2/3$						$\alpha_1 = 2/3, \alpha_2 = 1/2$						$\alpha_1 = 1, \alpha_2 = 1$						$\alpha_1 = 1, \alpha_2 = 2/3$						$\alpha_1 = 2/3, \alpha_2 = 1/2$																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																														
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00.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0

Note: The DGP is given by (44) with β_{11} and β_{22} both generated from normal distribution, and contains two latent factors with factor strengths $(\alpha_1, \alpha_2) = [(1, 1), (1, 2/3), (2/3, 1/2)]$. λ is the spatial autocorrelation coefficient of the error term. m_0 is the true number of factors and m is the number of selected PCs used to estimate factors. CD denotes the standard CD test statistic while CD^* denotes the bias-corrected CD test statistic. CD_{W+} denotes the power-enhanced randomized CD test statistic.

controlling for the size when the errors are Gaussian, but tends to over-reject when the errors are chi-squared distributed and $n > T$. Both adjusted versions of the CD_{W+} test continue to lack power against spatial or network alternatives.

8 Empirical application

It is well known that house price changes are spatially correlated, but it is unclear if such correlations are mainly due to common factors (national or regional) or arise from spatial spillover effects not related to the common factors, a phenomenon also referred to as the ripple effect. See, for example, Tsai (2015), Chiang and Tsai (2016), Holly et al. (2011), Bailey et al. (2016), and Aquaro et al. (2021). To test for the presence of ripple effects the influence of common factors must first be filtered out and this is often a challenging exercise due to the latent nature of regional and national factors. Therefore, to find if there exist local spillover effects, one needs to test for significant residual cross-sectional dependence once the effects of common factors are filtered out.

We consider quarterly data on real house prices at the level of Metropolitan Statistical Areas (MSAs) in the U.S. There are 381 MSAs, under the February 2013 definition provided by the U.S. Office of Management and Budget (OMB). We use quarterly data on real house price changes compiled by Yang (2021) which covers $n = 377$ MSAs from the contiguous United States over the period 1975Q1-2014Q4 ($T = 160$ quarters). To allow for possible regional factors, we also follow Bailey et al. (2016) and start with the Bureau of Economic Analysis eight regional classification, namely New England, Mideast, Great Lakes, Plains, Southeast, Southwest, Rocky Mountain and Far West. But due to the low number of MSAs in New England and Rocky Mountain regions, we combine New England and Mideast, and Southwest and Rocky Mountain as two regions. We end up with a six region classification ($R = 6$), each covering a reasonable number of MSAs.

We model house price changes and consider an extended factor model with deterministic seasonal dummies to allow for seasonal movements in house prices. Bailey et al. (2016) find evidence of regional factors in U.S. house price changes which might not be picked up when using PCA. Given this finding, our model includes observed regional and national factors, as well as latent factors. Specifically, we suppose

$$\pi_{irt} = a_{ir} + \sum_{j=1}^3 \beta_{ir,j} 1\{q_t = j\} + \delta_{ir,1} \bar{\pi}_{rt} + \delta_{ir,2} \bar{\pi}_t + \boldsymbol{\gamma}'_{ir} \mathbf{f}_t + u_{irt}, \quad (53)$$

where π_{irt} is the real house price changes in MSA i located in region $r = 1, 2, \dots, 6$, $1\{q_t = j\}$ is the index for quarter j , and $\mathbf{f}_t$ is the $m_0 \times 1$ vector of latent factors. $\bar{\pi}_{rt} = n_r^{-1} \sum_{i=1}^{n_r} \pi_{irt}$ and $\bar{\pi}_t = n^{-1} \sum_{r=1}^R \sum_{i=1}^{n_r} \pi_{irt}$ are proxies for the regional and national factors. To filter out the effects of seasonal dummies as well as observed factors, we first run the least squares regression of π_{irt} on an intercept and $(1\{q_t = j\}, \bar{\pi}_{rt}, \bar{\pi}_t)$ for each i to generate the residuals

$$\hat{v}_{irt} = \pi_{irt} - \hat{a}_{ir} - \sum_{j=1}^3 \hat{\beta}_{ir,j} 1\{q_t = j\} - \hat{\delta}_{ir,1} \bar{\pi}_{rt} - \hat{\delta}_{ir,2} \bar{\pi}_t, \quad (54)$$

and then apply PCA to $\{\hat{v}_{irt} : i = 1, \dots, n, r = 1, \dots, R, t = 1, \dots, T\}$ to obtain $\hat{\boldsymbol{\gamma}}_{ir}$ and $\hat{\mathbf{f}}_t$, yield-

ing the residuals

$$\hat{u}_{irt} = \pi_{irt} - \hat{a}_{ir} - \sum_{j=1}^3 \hat{\beta}_{ir,j} 1\{q_t = j\} - \hat{\delta}_{ir,1} \bar{\pi}_{rt} - \hat{\delta}_{ir,2} \bar{\pi}_t - \hat{\gamma}'_{ir} \hat{\mathbf{f}}_t. \quad (55)$$

For the case without adjusting for error serial correlation, the above residuals are used to compute CD , CD^* and CD_{W+} , given by (16), (31) and (49). For the case with serially correlated errors, the variance adjusted versions of the three CD statistics are generated by scaling original test statistics using (41), where $\tilde{\varepsilon}_{it,T}$ is replaced by the standardized residuals generated from (55), while the ARDL adjusted versions are computed using the residuals from the following dynamic panel data model with latent factors, $\mathbf{h}_t$:

$$\begin{aligned} \pi_{irt} = & a_{ir} + \rho_{ir} \pi_{irt-1} + \sum_{j=1}^3 \beta_{ir,j} 1\{q_t = j\} + \delta_{ir,1} \bar{\pi}_{rt} + \delta_{ir,2} \bar{\pi}_t \\ & + \sum_{j=1}^3 \lambda_{ir,j} 1\{q_{t-1} = j\} + \omega_{ir,1} \bar{\pi}_{rt-1} + \omega_{ir,2} \bar{\pi}_{t-1} + \mathbf{g}'_{ir} \mathbf{h}_t + \epsilon_{irt}. \end{aligned} \quad (56)$$

To estimate the number of latent factors, m , we consider the information criteria IC_{P1} and IC_{P2} proposed by Bai and Ng (2002), and the ER and GR criteria proposed by Ahn and Horenstein (2013). Given the spatial diversity of U.S. housing market, we set $m_{\max} = 10$, although once we allow for national and regional factors we would expect m_0 and its estimate, $\hat{m}$, to be relatively small. The estimated number of factors and the associated CD test statistics are summarized in Table 5. The first four columns of the table report $\hat{m}$, CD , CD^* and CD_{W+} statistics that are not adjusted for error serial correlation, whilst the middle and the final four columns report, respectively, the variance adjusted and ARDL adjusted versions of these statistics.

As can be seen in the case of no error serial correlations, there are large differences in the number of factors selected by the different criteria, with IC_{p1} selecting the assumed maximum number of factors, IC_{p2} selecting 4, and ER and GR both selecting 2 factors. These estimates are not affected when we allow for error serial correlations and consider variance adjustment. Alternatively, when ARDL adjustment is applied, the criteria select even fewer factors.

CD , CD^* and CD_{W+} tests all reject the null hypothesis of cross-sectional independence, irrespective of the choices of $\hat{m}$ and whether we allow for error serial correlation. In view of the theoretical and finite sample results reported in this paper, it is advisable to focus on the CD^* test results and recognize that the relatively large magnitudes obtained for CD and CD_{W+} test statistics could be due to their tendencies to over-rejection in presence of strong latent factors and non-Gaussian errors. Focusing on CD^* , we find that even with $\hat{m} = 10$ we still find that CD^* test strongly rejects the null of cross-sectional independence with CD^* statistic of 25.5, compared to 95 per cent critical value of 1.96. There is a clear evidence that in addition to latent factors, spatial modeling of the type carried out in Bailey et al. (2016) and Aquaro et al. (2021) is likely to be necessary to account for the remaining error cross-sectional dependence.

9 Concluding remarks

In this paper we have revisited the problem of testing error cross-sectional independence in panel data models with latent factors. Starting with a pure latent multi-factor model we show

Table 5: Tests of error cross-sectional dependence for the house price application

	Not adjusted for error serial correlation				Adjusted for error serial correlation							
					variance adjusted				ARDL adjusted			
	$\hat{m}$	CD	CD^*	CD_{W+}	$\hat{m}$	CD	CD^*	CD_{W+}	$\hat{m}$	CD	CD^*	CD_{W+}
IC_{p1}	10	22.2	53.8	1080.9	10	10.5	25.5	512.2	10	59.1	64.7	1171.9
IC_{p2}	4	112.4	122.4	1444.8	4	50.7	55.3	652.3	2	64.7	66.3	1598.4
ER	2	107.9	117.9	1588.4	2	47.3	51.8	697.1	1	60.3	61.4	1766.8
GR	2	107.9	117.9	1588.4	2	47.3	51.8	697.1	1	60.3	61.4	1766.8

Note: The test statistics are computed using the residuals from equation (53) for the first eight columns and the residuals from equation (56) for the last four columns. The panel regressions allow for seasonal dummies and national and regional effects. IC_{p1} , IC_{p2} denote the two information criteria by Bai and Ng (2002), while ER , GR denote the two criteria proposed by Ahn and Horenstein (2013). The maximum number of latent factors, $m_{\max}$, is set as 10. The number of selected factors is denoted by $\hat{m}$. CD denotes the standard CD test statistic, CD^* denotes the bias-corrected CD test statistic, and CD_{W+} denotes JR’s power-enhanced randomized CD statistic.

that the standard CD test proposed by Pesaran (2004) remains valid if the latent factors are weak, but over-reject when one or more of the latent factors are strong. The over-rejection of the CD test in the case of strong factors is also established by Juodis and Reese (2022), who propose a randomized test statistic to correct for over-rejection and add a screening component to achieve power. However, as we show, JR’s CD_{W+} test is not guaranteed to have the correct size and need not be powerful against spatial or network alternatives. Such alternatives are of particular interest in the analyses of ripple effects in housing markets, and clustering of firms within industries in capital or arbitrage asset pricing models. In fact, using Monte Carlo experiments we show that under non-Gaussian errors the JR test continues to over-reject when the cross section dimension (n) is larger than the time dimension (T), and often has power close to size against spatial alternatives. To overcome some of these shortcomings, we propose a simple bias-corrected CD test, labeled CD^* , which is shown to be asymptotically $N(0, 1)$ under the null when n and $T \rightarrow \infty$ such that $n/T \rightarrow \kappa$, for a fixed constant κ . In addition, CD^* is shown to have power against network type dependence. These results hold for pure latent factor models as well as for panel regression models with latent factors. To deal with possible error serial dependence, following Baltagi et al. (2016), we also consider a variance adjusted version of CD^* , as well as an alternative ARDL adjusted version that eliminates the error serial dependence before the application of the CD^* test procedure. Both of these approaches are shown to perform well within the Monte Carlo set up of the paper.

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Online Supplement to

How to Detect Network Dependence in Latent Factor Models? A Bias-Corrected CD Test

by

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This online supplement is in two parts. Section S1 provides mathematical appendix of the main paper. Section S2 reports Monte Carlo simulation results for the experiments discussed in Section 7 of the main paper.

Notations: For the $n \times n$ matrix $\mathbf{A} = (a_{ij})$, we denote its smallest and largest eigenvalues by $\mu_{\min}(\mathbf{A})$ and $\mu_{\max}(\mathbf{A})$, respectively, its trace by $\text{tr}(\mathbf{A}) = \sum_{i=1}^n a_{ii}$, its Frobenius norm by $\|\mathbf{A}\|_F = (\text{tr}(\mathbf{A}'\mathbf{A}))^{1/2}$, its spectral norm by $\|\mathbf{A}\| = \mu_{\max}^{1/2}(\mathbf{A}'\mathbf{A}) \leq \|\mathbf{A}\|_F$, its maximum absolute column sum norm by $\|\mathbf{A}\|_1 = \max_{1 \leq j \leq n} (\sum_{i=1}^n |a_{ij}|)$, and its maximum absolute row sum norm by $\|\mathbf{A}\|_\infty = \max_{1 \leq i \leq n} (\sum_{j=1}^n |a_{ij}|)$. We write $\mathbf{A} > \mathbf{0}$ when $\mathbf{A}$ is positive definite. For matrices $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$, let $\mathbf{A} \odot \mathbf{B} = \mathbf{B} \odot \mathbf{A}$ denote Hadamard product with elements $a_{ij}b_{ij}$. $\rightarrow_p$ denotes convergence in probability, $\rightarrow_d$ convergence in distribution, and $\overset{a}{\sim}$ asymptotic equivalence in distribution. $O_p(\cdot)$ and $o_p(\cdot)$ denote the stochastic order relations. In particular, $o_p(1)$ indicates terms that tend to zero in probability as $(n, T) \rightarrow \infty$, such that $n/T \rightarrow \kappa$, where $0 < \kappa < \infty$. C and c will be used to denote finite large and non-zero small positive numbers, respectively, that are bounded in n and T . They can take different values at different instances. $\delta_{nT} = \min(\sqrt{n}, \sqrt{T})$. If $\{f_n\}_{n=1}^\infty$ is any real sequence and $\{g_n\}_{n=1}^\infty$ is a sequence of positive real numbers, then $f_n = O(g_n)$, if there exists C such that $|f_n|/g_n \leq C$ for all n . $f_n = o(g_n)$ if $f_n/g_n \rightarrow 0$ as $n \rightarrow \infty$. If $\{f_n\}_{n=1}^\infty$ and $\{g_n\}_{n=1}^\infty$ are both positive sequences of real numbers, then $f_n = \Theta(g_n)$ if there exists $n_0 \geq 1$ and positive finite constants C_0 and C_1 , such that $\inf_{n \geq n_0} (f_n/g_n) \geq C_0$, and $\sup_{n \geq n_0} (f_n/g_n) \leq C_1$.

S1 Mathematical appendix

Section S1.1 provides proofs of Theorems 1-3 and Proposition 1 in the paper. Section S1.2 states and establishes the auxiliary lemmas used in these proofs. Section S1.3 derives the order of θ_n in terms of the factor strengths. Section S1.4 considers CD_{W+} test proposed by Juodis and Reese (2022), and discusses some of its properties.

S1.1 Proofs of the theorems

Proof of Theorem 1. Here we provide a proof for part (b) of Theorem 1. The proof for part (a) follows trivially by setting $\lambda_T = 0$. To this end we first note that the CD statistic given by (16) can be written as (for a proof see Lemma S.1)

$$CD = \left(\sqrt{\frac{n}{n-1}} \right) \frac{1}{\sqrt{2T}} \sum_{t=1}^T \left[\left(\frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{\hat{u}_{it}}{\hat{\sigma}_{i,T}} \right)^2 - 1 \right],$$

where $\hat{u}_{it}$ is defined by (13). Using Lemma S.15 we also note that

$$CD = \widetilde{CD} + o_p(1), \quad (\text{S.1})$$

where

$$\widetilde{CD} = \left(\sqrt{\frac{n}{n-1}} \right) \frac{1}{\sqrt{T}} \sum_{t=1}^T \left[\frac{\left(\frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{\hat{u}_{it}}{\omega_{i,T}} \right)^2 - 1}{\sqrt{2}} \right], \quad (\text{S.2})$$

with $\omega_{i,T} = (T^{-1} \sigma_i^2 \boldsymbol{\varepsilon}'_{io} \mathbf{M}_F \boldsymbol{\varepsilon}_{io})^{1/2}$, $\boldsymbol{\varepsilon}_{io} = (\varepsilon_{i1}, \varepsilon_{i2}, \dots, \varepsilon_{iT})'$ and $\mathbf{M}_F = \mathbf{I}_T - \mathbf{F}(\mathbf{F}'\mathbf{F})^{-1}\mathbf{F}'$. Also letting

$$\varepsilon_{it}(\lambda_T) = \varepsilon_{it} + \lambda_T \mathbf{w}'_{i0} \boldsymbol{\varepsilon}_{ot}, \quad (\text{S.3})$$

where $\lambda_T = c_\lambda T^{-1/2}$ with $c_\lambda \neq 0$, $\mathbf{w}_{i0} = (w_{i1}, w_{i2}, \dots, w_{in})'$ and $\boldsymbol{\varepsilon}_{ot} = (\varepsilon_{1t}, \varepsilon_{2t}, \dots, \varepsilon_{nt})'$, under (1) $\hat{u}_{it}$ can now be expressed as

$$\hat{u}_{it} = \sigma_i \varepsilon_{it}(\lambda_T) - \boldsymbol{\gamma}'_i (\hat{\mathbf{f}}_t - \mathbf{f}_t) - (\hat{\boldsymbol{\gamma}}_i - \boldsymbol{\gamma}_i)' \mathbf{f}_t - (\hat{\boldsymbol{\gamma}}_i - \boldsymbol{\gamma}_i)' (\hat{\mathbf{f}}_t - \mathbf{f}_t). \quad (\text{S.4})$$

Let $\boldsymbol{\delta}_{i,T} = \boldsymbol{\gamma}_i / \omega_{i,T}$, and $\hat{\boldsymbol{\delta}}_{i,T} = \hat{\boldsymbol{\gamma}}_i / \omega_{i,T}$. Then

$$\hat{u}_{it} / \omega_{i,T} = \sigma_i \varepsilon_{it}(\lambda_T) / \omega_{i,T} - \boldsymbol{\delta}'_{i,T} (\hat{\mathbf{f}}_t - \mathbf{f}_t) - (\hat{\boldsymbol{\delta}}_{i,T} - \boldsymbol{\delta}_{i,T})' \mathbf{f}_t - (\hat{\boldsymbol{\delta}}_{i,T} - \boldsymbol{\delta}_{i,T})' (\hat{\mathbf{f}}_t - \mathbf{f}_t). \quad (\text{S.5})$$

Also, subject to the normalization $n^{-1} \sum_{j=1}^n \hat{\boldsymbol{\gamma}}_j \hat{\boldsymbol{\gamma}}'_j = \mathbf{I}_{m_0}$ and $n^{-1} \sum_{j=1}^n \boldsymbol{\gamma}_j \boldsymbol{\gamma}'_j = \mathbf{I}_{m_0}$ we have

$$\hat{\mathbf{f}}_t = n^{-1} \sum_{j=1}^n \hat{\boldsymbol{\gamma}}_j y_{jt} = \left(n^{-1} \sum_{j=1}^n \hat{\boldsymbol{\gamma}}_j \boldsymbol{\gamma}'_j \right) \mathbf{f}_t + n^{-1} \sum_{j=1}^n \hat{\boldsymbol{\gamma}}_j \sigma_j \varepsilon_{jt}(\lambda_T),$$

and hence

$$\hat{\mathbf{f}}_t - \mathbf{f}_t = \left[n^{-1} \sum_{j=1}^n (\hat{\boldsymbol{\gamma}}_j - \boldsymbol{\gamma}_j) \boldsymbol{\gamma}'_j \right] \mathbf{f}_t + n^{-1} \sum_{j=1}^n (\hat{\boldsymbol{\gamma}}_j - \boldsymbol{\gamma}_j) \sigma_j \varepsilon_{jt}(\lambda_T) + n^{-1} \sum_{j=1}^n \boldsymbol{\gamma}_j \sigma_j \varepsilon_{jt}(\lambda_T).$$

Using this result in (S.5) we obtain

$$\begin{aligned} \hat{u}_{it} / \omega_{i,T} &= \sigma_i \varepsilon_{it}(\lambda_T) / \omega_{i,T} - \boldsymbol{\delta}'_{i,T} \left[n^{-1} \sum_{j=1}^n \boldsymbol{\gamma}_j \sigma_j \varepsilon_{jt}(\lambda_T) \right] \\ &\quad - \boldsymbol{\delta}'_{i,T} \left[n^{-1} \sum_{j=1}^n (\hat{\boldsymbol{\gamma}}_j - \boldsymbol{\gamma}_j) \boldsymbol{\gamma}'_j \right] \mathbf{f}_t - \boldsymbol{\delta}'_{i,T} \left[n^{-1} \sum_{j=1}^n (\hat{\boldsymbol{\gamma}}_j - \boldsymbol{\gamma}_j) \sigma_j \varepsilon_{jt}(\lambda_T) \right] \\ &\quad - (\hat{\boldsymbol{\delta}}_{i,T} - \boldsymbol{\delta}_{i,T})' \mathbf{f}_t - (\hat{\boldsymbol{\delta}}_{i,T} - \boldsymbol{\delta}_{i,T})' (\hat{\mathbf{f}}_t - \mathbf{f}_t), \end{aligned} \quad (\text{S.6})$$

and summing over i yields

$$\begin{aligned} n^{-1/2} \sum_{i=1}^n \hat{u}_{it} / \omega_{i,T} &= n^{-1/2} \sum_{i=1}^n \sigma_i \varepsilon_{it}(\lambda_T) / \omega_{i,T} - \boldsymbol{\varphi}'_{nT} \left(n^{-1/2} \sum_{i=1}^n \boldsymbol{\gamma}_i \sigma_i \varepsilon_{it}(\lambda_T) \right) \\ &\quad - \boldsymbol{\varphi}'_{nT} \left[n^{-1/2} \sum_{i=1}^n (\hat{\boldsymbol{\gamma}}_i - \boldsymbol{\gamma}_i) \boldsymbol{\gamma}'_i \right] \mathbf{f}_t - \boldsymbol{\varphi}'_{nT} \left[n^{-1/2} \sum_{i=1}^n (\hat{\boldsymbol{\gamma}}_i - \boldsymbol{\gamma}_i) \sigma_i \varepsilon_{it}(\lambda_T) \right] \\ &\quad - \left[n^{-1/2} \sum_{i=1}^n (\hat{\boldsymbol{\delta}}_{i,T} - \boldsymbol{\delta}_{i,T})' \right] \mathbf{f}_t - \left[n^{-1/2} \sum_{i=1}^n (\hat{\boldsymbol{\delta}}_{i,T} - \boldsymbol{\delta}_{i,T})' \right]' (\hat{\mathbf{f}}_t - \mathbf{f}_t), \end{aligned}$$

where $\boldsymbol{\varphi}_{nT} = n^{-1} \sum_{i=1}^n \boldsymbol{\delta}_{i,T}$. Written more compactly

$$h_{t,nT}(\lambda_T) = n^{-1/2} \sum_{i=1}^n \hat{u}_{it}/\omega_{i,T} = \psi_{t,nT}(\lambda_T) - s_{t,nT}(\lambda_T), \quad (\text{S.7})$$

where

$$\psi_{t,nT}(\lambda_T) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{a_{i,nT} \sigma_i \varepsilon_{it}(\lambda_T)}{\omega_{i,T}}, \quad a_{i,nT} = 1 - \omega_{i,T} \boldsymbol{\varphi}'_{nT} \boldsymbol{\gamma}_i, \quad (\text{S.8})$$

$$s_{t,nT}(\lambda_T) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \left[\boldsymbol{\varphi}'_{nT} (\hat{\boldsymbol{\gamma}}_i - \boldsymbol{\gamma}_i) \sigma_i \varepsilon_{it}(\lambda_T) + \left(\hat{\boldsymbol{\delta}}_{i,T} - \boldsymbol{\delta}_{i,T} \right)' \hat{\mathbf{f}}_t + \boldsymbol{\varphi}'_{nT} (\hat{\boldsymbol{\gamma}}_i - \boldsymbol{\gamma}_i) \boldsymbol{\gamma}'_i \mathbf{f}_t \right]. \quad (\text{S.9})$$

Further, let

$$\xi_{t,n}(\lambda_T) = \frac{1}{\sqrt{n}} \sum_{i=1}^n a_{i,n} \varepsilon_{it}(\lambda_T), \quad a_{i,n} = 1 - \sigma_i \boldsymbol{\varphi}'_n \boldsymbol{\gamma}_i, \quad (\text{S.10})$$

where $\boldsymbol{\varphi}_n = n^{-1} \sum_{i=1}^n \boldsymbol{\delta}_i$, and $\boldsymbol{\delta}_i = \boldsymbol{\gamma}_i/\sigma_i$. Then $\psi_{t,nT}(\lambda_T)$, given by (S.8), can be written as

$$\begin{aligned} \psi_{t,nT}(\lambda_T) &= \xi_{t,n}(\lambda_T) + \frac{1}{\sqrt{n}} \sum_{i=1}^n (1 - \omega_{i,T} \boldsymbol{\varphi}'_{nT} \boldsymbol{\gamma}_i) \frac{\sigma_i \varepsilon_{it}(\lambda_T)}{\omega_{i,T}} - \frac{1}{\sqrt{n}} \sum_{i=1}^n (1 - \sigma_i \boldsymbol{\varphi}'_n \boldsymbol{\gamma}_i) \varepsilon_{it}(\lambda_T) \\ &= \xi_{t,n}(\lambda_T) - \frac{1}{\sqrt{n}} \sum_{i=1}^n \boldsymbol{\varphi}'_{nT} \boldsymbol{\gamma}_i \sigma_i \varepsilon_{it}(\lambda_T) + \frac{1}{\sqrt{n}} \sum_{i=1}^n \sigma_i \boldsymbol{\varphi}'_n \boldsymbol{\gamma}_i \varepsilon_{it}(\lambda_T) + \frac{1}{\sqrt{n}} \sum_{i=1}^n \left(\frac{\sigma_i \varepsilon_{it}(\lambda_T)}{\omega_{i,T}} - \varepsilon_{it}(\lambda_T) \right) \\ &= \xi_{t,n}(\lambda_T) + \frac{1}{\sqrt{n}} \sum_{i=1}^n \zeta_{it}(\lambda_T) - (\boldsymbol{\varphi}_{nT} - \boldsymbol{\varphi}_n)' \frac{1}{\sqrt{n}} \sum_{i=1}^n \boldsymbol{\gamma}_i \sigma_i \varepsilon_{it}(\lambda_T), \end{aligned}$$

where

$$\zeta_{it}(\lambda_T) = \left[\frac{1}{(T^{-1} \boldsymbol{\varepsilon}'_{i0} \mathbf{M}_F \boldsymbol{\varepsilon}_{i0})^{1/2}} - 1 \right] \varepsilon_{it}(\lambda_T). \quad (\text{S.11})$$

Writing $\psi_{t,nT}(\lambda_T)$ more compactly we have

$$\psi_{t,nT}(\lambda_T) = \xi_{t,n}(\lambda_T) + v_{t,nT}(\lambda_T) - (\boldsymbol{\varphi}_{nT} - \boldsymbol{\varphi}_n)' \boldsymbol{\kappa}_{t,n}(\lambda_T), \quad (\text{S.12})$$

where

$$\xi_{t,n}(\lambda_T) = \frac{1}{\sqrt{n}} \sum_{i=1}^n a_{i,n} \varepsilon_{it}(\lambda_T), \quad a_{i,n} = 1 - \sigma_i \boldsymbol{\varphi}'_n \boldsymbol{\gamma}_i, \quad (\text{S.13})$$

$$\boldsymbol{\kappa}_{t,n}(\lambda_T) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \boldsymbol{\gamma}_i \sigma_i \varepsilon_{it}(\lambda_T), \quad (\text{S.14})$$

$$v_{t,nT}(\lambda_T) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \zeta_{it}(\lambda_T). \quad (\text{S.15})$$

Using (S.7) in (S.2) and after some algebra we have (where we have made the dependence of $\widetilde{CD}$ on λ_T explicit)

$$\widetilde{CD}(\lambda_T) = \left(\sqrt{\frac{n}{n-1}} \right) \frac{1}{\sqrt{T}} \sum_{t=1}^T \left(\frac{\psi_{t,nT}^2(\lambda_T) - 1}{\sqrt{2}} \right) + \left(\sqrt{\frac{n}{n-1}} \right) (p_{nT}(\lambda_T) - q_{nT}(\lambda_T)),$$

where $\psi_{t,nT}(\lambda_T)$ is defined by (S.12),

$$p_{nT}(\lambda_T) = T^{-1/2} \sum_{t=1}^T s_{t,nT}^2(\lambda_T), \quad (\text{S.16})$$

and

$$q_{nT}(\lambda_T) = T^{-1/2} \sum_{t=1}^T \psi_{t,nT}(\lambda_T) s_{t,nT}(\lambda_T). \quad (\text{S.17})$$

By Lemma S.10 in Section S1.2 $p_{nT}(\lambda_T) = o_p(1)$, and $q_{nT}(\lambda_T) = o_p(1)$. Hence

$$\widetilde{CD}(\lambda_T) = \frac{1}{\sqrt{T}} \sum_{t=1}^T \left(\frac{\psi_{t,nT}^2(\lambda_T) - 1}{\sqrt{2}} \right) + o_p(1). \quad (\text{S.18})$$

Now consider $T^{-1/2} \sum_{t=1}^T \psi_{t,nT}^2(\lambda_T)$ and using (S.12) note that

$$\begin{aligned} \frac{1}{\sqrt{T}} \sum_{t=1}^T \psi_{t,nT}^2(\lambda_T) &= \left(\frac{1}{\sqrt{T}} \sum_{t=1}^T \xi_{t,n}^2(\lambda_T) \right) \left(1 + \frac{2 \sum_{t=1}^T \xi_{t,n}(\lambda_T) v_{t,nT}(\lambda_T)}{\sum_{t=1}^T \xi_{t,n}^2(\lambda_T)} \right) + \frac{1}{\sqrt{T}} \sum_{t=1}^T v_{t,nT}^2(\lambda_T) \\ &\quad + \sqrt{T} (\boldsymbol{\varphi}_{nT} - \boldsymbol{\varphi}_n)' \left(\frac{\sum_{t=1}^T \boldsymbol{\kappa}_{t,n}(\lambda_T) \boldsymbol{\kappa}_{t,n}'(\lambda_T)}{T} \right) (\boldsymbol{\varphi}_{nT} - \boldsymbol{\varphi}_n) \\ &\quad - 2\sqrt{T} (\boldsymbol{\varphi}_{nT} - \boldsymbol{\varphi}_n)' \left(\frac{1}{T} \sum_{t=1}^T \boldsymbol{\kappa}_{t,n}(\lambda_T) v_{t,nT}(\lambda_T) \right) \\ &\quad - 2\sqrt{T} (\boldsymbol{\varphi}_{nT} - \boldsymbol{\varphi}_n)' \left(\frac{1}{T} \sum_{t=1}^T \boldsymbol{\kappa}_{t,n}(\lambda_T) \xi_{t,n}(\lambda_T) \right). \end{aligned}$$

By Lemma S.17, it follows

$$\frac{1}{\sqrt{T}} \sum_{t=1}^T \psi_{t,nT}^2(\lambda_T) = \left(\frac{1}{\sqrt{T}} \sum_{t=1}^T \xi_{t,n}^2(\lambda_T) \right) \left(1 + \frac{2 \sum_{t=1}^T \xi_{t,n}(\lambda_T) v_{t,nT}(\lambda_T)}{\sum_{t=1}^T \xi_{t,n}^2(\lambda_T)} \right) + o_p(1). \quad (\text{S.19})$$

Consider now the bias-corrected version of $\widetilde{CD}(\lambda_T)$ defined by

$$\widetilde{CD}^*(\lambda_T) = \frac{\widetilde{CD}(\lambda_T) + \sqrt{\frac{T}{2}} \theta_n}{1 - \theta_n} \quad (\text{S.20})$$

where $\theta_n = 1 - \frac{1}{n} \sum_{i=1}^n a_{i,n}^2$, and $a_{i,n} = 1 - \sigma_i \boldsymbol{\varphi}_n' \boldsymbol{\gamma}_i$. Using (S.18) in (S.20), we have

$$\begin{aligned} \widetilde{CD}^*(\lambda_T) &= \frac{\frac{1}{\sqrt{T}} \sum_{t=1}^T \left(\frac{\psi_{t,nT}^2(\lambda_T) - 1}{\sqrt{2}} \right) + \sqrt{\frac{T}{2}} \theta_n}{1 - \theta_n} + o_p(1) \\ &= \frac{\frac{1}{\sqrt{T}} \sum_{t=1}^T \left(\frac{\psi_{t,nT}^2(\lambda_T)}{\sqrt{2}} \right) - \sqrt{\frac{T}{2}} (1 - \theta_n)}{1 - \theta_n} + o_p(1). \end{aligned}$$

Now using (S.19) in the above and after some re-arrangement of the terms we obtain

$$\widetilde{CD}^*(\lambda_T) = \frac{\left[\frac{1}{\sqrt{T}} \sum_{t=1}^T \left(\frac{\xi_{t,n}^2(\lambda_T) - (1-\theta_n)}{\sqrt{2}} \right) \right] \left(1 + \frac{2}{\sqrt{T}} w_{nT}(\lambda_T) \right)}{1 - \theta_n} + \sqrt{2} w_{nT}(\lambda_T) + o_p(1),$$

where

$$w_{nT}(\lambda_T) = \frac{T^{-1/2} \sum_{t=1}^T \xi_{t,n}(\lambda_T) v_{t,nT}(\lambda_T)}{T^{-1} \sum_{t=1}^T \xi_{t,n}^2(\lambda_T)}.$$

By Lemma S.18, $w_{nT}(\lambda_T) = o_p(1)$. Hence

$$\widetilde{CD}^*(\lambda_T) = \frac{\frac{1}{\sqrt{T}} \sum_{t=1}^T \left(\frac{\xi_{t,n}^2(\lambda_T) - (1-\theta_n)}{\sqrt{2}} \right)}{1 - \theta_n} + o_p(1).$$

In particular, since $\varepsilon_{it}(\lambda_T) = \varepsilon_{it} + \lambda_T \mathbf{w}'_{i0} \varepsilon_{ot}$, then

$$\begin{aligned} \xi_{t,n}(\lambda_T) &= \frac{1}{\sqrt{n}} \sum_{i=1}^n a_{i,n} \varepsilon_{it}(\lambda_T) \\ &= \frac{1}{\sqrt{n}} \sum_{i=1}^n a_{i,n} \varepsilon_{it} + \lambda_T \frac{1}{\sqrt{n}} \sum_{i=1}^n a_{i,n} \mathbf{w}'_{i0} \varepsilon_{ot} \\ &\equiv A_{nt} + \lambda_T B_{nt}. \end{aligned}$$

Using this result, $\widetilde{CD}^*(\lambda_T)$ can be written as

$$\begin{aligned} \widetilde{CD}^*(\lambda_T) &= \frac{\frac{1}{\sqrt{T}} \sum_{t=1}^T \left(\frac{\xi_{t,n}^2(\lambda_T) - (1-\theta_n)}{\sqrt{2}} \right)}{1 - \theta_n} + o_p(1) \\ &= \frac{\frac{1}{\sqrt{T}} \sum_{t=1}^T \left(\frac{A_{nt}^2 + 2\lambda_T A_{nt} B_{nt} + \lambda_T^2 B_{nt}^2 - (1-\theta_n)}{\sqrt{2}} \right)}{1 - \theta_n} + o_p(1) \\ &= \frac{\frac{1}{\sqrt{T}} \sum_{t=1}^T \left(\frac{A_{nt}^2 - (1-\theta_n)}{\sqrt{2}} \right)}{1 - \theta_n} + \frac{\sqrt{2}\lambda_T}{(1 - \theta_n) \sqrt{T}} \sum_{t=1}^T A_{nt} B_{nt} + \frac{\frac{\lambda_T^2}{\sqrt{2}\sqrt{T}} \sum_{t=1}^T B_{nt}^2}{1 - \theta_n} + o_p(1) \\ &= \widetilde{CD}^*(0) + \phi_{nT} + g_{nT} + o_p(1). \end{aligned} \tag{S.21}$$

The second term of (S.21) can be written as

$$\begin{aligned} \phi_{nT} &= \frac{\sqrt{2}\lambda_T}{(1 - \theta_n) \sqrt{T}} \sum_{t=1}^T A_{nt} B_{nt} \\ &= \phi_n + \frac{\sqrt{2}c_\lambda}{(1 - \theta_n)} T^{-1} \sum_{t=1}^T [A_{nt} B_{nt} - E(A_{nt} B_{nt})] \end{aligned}$$

where $\phi_n = \frac{\sqrt{2}c_\lambda}{(1-\theta_n)} T^{-1} \sum_{t=1}^T E(A_{nt}B_{nt})$. Further

$$\begin{aligned} E(A_{nt}B_{nt}) &= E\left(\frac{1}{\sqrt{n}} \sum_{i=1}^n a_{i,n} \varepsilon_{it}\right) \left(\frac{1}{\sqrt{n}} \sum_{j=1}^n a_{j,n} \mathbf{w}'_{j0} \varepsilon_{ot}\right) \\ &= \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n a_{j,n} a_{i,n} E(\mathbf{w}'_{j0} \varepsilon_{ot} \varepsilon_{it}) = \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n a_{j,n} a_{i,n} w_{ji} \\ &= n^{-1} \mathbf{a}'_n \mathbf{W} \mathbf{a}_n \end{aligned}$$

with $\mathbf{a}_n = (a_{1,n}, a_{2,n}, \dots, a_{n,n})'$. Hence

$$\phi_n = \frac{\sqrt{2}c_\lambda}{(1-\theta_n)} \left(T^{-1} \sum_{t=1}^T E(A_{nt}B_{nt}) \right) = \frac{\sqrt{2}c_\lambda}{(1-\theta_n)} n^{-1} \mathbf{a}'_n \mathbf{W} \mathbf{a}_n. \quad (\text{S.22})$$

Under part (a) of Assumption 2, $\varepsilon_{it} \sim IID(0, 1)$ for all i and t , with $E(\varepsilon_{it}^8) < C$, it then follows that $A_{nt}B_{nt} - E(A_{nt}B_{nt})$ will be serially independent with zero means and finite variances and by weak law of large numbers $T^{-1} \sum_{t=1}^T [A_{nt}B_{nt} - E(A_{nt}B_{nt})] = o_p(1)$. Therefore

$$\phi_{nT} = \phi_n + o_p(1). \quad (\text{S.23})$$

Similarly, for the third term of (S.21) we first note that

$$\begin{aligned} T^{-1} \sum_{t=1}^T B_{nt}^2 &\rightarrow_p E(B_{nt}^2) = \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n a_{i,n} a_{j,n} \mathbf{w}'_{i0} E(\varepsilon_{ot} \varepsilon'_{ot}) \mathbf{w}_{j0} \\ &= \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n a_{i,n} a_{j,n} \mathbf{w}'_{i0} \mathbf{w}_{j0} = \frac{1}{n} \sum_{s=1}^n \sum_{i=1}^n \sum_{j=1}^n a_{i,n} w_{is} a_{j,n} w_{js} \\ &= \frac{1}{n} \sum_{s=1}^n \left(\sum_{i=1}^n a_{i,n} w_{is} \right)^2 < C, \end{aligned}$$

and hence

$$g_{nT} = \frac{\frac{\lambda_T^2}{\sqrt{2}\sqrt{T}} \sum_{t=1}^T B_{nt}^2}{1 - \theta_n} = \frac{c_\lambda^2}{\sqrt{2T}(1 - \theta_n)} \left(T^{-1} \sum_{t=1}^T B_{nt}^2 \right) = O_p\left(\frac{1}{\sqrt{T}}\right). \quad (\text{S.24})$$

Using (S.23) and (S.24) in (S.21) now yields

$$\widetilde{CD}^*(\lambda_T) = \widetilde{CD}^*(0) + \phi_n + o_p(1). \quad (\text{S.25})$$

Consider the first term of (S.21), $\widetilde{CD}^*(0)$, and note that A_{nt} can be written as $A_{nt} = n^{-1/2} \mathbf{a}'_n \varepsilon_{ot}$, where $\mathbf{a}'_n = (a_{1,n}, a_{2,n}, \dots, a_{n,n})$, and we have (recall that $a_i = 1 - \sigma_i \varphi'_n \gamma_i$)

$$E(A_{nt}^2) = n^{-1} E(\varepsilon'_{ot} \mathbf{a}_n \mathbf{a}'_n \varepsilon_{ot}) = n^{-1} \mathbf{a}'_n \mathbf{a}_n = n^{-1} \sum_{i=1}^n a_{i,n}^2 = \frac{1}{n} \sum_{i=1}^n (1 - \sigma_i \varphi'_n \gamma_i)^2 = 1 - \theta_n > 0,$$

and (using result (S.7) of Lemma 6 in Pesaran and Yamagata (2024))

$$\begin{aligned} E(A_{nt}^4) &= n^2 E[(\boldsymbol{\varepsilon}'_{ot} \mathbf{a}_n \mathbf{a}'_n \boldsymbol{\varepsilon}_{ot})^2] = \kappa_2 n^{-2} \text{tr}[\mathbf{A} \odot \mathbf{A}] + n^{-2} [\text{tr}(\mathbf{A})]^2 + 2n^{-2} \text{tr}(\mathbf{A}^2) \\ &= \kappa_2 n^{-2} \sum_{i=1}^n a_{i,n}^4 + 3 \left(n^{-1} \sum_{i=1}^n a_{i,n}^2 \right)^2, \end{aligned}$$

where $\kappa_2 = E(\varepsilon_{it}^4) - 3$ and $\mathbf{A} = \mathbf{a}_n \mathbf{a}'_n$. Hence

$$\text{Var}(A_{nt}^2) = 2 \left(\frac{1}{n} \sum_{i=1}^n a_{i,n}^2 \right)^2 - \frac{\kappa_2}{n} \left(\frac{1}{n} \sum_{i=1}^n a_{i,n}^4 \right).$$

Furthermore, since $\sup_i |a_{i,n}| = \sup_i |1 - \sigma_i \boldsymbol{\varphi}'_n \boldsymbol{\gamma}_i| < 1 + (\sup_i \sigma_i) (\sup_i \|\boldsymbol{\gamma}_i\|) \|\boldsymbol{\varphi}_n\| < C$, then $n^{-2} \sum_{i=1}^n a_{i,n}^4 = O(n^{-1})$, and $\text{Var}(A_{nt}^2) = 2(1 - \theta_n)^2 + O(n^{-1})$. Using the above results it now readily follows that,

$$\widetilde{CD}^*(0) = \frac{\frac{1}{\sqrt{T}} \sum_{t=1}^T \left(\frac{A_{nt}^2 - (1 - \theta_n)}{\sqrt{2}} \right)}{1 - \theta_n} = \frac{1}{\sqrt{T}} \sum_{t=1}^T \left(\frac{A_{nt}^2 - E(A_{nt}^2)}{\sqrt{\text{Var}(A_{nt}^2) + O(n^{-1})}} \right). \quad (\text{S.26})$$

Since under part (a) of Assumption 2, $\varepsilon_{it} \sim IID(0, 1)$ for all i and t , then $A_{nt} = \frac{1}{\sqrt{n}} \sum_{i=1}^n a_{i,n} \varepsilon_{it}$, and A_{nt}^2 are also independently distributed over t with finite second order moments. Then by Lindeberg-Lévy central limit theorem it follows that $\widetilde{CD}^*(0) \rightarrow_d N(0, 1)$, as n and $T \rightarrow \infty$. Using this result in (S.25) we further have $\widetilde{CD}^*(\lambda_T) \rightarrow_d N(\phi, 1)$, where $\phi = \lim_{n \rightarrow \infty} \phi_n$. By Lemma S.15 we have $CD = \widetilde{CD} + o_p(1)$, then it follows

$$\begin{aligned} CD^*(\theta_n) &= \frac{CD + \sqrt{\frac{T}{2}} \theta_n}{1 - \theta_n} = \frac{\widetilde{CD} + \sqrt{\frac{T}{2}} \theta_n}{1 - \theta_n} + o_p(1) \\ &= \widetilde{CD}^*(\lambda_T) + o_p(1), \end{aligned}$$

where the final line holds by (S.20). Now result (26) can be established as required. ■

Proof of Proposition 1. Note that θ_n define by (22) can be written as $\theta_n = 2g_n - \boldsymbol{\varphi}'_n \mathbf{H}_n \boldsymbol{\varphi}_n$, where $g_n = n^{-1} \sum_{i=1}^n \sigma_i \boldsymbol{\varphi}'_n \boldsymbol{\gamma}_i$, $\mathbf{H}_n = n^{-1} \sum_{i=1}^n \sigma_i (\boldsymbol{\gamma}_i \boldsymbol{\gamma}'_i)$, $\boldsymbol{\varphi}_n = n^{-1} \sum_{i=1}^n \boldsymbol{\delta}_i$, and $\boldsymbol{\delta}_i = \boldsymbol{\gamma}_i / \sigma_i$. Similarly using (28) we have $\hat{\theta}_{nT} = 2\hat{g}_{nT} - \hat{\boldsymbol{\varphi}}'_{nT} \hat{\mathbf{H}}_{nT} \hat{\boldsymbol{\varphi}}_{nT}$, where $\hat{g}_{nT} = n^{-1} \sum_{i=1}^n \hat{\sigma}_{i,T} \hat{\boldsymbol{\varphi}}'_{nT} \hat{\boldsymbol{\gamma}}_i$, $\hat{\mathbf{H}}_{nT} = n^{-1} \sum_{i=1}^n \hat{\sigma}_{i,T}^2 (\hat{\boldsymbol{\gamma}}_i \hat{\boldsymbol{\gamma}}'_i)$, $\hat{\boldsymbol{\varphi}}_{nT} = n^{-1} \sum_{i=1}^n \hat{\boldsymbol{\delta}}_{i,nT}$, and $\hat{\boldsymbol{\delta}}_{i,nT} = \hat{\boldsymbol{\gamma}}_i / \hat{\sigma}_{i,T}$. Then

$$\sqrt{T} (\hat{\theta}_{nT} - \theta_n) = 2\sqrt{T} (\hat{g}_{nT} - g_n) - \sqrt{T} (\hat{\boldsymbol{\varphi}}'_{nT} \hat{\mathbf{H}}_{nT} \hat{\boldsymbol{\varphi}}_{nT} - \boldsymbol{\varphi}'_n \mathbf{H}_n \boldsymbol{\varphi}_n). \quad (\text{S.27})$$

Consider the first term of the above

$$\begin{aligned} 2\sqrt{T} (\hat{g}_{nT} - g_n) &= 2\sqrt{T} (\hat{\boldsymbol{\varphi}}_{nT} - \boldsymbol{\varphi}_n) \left(\frac{1}{n} \sum_{i=1}^n \sigma_i \boldsymbol{\gamma}_i \right) \\ &\quad + 2\sqrt{T} \left[\hat{\boldsymbol{\varphi}}'_{nT} \left(\frac{1}{n} \sum_{i=1}^n \hat{\sigma}_{i,T} \hat{\boldsymbol{\gamma}}_i - \frac{1}{n} \sum_{i=1}^n \sigma_i \boldsymbol{\gamma}_i \right) \right], \end{aligned} \quad (\text{S.28})$$

and since σ_i and γ_i are bounded then $n^{-1} \sum_{i=1}^n \sigma_i \gamma_i = O(1)$. Also by (S.114) of Lemma S.11 we have $\sqrt{T}(\hat{\varphi}_{nT} - \varphi_n) = o_p(1)$, and hence the first term of the above is $o_p(1)$. To establish the probability order of the second term of (S.28), we first note that

$$2\sqrt{T} \left[\hat{\varphi}'_{nT} \left(\frac{1}{n} \sum_{i=1}^n \hat{\sigma}_{i,T} \hat{\gamma}_i - \frac{1}{n} \sum_{i=1}^n \sigma_i \gamma_i \right) \right] = 2\sqrt{T} \left[\varphi'_n \left(\frac{1}{n} \sum_{i=1}^n \hat{\sigma}_{i,T} \hat{\gamma}_i - \frac{1}{n} \sum_{i=1}^n \sigma_i \gamma_i \right) \right] + 2\sqrt{T} \left[(\hat{\varphi}_{nT} - \varphi_n)' \left(\frac{1}{n} \sum_{i=1}^n \hat{\sigma}_{i,T} \hat{\gamma}_i - \frac{1}{n} \sum_{i=1}^n \sigma_i \gamma_i \right) \right]. \quad (\text{S.29})$$

But by (S.184) $\varphi_n = O_p(1)$, and by (S.156) $n^{-1} \sum_{i=1}^n (\hat{\sigma}_{i,T} \hat{\gamma}_i - \sigma_i \gamma_i) = O_p(\ln(n)/T)$, which also establishes that the second term of (S.29) is $o_p(1)$. Therefore overall we have

$$\sqrt{T}(\hat{g}_{nT} - g_n) = o_p(1). \quad (\text{S.30})$$

Consider now the second term of (S.27) and note that

$$\begin{aligned} \sqrt{T} \left(\hat{\varphi}'_{nT} \hat{\mathbf{H}}_{nT} \hat{\varphi}_{nT} - \varphi'_n \mathbf{H}_n \varphi_n \right) &= \sqrt{T} (\hat{\varphi}_{nT} - \varphi_n)' \hat{\mathbf{H}}_{nT} (\hat{\varphi}_{nT} - \varphi_n) \\ &\quad + 2\sqrt{T} (\hat{\varphi}_{nT} - \varphi_n)' \hat{\mathbf{H}}_{nT} \varphi_n + \sqrt{T} \varphi'_n (\hat{\mathbf{H}}_{nT} - \mathbf{H}_n) \varphi_n, \end{aligned} \quad (\text{S.31})$$

where $\hat{\mathbf{H}}_{nT} = \frac{1}{n} \sum_{i=1}^n \hat{\sigma}_{i,T}^2 (\hat{\gamma}_i \hat{\gamma}_i')$, and

$$\begin{aligned} \sqrt{T} (\hat{\mathbf{H}}_{nT} - \mathbf{H}_n) &= \frac{\sqrt{T}}{n} \sum_{i=1}^n \sigma_i^2 (\hat{\gamma}_i \hat{\gamma}_i' - \gamma_i \gamma_i') + \frac{\sqrt{T}}{n} \sum_{i=1}^n (\hat{\sigma}_{i,T}^2 - \sigma_i^2) (\hat{\gamma}_i \hat{\gamma}_i' - \gamma_i \gamma_i') \\ &\quad + \frac{\sqrt{T}}{n} \sum_{i=1}^n (\hat{\sigma}_{i,T}^2 - \omega_{i,T}^2) \gamma_i \gamma_i' + \frac{\sqrt{T}}{n} \sum_{i=1}^n (\omega_{i,T}^2 - \sigma_i^2) \gamma_i \gamma_i' \\ &= \sum_{j=1}^4 \mathbf{D}_{j,nT}. \end{aligned} \quad (\text{S.32})$$

The first two terms of (S.31) are $o_p(1)$, since $\|\varphi_n\| < C$, and $\sqrt{T}(\hat{\varphi}_{nT} - \varphi_n) = o_p(1)$, and $n^{-1} \sum_{i=1}^n \hat{\sigma}_{i,T}^2 (\hat{\gamma}_i \hat{\gamma}_i') = O_p(1)$. To establish the probability order of the third term of (S.31), since $\|\varphi_n\| < C$ it is sufficient to consider the four terms of $\sqrt{T}(\hat{\mathbf{H}}_{nT} - \mathbf{H}_n)$. It is clear that $\mathbf{D}_{2,nT}$ is dominated by $\mathbf{D}_{1,nT}$ and by (S.155) of Lemma S.16,

$$\mathbf{D}_{1,nT} = \frac{\sqrt{T}}{n} \sum_{i=1}^n \sigma_i^2 (\hat{\gamma}_i \hat{\gamma}_i' - \gamma_i \gamma_i') = O_p \left(\sqrt{\frac{\ln(n)}{n}} \right) = o_p(1).$$

Using (S.83) of Lemma S.8 and replacing b_{ni} with $\gamma_{ij} \gamma_{ij}'$ for $j, j' = 1, 2, \dots, m_0$, it then follows that

$$\mathbf{D}_{3,nT} = \frac{\sqrt{T}}{n} \sum_{i=1}^n (\hat{\sigma}_{i,T}^2 - \omega_{i,T}^2) \gamma_i \gamma_i' = O_p \left(\frac{\ln(n)}{\sqrt{T}} \right) = o_p(1).$$

Finally, denote the (j, j') element of $\mathbf{D}_{4,nT}$ by $d_{4,nT}(j, j')$ and note that

$$d_{4,nT}(j, j') = \frac{1}{n} \sum_{i=1}^n (\sigma_i^2 \gamma_{ij} \gamma_{ij'}) \sqrt{T} \left(\frac{\boldsymbol{\varepsilon}_{io}' \mathbf{M}_F \boldsymbol{\varepsilon}_{io}}{T} - 1 \right), \text{ for } j, j' = 1, 2, \dots, m_0.$$

But under Assumptions 2 and 3, $|\sigma_i^2 \gamma_{ij} \gamma_{ij'}| < C$, and $\sqrt{T} (T^{-1} \boldsymbol{\varepsilon}_{io}' \mathbf{M}_F \boldsymbol{\varepsilon}_{io} - 1)$, for $i = 1, 2, \dots, n$ are identically and independently distributed across i , with mean $1/\sqrt{T}$ and a finite variance^{S1}. Then by standard law of large numbers, for each (j, j') , $d_{4,nT}(j, j') \rightarrow_p 0$, as n and $T \rightarrow \infty$, and hence we also have $\mathbf{D}_{4,nT} = o_p(1)$. Overall, $\hat{\mathbf{H}}_{nT} - \mathbf{H}_n = o_p(1)$, and we have $\sqrt{T} (\hat{\boldsymbol{\varphi}}_{nT}' \hat{\mathbf{H}}_{nT} \hat{\boldsymbol{\varphi}}_{nT} - \boldsymbol{\varphi}_n' \mathbf{H}_n \boldsymbol{\varphi}_n) = o_p(1)$. Using this result and (S.30) in (S.27) now yields $\sqrt{T} (\hat{\theta}_{nT} - \theta_n) = o_p(1)$, as required. ■

Proof of Theorem 2. Recall from (31) that CD^* is given by

$$CD^* = \frac{CD + \sqrt{\frac{T}{2}} \hat{\theta}_{nT}}{1 - \hat{\theta}_{nT}},$$

where $\hat{\theta}_{nT} = 1 - \frac{1}{n} \sum_{i=1}^n \hat{a}_{i,n}^2$, $\hat{a}_{i,n} = 1 - \hat{\sigma}_{i,T} (\boldsymbol{\varphi}_{nT}' \hat{\boldsymbol{\gamma}}_i)$, and $\hat{\boldsymbol{\varphi}}_{nT} = n^{-1} \sum_{i=1}^n \hat{\boldsymbol{\gamma}}_i / \hat{\sigma}_{i,T}$, subject to the normalization $n^{-1} \sum_{i=1}^n \hat{\boldsymbol{\gamma}}_i \hat{\boldsymbol{\gamma}}_i' = \mathbf{I}_{m_0}$. By result (30) of Proposition 1, $\sqrt{T} (\hat{\theta}_{nT} - \theta_n) = o_p(1)$, and hence

$$\begin{aligned} CD^* &= \frac{CD + \sqrt{\frac{T}{2}} \theta_n + \sqrt{\frac{T}{2}} (\hat{\theta}_{nT} - \theta_n)}{1 - \theta_n - (\hat{\theta}_{nT} - \theta_n)} + o_p(1) \\ &= \frac{CD + \sqrt{\frac{T}{2}} \theta_n}{1 - \theta_n} + o_p(1) = CD^*(\theta_n) + o_p(1). \end{aligned}$$

Theorem 2 is then established by following Theorem 1. ■

Proof of Theorem 3. Let $v_{it} = y_{it} - \boldsymbol{\alpha}_i' \mathbf{d}_t - \boldsymbol{\beta}_i' \mathbf{x}_{it}$, and $u_{it} = y_{it} - \boldsymbol{\alpha}_i' \mathbf{d}_t - \boldsymbol{\beta}_i' \mathbf{x}_{it} - \boldsymbol{\gamma}_i' \mathbf{f}_t = v_{it} - \boldsymbol{\gamma}_i' \mathbf{f}_t$, and consider the following two optimization problems

$$\min_{\mathbf{F}, \mathbf{F}} \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T (v_{it} - \boldsymbol{\gamma}_i' \mathbf{f}_t)^2, \quad (\text{S.33})$$

$$\min_{\mathbf{F}, \mathbf{F}} \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T (\hat{v}_{it} - \boldsymbol{\gamma}_i' \mathbf{f}_t)^2, \quad (\text{S.34})$$

where

$$\begin{aligned} \hat{v}_{it} &= y_{it} - \hat{\boldsymbol{\alpha}}_{CCE,i}' \mathbf{d}_t - \hat{\boldsymbol{\beta}}_{CCE,i}' \mathbf{x}_{it} \\ &= y_{it} - \boldsymbol{\alpha}_i' \mathbf{d}_t - \boldsymbol{\beta}_i' \mathbf{x}_{it} - (\hat{\boldsymbol{\alpha}}_{CCE,i} - \boldsymbol{\alpha}_i)' \mathbf{d}_t - (\hat{\boldsymbol{\beta}}_{CCE,i} - \boldsymbol{\beta}_i)' \mathbf{x}_{it} \\ &= v_{it} - (\hat{\boldsymbol{\alpha}}_{CCE,i} - \boldsymbol{\alpha}_i)' \mathbf{d}_t - (\hat{\boldsymbol{\beta}}_{CCE,i} - \boldsymbol{\beta}_i)' \mathbf{x}_{it}. \end{aligned} \quad (\text{S.35})$$

^{S1}The mean and variance of $\sqrt{T} (T^{-1} \boldsymbol{\varepsilon}_{io}' \mathbf{M}_F \boldsymbol{\varepsilon}_{io} - 1)$ can be obtained using (S.124) and (S.125) in Lemma S.13.

We need to show that solving problem (S.34) is asymptotically equivalent to solving problem (S.33). First, using the results in Pesaran and Tosetti (2011) and the fact that $\mathbf{d}_t$ and $\mathbf{x}_{it}$ are (stochastically) bounded^{S2},

$$\mathbf{d}_t' (\hat{\boldsymbol{\alpha}}_{CCE,i} - \boldsymbol{\alpha}_i) = O_p \left(\frac{1}{\sqrt{T}} \right) + O_p \left(\frac{1}{n} \right) + O_p \left(\frac{1}{\sqrt{nT}} \right), \quad (\text{S.36})$$

$$\mathbf{x}_{it}' (\hat{\boldsymbol{\beta}}_{CCE,i} - \boldsymbol{\beta}_i) = O_p \left(\frac{1}{\sqrt{T}} \right) + O_p \left(\frac{1}{n} \right) + O_p \left(\frac{1}{\sqrt{nT}} \right), \quad (\text{S.37})$$

then rewrite the criterion for (S.34) with (S.35),

$$\begin{aligned} & \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T \left(\hat{v}_{it} - \gamma_i' \mathbf{f}_t \right)^2 \\ &= \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T \left(v_{it} - \gamma_i' \mathbf{f}_t - (\hat{\boldsymbol{\alpha}}_{CCE,i} - \boldsymbol{\alpha}_i)' \mathbf{d}_t - (\hat{\boldsymbol{\beta}}_{CCE,i} - \boldsymbol{\beta}_i)' \mathbf{x}_{it} \right)^2 \\ &= \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T \left(v_{it} - \gamma_i' \mathbf{f}_t \right)^2 + \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T (\hat{\boldsymbol{\alpha}}_{CCE,i} - \boldsymbol{\alpha}_i)' \mathbf{d}_t \mathbf{d}_t' (\hat{\boldsymbol{\alpha}}_{CCE,i} - \boldsymbol{\alpha}_i) \\ &+ \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T (\hat{\boldsymbol{\beta}}_{CCE,i} - \boldsymbol{\beta}_i)' \mathbf{x}_{it} \mathbf{x}_{it}' (\hat{\boldsymbol{\beta}}_{CCE,i} - \boldsymbol{\beta}_i) - \frac{2}{nT} \sum_{i=1}^n \sum_{t=1}^T \left(v_{it} - \gamma_i' \mathbf{f}_t \right) (\hat{\boldsymbol{\alpha}}_{CCE,i} - \boldsymbol{\alpha}_i)' \mathbf{d}_t \\ &- \frac{2}{nT} \sum_{i=1}^n \sum_{t=1}^T \left(v_{it} - \gamma_i' \mathbf{f}_t \right) (\hat{\boldsymbol{\beta}}_{CCE,i} - \boldsymbol{\beta}_i)' \mathbf{x}_{it} - \frac{2}{nT} \sum_{i=1}^n \sum_{t=1}^T (\hat{\boldsymbol{\alpha}}_{CCE,i} - \boldsymbol{\alpha}_i)' \mathbf{d}_t \mathbf{x}_{it}' (\hat{\boldsymbol{\beta}}_{CCE,i} - \boldsymbol{\beta}_i) \\ &= \sum_{j=1}^6 A_{j,nT}. \end{aligned} \quad (\text{S.38})$$

Therefore, using (S.36) and (S.37) it is obvious $A_{2,nT} = O_p \left(\frac{1}{\sqrt{T}} \right) + O_p \left(\frac{1}{n} \right) + O_p \left(\frac{1}{\sqrt{nT}} \right)$ and $A_{3,nT} = O_p \left(\frac{1}{\sqrt{T}} \right) + O_p \left(\frac{1}{n} \right) + O_p \left(\frac{1}{\sqrt{nT}} \right)$. Also consider the fourth term of (S.38) and by Cauchy-Schwarz inequality,

$$\begin{aligned} |A_{4,nT}| &= 2 \left| \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T u_{it} (\hat{\boldsymbol{\alpha}}_{CCE,i} - \boldsymbol{\alpha}_i)' \mathbf{d}_t \right| \\ &\leq 2 \left(\frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T u_{it}^2 \right)^{1/2} \left(\frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T \|(\hat{\boldsymbol{\alpha}}_{CCE,i} - \boldsymbol{\alpha}_i)' \mathbf{d}_t\|^2 \right)^{1/2}, \end{aligned}$$

where $u_{it} = v_{it} - \gamma_i' \mathbf{f}_t = y_{it} - \boldsymbol{\alpha}_i' \mathbf{d}_t - \boldsymbol{\beta}_i' \mathbf{x}_{it} - \gamma_i' \mathbf{f}_t$, so given (S.36) we have $A_{4,nT} = O_p \left(\frac{1}{\sqrt{T}} \right) + O_p \left(\frac{1}{n} \right) + O_p \left(\frac{1}{\sqrt{nT}} \right)$. Similarly, we can show $A_{5,nT}$ and $A_{6,nT}$ share the same probability order as $A_{4,nT}$. Since in both optimization problems γ_i and $\mathbf{f}_t$ are only identified up to $m_0 \times m_0$ rotation

^{S2}See equation (31) in Pesaran and Tosetti (2011).

matrices, it follows

$$\begin{aligned} \min_{\mathbf{\Gamma}, \mathbf{F}} \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T \left(v_{it} - \gamma'_i \mathbf{f}_t \right)^2 &\equiv \min_{\mathbf{\Gamma}, \mathbf{F}} \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T \left(\hat{v}_{it} - \gamma'_i \mathbf{f}_t \right)^2 \\ &\quad + O_p \left(\frac{1}{\sqrt{T}} \right) + O_p \left(\frac{1}{n} \right) + O_p \left(\frac{1}{\sqrt{nT}} \right). \end{aligned}$$

Hence, PCs based on $\hat{v}_{it}$ are asymptotically equivalent to those based on v_{it} . The remaining proof of Theorem 3 can follow from the proof of Theorem 2. ■

S1.2 Statement and Proofs of the Lemmas

This section provides auxiliary lemmas and the associated proofs, which are required to establish the main results of the paper.

Lemma S.1 *The CD statistic defined by (16) can be written equivalently as,*

$$CD = \left(\sqrt{\frac{n}{n-1}} \right) \frac{1}{\sqrt{2T}} \sum_{t=1}^T \left[\left(\frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{\hat{u}_{it}}{\hat{\sigma}_{i,T}} \right)^2 - 1 \right]. \quad (\text{S.39})$$

Proof. Using $\hat{\rho}_{ij,T} = \left(\frac{1}{T} \sum_{t=1}^T \hat{u}_{it} \hat{u}_{jt} \right) / \hat{\sigma}_{i,T} \hat{\sigma}_{j,T}$ in (16) we have:

$$CD = \sqrt{\frac{2T}{n(n-1)}} \sum_{i=1}^{n-1} \sum_{j=i+1}^n \frac{\frac{1}{T} \sum_{t=1}^T \hat{u}_{it} \hat{u}_{jt}}{\hat{\sigma}_{i,T} \hat{\sigma}_{j,T}} = \sqrt{\frac{2T}{n(n-1)}} \frac{1}{T} \sum_{t=1}^T \left(\sum_{i=1}^{n-1} \sum_{j=i+1}^n \left(\frac{\hat{u}_{it}}{\hat{\sigma}_{i,T}} \right) \left(\frac{\hat{u}_{jt}}{\hat{\sigma}_{j,T}} \right) \right). \quad (\text{S.40})$$

Further, we note that

$$\frac{1}{n} \sum_{i=1}^{n-1} \sum_{j=i+1}^n \left(\frac{\hat{u}_{it}}{\hat{\sigma}_{i,T}} \right) \left(\frac{\hat{u}_{jt}}{\hat{\sigma}_{j,T}} \right) = \frac{1}{2} \left[\left(\frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{\hat{u}_{it}}{\hat{\sigma}_{i,T}} \right)^2 - \frac{1}{n} \sum_{i=1}^n \left(\frac{\hat{u}_{it}}{\hat{\sigma}_{i,T}} \right)^2 \right].$$

Then using this result in (S.40), and after some algebra, we have

$$\begin{aligned} CD &= \sqrt{\frac{2Tn^2}{n(n-1)}} \frac{1}{2T} \sum_{t=1}^T \left[\left(\frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{\hat{u}_{it}}{\hat{\sigma}_{i,T}} \right)^2 - \frac{1}{n} \sum_{i=1}^n \left(\frac{\hat{u}_{it}}{\hat{\sigma}_{i,T}} \right)^2 \right] \\ &= \sqrt{\frac{2Tn^2}{n(n-1)}} \frac{1}{2} \left[\frac{1}{T} \sum_{t=1}^T \left(\frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{\hat{u}_{it}}{\hat{\sigma}_{i,T}} \right)^2 - \frac{1}{n} \sum_{i=1}^n \frac{1}{T} \sum_{t=1}^T \left(\frac{\hat{u}_{it}}{\hat{\sigma}_{i,T}} \right)^2 \right] \\ &= \left(\sqrt{\frac{n}{n-1}} \right) \frac{1}{\sqrt{2T}} \sum_{t=1}^T \left[\left(\frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{\hat{u}_{it}}{\hat{\sigma}_{i,T}} \right)^2 - 1 \right], \end{aligned}$$

as required. ■

Lemma S.2 Consider the latent factor model given by (1) and (2). The latent factors, $\mathbf{f}_t$, and their loadings, $\boldsymbol{\gamma}_i$, are estimated by principal components, $\hat{\mathbf{f}}_t$ and $\hat{\boldsymbol{\gamma}}_i$, given by (12). Suppose that Assumptions 1-4 hold and $(n, T) \rightarrow \infty$, such that $n/T \rightarrow \kappa$ for $0 < \kappa < \infty$. Then

$$\|\hat{\mathbf{F}} - \mathbf{F}\|_F = O_p \left(\frac{\sqrt{T}}{\delta_{nT}} \right), \quad (\text{S.41})$$

$$\|\hat{\mathbf{\Gamma}} - \mathbf{\Gamma}\|_F = O_p \left(\frac{\sqrt{n}}{\delta_{nT}} \right), \quad (\text{S.42})$$

$$\|\mathbf{U}(\lambda_T)'(\hat{\mathbf{F}} - \mathbf{F})\|_F = O_p \left(\frac{\sqrt{nT}}{\delta_{nT}} \right), \quad (\text{S.43})$$

$$\|\mathbf{\Gamma}'(\hat{\mathbf{\Gamma}} - \mathbf{\Gamma})\|_F = O_p \left(\frac{n}{\delta_{nT}} \right), \quad (\text{S.44})$$

$$\|\mathbf{F}'(\hat{\mathbf{F}} - \mathbf{F})\|_F = O_p \left(\frac{T}{\delta_{nT}} \right), \quad (\text{S.45})$$

$$(\hat{\mathbf{F}} - \mathbf{F})' \mathbf{F} = O_p \left(\frac{T}{\delta_{nT}^2} \right), \quad (\text{S.46})$$

$$(\hat{\mathbf{F}} - \mathbf{F})' \hat{\mathbf{F}} = O_p \left(\frac{T}{\delta_{nT}^2} \right), \quad (\text{S.47})$$

$$(\hat{\mathbf{\Gamma}} - \mathbf{\Gamma})' \mathbf{u}_{ot}(\lambda_T) = O_p \left(\frac{n}{\delta_{nT}^2} \right), \quad (\text{S.48})$$

where $\mathbf{F} = (\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_T)'$, $\hat{\mathbf{F}} = (\hat{\mathbf{f}}_1, \hat{\mathbf{f}}_2, \dots, \hat{\mathbf{f}}_T)'$, $\mathbf{\Gamma} = (\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \dots, \boldsymbol{\gamma}_n)'$, $\hat{\mathbf{\Gamma}} = (\hat{\boldsymbol{\gamma}}_1, \hat{\boldsymbol{\gamma}}_2, \dots, \hat{\boldsymbol{\gamma}}_n)'$, $\mathbf{U}(\lambda_T) = (\mathbf{u}_{o1}(\lambda_T), \mathbf{u}_{o2}(\lambda_T), \dots, \mathbf{u}_{oT}(\lambda_T))'$, $\mathbf{u}_{ot}(\lambda_T) = (\sigma_1 \varepsilon_{1t}(\lambda_T), \sigma_2 \varepsilon_{2t}(\lambda_T), \dots, \sigma_n \varepsilon_{nt}(\lambda_T))'$, and $\varepsilon_{it}(\lambda_T) = \varepsilon_{it} + \lambda_T \sum_{j=1}^n w_{ij} \varepsilon_{jt}$.

Proof. Since Assumptions 1-4 are a sub-set of assumptions made by Bai (2003), so results (S.41) to (S.44), (S.46) and (S.47) follow directly from Lemmas B.1, B.2 and B.3, and Theorems 1 and 2 of Bai (2003). Results (S.45) and (S.48) can be established analogously. ■

Lemma S.3 Suppose that Assumptions 1-3 hold and $(n, T) \rightarrow \infty$, such that $n/T \rightarrow \kappa$, for $0 < \kappa < \infty$. Then

$$\sup_i (T^{-1} \|\boldsymbol{\varepsilon}_{io}\|^2) = O_p(1), \quad (\text{S.49})$$

$$\sup_i \left\| \frac{\mathbf{F}' \boldsymbol{\varepsilon}_{io}}{T} \right\| = O_p \left(\sqrt{\frac{\ln(n)}{T}} \right), \quad (\text{S.50})$$

$$\sup_t \left\| \frac{\mathbf{\Gamma}' \boldsymbol{\varepsilon}_{ot}}{n} \right\| = O_p \left(\sqrt{\frac{\ln(T)}{n}} \right), \quad (\text{S.51})$$

$$\sup_i \left\| \frac{1}{nT} \sum_{t=1}^T \sum_{j=1}^n \sigma_j \boldsymbol{\gamma}_j' \varepsilon_{it} \varepsilon_{jt} \right\| = O_p \left(\sqrt{\frac{\ln(n)}{nT}} \right), \quad (\text{S.52})$$

where $\boldsymbol{\varepsilon}_{io} = (\varepsilon_{i1}, \varepsilon_{i2}, \dots, \varepsilon_{iT})'$ and $\boldsymbol{\varepsilon}_{ot} = (\varepsilon_{1t}, \varepsilon_{2t}, \dots, \varepsilon_{nt})'$.

Proof. Consider (S.49) and note

$$T^{-1} \|\boldsymbol{\varepsilon}_{i\cdot}\|^2 = \frac{1}{T} \sum_{t=1}^T [\varepsilon_{it}^2 - E(\varepsilon_{it}^2)] + \frac{1}{T} \sum_{t=1}^T E(\varepsilon_{it}^2) = \frac{1}{T} \sum_{t=1}^T z_{it} + 1,$$

where $z_{it} = \varepsilon_{it}^2 - E(\varepsilon_{it}^2)$. Then

$$\sup_i (T^{-1} \|\boldsymbol{\varepsilon}_{i\cdot}\|^2) \leq \sup_i \left| \frac{1}{T} \sum_{t=1}^T z_{it} \right| + 1. \quad (\text{S.53})$$

To establish the probability of the first term, consider the filtration $\mathcal{I}_{it}^{(1)} = \{\varepsilon_{i\tau} : \tau = t-1, t-2, \dots\}$ and, given the serial independence of ε_{it} , note that $E(z_{it} | \mathcal{I}_{i,t-1}^{(1)}) = 0$, so z_{it} is a martingale difference process with respect to $\mathcal{I}_{i,t-1}^{(1)}$. In addition, $\text{Var}(z_{it}) = \text{Var}(\varepsilon_{it}^2) = E(\varepsilon_{it}^4) - (E\varepsilon_{it}^2)^2$ which is bounded by assumption. Also, as ε_{it} is sub-exponential by part (a) of Assumption 2, ε_{it}^2 (and hence z_{it}) is sub-exponential, and there exist positive constants C_4 , C_5 and r_3 such that

$$\sup_i \Pr(|z_{it}| > a) \leq C_4 \exp(-C_5 a^{r_3}), \text{ for all } a > 0.$$

Then by Lemma A3 in the theory online supplement of Chudik et al. (2018), for $\varsigma_T = \Theta(T^\mu)$ and $0 < \mu < (r_3 + 1) / (r_3 + 2)$, there exists a positive constant C_6 such that

$$\Pr\left(\left|\sum_{t=1}^T z_{it}\right| > \varsigma_T\right) \leq \exp(-C_6 T^{-1} \varsigma_T^2),$$

and if $\mu > (r_3 + 1) / (r_3 + 2)$ there exists a positive constant C_7 such that

$$\Pr\left(\left|\sum_{t=1}^T z_{it}\right| > \varsigma_T\right) \leq \exp\left(-C_7 (\varsigma_T)^{\frac{r_3}{r_3+1}}\right).$$

By Boole's inequality

$$\Pr\left(\sup_i \left|\sum_{t=1}^T z_{it}\right| > \varsigma_T\right) \leq \exp(\ln(n) - C_6 T^{-1} \varsigma_T^2), \text{ if } 0 < \mu < (r_3 + 1) / (r_3 + 2)$$

and

$$\Pr\left(\sup_i \left|\sum_{t=1}^T z_{it}\right| > \varsigma_T\right) \leq \exp\left(\ln(n) - C_7 (\varsigma_T)^{\frac{r_3}{r_3+1}}\right), \text{ if } \mu > (r_3 + 1) / (r_3 + 2).$$

Let $\varsigma_T = C_8 \sqrt{T \ln(n)}$ where C_8 is a finite but sufficiently large constant. Then for $0 < \mu < (r_3 + 1) / (r_3 + 2)$, we have

$$\begin{aligned} \Pr\left(\sup_i \left|\frac{1}{T} \sum_{t=1}^T z_{it}\right| > C_8 \sqrt{\frac{\ln(n)}{T}}\right) &= \Pr\left(\sup_i \left|\sum_{t=1}^T \varepsilon_{it} - E(\varepsilon_{it}^2)\right| > C_8 \sqrt{T \ln(n)}\right) \\ &\leq \exp[\ln(n) - C_6 T^{-1} C_8^2 T \ln(n)] \\ &= \exp[\ln(n) - C_6 C_8^2 \ln(n)], \end{aligned}$$

which is $o(1)$ given C_8 is sufficiently large. Also for $\mu \geq (r_3 + 1) / (r_3 + 2)$, we have

$$\begin{aligned} \Pr \left(\sup_i \left| \frac{1}{T} \sum_{t=1}^T \varepsilon_{it} \right| > C_8 \sqrt{\frac{\ln(n)}{T}} \right) &= \Pr \left(\sup_i \left| \sum_{t=1}^T \varepsilon_{it} \right| > C_8 \sqrt{T \ln(n)} \right) \\ &\leq \exp \left[\ln(n) - C_7 \left(C_8 \sqrt{T \ln(n)} \right)^{\frac{r_3}{r_3+1}} \right], \end{aligned}$$

which is also $o(1)$ as n and T are of the same order of magnitude and sufficiently we have

$$\frac{\ln(n)}{\left[\sqrt{n \ln(n)} \right]^{\frac{r_3}{r_3+1}}} = \frac{[\ln(n)]^{\frac{r_3+2}{2(r_3+1)}}}{n^{\frac{r_3}{2(r_3+1)}}} = \left[\frac{(\ln(n))^{\frac{r_3+2}{r_3}}}{n} \right]^{\frac{r_3}{2(r_3+1)}} \rightarrow 0.$$

Therefore, $\sup_i \left| T^{-1} \sum_{t=1}^T z_{it} \right| = O_p \left(\sqrt{\frac{\ln(n)}{T}} \right)$ and (S.54) follows from (S.53). Next, consider (S.50) and note by Assumptions 1 and 2, $\mathbf{f}_t$ is independent from $\varepsilon_{it'}$ for all $t, t' = 1, 2, \dots, T$, also ε_{it} is serially independent, then for a suitable choice of $\mathcal{I}_{i,t-1}^{(2)} = \{\mathbf{f}_{\tau \varepsilon_{i\tau}} : \tau = t-1, t-2, \dots\}$ and $i = 1, 2, \dots, n$, $E(\mathbf{f}_t \varepsilon_{it} | \mathcal{I}_{i,t-1}^{(2)}) = E(\mathbf{f}_t | \mathcal{I}_{i,t-1}^{(2)}) E(\varepsilon_{it}) = \mathbf{0}$ so $\mathbf{f}_t \varepsilon_{it}$ is a martingale difference sequence with respect to the filtration $\mathcal{I}_{i,t-1}^{(2)}$. In addition, $E(\mathbf{f}_t \varepsilon_{it}) = E(\mathbf{f}_t) E(\varepsilon_{it}) = \mathbf{0}$ and $\text{Var}(\mathbf{f}_t \varepsilon_{it}) = E(\mathbf{f}_t \mathbf{f}_t') E(\varepsilon_{it}^2)$, which is bounded by Assumptions 1 and 2. Also by assumptions both $\mathbf{f}_t$ and ε_{it} are sub-exponential, then it also follows that $\mathbf{f}_t \varepsilon_{it}$ is sub-exponential. Hence, the method of proof used above can also be applied to $\left\| \sum_{t=1}^T \mathbf{f}_t \varepsilon_{it} \right\|$, and result (S.50) follows. Similarly, (S.51) can be established by the symmetry of the standard factor models in γ_i and $\mathbf{f}_t$. Now consider (S.52), and note that we have the following decomposition,

$$\begin{aligned} \mathbf{q}_{i,nT} &= \frac{1}{nT} \sum_{t=1}^T \sum_{j=1}^n \sigma_j \gamma_j \varepsilon_{it} \varepsilon_{jt} \\ &= \frac{1}{nT} \sum_{t=1}^T \sum_{j=1}^n [\sigma_j \gamma_j \varepsilon_{it} \varepsilon_{jt} - \sigma_j \gamma_j E(\varepsilon_{it} \varepsilon_{jt})] + \frac{1}{nT} \sum_{t=1}^T \sum_{j=1}^n \sigma_j \gamma_j E(\varepsilon_{it} \varepsilon_{jt}) = \mathbf{q}_{i,nT}^{(a)} + \mathbf{q}_{i,nT}^{(b)}. \end{aligned}$$

Since $E(\varepsilon_{it} \varepsilon_{jt}) = 0$ if $i \neq j$, and $E(\varepsilon_{it} \varepsilon_{jt}) = 1$, if $i = j$, then

$$\mathbf{q}_{i,nT}^{(b)} = \frac{1}{nT} \sum_{t=1}^T \sum_{j=1}^n \sigma_j \gamma_j E(\varepsilon_{it} \varepsilon_{jt}) = n^{-1} \sigma_j \gamma_j,$$

and $\sup_i \left\| \mathbf{q}_{i,nT}^{(b)} \right\| = O(n^{-1})$. Consider now the first term and note that

$$\mathbf{q}_{i,nT}^{(a)} = \frac{1}{nT} \sum_{t=1}^T \sum_{j=1}^n \mathbf{s}_{i,jt},$$

where $\mathbf{s}_{i,jt} = \sigma_j \gamma_j [\varepsilon_{it} \varepsilon_{jt} - E(\varepsilon_{it} \varepsilon_{jt})]$. Since by assumption ε_{it} are independently distributed over all i and t , then $E(\mathbf{s}_{i,jt} | \mathcal{I}_{i,t-1}^{(3)}) = \mathbf{0}$, where $\mathcal{I}_{i,t-1}^{(3)} = \{\varepsilon_{i\tau} \varepsilon_{j\tau} : j = 1, 2, \dots, n \text{ and } \tau =$

$t-1, t-2, \dots\}$. Hence $\mathbf{s}_{i,jt}$ is a martingale difference process with respect to the filtration, $\mathcal{I}_{i,t-1}^{(3)}$. The variance of $\mathbf{s}_{i,jt}$ is $(\sigma_j^2 \boldsymbol{\gamma}_j \boldsymbol{\gamma}_j') \text{Var}(\varepsilon_{it} \varepsilon_{jt})$ where $\text{Var}(\varepsilon_{it} \varepsilon_{jt}) = 1$ if $i \neq j$, and $\text{Var}(\varepsilon_{it} \varepsilon_{jt}) = \text{Var}(\varepsilon_{it}^2) = E(\varepsilon_{it}^4) - 1$ if $i = j$, so that by Assumption 2, $\|\text{Var}(\mathbf{s}_{i,jt})\| < C$. Also, since by assumption ε_{it} is sub-exponential, then it follows that $\mathbf{s}_{i,jt}$ is also sub-exponential, and the above method of proof can be applied to all elements of $\mathbf{q}_{i,nT}^{(a)}$. Specifically $\sup_i \|\mathbf{q}_{i,nT}^{(a)}\| = O_p \left(\sqrt{\frac{\ln(n)}{nT}} \right)$,

$$\sup_i \|\mathbf{q}_{i,nT}\| = O_p \left(\sqrt{\frac{\ln(n)}{nT}} \right) + O(n^{-1}) = O_p \left(\sqrt{\frac{\ln(n)}{nT}} \right),$$

and result (S.52) follows, as required. $\blacksquare$

Lemma S.4 *Consider the latent factor model given by (1) and (2). The latent factors, $\mathbf{f}_t$, and their loadings, $\boldsymbol{\gamma}_i$, are estimated by principal components, $\hat{\mathbf{f}}_t$ and $\hat{\boldsymbol{\gamma}}_i$, given by (12). Suppose that Assumptions 1-4 hold and $(n, T) \rightarrow \infty$, such that $n/T \rightarrow \kappa$, for $0 < \kappa < \infty$. Then*

$$\sup_i (T^{-1} \|\boldsymbol{\varepsilon}_{i0}(\lambda_T)\|^2) = O_p(1), \quad (\text{S.54})$$

$$\sup_i \left\| \frac{\mathbf{F}' \boldsymbol{\varepsilon}_{i0}(\lambda_T)}{T} \right\| = O_p \left(\sqrt{\frac{\ln(n)}{T}} \right), \quad (\text{S.55})$$

$$\sup_t \left\| \frac{\mathbf{F}' \boldsymbol{\varepsilon}_{ot}(\lambda_T)}{n} \right\| = O_p \left(\sqrt{\frac{\ln(T)}{n}} \right), \quad (\text{S.56})$$

$$\sup_i \left\| \frac{1}{nT} \sum_{t=1}^T \sum_{j=1}^n \sigma_j \boldsymbol{\gamma}_j \varepsilon_{it}(\lambda_T) \varepsilon_{jt}(\lambda_T) \right\| = O_p \left(\sqrt{\frac{\ln(n)}{nT}} \right), \quad (\text{S.57})$$

$$\sup_i \|\hat{\boldsymbol{\gamma}}_i - \boldsymbol{\gamma}_i\| = O_p \left(\sqrt{\frac{\ln(n)}{T}} \right), \quad (\text{S.58})$$

$$\sup_t \|\hat{\mathbf{f}}_t - \mathbf{f}_t\| = O_p \left(\sqrt{\frac{\ln(T)}{n}} \right), \quad (\text{S.59})$$

where $\boldsymbol{\varepsilon}_{i0}(\lambda_T) = (\varepsilon_{i1}(\lambda_T), \varepsilon_{i2}(\lambda_T), \dots, \varepsilon_{iT}(\lambda_T))'$ and $\boldsymbol{\varepsilon}_{ot}(\lambda_T) = (\varepsilon_{1t}(\lambda_T), \varepsilon_{2t}(\lambda_T), \dots, \varepsilon_{nt}(\lambda_T))'$.

Proof. Consider (S.54) and note by definition

$$\varepsilon_{it}(\lambda_T) = \varepsilon_{it} + \lambda_T \mathbf{w}_{i0}' \boldsymbol{\varepsilon}_{ot}, \quad (\text{S.60})$$

where $\mathbf{w}_{i0} = (w_{i1}, w_{i2}, \dots, w_{in})'$ and $\boldsymbol{\varepsilon}_{ot} = (\varepsilon_{1t}, \varepsilon_{2t}, \dots, \varepsilon_{nt})'$, such that

$$\varepsilon_{it}^2(\lambda_T) = \varepsilon_{it}^2 + 2\lambda_T \mathbf{w}_{i0}' \boldsymbol{\varepsilon}_{ot} \varepsilon_{it} + \lambda_T^2 \mathbf{w}_{i0}' \boldsymbol{\varepsilon}_{ot} \boldsymbol{\varepsilon}_{ot}' \mathbf{w}_{i0},$$

and hence

$$\frac{1}{T} \sum_{t=1}^T \varepsilon_{it}^2(\lambda_T) = \frac{1}{T} \sum_{t=1}^T \varepsilon_{it}^2 + 2\lambda_T \mathbf{w}_{i0}' \left(\frac{1}{T} \sum_{t=1}^T \boldsymbol{\varepsilon}_{ot} \varepsilon_{it} \right) + \lambda_T^2 \mathbf{w}_{i0}' \mathbf{V}_{\varepsilon T} \mathbf{w}_{i0},$$

where $\mathbf{V}_{\varepsilon T} = T^{-1} \sum_{t=1}^T \boldsymbol{\varepsilon}_{ot} \boldsymbol{\varepsilon}'_{ot}$ and $\|\mathbf{V}_{\varepsilon T}\| = O_p(1)$ by part (b) of Assumption 2. Also

$$\left| \frac{1}{T} \sum_{t=1}^T \varepsilon_{it}^2(\lambda_T) \right| \leq \left| \frac{1}{T} \sum_{t=1}^T \varepsilon_{it}^2 \right| + 2|\lambda_T| \|\mathbf{w}_{i0}\| \left\| \frac{1}{T} \sum_{t=1}^T \boldsymbol{\varepsilon}_{ot} \varepsilon_{it} \right\| + \lambda_T^2 \|\mathbf{w}_{i0}\|^2 \|\mathbf{V}_{\varepsilon T}\|.$$

Denote $\mathbf{e}_i$ as $n \times 1$ selection vector with 1 on its i^{th} element and zeros elsewhere, and note that

$$\sup_i \left\| \frac{1}{T} \sum_{t=1}^T \boldsymbol{\varepsilon}_{ot} \varepsilon_{it} \right\| = \sup_i \left\| \frac{1}{T} \sum_{t=1}^T \boldsymbol{\varepsilon}_{ot} \boldsymbol{\varepsilon}'_{ot} \mathbf{e}_i \right\| \leq \left\| \frac{1}{T} \sum_{t=1}^T \boldsymbol{\varepsilon}_{ot} \boldsymbol{\varepsilon}'_{ot} \right\| \left(\sup_i \|\mathbf{e}_i\| \right) = \|\mathbf{V}_{\varepsilon T}\|.$$

Using this result we now have (recalling that $\lambda_T = c_\lambda T^{-1/2}$)

$$\begin{aligned} \sup_i \left| \frac{1}{T} \sum_{t=1}^T \varepsilon_{it}^2(\lambda_T) \right| &\leq \sup_i \left| \frac{1}{T} \sum_{t=1}^T \varepsilon_{it}^2 \right| + 2c_\lambda T^{-1/2} \|\mathbf{V}_{\varepsilon T}\| \left(\sup_i \|\mathbf{w}_{i0}\| \right) + \\ &\quad c_\lambda^2 T^{-1} \|\mathbf{V}_{\varepsilon T}\| \left(\sup_i \|\mathbf{w}_{i0}\|^2 \right). \end{aligned}$$

Therefore, since $\sup_i \|\mathbf{w}_{i0}\| < C$ and by Assumption $\|\mathbf{V}_{\varepsilon T}\| = O_p(1)$, then

$$\sup_i \left| \frac{1}{T} \sum_{t=1}^T \varepsilon_{it}^2(\lambda_T) \right| = \sup_i \left| \frac{1}{T} \sum_{t=1}^T \varepsilon_{it}^2 \right| + O_p\left(\frac{1}{\sqrt{T}}\right). \quad (\text{S.61})$$

Result (S.54) now follows from (S.49). Similarly, to establish (S.55) note that

$$\frac{\mathbf{F}' \boldsymbol{\varepsilon}_{io}(\lambda_T)}{T} = \frac{1}{T} \sum_{t=1}^T \mathbf{f}_t \varepsilon_{it}(\lambda_T) = \frac{1}{T} \sum_{t=1}^T \mathbf{f}_t \varepsilon_{it} + \frac{\lambda_T}{T} \sum_{t=1}^T \mathbf{f}_t \boldsymbol{\varepsilon}'_{ot} \mathbf{w}_{i0},$$

and

$$\left\| \frac{\mathbf{F}' \boldsymbol{\varepsilon}_{io}(\lambda_T)}{T} \right\| \leq \left\| \frac{1}{T} \sum_{t=1}^T \mathbf{f}_t \varepsilon_{it} \right\| + |\lambda_T| \|\mathbf{w}_{i0}\| \left\| \frac{1}{T} \sum_{t=1}^T \mathbf{f}_t \boldsymbol{\varepsilon}'_{ot} \right\|.$$

Applying the supremum operator to both sides yields

$$\sup_i \left\| \frac{\mathbf{F}' \boldsymbol{\varepsilon}_{io}(\lambda_T)}{T} \right\| \leq \sup_i \left\| \frac{1}{T} \sum_{t=1}^T \mathbf{f}_t \varepsilon_{it} \right\| + |\lambda_T| \left(\sup_i \|\mathbf{w}_{i0}\| \right) \left\| \frac{1}{T} \sum_{t=1}^T \mathbf{f}_t \boldsymbol{\varepsilon}'_{ot} \right\|.$$

Also

$$\begin{aligned} E \left\| \frac{1}{T} \sum_{t=1}^T \mathbf{f}_t \boldsymbol{\varepsilon}'_{ot} \right\|^2 &\leq E \left\| \frac{1}{T} \sum_{t=1}^T \mathbf{f}_t \boldsymbol{\varepsilon}'_{ot} \right\|_F^2 = \text{tr} \left[E \left[\left(\frac{1}{T} \sum_{t=1}^T \mathbf{f}_t \boldsymbol{\varepsilon}'_{ot} \right) \left(\frac{1}{T} \sum_{t=1}^T \mathbf{f}_t \boldsymbol{\varepsilon}'_{ot} \right)' \right] \right] \\ &= \text{tr} \left(\frac{1}{T^2} \sum_{t=1}^T \sum_{t'=1}^T E(\mathbf{f}_t \boldsymbol{\varepsilon}'_{ot} \boldsymbol{\varepsilon}_{ot'} \mathbf{f}_{t'}') \right) = \frac{1}{T^2} \text{tr} \left(\sum_{t=1}^T E(\mathbf{f}_t \mathbf{f}_t') E(\boldsymbol{\varepsilon}'_{ot} \boldsymbol{\varepsilon}_{ot}) \right) \\ &= \frac{n}{T} \text{tr}(\boldsymbol{\Sigma}_{ff}) = O\left(\frac{nm_0}{T}\right), \end{aligned}$$

which establishes that $T^{-1} \sum_{t=1}^T \mathbf{f}_t \boldsymbol{\varepsilon}'_{ot} = O_p(1)$ since by assumption n and T have the same orders of magnitudes. Given $\lambda_T = c_\lambda T^{-1/2}$ and $\sup_i \|\mathbf{w}_{i0}\| < C$, we now have

$$\sup_i \left\| \frac{\mathbf{F}' \boldsymbol{\varepsilon}_{i0}(\lambda_T)}{T} \right\| = \sup_i \left\| \frac{1}{T} \sum_{t=1}^T \mathbf{f}_t \varepsilon_{it} \right\| + O_p\left(\frac{1}{\sqrt{T}}\right),$$

and (S.55) follows using (S.50). Similarly, (S.56) can be established using result (S.51). Next, consider (S.57) and using the definition of $\varepsilon_{it}(\lambda_T)$ in (S.60) yields

$$\begin{aligned} \frac{1}{nT} \sum_{t=1}^T \sum_{j=1}^n \sigma_j \gamma_j \varepsilon_{it}(\lambda_T) \varepsilon_{jt}(\lambda_T) &= \frac{1}{nT} \sum_{t=1}^T \sum_{j=1}^n \sigma_j \gamma_j \varepsilon_{it} \varepsilon_{jt} + \frac{\lambda_T}{nT} \sum_{t=1}^T \sum_{j=1}^n \sigma_j \gamma_j \mathbf{w}'_{j0} \boldsymbol{\varepsilon}_{ot} \varepsilon_{it} + \\ &\quad \frac{\lambda_T}{nT} \sum_{t=1}^T \sum_{j=1}^n \sigma_j \gamma_j \mathbf{w}'_{i0} \boldsymbol{\varepsilon}_{ot} \varepsilon_{jt} + \frac{\lambda_T^2}{n} \sum_{j=1}^n \sigma_j \gamma_j \mathbf{w}'_{i0} \left(\frac{1}{T} \sum_{t=1}^T \boldsymbol{\varepsilon}_{ot} \boldsymbol{\varepsilon}'_{ot} \right) \mathbf{w}_{j0}, \end{aligned}$$

which implies

$$\begin{aligned} \left\| \frac{1}{nT} \sum_{t=1}^T \sum_{j=1}^n \sigma_j \gamma_j \varepsilon_{it}(\lambda_T) \varepsilon_{jt}(\lambda_T) \right\| &\leq \left\| \frac{1}{nT} \sum_{t=1}^T \sum_{j=1}^n \sigma_j \gamma_j \varepsilon_{it} \varepsilon_{jt} \right\| + \left\| \frac{\lambda_T}{nT} \sum_{t=1}^T \sum_{j=1}^n \sigma_j \gamma_j \mathbf{w}'_{j0} \boldsymbol{\varepsilon}_{ot} \varepsilon_{it} \right\| + \\ &\quad \left\| \frac{\lambda_T}{nT} \sum_{t=1}^T \sum_{j=1}^n \sigma_j \gamma_j \mathbf{w}'_{i0} \boldsymbol{\varepsilon}_{ot} \varepsilon_{jt} \right\| + \left\| \frac{\lambda_T^2}{n} \sum_{j=1}^n \sigma_j \gamma_j \mathbf{w}'_{i0} \left(\frac{1}{T} \sum_{t=1}^T \boldsymbol{\varepsilon}_{ot} \boldsymbol{\varepsilon}'_{ot} \right) \mathbf{w}_{j0} \right\|. \end{aligned}$$

Taking the supremum on both sides of this inequality yields

$$\begin{aligned} &\sup_i \left\| \frac{1}{nT} \sum_{t=1}^T \sum_{j=1}^n \sigma_j \gamma_j \varepsilon_{it}(\lambda_T) \varepsilon_{jt}(\lambda_T) \right\| \\ &\leq \sup_i \left\| \frac{1}{nT} \sum_{t=1}^T \sum_{j=1}^n \sigma_j \gamma_j \varepsilon_{it} \varepsilon_{jt} \right\| + \sup_i \left\| \frac{\lambda_T}{nT} \sum_{t=1}^T \sum_{j=1}^n \sigma_j \gamma_j \mathbf{w}'_{j0} \boldsymbol{\varepsilon}_{ot} \varepsilon_{it} \right\| + \\ &\sup_i \left\| \frac{\lambda_T}{nT} \sum_{t=1}^T \sum_{j=1}^n \sigma_j \gamma_j \mathbf{w}'_{i0} \boldsymbol{\varepsilon}_{ot} \varepsilon_{jt} \right\| + \sup_i \left\| \frac{\lambda_T^2}{n} \sum_{j=1}^n \sigma_j \gamma_j \mathbf{w}'_{i0} \left(\frac{1}{T} \sum_{t=1}^T \boldsymbol{\varepsilon}_{ot} \boldsymbol{\varepsilon}'_{ot} \right) \mathbf{w}_{j0} \right\|. \quad (\text{S.62}) \end{aligned}$$

Note that $\varepsilon_{it} = \boldsymbol{\varepsilon}'_{ot} \mathbf{e}_i$ where $\mathbf{e}_i$ is an $n \times 1$ selection vector with 1 on its i^{th} element and zero elsewhere. Then the second term of the above can be bounded as

$$\begin{aligned} \sup_i \left\| \frac{\lambda_T}{nT} \sum_{t=1}^T \sum_{j=1}^n \sigma_j \gamma_j \mathbf{w}'_{j0} \boldsymbol{\varepsilon}_{ot} \varepsilon_{it} \right\| &= \sup_i \left\| \frac{\lambda_T}{n} \sum_{j=1}^n \sigma_j \gamma_j \mathbf{w}'_{j0} \left(\frac{1}{T} \sum_{t=1}^T \boldsymbol{\varepsilon}_{ot} \boldsymbol{\varepsilon}'_{ot} \right) \mathbf{e}_i \right\| \\ &\leq |\lambda_T| \left\| \frac{1}{n} \sum_{j=1}^n \sigma_j \gamma_j \mathbf{w}'_{j0} \left(\frac{1}{T} \sum_{t=1}^T \boldsymbol{\varepsilon}_{ot} \boldsymbol{\varepsilon}'_{ot} \right) \right\| \left(\sup_i \|\mathbf{e}_i\| \right) \\ &\leq |\lambda_T| \left\| \frac{1}{n} \sum_{j=1}^n \sigma_j \gamma_j \mathbf{w}'_{j0} \right\| \|\mathbf{V}_{\varepsilon T}\|. \end{aligned}$$

Since $\sup_i \sigma_i^2 < C$, and by Assumptions 3-4, $\sup_{s,i} \gamma_{si}^2 < C$ and $\sup_i \sum_{j=1}^n |w_{ji}| < C$, then it follows

$$\begin{aligned} \left\| \frac{1}{n} \sum_{j=1}^n \sigma_j \gamma_j \mathbf{w}'_{j0} \right\|^2 &\leq \left\| \frac{1}{n} \sum_{j=1}^n \sigma_j \gamma_j \mathbf{w}'_{j0} \right\|_F^2 = \frac{1}{n^2} \sum_{s=1}^{m_0} \sum_{i=1}^n \left(\sum_{j=1}^n \sigma_j \gamma_{sj} w_{ji} \right)^2 \\ &\leq \frac{1}{n^2} \sum_{s=1}^{m_0} \sum_{i=1}^n \left(\sum_{j=1}^n |\sigma_j \gamma_{sj}| |w_{ji}| \right)^2 \\ &\leq \left(\sup_i \sigma_i^2 \right) \left(\sup_{s,i} \gamma_{si}^2 \right) \left[\frac{1}{n} \sum_{s=1}^{m_0} \left(\frac{1}{n} \sum_{i=1}^n \left(\sum_{j=1}^n |w_{ji}| \right)^2 \right) \right] \\ &= O\left(\frac{1}{n}\right). \end{aligned}$$

In addition, $\lambda_T = c_\lambda T^{-1/2}$ and $\|\mathbf{V}_{\varepsilon T}\| = O_p(n/T)$ with $0 < n/T < C$. Hence, we have $\sup_i \left\| \frac{\lambda_T}{nT} \sum_{t=1}^T \sum_{j=1}^n \sigma_j \gamma_j \mathbf{w}'_{j0} \boldsymbol{\varepsilon}_{ot} \varepsilon_{it} \right\| = O_p((nT)^{-1/2})$. Similarly the third term of (S.62) is also $O_p((nT)^{-1/2})$. For the fourth term of (S.62),

$$\begin{aligned} \sup_i \left\| \frac{\lambda_T^2}{n} \sum_{j=1}^n \sigma_j \gamma_j \mathbf{w}'_{i0} \left(\frac{1}{T} \sum_{t=1}^T \boldsymbol{\varepsilon}_{ot} \boldsymbol{\varepsilon}'_{ot} \right) \mathbf{w}_{j0} \right\| &\leq \lambda_T^2 \left(\sup_i \|\mathbf{w}_{i0}\| \right) \left(\frac{1}{n} \sum_{j=1}^n \sigma_j \|\gamma_j\| \|\mathbf{w}_{j0}\| \right) \|\mathbf{V}_{\varepsilon T}\| \\ &= O_p\left(\frac{1}{T}\right). \end{aligned}$$

Using the above results in (S.62) we now have

$$\sup_i \left\| \frac{1}{nT} \sum_{t=1}^T \sum_{j=1}^n \sigma_j \gamma_j \varepsilon_{it} (\lambda_T) \varepsilon_{jt} (\lambda_T) \right\| \leq \sup_i \left\| \frac{1}{nT} \sum_{t=1}^T \sum_{j=1}^n \sigma_j \gamma_j \varepsilon_{it} \varepsilon_{jt} \right\| + O_p\left(\sqrt{\frac{\ln(n)}{nT}}\right),$$

and (S.57) follows using (S.52) to establish the order of the first term of the above. To establish (S.58) note that by definition of $\hat{\gamma}_i$,

$$\begin{aligned} \hat{\gamma}_i - \gamma_i &= \left(\hat{\mathbf{F}}' \hat{\mathbf{F}} \right)^{-1} \hat{\mathbf{F}}' (\mathbf{F} \gamma_i + \sigma_i \boldsymbol{\varepsilon}_{io}(\lambda_T)) - \gamma_i = \left(\hat{\mathbf{F}}' \hat{\mathbf{F}} \right)^{-1} \hat{\mathbf{F}}' \left[(\mathbf{F} - \hat{\mathbf{F}}) \gamma_i + \hat{\mathbf{F}} \gamma_i + \sigma_i \boldsymbol{\varepsilon}_{io}(\lambda_T) \right] - \gamma_i \\ &= \left(\frac{\hat{\mathbf{F}}' \hat{\mathbf{F}}}{T} \right)^{-1} \left(\frac{\hat{\mathbf{F}} - \mathbf{F} + \mathbf{F}}{T} \right)' \left[(\mathbf{F} - \hat{\mathbf{F}}) \gamma_i + \sigma_i \boldsymbol{\varepsilon}_{io}(\lambda_T) \right] \\ &= \left(\frac{\hat{\mathbf{F}}' \hat{\mathbf{F}}}{T} \right)^{-1} \left[\frac{(\hat{\mathbf{F}} - \mathbf{F})' (\mathbf{F} - \hat{\mathbf{F}})}{T} \right] \gamma_i + \left(\frac{\hat{\mathbf{F}}' \hat{\mathbf{F}}}{T} \right)^{-1} \left[\frac{\mathbf{F}' (\mathbf{F} - \hat{\mathbf{F}})}{T} \right] \gamma_i \\ &\quad + \left(\frac{\hat{\mathbf{F}}' \hat{\mathbf{F}}}{T} \right)^{-1} \left[\frac{\sigma_i (\hat{\mathbf{F}} - \mathbf{F})' \boldsymbol{\varepsilon}_{io}(\lambda_T)}{T} \right] + \left(\frac{\hat{\mathbf{F}}' \hat{\mathbf{F}}}{T} \right)^{-1} \left(\frac{\sigma_i \mathbf{F}' \boldsymbol{\varepsilon}_{io}(\lambda_T)}{T} \right) = \sum_{j=1}^4 \mathbf{a}_{j,iT}, \end{aligned}$$

and

$$\|\hat{\gamma}_i - \gamma_i\| \leq \sum_{j=1}^4 \|\mathbf{a}_{j,iT}\|. \quad (\text{S.63})$$

Firstly we have

$$\begin{aligned}\|\mathbf{a}_{1,iT}\| &\leq \left\| \left(\frac{\hat{\mathbf{F}}'\hat{\mathbf{F}}}{T} \right)^{-1} \right\| \left\| \left[\frac{(\hat{\mathbf{F}} - \mathbf{F})'(\mathbf{F} - \hat{\mathbf{F}})}{T} \right] \right\| \|\boldsymbol{\gamma}_i\| \\ &\leq \left\| \left(\frac{\hat{\mathbf{F}}'\hat{\mathbf{F}}}{T} \right)^{-1} \right\| \left(\frac{\|\hat{\mathbf{F}} - \mathbf{F}\|^2}{T} \right) \|\boldsymbol{\gamma}_i\|,\end{aligned}$$

which implies

$$\sup_i \|\mathbf{a}_{1,iT}\| \leq \left\| \left(\frac{\hat{\mathbf{F}}'\hat{\mathbf{F}}}{T} \right)^{-1} \right\| \left(\frac{\|\hat{\mathbf{F}} - \mathbf{F}\|^2}{T} \right) \sup_i \|\boldsymbol{\gamma}_i\|.$$

Using (S.87) and (S.88) in Lemma S.8, and (S.41) in Lemma S.2, we first note that

$$\frac{\hat{\mathbf{F}}'\hat{\mathbf{F}}}{T} = O_p(1), \quad \left(\frac{\hat{\mathbf{F}}'\hat{\mathbf{F}}}{T} \right)^{-1} = O_p(1), \text{ and } T^{-1} \|\hat{\mathbf{F}} - \mathbf{F}\|_F^2 = O_p\left(\frac{1}{\delta_{nT}^2}\right). \quad (\text{S.64})$$

Also $\sup_i \|\boldsymbol{\gamma}_i\| < C$. Then

$$\sup_i \|\mathbf{a}_{1,iT}\| = O_p\left(\frac{1}{\delta_{nT}^2}\right). \quad (\text{S.65})$$

Similarly, $\|\mathbf{a}_{2,iT}\| \leq \left\| \left(\frac{\hat{\mathbf{F}}'\hat{\mathbf{F}}}{T} \right)^{-1} \right\| \left\| \frac{\mathbf{F}'(\mathbf{F} - \hat{\mathbf{F}})}{T} \right\| \|\boldsymbol{\gamma}_i\|$, and using (S.46) and (S.64) it follows that

$$\sup_i \|\mathbf{a}_{2,iT}\| \leq \left\| \left(\frac{\hat{\mathbf{F}}'\hat{\mathbf{F}}}{T} \right)^{-1} \right\| \left(\frac{\|\mathbf{F}'(\mathbf{F} - \hat{\mathbf{F}})\|}{T} \right) \sup_i \|\boldsymbol{\gamma}_i\| = O_p\left(\frac{1}{\delta_{nT}^2}\right). \quad (\text{S.66})$$

Regarding $\mathbf{a}_{3,iT}$, note that by Cauchy-Schwarz inequality

$$\|\mathbf{a}_{3,iT}\| \leq \left\| \left(\frac{\hat{\mathbf{F}}'\hat{\mathbf{F}}}{T} \right)^{-1} \right\| \left\| \frac{\sigma_i (\hat{\mathbf{F}} - \mathbf{F})' \boldsymbol{\varepsilon}_{i0}(\lambda_T)}{T} \right\| \leq \left\| \left(\frac{\hat{\mathbf{F}}'\hat{\mathbf{F}}}{T} \right)^{-1} \right\| \left(\frac{\|\hat{\mathbf{F}} - \mathbf{F}\|^2}{T} \right)^{1/2} \left(\frac{\sigma_i^2 \|\boldsymbol{\varepsilon}_{i0}(\lambda_T)\|^2}{T} \right)^{1/2}$$

and

$$\sup_i \|\mathbf{a}_{3,iT}\| \leq \left(\sup_i \sigma_i \right) \left\| \left(\frac{\hat{\mathbf{F}}'\hat{\mathbf{F}}}{T} \right)^{-1} \right\| \left(\frac{\|\hat{\mathbf{F}} - \mathbf{F}\|^2}{T} \right)^{1/2} \left[\sup_i \left(\frac{\|\boldsymbol{\varepsilon}_{i0}(\lambda_T)\|^2}{T} \right) \right]^{1/2}.$$

Now using (S.54) and (S.64) it follows that

$$\sup_i \|\mathbf{a}_{3,iT}\| = O_p\left(\frac{1}{\delta_{nT}}\right). \quad (\text{S.67})$$

Next, note

$$\|\mathbf{a}_{4,iT}\| \leq \left\| \left(\frac{\hat{\mathbf{F}}' \hat{\mathbf{F}}}{T} \right)^{-1} \right\| \left\| \frac{\sigma_i \mathbf{F}' \boldsymbol{\varepsilon}_{io}(\lambda_T)}{T} \right\|$$

then by (S.55) we also have

$$\sup_i \|\mathbf{a}_{4,iT}\| \leq \left(\sup_i \sigma_i \right) \left\| \left(\frac{\hat{\mathbf{F}}' \hat{\mathbf{F}}}{T} \right)^{-1} \right\| \left(\sup_i \left\| \frac{\mathbf{F}' \boldsymbol{\varepsilon}_{io}(\lambda_T)}{T} \right\| \right) = O_p \left(\sqrt{\frac{\ln(n)}{T}} \right). \quad (\text{S.68})$$

Hence using (S.65)-(S.68) in (S.63) we have

$$\sup_i \|\hat{\boldsymbol{\gamma}}_i - \boldsymbol{\gamma}_i\| \leq \sum_{j=1}^4 \sup_i \|\mathbf{a}_{j,iT}\| = O_p \left(\sqrt{\frac{\ln(n)}{T}} \right),$$

as required. Result (S.59) follows by symmetry. ■

Lemma S.5 Consider $\varepsilon_{it}(\lambda_T) = \varepsilon_{it} + \lambda_T \mathbf{w}'_{i0} \boldsymbol{\varepsilon}_{ot}$, where $\boldsymbol{\varepsilon}_{ot} = (\varepsilon_{1t}, \varepsilon_{2t}, \dots, \varepsilon_{nt})'$, $\varepsilon_{it} \sim IID(0, 1)$ for all i and t , $\mathbf{w}_{i0} = (w_{i1}, w_{i2}, \dots, w_{in})'$, and $\mathbf{W} = (w_{ij})$ satisfy the bounded conditions $\|\mathbf{W}\|_1 = \sup_j \sum_{i=1}^n |w_{ij}| < C$, and $\|\mathbf{W}\|_\infty = \sup_i \sum_{j=1}^n |w_{ij}| < C$. Then for all $|\lambda_T| < C$ we have

$$\sup_j \sum_{i=1}^n |E[\varepsilon_{it}(\lambda_T) \varepsilon_{jt}(\lambda_T)]| < C, \quad \sup_i \sum_{j=1}^n |E[\varepsilon_{it}(\lambda_T) \varepsilon_{jt}(\lambda_T)]| < C,$$

and

$$n^{-1} \sum_{i=1}^n \sum_{j=1}^n |E[\varepsilon_{it}(\lambda_T) \varepsilon_{jt}(\lambda_T)]| < C. \quad (\text{S.69})$$

Proof. Let $\boldsymbol{\varepsilon}_{ot}(\lambda_T) = \boldsymbol{\varepsilon}_{ot} + \lambda_T \mathbf{W} \boldsymbol{\varepsilon}_{ot}$, where $\mathbf{W}' = (\mathbf{w}_{10}, \mathbf{w}_{20}, \dots, \mathbf{w}_{n0})$. Then

$$\boldsymbol{\varepsilon}_{ot}(\lambda_T) \boldsymbol{\varepsilon}'_{ot}(\lambda_T) = \boldsymbol{\varepsilon}_{ot} \boldsymbol{\varepsilon}'_{ot} + \lambda_T^2 \mathbf{W} \boldsymbol{\varepsilon}_{ot} \boldsymbol{\varepsilon}'_{ot} \mathbf{W}' + \lambda_T \mathbf{W} \boldsymbol{\varepsilon}_{ot} \boldsymbol{\varepsilon}'_{ot} + \lambda_T \boldsymbol{\varepsilon}_{ot} \boldsymbol{\varepsilon}'_{ot} \mathbf{W}',$$

and

$$\mathbf{V}_\varepsilon(\lambda_T) = E[\boldsymbol{\varepsilon}_{ot}(\lambda_T) \boldsymbol{\varepsilon}'_{ot}(\lambda_T)] = \mathbf{I}_n + \lambda_T (\mathbf{W} + \mathbf{W}') + \lambda_T^2 \mathbf{W} \mathbf{W}'.$$

Consider the maximum absolute column sum norm of $\mathbf{V}_\varepsilon(\lambda_T)$ and note that

$$\|\mathbf{V}_\varepsilon(\lambda_T)\|_1 = \sup_j \sum_{i=1}^n |E[\varepsilon_{it}(\lambda_T) \varepsilon_{jt}(\lambda_T)]| < 1 + |\lambda_T| (\|\mathbf{W}\|_1 + \|\mathbf{W}\|_\infty) + \lambda_T^2 \|\mathbf{W}\|_1 \|\mathbf{W}\|_\infty < C.$$

Similarly for the maximum absolute row sum norm of $\mathbf{V}_\varepsilon(\lambda_T)$

$$\|\mathbf{V}_\varepsilon(\lambda_T)\|_\infty = \sup_i \sum_{j=1}^n |E[\varepsilon_{it}(\lambda_T) \varepsilon_{jt}(\lambda_T)]| < 1 + |\lambda_T| (\|\mathbf{W}\|_\infty + \|\mathbf{W}\|_1) + \lambda_T^2 \|\mathbf{W}\|_\infty \|\mathbf{W}\|_1 < C,$$

and result (S.69) follows. ■

Lemma S.6 Consider the latent factor model given by (1) and (2). Suppose that Assumptions 1-4 hold and $(n, T) \rightarrow \infty$, such that $n/T \rightarrow \kappa$, for $0 < \kappa < \infty$. Then for the estimator of factors, we have

$$\sup_i \left\| \frac{(\hat{\mathbf{F}} - \mathbf{F})' \boldsymbol{\varepsilon}_{i\circ}(\lambda_T)}{T} \right\| = O_p \left(\sqrt{\frac{\ln(n)}{nT}} \right). \quad (\text{S.70})$$

Proof. By (A.1) of Bai (2003) we note that

$$\hat{\mathbf{f}}_t - \mathbf{f}_t = \frac{1}{T} \sum_{t'=1}^T \hat{\mathbf{f}}_{t'} \eta_{tt'} + \frac{1}{T} \sum_{t'=1}^T \hat{\mathbf{f}}_{t'} \zeta_{tt'} + \frac{1}{T} \sum_{t'=1}^T \hat{\mathbf{f}}_{t'} \boldsymbol{\varkappa}_{tt'} + \frac{1}{T} \sum_{t'=1}^T \hat{\mathbf{f}}_{t'} \xi_{tt'}$$

where $\eta_{tt'} = n^{-1} \sum_{i=1}^n \sigma_i^2 E(\varepsilon_{it}(\lambda_T) \varepsilon_{it'}(\lambda_T))$, $\zeta_{tt'} = n^{-1} \sum_{i=1}^n \sigma_i^2 \varepsilon_{it}^2(\lambda_T) - \eta_{tt'}$, $\boldsymbol{\varkappa}_{tt'} = n^{-1} \sum_{i=1}^n \sigma_i \mathbf{f}_{t'}' \boldsymbol{\gamma}_i' \varepsilon_{it}(\lambda_T)$, and $\xi_{tt'} = n^{-1} \sum_{i=1}^n \sigma_i \mathbf{f}_{t'}' \boldsymbol{\gamma}_i' \varepsilon_{it'}(\lambda_T)$. Hence

$$\begin{aligned} \frac{1}{T} (\hat{\mathbf{F}} - \mathbf{F})' \boldsymbol{\varepsilon}_{i\circ}(\lambda_T) &= \frac{1}{T} \sum_{t=1}^T (\hat{\mathbf{f}}_t - \mathbf{f}_t) \varepsilon_{it}(\lambda_T) \\ &= \frac{1}{T^2} \sum_{t=1}^T \sum_{t'=1}^T \hat{\mathbf{f}}_{t'} \eta_{tt'} \varepsilon_{it}(\lambda_T) + \frac{1}{T^2} \sum_{t=1}^T \sum_{t'=1}^T \hat{\mathbf{f}}_{t'} \zeta_{tt'} \varepsilon_{it}(\lambda_T) \\ &\quad + \frac{1}{T^2} \sum_{t=1}^T \sum_{t'=1}^T \hat{\mathbf{f}}_{t'} \boldsymbol{\varkappa}_{tt'} \varepsilon_{it}(\lambda_T) + \frac{1}{T^2} \sum_{t=1}^T \sum_{t'=1}^T \hat{\mathbf{f}}_{t'} \xi_{tt'} \varepsilon_{it}(\lambda_T) \\ &= \sum_{j=1}^4 \mathbf{b}_{j,iT}, \end{aligned}$$

and

$$\sup_i \left\| \frac{(\hat{\mathbf{F}} - \mathbf{F})' \boldsymbol{\varepsilon}_{i\circ}(\lambda_T)}{T} \right\| \leq \sum_{j=1}^4 \sup_i \|\mathbf{b}_{j,iT}\|. \quad (\text{S.71})$$

Firstly, consider $\mathbf{b}_{1,iT}$ and note that

$$\begin{aligned} \mathbf{b}_{1,iT} &= \frac{1}{T^2} \sum_{t=1}^T \sum_{t'=1}^T (\hat{\mathbf{f}}_{t'} - \mathbf{f}_{t'}) \eta_{tt'} \varepsilon_{it}(\lambda_T) + \frac{1}{T^2} \sum_{t=1}^T \sum_{t'=1}^T \mathbf{f}_{t'} \eta_{tt'} \varepsilon_{it}(\lambda_T) \\ &= \mathbf{b}_{1,1,iT} + \mathbf{b}_{1,2,iT}, \end{aligned}$$

so

$$\|\mathbf{b}_{1,iT}\| \leq \|\mathbf{b}_{1,1,iT}\| + \|\mathbf{b}_{1,2,iT}\|.$$

Note for the first term on the right hand side,

$$\begin{aligned}
\|\mathbf{b}_{1,1,iT}\| &= \left\| \frac{1}{T} \sum_{t'=1}^T (\hat{\mathbf{f}}_{t'} - \mathbf{f}_{t'}) \left(\frac{1}{T} \sum_{t=1}^T \eta_{tt'} \varepsilon_{it}(\lambda_T) \right) \right\| \\
&\leq \left(\frac{1}{T} \sum_{t'=1}^T \|\hat{\mathbf{f}}_{t'} - \mathbf{f}_{t'}\|^2 \right)^{1/2} \left[\frac{1}{T} \sum_{t'=1}^T \left(\frac{1}{T} \sum_{t=1}^T \eta_{tt'} \varepsilon_{it}(\lambda_T) \right)^2 \right]^{1/2} \\
&\leq \frac{1}{\sqrt{T}} \left(\frac{1}{T} \sum_{t'=1}^T \|\hat{\mathbf{f}}_{t'} - \mathbf{f}_{t'}\|^2 \right)^{1/2} \left(\frac{1}{T} \sum_{t'=1}^T \sum_{t=1}^T \eta_{tt'}^2 \right)^{1/2} \left(\frac{1}{T} \sum_{t=1}^T \varepsilon_{it}^2(\lambda_T) \right)^{1/2},
\end{aligned}$$

where the second and third lines hold by Cauchy-Schwarz inequality. By definition of $\eta_{tt'}$ and serial independence of $\varepsilon_{it}(\lambda_T)$,

$$\eta_{tt'} = \begin{cases} \bar{\sigma}_n^2, & \text{for } t = t' \\ 0, & \text{otherwise,} \end{cases}$$

where $\bar{\sigma}_n^2 = n^{-1} \sum_{i=1}^n \sigma_i^2 E(\varepsilon_{it}^2(\lambda_T))$. Under assumption on weight $\{w_{ij}\}$,

$$E(\varepsilon_{it}^2(\lambda_T)) = 1 + \lambda_T^2 \left(\sum_{j=1}^n w_{ij}^2 \right) < C$$

therefore

$$\frac{1}{T} \sum_{t'=1}^T \sum_{t=1}^T \eta_{tt'}^2 = (\bar{\sigma}_n^2)^2 < C. \quad (\text{S.72})$$

Given results (S.41), (S.54) and (S.72), it follows that

$$\begin{aligned}
\sup_i \|\mathbf{b}_{1,1,iT}\| &\leq \frac{1}{\sqrt{T}} \left(\frac{1}{T} \sum_{t'=1}^T \|\hat{\mathbf{f}}_{t'} - \mathbf{f}_{t'}\|^2 \right)^{1/2} \left(\frac{1}{T} \sum_{t'=1}^T \sum_{t=1}^T \eta_{tt'}^2 \right)^{1/2} \left(\sup_i \frac{1}{T} \sum_{t=1}^T \varepsilon_{it}^2(\lambda_T) \right)^{1/2} \\
&= O_p(T^{-1/2} \delta_{nT}^{-1}).
\end{aligned}$$

Now consider $\mathbf{b}_{1,2,iT}$, note that using properties of $\eta_{tt'}$, we have

$$\frac{1}{T^2} \sum_{t=1}^T \sum_{t'=1}^T \mathbf{f}_{t'} \eta_{tt'} \varepsilon_{it}(\lambda_T) = \frac{1}{T^2} \sum_{t=1}^T \mathbf{f}_t \eta_{tt} \varepsilon_{it}(\lambda_T) + \frac{1}{T^2} \sum_{t=1}^T \sum_{t' \neq t}^T \mathbf{f}_{t'} \eta_{tt'} \varepsilon_{it}(\lambda_T) = \bar{\sigma}_n^2 \left(\frac{1}{T^2} \sum_{t=1}^T \mathbf{f}_t \varepsilon_{it}(\lambda_T) \right),$$

and hence

$$\|\mathbf{b}_{1,2,iT}\| = \left\| \frac{1}{T^2} \sum_{t=1}^T \sum_{t'=1}^T \mathbf{f}_{t'} \eta_{tt'} \varepsilon_{it}(\lambda_T) \right\| \leq \frac{1}{T} \bar{\sigma}_n^2 \left(\left\| \frac{1}{T} \sum_{t=1}^T \mathbf{f}_t \varepsilon_{it}(\lambda_T) \right\| \right).$$

Then using (S.55) we have

$$\sup_i \|\mathbf{b}_{1,2,iT}\| \leq \frac{1}{T} \bar{\sigma}_n^2 \sup_i \left\| \frac{1}{T} \sum_{t=1}^T \mathbf{f}_t \varepsilon_{it}(\lambda_T) \right\| = O_p \left(\frac{1}{T} \sqrt{\frac{\ln(n)}{T}} \right).$$

Hence, combining the probability orders of $\sup_i \|\mathbf{b}_{1,1,iT}\|$ and $\sup_i \|\mathbf{b}_{1,2,iT}\|$, we have

$$\sup_i \|\mathbf{b}_{1,nT}\| \leq \sup_i \|\mathbf{b}_{1,1,iT}\| + \sup_i \|\mathbf{b}_{1,2,iT}\| = O_p \left(\frac{1}{T^{1/2} \delta_{nT}} \right). \quad (\text{S.73})$$

Next consider $\mathbf{b}_{2,iT}$ in (S.71), which can be written as

$$\begin{aligned} \mathbf{b}_{2,iT} &= \frac{1}{T^2} \sum_{t=1}^T \sum_{t'=1}^T \left(\hat{\mathbf{f}}_{t'} - \mathbf{f}_{t'} \right) \zeta_{tt'} \varepsilon_{it} (\lambda_T) + \frac{1}{T^2} \sum_{t=1}^T \sum_{t'=1}^T \mathbf{f}_{t'} \zeta_{tt'} \varepsilon_{it} (\lambda_T) \\ &= \mathbf{b}_{2,1,iT} + \mathbf{b}_{2,2,iT}. \end{aligned}$$

For the first term we can apply Cauchy-Schwarz inequality to obtain

$$\begin{aligned} \|\mathbf{b}_{2,1,iT}\| &= \left\| \frac{1}{T^2} \sum_{t'=1}^T \left(\hat{\mathbf{f}}_{t'} - \mathbf{f}_{t'} \right) \left(\sum_{t=1}^T \zeta_{tt'} \varepsilon_{it} (\lambda_T) \right) \right\| \\ &\leq \left(\frac{1}{T} \sum_{t'=1}^T \|\hat{\mathbf{f}}_{t'} - \mathbf{f}_{t'}\|^2 \right)^{1/2} \left[\frac{1}{T^3} \sum_{t'=1}^T \left(\sum_{t=1}^T \zeta_{tt'} \varepsilon_{it} (\lambda_T) \right)^2 \right]^{1/2} \\ &\leq \left(\frac{1}{T} \sum_{t'=1}^T \|\hat{\mathbf{f}}_{t'} - \mathbf{f}_{t'}\|^2 \right)^{1/2} \left(\frac{1}{T^2} \sum_{t'=1}^T \sum_{t=1}^T \zeta_{tt'}^2 \right)^{1/2} \left(\frac{1}{T} \sum_{t=1}^T \varepsilon_{it}^2 (\lambda_T) \right)^{1/2}. \end{aligned}$$

Since

$$\zeta_{tt'} = n^{-1} \sum_{i=1}^n \sigma_i^2 \varepsilon_{it}^2 (\lambda_T) - \eta_{tt'} = \frac{1}{n} \sum_{j=1}^n \sigma_j^2 [\varepsilon_{jt} (\lambda_T) \varepsilon_{jt'} (\lambda_T) - E(\varepsilon_{jt} (\lambda_T) \varepsilon_{jt'} (\lambda_T))],$$

then it follows that

$$\begin{aligned} E \left(\frac{1}{T^2} \sum_{t'=1}^T \sum_{t=1}^T \zeta_{tt'}^2 \right) &= \frac{1}{T^2} \sum_{t'=1}^T \sum_{t=1}^T E(\zeta_{tt'}^2) \\ &= \frac{1}{T^2 n^2} \sum_{t'=1}^T \sum_{t=1}^T E \left(\sum_{j=1}^n \sigma_j^2 [\varepsilon_{jt} (\lambda_T) \varepsilon_{jt'} (\lambda_T) - E(\varepsilon_{jt} (\lambda_T) \varepsilon_{jt'} (\lambda_T))] \right)^2 \\ &= \frac{1}{T^2 n^2} \sum_{t=1}^T \sum_{t'=1}^T \sum_{j=1}^n \sum_{j'=1}^n \sigma_j^2 \sigma_{j'}^2 E(\varepsilon_{jt} (\lambda_T) \varepsilon_{jt'} (\lambda_T) \varepsilon_{j't} (\lambda_T) \varepsilon_{j't'} (\lambda_T)) \\ &\quad - \frac{1}{T^2 n^2} \sum_{t=1}^T \sum_{t'=1}^T \sum_{j=1}^n \sum_{j'=1}^n \sigma_j^2 \sigma_{j'}^2 E(\varepsilon_{jt} (\lambda_T) \varepsilon_{jt'} (\lambda_T)) E(\varepsilon_{j't} (\lambda_T) \varepsilon_{j't'} (\lambda_T)). \end{aligned} \quad (\text{S.74})$$

For the first term of (S.74), given the serial independence of $\varepsilon_{it}(\lambda_T)$ and note $E(\varepsilon_{it}(\lambda_T)) = E(\varepsilon_{it} + \lambda_T \mathbf{w}'_{i0} \boldsymbol{\varepsilon}_{ot}) = 0$, some algebra yields

$$\begin{aligned}
& \frac{1}{T^2 n^2} \sum_{t=1}^T \sum_{t'=1}^T \sum_{j=1}^n \sum_{j'=1}^n \sigma_j^2 \sigma_{j'}^2 E(\varepsilon_{jt}(\lambda_T) \varepsilon_{jt'}(\lambda_T) \varepsilon_{j't}(\lambda_T) \varepsilon_{j't'}(\lambda_T)) \\
&= \frac{1}{T^2 n^2} \sum_{t=1}^T \sum_{j=1}^n \sigma_j^4 E(\varepsilon_{jt}^4(\lambda_T)) + \frac{1}{T^2 n^2} \sum_{t=1}^T \sum_{t' \neq t}^T \sum_{j=1}^n \sigma_j^4 E(\varepsilon_{jt}^2(\lambda_T)) E(\varepsilon_{jt'}^2(\lambda_T)) + \\
& \frac{1}{T^2 n^2} \sum_{t=1}^T \sum_{j=1}^n \sum_{j' \neq j}^n \sigma_j^2 \sigma_{j'}^2 E(\varepsilon_{jt}^2(\lambda_T) \varepsilon_{j't}^2(\lambda_T)) + \\
& \frac{1}{T^2 n^2} \sum_{t=1}^T \sum_{t' \neq t}^T \sum_{j=1}^n \sum_{j' \neq j}^n \sigma_j^2 \sigma_{j'}^2 E(\varepsilon_{jt}(\lambda_T) \varepsilon_{jt'}(\lambda_T) \varepsilon_{j't}(\lambda_T) \varepsilon_{j't'}(\lambda_T)). \tag{S.75}
\end{aligned}$$

To show the order of (S.75), note $\varepsilon_{it} = \mathbf{e}'_i \boldsymbol{\varepsilon}_{ot}$ where $\mathbf{e}_i$ is an $n \times 1$ selection vector with 1 on its i^{th} element and zero elsewhere, then

$$\begin{aligned}
\varepsilon_{jt}^4(\lambda_T) &= (\varepsilon_{it} + \lambda_T \mathbf{w}'_{i0} \boldsymbol{\varepsilon}_{ot})^4 = [(\mathbf{e}_i + \lambda_T \mathbf{w}_{i0})' \boldsymbol{\varepsilon}_{ot}]^4 \\
&= [\boldsymbol{\varepsilon}'_{ot} (\mathbf{e}_i + \lambda_T \mathbf{w}_{i0}) (\mathbf{e}_i + \lambda_T \mathbf{w}_{i0})' \boldsymbol{\varepsilon}_{ot}]^2 = (\boldsymbol{\varepsilon}'_{ot} \mathbf{A}_i \boldsymbol{\varepsilon}_{ot})^2,
\end{aligned}$$

with $\mathbf{A}_i = (\mathbf{e}_i + \lambda_T \mathbf{w}_{i0}) (\mathbf{e}_i + \lambda_T \mathbf{w}_{i0})'$, then based on result (S.7) of Lemma 6 in Pesaran and Yamagata (2024) and noting $w_{ii} = 0$ for all i ,

$$\begin{aligned}
E(\varepsilon_{jt}^4(\lambda_T)) &= E(\boldsymbol{\varepsilon}'_{ot} \mathbf{A}_i \boldsymbol{\varepsilon}_{ot})^2 \\
&= \kappa_2 \text{tr}[(\mathbf{A}_i \odot \mathbf{A}_i)] + [\text{tr}(\mathbf{A}_i)]^2 + 2 \text{tr}(\mathbf{A}_i^2) \\
&= \kappa_2 \left(1 + \lambda_T^4 \sum_{j=1}^n w_{ij}^4 \right) + 3 \left(1 + \lambda_T^2 \sum_{j=1}^n w_{ij}^2 \right) \\
&= O(1) \tag{S.76}
\end{aligned}$$

where $\kappa_2 = E(\varepsilon_{it}^4) - 3$ and the last line holds as by condition (10),

$$\sum_{j=1}^n w_{ij}^2 \leq \left(\sum_{j=1}^n |w_{ij}| \right)^2 < C, \sum_{j=1}^n w_{ij}^4 \leq \left(\sum_{j=1}^n |w_{ij}| \right)^4 < C.$$

It therefore follows that $E(\varepsilon_{jt}^2(\lambda_T))$ and $E(\varepsilon_{jt}^4(\lambda_T))$ are bounded so that the first two terms of (S.75) satisfy

$$\begin{aligned}
& \frac{1}{T^2 n^2} \sum_{t=1}^T \sum_{j=1}^n \sigma_j^4 E(\varepsilon_{jt}^4(\lambda_T)) = O\left(\frac{1}{nT}\right), \\
& \frac{1}{T^2 n^2} \sum_{t=1}^T \sum_{t' \neq t}^T \sum_{j=1}^n \sigma_j^4 E(\varepsilon_{jt}^2(\lambda_T)) E(\varepsilon_{jt'}^2(\lambda_T)) = O\left(\frac{1}{n}\right).
\end{aligned}$$

In addition, the third term of (S.75) can be bounded using Cauchy-Schwarz inequality,

$$\begin{aligned} & \frac{1}{T^2 n^2} \sum_{t=1}^T \sum_{j=1}^n \sum_{j' \neq j}^n \sigma_j^2 \sigma_{j'}^2 E(\varepsilon_{jt}^2(\lambda_T) \varepsilon_{j't}^2(\lambda_T)) \\ &= \frac{1}{T^2 n^2} \sum_{t=1}^T \sum_{j=1}^n \sum_{j' \neq j}^n \sigma_j^2 \sigma_{j'}^2 [E(\varepsilon_{jt}^4(\lambda_T))]^{1/2} \times [E(\varepsilon_{j't}^4(\lambda_T))]^{1/2} = O\left(\frac{1}{T}\right). \end{aligned}$$

The fourth term of (S.75) can be expanded based on the serial independence of $\varepsilon_{it}(\lambda_T)$, so that

$$\begin{aligned} & \frac{1}{T^2 n^2} \sum_{t=1}^T \sum_{t' \neq t}^T \sum_{j=1}^n \sum_{j' \neq j}^n \sigma_j^2 \sigma_{j'}^2 E(\varepsilon_{jt}(\lambda_T) \varepsilon_{jt'}(\lambda_T) \varepsilon_{j't}(\lambda_T) \varepsilon_{j't'}(\lambda_T)) \\ &= \frac{1}{T^2 n^2} \sum_{j=1}^n \sum_{j' \neq j}^n \sigma_j^2 \sigma_{j'}^2 \left[\sum_{t=1}^T E(\varepsilon_{jt}(\lambda_T) \varepsilon_{j't}(\lambda_T)) \right] \left[\sum_{t' \neq t}^T E(\varepsilon_{j't'}(\lambda_T) \varepsilon_{j't'}(\lambda_T)) \right]. \end{aligned}$$

Also note by definition of $\varepsilon_{jt}(\lambda_T)$, for $j \neq j'$,

$$E(\varepsilon_{jt}(\lambda_T) \varepsilon_{j't}(\lambda_T)) = \lambda_T g_{jj'} + \lambda_T^2 \mathbf{w}'_{j0} \mathbf{w}_{j'0}$$

where $g_{jj'} = w_{jj'} + w_{j'j}$ and $\mathbf{w}_{j0} = (w_{j1}, w_{j2}, \dots, w_{jn})'$, so that

$$\left[\sum_{t=1}^T E(\varepsilon_{jt}(\lambda_T) \varepsilon_{j't}(\lambda_T)) \right] \left[\sum_{t' \neq t}^T E(\varepsilon_{j't'}(\lambda_T) \varepsilon_{j't'}(\lambda_T)) \right] = T(T-1) (\lambda_T g_{jj'} + \lambda_T^2 \mathbf{w}'_{j0} \mathbf{w}_{j'0})^2.$$

Then it follows

$$\begin{aligned} & \frac{1}{T^2 n^2} \sum_{t=1}^T \sum_{t' \neq t}^T \sum_{j=1}^n \sum_{j' \neq j}^n \sigma_j^2 \sigma_{j'}^2 E(\varepsilon_{jt}(\lambda_T) \varepsilon_{jt'}(\lambda_T) \varepsilon_{j't}(\lambda_T) \varepsilon_{j't'}(\lambda_T)) \\ &= \frac{T(T-1)}{T^2 n^2} \sum_{j=1}^n \sum_{j' \neq j}^n \sigma_j^2 \sigma_{j'}^2 (\lambda_T g_{jj'} + \lambda_T^2 \mathbf{w}'_{j0} \mathbf{w}_{j'0})^2 \\ &\leq \frac{2T(T-1)}{T^2 n^2} \sum_{j=1}^n \sum_{j' \neq j}^n \sigma_j^2 \sigma_{j'}^2 (\lambda_T g_{jj'})^2 + \frac{2T(T-1)}{T^2 n^2} \sum_{j=1}^n \sum_{j' \neq j}^n \sigma_j^2 \sigma_{j'}^2 (\lambda_T^2 \mathbf{w}'_{j0} \mathbf{w}_{j'0})^2. \end{aligned}$$

Since $\sum_{j'=1}^n |w_{jj'}|^2 \leq \left(\sum_{j'=1}^n |w_{jj'}| \right)^2 \leq \|\mathbf{w}_{j0}\|^2 < C$ by (10), $\lambda_T = c_\lambda T^{-1/2}$ and $|\mathbf{w}'_{j0} \mathbf{w}_{j'0}| \leq \|\mathbf{w}_{j0}\| \|\mathbf{w}_{j'0}\| < C$, we further have

$$\begin{aligned} & \frac{2T(T-1)}{T^2 n^2} \sum_{j=1}^n \sum_{j' \neq j}^n \sigma_j^2 \sigma_{j'}^2 (\lambda_T g_{jj'})^2 = \frac{2T(T-1) \lambda_T^2}{T^2 n^2} \sum_{j=1}^n \sum_{j' \neq j}^n \sigma_j^2 \sigma_{j'}^2 (w_{jj'} + w_{j'j})^2 \\ &\leq \frac{4T(T-1) \lambda_T^2}{T^2 n^2} \sum_{j=1}^n \sum_{j' \neq j}^n \sigma_j^2 \sigma_{j'}^2 (|w_{jj'}|^2 + |w_{j'j}|^2) \\ &\leq \frac{4T(T-1) \lambda_T^2}{T^2 n^2} \left(\sup_i \sigma_i^4 \right) \sum_{j=1}^n \sum_{j' \neq j}^n (|w_{jj'}|^2 + |w_{j'j}|^2) \\ &= O\left(\frac{1}{nT}\right), \end{aligned}$$

and

$$\begin{aligned}
\frac{2T(T-1)}{T^2 n^2} \sum_{j=1}^n \sum_{j' \neq j}^n \sigma_j^2 \sigma_{j'}^2 (\lambda_T^2 \mathbf{w}'_{j0} \mathbf{w}_{j'0})^2 &= \frac{2T(T-1) \lambda_T^4}{T^2 n^2} \sum_{j=1}^n \sum_{j' \neq j}^n \sigma_j^2 \sigma_{j'}^2 (\mathbf{w}'_{j0} \mathbf{w}_{j'0})^2 \\
&\leq \frac{2T(T-1) \lambda_T^4}{T^2} \left(\sup_i \sigma_i^4 \right) \left(\sup_{j,j'} |\mathbf{w}'_{j0} \mathbf{w}_{j'0}| \right)^2 \\
&= O_p \left(\frac{1}{T^2} \right).
\end{aligned}$$

Overall, using (S.75) we are able to show the first term of (S.74) is $O(T^{-1})$ as $n, T \rightarrow \infty$ such that $n/T = \kappa$ where $0 < \kappa < \infty$. Compared to that, the second term of (S.74) satisfies

$$\begin{aligned}
&\frac{1}{T^2 n^2} \sum_{t=1}^T \sum_{t'=1}^T \sum_{j=1}^n \sum_{j'=1}^n \sigma_j^2 \sigma_{j'}^2 E(\varepsilon_{jt}(\lambda_T) \varepsilon_{j't'}(\lambda_T)) E(\varepsilon_{j't}(\lambda_T) \varepsilon_{j't'}(\lambda_T)) \\
&= \frac{1}{T^2 n^2} \sum_{t=1}^T \sum_{j=1}^n \sum_{j'=1}^n \sigma_j^2 \sigma_{j'}^2 E(\varepsilon_{jt}^2(\lambda_T)) E(\varepsilon_{j't}^2(\lambda_T)) = O\left(\frac{1}{T}\right).
\end{aligned}$$

As we have shown the orders of the two terms in (S.74), it follows that $E\left(\frac{1}{T^2} \sum_{t'=1}^T \sum_{t=1}^T \zeta_{tt'}^2\right) = O\left(\frac{1}{T}\right)$, so by Markov inequality $T^{-2} \sum_{t'=1}^T \sum_{t=1}^T \zeta_{tt'}^2 = O_p\left(\frac{1}{T}\right)$. Based on this result and using (S.41) and (S.54), it follows that

$$\begin{aligned}
\sup_i \|\mathbf{b}_{2,1,iT}\| &\leq \left(\frac{1}{T} \sum_{t'=1}^T \|\hat{\mathbf{f}}_{t'} - \mathbf{f}_{t'}\|^2 \right)^{1/2} \left(\frac{1}{T^2} \sum_{t'=1}^T \sum_{t=1}^T \zeta_{tt'}^2 \right)^{1/2} \left(\sup_i \frac{1}{T} \sum_{t=1}^T \varepsilon_{it}^2(\lambda_T) \right)^{1/2} \\
&= O_p\left(\frac{1}{\delta_{nT}}\right) \times O_p\left(\frac{1}{\sqrt{n}}\right) = O_p\left(\frac{1}{\sqrt{n}\delta_{nT}}\right).
\end{aligned}$$

Note that $\mathbf{b}_{2,2,iT}$ can be written as

$$\mathbf{b}_{2,2,iT} = \frac{1}{\sqrt{nT}} \frac{1}{T} \sum_{t=1}^T \mathbf{z}_t \varepsilon_{it}(\lambda_T),$$

where

$$\mathbf{z}_t = \frac{1}{\sqrt{nT}} \sum_{t'=1}^T \sum_{k=1}^n \sigma_k \mathbf{f}_{t'} [\varepsilon_{kt'}(\lambda_T) \varepsilon_{kt}(\lambda_T) - E(\varepsilon_{kt'}(\lambda_T) \varepsilon_{kt}(\lambda_T))]$$

and $\|\mathbf{z}_t\|^2 = O_p(1)$. Then by Cauchy-Schwarz inequality,

$$\|\mathbf{b}_{2,2,iT}\| = \frac{1}{\sqrt{nT}} \left\| \frac{1}{T} \sum_{t=1}^T \mathbf{z}_t \varepsilon_{it}(\lambda_T) \right\| \leq \frac{1}{\sqrt{nT}} \left(\frac{1}{T} \sum_{t=1}^T \|\mathbf{z}_t\|^2 \right)^{1/2} \left(\frac{1}{T} \sum_{t=1}^T \varepsilon_{it}^2(\lambda_T) \right)^{1/2},$$

and in view of (S.54) we have

$$\sup_i \|\mathbf{b}_{2,2,iT}\| \leq \frac{1}{\sqrt{nT}} \left(\frac{1}{T} \sum_{t=1}^T \|\mathbf{z}_t\|^2 \right)^{1/2} \sup_i \left(\frac{1}{T} \sum_{t=1}^T \varepsilon_{it}^2(\lambda_T) \right)^{1/2} = O_p\left(\frac{1}{\sqrt{nT}}\right).$$

Hence

$$\sup_i \|\mathbf{b}_{2,iT}\| \leq \sup_i \|\mathbf{b}_{2,1,iT}\| + \sup_i \|\mathbf{b}_{2,2,iT}\| = O_p\left(\frac{1}{\sqrt{n}\delta_{nT}}\right) + O_p\left(\frac{1}{\sqrt{nT}}\right).$$

Now consider $\mathbf{b}_{3,iT}$ in (S.71) and note that

$$\begin{aligned} \mathbf{b}_{3,iT} &= \frac{1}{T^2} \sum_{t=1}^T \sum_{t'=1}^T (\hat{\mathbf{f}}_{t'} - \mathbf{f}_{t'}) \boldsymbol{\varkappa}_{tt'} \varepsilon_{it}(\lambda_T) + \frac{1}{T^2} \sum_{t=1}^T \sum_{t'=1}^T \mathbf{f}_{t'} \boldsymbol{\varkappa}_{tt'} \varepsilon_{it}(\lambda_T) \\ &= \mathbf{b}_{3,1,iT} + \mathbf{b}_{3,2,iT}, \end{aligned}$$

where

$$\begin{aligned} \|\mathbf{b}_{3,1,iT}\| &= \left\| \frac{1}{T^2} \sum_{t'=1}^T (\hat{\mathbf{f}}_{t'} - \mathbf{f}_{t'}) \left(\sum_{t=1}^T \boldsymbol{\varkappa}_{tt'} \varepsilon_{it}(\lambda_T) \right) \right\| \\ &\leq \left(\frac{1}{T} \sum_{t'=1}^T \|\hat{\mathbf{f}}_{t'} - \mathbf{f}_{t'}\|^2 \right)^{1/2} \left[\frac{1}{T^3} \sum_{t'=1}^T \left(\sum_{t=1}^T \boldsymbol{\varkappa}_{tt'} \varepsilon_{it}(\lambda_T) \right)^2 \right]^{1/2} \\ &\leq \left(\frac{1}{T} \sum_{t'=1}^T \|\hat{\mathbf{f}}_{t'} - \mathbf{f}_{t'}\|^2 \right)^{1/2} \left(\frac{1}{T^2} \sum_{t'=1}^T \sum_{t=1}^T \boldsymbol{\varkappa}_{tt'}^2 \right)^{1/2} \left(\frac{1}{T} \sum_{t=1}^T \varepsilon_{it}^2(\lambda_T) \right)^{1/2}. \end{aligned}$$

By definition of $\boldsymbol{\varkappa}_{tt'}$ and given (S.69) it follows that

$$\begin{aligned} E \left(\frac{1}{T^2} \sum_{t=1}^T \sum_{t'=1}^T \boldsymbol{\varkappa}_{tt'}^2 \right) &= \frac{1}{T^2} \sum_{t'=1}^T \sum_{t=1}^T E \left(\frac{1}{n} \sum_{j=1}^n \sigma_j \mathbf{f}_{t'}' \boldsymbol{\gamma}_j \varepsilon_{jt}(\lambda_T) \right)^2 \\ &= \frac{1}{T^2 n^2} \sum_{t'=1}^T \sum_{t=1}^T \sum_{j=1}^n \sum_{j'=1}^n \sigma_j \sigma_{j'} E(\mathbf{f}_{t'}' \boldsymbol{\gamma}_j \mathbf{f}_t \boldsymbol{\gamma}_{j'}) E(\varepsilon_{jt}(\lambda_T) \varepsilon_{j't}(\lambda_T)) \\ &\leq \left(\sup_i \|\boldsymbol{\gamma}_i\|^2 \right) \left(\sup_t E \|\mathbf{f}_t\|^2 \right) \frac{1}{T^2 n^2} \sum_{t'=1}^T \sum_{t=1}^T \sum_{j=1}^n \sum_{j'=1}^n \sigma_j \sigma_{j'} E(\varepsilon_{jt}(\lambda_T) \varepsilon_{j't}(\lambda_T)) \\ &\leq \left(\sup_i \|\boldsymbol{\gamma}_i\|^2 \right) \left(\sup_t E \|\mathbf{f}_t\|^2 \right) \left(\sup_i \sigma_i^2 \right) \frac{1}{nT} \sum_{t=1}^T \left(\frac{1}{n} \sum_{j=1}^n \sum_{j'=1}^n |E(\varepsilon_{jt}(\lambda_T) \varepsilon_{j't}(\lambda_T))| \right) \\ &= O\left(\frac{1}{n}\right), \end{aligned}$$

and by Markov inequality we have $T^{-2} \sum_{t'=1}^T \sum_{t=1}^T \boldsymbol{\varkappa}_{tt'}^2 = O_p(n^{-1})$. Then given this result and (S.41), (S.54), it follows

$$\begin{aligned} \sup_i \|\mathbf{b}_{3,1,iT}\| &= \left(\frac{1}{T} \sum_{t'=1}^T \|\hat{\mathbf{f}}_{t'} - \mathbf{f}_{t'}\|^2 \right)^{1/2} \left(\frac{1}{T^2} \sum_{t'=1}^T \sum_{t=1}^T \boldsymbol{\varkappa}_{tt'}^2 \right)^{1/2} \left(\sup_i \frac{1}{T} \sum_{t=1}^T \varepsilon_{it}^2(\lambda_T) \right)^{1/2} \\ &= O_p\left(\frac{1}{\delta_{nT}}\right) \times O_p\left(\frac{1}{\sqrt{n}}\right) = O_p\left(\frac{1}{\sqrt{n}\delta_{nT}}\right). \end{aligned}$$

Next consider $\|\mathbf{b}_{3,2,iT}\|$, and note that

$$\frac{1}{T^2} \sum_{t=1}^T \sum_{t'=1}^T \mathbf{f}_{t'} \boldsymbol{\kappa}_{tt'} \varepsilon_{it}(\lambda_T) = \left(\frac{1}{T} \sum_{t'=1}^T \mathbf{f}_{t'} \mathbf{f}_{t'}' \right) \left(\frac{1}{Tn} \sum_{t=1}^T \sum_{j=1}^n \sigma_j \boldsymbol{\gamma}_j \varepsilon_{jt}(\lambda_T) \varepsilon_{it}(\lambda_T) \right),$$

and therefore

$$\|\mathbf{b}_{3,2,iT}\| = \left\| \frac{1}{T^2} \sum_{t=1}^T \sum_{t'=1}^T \mathbf{f}_{t'} \boldsymbol{\kappa}_{tt'} \varepsilon_{it}(\lambda_T) \right\| \leq \left(\frac{1}{T} \sum_{t'=1}^T \|\mathbf{f}_{t'} \mathbf{f}_{t'}'\| \right) \left(\left\| \frac{1}{nT} \sum_{t=1}^T \sum_{j=1}^n \sigma_j \boldsymbol{\gamma}_j \varepsilon_{jt}(\lambda_T) \varepsilon_{it}(\lambda_T) \right\| \right).$$

Then given (S.57),

$$\sup_i \|\mathbf{b}_{3,2,iT}\| \leq \left(\frac{1}{T} \sum_{t'=1}^T \|\mathbf{f}_{t'} \mathbf{f}_{t'}'\| \right) \left(\sup_i \left\| \frac{1}{nT} \sum_{t=1}^T \sum_{j=1}^n \sigma_j \boldsymbol{\gamma}_j \varepsilon_{jt}(\lambda_T) \varepsilon_{it}(\lambda_T) \right\| \right) = O_p \left(\sqrt{\frac{\ln(n)}{nT}} \right).$$

Hence,

$$\sup_i \|\mathbf{b}_{3,iT}\| \leq \sup_i \|\mathbf{b}_{3,1,iT}\| + \sup_i \|\mathbf{b}_{3,2,iT}\| = O_p \left(\sqrt{\frac{\ln(n)}{nT}} \right).$$

Similarly, the probability order of $\sup_i \|\mathbf{b}_{4,iT}\|$ can also be shown to be $O_p \left(\sqrt{\ln(n) / (nT)} \right)$.

Overall, result (S.70) follows as we can use (S.71) to show

$$\begin{aligned} \sup_i \left\| \frac{(\hat{\mathbf{F}} - \mathbf{F})' \boldsymbol{\varepsilon}_{i0}(\lambda_T)}{T} \right\| &\leq \sup_i \left(\sum_{j=1}^4 \|\mathbf{b}_{j,iT}\| \right) \leq \sum_{j=1}^4 \sup_i \|\mathbf{b}_{j,iT}\| \\ &= O_p \left(\frac{1}{\sqrt{T} \delta_{nT}} \right) + O_p \left(\frac{1}{\sqrt{n} \delta_{nT}} \right) + O_p \left(\frac{1}{\sqrt{nT}} \right) + O_p \left(\sqrt{\frac{\ln(n)}{nT}} \right) \\ &= O_p \left(\sqrt{\frac{\ln(n)}{nT}} \right). \end{aligned}$$

■

Lemma S.7 Denote $\boldsymbol{\varepsilon}_{i0} = (\varepsilon_{i1}, \varepsilon_{i2}, \dots, \varepsilon_{iT})'$ and $\mathbf{b}_i = (b_{i1}, b_{i2}, \dots, b_{iT})'$ with $b_{it} = \mathbf{w}_{i0}' \boldsymbol{\varepsilon}_{0t}$, $\mathbf{w}_{i0} = (w_{i1}, w_{i2}, \dots, w_{in})'$ and $\boldsymbol{\varepsilon}_{0t} = (\varepsilon_{1t}, \varepsilon_{2t}, \dots, \varepsilon_{nt})'$. Suppose that Assumptions 2 and 4 hold. Then as $(n, T) \rightarrow \infty$, such that $n/T \rightarrow \kappa$, for $0 < \kappa < \infty$,

$$\sup_i \left| \frac{\boldsymbol{\varepsilon}_{i0}' \mathbf{b}_i}{T} \right| = O_p \left(\sqrt{\frac{\ln(n)}{T}} \right), \quad (\text{S.77})$$

$$\sup_i \left| \frac{\mathbf{b}_i' \mathbf{b}_i}{T} \right| = O_p(1). \quad (\text{S.78})$$

Proof. Consider (S.77) and denote $\mathcal{I}_{i,t-1}^{(eb)} = \{\varepsilon_{i\tau} b_{i\tau} : \tau = t-1, t-2, \dots\}$ and $i = 1, 2, \dots, n$. By assumption, ε_{it} is cross-sectionally independent, and is independent from b_{it} as $w_{ii} = 0$ for all i . Then $E \left(\varepsilon_{it} b_{it} | \mathcal{I}_{i,t-1}^{(eb)} \right) = E(\varepsilon_{it} b_{it}) = 0$. Also

$$\text{Var}(\varepsilon_{it} b_{it}) = \text{Var}(\varepsilon_{it}) \text{Var}(b_{it}) = E(b_{it}^2),$$

which is bounded as by condition (10),

$$E(b_{it}^2) = E(\mathbf{w}'_{i0}\boldsymbol{\varepsilon}_{ot})^2 = \mathbf{w}'_{i0}E(\boldsymbol{\varepsilon}_{ot}\boldsymbol{\varepsilon}'_{ot})\mathbf{w}_{i0} = \sum_{s=1}^n w_{is}^2 < C. \quad (\text{S.79})$$

In addition, since $\varepsilon_{it}(\lambda_T) = \varepsilon_{it} + \lambda_T b_{it}$ is sub-exponential for any $|\lambda_T| < C$, then it follows that ε_{it} and b_{it} are both sub-exponential, and hence $\varepsilon_{it}b_{it}$ is also sub-exponential. Hence, result (S.77) can be established by applying the method of proof used for result (S.49). Now consider (S.78) and note

$$\frac{\mathbf{b}'_i \mathbf{b}_i}{T} = \frac{1}{T} \sum_{t=1}^T [b_{it}^2 - E(b_{it}^2)] + \frac{1}{T} \sum_{t=1}^T E(b_{it}^2),$$

which further implies

$$\sup_i \left| \frac{\mathbf{b}'_i \mathbf{b}_i}{T} \right| \leq \sup_i \left| \frac{1}{T} \sum_{t=1}^T [b_{it}^2 - E(b_{it}^2)] \right| + \sup_i \left(\frac{1}{T} \sum_{t=1}^T E(b_{it}^2) \right).$$

Denote $\mathcal{I}_{i,t-1}^{(b)} = \{b_{i\tau} : \tau = t-1, t-2, \dots\}$ and $i = 1, 2, \dots, n$, then $E(b_{it}^2 - E(b_{it}^2) | \mathcal{I}_{i,t-1}^{(b)}) = E(b_{it}^2 - E(b_{it}^2)) = 0$. Besides, $\text{Var}(b_{it}^2) = E(b_{it}^4) - [E(b_{it}^2)]^2$ is bounded as (S.76) shows $E(b_{it}^4) = E(\mathbf{w}'_{i0}\boldsymbol{\varepsilon}_{ot})^4 = O(1)$. Also $b_{it}^2 - E(b_{it}^2)$ is sub-exponential given that it is already established that b_{it} is sub-exponential. Therefore, it follows that $\sup_i \left| \frac{1}{T} \sum_{t=1}^T [b_{it}^2 - E(b_{it}^2)] \right| = O_p\left(\sqrt{\frac{\ln(n)}{T}}\right)$ by applying the method of proof used for result (S.54). Also by (S.79), we have $\sup_i \left(\frac{1}{T} \sum_{t=1}^T E(b_{it}^2) \right) = O(1)$. Result (S.78) now follows straightforwardly. ■

Lemma S.8 Consider the latent factor model given by (1) and (2). Let $\hat{\sigma}_{i,T} = (T^{-1}\mathbf{e}'_i\mathbf{e}_i)^{1/2}$, where $\mathbf{e}_i = \mathbf{M}_{\hat{\mathbf{F}}}\mathbf{y}_i$, $\mathbf{M}_{\hat{\mathbf{F}}} = \mathbf{I}_T - \hat{\mathbf{F}}(\hat{\mathbf{F}}'\hat{\mathbf{F}})^{-1}\hat{\mathbf{F}}'$, $\mathbf{y}_i = (y_{i1}, y_{i2}, \dots, y_{iT})'$, and $\hat{\mathbf{F}}$ is given by (12). Also let $\omega_{i,T} = (T^{-1}\sigma_i^2\boldsymbol{\varepsilon}'_{io}\mathbf{M}_F\boldsymbol{\varepsilon}_{io})^{1/2}$, where $\mathbf{M}_F = \mathbf{I}_T - \mathbf{F}(\mathbf{F}'\mathbf{F})^{-1}\mathbf{F}'$. Suppose that Assumptions 1-4 hold and $(n, T) \rightarrow \infty$, such that $n/T \rightarrow \kappa$, for $0 < \kappa < \infty$. Then

$$\sup_i |\hat{\sigma}_{i,T}^2 - \omega_{i,T}^2| = O_p\left(\frac{\ln(n)}{T}\right), \quad (\text{S.80})$$

$$\sup_i |\hat{\sigma}_{i,T} - \omega_{i,T}| = O_p\left(\frac{\ln(n)}{T}\right), \quad (\text{S.81})$$

$$\sup_i \left| \frac{1}{\hat{\sigma}_{i,T}} - \frac{1}{\omega_{i,T}} \right| = O_p\left(\frac{\ln(n)}{T}\right), \quad (\text{S.82})$$

and

$$\frac{1}{n} \sum_{i=1}^n |\hat{\sigma}_{i,T}^2 - \omega_{i,T}^2| = O_p\left(\frac{\ln(n)}{T}\right), \quad (\text{S.83})$$

$$\frac{1}{n} \sum_{i=1}^n |\hat{\sigma}_{i,T} - \omega_{i,T}| = O_p\left(\frac{\ln(n)}{T}\right), \quad (\text{S.84})$$

$$\frac{1}{n} \sum_{i=1}^n \left| \frac{1}{\hat{\sigma}_{i,T}} - \frac{1}{\omega_{i,T}} \right| = O_p\left(\frac{\ln(n)}{T}\right). \quad (\text{S.85})$$

Proof. Note that by model (1), $\mathbf{y}_i = \mathbf{F}\boldsymbol{\gamma}_i + \sigma_i \boldsymbol{\varepsilon}_{i0}(\lambda_T)$ where $\boldsymbol{\varepsilon}_{i0}(\lambda_T) = (\varepsilon_{i1}(\lambda_T), \varepsilon_{i2}(\lambda_T), \dots, \varepsilon_{iT}(\lambda_T))'$, which implies

$$\begin{aligned}\mathbf{e}_i &= \mathbf{M}_{\hat{\mathbf{F}}} \mathbf{y}_i = \mathbf{M}_{\hat{\mathbf{F}}} (\mathbf{F}\boldsymbol{\gamma}_i + \sigma_i \boldsymbol{\varepsilon}_{i0}(\lambda_T)) \\ &= \sigma_i \mathbf{M}_F \boldsymbol{\varepsilon}_{i0}(\lambda_T) + \sigma_i (\mathbf{M}_{\hat{\mathbf{F}}} - \mathbf{M}_F) \boldsymbol{\varepsilon}_{i0}(\lambda_T) + \mathbf{M}_{\hat{\mathbf{F}}} \mathbf{F}\boldsymbol{\gamma}_i.\end{aligned}$$

Then $\hat{\sigma}_{i,T}^2$ can be decomposed as

$$\begin{aligned}\hat{\sigma}_{i,T}^2 - \omega_{i,T}^2 &= \frac{\boldsymbol{\gamma}_i' \mathbf{F}' \mathbf{M}_{\hat{\mathbf{F}}} \mathbf{F} \boldsymbol{\gamma}_i}{T} + \frac{\sigma_i^2 \boldsymbol{\varepsilon}_{i0}(\lambda_T)' (\mathbf{M}_{\hat{\mathbf{F}}} - \mathbf{M}_F) (\mathbf{M}_{\hat{\mathbf{F}}} - \mathbf{M}_F) \boldsymbol{\varepsilon}_{i0}(\lambda_T)}{T} \\ &\quad + \frac{2\sigma_i^2 \boldsymbol{\varepsilon}_{i0}(\lambda_T)' \mathbf{M}_F (\mathbf{M}_{\hat{\mathbf{F}}} - \mathbf{M}_F) \boldsymbol{\varepsilon}_{i0}(\lambda_T)}{T} + \frac{2\sigma_i^2 \boldsymbol{\varepsilon}_{i0}(\lambda_T)' \mathbf{M}_F \mathbf{M}_{\hat{\mathbf{F}}} \mathbf{F} \boldsymbol{\gamma}_i}{T} \\ &\quad + \frac{2\sigma_i \boldsymbol{\varepsilon}_{i0}(\lambda_T)' (\mathbf{M}_{\hat{\mathbf{F}}} - \mathbf{M}_F) \mathbf{M}_{\hat{\mathbf{F}}} \mathbf{F} \boldsymbol{\gamma}_i}{T} + \frac{\sigma_i^2 (\boldsymbol{\varepsilon}_{i0}(\lambda_T) + \boldsymbol{\varepsilon}_{i0})' \mathbf{M}_F (\boldsymbol{\varepsilon}_{i0}(\lambda_T) - \boldsymbol{\varepsilon}_{i0})}{T} \\ &= \sum_{j=1}^6 B_{j,iT},\end{aligned}$$

and hence

$$|\hat{\sigma}_{i,T}^2 - \omega_{i,T}^2| \leq \sum_{j=1}^6 |B_{j,iT}|. \quad (\text{S.86})$$

Starting with the first term, and note

$$\begin{aligned}|B_{1,iT}| &= \left| \frac{\boldsymbol{\gamma}_i' \mathbf{F}' \mathbf{M}_{\hat{\mathbf{F}}} \mathbf{F} \boldsymbol{\gamma}_i}{T} \right| = \left| \frac{\boldsymbol{\gamma}_i' (\mathbf{F} - \hat{\mathbf{F}})' \mathbf{M}_{\hat{\mathbf{F}}} (\mathbf{F} - \hat{\mathbf{F}}) \boldsymbol{\gamma}_i}{T} \right| \\ &\leq \|\boldsymbol{\gamma}_i\|^2 \|\mathbf{M}_{\hat{\mathbf{F}}}\| \left(\frac{\|\hat{\mathbf{F}} - \mathbf{F}\|^2}{T} \right).\end{aligned}$$

Since $\sup_i \|\boldsymbol{\gamma}_i\| < C$, then using (S.41) of Lemma S.2 we have (note that $\|\mathbf{M}_{\hat{\mathbf{F}}}\| = 1$)

$$\sup_i |B_{1,iT}| \leq \left(\sup_i \|\boldsymbol{\gamma}_i\|^2 \right) \|\mathbf{M}_{\hat{\mathbf{F}}}\| \left(\frac{\|\hat{\mathbf{F}} - \mathbf{F}\|^2}{T} \right) = O_p \left(\frac{1}{\delta_{nT}^2} \right).$$

To establish the probability orders of the remaining terms of (S.83), we first observe that

$$\frac{\hat{\mathbf{F}}' \hat{\mathbf{F}}}{T} - \frac{\mathbf{F}' \mathbf{F}}{T} = \frac{(\hat{\mathbf{F}} - \mathbf{F})' (\hat{\mathbf{F}} - \mathbf{F})}{T} + \frac{(\hat{\mathbf{F}} - \mathbf{F})' \mathbf{F}}{T} + \frac{\mathbf{F}' (\hat{\mathbf{F}} - \mathbf{F})}{T},$$

hence using results (S.41) and (S.46) it follows

$$\begin{aligned}\left\| \frac{\hat{\mathbf{F}}' \hat{\mathbf{F}}}{T} - \frac{\mathbf{F}' \mathbf{F}}{T} \right\| &\leq \frac{\|\hat{\mathbf{F}} - \mathbf{F}\|^2}{T} + \frac{\|(\hat{\mathbf{F}} - \mathbf{F})' \mathbf{F}\|}{T} + \frac{\|\mathbf{F}' (\hat{\mathbf{F}} - \mathbf{F})\|}{T} \\ &= O_p \left(\frac{1}{\delta_{nT}^2} \right) + O_p \left(\frac{1}{\delta_{nT}} \right) = O_p \left(\frac{1}{\delta_{nT}} \right).\end{aligned} \quad (\text{S.87})$$

By assumption $T^{-1}\mathbf{F}'\mathbf{F}$ is a positive definite matrix, then

$$\begin{aligned} \left\| \left(\frac{\hat{\mathbf{F}}'\hat{\mathbf{F}}}{T} \right)^{-1} - \left(\frac{\mathbf{F}'\mathbf{F}}{T} \right)^{-1} \right\| &\leq \left\| \left(\frac{\hat{\mathbf{F}}'\hat{\mathbf{F}}}{T} \right)^{-1} \right\| \left\| \frac{\hat{\mathbf{F}}'\hat{\mathbf{F}}}{T} - \frac{\mathbf{F}'\mathbf{F}}{T} \right\| \left\| \left(\frac{\mathbf{F}'\mathbf{F}}{T} \right)^{-1} \right\| \\ &= O_p \left(\frac{1}{\delta_{nT}} \right), \end{aligned} \quad (\text{S.88})$$

so it follows that

$$\frac{\hat{\mathbf{F}}'\hat{\mathbf{F}}}{T} = \frac{\mathbf{F}'\mathbf{F}}{T} + O_p \left(\frac{1}{\delta_{nT}} \right), \text{ and } \left(\frac{\hat{\mathbf{F}}'\hat{\mathbf{F}}}{T} \right)^{-1} = \left(\frac{\mathbf{F}'\mathbf{F}}{T} \right)^{-1} + O_p \left(\frac{1}{\delta_{nT}} \right). \quad (\text{S.89})$$

Consider now $B_{2,iT}$, and note that

$$\begin{aligned} &\sigma_i^2 \boldsymbol{\varepsilon}_{io}(\lambda_T)' (\mathbf{M}_{\hat{F}} - \mathbf{M}_F) (\mathbf{M}_{\hat{F}} - \mathbf{M}_F) \boldsymbol{\varepsilon}_{io}(\lambda_T) \\ &= \sigma_i^2 \boldsymbol{\varepsilon}_{io}(\lambda_T)' \left[\hat{\mathbf{F}} \left(\hat{\mathbf{F}}'\hat{\mathbf{F}} \right)^{-1} \hat{\mathbf{F}}' - \mathbf{F} \left(\mathbf{F}'\mathbf{F} \right)^{-1} \mathbf{F}' \right] \boldsymbol{\varepsilon}_{io}(\lambda_T) + \\ &\sigma_i^2 \boldsymbol{\varepsilon}_{io}(\lambda_T)' \left(\mathbf{I}_{m_0} - \hat{\mathbf{F}} \left(\hat{\mathbf{F}}'\hat{\mathbf{F}} \right)^{-1} \hat{\mathbf{F}}' \right) \mathbf{F} \left(\mathbf{F}'\mathbf{F} \right)^{-1} \mathbf{F}' \boldsymbol{\varepsilon}_{io}(\lambda_T) + \\ &\sigma_i^2 \boldsymbol{\varepsilon}_{io}(\lambda_T)' \mathbf{F} \left(\mathbf{F}'\mathbf{F} \right)^{-1} \mathbf{F}' \left(\mathbf{I}_{m_0} - \hat{\mathbf{F}} \left(\hat{\mathbf{F}}'\hat{\mathbf{F}} \right)^{-1} \hat{\mathbf{F}}' \right) \boldsymbol{\varepsilon}_{io}(\lambda_T), \end{aligned}$$

where the first term can be further decomposed as

$$\begin{aligned} \hat{\mathbf{F}} \left(\hat{\mathbf{F}}'\hat{\mathbf{F}} \right)^{-1} \hat{\mathbf{F}}' - \mathbf{F} \left(\mathbf{F}'\mathbf{F} \right)^{-1} \mathbf{F}' &= \left(\hat{\mathbf{F}} - \mathbf{F} \right) \left(\hat{\mathbf{F}}'\hat{\mathbf{F}} \right)^{-1} \left(\hat{\mathbf{F}} - \mathbf{F} \right)' + \left[\mathbf{F} \left(\hat{\mathbf{F}}'\hat{\mathbf{F}} \right)^{-1} \mathbf{F}' - \mathbf{F} \left(\mathbf{F}'\mathbf{F} \right)^{-1} \mathbf{F}' \right] \\ &\quad + \mathbf{F} \left(\hat{\mathbf{F}}'\hat{\mathbf{F}} \right)^{-1} \left(\hat{\mathbf{F}} - \mathbf{F} \right)' + \left(\hat{\mathbf{F}} - \mathbf{F} \right) \left(\hat{\mathbf{F}}'\hat{\mathbf{F}} \right)^{-1} \mathbf{F}'. \end{aligned}$$

Hence

$$B_{2,iT} = \sum_{j=1}^6 B_{2,j,iT}$$

where

$$\begin{aligned} B_{2,1,iT} &= T^{-1} \sigma_i^2 \boldsymbol{\varepsilon}_{io}(\lambda_T)' \left(\hat{\mathbf{F}} - \mathbf{F} \right) \left(\hat{\mathbf{F}}'\hat{\mathbf{F}} \right)^{-1} \left(\hat{\mathbf{F}} - \mathbf{F} \right)' \boldsymbol{\varepsilon}_{io}(\lambda_T), \\ B_{2,2,iT} &= T^{-1} \sigma_i^2 \boldsymbol{\varepsilon}_{io}(\lambda_T)' \left[\mathbf{F} \left(\hat{\mathbf{F}}'\hat{\mathbf{F}} \right)^{-1} \mathbf{F}' - \mathbf{F} \left(\mathbf{F}'\mathbf{F} \right)^{-1} \mathbf{F}' \right] \boldsymbol{\varepsilon}_{io}(\lambda_T), \\ B_{2,3,iT} &= B_{2,4,iT} = T^{-1} \sigma_i^2 \boldsymbol{\varepsilon}_{io}(\lambda_T)' \mathbf{F} \left(\hat{\mathbf{F}}'\hat{\mathbf{F}} \right)^{-1} \left(\hat{\mathbf{F}} - \mathbf{F} \right)' \boldsymbol{\varepsilon}_{io}(\lambda_T), \\ B_{2,5,iT} &= B_{2,6,iT} = T^{-1} \sigma_i^2 \boldsymbol{\varepsilon}_{io}(\lambda_T)' \left(\mathbf{I}_{m_0} - \hat{\mathbf{F}} \left(\hat{\mathbf{F}}'\hat{\mathbf{F}} \right)^{-1} \hat{\mathbf{F}}' \right) \mathbf{F} \left(\mathbf{F}'\mathbf{F} \right)^{-1} \mathbf{F}' \boldsymbol{\varepsilon}_{io}(\lambda_T), \end{aligned}$$

which implies

$$B_{2,iT} \leq \sum_{j=1}^6 |B_{2,j,iT}|. \quad (\text{S.90})$$

For $B_{2,1,iT}$, note that

$$\begin{aligned} |B_{2,1,iT}| &= \left| T^{-1} \sigma_i^2 \boldsymbol{\varepsilon}_{io}(\lambda_T)' (\hat{\mathbf{F}} - \mathbf{F}) (\hat{\mathbf{F}}' \hat{\mathbf{F}})^{-1} (\hat{\mathbf{F}} - \mathbf{F})' \boldsymbol{\varepsilon}_{io}(\lambda_T) \right| \\ &\leq \sigma_i^2 \left(\frac{1}{T} \|\boldsymbol{\varepsilon}_{io}(\lambda_T)\|^2 \right) \left(\frac{\|\hat{\mathbf{F}} - \mathbf{F}\|^2}{T} \right) \left\| \left(\frac{\hat{\mathbf{F}}' \hat{\mathbf{F}}}{T} \right)^{-1} \right\|. \end{aligned}$$

Then by (S.41) and (S.54) we have

$$\sup_i |B_{2,1,iT}| \leq \left(\sup_i \sigma_i^2 \right) \left(\sup_i \frac{1}{T} \|\boldsymbol{\varepsilon}_{io}(\lambda_T)\|^2 \right) \left(\frac{\|\hat{\mathbf{F}} - \mathbf{F}\|^2}{T} \right) \left\| \left(\frac{\hat{\mathbf{F}}' \hat{\mathbf{F}}}{T} \right)^{-1} \right\| = O_p \left(\frac{1}{\delta_{nT}^2} \right).$$

Next, consider $B_{2,2,iT}$ and note that

$$\begin{aligned} |B_{2,2,iT}| &= \left| \frac{1}{T} \sigma_i^2 \boldsymbol{\varepsilon}_{io}(\lambda_T)' \left[\mathbf{F} (\hat{\mathbf{F}}' \hat{\mathbf{F}})^{-1} \mathbf{F}' - \mathbf{F} (\mathbf{F}' \mathbf{F})^{-1} \mathbf{F}' \right] \boldsymbol{\varepsilon}_{io}(\lambda_T) \right| \\ &= \sigma_i^2 \left| \left(\frac{\boldsymbol{\varepsilon}_{io}(\lambda_T)' \mathbf{F}}{T} \right) \left[\left(\frac{\hat{\mathbf{F}}' \hat{\mathbf{F}}}{T} \right)^{-1} - \left(\frac{\mathbf{F}' \mathbf{F}}{T} \right)^{-1} \right] \left(\frac{\mathbf{F}' \boldsymbol{\varepsilon}_{io}(\lambda_T)}{T} \right) \right| \\ &\leq \sigma_i^2 \left(\left\| \frac{\boldsymbol{\varepsilon}_{io}(\lambda_T)' \mathbf{F}}{T} \right\|^2 \right) \left\| \left(\frac{\hat{\mathbf{F}}' \hat{\mathbf{F}}}{T} \right)^{-1} - \left(\frac{\mathbf{F}' \mathbf{F}}{T} \right)^{-1} \right\|. \end{aligned}$$

Using (S.55) (from Lemma S.4) and (S.89)), we have

$$\begin{aligned} \sup_i |B_{2,2,iT}| &\leq \left(\sup_i \sigma_i^2 \right) \left(\sup_i \left\| \frac{\boldsymbol{\varepsilon}_{io}(\lambda_T)' \mathbf{F}}{T} \right\|^2 \right) \left\| \left(\frac{\hat{\mathbf{F}}' \hat{\mathbf{F}}}{T} \right)^{-1} - \left(\frac{\mathbf{F}' \mathbf{F}}{T} \right)^{-1} \right\| \\ &= O_p \left(\frac{\ln(n)}{T} \right) \times O_p \left(\frac{1}{\delta_{nT}} \right) = O_p \left(\frac{\ln(n)}{T \delta_{nT}} \right). \end{aligned}$$

Similarly

$$\begin{aligned} |B_{2,3,iT}| &= \sigma_i^2 \left| \left(\frac{\boldsymbol{\varepsilon}_{io}(\lambda_T)' \mathbf{F}}{T} \right) \left(\frac{\hat{\mathbf{F}}' \hat{\mathbf{F}}}{T} \right)^{-1} \left[\frac{(\hat{\mathbf{F}} - \mathbf{F})' \boldsymbol{\varepsilon}_{io}(\lambda_T)}{T} \right] \right| \\ &\leq \sigma_i^2 \left\| \frac{\boldsymbol{\varepsilon}_{io}(\lambda_T)' \mathbf{F}}{T} \right\| \left\| \left(\frac{\hat{\mathbf{F}}' \hat{\mathbf{F}}}{T} \right)^{-1} \right\| \left\| \frac{(\hat{\mathbf{F}} - \mathbf{F})' \boldsymbol{\varepsilon}_{io}(\lambda_T)}{T} \right\|, \end{aligned} \tag{S.91}$$

and using (S.55), (S.70) and (S.89) we have

$$\begin{aligned} \sup_i |B_{2,3,iT}| &\leq \left\| \left(\frac{\hat{\mathbf{F}}' \hat{\mathbf{F}}}{T} \right)^{-1} \right\| \left(\sup_i \left\| \frac{\boldsymbol{\varepsilon}_{io}(\lambda_T)' \mathbf{F}}{T} \right\| \right) \left(\sup_i \left\| \frac{(\hat{\mathbf{F}} - \mathbf{F})' \boldsymbol{\varepsilon}_{io}(\lambda_T)}{T} \right\| \right) \\ &= O_p \left(\sqrt{\frac{\ln(n)}{T}} \right) \times O_p \left(\sqrt{\frac{\ln(n)}{nT}} \right). \end{aligned}$$

Next,

$$|B_{2,5,iT}| = \left| T^{-1} \sigma_i^2 \boldsymbol{\varepsilon}_{io}(\lambda_T)' \left(\mathbf{I}_{m_0} - \hat{\mathbf{F}} \left(\hat{\mathbf{F}}' \hat{\mathbf{F}} \right)^{-1} \hat{\mathbf{F}}' \right) \mathbf{F} \left(\mathbf{F}' \mathbf{F} \right)^{-1} \mathbf{F}' \boldsymbol{\varepsilon}_{io}(\lambda_T) \right| \\ \leq \sigma_i^2 \left| \frac{\boldsymbol{\varepsilon}_{io}(\lambda_T)' \mathbf{P}_F \boldsymbol{\varepsilon}_{io}(\lambda_T)}{T} \right| + \sigma_i^2 \left| \frac{\boldsymbol{\varepsilon}_{io}'(\lambda_T) \mathbf{P}_F (\mathbf{P}_{\hat{\mathbf{F}}} - \mathbf{P}_F) \boldsymbol{\varepsilon}_{io}(\lambda_T)}{T} \right|, \quad (\text{S.92})$$

where $\mathbf{P}_F = \mathbf{F} (\mathbf{F}' \mathbf{F})^{-1} \mathbf{F}'$ and $\mathbf{P}_{\hat{\mathbf{F}}} = \hat{\mathbf{F}} (\hat{\mathbf{F}}' \hat{\mathbf{F}})^{-1} \hat{\mathbf{F}}'$. But

$$\left| \frac{\boldsymbol{\varepsilon}_{io}(\lambda_T)' \mathbf{P}_F \boldsymbol{\varepsilon}_{io}(\lambda_T)}{T} \right| = \left(\frac{\boldsymbol{\varepsilon}_{io}(\lambda_T)' \mathbf{F}}{T} \right) \left(\frac{\mathbf{F}' \mathbf{F}}{T} \right)^{-1} \left(\frac{\mathbf{F}' \boldsymbol{\varepsilon}_{io}(\lambda_T)}{T} \right) \leq \left\| \left(\frac{\mathbf{F}' \mathbf{F}}{T} \right)^{-1} \right\| \left\| \frac{\boldsymbol{\varepsilon}_{io}(\lambda_T)' \mathbf{F}}{T} \right\|^2,$$

and again using (S.55)

$$\sup_i \left| \frac{\boldsymbol{\varepsilon}_{io}(\lambda_T)' \mathbf{P}_F \boldsymbol{\varepsilon}_{io}(\lambda_T)}{T} \right| \leq \left\| \left(\frac{\mathbf{F}' \mathbf{F}}{T} \right)^{-1} \right\| \left(\sup_i \left\| \frac{\boldsymbol{\varepsilon}_{io}(\lambda_T)' \mathbf{F}}{T} \right\|^2 \right) = O_p \left(\frac{\ln(n)}{T} \right).$$

In addition, in (S.92) the probability order of the first term dominates the second term. So we have

$$\sup_i |B_{2,5,iT}| \leq \left(\sup_i \sigma_i^2 \right) \left(\sup_i \left| \frac{\boldsymbol{\varepsilon}_{io}(\lambda_T)' \mathbf{P}_F \boldsymbol{\varepsilon}_{io}(\lambda_T)}{T} \right| \right) = O_p \left(\frac{\ln(n)}{T} \right).$$

Overall, we can use (S.90) to establish

$$\sup_i |B_{2,iT}| \leq \sup_i \left(\sum_{j=1}^6 |B_{2,j,iT}| \right) \leq \sum_{j=1}^6 \sup_i |B_{2,j,iT}| = O_p \left(\frac{\ln(n)}{T} \right).$$

Next, consider $B_{3,iT}$ and note that

$$B_{3,iT} = \frac{2\sigma_i^2 \boldsymbol{\varepsilon}_{io}(\lambda_T)' (\mathbf{I}_{m_0} - \mathbf{P}_F) (\mathbf{P}_F - \mathbf{P}_{\hat{\mathbf{F}}}) \boldsymbol{\varepsilon}_{io}(\lambda_T)}{T} = \frac{2\sigma_i^2 \boldsymbol{\varepsilon}_{io}(\lambda_T)' (\mathbf{P}_F - \mathbf{P}_{\hat{\mathbf{F}}} - \mathbf{P}_F + \mathbf{P}_F \mathbf{P}_{\hat{\mathbf{F}}}) \boldsymbol{\varepsilon}_{io}(\lambda_T)}{T} \\ = \frac{2\sigma_i^2 \boldsymbol{\varepsilon}_{io}(\lambda_T)' \mathbf{P}_{\hat{\mathbf{F}}} \boldsymbol{\varepsilon}_{io}(\lambda_T)}{T} + \frac{2\sigma_i^2 \boldsymbol{\varepsilon}_{io}(\lambda_T)' \mathbf{P}_F \mathbf{P}_{\hat{\mathbf{F}}} \boldsymbol{\varepsilon}_{io}(\lambda_T)}{T} = B_{3,1,nT} + B_{3,2,nT}.$$

But

$$|B_{3,1,iT}| = 2 \left| \sigma_i^2 \left(\frac{\boldsymbol{\varepsilon}_{io}(\lambda_T)' \hat{\mathbf{F}}}{T} \right) \left(\frac{\hat{\mathbf{F}}' \hat{\mathbf{F}}}{T} \right)^{-1} \left(\frac{\hat{\mathbf{F}}' \boldsymbol{\varepsilon}_{io}(\lambda_T)}{T} \right) \right| \leq 2\sigma_i^2 \left\| \frac{\boldsymbol{\varepsilon}_{io}(\lambda_T)' \hat{\mathbf{F}}}{T} \right\|^2 \left\| \left(\frac{\hat{\mathbf{F}}' \hat{\mathbf{F}}}{T} \right)^{-1} \right\|^2 \\ \leq 4\sigma_i^2 \left(\left\| \frac{\boldsymbol{\varepsilon}_{io}(\lambda_T)' \mathbf{F}}{T} \right\|^2 + \left\| \frac{\boldsymbol{\varepsilon}_{io}(\lambda_T)' (\hat{\mathbf{F}} - \mathbf{F})}{T} \right\|^2 \right) \left\| \left(\frac{\hat{\mathbf{F}}' \hat{\mathbf{F}}}{T} \right)^{-1} \right\|^2,$$

so using (S.55) and (S.70), we have

$$\begin{aligned}
\sup_i |B_{3,1,iT}| &\leq 4 \left(\sup_i \sigma_i^2 \right) \sup_i \left(\left\| \frac{\boldsymbol{\varepsilon}_{io}(\lambda_T)' \mathbf{F}}{T} \right\|^2 + \left\| \frac{\boldsymbol{\varepsilon}_{io}(\lambda_T)' (\hat{\mathbf{F}} - \mathbf{F})}{T} \right\|^2 \right) \left\| \left(\frac{\hat{\mathbf{F}}' \hat{\mathbf{F}}}{T} \right)^{-1} \right\|^2 \\
&\leq 4 \left(\sup_i \sigma_i^2 \right) \left(\sup_i \left\| \frac{\boldsymbol{\varepsilon}_{io}(\lambda_T)' \mathbf{F}}{T} \right\|^2 + \sup_i \left\| \frac{\boldsymbol{\varepsilon}_{io}(\lambda_T)' (\hat{\mathbf{F}} - \mathbf{F})}{T} \right\|^2 \right) \times \left\| \left(\frac{\hat{\mathbf{F}}' \hat{\mathbf{F}}}{T} \right)^{-1} \right\|^2 \\
&= O_p \left(\frac{\ln(n)}{T} \right).
\end{aligned}$$

Also,

$$\begin{aligned}
|B_{3,2,iT}| &= 2\sigma_i^2 \left| \left(\frac{\boldsymbol{\varepsilon}_{io}(\lambda_T)' \mathbf{F}}{T} \right) \left(\frac{\mathbf{F}' \mathbf{F}}{T} \right)^{-1} \left(\frac{\mathbf{F}' \hat{\mathbf{F}}}{T} \right) \left(\frac{\hat{\mathbf{F}}' \hat{\mathbf{F}}}{T} \right)^{-1} \left(\frac{\hat{\mathbf{F}}' \boldsymbol{\varepsilon}_{io}(\lambda_T)}{T} \right) \right| \\
&\leq 2\sigma_i^2 \left| \left(\frac{\boldsymbol{\varepsilon}_{io}(\lambda_T)' \mathbf{F}}{T} \right) \left(\frac{\mathbf{F}' \mathbf{F}}{T} \right)^{-1} \left(\frac{\mathbf{F}' \hat{\mathbf{F}}}{T} \right) \left(\frac{\hat{\mathbf{F}}' \hat{\mathbf{F}}}{T} \right)^{-1} \left[\frac{\mathbf{F}' \boldsymbol{\varepsilon}_{io}(\lambda_T)}{T} + \frac{(\hat{\mathbf{F}} - \mathbf{F})' \boldsymbol{\varepsilon}_{io}(\lambda_T)}{T} \right] \right| \\
&\leq 2\sigma_i^2 \left\| \frac{\boldsymbol{\varepsilon}_{io}(\lambda_T)' \mathbf{F}}{T} \right\| \left(\left\| \frac{\mathbf{F}' \boldsymbol{\varepsilon}_{io}(\lambda_T)}{T} \right\| + \left\| \frac{(\hat{\mathbf{F}} - \mathbf{F})' \boldsymbol{\varepsilon}_{io}(\lambda_T)}{T} \right\| \right) \left\| \left(\frac{\mathbf{F}' \mathbf{F}}{T} \right)^{-1} \right\| \times \\
&\quad \left\| \left(\frac{\hat{\mathbf{F}}' \hat{\mathbf{F}}}{T} \right)^{-1} \right\| \left\| \frac{\mathbf{F}' \hat{\mathbf{F}}}{T} \right\|.
\end{aligned}$$

Again using (S.55) and (S.70) yields

$$\begin{aligned}
&\sup_i \sigma_i^2 \left\| \frac{\boldsymbol{\varepsilon}_{io}(\lambda_T)' \mathbf{F}}{T} \right\| \left(\left\| \frac{\mathbf{F}' \boldsymbol{\varepsilon}_{io}(\lambda_T)}{T} \right\| + \left\| \frac{(\hat{\mathbf{F}} - \mathbf{F})' \boldsymbol{\varepsilon}_{io}(\lambda_T)}{T} \right\| \right) \\
&\leq \left(\sup_i \sigma_i^2 \right) \left[\sup_i \left\| \frac{\boldsymbol{\varepsilon}_{io}(\lambda_T)' \mathbf{F}}{T} \right\|^2 + \sup_i \left\| \frac{\boldsymbol{\varepsilon}_{io}(\lambda_T)' \mathbf{F}}{T} \right\| \left(\sup_i \left\| \frac{(\hat{\mathbf{F}} - \mathbf{F})' \boldsymbol{\varepsilon}_{io}(\lambda_T)}{T} \right\| \right) \right] \\
&= O_p \left(\frac{\ln(n)}{T} \right) + O_p \left(\sqrt{\frac{\ln(n)}{T}} \right) \times O_p \left(\sqrt{\frac{\ln(n)}{nT}} \right) \\
&= O_p \left(\frac{\ln(n)}{T} \right),
\end{aligned}$$

while by result (S.46),

$$\left\| \frac{\mathbf{F}' \hat{\mathbf{F}}}{T} \right\| \leq \left\| \frac{\mathbf{F}' \mathbf{F}}{T} \right\| + \left\| \frac{\mathbf{F}' (\hat{\mathbf{F}} - \mathbf{F})}{T} \right\| = O_p(1).$$

Using these results as well as (S.89) then yields

$$\begin{aligned} \sup_i |B_{3,2,iT}| &\leq 2 \left[\sup_i \sigma_i^2 \left\| \frac{\boldsymbol{\varepsilon}_{i0}(\lambda_T)' \mathbf{F}}{T} \right\| \left(\left\| \frac{\mathbf{F}' \boldsymbol{\varepsilon}_{i0}(\lambda_T)}{T} \right\| + \left\| \frac{(\hat{\mathbf{F}} - \mathbf{F})' \boldsymbol{\varepsilon}_{i0}(\lambda_T)}{T} \right\| \right) \right] \times \\ &\quad \left\| \left(\frac{\mathbf{F}' \mathbf{F}}{T} \right)^{-1} \right\| \left\| \left(\frac{\hat{\mathbf{F}}' \hat{\mathbf{F}}}{T} \right)^{-1} \right\| \left\| \frac{\mathbf{F}' \hat{\mathbf{F}}}{T} \right\| \\ &= O_p \left(\frac{\ln(n)}{T} \right). \end{aligned}$$

Overall,

$$\sup_i |B_{3,iT}| \leq \sup_i (|B_{3,1,iT}| + |B_{3,2,iT}|) \leq \sup_i |B_{3,1,iT}| + \sup_i |B_{3,2,iT}| = O_p \left(\frac{\ln(n)}{T} \right).$$

Now consider $B_{4,iT}$, and note that

$$\begin{aligned} B_{4,iT} &= \frac{2\sigma_i \boldsymbol{\gamma}_i' (\mathbf{F} - \hat{\mathbf{F}})' \mathbf{M}_{\hat{\mathbf{F}}} \mathbf{M}_F \boldsymbol{\varepsilon}_{i0}(\lambda_T)}{T} \\ &= - \frac{2\sigma_i \boldsymbol{\gamma}_i' (\hat{\mathbf{F}} - \mathbf{F})' \boldsymbol{\varepsilon}_{i0}(\lambda_T)}{T} + \frac{2\sigma_i \boldsymbol{\gamma}_i' (\hat{\mathbf{F}} - \mathbf{F})' \hat{\mathbf{F}} (\hat{\mathbf{F}}' \hat{\mathbf{F}})^{-1} \hat{\mathbf{F}}' \boldsymbol{\varepsilon}_{i0}(\lambda_T)}{T} \\ &\quad - \frac{2\sigma_i \boldsymbol{\gamma}_i' (\hat{\mathbf{F}} - \mathbf{F})' \mathbf{M}_{\hat{\mathbf{F}}} (\hat{\mathbf{F}} - \mathbf{F}) (\mathbf{F}' \mathbf{F})^{-1} \mathbf{F}' \boldsymbol{\varepsilon}_{i0}(\lambda_T)}{T} \\ &= \sum_{j=1}^3 B_{4,j,1T}. \end{aligned}$$

For the first term,

$$|B_{4,1,iT}| \leq 2 \left| \frac{\sigma_i \boldsymbol{\gamma}_i' (\hat{\mathbf{F}} - \mathbf{F})' \boldsymbol{\varepsilon}_{i0}(\lambda_T)}{T} \right| \leq 2\sigma_i \|\boldsymbol{\gamma}_i\| \left\| \frac{(\hat{\mathbf{F}} - \mathbf{F})' \boldsymbol{\varepsilon}_{i0}(\lambda_T)}{T} \right\|,$$

so using (S.70) and the boundedness of σ_i and $\boldsymbol{\gamma}_i$, we have

$$\sup_i |B_{4,1,iT}| \leq 2 \left(\sup_i \sigma_i \right) \left(\sup_i \|\boldsymbol{\gamma}_i\| \right) \left(\left\| \frac{(\hat{\mathbf{F}} - \mathbf{F})' \boldsymbol{\varepsilon}_{i0}(\lambda_T)}{T} \right\| \right) = O_p \left(\sqrt{\frac{\ln(n)}{nT}} \right).$$

Similarly,

$$\begin{aligned} |B_{4,2,iT}| &= \left| \frac{2\sigma_i \boldsymbol{\gamma}_i' (\hat{\mathbf{F}} - \mathbf{F})' \hat{\mathbf{F}} (\hat{\mathbf{F}}' \hat{\mathbf{F}})^{-1} \hat{\mathbf{F}}' \boldsymbol{\varepsilon}_{i0}(\lambda_T)}{T} \right| \\ &\leq 2\sigma_i \|\boldsymbol{\gamma}_i\| \left\| \frac{(\hat{\mathbf{F}} - \mathbf{F})' \hat{\mathbf{F}}}{T} \right\| \left\| \left(\frac{\hat{\mathbf{F}}' \hat{\mathbf{F}}}{T} \right)^{-1} \right\| \left(\left\| \frac{\mathbf{F}' \boldsymbol{\varepsilon}_{i0}(\lambda_T)}{T} \right\| + \left\| \frac{(\hat{\mathbf{F}} - \mathbf{F})' \boldsymbol{\varepsilon}_{i0}(\lambda_T)}{T} \right\| \right), \end{aligned}$$

and using (S.55) and (S.70) we have

$$\begin{aligned} \sup_i |B_{4,2,iT}| &\leq 2 \left(\sup_i \sigma_i \right) \left(\sup_i \|\gamma_i\| \right) \left(\sup_i \frac{\|\mathbf{F}' \boldsymbol{\varepsilon}_{i\circ}(\lambda_T)\|}{T} + \sup_i \frac{\left\| \left(\hat{\mathbf{F}} - \mathbf{F} \right)' \boldsymbol{\varepsilon}_{i\circ}(\lambda_T) \right\|}{T} \right) \times \\ &\quad \left\| \frac{\left(\hat{\mathbf{F}} - \mathbf{F} \right)' \hat{\mathbf{F}}}{T} \right\| \left\| \left(\frac{\hat{\mathbf{F}}' \hat{\mathbf{F}}}{T} \right)^{-1} \right\| = O_p \left(\sqrt{\frac{\ln(n)}{T}} \right) \times O_p \left(\frac{1}{\delta_{nT}^2} \right). \end{aligned}$$

Moreover,

$$\begin{aligned} |B_{4,3,iT}| &= \left| \frac{2\sigma_i \gamma_i' \left(\hat{\mathbf{F}} - \mathbf{F} \right)' \mathbf{M}_{\hat{\mathbf{F}}} \left(\hat{\mathbf{F}} - \mathbf{F} \right) (\mathbf{F}' \mathbf{F})^{-1} \mathbf{F}' \boldsymbol{\varepsilon}_{i\circ}(\lambda_T)}{T} \right| \\ &\leq 2\sigma_i \|\gamma_i\| \left(\frac{\|\hat{\mathbf{F}} - \mathbf{F}\|^2}{T} \right) \left\| \left(\frac{\mathbf{F}' \mathbf{F}}{T} \right)^{-1} \right\| \left\| \frac{\mathbf{F}' \boldsymbol{\varepsilon}_{i\circ}(\lambda_T)}{T} \right\|, \end{aligned}$$

then taking the supremum,

$$\begin{aligned} \sup_i |B_{4,3,iT}| &\leq 2 \left(\sup_i \sigma_i \right) \left(\sup_i \|\gamma_i\| \right) \left(\sup_i \left\| \frac{\mathbf{F}' \boldsymbol{\varepsilon}_{i\circ}(\lambda_T)}{T} \right\| \right) \left(\frac{\|\hat{\mathbf{F}} - \mathbf{F}\|^2}{T} \right) \left\| \left(\frac{\mathbf{F}' \mathbf{F}}{T} \right)^{-1} \right\| \\ &= O_p \left(\sqrt{\frac{\ln(n)}{T}} \right) \times O_p \left(\frac{1}{\delta_{nT}^2} \right). \end{aligned}$$

Hence, it follows that

$$\sup_i |B_{4,iT}| \leq \sup_i \sum_{j=1}^3 |B_{4,j,iT}| \leq \sum_{j=1}^3 \sup_i |B_{4,j,iT}| = O_p \left(\sqrt{\frac{\ln(n)}{nT}} \right).$$

Similarly, $\sup_i |B_{5,iT}| = O_p \left(\sqrt{\ln(n)/(nT)} \right)$. For the final term $B_{6,iT}$, note

$$\begin{aligned} B_{6,iT} &= \frac{\sigma_i^2 (\boldsymbol{\varepsilon}_{i\circ}(\lambda_T) + \boldsymbol{\varepsilon}_{i\circ})' (\boldsymbol{\varepsilon}_{i\circ}(\lambda_T) - \boldsymbol{\varepsilon}_{i\circ})}{T} - \frac{\sigma_i^2 (\boldsymbol{\varepsilon}_{i\circ}(\lambda_T) + \boldsymbol{\varepsilon}_{i\circ})' \mathbf{F}}{T} \left(\frac{\mathbf{F}' \mathbf{F}}{T} \right)^{-1} \frac{\mathbf{F}' (\boldsymbol{\varepsilon}_{i\circ}(\lambda_T) - \boldsymbol{\varepsilon}_{i\circ})}{T} \\ &= B_{6,1,iT} + B_{6,2,iT}. \end{aligned}$$

Note since $\boldsymbol{\varepsilon}_{i\circ}(\lambda_T) = \boldsymbol{\varepsilon}_{i\circ} + \lambda_T \mathbf{b}_i$ where $\boldsymbol{\varepsilon}_{i\circ} = (\varepsilon_{i1}, \varepsilon_{i2}, \dots, \varepsilon_{iT})'$ and $\mathbf{b}_i = (b_{i1}, b_{i2}, \dots, b_{iT})'$ with $b_{it} = \mathbf{w}_{i0}' \boldsymbol{\varepsilon}_{ot}$, $\mathbf{w}_{i0} = (w_{i1}, w_{is}, \dots, w_{in})'$ and $\boldsymbol{\varepsilon}_{ot} = (\varepsilon_{1t}, \varepsilon_{2t}, \dots, \varepsilon_{nt})'$, then

$$B_{6,1,iT} = \sigma_i^2 \left(\frac{2\lambda_T \boldsymbol{\varepsilon}_{i\circ}' \mathbf{b}_i}{T} + \frac{\lambda_T^2 \mathbf{b}_i' \mathbf{b}_i}{T} \right),$$

and it further implies

$$\sup_i |B_{6,1,iT}| \leq \left(\sup_i \sigma_i^2 \right) \left(2 |\lambda_T| \sup_i \left| \frac{\boldsymbol{\varepsilon}'_{io} \mathbf{b}_i}{T} \right| + \lambda_T^2 \sup_i \left| \frac{\mathbf{b}'_i \mathbf{b}_i}{T} \right| \right) = O_p \left(\frac{\sqrt{\ln(n)}}{T} \right) + O_p \left(\frac{1}{T} \right),$$

where the final equation holds by results (S.77) and (S.78). For $B_{6,2,iT}$, note

$$|B_{6,2,iT}| \leq \sigma_i^2 \left\| \frac{(\boldsymbol{\varepsilon}_{io}(\lambda_T) + \boldsymbol{\varepsilon}_{io})' \mathbf{F}}{T} \right\| \left\| \left(\frac{\mathbf{F}' \mathbf{F}}{T} \right)^{-1} \right\| \left\| \frac{\mathbf{F}' (\boldsymbol{\varepsilon}_{io}(\lambda_T) - \boldsymbol{\varepsilon}_{io})}{T} \right\|$$

so that

$$\sup_i |B_{6,2,iT}| \leq \left(\sup_i \sigma_i^2 \right) \left(\sup_i \left\| \frac{(\boldsymbol{\varepsilon}_{io}(\lambda_T) + \boldsymbol{\varepsilon}_{io})' \mathbf{F}}{T} \right\| \right) \left\| \left(\frac{\mathbf{F}' \mathbf{F}}{T} \right)^{-1} \right\| \left(\sup_i \left\| \frac{\mathbf{F}' (\boldsymbol{\varepsilon}_{io}(\lambda_T) - \boldsymbol{\varepsilon}_{io})}{T} \right\| \right).$$

As result (S.54) shows $\sup_i |T^{-1} \boldsymbol{\varepsilon}_{io}(\lambda_T)' \mathbf{F}| = O_p \left(\sqrt{\ln(n)/T} \right)$, we have

$$\begin{aligned} \sup_i \left\| \frac{(\boldsymbol{\varepsilon}_{io}(\lambda_T) + \boldsymbol{\varepsilon}_{io})' \mathbf{F}}{T} \right\| &\leq \sup_i \left\| \frac{\boldsymbol{\varepsilon}_{io}(\lambda_T)' \mathbf{F}}{T} \right\| + \sup_i \left\| \frac{\boldsymbol{\varepsilon}'_{io} \mathbf{F}}{T} \right\| = O_p \left(\sqrt{\ln(n)/T} \right), \\ \sup_i \left\| \frac{\mathbf{F}' (\boldsymbol{\varepsilon}_{io}(\lambda_T) - \boldsymbol{\varepsilon}_{io})}{T} \right\| &\leq \sup_i \left\| \frac{\boldsymbol{\varepsilon}_{io}(\lambda_T)' \mathbf{F}}{T} \right\| + \sup_i \left\| \frac{\boldsymbol{\varepsilon}'_{io} \mathbf{F}}{T} \right\| = O_p \left(\sqrt{\ln(n)/T} \right). \end{aligned}$$

Hence it follows $\sup_i |B_{6,2,iT}| = O_p(\ln(n)/T)$. Overall, we are able to show

$$\sup_i |B_{6,iT}| \leq \sup_i |B_{6,1,iT}| + \sup_i |B_{6,2,iT}| = O_p \left(\frac{\ln(n)}{T} \right).$$

Using the above results and noting that n and T are assumed to be of the same order of magnitude, then (S.80) follows noting that

$$\sup_i |\hat{\sigma}_{i,T}^2 - \omega_{i,T}^2| \leq \sup_i \sum_{j=1}^6 |B_{j,iT}| \leq \sum_{j=1}^6 \sup_i |B_{j,iT}| = O_p \left(\frac{\ln(n)}{T} \right).$$

To establish (S.81) note that

$$\sup_i |\hat{\sigma}_{i,T} - \omega_{i,T}| = \sup_i \frac{|\hat{\sigma}_{i,T}^2 - \omega_{i,T}^2|}{\hat{\sigma}_{i,T} + \omega_{i,T}} = \sup_i \left(\frac{1}{\hat{\sigma}_{i,T} + \omega_{i,T}} \right) \sup_i |\hat{\sigma}_{i,T}^2 - \omega_{i,T}^2|$$

Since $\hat{\sigma}_{i,T}^2 > 0$ (by construction)

$$\sup_i |\hat{\sigma}_{i,T} - \omega_{i,T}| \leq \sup_i \left(\frac{1}{\omega_{i,T}} \right) \sup_i |\hat{\sigma}_{i,T}^2 - \omega_{i,T}^2|.$$

Note by definition of $\omega_{i,T}$, we have $\omega_{i,T}^{-2} = \sigma_i^{-2} (T^{-1} \boldsymbol{\varepsilon}'_{io} \mathbf{M}_F \boldsymbol{\varepsilon}_{io})^{-1}$, so that

$$\begin{aligned} \sup_i \left(\frac{1}{\omega_{i,T}} \right) &= \sup_i \left(\frac{1}{\omega_{i,T}^2} \right)^{1/2} \leq \left(\sup_i \frac{1}{\sigma_i^2} \right)^{1/2} \left(\sup_i \frac{1}{T^{-1} \boldsymbol{\varepsilon}'_{io} \mathbf{M}_F \boldsymbol{\varepsilon}_{io}} \right)^{1/2} \\ &= \left(\frac{1}{\inf_i \sigma_i^2} \right)^{1/2} \left(\frac{1}{\inf_i (T^{-1} \boldsymbol{\varepsilon}'_{io} \mathbf{M}_F \boldsymbol{\varepsilon}_{io})} \right)^{1/2}. \end{aligned}$$

Since $\inf_i \sigma_i^2 > c > 0$ and by condition (5) in Assumption 2, $\inf_i T^{-1} \boldsymbol{\epsilon}'_{i0} \mathbf{M}_F \boldsymbol{\epsilon}_{i0} > c > 0$ as $T \rightarrow \infty$, then it readily follows $\sup_i \omega_{i,T}^{-1} < C < \infty$ and result (S.81) can be established by (S.80). Similarly, note that

$$\sup_i \left| \frac{1}{\hat{\sigma}_{i,T}} - \frac{1}{\omega_{i,T}} \right| \leq \sup_i \left(\frac{1}{\hat{\sigma}_{i,T}} \right) \sup_i \left(\frac{1}{\omega_{i,T}} \right) \sup_i |\hat{\sigma}_{i,T} - \omega_{i,T}|,$$

where $\sup_i \left(\frac{1}{\hat{\sigma}_{i,T}} \right) < C$, by construction. Hence, result (S.82) can be established using (S.81). Finally, results (S.83), (S.84) and (S.85) follow using (S.80), (S.81) and (S.82), respectively. ■

Lemma S.9 *Consider the latent factor model given by (1) and (2). The latent factors, $\mathbf{f}_t$, and their loadings, $\boldsymbol{\gamma}_i$, are estimated by principal components, $\hat{\mathbf{f}}_t$ and $\hat{\boldsymbol{\gamma}}_i$, given by (12). Suppose that Assumptions 1-4 hold and $(n, T) \rightarrow \infty$, such that $n/T \rightarrow \kappa$, for $0 < \kappa < \infty$. Then*

$$\mathbf{d}_{1,nT} = \frac{1}{n} \sum_{i=1}^n b_{in} (\hat{\boldsymbol{\gamma}}_i - \boldsymbol{\gamma}_i) = O_p \left(\sqrt{\frac{\ln(n)}{nT}} \right), \quad (\text{S.93})$$

$$\mathbf{d}_{2,nT} = \frac{1}{n} \sum_{i=1}^n (\hat{\boldsymbol{\delta}}_{i,T} - \boldsymbol{\delta}_{i,T}) = O_p \left(\sqrt{\frac{\ln(n)}{nT}} \right), \quad (\text{S.94})$$

$$\mathbf{d}_{3,nT} = \frac{1}{n} \sum_{i=1}^n b_{in} (\hat{\boldsymbol{\gamma}}_i - \boldsymbol{\gamma}_i) \boldsymbol{\gamma}_i' = O_p \left(\sqrt{\frac{\ln(n)}{nT}} \right), \quad (\text{S.95})$$

$$\mathbf{d}_{4,nT} = \frac{1}{n} \sum_{i=1}^n (\omega_{i,T} - \sigma_i) (\hat{\boldsymbol{\gamma}}_i - \boldsymbol{\gamma}_i) = O_p \left(\frac{1}{\delta_{nT}^2} \right), \quad (\text{S.96})$$

$$\mathbf{d}_{5,nT} = \frac{1}{n} \sum_{i=1}^n \left(\frac{1}{\omega_{i,T}} - \frac{1}{\sigma_i} \right) (\hat{\boldsymbol{\gamma}}_i - \boldsymbol{\gamma}_i) = O_p \left(\frac{1}{\delta_{nT}^2} \right), \quad (\text{S.97})$$

$$\mathbf{d}_{6,nT} = \frac{1}{n} \sum_{i=1}^n (\hat{\sigma}_{i,T} - \omega_{i,T}) (\hat{\boldsymbol{\gamma}}_i - \boldsymbol{\gamma}_i) = O_p \left[\left(\frac{\ln(n)}{T} \right)^{3/2} \right], \quad (\text{S.98})$$

$$\mathbf{d}_{7,nT} = \frac{1}{n} \sum_{i=1}^n \left(\frac{1}{\hat{\sigma}_{i,T}} - \frac{1}{\omega_{i,T}} \right) (\hat{\boldsymbol{\gamma}}_i - \boldsymbol{\gamma}_i) = O_p \left[\left(\frac{\ln(n)}{T} \right)^{3/2} \right], \quad (\text{S.99})$$

where $\{b_{in}\}_{i=1}^n$ is a sequence of fixed values bounded in n , such that $n^{-1} \sum_{i=1}^n b_{in}^2 = O(1)$, $\boldsymbol{\delta}_{i,T} = \boldsymbol{\gamma}_i / \omega_{i,T}$, $\hat{\boldsymbol{\delta}}_{i,T} = \hat{\boldsymbol{\gamma}}_i / \omega_{i,T}$, and $\omega_{i,T} = (T^{-1} \sigma_i^2 \boldsymbol{\epsilon}'_{i0} \mathbf{M}_F \boldsymbol{\epsilon}_{i0})^{1/2}$.

Proof. Note that in general

$$\begin{aligned} \hat{\boldsymbol{\gamma}}_i - \boldsymbol{\gamma}_i &= \left(\frac{\hat{\mathbf{F}}' \hat{\mathbf{F}}}{T} \right)^{-1} \left(\frac{\hat{\mathbf{F}}' \mathbf{F} \boldsymbol{\gamma}_i}{T} + \frac{\sigma_i \hat{\mathbf{F}}' \boldsymbol{\epsilon}_{i0} (\lambda_T)}{T} \right) - \left(\frac{\hat{\mathbf{F}}' \hat{\mathbf{F}}}{T} \right)^{-1} \left(\frac{\hat{\mathbf{F}}' \hat{\mathbf{F}}}{T} \right) \boldsymbol{\gamma}_i \\ &= - \left(\frac{\hat{\mathbf{F}}' \hat{\mathbf{F}}}{T} \right)^{-1} \left[\frac{\hat{\mathbf{F}}' (\hat{\mathbf{F}} - \mathbf{F}) \boldsymbol{\gamma}_i}{T} \right] + \left(\frac{\hat{\mathbf{F}}' \hat{\mathbf{F}}}{T} \right)^{-1} \left(\frac{\sigma_i \hat{\mathbf{F}}' \boldsymbol{\epsilon}_{i0} (\lambda_T)}{T} \right), \end{aligned} \quad (\text{S.100})$$

and we have

$$\begin{aligned} \mathbf{d}_{1,nT} &= \frac{1}{n} \sum_{i=1}^n b_{in} (\hat{\gamma}_i - \gamma_i) \\ &= - \left(\frac{\hat{\mathbf{F}}' \hat{\mathbf{F}}}{T} \right)^{-1} \left[\frac{\hat{\mathbf{F}}' (\hat{\mathbf{F}} - \mathbf{F})}{T} \right] \left(\frac{1}{n} \sum_{i=1}^n b_{in} \gamma_i \right) + \left(\frac{\hat{\mathbf{F}}' \hat{\mathbf{F}}}{T} \right)^{-1} T^{-1} \left(\frac{1}{n} \sum_{i=1}^n b_{in} \sigma_i \hat{\mathbf{F}}' \boldsymbol{\varepsilon}_{i0}(\lambda_T) \right). \end{aligned} \quad (\text{S.101})$$

Since by assumption $\|\gamma_i\| < C$, we have

$$\begin{aligned} \left\| \frac{1}{n} \sum_{i=1}^n b_{in} \gamma_i \right\| &\leq \left(\frac{1}{n} \sum_{i=1}^n b_{in}^2 \right)^{1/2} \left\| \frac{1}{n} \sum_{i=1}^n \gamma_i \gamma_i' \right\|^{1/2} \\ &\leq \left(\frac{1}{n} \sum_{i=1}^n b_{in}^2 \right)^{1/2} \left(\frac{1}{n} \sum_{i=1}^n \|\gamma_i\|^2 \right)^{1/2} < C. \end{aligned}$$

Also by results (S.64) and (S.46), the first term of (S.101) is $O_p(\delta_{nT}^{-2})$. For the second term of (S.101), since $(T^{-1} \hat{\mathbf{F}}' \hat{\mathbf{F}})^{-1} = O_p(1)$, we note that

$$T^{-1} (\hat{\mathbf{F}} - \mathbf{F} + \mathbf{F})' \left(\frac{1}{n} \sum_{i=1}^n b_{in} \sigma_i \boldsymbol{\varepsilon}_{i0}(\lambda_T) \right) = \frac{1}{n} \sum_{i=1}^n b_{in} \left(\frac{\hat{\mathbf{F}} - \mathbf{F}}{T} \right)' \sigma_i \boldsymbol{\varepsilon}_{i0}(\lambda_T) + \frac{1}{Tn} \sum_{i=1}^n b_{in} \mathbf{F}' \sigma_i \boldsymbol{\varepsilon}_{i0}(\lambda_T).$$

Using result (S.70), we have

$$\begin{aligned} &\left\| \frac{1}{n} \sum_{i=1}^n b_{in} \left(\frac{\hat{\mathbf{F}} - \mathbf{F}}{T} \right)' \sigma_i \boldsymbol{\varepsilon}_{i0}(\lambda_T) \right\| \\ &\leq \frac{1}{n} \sum_{i=1}^n |b_{in} \sigma_i| \left\| \left(\frac{\hat{\mathbf{F}} - \mathbf{F}}{T} \right)' \boldsymbol{\varepsilon}_{i0}(\lambda_T) \right\| \leq \left(\sup_i |b_{in}| \right) \left(\sup_i \sigma_i \right) \left(\sup_i \left\| \frac{(\hat{\mathbf{F}} - \mathbf{F})' \boldsymbol{\varepsilon}_{i0}(\lambda_T)}{T} \right\| \right) \\ &= O_p \left(\sqrt{\frac{\ln(n)}{nT}} \right). \end{aligned} \quad (\text{S.102})$$

Under part (a) of Assumption 2 and by the serial independence of ε_{it} ,

$$\begin{aligned} E \left\| \frac{1}{\sqrt{nT}} \sum_{i=1}^n \sum_{t=1}^T b_{in} \mathbf{f}_t \sigma_i \boldsymbol{\varepsilon}_{i0}(\lambda_T) \right\|^2 &= \frac{1}{nT} \sum_{i=1}^n \sum_{j=1}^n \sum_{t=1}^T b_{in} b_{jn} \sigma_i \sigma_j E \|\mathbf{f}_t\|^2 E(\varepsilon_{it}(\lambda_T) \varepsilon_{jt}(\lambda_T)) \\ &\leq E \|\mathbf{f}_t\|^2 \left(\sup_i b_{in}^2 \right) \left(\sup_i \sigma_i^2 \right) \left[\frac{1}{nT} \sum_{t=1}^T \sum_{i=1}^n \sum_{j=1}^n |E(\varepsilon_{it}(\lambda_T) \varepsilon_{jt}(\lambda_T))| \right] \end{aligned}$$

which is $O(1)$ based on (S.69) and the boundedness of $E \|\mathbf{f}_t\|^2$, b_{in}^2 and σ_i^2 required by assumptions. So it follows

$$\frac{1}{nT} \sum_{i=1}^n b_{in} \mathbf{F}' \sigma_i \boldsymbol{\varepsilon}_{i0}(\lambda_T) = \frac{1}{\sqrt{nT}} \left(\frac{1}{\sqrt{nT}} \sum_{i=1}^n \sum_{t=1}^T b_{in} \mathbf{f}_t \sigma_i \boldsymbol{\varepsilon}_{i0}(\lambda_T) \right) = O_p \left(\frac{1}{\sqrt{nT}} \right). \quad (\text{S.103})$$

Result (S.93) now follows using (S.102) and (S.103) in (S.101), and noting that by assumption n and T are of the same order. Consider now (S.94), and note that it can be written as

$$\begin{aligned} \mathbf{d}_{2,nT} &= \frac{1}{n} \sum_{i=1}^n \left(\frac{\hat{\gamma}_i}{\omega_{i,T}} - \frac{\gamma_i}{\omega_{i,T}} \right) \\ &= \frac{1}{n} \sum_{i=1}^n \left(\frac{\hat{\gamma}_i - \gamma_i}{\sigma_i} \right) \left(1 - \frac{\omega_{i,T} - \sigma_i}{\omega_{i,T}} \right) \\ &= \frac{1}{n} \sum_{i=1}^n \left(\frac{\hat{\gamma}_i - \gamma_i}{\sigma_i} \right) - \frac{1}{n} \sum_{i=1}^n \left(\frac{\hat{\gamma}_i - \gamma_i}{\sigma_i} \right) \left(1 - \frac{T}{\boldsymbol{\varepsilon}'_{i0} \mathbf{M}_F \boldsymbol{\varepsilon}_{i0}} \right). \end{aligned}$$

The first term of the above has the same form as (S.93), and becomes identical to it if we replace a_i in (S.93) with $1/\sigma_i$, since by assumption $\inf_i(\sigma_i) > c$. Hence, the order of the first term is $O_p(\sqrt{\ln(n)/(nT)})$. Also the second term is dominated by the first term, since $1 - (T^{-1} \boldsymbol{\varepsilon}'_{i0} \mathbf{M}_F \boldsymbol{\varepsilon}_{i0})^{-1} = O_p(T^{-1})$ based on result (S.133). Therefore, (S.94) is established as required. For (S.95), note that

$$\begin{aligned} \mathbf{d}_{3,nT} &= \frac{1}{n} \sum_{i=1}^n b_{in} \left[- \left(\hat{\mathbf{F}}' \hat{\mathbf{F}} \right)^{-1} \hat{\mathbf{F}}' \left(\hat{\mathbf{F}} - \mathbf{F} \right) \gamma_i + \sigma_i \left(\hat{\mathbf{F}}' \hat{\mathbf{F}} \right)^{-1} \hat{\mathbf{F}}' \boldsymbol{\varepsilon}_{i0}(\lambda_T) \right] \gamma'_i \\ &= - \left(\frac{\hat{\mathbf{F}}' \hat{\mathbf{F}}}{T} \right)^{-1} \frac{\hat{\mathbf{F}}' \left(\hat{\mathbf{F}} - \mathbf{F} \right)}{T} \left(\frac{1}{n} \sum_{i=1}^n b_{in} \gamma_i \gamma'_i \right) + \left(\frac{\hat{\mathbf{F}}' \hat{\mathbf{F}}}{T} \right)^{-1} T^{-1} \hat{\mathbf{F}}' \left(\frac{1}{n} \sum_{i=1}^n b_{in} \sigma_i \boldsymbol{\varepsilon}_{i0}(\lambda_T) \gamma'_i \right) \\ &= - \left(\frac{\hat{\mathbf{F}}' \hat{\mathbf{F}}}{T} \right)^{-1} \frac{\hat{\mathbf{F}}' \left(\hat{\mathbf{F}} - \mathbf{F} \right)}{T} \left(\frac{1}{n} \sum_{i=1}^n b_{in} \gamma_i \gamma'_i \right) + \left(\frac{\hat{\mathbf{F}}' \hat{\mathbf{F}}}{T} \right)^{-1} \left(\frac{1}{n} \sum_{i=1}^n T^{-1} b_{in} \sigma_i \left(\hat{\mathbf{F}} - \mathbf{F} \right)' \boldsymbol{\varepsilon}_{i0}(\lambda_T) \gamma'_i \right) \\ &\quad + \left(\frac{\hat{\mathbf{F}}' \hat{\mathbf{F}}}{T} \right)^{-1} \left(\frac{1}{n} \sum_{i=1}^n T^{-1} b_{in} \sigma_i \mathbf{F}' \boldsymbol{\varepsilon}_{i0}(\lambda_T) \gamma'_i \right). \end{aligned} \tag{S.104}$$

Recall that $(T^{-1} \hat{\mathbf{F}}' \hat{\mathbf{F}})^{-1} = O_p(1)$, and $n^{-1} \sum_{i=1}^n \gamma_i \gamma'_i = O_p(1)$. Also note that b_{in} is bounded in n . Then using (S.47) it follows that (n and T being of the same order)

$$\left(\frac{\hat{\mathbf{F}}' \hat{\mathbf{F}}}{T} \right)^{-1} \frac{\hat{\mathbf{F}}' \left(\hat{\mathbf{F}} - \mathbf{F} \right)}{T} \left(n^{-1} \sum_{i=1}^n b_{in} \gamma_i \gamma'_i \right) = O_p \left(\frac{1}{\min(n, T)} \right) = O_p(\delta_{nT}^{-2}).$$

Similarly, using (S.70)

$$\left(\frac{\hat{\mathbf{F}}' \hat{\mathbf{F}}}{T} \right)^{-1} \left(n^{-1} \sum_{i=1}^n T^{-1} \left(\hat{\mathbf{F}} - \mathbf{F} \right)' b_{in} \sigma_i \boldsymbol{\varepsilon}_{i0}(\lambda_T) \gamma'_i \right) = O_p \left(\sqrt{\frac{\ln(n)}{nT}} \right).$$

The last term of (S.104) can be written as $(nT)^{-1/2} \left(T^{-1} \hat{\mathbf{F}}' \hat{\mathbf{F}} \right)^{-1} (n^{-1/2} T^{-1/2} \sum_{i=1}^n b_{in} \sigma_i \mathbf{F}' \boldsymbol{\varepsilon}_{i0}(\lambda_T) \gamma'_i)$, where $n^{-1/2} T^{-1/2} \sum_{i=1}^n b_{in} \sigma_i \mathbf{F}' \boldsymbol{\varepsilon}_{i0}(\lambda_T) \gamma'_i$ is an $m_0 \times m_0$ matrix with its (j, j') element given by

$n^{-1/2}T^{-1/2} \sum_{i=1}^n \sum_{t=1}^T b_{in} \sigma_i f_{jt} \varepsilon_{it} (\lambda_T) \gamma_{ij'}$ for $j, j' = 1, 2, \dots, m_0$. It can be further shown

$$\begin{aligned}
& E \left(n^{-1/2} T^{-1/2} \sum_{i=1}^n \sum_{t=1}^T b_{in} \sigma_i f_{jt} \gamma_{ij'} \varepsilon_{it} (\lambda_T) \right)^2 \\
&= \frac{1}{nT} \sum_{i=1}^n \sum_{i'=1}^n \sum_{t=1}^T b_{in} b_{i'n} \sigma_i \sigma_{i'} \gamma_{ij'} \gamma_{i'j'} E(f_{jt}^2) E(\varepsilon_{it}(\lambda_T) \varepsilon_{i't}(\lambda_T)) \\
&\leq \frac{1}{nT} \sum_{i=1}^n \sum_{i'=1}^n \sum_{t=1}^T |b_{in} b_{i'n} \sigma_i \sigma_{i'} \gamma_{ij'} \gamma_{i'j'}| E(f_{jt}^2) |E(\varepsilon_{it}(\lambda_T) \varepsilon_{i't}(\lambda_T))| \\
&\leq \left(\sup_i b_{in}^2 \right) \left(\sup_i \gamma_{ij'}^2 \right) \left(\sup_i \sigma_i^2 \right) E(f_{jt}^2) \left[\frac{1}{nT} \sum_{i=1}^n \sum_{i'=1}^n \sum_{t=1}^T |E(\varepsilon_{it}(\lambda_T) \varepsilon_{i't}(\lambda_T))| \right]
\end{aligned}$$

which is $O(1)$ based on (S.69) and the boundedness of b_{in}^2 , $\gamma_{ij'}^2$, σ_i^2 and $E(f_{jt}^2)$. Consequentially, $n^{-1/2}T^{-1/2} \sum_{i=1}^n b_{in} \sigma_i \mathbf{F}' \boldsymbol{\varepsilon}_{i\circ} (\lambda_T) \boldsymbol{\gamma}'_i = O_p(1)$ and the last term of (S.104) are also $O_p(\delta_{nT}^{-2})$. Thus result (S.95) is established, as required. To prove (S.96) we first write it as

$$\mathbf{d}_{4,nT} = \left(\sqrt{\frac{1}{T}} \right) \frac{1}{n} \sum_{i=1}^n q_{iT} (\hat{\gamma}_i - \gamma_i),$$

where

$$q_{iT} = \sqrt{T} (\omega_{i,T} - \sigma_i) = \sigma_i \sqrt{T} \left[\left(\frac{\boldsymbol{\varepsilon}'_{i\circ} \mathbf{M}_F \boldsymbol{\varepsilon}_{i\circ}}{T} \right)^{1/2} - 1 \right],$$

and conditional on $\mathbf{F}$ and σ_i , q_{iT} are independently distributed across i . Using results in Lemma S.14 it is easily seen that $E(q_{iT}) = O(T^{-1/2})$ and $Var(q_{iT}) = O(1)$, and hence $n^{-1} \sum_{i=1}^n q_{iT}^2 = O_p(1)$. Also by Cauchy-Schwarz inequality we have

$$\|\mathbf{d}_{4,nT}\| \leq \left(\sqrt{\frac{1}{T}} \right) \left(n^{-1} \sum_{i=1}^n q_{iT}^2 \right)^{1/2} \left(n^{-1/2} \|\hat{\mathbf{\Gamma}} - \mathbf{\Gamma}\| \right),$$

where $T^{-1}n = \Theta(1)$, and by (S.42) $n^{-1/2} \|\hat{\mathbf{\Gamma}} - \mathbf{\Gamma}\| = O_p(\delta_{nT}^{-1})$, and (S.96) is established. Result (S.97) follows similarly, with q_{iT} re-defined as $q_{iT} = \sigma_i^{-1} \sqrt{T} \left[\left(\frac{\boldsymbol{\varepsilon}'_{i\circ} \mathbf{M}_F \boldsymbol{\varepsilon}_{i\circ}}{T} \right)^{-1/2} - 1 \right]$, and noting that $\sup_i (1/\sigma_i^2) < C$, and using results in Lemma S.14. Result (S.98) is established as

$$\begin{aligned}
\|\mathbf{d}_{6,nT}\| &= \left\| \frac{1}{n} \sum_{i=1}^n (\hat{\sigma}_{i,T} - \omega_{i,T}) (\hat{\gamma}_i - \gamma_i) \right\| \leq \frac{1}{n} \sum_{i=1}^n \|(\hat{\sigma}_{i,T} - \omega_{i,T}) (\hat{\gamma}_i - \gamma_i)\| \\
&\leq \left(\frac{1}{n} \sum_{i=1}^n |\hat{\sigma}_{i,T} - \omega_{i,T}| \right) \left(\sup_i \|\hat{\gamma}_i - \gamma_i\| \right) \\
&= O_p \left[\left(\frac{\ln(n)}{T} \right)^{3/2} \right]
\end{aligned}$$

where by (S.58) $\sup_i \|\hat{\gamma}_i - \gamma_i\| = O_p\left(\sqrt{\ln(n)/T}\right)$, and by (S.84) $n^{-1} \sum_{i=1}^n |\hat{\sigma}_{i,T} - \omega_{i,T}| = O_p(\ln(n)/T)$. Similarly by (S.42) and (S.85) we have

$$\begin{aligned} \|\mathbf{d}_{7,nT}\| &= \left\| \frac{1}{n} \sum_{i=1}^n \left(\frac{1}{\hat{\sigma}_{i,T}} - \frac{1}{\omega_{i,T}} \right) (\hat{\gamma}_i - \gamma_i) \right\| \leq \frac{1}{n} \sum_{i=1}^n \left\| \left(\frac{1}{\hat{\sigma}_{i,T}} - \frac{1}{\omega_{i,T}} \right) (\hat{\gamma}_i - \gamma_i) \right\| \\ &\leq \left(\frac{1}{n} \sum_{i=1}^n \left| \frac{1}{\hat{\sigma}_{i,T}} - \frac{1}{\omega_{i,T}} \right| \right) \left(\sup_i \|\hat{\gamma}_i - \gamma_i\| \right) \\ &= O_p \left[\left(\frac{\ln(n)}{T} \right)^{3/2} \right]. \end{aligned}$$

■

Lemma S.10 *Consider the latent factor model given by (1) and (2). Suppose that Assumptions 1-4 hold and $(n, T) \rightarrow \infty$, such that $n/T \rightarrow \kappa$, for $0 < \kappa < \infty$. Then*

$$p_{nT}(\lambda_T) = \frac{1}{\sqrt{T}} \sum_{t=1}^T s_{t,nT}^2(\lambda_T) = o_p(1), \quad (\text{S.105})$$

$$q_{nT}(\lambda_T) = \frac{1}{\sqrt{T}} \sum_{t=1}^T \psi_{t,nT}(\lambda_T) s_{t,nT}(\lambda_T) = o_p(1), \quad (\text{S.106})$$

where $\psi_{t,nT}(\lambda_T)$ and $s_{t,nT}(\lambda_T)$ are defined by (S.8) and (S.9), respectively.

Proof. Using (S.9), recall that

$$\begin{aligned} s_{t,nT}(\lambda_T) &= \boldsymbol{\varphi}'_{nT} \left[n^{-1/2} \sum_{i=1}^n (\hat{\gamma}_i - \gamma_i) \sigma_i \varepsilon_{it}(\lambda_T) \right] + \boldsymbol{\varphi}'_{nT} \left[n^{-1/2} \sum_{i=1}^n (\hat{\gamma}_i - \gamma_i) \gamma'_i \right] \mathbf{f}_t \\ &\quad + \left[n^{-1/2} \sum_{i=1}^n (\hat{\boldsymbol{\delta}}_{i,T} - \boldsymbol{\delta}_{i,T})' \right] \mathbf{f}_t + \left[n^{-1/2} \sum_{i=1}^n (\hat{\boldsymbol{\delta}}_{i,T} - \boldsymbol{\delta}_{i,T})' \right] (\hat{\mathbf{f}}_t - \mathbf{f}_t). \end{aligned} \quad (\text{S.107})$$

We also note that using (S.12), $\psi_{t,nT}(\lambda_T)$ can be written as

$$\psi_{t,nT}(\lambda_T) = \xi_{t,n}(\lambda_T) - (\boldsymbol{\varphi}_{nT} - \boldsymbol{\varphi}_n)' \boldsymbol{\kappa}_{t,n}(\lambda_T) + v_{t,nT}(\lambda_T) \quad (\text{S.108})$$

where

$$\xi_{t,n}(\lambda_T) = \frac{1}{\sqrt{n}} \sum_{i=1}^n a_{i,n} \varepsilon_{it}(\lambda_T), \quad a_{i,n} = 1 - \sigma_i \boldsymbol{\varphi}'_n \gamma_i, \quad (\text{S.109})$$

$$\boldsymbol{\kappa}_{t,n}(\lambda_T) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \gamma_i \sigma_i \varepsilon_{it}(\lambda_T), \quad (\text{S.110})$$

$$v_{t,nT}(\lambda_T) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \left[\frac{1}{(\boldsymbol{\varepsilon}'_{i0} \mathbf{M}_F \boldsymbol{\varepsilon}_{i0}/T)^{1/2}} - 1 \right] \varepsilon_{it}(\lambda_T). \quad (\text{S.111})$$

After squaring $s_{t,nT}(\lambda_T)$, we end up with $p_{nT}(\lambda_T) = \sum_{j=1}^{10} A_{j,nT}(\lambda_T)$, composed of four squared terms and six cross product terms. For the first square term we have

$$A_{1,nT}(\lambda_T) = \sqrt{T} \boldsymbol{\varphi}'_{nT} \left(\frac{1}{T} \sum_{t=1}^T \mathbf{b}_{t,n}(\lambda_T) \mathbf{b}'_{t,n}(\lambda_T) \right) \boldsymbol{\varphi}_{nT},$$

where $\mathbf{b}_{t,n}(\lambda_T) = n^{-1/2} \sum_{i=1}^n (\hat{\gamma}_i - \gamma_i) \sigma_i \varepsilon_{it}(\lambda_T)$. Let $\mathbf{u}_{ot}(\lambda_T) = (\sigma_1 \varepsilon_{1t}(\lambda_T), \sigma_2 \varepsilon_{2t}(\lambda_T), \dots, \sigma_n \varepsilon_{nt}(\lambda_T))'$ so that $\mathbf{b}_{t,n}(\lambda_T) = n^{-1/2} (\hat{\mathbf{\Gamma}} - \mathbf{\Gamma})' \mathbf{u}_{ot}(\lambda_T)$. Then

$$|A_{1,nT}(\lambda_T)| \leq \frac{\sqrt{T}}{n} \|\boldsymbol{\varphi}_{nT}\|^2 \|\hat{\mathbf{\Gamma}} - \mathbf{\Gamma}\|^2 \|\mathbf{V}_T(\lambda_T)\|,$$

where $\mathbf{V}_T(\lambda_T) = T^{-1} \sum_{t=1}^T \mathbf{u}_{ot}(\lambda_T) \mathbf{u}'_{ot}(\lambda_T)$. From (2) we have

$$\mathbf{u}_{ot}(\lambda_T) = \mathbf{D}_0 (\boldsymbol{\varepsilon}_{ot} + \lambda_T \mathbf{W} \boldsymbol{\varepsilon}_{ot})$$

where $\mathbf{D}_0 = \text{diag}(\sigma_1, \sigma_2, \dots, \sigma_n)$, $\boldsymbol{\varepsilon}_{ot} = (\varepsilon_{1t}, \varepsilon_{2t}, \dots, \varepsilon_{nt})'$ and $\mathbf{W} = (w_{ij})$ is the connection matrix. Therefore

$$\mathbf{u}_{ot}(\lambda_T) \mathbf{u}'_{ot}(\lambda_T) = \mathbf{D}_0 (\boldsymbol{\varepsilon}_{ot} \boldsymbol{\varepsilon}'_{ot} + \lambda_T^2 \mathbf{W} \boldsymbol{\varepsilon}_{ot} \boldsymbol{\varepsilon}'_{ot} \mathbf{W}' + \lambda_T \boldsymbol{\varepsilon}_{ot} \boldsymbol{\varepsilon}'_{ot} \mathbf{W}' + \lambda_T \mathbf{W} \boldsymbol{\varepsilon}_{ot} \boldsymbol{\varepsilon}'_{ot}) \mathbf{D}_0$$

and

$$\mathbf{V}_T(\lambda_T) = \mathbf{D}_0 (\mathbf{V}_{\varepsilon T} + \lambda_T^2 \mathbf{W} \mathbf{V}_{\varepsilon T} \mathbf{W}' + \lambda_T \mathbf{V}_{\varepsilon T} \mathbf{W}' + \lambda_T \mathbf{W} \mathbf{V}_{\varepsilon T}) \mathbf{D}_0.$$

where $\mathbf{V}_{\varepsilon T} = T^{-1} \sum_{t=1}^T \boldsymbol{\varepsilon}_{ot} \boldsymbol{\varepsilon}'_{ot}$. It follows

$$\|\mathbf{V}_T(\lambda_T)\| \leq \|\mathbf{V}_{\varepsilon T}\| \|\mathbf{D}_0\|^2 (\|\mathbf{I}_n\| + \lambda_T^2 \|\mathbf{W}\| \|\mathbf{W}'\| + |\lambda_T| \|\mathbf{W}'\| + |\lambda_T| \|\mathbf{W}\|) = O_p\left(\frac{n}{T}\right) \quad (\text{S.112})$$

where the equation holds as $\|\mathbf{D}_0\|$ and $\|\mathbf{W}\|$ are both bounded while it is implied by part (b) of Assumption 2 that $\|\mathbf{V}_{\varepsilon T}\| = \mu_{\max}(\mathbf{V}_{\varepsilon T}) = O_p\left(\frac{n}{T}\right)$. In addition, $\|\boldsymbol{\varphi}_{nT}\| = O_p(1)$ while (S.42) gives $\|\hat{\mathbf{\Gamma}} - \mathbf{\Gamma}\|_F = O_p\left(\frac{\sqrt{n}}{\delta_{nT}}\right)$. Then $|A_{1,nT}(\lambda_T)| = \frac{\sqrt{T}}{n} O_p\left(\frac{\sqrt{n}}{\delta_{nT}}\right) = \sqrt{\frac{T}{n}} O_p\left(\frac{1}{\delta_{nT}}\right) = O_p\left(\frac{1}{\delta_{nT}}\right)$, since n and T are of the same order of magnitude. For the second squared term we have

$$A_{2,nT}(\lambda_T) = \sqrt{T} \left[n^{-1/2} \sum_{i=1}^n (\hat{\boldsymbol{\delta}}_{i,T} - \boldsymbol{\delta}_{i,T})' \right] (T^{-1} \mathbf{F}' \mathbf{F}) \left[n^{-1/2} \sum_{i=1}^n (\hat{\boldsymbol{\delta}}_{i,T} - \boldsymbol{\delta}_{i,T}) \right].$$

By assumption $T^{-1} \mathbf{F}' \mathbf{F} = O_p(1)$, and using (S.94) $n^{-1/2} \sum_{i=1}^n (\hat{\boldsymbol{\delta}}_{i,T} - \boldsymbol{\delta}_{i,T}) = O_p\left(\sqrt{\ln(n)/T}\right)$.

Hence, $A_{2,nT}(\lambda_T) = O_p\left(\ln(n)/\sqrt{T}\right) = o_p(1)$. Similarly,

$$\begin{aligned} A_{3,nT}(\lambda_T) &= \sqrt{T} \boldsymbol{\varphi}'_{nT} \left[n^{-1/2} \sum_{i=1}^n (\hat{\gamma}_i - \gamma_i) \gamma'_i \right] \left(T^{-1} \sum_{t=1}^T \mathbf{f}_t \mathbf{f}'_t \right) \left[n^{-1/2} \sum_{i=1}^n \gamma_i (\hat{\gamma} - \gamma_i)' \right] \boldsymbol{\varphi}_{nT} \\ &= \sqrt{T} \boldsymbol{\varphi}'_{nT} \left[n^{-1/2} \sum_{i=1}^n (\hat{\gamma}_i - \gamma_i) \gamma'_i \right] (T^{-1} \mathbf{F}' \mathbf{F}) \left[n^{-1/2} \sum_{i=1}^n \gamma_i (\hat{\gamma} - \gamma_i)' \right] \boldsymbol{\varphi}_{nT}, \end{aligned}$$

where $\|\boldsymbol{\varphi}_{nT}\|$ is bounded, and by (S.95) $n^{-1/2} \sum_{i=1}^n (\hat{\gamma}_i - \gamma_i) \gamma'_i = O_p \left(\sqrt{\ln(n)/T} \right)$. Hence

$$\begin{aligned} A_{3,nT}(\lambda_T) &= \sqrt{T} \boldsymbol{\varphi}'_{nT} O_p \left(\sqrt{\frac{\ln(n)}{T}} \right) (T^{-1} \mathbf{F}' \mathbf{F}) O_p \left(\sqrt{\frac{\ln(n)}{T}} \right) \boldsymbol{\varphi}_{nT} \\ &= O_p \left(\frac{\ln(n)}{\sqrt{T}} \right) = o_p(1). \end{aligned}$$

Next

$$A_{4,nT}(\lambda_T) = \sqrt{T} \left[\frac{1}{\sqrt{n}} \sum_{i=1}^n (\hat{\boldsymbol{\delta}}_{i,T} - \boldsymbol{\delta}_{i,T}) \right]' \left(\frac{\|\hat{\mathbf{F}} - \mathbf{F}\|^2}{T} \right) \left[\frac{1}{\sqrt{n}} \sum_{i=1}^n (\hat{\boldsymbol{\delta}}_{i,T} - \boldsymbol{\delta}_{i,T}) \right],$$

where by (S.41) $T^{-1} \|\hat{\mathbf{F}} - \mathbf{F}\|_F^2 = O_p(\delta_{nT}^{-2})$, and by (S.94) $n^{-1/2} \sum_{i=1}^n (\hat{\boldsymbol{\delta}}_{i,T} - \boldsymbol{\delta}_{i,T}) = O_p \left(\sqrt{\ln(n)/T} \right)$.

Hence, $A_{4,nT}(\lambda_T) = \sqrt{T} O_p(\delta_{nT}^{-2}) \times O_p(\ln(n)/T) = o_p(1)$. The probability orders of the cross product terms of $p_{nT}(\lambda_T)$, namely $A_{5,NT}(\lambda_T), \dots, A_{10,NT}(\lambda_T)$, are also easily seen to be $o_p(1)$, by application of the Cauchy-Schwarz inequality to the product pairs of the terms $A_{1,NT}(\lambda_T), A_{2,NT}(\lambda_T), A_{3,NT}(\lambda_T)$, and $A_{4,NT}(\lambda_T)$. Thus, overall $p_{nT}(\lambda_T) = o_p(1)$, as required. Consider now $q_{nT}(\lambda_T)$ and note that it can be written as (using (S.108) in (S.106))

$$\begin{aligned} q_{nT}(\lambda_T) &= \frac{1}{\sqrt{T}} \sum_{t=1}^T s_{t,nT}(\lambda_T) \xi_{t,n}(\lambda_T) - (\boldsymbol{\varphi}_{nT} - \boldsymbol{\varphi}_n)' \frac{1}{\sqrt{T}} \sum_{t=1}^T s_{t,nT}(\lambda_T) \boldsymbol{\kappa}_{t,n}(\lambda_T) \\ &\quad + \frac{1}{\sqrt{T}} \sum_{t=1}^T s_{t,nT}(\lambda_T) v_{t,nT}(\lambda_T), \end{aligned}$$

where $\xi_{t,n}(\lambda_T)$, $v_{t,nT}(\lambda_T)$ and $\boldsymbol{\kappa}_{t,n}(\lambda_T)$ are given by (S.109), (S.110) and (S.111), respectively. Consider the first term of the above and using (S.109) write it as

$$\begin{aligned} \frac{1}{\sqrt{T}} \sum_{t=1}^T s_{t,nT}(\lambda_T) \xi_{t,n}(\lambda_T) &= \boldsymbol{\varphi}'_{nT} \left[\frac{1}{\sqrt{n}} \sum_{i=1}^n (\hat{\gamma}_i - \gamma_i) \frac{1}{\sqrt{T}} \sum_{t=1}^T \xi_{t,n}(\lambda_T) \sigma_i \varepsilon_{it}(\lambda_T) \right] \\ &\quad + \boldsymbol{\varphi}'_{nT} \left[n^{-1/2} \sum_{i=1}^n (\hat{\gamma}_i - \gamma_i) \gamma'_i \right] \left(\frac{1}{\sqrt{T}} \sum_{t=1}^T \xi_{t,n}(\lambda_T) \mathbf{f}_t \right) \\ &\quad + \left[n^{-1/2} \sum_{i=1}^n (\hat{\boldsymbol{\delta}}_{i,T} - \boldsymbol{\delta}_{i,T})' \right] \left(\frac{1}{\sqrt{T}} \sum_{t=1}^T \xi_{t,n}(\lambda_T) \mathbf{f}_t \right) \\ &\quad + \left[n^{-1/2} \sum_{i=1}^n (\hat{\boldsymbol{\delta}}_{i,T} - \boldsymbol{\delta}_{i,T})' \right] \left[\frac{1}{\sqrt{T}} \sum_{t=1}^T \xi_{t,n}(\lambda_T) (\hat{\mathbf{f}}_t - \mathbf{f}_t) \right] \\ &= \sum_{j=1}^4 B_{j,nT}(\lambda_T). \end{aligned}$$

Using (S.109), $B_{1,nT}(\lambda_T)$ can be written as

$$B_{1,nT}(\lambda_T) = \boldsymbol{\varphi}'_{nT} \left[\frac{\sqrt{T}}{\sqrt{n}} \sum_{i=1}^n (\hat{\gamma}_i - \gamma_i) \frac{1}{\sqrt{n}} \sum_{j=1}^n a_{j,n} \left(\frac{1}{T} \sum_{t=1}^T \sigma_i \varepsilon_{jt}(\lambda_T) \varepsilon_{it}(\lambda_T) \right) \right]$$

where $a_{i,n} = 1 - \sigma_i \boldsymbol{\varphi}'_n \boldsymbol{\gamma}_i$. Since $\varepsilon_{it}(\lambda_T)$ are independently distributed over t and weakly cross-sectionally dependent; and n and T are of the same order, and $\boldsymbol{\varphi}_{nT} = O_p(1)$, then

$$B_{1,nT}(\lambda_T) = O_p \left(\frac{1}{\sqrt{n}} \sum_{i=1}^n a_{i,n} \sigma_i (\hat{\gamma}_i - \gamma_i) \right).$$

Further, letting $b_{in} = a_{i,n} \sigma_i$ and noting that $n^{-1} \sum_{i=1}^n \sigma_i^2 (1 - \sigma_i \boldsymbol{\varphi}'_n \boldsymbol{\gamma}_i)^2 < C$, it follows from (S.93) that $n^{-1/2} \sum_{i=1}^n a_{i,n} \sigma_i (\hat{\gamma}_i - \gamma_i) = O_p(\sqrt{\ln(n)/T})$, which in turn establishes that $B_{1,nT}(\lambda_T) = o_p(1)$. Similarly, using (S.109), $B_{2,nT}(\lambda_T)$ can be written as

$$B_{2,nT}(\lambda_T) = \boldsymbol{\varphi}'_{nT} \left[n^{-1/2} \sum_{i=1}^n (\hat{\gamma}_i - \gamma_i) \boldsymbol{\gamma}'_i \right] \left(\frac{1}{\sqrt{nT}} \sum_{j=1}^n \sum_{t=1}^T a_{j,n} \mathbf{f}_t \varepsilon_{jt}(\lambda_T) \right).$$

Recall that $\boldsymbol{\varphi}_{nT} = O_p(1)$, and by (S.95) $n^{-1/2} \sum_{i=1}^n (\hat{\gamma}_i - \gamma_i) \boldsymbol{\gamma}'_i = O_p(\sqrt{\ln(n)/T})$. Also, under parts (a) and (c) of Assumption 2 $\frac{1}{\sqrt{nT}} \sum_{j=1}^n \sum_{t=1}^T a_{j,n} \mathbf{f}_t \varepsilon_{jt}(\lambda_T) = O_p(1)$. Then it follows that $B_{2,nT}(\lambda_T) = o_p(1)$. Similarly, it is established that $B_{3,nT}(\lambda_T) = o_p(1)$, noting that by (S.94) we have $n^{-1/2} \sum_{i=1}^n (\hat{\boldsymbol{\delta}}_{i,T} - \boldsymbol{\delta}_{i,T}) = O_p(\sqrt{\ln(n)/T})$. The fourth term, $B_{4,nT}(\lambda_T)$, is dominated by the third term and is also $o_p(1)$. Thus overall, $T^{-1/2} \sum_{t=1}^T s_{t,nT}(\lambda_T) \xi_{t,n}(\lambda_T) = o_p(1)$. Using the same line of reasoning, it is also readily established that $T^{-1/2} \sum_{t=1}^T s_{t,nT}(\lambda_T) \boldsymbol{\kappa}_{t,n}(\lambda_T) = o_p(1)$, considering that, $\boldsymbol{\kappa}_{t,n}(\lambda_T) = n^{-1/2} \sum_{i=1}^n \boldsymbol{\gamma}_i \sigma_i \varepsilon_{it}(\lambda_T)$ has the same format as $\xi_{t,n}(\lambda_T)$, and in addition by (S.113) $\boldsymbol{\varphi}_{nT} - \boldsymbol{\varphi}_n = O_p(n^{-1/2} T^{-1/2}) + O_p(T^{-1})$. Finally, the last term of q_{nT} is given by

$$\begin{aligned} \frac{1}{\sqrt{T}} \sum_{t=1}^T s_{t,nT}(\lambda_T) v_{t,nT}(\lambda_T) &= \frac{1}{\sqrt{T}} \sum_{t=1}^T v_{t,nT}(\lambda_T) \boldsymbol{\varphi}'_{nT} \left[n^{-1/2} \sum_{i=1}^n (\hat{\gamma}_i - \gamma_i) \sigma_i \varepsilon_{it}(\lambda_T) \right] \\ &\quad + \frac{1}{\sqrt{T}} \sum_{t=1}^T v_{t,nT}(\lambda_T) \boldsymbol{\varphi}'_{nT} \left[n^{-1/2} \sum_{i=1}^n (\hat{\gamma}_i - \gamma_i) \boldsymbol{\gamma}'_i \right] \mathbf{f}_t \\ &\quad + \frac{1}{\sqrt{T}} \sum_{t=1}^T v_{t,nT}(\lambda_T) \left[n^{-1/2} \sum_{i=1}^n (\hat{\boldsymbol{\delta}}_{i,T} - \boldsymbol{\delta}_{i,T})' \right] \mathbf{f}_t \\ &\quad + \frac{1}{\sqrt{T}} \sum_{t=1}^T v_{t,nT}(\lambda_T) \left[n^{-1/2} \sum_{i=1}^n (\hat{\boldsymbol{\delta}}_{i,T} - \boldsymbol{\delta}_{i,T})' \right] (\hat{\mathbf{f}}_t - \mathbf{f}_t) \\ &= \sum_{j=1}^4 C_{j,nT}(\lambda_T). \end{aligned}$$

Using (S.111) we have

$$\begin{aligned} C_{1,nT}(\lambda_T) &= \frac{1}{\sqrt{T}} \sum_{t=1}^T \frac{1}{\sqrt{n}} \sum_{j=1}^n \left(\frac{1}{(\boldsymbol{\varepsilon}'_{j\circ} \mathbf{M}_F \boldsymbol{\varepsilon}_{j\circ}/T)^{1/2}} - 1 \right) \varepsilon_{jt}(\lambda_T) \boldsymbol{\varphi}'_{nT} \left[n^{-1/2} \sum_{i=1}^n (\hat{\gamma}_i - \gamma_i) \sigma_i \varepsilon_{it}(\lambda_T) \right] \\ &= \sqrt{\frac{T}{n}} \boldsymbol{\varphi}'_{nT} \sum_{j=1}^n \left[n^{-1/2} \sum_{i=1}^n (\hat{\gamma}_i - \gamma_i) \frac{1}{T} \sum_{t=1}^T \sigma_i \left(\frac{1}{(\boldsymbol{\varepsilon}'_{j\circ} \mathbf{M}_F \boldsymbol{\varepsilon}_{j\circ}/T)^{1/2}} - 1 \right) \varepsilon_{it}(\lambda_T) \varepsilon_{jt}(\lambda_T) \right]. \end{aligned}$$

Again, since $\varepsilon_{it}(\lambda_T)$ is distributed independently over t and weakly cross-sectionally dependent, then

$$\frac{1}{T} \sum_{t=1}^T \sigma_i \left(\frac{1}{(\boldsymbol{\varepsilon}'_{j_0} \mathbf{M}_F \boldsymbol{\varepsilon}_{j_0} / T)^{1/2}} - 1 \right) \varepsilon_{it}(\lambda_T) \varepsilon_{jt}(\lambda_T) \rightarrow_p 0, \text{ if } i \neq j,$$

and

$$\begin{aligned} & \frac{1}{T} \sum_{t=1}^T \sigma_i \left(\frac{1}{(\boldsymbol{\varepsilon}'_{j_0} \mathbf{M}_F \boldsymbol{\varepsilon}_{j_0} / T)^{1/2}} - 1 \right) \varepsilon_{it}(\lambda_T) \varepsilon_{jt}(\lambda_T) \\ & \rightarrow_p \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=1}^T \sigma_i E \left\{ \left[\left(\frac{\boldsymbol{\varepsilon}'_{i_0} \mathbf{M}_F \boldsymbol{\varepsilon}_{i_0}}{T} \right)^{-1/2} - 1 \right] \varepsilon_{it}^2(\lambda_T) \right\}, \text{ if } i = j. \end{aligned}$$

Since $\varepsilon_{it}(\lambda_T) = \varepsilon_{it} + \lambda_T b_{it}$ with $b_{it} = \mathbf{w}'_{i0} \boldsymbol{\varepsilon}_{ot}$ and $w_{ii} = 0$ so that ε_{it} is independent from b_{it} , then

$$\begin{aligned} & E \left\{ \left[\left(\frac{\boldsymbol{\varepsilon}'_{i_0} \mathbf{M}_F \boldsymbol{\varepsilon}_{i_0}}{T} \right)^{-1/2} - 1 \right] \varepsilon_{it}^2(\lambda_T) \right\} \\ & = E \left\{ \left[\left(\frac{\boldsymbol{\varepsilon}'_{i_0} \mathbf{M}_F \boldsymbol{\varepsilon}_{i_0}}{T} \right)^{-1/2} - 1 \right] \varepsilon_{it}^2 \right\} + E \left[\left(\frac{\boldsymbol{\varepsilon}'_{i_0} \mathbf{M}_F \boldsymbol{\varepsilon}_{i_0}}{T} \right)^{-1/2} - 1 \right] \lambda_T^2 E(b_{it}^2) \\ & = O\left(\frac{1}{T}\right) + O\left(\frac{1}{T^2}\right) \end{aligned}$$

where the last line holds by (S.79), (S.133) and (S.136). Also by (S.93) $n^{-1/2} \sum_{i=1}^n (\hat{\gamma}_i - \gamma_i) = O_p(\sqrt{\ln(n)/T})$. As n and T being of the same order, it then follows that $C_{1,nT}(\lambda_T) = o_p(1)$. Similarly to $B_{2,nT}(\lambda_T)$, we have

$$\begin{aligned} C_{2,nT}(\lambda_T) &= \boldsymbol{\varphi}'_{nT} \left[n^{-1/2} \sum_{i=1}^n (\hat{\gamma}_i - \gamma_i) \gamma'_i \right] \left[\frac{1}{\sqrt{nT}} \sum_{j=1}^n \sum_{t=1}^T \left(\frac{1}{(\boldsymbol{\varepsilon}'_{j_0} \mathbf{M}_F \boldsymbol{\varepsilon}_{j_0} / T)^{1/2}} - 1 \right) \varepsilon_{jt}(\lambda_T) \mathbf{f}_t \right] \\ &= O_p \left(\sqrt{\frac{\ln(n)}{T}} \right) O_p(1) = o_p(1). \end{aligned}$$

The same line of reasoning as used for $B_{3,nT}(\lambda_T)$ and $B_{4,nT}(\lambda_T)$ can be used to establish $C_{j,nT}(\lambda_T) = o_p(1)$ for $j = 3$ and 4 . Hence, $T^{-1/2} \sum_{t=1}^T s_{t,nT}(\lambda_T) v_{t,nT}(\lambda_T) = o_p(1)$, and overall we have $q_{nT}(\lambda_T) = o_p(1)$, as required. ■

Lemma S.11 *Consider the latent factor model given by (1) and (2). Suppose that Assumptions 1-4 hold and $(n, T) \rightarrow \infty$, such that $n/T \rightarrow \kappa$, for $0 < \kappa < \infty$. Then*

$$\sqrt{T}(\boldsymbol{\varphi}_n - \boldsymbol{\varphi}_{nT}) = O_p(n^{-1/2}) + O_p(T^{-1/2}), \quad (\text{S.113})$$

$$\sqrt{T}(\hat{\boldsymbol{\varphi}}_{nT} - \boldsymbol{\varphi}_n) = o_p(1), \quad (\text{S.114})$$

where $\boldsymbol{\varphi}_n = n^{-1} \sum_{i=1}^n \gamma_i / \sigma_i$, $\boldsymbol{\varphi}_{nT} = n^{-1} \sum_{i=1}^n \gamma_i / \omega_{i,T}$ with $\omega_{i,T} = (T^{-1} \sigma_i^2 \boldsymbol{\varepsilon}'_{i_0} \mathbf{M}_F \boldsymbol{\varepsilon}_{i_0})^{1/2}$, $\hat{\boldsymbol{\varphi}}_{nT} = n^{-1} \sum_{i=1}^n \hat{\gamma}_i / \hat{\sigma}_{i,T}$, $\hat{\sigma}_{i,T} = (T^{-1} \mathbf{y}'_i \mathbf{M}_{\hat{\mathbf{F}}} \mathbf{y}_i)^{1/2}$ and $\hat{\gamma}_i$ and $\hat{\mathbf{F}}$ are the principal component estimators of γ_i and $\mathbf{F}$.

Proof. First note that

$$\begin{aligned}\sqrt{T}(\boldsymbol{\varphi}_n - \boldsymbol{\varphi}_{nT}) &= \frac{\sqrt{T}}{n} \sum_{i=1}^n \frac{\gamma_i}{\sigma_i} \left\{ \left(1 - \frac{\sigma_i}{\omega_{i,T}}\right) - \left[1 - E\left(\frac{\sigma_i}{\omega_{i,T}}\right)\right] \right\} + \frac{\sqrt{T}}{n} \sum_{i=1}^n \frac{\gamma_i}{\sigma_i} \left[1 - E\left(\frac{\sigma_i}{\omega_{i,T}}\right)\right], \\ &= \mathbf{g}_{1,nT} + \mathbf{g}_{2,nT},\end{aligned}$$

where

$$\begin{aligned}\mathbf{g}_{1,nT} &= -\frac{1}{n} \sum_{i=1}^n \sqrt{T} \left[\frac{\sigma_i}{\omega_{i,T}} - E\left(\frac{\sigma_i}{\omega_{i,T}}\right) \right] \frac{\gamma_i}{\sigma_i}, \\ \mathbf{g}_{2,nT} &= \frac{\sqrt{T}}{n} \sum_{i=1}^n \left[1 - E\left(\frac{\sigma_i}{\omega_{i,T}}\right) \right] \frac{\gamma_i}{\sigma_i}.\end{aligned}$$

Since $\sigma_i/\omega_{i,T} = (T^{-1}\boldsymbol{\varepsilon}'_{i0}\mathbf{M}_F\boldsymbol{\varepsilon}_{i0})^{-1/2}$, $\|\boldsymbol{\gamma}_i\| < C$, $\sigma_i < C$, then using result (S.133) we have $E\left(\frac{\sigma_i}{\omega_{i,T}}\right) = 1 + O(T^{-1})$, and $\mathbf{g}_{2,nT} = O(T^{-1/2})$. The first term can be written as $\mathbf{g}_{1,nT} = n^{-1} \sum_{i=1}^n \boldsymbol{\gamma}_i \sigma_i^{-1} \chi_{i,T}$, where $\chi_{i,T} = -\sqrt{T}[\sigma_i/\omega_{i,T} - E(\sigma_i/\omega_{i,T})]$. Conditional on $\mathbf{F}$ and σ_i , $\chi_{i,T}$ are distributed independently over i with mean zero and bounded variances:^{S3}

$$Var(\chi_{i,T}) = T \left[E\left(\frac{T}{\boldsymbol{\varepsilon}'_{i0}\mathbf{M}_F\boldsymbol{\varepsilon}_{i0}}\right) - \left[E\left(\frac{\sigma_i}{\omega_{i,T}}\right)\right]^2 \right] = T \left[1 + O\left(\frac{1}{T}\right) - \left[1 + O\left(\frac{1}{T}\right)\right]^2 \right] = O(1).$$

Hence, $\mathbf{g}_{1,nT} = O_p(n^{-1/2})$, and the desired result (S.113) follows. Consider now (S.114) and note that it can be decomposed as

$$\sqrt{T}(\hat{\boldsymbol{\varphi}}_{nT} - \boldsymbol{\varphi}_n) = \sqrt{T}(\boldsymbol{\varphi}_{nT} - \boldsymbol{\varphi}_n) + \sqrt{T}(\hat{\boldsymbol{\varphi}}_{nT} - \boldsymbol{\varphi}_{nT}), \quad (\text{S.115})$$

where it is already established that the first term is $o_p(1)$. Consider now the probability order of the second term and note that it can be written as

$$\begin{aligned}\sqrt{T}(\hat{\boldsymbol{\varphi}}_{nT} - \boldsymbol{\varphi}_{nT}) &= \frac{\sqrt{T}}{n} \sum_{i=1}^n \left(\frac{\hat{\gamma}_i}{\hat{\sigma}_{i,T}} - \frac{\gamma_i}{\omega_{i,T}} \right) \\ &= \frac{\sqrt{T}}{n} \sum_{i=1}^n \gamma_i \left(\frac{1}{\hat{\sigma}_{i,T}} - \frac{1}{\omega_{i,T}} \right) + \frac{\sqrt{T}}{n} \sum_{i=1}^n \frac{1}{\sigma_i} (\hat{\gamma}_i - \gamma_i) + \\ &\quad \frac{\sqrt{T}}{n} \sum_{i=1}^n \left(\frac{1}{\omega_{i,T}} - \frac{1}{\sigma_i} \right) (\hat{\gamma}_i - \gamma_i) + \frac{\sqrt{T}}{n} \sum_{i=1}^n \left(\frac{1}{\hat{\sigma}_{i,T}} - \frac{1}{\omega_{i,T}} \right) (\hat{\gamma}_i - \gamma_i).\end{aligned}$$

Now using (S.85) of Lemma S.8 we have

$$\frac{\sqrt{T}}{n} \sum_{i=1}^n \left(\frac{1}{\hat{\sigma}_{i,T}} - \frac{1}{\omega_{i,T}} \right) \gamma_i = O_p\left(\frac{\ln(n)}{\sqrt{T}}\right) = o_p(1).$$

Using this result as well as (S.93), (S.97) and (S.99), we have $\sqrt{T}(\hat{\boldsymbol{\varphi}}_{nT} - \boldsymbol{\varphi}_{nT}) = o_p(1)$, which in turn establishes (S.114), as required. ■

^{S3}When ε_{it} are normally distributed we have the exact result $E\left(\frac{T}{\boldsymbol{\varepsilon}'_{i0}\mathbf{M}_F\boldsymbol{\varepsilon}_{i0}}\right) = T/(T - m_0 - 2)$.

Lemma S.12 Suppose $\mathbf{M}_F = \mathbf{I}_T - \mathbf{F}(\mathbf{F}'\mathbf{F})^{-1}\mathbf{F}'$, where $\mathbf{F}$ is a $T \times m_0$ matrix, and $\boldsymbol{\tau}_T$ is a $T \times 1$ vector of ones. Then

$$\text{tr}(\mathbf{M}_F) = v, \text{tr}(\mathbf{M}_F \odot \mathbf{M}_F) = O(v), \quad (\text{S.116})$$

$$\text{tr}(\mathbf{M}_F \odot \mathbf{M}_F \odot \mathbf{M}_F) = O(v), \text{tr}(\mathbf{M}_F \odot \mathbf{M}_F \odot \mathbf{M}_F \odot \mathbf{M}_F) = O(v), \quad (\text{S.117})$$

$$\text{tr}[(\mathbf{I}_T \odot \mathbf{M}_F) \mathbf{M}_F] = O(v), \text{tr}[(\mathbf{M}_F \odot \mathbf{M}_F) \mathbf{M}_F] = O(v), \quad (\text{S.118})$$

$$\boldsymbol{\tau}_T' (\mathbf{M}_F \odot \mathbf{M}_F) \boldsymbol{\tau}_T = O(v), \boldsymbol{\tau}_T' (\mathbf{M}_F \odot \mathbf{M}_F \odot \mathbf{M}_F) \boldsymbol{\tau}_T = O(v), \quad (\text{S.119})$$

$$\boldsymbol{\tau}_T' (\mathbf{M}_F \odot \mathbf{M}_F \odot \mathbf{M}_F \odot \mathbf{M}_F) \boldsymbol{\tau}_T = O(v), \boldsymbol{\tau}_T' (\mathbf{I}_T \odot \mathbf{M}_F) (\mathbf{M}_F \odot \mathbf{M}_F \odot \mathbf{M}_F) \boldsymbol{\tau}_T = O(v), \quad (\text{S.120})$$

$$\boldsymbol{\tau}_T' (\mathbf{I}_T \odot \mathbf{M}_F) (\mathbf{M}_F \odot \mathbf{M}_F) (\mathbf{I}_T \odot \mathbf{M}_F) \boldsymbol{\tau}_T = O(v), \boldsymbol{\tau}_T' (\mathbf{I}_T \odot \mathbf{M}_F) (\mathbf{I}_T \odot \mathbf{M}_F) \boldsymbol{\tau}_T = O(v), \quad (\text{S.121})$$

$$\boldsymbol{\tau}_T' (\mathbf{I}_T \odot \mathbf{M}_F) \mathbf{M}_F (\mathbf{I}_T \odot \mathbf{M}_F) \boldsymbol{\tau}_T = O(v^{3/2}), \boldsymbol{\tau}_T' (\mathbf{M}_F \odot \mathbf{M}_F) \mathbf{M}_F (\mathbf{I}_T \odot \mathbf{M}_F) \boldsymbol{\tau}_T = O(v^{3/2}), \quad (\text{S.122})$$

$$\boldsymbol{\tau}_T' (\mathbf{I}_T \odot \mathbf{M}_F) \mathbf{M}_F (\mathbf{I}_T \odot \mathbf{M}_F \odot \mathbf{M}_F) \boldsymbol{\tau}_T = O(v^{3/2}), \quad (\text{S.123})$$

where $v = T - m_0$.

Proof. See Lemma 10 of Pesaran and Yamagata (2024). ■

Lemma S.13 Suppose the $T \times 1$ vector $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_T)'$ is $\boldsymbol{\varepsilon} \sim IID(\mathbf{0}, \mathbf{I}_T)$, $\sup_t E(|\varepsilon_t|^{8+s}) < C$ for some small $s > 0$, and $\mathbf{M}_F = \mathbf{I}_T - \mathbf{F}(\mathbf{F}'\mathbf{F})^{-1}\mathbf{F}'$, where the $T \times m_0$ matrix $\mathbf{F}$ is distributed independently of $\boldsymbol{\varepsilon}$, then

$$E\left(\frac{\boldsymbol{\varepsilon}'\mathbf{M}_F\boldsymbol{\varepsilon}}{v}\right) = 1, \quad (\text{S.124})$$

$$E\left[\left(\frac{\boldsymbol{\varepsilon}'\mathbf{M}_F\boldsymbol{\varepsilon}}{v}\right)^2\right] = 1 + O\left(\frac{1}{v}\right), \quad (\text{S.125})$$

$$E(q_v) = 0, E(q_v^4) = O(1), \quad (\text{S.126})$$

where $v = T - m_0$, and $q_v = \sqrt{v}\left(\frac{\boldsymbol{\varepsilon}'\mathbf{M}_F\boldsymbol{\varepsilon}}{v} - 1\right)$.

Proof. Denote $\boldsymbol{\tau}_T$ as a $T \times 1$ vector of ones and suppose $\gamma_1 = E(\varepsilon_t^3)$, $\gamma_2 = E(\varepsilon_t^4) - 3$, $\gamma_3 = E(\varepsilon_t^5) - 10\gamma_1$, $\gamma_4 = E(\varepsilon_t^6) - 15\gamma_2 - 10\gamma_1^2 - 15$, $\gamma_6 = E(\varepsilon_t^8) - 28\gamma_4 - 56\gamma_3\gamma_1 - 35\gamma_2^2 - 210\gamma_2 - 280\gamma_1^2 - 105$, which are all bounded as it is assumed $\sup_t E(|\varepsilon_t|^{8+\epsilon}) < C$. Since $\boldsymbol{\varepsilon} \sim IID(\mathbf{0}, \mathbf{I}_T)$ and $\mathbf{M}_F = (m_{tt'})$ is an idempotent matrix then results (S.6) to (S.9) of Lemma 6 in Pesaran and Yamagata (2024) apply and we have (since $\text{tr}(\mathbf{M}_F^s) = \text{tr}(\mathbf{M}_F) = v$, for $s = 1, 2, \dots$)

$$E(\boldsymbol{\varepsilon}'\mathbf{M}_F\boldsymbol{\varepsilon}) = v, \quad (\text{S.127})$$

$$E\left[(\boldsymbol{\varepsilon}'\mathbf{M}_F\boldsymbol{\varepsilon})^2\right] = v^2 + 2v + \gamma_2 \text{tr}(\mathbf{M}_F \odot \mathbf{M}_F), \quad (\text{S.128})$$

$$\begin{aligned} E\left[(\boldsymbol{\varepsilon}'\mathbf{M}_F\boldsymbol{\varepsilon})^3\right] &= v^3 + 6v^2 + 8v + \gamma_4 \text{tr}(\mathbf{M}_F \odot \mathbf{M}_F \odot \mathbf{M}_F) + 3\gamma_2(v+4) \text{tr}(\mathbf{M}_F \odot \mathbf{M}_F) \\ &\quad + 6\gamma_1^2 [\boldsymbol{\tau}_T' (\mathbf{I}_T \odot \mathbf{M}_F) \mathbf{M}_F (\mathbf{I}_T \odot \mathbf{M}_F) \boldsymbol{\tau}_T] + 4\gamma_1^2 [\boldsymbol{\tau}_T' (\mathbf{M}_F \odot \mathbf{M}_F \odot \mathbf{M}_F) \boldsymbol{\tau}_T] \end{aligned} \quad (\text{S.129})$$

$$E \left[(\boldsymbol{\varepsilon}' \mathbf{M}_F \boldsymbol{\varepsilon})^4 \right] = v^4 + 12v^3 + 44v^2 + 48v + \gamma_2 f_{\gamma_2} + \gamma_4 f_{\gamma_4} + \gamma_6 f_{\gamma_6} + \gamma_1^2 f_{\gamma_1^2} + \gamma_2^2 f_{\gamma_2^2} + \gamma_1 \gamma_3 f_{\gamma_1 \gamma_3}, \quad (\text{S.130})$$

where

$$f_{\gamma_2} = (6v^2 + 48v) \text{tr}(\mathbf{M}_F \odot \mathbf{M}_F) + 12 [\boldsymbol{\tau}'_T (\mathbf{M}_F \odot \mathbf{M}_F) \boldsymbol{\tau}_T \text{tr}(\mathbf{M}_F \odot \mathbf{M}_F)] \\ + 96 \text{tr}[(\mathbf{I}_T \odot \mathbf{M}_F) \mathbf{M}_F] + 48 \boldsymbol{\tau}'_T (\mathbf{I}_T \odot \mathbf{M}_F) (\mathbf{I}_T \odot \mathbf{M}_F) \boldsymbol{\tau}_T,$$

$$f_{\gamma_4} = (4v + 24) \text{tr}(\mathbf{M}_F \odot \mathbf{M}_F \odot \mathbf{M}_F),$$

$$f_{\gamma_6} = \text{tr}(\mathbf{M}_F \odot \mathbf{M}_F \odot \mathbf{M}_F \odot \mathbf{M}_F),$$

$$f_{\gamma_1^2} = 24v \boldsymbol{\tau}'_T (\mathbf{I}_T \odot \mathbf{M}_F) \mathbf{M}_F (\mathbf{I}_T \odot \mathbf{M}_F) \boldsymbol{\tau}_T + 48 \boldsymbol{\tau}'_T (\mathbf{I}_T \odot \mathbf{M}_F) \mathbf{M}_F (\mathbf{I}_T \odot \mathbf{M}_F) \boldsymbol{\tau}_T \\ + 16v \boldsymbol{\tau}'_T (\mathbf{M}_F \odot \mathbf{M}_F \odot \mathbf{M}_F) \boldsymbol{\tau}_T + 96 \boldsymbol{\tau}'_T (\mathbf{M}_F \odot \mathbf{M}_F) \mathbf{M}_F (\mathbf{I}_T \odot \mathbf{M}_F) \boldsymbol{\tau}_T \\ + 96 \text{tr}[\mathbf{M}_F (\mathbf{M}_F \odot \mathbf{M}_F) \mathbf{M}_F],$$

$$f_{\gamma_2^2} = 3 [\text{tr}(\mathbf{M}_F \odot \mathbf{M}_F)]^2 + 24 \boldsymbol{\tau}'_T (\mathbf{I}_T \odot \mathbf{M}_F) (\mathbf{M}_F \odot \mathbf{M}_F) (\mathbf{I}_T \odot \mathbf{M}_F) \boldsymbol{\tau}_T \\ + 8 \boldsymbol{\tau}'_T (\mathbf{M}_F \odot \mathbf{M}_F \odot \mathbf{M}_F \odot \mathbf{M}_F) \boldsymbol{\tau}_T,$$

$$f_{\gamma_1 \gamma_3} = 24 \boldsymbol{\tau}'_T (\mathbf{I}_T \odot \mathbf{M}_F) \mathbf{M}_F (\mathbf{I}_T \odot \mathbf{M}_F \odot \mathbf{M}_F) \boldsymbol{\tau}_T + 32 \boldsymbol{\tau}'_T (\mathbf{I}_T \odot \mathbf{M}_F) (\mathbf{M}_F \odot \mathbf{M}_F \odot \mathbf{M}_F) \boldsymbol{\tau}_T.$$

Result (S.124) follows from (S.127). To establish (S.125), using (S.128), we first note that

$$E \left(\frac{\boldsymbol{\varepsilon}' \mathbf{M}_F \boldsymbol{\varepsilon}}{v} \right)^2 = 1 + \frac{2}{v} + \gamma_2 \frac{\text{tr}(\mathbf{M}_F \odot \mathbf{M}_F)}{v^2}.$$

But by (S.116) $\text{tr}(\mathbf{M}_F \odot \mathbf{M}_F) = O(v)$ and by assumption γ_2 is bounded. Hence $E \left(\frac{\boldsymbol{\varepsilon}' \mathbf{M}_F \boldsymbol{\varepsilon}}{v} \right)^2 = 1 + O(\frac{1}{v})$, as required. To prove (S.126), noting that $E(\boldsymbol{\varepsilon}' \mathbf{M}_F \boldsymbol{\varepsilon}) = v$ then $E(q_v) = 0$. Also

$$q_v^4 = v^2 \left(\frac{\boldsymbol{\varepsilon}' \mathbf{M}_F \boldsymbol{\varepsilon}}{v} - 1 \right)^4 = v^2 \left[\left(\frac{\boldsymbol{\varepsilon}' \mathbf{M}_F \boldsymbol{\varepsilon}}{v} \right)^4 - 4 \left(\frac{\boldsymbol{\varepsilon}' \mathbf{M}_F \boldsymbol{\varepsilon}}{v} \right)^3 + 6 \left(\frac{\boldsymbol{\varepsilon}' \mathbf{M}_F \boldsymbol{\varepsilon}}{v} \right)^2 - 4 \left(\frac{\boldsymbol{\varepsilon}' \mathbf{M}_F \boldsymbol{\varepsilon}}{v} \right) + 1 \right],$$

and taking expectation yields

$$E(q_v^4) = v^2 \left[E \left[\left(\frac{\boldsymbol{\varepsilon}' \mathbf{M}_F \boldsymbol{\varepsilon}}{v} \right)^4 \right] - 4E \left[\left(\frac{\boldsymbol{\varepsilon}' \mathbf{M}_F \boldsymbol{\varepsilon}}{v} \right)^3 \right] + 6E \left[\left(\frac{\boldsymbol{\varepsilon}' \mathbf{M}_F \boldsymbol{\varepsilon}}{v} \right)^2 \right] - 4E \left(\frac{\boldsymbol{\varepsilon}' \mathbf{M}_F \boldsymbol{\varepsilon}}{v} \right) + 1 \right] \\ = \frac{1}{v^2} E[(\boldsymbol{\varepsilon}' \mathbf{M}_F \boldsymbol{\varepsilon})^4] - \frac{4}{v} E[(\boldsymbol{\varepsilon}' \mathbf{M}_F \boldsymbol{\varepsilon})^3] + 6E[(\boldsymbol{\varepsilon}' \mathbf{M}_F \boldsymbol{\varepsilon})^2] - 4vE(\boldsymbol{\varepsilon}' \mathbf{M}_F \boldsymbol{\varepsilon}) + v^2.$$

Now using the results in (S.127)-(S.130), and after some algebra, we obtain

$$E(q_v^4) = \frac{1}{v^2} \left[\begin{aligned} &12v^2 + 48v + 12\gamma_2 [\boldsymbol{\tau}'_T (\mathbf{M}_F \odot \mathbf{M}_F) \boldsymbol{\tau}_T] \text{tr}(\mathbf{M}_F \odot \mathbf{M}_F) \\ &+ 96\gamma_2 \text{tr}[(\mathbf{I}_T \odot \mathbf{M}_F) \mathbf{M}_F] + 48\gamma_2 [\boldsymbol{\tau}'_T (\mathbf{I}_T \odot \mathbf{M}_F) (\mathbf{I}_T \odot \mathbf{M}_F) \boldsymbol{\tau}_T] \\ &+ (4\gamma_4 v + 24\gamma_4) \text{tr}(\mathbf{M}_F \odot \mathbf{M}_F \odot \mathbf{M}_F) + \gamma_6 \text{tr}(\mathbf{M}_F \odot \mathbf{M}_F \odot \mathbf{M}_F \odot \mathbf{M}_F) \\ &+ 48\gamma_1^2 [\boldsymbol{\tau}'_T (\mathbf{I}_T \odot \mathbf{M}_F) \mathbf{M}_F (\mathbf{I}_T \odot \mathbf{M}_F) \boldsymbol{\tau}_T] + 96\gamma_1^2 [\boldsymbol{\tau}'_T (\mathbf{M}_F \odot \mathbf{M}_F) \mathbf{M}_F (\mathbf{I}_T \odot \mathbf{M}_F) \boldsymbol{\tau}_T] \\ &+ 96\gamma_1^2 \text{tr}[(\mathbf{M}_F \odot \mathbf{M}_F) \mathbf{M}_F] + 3\gamma_2^2 [\text{tr}(\mathbf{M}_F \odot \mathbf{M}_F)]^2 \\ &+ 24\gamma_2^2 [\boldsymbol{\tau}'_T (\mathbf{I}_T \odot \mathbf{M}_F) (\mathbf{M}_F \odot \mathbf{M}_F) (\mathbf{I}_T \odot \mathbf{M}_F) \boldsymbol{\tau}_T] + 8\gamma_2^2 [\boldsymbol{\tau}'_T (\mathbf{M}_F \odot \mathbf{M}_F \odot \mathbf{M}_F \odot \mathbf{M}_F) \boldsymbol{\tau}_T] \\ &+ 24\gamma_1 \gamma_3 [\boldsymbol{\tau}'_T (\mathbf{I}_T \odot \mathbf{M}_F) \mathbf{M}_F (\mathbf{I}_T \odot \mathbf{M}_F \odot \mathbf{M}_F) \boldsymbol{\tau}_T] \\ &+ 32\gamma_1 \gamma_3 [\boldsymbol{\tau}'_T (\mathbf{I}_T \odot \mathbf{M}_F) (\mathbf{M}_F \odot \mathbf{M}_F \odot \mathbf{M}_F) \boldsymbol{\tau}_T] \end{aligned} \right] \\ = \sum_{s=1}^{15} a_{s,v}. \quad (\text{S.131})$$

Further noting that $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5$, and γ_6 are all bounded, then using the results (S.116)-(S.123) we have

$$\begin{aligned}
a_{1,v} &= 12, a_{2,v} = \frac{48}{v}, \\
a_{3,v} &= \frac{12\gamma_2 [\boldsymbol{\tau}'_T (\mathbf{M}_F \odot \mathbf{M}_F) \boldsymbol{\tau}_T] \text{tr} (\mathbf{M}_F \odot \mathbf{M}_F)}{v^2} = O(1), \\
a_{4,v} &= \frac{96\gamma_2 \text{tr} [(\mathbf{I}_T \odot \mathbf{M}_F) \mathbf{M}_F]}{v^2} = O\left(\frac{1}{v}\right), \\
a_{5,v} &= \frac{48\gamma_2 [\boldsymbol{\tau}'_T (\mathbf{I}_T \odot \mathbf{M}_F) (\mathbf{I}_T \odot \mathbf{M}_F) \boldsymbol{\tau}_T]}{v^2} = O\left(\frac{1}{v}\right), \\
a_{6,v} &= \frac{(4\gamma_4 v + 24\gamma_4) \text{tr} (\mathbf{M}_F \odot \mathbf{M}_F \odot \mathbf{M}_F)}{v^2} = O(1), \\
a_{7,v} &= \frac{\gamma_6 \text{tr} (\mathbf{M}_F \odot \mathbf{M}_F \odot \mathbf{M}_F \odot \mathbf{M}_F)}{v^2} = O\left(\frac{1}{v}\right), \\
a_{8,v} &= \frac{48\gamma_1^2 [\boldsymbol{\tau}'_T (\mathbf{I}_T \odot \mathbf{M}_F) \mathbf{M}_F (\mathbf{I}_T \odot \mathbf{M}_F) \boldsymbol{\tau}_T]}{v^2} = O\left(\frac{1}{\sqrt{v}}\right), \\
a_{9,v} &= \frac{96\gamma_1^2 \boldsymbol{\tau}'_T (\mathbf{M}_F \odot \mathbf{M}_F) \mathbf{M}_F (\mathbf{I}_T \odot \mathbf{M}_F) \boldsymbol{\tau}_T}{v^2} = O\left(\frac{1}{\sqrt{v}}\right), \\
a_{10,v} &= \frac{96\gamma_1^2 \text{tr} [(\mathbf{M}_F \odot \mathbf{M}_F) \mathbf{M}_F]}{v^2} = O\left(\frac{1}{v}\right), \\
a_{11,v} &= \frac{3\gamma_2^2 [\text{tr} (\mathbf{M}_F \odot \mathbf{M}_F)]^2}{v^2} = O(1), \\
a_{12,v} &= \frac{24\gamma_2^2 [\boldsymbol{\tau}'_T (\mathbf{I}_T \odot \mathbf{M}_F) (\mathbf{M}_F \odot \mathbf{M}_F) (\mathbf{I}_T \odot \mathbf{M}_F) \boldsymbol{\tau}_T]}{v^2} = O\left(\frac{1}{v}\right), \\
a_{13,v} &= \frac{8\gamma_2^2 [\boldsymbol{\tau}'_T (\mathbf{M}_F \odot \mathbf{M}_F \odot \mathbf{M}_F \odot \mathbf{M}_F) \boldsymbol{\tau}_T]}{v^2} = O\left(\frac{1}{v}\right), \\
a_{14,v} &= \frac{24\gamma_1\gamma_3 [\boldsymbol{\tau}'_T (\mathbf{I}_T \odot \mathbf{M}_F) \mathbf{M}_F (\mathbf{I}_T \odot \mathbf{M}_F \odot \mathbf{M}_F) \boldsymbol{\tau}_T]}{v^2} = O\left(\frac{1}{\sqrt{v}}\right), \\
a_{15,v} &= \frac{32\gamma_1\gamma_3 [\boldsymbol{\tau}'_T (\mathbf{I}_T \odot \mathbf{M}_F) (\mathbf{M}_F \odot \mathbf{M}_F \odot \mathbf{M}_F) \boldsymbol{\tau}_T]}{v^2} = O\left(\frac{1}{v}\right).
\end{aligned}$$

Using these results in (S.131) it now follows that $E(q_v^4) = O(1)$, as required. ■

Lemma S.14 Suppose the $T \times 1$ vector $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_T)'$ is $\boldsymbol{\varepsilon} \sim IID(\mathbf{0}, \mathbf{I}_T)$, $\sup_t E(|\varepsilon_t|^{8+s}) < C$ for some small $s > 0$, and $\mathbf{M}_F = \mathbf{I}_T - \mathbf{F}(\mathbf{F}'\mathbf{F})^{-1}\mathbf{F}'$, where the $T \times m_0$ matrix $\mathbf{F}$ is distributed independently of $\boldsymbol{\varepsilon}$. Suppose there exists a finite integer v_0 such that for all $v > v_0$,

$$\frac{\boldsymbol{\varepsilon}' \mathbf{M}_F \boldsymbol{\varepsilon}}{v} > c > 0. \tag{S.132}$$

Then for $v > v_0$,

$$E \left[\left(\frac{\boldsymbol{\varepsilon}' \mathbf{M}_F \boldsymbol{\varepsilon}}{v} \right)^{1/2} \right] = 1 + O \left(\frac{1}{v} \right), \quad E \left[\left(\frac{\boldsymbol{\varepsilon}' \mathbf{M}_F \boldsymbol{\varepsilon}}{v} \right)^{-s/2} \right] = 1 + O \left(\frac{1}{v} \right), \quad \text{for } s = 1, 2, 3, 4, \quad (\text{S.133})$$

$$E \left[\varepsilon_t^2 \left(\frac{\boldsymbol{\varepsilon}' \mathbf{M}_F \boldsymbol{\varepsilon}}{v} \right) \right] = 1 + O \left(\frac{1}{v} \right), \quad (\text{S.134})$$

$$E \left[\varepsilon_t^2 \left(\frac{\boldsymbol{\varepsilon}' \mathbf{M}_F \boldsymbol{\varepsilon}}{v} \right)^{-1} \right] = 1 + O \left(\frac{1}{v} \right), \quad (\text{S.135})$$

$$E \left[\varepsilon_t^2 \left(\frac{\boldsymbol{\varepsilon}' \mathbf{M}_F \boldsymbol{\varepsilon}}{v} \right)^{-1/2} \right] = 1 + O \left(\frac{1}{v} \right), \quad (\text{S.136})$$

$$E \left[\varepsilon_t \varepsilon_{t'} \left(\frac{\boldsymbol{\varepsilon}' \mathbf{M}_F \boldsymbol{\varepsilon}}{v} \right)^{-1/2} \right] = O \left(\frac{1}{v} \right), \quad \text{for } t \neq t', \quad (\text{S.137})$$

$$E \left[\varepsilon_t \left(\frac{\boldsymbol{\varepsilon}' \mathbf{M}_F \boldsymbol{\varepsilon}}{v} \right)^{-1/2} \right] = O \left(\frac{1}{v} \right). \quad (\text{S.138})$$

Proof. To establish the results in (S.133) we first note that

$$\frac{\boldsymbol{\varepsilon}' \mathbf{M}_F \boldsymbol{\varepsilon}}{v} = 1 + \frac{1}{\sqrt{v}} q_v \quad (\text{S.139})$$

where

$$q_v = \sqrt{v} \left(\frac{\boldsymbol{\varepsilon}' \mathbf{M}_F \boldsymbol{\varepsilon}}{v} - 1 \right). \quad (\text{S.140})$$

Applying the Taylor Theorem to $(v^{-1} \boldsymbol{\varepsilon}' \mathbf{M}_F \boldsymbol{\varepsilon})^{1/2}$ we have

$$\left(\frac{\boldsymbol{\varepsilon}' \mathbf{M}_F \boldsymbol{\varepsilon}}{v} \right)^{1/2} = 1 + \frac{1}{2} \frac{q_v}{\sqrt{v}} - \frac{1}{8v} R_v, \quad (\text{S.141})$$

where

$$R_v = \left(1 + \frac{\bar{q}_v}{\sqrt{v}} \right)^{-3/2} q_v^2,$$

and $\bar{q}_v$ lies on the interval between 0 and q_v . Since $E(q_v) = 0$ as shown by result (S.126) of Lemma S.13, taking expectations of both sides of (S.141) yields

$$E \left(\frac{\boldsymbol{\varepsilon}' \mathbf{M}_F \boldsymbol{\varepsilon}}{v} \right)^{1/2} = 1 - \frac{1}{8v} E(R_v). \quad (\text{S.142})$$

It is, therefore, sufficient to show that $E(|R_v|) < C$. By Cauchy-Schwarz inequality we have

$$E(|R_v|) \leq \left[E \left(\left| 1 + \frac{\bar{q}_v}{\sqrt{v}} \right|^{-3} \right) \right]^{1/2} [E(q_v^4)]^{1/2}. \quad (\text{S.143})$$

Consider $\left|1 + \frac{\bar{q}_v}{\sqrt{v}}\right|$ and distinguish the cases (a) $q_v \geq 0$ or equivalently if $\frac{\boldsymbol{\varepsilon}'\mathbf{M}_F\boldsymbol{\varepsilon}}{v} \geq 1$ and (b) $q_v < 0$ or equivalently if $\frac{\boldsymbol{\varepsilon}'\mathbf{M}_F\boldsymbol{\varepsilon}}{v} < 1$. Under (a) $0 \leq \bar{q}_v < q_v$, we have $\left|1 + \frac{\bar{q}_v}{\sqrt{v}}\right| \geq 1$. Under (b) $q_v < \bar{q}_v < 0$, we have $\left|1 + \frac{\bar{q}_v}{\sqrt{v}}\right| > \left|1 + \frac{q_v}{\sqrt{v}}\right| = \frac{\boldsymbol{\varepsilon}'\mathbf{M}_F\boldsymbol{\varepsilon}}{v}$, and under condition (S.132) $\left|1 + \frac{\bar{q}_v}{\sqrt{v}}\right| > c > 0$. Hence, irrespective of whether $q_v \geq 0$ or not,

$$\left|1 + \frac{\bar{q}_v}{\sqrt{v}}\right| > c > 0, \quad (\text{S.144})$$

and we have $E\left(\left|1 + \frac{\bar{q}_v}{\sqrt{v}}\right|^{-3}\right) < C$. Also it is established that $E(q_v^4) = O(1)$ by result (S.126) of Lemma S.13. Using these results in (S.143) it follows that $E(|R_v|) < C$, and given (S.142) we can show

$$E\left[\left(\frac{\boldsymbol{\varepsilon}'\mathbf{M}_F\boldsymbol{\varepsilon}}{v}\right)^{1/2}\right] = 1 + O\left(\frac{1}{v}\right).$$

The other results in (S.133) can also be established similarly. Result (S.134) follows (S.7) in Lemma 6 of Pesaran and Yamagata (2024) by setting $\varepsilon_t^2 = \boldsymbol{\varepsilon}'\mathbf{A}_1\boldsymbol{\varepsilon}$, where $\mathbf{A}_1$ has only one non-zero element on its diagonal. Result (S.135) can be established using a result due to Lieberman (1994) (see Lemmas 5 and 21 in the online supplement of Pesaran and Yamagata (2024)). To establish (S.136), note that by applying the Taylor Theorem to $(v^{-1}\boldsymbol{\varepsilon}'\mathbf{M}_F\boldsymbol{\varepsilon})^{-1/2}$,

$$\varepsilon_t^2 \left(\frac{\boldsymbol{\varepsilon}'\mathbf{M}_F\boldsymbol{\varepsilon}}{v}\right)^{-1/2} = \varepsilon_t^2 - \frac{1}{2} \frac{q_v \varepsilon_t^2}{\sqrt{v}} + \frac{3}{8v} R_{e,v}$$

where

$$R_{e,v} = \left(1 + \frac{\bar{q}_v}{\sqrt{v}}\right)^{-5/2} q_v^2 \varepsilon_t^2,$$

and taking expectations yields

$$E\left[\varepsilon_t^2 \left(\frac{\boldsymbol{\varepsilon}'\mathbf{M}_F\boldsymbol{\varepsilon}}{v}\right)^{-1/2}\right] = E(\varepsilon_t^2) - \frac{1}{2} \frac{E(q_v \varepsilon_t^2)}{\sqrt{v}} + \frac{3}{8v} E(R_{e,v}). \quad (\text{S.145})$$

$E(\varepsilon_t^2) = 1$, and using (S.134) we have

$$E\left[\varepsilon_t^2 \left(\frac{q_v}{\sqrt{v}}\right)\right] = E\left[\varepsilon_t^2 \left(\frac{\boldsymbol{\varepsilon}'\mathbf{M}_F\boldsymbol{\varepsilon}}{v} - 1\right)\right] = O\left(\frac{1}{v}\right). \quad (\text{S.146})$$

By Cauchy-Schwarz inequality

$$\begin{aligned} E(|R_{e,v}|) &\leq \left[E\left[\varepsilon_t^4 \left(\left|1 + \frac{\bar{q}_v}{\sqrt{v}}\right|^{-5}\right)\right]\right]^{1/2} [E(q_v^4)]^{1/2} \\ &\leq \left[E\left(\left|1 + \frac{\bar{q}_v}{\sqrt{v}}\right|^{-10}\right)\right]^{1/4} [E(\varepsilon_t^8)]^{1/4} [E(q_v^4)]^{1/2}. \end{aligned} \quad (\text{S.147})$$

Given (S.144), it is easily seen that $E \left(\left| 1 + \frac{\bar{q}_v}{\sqrt{v}} \right|^{-10} \right) < C$. Also $E(\varepsilon_t^8) < C$ by assumption and $E(q_v^4) < C$ by (S.126). Hence, using these results in (S.147) it follows $E(|R_{e,v}|) < C$, which completes the proof of (S.136). To establish (S.137), using (S.141) note that for $t \neq t'$,

$$E \left[\varepsilon_t \varepsilon_{t'} \left(\frac{\boldsymbol{\varepsilon}' \mathbf{M}_F \boldsymbol{\varepsilon}}{v} \right)^{-1/2} \right] = E(\varepsilon_t \varepsilon_{t'}) - \frac{1}{2} \frac{E(q_v \varepsilon_t \varepsilon_{t'})}{\sqrt{v}} + \frac{3}{8v} E(R_{tt',v})$$

where

$$R_{tt',v} = \left(1 + \frac{\bar{q}_v}{\sqrt{v}} \right)^{-5/2} q_v^2 \varepsilon_t \varepsilon_{t'}.$$

Note that $E(\varepsilon_t \varepsilon_{t'}) = 0$ for $t \neq t'$ by serial independence of ε_t . In addition, using definition of q_v in (S.140) yields

$$\begin{aligned} \frac{E(q_v \varepsilon_t \varepsilon_{t'})}{\sqrt{v}} &= E \left[\left(\frac{\boldsymbol{\varepsilon}' \mathbf{M}_F \boldsymbol{\varepsilon}}{v} - 1 \right) \varepsilon_t \varepsilon_{t'} \right] \\ &= E \left[\left(\frac{\boldsymbol{\varepsilon}' \mathbf{M}_F \boldsymbol{\varepsilon}}{v} \right) \varepsilon_t \varepsilon_{t'} \right] = \frac{1}{2v} E[(\boldsymbol{\varepsilon}' \mathbf{A} \boldsymbol{\varepsilon})(\boldsymbol{\varepsilon}' \mathbf{B} \boldsymbol{\varepsilon})], \end{aligned}$$

where $\mathbf{A} = \mathbf{M}_F$ and $\mathbf{B} = (b_{tt'})$ with $b_{tt'}$ and $b_{t't}$ ($t \neq t'$) being the only non-zero elements. Now using (S.7) of Lemma 6 in Pesaran and Yamagata (2024) it follows that $E(q_v \varepsilon_t \varepsilon_{t'}) = 0$. Also by Cauchy-Schwarz inequality ,

$$\begin{aligned} E(|R_{tt',v}|) &\leq \left[E \left[\varepsilon_t^2 \varepsilon_{t'}^2 \left(\left| 1 + \frac{\bar{q}_v}{\sqrt{v}} \right|^{-5} \right) \right] \right]^{1/2} [E(q_v^4)]^{1/2} \\ &\leq \left[E \left(\left| 1 + \frac{\bar{q}_v}{\sqrt{v}} \right|^{-10} \right) \right]^{1/4} [E(\varepsilon_t^4) E(\varepsilon_{t'}^4)]^{1/4} [E(q_v^4)]^{1/2} < C, \end{aligned}$$

where the final inequality follows using the same line of argument used to bound $E(|R_{e,v}|)$ in (S.147). Overall, result (S.137) is established. Finally consider (S.138) and note that

$$E \left[\varepsilon_t \left(\frac{\boldsymbol{\varepsilon}' \mathbf{M}_F \boldsymbol{\varepsilon}}{v} \right)^{-1/2} \right] = E(\varepsilon_t) - \frac{1}{2} \frac{E(q_v \varepsilon_t)}{\sqrt{v}} + \frac{3}{8v} E(R_{t,v}),$$

where

$$R_{t,v} = \left(1 + \frac{\bar{q}_v}{\sqrt{v}} \right)^{-5/2} q_v^2 \varepsilon_t.$$

We have $E(\varepsilon_t) = 0$ and

$$\frac{E(q_v \varepsilon_t)}{\sqrt{v}} = E \left(\frac{\boldsymbol{\varepsilon}' \mathbf{M}_F \boldsymbol{\varepsilon} \varepsilon_t}{v} - \varepsilon_t \right) = E \left(\frac{\boldsymbol{\varepsilon}' \mathbf{M}_F \boldsymbol{\varepsilon} \varepsilon_t}{v} \right).$$

Denote $\{m_{jj'} : j, j' = 1, 2, \dots, T\}$ as the element of $\mathbf{M}_F$, such that based on part (c) of Assumption 2 $m_{jj'}$ is independent from ε_t and $E(m_{jj'}) = O(1)$. Then it follows that

$$\begin{aligned} E \left(\frac{\boldsymbol{\varepsilon}' \mathbf{M}_F \boldsymbol{\varepsilon} \varepsilon_t}{v} \right) &= \frac{1}{v} \sum_{j=1}^v \sum_{j'=1}^v E(\varepsilon_j m_{jj'} \varepsilon_{j'} \varepsilon_t) = \frac{1}{v} \sum_{j=1}^v \sum_{j'=1}^v E(\varepsilon_j \varepsilon_{j'} \varepsilon_t) E(m_{jj'}) = \frac{1}{v} E(\varepsilon_t^3) E(m_{tt}) \\ &= O \left(\frac{1}{v} \right), \end{aligned}$$

where the second equation holds due to the independence of ε_t and $m_{jj'}$, while the third equation holds due to the serial independence of ε_t . Besides by Cauchy-Schwarz inequality,

$$\begin{aligned} E(|R_{t,v}|) &\leq \left[E \left[\varepsilon_t^2 \left(\left| 1 + \frac{\bar{q}_v}{\sqrt{v}} \right|^{-5} \right) \right] \right]^{1/2} [E(q_v^4)]^{1/2} \\ &\leq \left[E \left(\left| 1 + \frac{\bar{q}_v}{\sqrt{v}} \right|^{-10} \right) \right]^{1/4} [E(\varepsilon_t^4)]^{1/4} [E(q_v^4)]^{1/2} < C, \end{aligned}$$

where the last inequality holds again using the same line of argument used to bound $E(|R_{e,v}|)$ in (S.147). Overall, result (S.138) is established. ■

Lemma S.15 *Consider the latent factor model given by (1) and (2). $\widetilde{CD}$ and CD statistics are defined by (S.2) and (S.39). Suppose that Assumptions 1-4 hold and $(n, T) \rightarrow \infty$, such that $n/T \rightarrow \kappa$, for $0 < \kappa < \infty$. Then*

$$CD = \widetilde{CD} + o_p(1). \quad (\text{S.148})$$

Proof. Using (S.2) and (S.39) we first note that

$$\left(\sqrt{\frac{2(n-1)}{n}} \right) (CD - \widetilde{CD}) = \frac{1}{\sqrt{T}} \sum_{t=1}^T \left[\left(\frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{\hat{u}_{it}}{\hat{\sigma}_{i,T}} \right)^2 - \left(\frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{\hat{u}_{it}}{\omega_{i,T}} \right)^2 \right]. \quad (\text{S.149})$$

Also note that

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{\hat{u}_{it}}{\hat{\sigma}_{i,T}} = h_{t,nT} + g_{t,nT} \quad (\text{S.150})$$

where (also see (S.7))

$$h_{t,nT} = \frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{\hat{u}_{it}}{\omega_{i,T}} = \frac{\mathbf{c}'_{nT} \hat{\mathbf{u}}_{ot}}{\sqrt{n}}, \text{ and } g_{t,nT} = \frac{1}{\sqrt{n}} \sum_{i=1}^n \hat{u}_{it} \left(\frac{1}{\hat{\sigma}_{i,T}} - \frac{1}{\omega_{i,T}} \right) = \frac{\mathbf{d}'_{nT} \hat{\mathbf{u}}_{ot}}{\sqrt{n}}$$

$\hat{\mathbf{u}}_{ot} = (\hat{u}_{1t}, \hat{u}_{2t}, \dots, \hat{u}_{nt})'$, $\mathbf{c}_{nT} = (\omega_{1,T}^{-1}, \omega_{2,T}^{-1}, \dots, \omega_{n,T}^{-1})'$, $\mathbf{d}_{nT} = (d_{1T}, d_{2T}, \dots, d_{nT})'$, and $d_{iT} = \hat{\sigma}_{i,T}^{-1} - \omega_{i,T}^{-1}$. Then squaring both sides of (S.150) and using the result in (S.149) we have

$$\begin{aligned} \left(\sqrt{\frac{2(n-1)}{n}} \right) (CD - \widetilde{CD}) &= \frac{1}{\sqrt{T}} \sum_{t=1}^T g_{t,nT}^2 + \frac{2}{\sqrt{T}} \sum_{t=1}^T h_{t,nT} g_{t,nT} \\ &= \sqrt{\frac{T}{n}} \left(\frac{1}{\sqrt{n}} \mathbf{d}'_{nT} \hat{\mathbf{V}}_T \mathbf{d}_{nT} + \frac{2}{\sqrt{n}} \mathbf{c}'_{nT} \hat{\mathbf{V}}_T \mathbf{d}_{nT} \right), \end{aligned} \quad (\text{S.151})$$

where $\hat{\mathbf{V}}_T = T^{-1} \sum_{t=1}^T \hat{\mathbf{u}}_{ot} \hat{\mathbf{u}}'_{ot}$. Now using (S.4), the error vector $\hat{\mathbf{u}}_{ot}$ can be written as

$$\hat{\mathbf{u}}_{ot} = \mathbf{u}_{ot}(\lambda_T) - \Gamma(\hat{\mathbf{f}}_t - \mathbf{f}_t) - (\hat{\Gamma} - \Gamma)\mathbf{f}_t - (\hat{\Gamma} - \Gamma)(\hat{\mathbf{f}}_t - \mathbf{f}_t).$$

where $\mathbf{u}_{ot}(\lambda_T) = (\sigma_1 \varepsilon_{1t}(\lambda_T), \sigma_2 \varepsilon_{2t}(\lambda_T), \dots, \sigma_n \varepsilon_{nt}(\lambda_T))'$. Using this expression we now have

$$\begin{aligned}\hat{\mathbf{V}}_T &= T^{-1} \sum_{t=1}^T \mathbf{u}_{ot}(\lambda_T) \mathbf{u}'_{ot}(\lambda_T) + \mathbf{\Gamma} \left[T^{-1} \sum_{t=1}^T (\hat{\mathbf{f}}_t - \mathbf{f}_t) (\hat{\mathbf{f}}_t - \mathbf{f}_t)' \right] \mathbf{\Gamma}' \\ &\quad + (\hat{\mathbf{\Gamma}} - \mathbf{\Gamma}) \left(T^{-1} \sum_{t=1}^T \mathbf{f}_t \mathbf{f}_t' \right) (\hat{\mathbf{\Gamma}} - \mathbf{\Gamma})' + (\hat{\mathbf{\Gamma}} - \mathbf{\Gamma}) \left[T^{-1} \sum_{t=1}^T (\hat{\mathbf{f}}_t - \mathbf{f}_t) (\hat{\mathbf{f}}_t - \mathbf{f}_t)' \right] (\hat{\mathbf{\Gamma}} - \mathbf{\Gamma})' \mathbf{\Gamma}' \\ &\quad - \left[T^{-1} \sum_{t=1}^T \mathbf{u}_{ot}(\lambda_T) (\hat{\mathbf{f}}_t - \mathbf{f}_t)' \right] - \left[T^{-1} \sum_{t=1}^T \mathbf{u}_{ot}(\lambda_T) \mathbf{f}_t' \right] (\hat{\mathbf{\Gamma}} - \mathbf{\Gamma})' \\ &\quad - \left[T^{-1} \sum_{t=1}^T \mathbf{u}_{ot}(\lambda_T) (\hat{\mathbf{f}}_t - \mathbf{f}_t)' \right] (\hat{\mathbf{\Gamma}} - \mathbf{\Gamma})' + \mathbf{\Gamma} \left[T^{-1} \sum_{t=1}^T (\hat{\mathbf{f}}_t - \mathbf{f}_t) \mathbf{f}_t' \right] (\hat{\mathbf{\Gamma}} - \mathbf{\Gamma})' \\ &\quad + \mathbf{\Gamma} \left[T^{-1} \sum_{t=1}^T (\hat{\mathbf{f}}_t - \mathbf{f}_t) (\hat{\mathbf{f}}_t - \mathbf{f}_t)' \right] (\hat{\mathbf{\Gamma}} - \mathbf{\Gamma})' + (\hat{\mathbf{\Gamma}} - \mathbf{\Gamma}) \left[T^{-1} \sum_{t=1}^T \mathbf{f}_t (\hat{\mathbf{f}}_t - \mathbf{f}_t)' \right] (\hat{\mathbf{\Gamma}} - \mathbf{\Gamma})',\end{aligned}$$

or in matrix forms

$$\begin{aligned}\hat{\mathbf{V}}_T &= \mathbf{V}_T(\lambda_T) + \mathbf{\Gamma} \left[T^{-1} (\hat{\mathbf{F}} - \mathbf{F})' (\hat{\mathbf{F}} - \mathbf{F}) \right] \mathbf{\Gamma}' + (\hat{\mathbf{\Gamma}} - \mathbf{\Gamma}) \mathbf{\Sigma}_{T,ff} (\hat{\mathbf{\Gamma}} - \mathbf{\Gamma})' \\ &\quad + (\hat{\mathbf{\Gamma}} - \mathbf{\Gamma}) \left[T^{-1} (\hat{\mathbf{F}} - \mathbf{F})' (\hat{\mathbf{F}} - \mathbf{F}) \right] (\hat{\mathbf{\Gamma}} - \mathbf{\Gamma})' - T^{-1} \mathbf{U}'(\lambda_T) (\hat{\mathbf{F}} - \mathbf{F}) \mathbf{\Gamma}' \\ &\quad - T^{-1} \mathbf{U}'(\lambda_T) \mathbf{F} (\hat{\mathbf{\Gamma}} - \mathbf{\Gamma})' - T^{-1} \mathbf{U}'(\lambda_T) (\hat{\mathbf{F}} - \mathbf{F}) (\hat{\mathbf{\Gamma}} - \mathbf{\Gamma})' + \mathbf{\Gamma} \left[T^{-1} \mathbf{F}' (\hat{\mathbf{F}} - \mathbf{F}) \right] (\hat{\mathbf{\Gamma}} - \mathbf{\Gamma})' \\ &\quad + \mathbf{\Gamma} \left[T^{-1} (\hat{\mathbf{F}} - \mathbf{F})' (\hat{\mathbf{F}} - \mathbf{F}) \right] (\hat{\mathbf{\Gamma}} - \mathbf{\Gamma})' + (\hat{\mathbf{\Gamma}} - \mathbf{\Gamma}) \left[T^{-1} \mathbf{F}' (\hat{\mathbf{F}} - \mathbf{F}) \right] (\hat{\mathbf{\Gamma}} - \mathbf{\Gamma})'\end{aligned}$$

where $\mathbf{V}_T(\lambda_T) = T^{-1} \sum_{t=1}^T \mathbf{u}_{ot}(\lambda_T) \mathbf{u}'_{ot}(\lambda_T)$ and $\mathbf{U}(\lambda_T) = (\mathbf{u}_{o1}(\lambda_T), \mathbf{u}_{o2}(\lambda_T), \dots, \mathbf{u}_{on}(\lambda_T))'$. By (S.112) we have $\|\mathbf{V}_T(\lambda_T)\| = \mu_{\max}(\mathbf{V}_T(\lambda_T)) = O_p(\frac{n}{T})$. Also by results in Lemma S.2 all other terms of the $\hat{\mathbf{V}}_T$ are either $O_p(1)$ or of lower order, and we also have $\hat{\mathbf{V}}_T = O_p(1)$ since n and T are of the same order. Consider now the terms in (S.151) and note that

$$\left(\sqrt{\frac{2(n-1)}{n}} \right) |CD - \widetilde{CD}| < C \|\hat{\mathbf{V}}_T\| \left[\left(\frac{1}{\sqrt{n}} \|\mathbf{d}_{nT}\|^2 \right) + \left(\frac{2}{\sqrt{n}} \|\mathbf{c}_{nT}\| \right) \|\mathbf{d}_{nT}\| \right].$$

But

$$\begin{aligned}\frac{1}{\sqrt{n}} \|\mathbf{c}_{nT}\| &= \left(n^{-1} \sum_{i=1}^n \omega_{i,T}^{-2} \right)^{1/2}, \\ \|\mathbf{d}_{nT}\| &= \left(\sum_{i=1}^n (\hat{\sigma}_{i,T}^{-1} - \omega_{i,T}^{-1})^2 \right)^{1/2}.\end{aligned}$$

By part (c) of Assumption 2 $E(\omega_{i,T}^{-2}) < C < \infty$, such that

$$\frac{1}{n} E \|\mathbf{c}_{nT}\|^2 = n^{-1} \sum_{i=1}^n E(\omega_{i,T}^{-2}) < C$$

and hence $n^{-1/2} \|\mathbf{c}_{nT}\| = O_p(1)$. Also by (S.82) we have

$$\sup_i |\hat{\sigma}_{i,T}^{-1} - \omega_{i,T}^{-1}|^2 = \left(\sup_i |\hat{\sigma}_{i,T}^{-1} - \omega_{i,T}^{-1}| \right)^2 = O_p \left[\left(\frac{\ln(n)}{T} \right)^2 \right]$$

therefore

$$\|\mathbf{d}_{nT}\|^2 = \sum_{i=1}^n (\hat{\sigma}_{i,T}^{-1} - \omega_{i,T}^{-1})^2 \leq n \sup_i (\hat{\sigma}_{i,T}^{-1} - \omega_{i,T}^{-1})^2 = O_p \left[n \left(\frac{\ln(n)}{T} \right)^2 \right] = o_p(1), \quad (\text{S.152})$$

recalling that n and T are of the same order. Hence, $|CD - \widetilde{CD}| = o_p(1)$, as required. ■

Lemma S.16 *Suppose the data are generated by the latent factor model given by (1) and (2). The latent factors, $\mathbf{f}_t$, and their loadings, $\boldsymbol{\gamma}_i$, are estimated by principal components, $\hat{\mathbf{f}}_t$ and $\hat{\boldsymbol{\gamma}}_i$, given by (12). Suppose that Assumptions 1-4 hold and $(n, T) \rightarrow \infty$, such that $n/T \rightarrow \kappa$, for $0 < \kappa < \infty$. Then*

$$\frac{1}{n} \sum_{i=1}^n \frac{\hat{\boldsymbol{\gamma}}_i - \boldsymbol{\gamma}_i}{\sigma_i} = O_p \left(\sqrt{\frac{\ln(n)}{nT}} \right), \quad (\text{S.153})$$

$$\frac{1}{n} \sum_{i=1}^n (\hat{\boldsymbol{\gamma}}_i - \boldsymbol{\gamma}_i) \sigma_i = O_p \left(\sqrt{\frac{\ln(n)}{nT}} \right), \quad (\text{S.154})$$

$$\frac{1}{n} \sum_{i=1}^n \sigma_i^2 (\hat{\boldsymbol{\gamma}}_i \hat{\boldsymbol{\gamma}}_i' - \boldsymbol{\gamma}_i \boldsymbol{\gamma}_i') = O_p \left(\sqrt{\frac{\ln(n)}{nT}} \right), \quad (\text{S.155})$$

$$\frac{1}{n} \sum_{i=1}^n \hat{\sigma}_{i,T} \hat{\boldsymbol{\gamma}}_i - \frac{1}{n} \sum_{i=1}^n \sigma_i \boldsymbol{\gamma}_i = O_p \left(\frac{\ln(n)}{T} \right). \quad (\text{S.156})$$

Proof. Results (S.153) and (S.154) follow directly from (S.93) by setting $b_{in} = \sigma_i^{-1}$ and $b_{in} = \sigma_i$, respectively. To prove (S.155), note that

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^n \sigma_i^2 (\hat{\boldsymbol{\gamma}}_i \hat{\boldsymbol{\gamma}}_i' - \boldsymbol{\gamma}_i \boldsymbol{\gamma}_i') &= \frac{1}{n} \sum_{i=1}^n \sigma_i^2 (\hat{\boldsymbol{\gamma}}_i - \boldsymbol{\gamma}_i) (\hat{\boldsymbol{\gamma}}_i - \boldsymbol{\gamma}_i)' \\ &\quad + \frac{1}{n} \sum_{i=1}^n \sigma_i^2 (\hat{\boldsymbol{\gamma}}_i - \boldsymbol{\gamma}_i) \boldsymbol{\gamma}_i' + \frac{1}{n} \sum_{i=1}^n \sigma_i^2 \boldsymbol{\gamma}_i (\hat{\boldsymbol{\gamma}}_i - \boldsymbol{\gamma}_i)'. \end{aligned} \quad (\text{S.157})$$

Since σ_i^2 is bounded, then

$$\begin{aligned} \left\| \frac{1}{n} \sum_{i=1}^n \sigma_i^2 (\hat{\boldsymbol{\gamma}}_i - \boldsymbol{\gamma}_i) (\hat{\boldsymbol{\gamma}}_i - \boldsymbol{\gamma}_i)' \right\| &\leq \left(\sup_i \sigma_i^2 \right) \left\| \frac{1}{n} \sum_{i=1}^n (\hat{\boldsymbol{\gamma}}_i - \boldsymbol{\gamma}_i) (\hat{\boldsymbol{\gamma}}_i - \boldsymbol{\gamma}_i)' \right\| \\ &\leq \left(\sup_i \sigma_i^2 \right) \left(\frac{1}{n} \sum_{i=1}^n \|\hat{\boldsymbol{\gamma}}_i - \boldsymbol{\gamma}_i\|^2 \right), \end{aligned}$$

and using (S.42) it follows that

$$\frac{1}{n} \sum_{i=1}^n \sigma_i^2 (\hat{\boldsymbol{\gamma}}_i - \boldsymbol{\gamma}_i) (\hat{\boldsymbol{\gamma}}_i - \boldsymbol{\gamma}_i)' = O_p \left(\frac{1}{\delta_{nT}^2} \right).$$

Now using (S.95), setting $b_{in} = \sigma_i$, we have

$$\left\| \frac{1}{n} \sum_{i=1}^n \sigma_i^2 (\hat{\gamma}_i - \gamma_i) \gamma_i' \right\| = \left\| \frac{1}{n} \sum_{i=1}^n \sigma_i^2 \gamma_i (\hat{\gamma}_i - \gamma_i)' \right\| = O_p \left(\sqrt{\frac{\ln(n)}{nT}} \right),$$

and (S.155) follows. Finally, consider (S.156) and note that

$$\begin{aligned} & \frac{1}{n} \sum_{i=1}^n \hat{\sigma}_{i,T} \hat{\gamma}_i - \frac{1}{n} \sum_{i=1}^n \sigma_i \gamma_i \\ &= \frac{1}{n} \sum_{i=1}^n [(\hat{\sigma}_{i,T} - \omega_{i,T}) + \omega_{i,T}] (\hat{\gamma}_i - \gamma_i + \gamma_i) - \frac{1}{n} \sum_{i=1}^n \sigma_i \gamma_i \\ &= \frac{1}{n} \sum_{i=1}^n \gamma_i (\omega_{i,T} - \sigma_i) + \frac{1}{n} \sum_{i=1}^n \gamma_i (\hat{\sigma}_{i,T} - \omega_{i,T}) + \frac{1}{n} \sum_{i=1}^n \sigma_i (\hat{\gamma}_i - \gamma_i) \\ &+ \frac{1}{n} \sum_{i=1}^n (\omega_{i,T} - \sigma_i) (\hat{\gamma}_i - \gamma_i) + \frac{1}{n} \sum_{i=1}^n (\hat{\sigma}_{i,T} - \omega_{i,T}) (\hat{\gamma}_i - \gamma_i) \\ &= \mathbf{A}_{1,nT} + \mathbf{A}_{2,nT} + \mathbf{A}_{3,nT} + \mathbf{A}_{4,nT} + \mathbf{A}_{5,nT}. \end{aligned} \tag{S.158}$$

Recall also that under Assumptions 2 and 3 σ_i and γ_i are bounded and $\omega_{i,T}^2 = T^{-1} \sigma_i^2 \boldsymbol{\epsilon}_{i0}' \mathbf{M}_F \boldsymbol{\epsilon}_{i0}$, for $i = 1, 2, \dots, n$ are distributed independently across i , and from σ_i and γ_i . Starting with $\mathbf{A}_{1,nT}$, and using (S.133) we have

$$E \left(\sqrt{nT} \mathbf{A}_{1,nT} \right) = \frac{\sqrt{nT}}{n} \sum_{i=1}^n (\gamma_i \sigma_i) E \left[\left(\frac{\boldsymbol{\epsilon}_{i0}' \mathbf{M}_F \boldsymbol{\epsilon}_{i0}}{T} \right)^{1/2} - 1 \right] = O \left(\frac{\sqrt{nT}}{T} \right).$$

Since n and T are assumed to be of the same order then $E \left(\sqrt{nT} \mathbf{A}_{1,nT} \right) = O(1)$. Also, using result (S.133)

$$Var \left(\sqrt{nT} \mathbf{A}_{1,nT} \right) = \frac{nT}{n^2} \sum_{i=1}^n (\sigma_i^2 \gamma_i \gamma_i') Var \left[\left(\frac{\boldsymbol{\epsilon}_{i0}' \mathbf{M}_F \boldsymbol{\epsilon}_{i0}}{T} \right)^{1/2} \right] = O(1).$$

Therefore, $\sqrt{nT} \mathbf{A}_{1,nT} = O_p(1)$ and it follows that $\mathbf{A}_{1,nT} = O_p \left[(nT)^{-1/2} \right]$. Further, using (S.84) setting $b_{in} = \gamma_{ij}$, for $j = 1, 2, \dots, m_0$, then it follows that

$$\mathbf{A}_{2,nT} = \frac{1}{n} \sum_{i=1}^n \gamma_i (\hat{\sigma}_{i,T} - \omega_{i,T}) = O_p \left(\frac{\ln(n)}{T} \right).$$

Since $\mathbf{A}_{3,nT}$ is the same as the result in (S.154), which is already established, then $\mathbf{A}_{3,nT} = O_p \left(\sqrt{\ln(n) / (nT)} \right)$. Using result (S.96) it follows that

$$\mathbf{A}_{4,nT} = \frac{1}{n} \sum_{i=1}^n (\omega_{i,T} - \sigma_i) (\hat{\gamma}_i - \gamma_i) = O_p \left(\frac{1}{\delta_{nT}^2} \right).$$

Using result (S.98) we have

$$\mathbf{A}_{5,nT} = \frac{1}{n} \sum_{i=1}^n (\hat{\sigma}_{i,T} - \omega_{i,T}) (\hat{\gamma}_i - \gamma_i) = O_p \left[\left(\frac{\ln(n)}{T} \right)^{3/2} \right].$$

Result (S.156) now follows straightforwardly. ■

Lemma S.17 *Suppose the data are generated by the latent factor model given by (1) and (2). Denote $\boldsymbol{\varphi}_n = n^{-1} \sum_{i=1}^n \gamma_i / \sigma_i$, $\boldsymbol{\varphi}_{nT} = n^{-1} \sum_{i=1}^n \gamma_i / \omega_{i,T}$ with $\omega_{i,T} = (T^{-1} \sigma_i^2 \boldsymbol{\epsilon}'_{i0} \mathbf{M}_F \boldsymbol{\epsilon}_{i0})^{1/2}$ where $\boldsymbol{\epsilon}_{i0} = (\varepsilon_{i1}, \varepsilon_{i2}, \dots, \varepsilon_{iT})'$ and $\mathbf{M}_F = \mathbf{I}_T - \mathbf{F}(\mathbf{F}'\mathbf{F})^{-1}\mathbf{F}'$. Suppose that Assumptions 1-4 hold and $(n, T) \rightarrow \infty$, such that $n/T \rightarrow \kappa$, for $0 < \kappa < \infty$. Then*

$$\begin{aligned} g_{2,nT}(\lambda_T) &= \frac{1}{\sqrt{T}} \sum_{t=1}^T v_{t,nT}^2(\lambda_T) = o_p(1), \\ g_{3,nT}(\lambda_T) &= \sqrt{T} (\boldsymbol{\varphi}_{nT} - \boldsymbol{\varphi}_n)' \left(\frac{\sum_{t=1}^T \boldsymbol{\kappa}_{t,n}(\lambda_T) \boldsymbol{\kappa}'_{t,n}(\lambda_T)}{T} \right) (\boldsymbol{\varphi}_{nT} - \boldsymbol{\varphi}_n) = o_p(1), \\ g_{4,nT}(\lambda_T) &= \sqrt{T} (\boldsymbol{\varphi}_{nT} - \boldsymbol{\varphi}_n)' \left(\frac{1}{T} \sum_{t=1}^T \boldsymbol{\kappa}_{t,n}(\lambda_T) v_{t,nT}(\lambda_T) \right) = o_p(1), \\ g_{5,nT}(\lambda_T) &= \sqrt{T} (\boldsymbol{\varphi}_{nT} - \boldsymbol{\varphi}_n)' \left(\frac{1}{T} \sum_{t=1}^T \boldsymbol{\kappa}_{t,n}(\lambda_T) \xi_{t,n}(\lambda_T) \right) = o_p(1), \end{aligned}$$

where

$$\begin{aligned} v_{t,nT}(\lambda_T) &= \frac{1}{\sqrt{n}} \sum_{i=1}^n \left(\frac{1}{(\boldsymbol{\epsilon}'_{i0} \mathbf{M}_F \boldsymbol{\epsilon}_{i0}/T)^{1/2}} - 1 \right) \varepsilon_{it}(\lambda_T), \\ \boldsymbol{\kappa}_{t,n}(\lambda_T) &= \frac{1}{\sqrt{n}} \sum_{i=1}^n \gamma_i \sigma_i \varepsilon_{it}(\lambda_T), \\ \xi_{t,n}(\lambda_T) &= \frac{1}{\sqrt{n}} \sum_{i=1}^n a_{i,n} \varepsilon_{it}(\lambda_T), \quad a_{i,n} = 1 - \sigma_i \boldsymbol{\varphi}_n' \gamma_i. \end{aligned}$$

Proof. Starting with the $g_{2,nT}(\lambda_T)$, note that

$$g_{2,nT}(\lambda_T) = \sqrt{T} \left(\frac{1}{T} \sum_{t=1}^T v_{t,nT}^2(\lambda_T) \right) \leq \sqrt{T} \left(\sup_t v_{t,nT}^2(\lambda_T) \right). \quad (\text{S.159})$$

Meanwhile, we have

$$|v_{t,nT}(\lambda_T)| \leq \frac{1}{\sqrt{n}} \sum_{i=1}^n \left| \left(\frac{1}{(\boldsymbol{\epsilon}'_{i0} \mathbf{M}_F \boldsymbol{\epsilon}_{i0}/T)^{1/2}} - 1 \right) \right| |\varepsilon_{it}(\lambda_T)|,$$

and

$$\sup_t |v_{t,nT}(\lambda_T)| \leq \left[\frac{1}{\sqrt{n}} \sum_{i=1}^n \left| \left(\frac{1}{(\boldsymbol{\epsilon}'_{i0} \mathbf{M}_F \boldsymbol{\epsilon}_{i0}/T)^{1/2}} - 1 \right) \right| \right] \left(\sup_{i,t} |\varepsilon_{it}(\lambda_T)| \right). \quad (\text{S.160})$$

Consider the first term of the product in (S.160) and note since $\boldsymbol{\varepsilon}_{i_o}$ is cross-sectionally independent conditional then

$$E \left[\frac{1}{\sqrt{n}} \sum_{i=1}^n \left| \left(\frac{1}{(\boldsymbol{\varepsilon}'_{i_o} \mathbf{M}_F \boldsymbol{\varepsilon}_{i_o} / T)^{1/2}} - 1 \right) \right| \right]^2 = \frac{1}{n} \sum_{i=1}^n E \left(\frac{1}{(\boldsymbol{\varepsilon}'_{i_o} \mathbf{M}_F \boldsymbol{\varepsilon}_{i_o} / T)^{1/2}} - 1 \right)^2.$$

Also using (S.133)

$$\begin{aligned} E \left(\frac{1}{(\boldsymbol{\varepsilon}'_{i_o} \mathbf{M}_F \boldsymbol{\varepsilon}_{i_o} / T)^{1/2}} - 1 \right)^2 &= E \left(\frac{1}{\boldsymbol{\varepsilon}'_{i_o} \mathbf{M}_F \boldsymbol{\varepsilon}_{i_o} / T} \right) - 2E \left(\frac{1}{(\boldsymbol{\varepsilon}'_{i_o} \mathbf{M}_F \boldsymbol{\varepsilon}_{i_o} / T)^{1/2}} \right) + 1 \\ &= \left[E \left(\frac{1}{\boldsymbol{\varepsilon}'_{i_o} \mathbf{M}_F \boldsymbol{\varepsilon}_{i_o} / T} \right) - 1 \right] - 2 \left[E \left(\frac{1}{(\boldsymbol{\varepsilon}'_{i_o} \mathbf{M}_F \boldsymbol{\varepsilon}_{i_o} / T)^{1/2}} \right) - 1 \right] \\ &= O \left(\frac{1}{T} \right). \end{aligned}$$

Therefore by Markov inequality, we have

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n \left| \left(\frac{1}{(\boldsymbol{\varepsilon}'_{i_o} \mathbf{M}_F \boldsymbol{\varepsilon}_{i_o} / T)^{1/2}} - 1 \right) \right| = O_p \left(\frac{1}{\sqrt{T}} \right). \quad (\text{S.161})$$

Now consider the second term of the product in (S.160) and note there exist $C_{\varepsilon,1}$, $C_{\varepsilon,2}$ and $r_\varepsilon > 0$ such that

$$\Pr \left(\sup_{i,t} |\varepsilon_{it}(\lambda_T)| > a_\varepsilon \right) \leq nT \sup_{i,t} \Pr (|\varepsilon_{it}(\lambda_T)| > a_\varepsilon) \leq nT C_{\varepsilon,1} \exp(-C_{\varepsilon,2} (a_\varepsilon)^{r_\varepsilon}).$$

Hence, by letting $a_\varepsilon = \Theta(\ln(nT))$, we have

$$\Pr \left(\sup_{i,t} |\varepsilon_{it}(\lambda_T)| > a_\varepsilon \right) \leq C_{\varepsilon,1} \exp(\ln(nT) - C_{\varepsilon,2} (a_\varepsilon)^{r_\varepsilon}) = O(1),$$

which further implies $\sup_{i,t} |\varepsilon_{it}(\lambda_T)| = O_p(\ln(nT))$. Invoking this result and (S.161) in (S.160) now yields

$$\sup_t |v_{t,nT}(\lambda_T)| = O_p \left(\frac{\ln(nT)}{\sqrt{T}} \right). \quad (\text{S.162})$$

Then consider (S.159) and it follows

$$g_{2,nT}(\lambda_T) \leq \sqrt{T} \left(\sup_t v_{t,nT}^2(\lambda_T) \right) \leq \sqrt{T} \left(\sup_t |v_{t,nT}(\lambda_T)| \right)^2 = O_p \left(\frac{[\ln(nT)]^2}{\sqrt{T}} \right) = o_p(1), \quad (\text{S.163})$$

as n and T are of the same order of magnitude. Consider $g_{3,nT}(\lambda_T)$ and note that the result of Lemma S.11 holds such that

$$\sqrt{T}(\boldsymbol{\varphi}_n - \boldsymbol{\varphi}_{nT}) = O_p(n^{-1/2}) + O_p(T^{-1/2}). \quad (\text{S.164})$$

Furthermore, we have

$$\begin{aligned}
E \|\boldsymbol{\kappa}_{t,n}(\lambda_T)\|^2 &= \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n \boldsymbol{\gamma}'_i \boldsymbol{\gamma}_j \sigma_i \sigma_j E(\varepsilon_{it}(\lambda_T) \varepsilon_{jt}(\lambda_T)) \\
&\leq \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n \|\boldsymbol{\gamma}_i\| \|\boldsymbol{\gamma}_j\| |\sigma_i \sigma_j| |E(\varepsilon_{it}(\lambda_T) \varepsilon_{jt}(\lambda_T))| \\
&\leq \left(\sup_i \|\boldsymbol{\gamma}_i\|^2 \right) \left(\sup_i \sigma_i^2 \right) \left[\frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n |E(\varepsilon_{it}(\lambda_T) \varepsilon_{jt}(\lambda_T))| \right].
\end{aligned}$$

Given (S.69) and boundedness of $\boldsymbol{\gamma}_i$ and σ_i , it follows

$$E \|\boldsymbol{\kappa}_{t,n}(\lambda_T)\|^2 \leq \left(\sup_i \|\boldsymbol{\gamma}_i\|^2 \right) \left(\sup_i \sigma_i^2 \right) \left[\frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n |E(\varepsilon_{it}(\lambda_T) \varepsilon_{jt}(\lambda_T))| \right] = O(1) \quad (\text{S.165})$$

and $\boldsymbol{\kappa}_{t,n}(\lambda_T)$ is therefore $O_p(1)$. In addition, it is clear that $\boldsymbol{\kappa}_{t,n}(\lambda_T)$ is serially independent, and we further have $T^{-1} \sum_{t=1}^T \boldsymbol{\kappa}_{t,n}(\lambda_T) \boldsymbol{\kappa}'_{t,n}(\lambda_T) = O_p(1)$. Using this result together with (S.164) we then have

$$g_{3,nT}(\lambda_T) = o_p(1). \quad (\text{S.166})$$

Next for $g_{4,nT}(\lambda_T)$, by Cauchy-Schwarz inequality, we have

$$\left\| \frac{1}{T} \sum_{t=1}^T \boldsymbol{\kappa}_{t,n}(\lambda_T) v_{t,nT}(\lambda_T) \right\| \leq \left(\frac{1}{T} \sum_{t=1}^T \|\boldsymbol{\kappa}_{t,n}(\lambda_T)\|^2 \right)^{1/2} \left(\frac{1}{T} \sum_{t=1}^T v_{t,nT}^2(\lambda_T) \right)^{1/2} = O_p\left(\frac{\ln(nT)}{\sqrt{T}}\right)$$

where the equation holds by (S.163) and (S.165). Then using the above results it also follows that

$$g_{4,nT}(\lambda_T) = \sqrt{T}(\boldsymbol{\varphi}_{nT} - \boldsymbol{\varphi}_n)' \left(\frac{1}{T} \sum_{t=1}^T \boldsymbol{\kappa}_{t,n}(\lambda_T) v_{t,nT}(\lambda_T) \right) = o_p(1). \quad (\text{S.167})$$

Similarly, note $\xi_{t,n}(\lambda_T) = n^{-1/2} \sum_{i=1}^n a_{i,n} \varepsilon_{it}(\lambda_T)$ and by Cauchy-Schwarz inequality,

$$\frac{1}{T} \sum_{t=1}^T \boldsymbol{\kappa}_{t,n}(\lambda_T) \xi_{t,n}(\lambda_T) \leq \left(\frac{1}{T} \sum_{t=1}^T \|\boldsymbol{\kappa}_{t,n}(\lambda_T)\|^2 \right)^{1/2} \left(\frac{1}{T} \sum_{t=1}^T \xi_{t,n}^2(\lambda_T) \right)^{1/2}$$

where given (S.69), $\sup_i a_{i,n}^2 < C$ and $\sup_i \sigma_i^{-2} < C$, it further follows

$$\begin{aligned}
E(\xi_{t,n}^2(\lambda_T)) &= \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n a_{i,n} a_{j,n} E(\varepsilon_{it}(\lambda_T) \varepsilon_{jt}(\lambda_T)) \\
&< \left(\sup_i a_{i,n}^2 \right) \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n |E(\varepsilon_{it}(\lambda_T) \varepsilon_{jt}(\lambda_T))| \\
&< C.
\end{aligned} \quad (\text{S.168})$$

Hence, $T^{-1} \sum_{t=1}^T \boldsymbol{\kappa}_{t,n}(\lambda_T) \xi_{t,n}(\lambda_T) = O_p(1)$, and again using (S.164) it follows that

$$g_{5,nT}(\lambda_T) = \sqrt{T}(\boldsymbol{\varphi}_{nT} - \boldsymbol{\varphi}_n)' \left(\frac{1}{T} \sum_{t=1}^T \boldsymbol{\kappa}_{t,n}(\lambda_T) \xi_{t,n}(\lambda_T) \right) = o_p(1). \quad (\text{S.169})$$

■

Lemma S.18 Suppose the data are generated by the latent factor model given by (1) and (2). Further denote

$$w_{nT}(\lambda_T) = \frac{T^{-1/2} \sum_{t=1}^T \xi_{t,n}(\lambda_T) v_{t,nT}(\lambda_T)}{T^{-1} \sum_{t=1}^T \xi_{t,n}^2(\lambda_T)},$$

where

$$\begin{aligned} \xi_{t,n}(\lambda_T) &= \frac{1}{\sqrt{n}} \sum_{i=1}^n a_{i,n} \varepsilon_{it}(\lambda_T), a_{i,n} = 1 - \sigma_i \boldsymbol{\varphi}'_n \boldsymbol{\gamma}_i, \\ v_{t,nT}(\lambda_T) &= \frac{1}{\sqrt{n}} \sum_{i=1}^n \zeta_{it} \varepsilon_{it}(\lambda_T), \zeta_{it} = \frac{1}{(T^{-1} \boldsymbol{\varepsilon}'_{i\circ} \mathbf{M}_F \boldsymbol{\varepsilon}_{i\circ})^{1/2}} - 1, \end{aligned}$$

with $\boldsymbol{\varphi}_n = n^{-1} \sum_{i=1}^n \boldsymbol{\gamma}_i / \sigma_i$, $\boldsymbol{\varepsilon}_{i\circ} = (\varepsilon_{i1}, \varepsilon_{i2}, \dots, \varepsilon_{iT})'$, $\mathbf{M}_F = \mathbf{I}_T - \mathbf{F}(\mathbf{F}'\mathbf{F})^{-1}\mathbf{F}'$. Suppose that Assumptions 1-4 hold and $(n, T) \rightarrow \infty$, such that $n/T \rightarrow \kappa$, and $0 < \kappa < \infty$. Then $w_{nT}(\lambda_T) = o_p(1)$.

Proof. Consider first the denominator of $w_{nT}(\lambda_T)$ and using (S.168) yields

$$E \left(\frac{1}{T} \sum_{t=1}^T \xi_{t,n}^2(\lambda_T) \right) = \frac{1}{T} \sum_{t=1}^T E(\xi_{t,n}^2(\lambda_T)) < C.$$

Hence, by Markov inequality it is obvious that the denominator of $w_{nT}(\lambda_T)$ is $O_p(1)$. Consider now the numerator of $w_{nT}(\lambda_T)$, which is denoted as $r_{nT}(\lambda_T)$. For simplicity, let

$$\xi_{t,n} = \frac{1}{\sqrt{n}} \sum_{i=1}^n a_{i,n} \varepsilon_{it}, v_{t,nT} = \frac{1}{\sqrt{n}} \sum_{i=1}^n \zeta_{it} \varepsilon_{it},$$

then

$$\begin{aligned} r_{nT}(\lambda_T) &= \frac{1}{\sqrt{T}} \sum_{t=1}^T \xi_{t,n} v_{t,nT} + \frac{1}{\sqrt{T}} \sum_{t=1}^T [\xi_{t,n}(\lambda_T) - \xi_{t,n}] v_{t,nT}(\lambda_T) \\ &\quad + \frac{1}{\sqrt{T}} \sum_{t=1}^T \xi_{t,n}(\lambda_T) [v_{t,nT}(\lambda_T) - v_{t,nT}] - \frac{1}{\sqrt{T}} \sum_{t=1}^T [\xi_{t,n}(\lambda_T) - \xi_{t,n}] [v_{t,nT}(\lambda_T) - v_{t,nT}] \\ &= r_{1,nT} + \sum_{j=2}^4 r_{j,nT}(\lambda_T). \end{aligned}$$

For the first term of $r_{nT}(\lambda_T)$, note

$$\xi_{t,n} v_{t,nT} = \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n a_{in} \varepsilon_{it} \zeta_{jt},$$

where by parts (a) and (c) of Assumption 2, ε_{it} and ζ_{jt} are distributed independently for $i \neq j$. Therefore,

$$E(\xi_{t,n} v_{t,nT}) = \frac{1}{n} \sum_{i=1}^n a_{in} E \left[\varepsilon_{it}^2 \left(\frac{1}{(T^{-1} \boldsymbol{\varepsilon}'_{i\circ} \mathbf{M}_F \boldsymbol{\varepsilon}_{i\circ})^{1/2}} - 1 \right) \right].$$

Furthermore, since a_{in} is bounded, and

$$E \left[\varepsilon_{it}^2 \left(\frac{1}{(T^{-1} \boldsymbol{\varepsilon}'_{io} \mathbf{M}_F \boldsymbol{\varepsilon}_{io})^{1/2}} - 1 \right) \right] = O \left(\frac{1}{T} \right)$$

based on result (S.136), then $E(\xi_{t,n} v_{t,nT}) = O(T^{-1})$. Using these results in $r_{1,nT}$ we have

$$E(r_{1,nT}) = \frac{1}{\sqrt{T}} \sum_{t=1}^T E(\xi_{t,n} v_{t,nT}) = O \left(\frac{1}{\sqrt{T}} \right). \quad (\text{S.170})$$

Also

$$\begin{aligned} \text{Var}(r_{1,nT}) &= E(r_{1,nT}^2) - [E(r_{1,nT})]^2 \\ &= \frac{1}{T} \sum_{t=1}^T \sum_{t'=1}^T E(\xi_{t,n} v_{t,nT} \xi_{t',n} v_{t',nT}) + O(T^{-1}). \end{aligned} \quad (\text{S.171})$$

Consider now the first term of $\text{Var}(r_{1,nT})$, and using (S.11), note that

$$\begin{aligned} &E(\xi_{t,n} v_{t,nT} \xi_{t',n} v_{t',nT}) \\ &\equiv E \left[\left(\frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n a_{in} a_{jn} \varepsilon_{it} \varepsilon_{jt'} \right) \left(\frac{1}{n} \sum_{r=1}^n \sum_{s=1}^n \zeta_{rt} \zeta_{st'} \right) \right] \\ &= \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n \sum_{r=1}^n \sum_{s=1}^n a_{in} a_{jn} E \left\{ \left(\frac{1}{(T^{-1} \boldsymbol{\varepsilon}'_{ro} \mathbf{M}_F \boldsymbol{\varepsilon}_{ro})^{1/2}} - 1 \right) \left(\frac{1}{(T^{-1} \boldsymbol{\varepsilon}'_{so} \mathbf{M}_F \boldsymbol{\varepsilon}_{so})^{1/2}} - 1 \right) \varepsilon_{it} \varepsilon_{jt'} \varepsilon_{rt} \varepsilon_{st'} \right\}. \end{aligned}$$

Since by part (a) of Assumption 2, ε'_{it} s are cross-sectionally independent and after some algebra we have

$$\frac{1}{T} \sum_{t=1}^T \sum_{t'=1}^T E(\xi_{t,n} v_{t,nT} \xi_{t',n} v_{t',nT}) = \sum_{l=1}^6 D_{l,nT}, \quad (\text{S.172})$$

where

$$\begin{aligned} D_{1,nT} &= \frac{1}{T n^2} \sum_{t=1}^T \sum_{t'=1}^T \sum_{i=1}^n a_{in}^2 E \left[\varepsilon_{it}^2 \varepsilon_{it'}^2 \left(\frac{1}{(T^{-1} \boldsymbol{\varepsilon}'_{io} \mathbf{M}_F \boldsymbol{\varepsilon}_{io})^{1/2}} - 1 \right)^2 \right], \\ D_{2,nT} &= \frac{1}{T n^2} \sum_{t=1}^T \sum_{t'=1}^T \sum_{i=1}^n \sum_{r \neq i}^n a_{in}^2 E \left[\varepsilon_{it} \varepsilon_{it'}^2 \left(\frac{1}{(T^{-1} \boldsymbol{\varepsilon}'_{io} \mathbf{M}_F \boldsymbol{\varepsilon}_{io})^{1/2}} - 1 \right) \right] \times \\ &\quad E \left[\left(\frac{1}{(T^{-1} \boldsymbol{\varepsilon}'_{ro} \mathbf{M}_F \boldsymbol{\varepsilon}_{ro})^{1/2}} - 1 \right) \varepsilon_{rt} \right], \\ D_{3,nT} &= \frac{1}{T n^2} \sum_{t=1}^T \sum_{t'=1}^T \sum_{i=1}^n \sum_{s \neq i}^n a_{in}^2 E \left[\varepsilon_{it}^2 \varepsilon_{it'} \left(\frac{1}{(T^{-1} \boldsymbol{\varepsilon}'_{io} \mathbf{M}_F \boldsymbol{\varepsilon}_{io})^{1/2}} - 1 \right) \right] \times \\ &\quad E \left[\left(\frac{1}{(T^{-1} \boldsymbol{\varepsilon}'_{so} \mathbf{M}_F \boldsymbol{\varepsilon}_{so})^{1/2}} - 1 \right) \varepsilon_{st'} \right], \end{aligned}$$

$$D_{4,nT} = \frac{1}{Tn^2} \sum_{t=1}^T \sum_{t'=1}^T \sum_{i=1}^n \sum_{r \neq i}^n a_{in}^2 E(\varepsilon_{it} \varepsilon_{it'}) E \left[\left(\frac{1}{(T^{-1} \boldsymbol{\varepsilon}'_{r\circ} \mathbf{M}_F \boldsymbol{\varepsilon}_{r\circ})^{1/2}} - 1 \right)^2 \varepsilon_{rt} \varepsilon_{rt'} \right],$$

$$D_{5,nT} = \frac{1}{Tn^2} \sum_{t=1}^T \sum_{t'=1}^T \sum_{i=1}^n \sum_{j \neq i}^n a_{in} a_{jn} E \left[\varepsilon_{it}^2 \left(\frac{1}{(T^{-1} \boldsymbol{\varepsilon}'_{i\circ} \mathbf{M}_F \boldsymbol{\varepsilon}_{i\circ})^{1/2}} - 1 \right) \right] \times \\ E \left[\varepsilon_{jt'}^2 \left(\frac{1}{(T^{-1} \boldsymbol{\varepsilon}'_{j\circ} \mathbf{M}_F \boldsymbol{\varepsilon}_{j\circ})^{1/2}} - 1 \right) \right],$$

and

$$D_{6,nT} = \frac{1}{Tn^2} \sum_{t=1}^T \sum_{t'=1}^T \sum_{i=1}^n \sum_{j \neq i}^n a_{in} a_{jn} E \left[\varepsilon_{it} \varepsilon_{it'} \left(\frac{1}{(T^{-1} \boldsymbol{\varepsilon}'_{i\circ} \mathbf{M}_F \boldsymbol{\varepsilon}_{i\circ})^{1/2}} - 1 \right) \right] \times \\ E \left[\varepsilon_{jt} \varepsilon_{jt'} \left(\frac{1}{(T^{-1} \boldsymbol{\varepsilon}'_{j\circ} \mathbf{M}_F \boldsymbol{\varepsilon}_{j\circ})^{1/2}} - 1 \right) \right].$$

For $D_{1,nT}$ we have

$$D_{1,nT} = \frac{1}{n^2} \sum_{i=1}^n a_{in}^2 \frac{1}{T} \sum_{t=1}^T E \left[\varepsilon_{it}^4 \left(\frac{1}{(T^{-1} \boldsymbol{\varepsilon}'_{i\circ} \mathbf{M}_F \boldsymbol{\varepsilon}_{i\circ})^{1/2}} - 1 \right)^2 \right] \\ + \frac{1}{n^2} \sum_{i=1}^n a_{in}^2 \frac{1}{T} \sum_{t' \neq t}^T E \left[\varepsilon_{it}^2 \varepsilon_{it'}^2 \left(\frac{1}{(T^{-1} \boldsymbol{\varepsilon}'_{i\circ} \mathbf{M}_F \boldsymbol{\varepsilon}_{i\circ})^{1/2}} - 1 \right)^2 \right] = D_{1,1,nT} + D_{1,2,nT}.$$

It is clear that $D_{1,1,nT} = O(n^{-1})$. Also by Cauchy-Schwarz inequality

$$E \left[\varepsilon_{it}^2 \varepsilon_{it'}^2 \left(\frac{1}{(T^{-1} \boldsymbol{\varepsilon}'_{i\circ} \mathbf{M}_F \boldsymbol{\varepsilon}_{i\circ})^{1/2}} - 1 \right)^2 \right] \leq [E(\varepsilon_{it}^4 \varepsilon_{it'}^4)]^{1/2} \left[E \left(\frac{1}{(T^{-1} \boldsymbol{\varepsilon}'_{i\circ} \mathbf{M}_F \boldsymbol{\varepsilon}_{i\circ})^{1/2}} - 1 \right)^4 \right]^{1/2},$$

where by part (a) of Assumption 2 we have $E(\varepsilon_{it}^4 \varepsilon_{it'}^4) = E(\varepsilon_{it}^4) E(\varepsilon_{it'}^4) = O(1)$. Furthermore, in view of (S.133)

$$E \left(\frac{1}{(T^{-1} \boldsymbol{\varepsilon}'_{i\circ} \mathbf{M}_F \boldsymbol{\varepsilon}_{i\circ})^{1/2}} - 1 \right)^4 = E \left[\frac{1}{(T^{-1} \boldsymbol{\varepsilon}'_{i\circ} \mathbf{M}_F \boldsymbol{\varepsilon}_{i\circ})^2} \right] - 4E \left[\frac{1}{(T^{-1} \boldsymbol{\varepsilon}'_{i\circ} \mathbf{M}_F \boldsymbol{\varepsilon}_{i\circ})^{3/2}} \right] + 6E \left[\frac{1}{T^{-1} \boldsymbol{\varepsilon}'_{i\circ} \mathbf{M}_F \boldsymbol{\varepsilon}_{i\circ}} \right] \\ - 4E \left[\frac{1}{(T^{-1} \boldsymbol{\varepsilon}'_{i\circ} \mathbf{M}_F \boldsymbol{\varepsilon}_{i\circ})^{1/2}} \right] + 1 = O \left(\frac{1}{T} \right).$$

Using this result it follows that $D_{1,2,nT} = O(\sqrt{T}n^{-1})$, and overall

$$D_{1,nT} = O(\sqrt{T}n^{-1}). \quad (\text{S.173})$$

Consider now $D_{2,nT}$ and note that by (S.138) we have

$$E \left[\left(\frac{1}{(T^{-1} \boldsymbol{\epsilon}'_{r_o} \mathbf{M}_F \boldsymbol{\epsilon}_{r_o})^{1/2}} - 1 \right) \varepsilon_{rt} \right] = E \left[\frac{\varepsilon_{rt}}{(T^{-1} \boldsymbol{\epsilon}'_{r_o} \mathbf{M}_F \boldsymbol{\epsilon}_{r_o})^{1/2}} \right] = O \left(\frac{1}{T} \right).$$

Also by Cauchy-Schwarz inequality,

$$\left| E \left[\varepsilon_{it} \varepsilon_{it'}^2 \left(\frac{1}{(T^{-1} \boldsymbol{\epsilon}'_{i_o} \mathbf{M}_F \boldsymbol{\epsilon}_{i_o})^{1/2}} - 1 \right) \right] \right| \leq [E (\varepsilon_{it'}^4)]^{1/2} \left[E \left[\varepsilon_{it}^2 \left(\frac{1}{(T^{-1} \boldsymbol{\epsilon}'_{i_o} \mathbf{M}_F \boldsymbol{\epsilon}_{i_o})^{1/2}} - 1 \right)^2 \right] \right]^{1/2},$$

which is $O(T^{-1/2})$, since $E(\varepsilon_{it'}^4) = O(1)$ by part (a) of Assumption 2 and given results (S.135) and (S.136)

$$\begin{aligned} E \left[\varepsilon_{it}^2 \left(\frac{1}{(T^{-1} \boldsymbol{\epsilon}'_{i_o} \mathbf{M}_F \boldsymbol{\epsilon}_{i_o})^{1/2}} - 1 \right)^2 \right] &= E \left[\frac{\varepsilon_{it}^2}{(T^{-1} \boldsymbol{\epsilon}'_{i_o} \mathbf{M}_F \boldsymbol{\epsilon}_{i_o})} \right] - 2E \left[\frac{\varepsilon_{it}^2}{(T^{-1} \boldsymbol{\epsilon}'_{i_o} \mathbf{M}_F \boldsymbol{\epsilon}_{i_o})^{1/2}} \right] + 1 \\ &= \left[E \left[\frac{\varepsilon_{it}^2}{(T^{-1} \boldsymbol{\epsilon}'_{i_o} \mathbf{M}_F \boldsymbol{\epsilon}_{i_o})} \right] - 1 \right] - 2 \left[E \left[\frac{\varepsilon_{it}^2}{(T^{-1} \boldsymbol{\epsilon}'_{i_o} \mathbf{M}_F \boldsymbol{\epsilon}_{i_o})^{1/2}} \right] - 1 \right] \\ &= O \left(\frac{1}{T} \right). \end{aligned} \quad (\text{S.174})$$

Using these results it now follows that

$$\begin{aligned} |D_{2,nT}| &\leq \frac{1}{Tn^2} \sum_{t=1}^T \sum_{t'=1}^T \sum_{i=1}^n \sum_{r \neq i}^n a_{in}^2 \left| E \left[\varepsilon_{it} \varepsilon_{it'}^2 \left(\frac{1}{(T^{-1} \boldsymbol{\epsilon}'_{i_o} \mathbf{M}_F \boldsymbol{\epsilon}_{i_o})^{1/2}} - 1 \right) \right] \right| \times \\ &\quad \left| E \left[\left(\frac{1}{(T^{-1} \boldsymbol{\epsilon}'_{r_o} \mathbf{M}_F \boldsymbol{\epsilon}_{r_o})^{1/2}} - 1 \right) \varepsilon_{rt} \right] \right| \\ &= O \left(\frac{1}{\sqrt{T}} \right). \end{aligned} \quad (\text{S.175})$$

Similarly, it follows that

$$D_{3,nT} = O(T^{-1/2}). \quad (\text{S.176})$$

Consider the fourth term of (S.172), and using the serial independence of ε_{it} ,

$$\begin{aligned} D_{4,nT} &= \frac{1}{Tn^2} \sum_{t=1}^T \sum_{t'=1}^T \sum_{i=1}^n \sum_{r \neq i}^n a_{in}^2 E(\varepsilon_{it} \varepsilon_{it'}) E \left[\left(\frac{1}{(T^{-1} \boldsymbol{\epsilon}'_{r_o} \mathbf{M}_F \boldsymbol{\epsilon}_{r_o})^{1/2}} - 1 \right)^2 \varepsilon_{rt} \varepsilon_{rt'} \right] \\ &= \frac{1}{Tn^2} \sum_{t=1}^T \sum_{i=1}^n \sum_{r \neq i}^n a_{in}^2 E(\varepsilon_{it}^2) E \left[\left(\frac{1}{(T^{-1} \boldsymbol{\epsilon}'_{r_o} \mathbf{M}_F \boldsymbol{\epsilon}_{r_o})^{1/2}} - 1 \right)^2 \varepsilon_{rt}^2 \right] \\ &= O \left(\frac{1}{T} \right) \end{aligned} \quad (\text{S.177})$$

in which the final equation holds given $E(\varepsilon_{it}^2) = 1$ by part (a) of Assumption 2 and (S.174). To derive the order of $D_{5,nT}$, using result (S.136) we have

$$E \left[\frac{\varepsilon_{it}^2}{(T^{-1} \boldsymbol{\varepsilon}'_{i0} \mathbf{M}_F \boldsymbol{\varepsilon}_{i0})^{1/2}} - \varepsilon_{it}^2 \right] = O \left(\frac{1}{T} \right). \quad (\text{S.178})$$

Also since $\varepsilon_{it}^2 \left(\frac{1}{(T^{-1} \boldsymbol{\varepsilon}'_{i0} \mathbf{M}_F \boldsymbol{\varepsilon}_{i0})^{1/2}} - 1 \right)$ is independently distributed across i , then

$$\begin{aligned} D_{5,nT} &= \frac{1}{Tn^2} \sum_{t=1}^T \sum_{t'=1}^T \sum_{i=1}^n \sum_{j \neq i}^n a_{in} a_{jn} E \left[\varepsilon_{it}^2 \left(\frac{1}{(T^{-1} \boldsymbol{\varepsilon}'_{i0} \mathbf{M}_F \boldsymbol{\varepsilon}_{i0})^{1/2}} - 1 \right) \right] E \left[\varepsilon_{jt'}^2 \left(\frac{1}{(T^{-1} \boldsymbol{\varepsilon}'_{j0} \mathbf{M}_F \boldsymbol{\varepsilon}_{j0})^{1/2}} - 1 \right) \right] \\ &= O \left(\frac{1}{T} \right). \end{aligned} \quad (\text{S.179})$$

To establish the order of $D_{6,nT}$, using (S.137) and the serial independence of ε_{it} we have

$$E \left[\frac{\varepsilon_{it} \varepsilon_{it'}}{(T^{-1} \boldsymbol{\varepsilon}'_{i0} \mathbf{M}_F \boldsymbol{\varepsilon}_{i0})^{1/2}} - \varepsilon_{it} \varepsilon_{it'} \right] = O \left(\frac{1}{T} \right), \text{ for } t \neq t'.$$

Using this result and (S.178) we have

$$\begin{aligned} D_{6,nT} &= \frac{1}{Tn^2} \sum_{t=1}^T \sum_{i=1}^n \sum_{j \neq i}^n a_{in} a_{jn} E \left[\varepsilon_{it}^2 \left(\frac{1}{(T^{-1} \boldsymbol{\varepsilon}'_{i0} \mathbf{M}_F \boldsymbol{\varepsilon}_{i0})^{1/2}} - 1 \right) \right] E \left[\varepsilon_{jt}^2 \left(\frac{1}{(T^{-1} \boldsymbol{\varepsilon}'_{j0} \mathbf{M}_F \boldsymbol{\varepsilon}_{j0})^{1/2}} - 1 \right) \right] \\ &\quad + \frac{1}{Tn^2} \sum_{t=1}^T \sum_{t' \neq t}^T \sum_{i=1}^n \sum_{j \neq i}^n a_{in} a_{jn} E \left[\varepsilon_{it} \varepsilon_{it'} \left(\frac{1}{(T^{-1} \boldsymbol{\varepsilon}'_{i0} \mathbf{M}_F \boldsymbol{\varepsilon}_{i0})^{1/2}} - 1 \right) \right] \times \\ &\quad E \left[\varepsilon_{jt} \varepsilon_{jt'} \left(\frac{1}{(T^{-1} \boldsymbol{\varepsilon}'_{j0} \mathbf{M}_F \boldsymbol{\varepsilon}_{j0})^{1/2}} - 1 \right) \right] \\ &= O \left(\frac{1}{T} \right). \end{aligned} \quad (\text{S.180})$$

Hence, using (S.173), (S.175), (S.176), (S.177), (S.179) and (S.180) in (S.172) and then in (S.171), we have $Var(r_{1,nT}) = O \left(\frac{1}{\sqrt{T}} \right) + O \left(\frac{\sqrt{T}}{n} \right)$, and by Chebyshv's inequality

$$r_{1,nT} = O \left(\frac{1}{\sqrt{T}} \right) + O_p \left(\frac{\sqrt{T}}{n} \right).$$

For the second term of $r_{nT}(\lambda_T)$, as $\xi_{t,n}(\lambda_T) = \frac{1}{\sqrt{n}} \sum_{i=1}^n a_{i,n} \varepsilon_{it}(\lambda_T)$ where $\varepsilon_{it}(\lambda_T) = \varepsilon_{it} + \lambda_T b_{it}$ with $\lambda_T = c_\lambda T^{-1/2}$ and $b_{it} = \sum_{s=1}^n w_{is} \varepsilon_{st}$, then

$$\begin{aligned} |r_{2,nT}(\lambda_T)| &\leq \frac{1}{\sqrt{T}} \sum_{t=1}^T |[\xi_{t,n}(\lambda_T) - \xi_{t,n}] v_{t,nT}(\lambda_T)| \\ &\leq \left(\frac{1}{\sqrt{T}} \sum_{t=1}^T [\xi_{t,n}(\lambda_T) - \xi_{t,n}] \right) \left(\sup_t |v_{t,nT}(\lambda_T)| \right) \\ &\leq \left(\frac{1}{T} \sum_{t=1}^T \left| \frac{c_\lambda}{\sqrt{n}} \sum_{i=1}^n a_{i,n} b_{it} \right| \right) \left(\sup_t |v_{t,nT}(\lambda_T)| \right). \end{aligned}$$

Since $\sup_t |v_{t,nT}(\lambda_T)| = O_p\left(\frac{\ln(nT)}{\sqrt{T}}\right)$ as shown by (S.162), while $c_\lambda n^{-1/2} \sum_{i=1}^n a_{i,n} b_{it}$ is serially independent and

$$\begin{aligned}
E \left(\frac{c_\lambda}{\sqrt{n}} \sum_{i=1}^n a_{i,n} b_{it} \right)^2 &= \frac{c_\lambda^2}{n} \sum_{i=1}^n \sum_{j=1}^n a_{i,n} a_{j,n} \mathbf{w}'_{i0} E(\boldsymbol{\varepsilon}_{ot} \boldsymbol{\varepsilon}'_{ot}) \mathbf{w}_{j0} \\
&= \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n a_{i,n} a_{j,n} \mathbf{w}'_{i0} \mathbf{w}_{j0} = \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n a_{i,n} a_{j,n} \sum_{s=1}^n w_{is} w_{js} \\
&= \frac{1}{n} \sum_{s=1}^n \sum_{i=1}^n \sum_{j=1}^n a_{i,n} w_{is} a_{j,n} w_{js} \\
&= \frac{1}{n} \sum_{s=1}^n \left(\sum_{i=1}^n a_{i,n} w_{is} \right)^2 < C,
\end{aligned} \tag{S.181}$$

then it follows

$$r_{2,nT}(\lambda_T) = O_p\left(\frac{\ln(nT)}{\sqrt{T}}\right) = o_p(1).$$

Now consider the third term of $r_{nT}(\lambda_T)$ and note

$$\begin{aligned}
|r_{3,nT}(\lambda_T)| &= \frac{1}{\sqrt{T}} \sum_{t=1}^T \xi_{t,n}(\lambda_T) [v_{t,nT}(\lambda_T) - v_{t,nT}] \\
&= \frac{1}{T} \sum_{t=1}^T \xi_{t,n}(\lambda_T) \left[\frac{c_\lambda}{\sqrt{n}} \sum_{i=1}^n \left(\frac{1}{(T^{-1} \boldsymbol{\varepsilon}'_{io} \mathbf{M}_F \boldsymbol{\varepsilon}_{io})^{1/2}} - 1 \right) b_{it} \right] \\
&\leq \left(\frac{1}{T} \sum_{t=1}^T |\xi_{t,n}(\lambda_T)| \right) \sup_t \left| \frac{c_\lambda}{\sqrt{n}} \sum_{i=1}^n \left(\frac{1}{(T^{-1} \boldsymbol{\varepsilon}'_{io} \mathbf{M}_F \boldsymbol{\varepsilon}_{io})^{1/2}} - 1 \right) b_{it} \right|.
\end{aligned}$$

Clearly, $\xi_{t,n}(\lambda_T)$ is serially independent and we have shown $\xi_{t,n}(\lambda_T) = O_p(1)$ by (S.168). Since by part (a) of Assumption 2 and Lemma S.7, b_{it} is sub-exponential, then $\sup_{i,t} |b_{it}| = O_p[\ln(nT)]$ and using (S.161) it follows that

$$\begin{aligned}
\sup_t \left| \frac{c_\lambda}{\sqrt{n}} \sum_{i=1}^n \left(\frac{1}{(T^{-1} \boldsymbol{\varepsilon}'_{io} \mathbf{M}_F \boldsymbol{\varepsilon}_{io})^{1/2}} - 1 \right) b_{it} \right| &\leq \left(\sup_{i,t} |b_{it}| \right) \left[\frac{c_\lambda}{\sqrt{n}} \sum_{i=1}^n \left| \frac{1}{(T^{-1} \boldsymbol{\varepsilon}'_{io} \mathbf{M}_F \boldsymbol{\varepsilon}_{io})^{1/2}} - 1 \right| \right] \\
&= O_p\left(\frac{\ln(nT)}{\sqrt{T}}\right).
\end{aligned} \tag{S.182}$$

Overall we have $r_{3,nT}(\lambda_T) = O_p\left(\frac{\ln(nT)}{\sqrt{T}}\right) = o_p(1)$. Now consider $r_{4,nT}(\lambda_T)$ and note that

$$\begin{aligned}
& \frac{1}{\sqrt{T}} \sum_{t=1}^T [\xi_{t,n}(\lambda_T) - \xi_{t,n}] [v_{t,nT}(\lambda_T) - v_{t,nT}] \\
& \leq \left(\frac{1}{\sqrt{T}} \sum_{t=1}^T [\xi_{t,n}(\lambda_T) - \xi_{t,n}]^2 \right)^{1/2} \left(\frac{1}{\sqrt{T}} \sum_{t=1}^T [v_{t,nT}(\lambda_T) - v_{t,nT}]^2 \right)^{1/2} \\
& = \left[\frac{1}{T\sqrt{T}} \sum_{t=1}^T \left| \frac{c_\lambda}{\sqrt{n}} \sum_{i=1}^n a_{i,n} b_{it} \right|^2 \right]^{1/2} \left[\frac{1}{T\sqrt{T}} \sum_{t=1}^T \left[\frac{1}{\sqrt{n}} \sum_{i=1}^n \left(\frac{1}{(T^{-1} \boldsymbol{\epsilon}'_{io} \mathbf{M}_F \boldsymbol{\epsilon}_{io})^{1/2}} - 1 \right) c_\lambda b_{it} \right]^2 \right]^{1/2} \\
& = o_p(1),
\end{aligned}$$

where the final line holds by (S.181) and (S.182). Overall, we have shown $r_{nT}(\lambda_T) = o_p(1)$ and now it follows $w_{nT}(\lambda_T)$ is $o_p(1)$. ■

S1.3 Derivation of θ_n in terms of factor strengths

Consider θ_n defined by (22), and note that it can be written as

$$\begin{aligned}
\theta_n &= 1 - \frac{1}{n} \sum_{i=1}^n a_{i,n}^2 = 1 - \frac{1}{n} \sum_{i=1}^n (1 - \sigma_i \boldsymbol{\varphi}'_n \boldsymbol{\gamma}_i)^2 \\
&= 2 \boldsymbol{\varphi}'_n \left(\frac{1}{n} \sum_{i=1}^n \sigma_i \boldsymbol{\gamma}_i \right) - \boldsymbol{\varphi}'_n \left(\frac{1}{n} \sum_{i=1}^n \sigma_i^2 \boldsymbol{\gamma}_i \boldsymbol{\gamma}'_i \right) \boldsymbol{\varphi}_n,
\end{aligned} \tag{S.183}$$

where $\boldsymbol{\varphi}_n = n^{-1} \sum_{i=1}^n \boldsymbol{\gamma}_i / \sigma_i$. Then

$$|\theta_n| \leq \sup_i (\sigma_i^2) \|\boldsymbol{\varphi}_n\|_1^2 \left(\frac{1}{n} \sum_{i=1}^n \|\boldsymbol{\gamma}_i\|_1^2 \right) + 2 \sup_i (\sigma_i) \|\boldsymbol{\varphi}_n\|_1 \left(\frac{1}{n} \sum_{i=1}^n \|\boldsymbol{\gamma}_i\|_1 \right).$$

$\|\boldsymbol{\gamma}_i\|_1 = \sum_{j=1}^{m_0} |\gamma_{ij}|$, and

$$\|\boldsymbol{\varphi}_n\|_1 \leq \inf_i (\sigma_i) \left(\frac{1}{n} \sum_{i=1}^n \|\boldsymbol{\gamma}_i\|_1 \right) = \inf_i (\sigma_i) \left[\sum_{j=1}^{m_0} \left(\frac{1}{n} \sum_{i=1}^n |\gamma_{ij}| \right) \right]. \tag{S.184}$$

Since by assumption $\inf_i (\sigma_i) > c > 0$, and $\sup_i (\sigma_i^2) < C < \infty$, then the order of $|\theta_n|$ is determined by $\sum_{j=1}^{m_0} \left(\frac{1}{n} \sum_{i=1}^n |\gamma_{ij}| \right)$, where m_0 is a fixed integer. Hence, $|\theta_n| = \Theta(n^{\alpha-1})$ as required, where $\alpha = \max_j (\alpha_j)$, and α_j is defined by $\sum_{i=1}^n |\gamma_{ij}| = \Theta(n^{\alpha_j})$. See (24).

S1.4 Juodis and Reese's CD_{W+} test

The CD_{W+} test, proposed by Juodis and Reese (2022, JR), is used in our Monte Carlo experiments and is defined by $CD_{W+} = CD_W + \Delta_{nT}$, where CD_W is the randomized component given by (49) and Δ_{nT} is the so-called screening component defined by (51) in the paper. Here we provide some theoretical insights regarding the size and power of the CD_{W+} test. First,

for the CD_{W+} test to have the correct size under $H_0 : \rho_{ij} = 0$, for all $i \neq j$, the screening component Δ_{nT} of the test given by (51) must converge to zero as n and $T \rightarrow \infty$, jointly. To our knowledge, the conditions under which this holds are not investigated by JR. Whilst it is beyond the scope of the present paper to investigate the limiting properties of Δ_{nT} in the case of a general factor model, using results presented in Bailey et al. (2019) (BPS), we will provide sufficient conditions for $\Delta_{nT} \rightarrow_p 0$ in the case of the simple null model given by $y_{it} = \mu_i + \sigma_i \varepsilon_{it}$. By the Cauchy-Schwarz inequality we first note that for all $i \neq j$,

$$\begin{aligned} & E \left[|\hat{\rho}_{ij,T}| I \left(|\hat{\rho}_{ij,T}| > 2\sqrt{\frac{\ln(n)}{T}} \right) | \rho_{ij} = 0 \right] \\ & \leq [E(|\hat{\rho}_{ij,T}|^2 | \rho_{ij} = 0)]^{1/2} \left[P \left(|\hat{\rho}_{ij,T}| > 2\sqrt{\frac{\ln(n)}{T}} | \rho_{ij} = 0 \right) \right]^{1/2}, \end{aligned} \quad (\text{S.185})$$

where $\rho_{ij} = E(\varepsilon_{it}\varepsilon_{jt})$. Hence

$$\begin{aligned} & E(\Delta_{nT} | \rho_{ij} = 0, \text{ for all } i \neq j) \\ & \leq \frac{n(n-1)}{2} \sup_{i \neq j} [E(|\hat{\rho}_{ij,T}|^2 | \rho_{ij} = 0)]^{1/2} \sup_{i \neq j} \left[P \left(|\hat{\rho}_{ij,T}| > 2\sqrt{\frac{\ln(n)}{T}} | \rho_{ij} = 0 \right) \right]^{1/2}. \end{aligned} \quad (\text{S.186})$$

Now using results (9) and (10) of BPS, we have

$$E[|\hat{\rho}_{ij,T}|^2 | \rho_{ij} = 0] = O\left(\frac{1}{T}\right), \quad (\text{S.187})$$

and using result (A.4) in the online supplement of BPS, we also have

$$\sup_{i \neq j} P \left[|\hat{\rho}_{ij,T}| > \frac{C_p(n, \delta)}{\sqrt{T}} | \rho_{ij} = 0 \right] = O \left(e^{-\frac{1}{2} \frac{C_p^2(n, \delta)}{\varphi_{\max}}} \right) + O \left(T^{-\frac{(s-1)}{2}} \right),$$

where $C_p(n, \delta) = \Phi^{-1}(1 - \frac{p}{2n\delta})$, $0 < p < 1$, $\Phi^{-1}(\cdot)$ is the inverse of the cumulative distribution of a standard normal variable, $\delta > 0$, $\varphi_{\max} = \sup_{i \neq j} E(\varepsilon_{it}^2 \varepsilon_{jt}^2)$, and s is such that $\sup_{i \neq j} E|\varepsilon_{it}|^{2s} < C$, for some integer $s \geq 3$ (see Assumption 2 of BPS). Also using results in Lemma 2 in the online supplement of BPS, we have

$$\lim_{n \rightarrow \infty} \frac{C_p^2(n, \delta)}{\ln(n)} = 2\delta, \text{ and } e^{-\frac{1}{2} \frac{C_p^2(n, \delta)}{\varphi_{\max}}} = O(n^{-\delta/\varphi_{\max}}).$$

Therefore, $T^{-1/2}C_p(n, \delta)$, and $2\sqrt{T^{-1}\ln(n)}$ have the same limiting properties if we set $\delta = 2$. Overall, it then follows that

$$\sup_{i \neq j} P \left(|\hat{\rho}_{ij,T}| > 2\sqrt{\frac{\ln(n)}{T}} | \rho_{ij} = 0 \right) = O \left(n^{-\frac{2}{\varphi_{\max}}} \right) + O \left(T^{-\frac{(s-1)}{2}} \right). \quad (\text{S.188})$$

Using (S.187) and (S.188) in (S.186), we now have

$$E(\Delta_{nT} | \rho_{ij} = 0, \text{ for all } i \neq j) = O \left(\frac{n^{2-\frac{1}{\varphi_{\max}}}}{\sqrt{T}} \right) + O \left(n^2 T^{-\frac{(s+1)}{4}} \right). \quad (\text{S.189})$$

Therefore, $\Delta_{nT} \rightarrow_p 0$, if $n^2 T^{-\frac{s+1}{4}} \rightarrow 0$ and $T^{-1/2} n^{2-\frac{1}{\varphi_{\max}}} \rightarrow 0$. It is easily seen that both of these conditions will be met as n and $T \rightarrow \infty$ and $n = o(\sqrt{T})$ if ε_{it} is Gaussian, since under Gaussian errors, $\varphi_{\max} = 1$ and s can be taken to be sufficiently large. But, in general the expansion rate of T relative to n required to ensure $\Delta_{nT} \rightarrow_p 0$ will also depend on the degree to which $E(\varepsilon_{it}^2 \varepsilon_{jt}^2)$ exceeds unity. For example, if ε_{it} has a multivariate t -distribution with degrees of freedom $v > 4$, then letting $T = n^d$, $d > 0$, and using results in Lemma 5 of BPS's online supplement, we have

$$\varphi_{\max} = \sup_{i \neq j} E(\varepsilon_{it}^2 \varepsilon_{jt}^2 | \rho_{ij} = 0) = \frac{v-2}{v-4}.$$

Hence, $E(\Delta_{nT} | \rho_{ij} = 0, \text{ for all } i \neq j)$ defined by (S.189) tends to 0 if $n^{2-\frac{v-4}{v-2}-d/2} \rightarrow 0$, or if $d > \frac{2v}{v-2}$. Assumption 1 of JR requires $E|\varepsilon_{it}|^{8+\epsilon} < C$, for some small positive ϵ , and for this to be satisfied in the case of t -distributed errors we need $v > 9$, which yields $d > 2$ when $v = 10$, requiring T to rise faster than n .

Finally, for the CD_{W+} test to have power it is also necessary to show that Δ_{nT} diverges in n and T sufficiently fast under alternative hypotheses of interest, namely spatial or network dependence. In Section 7, we provide some Monte Carlo evidence on this issue. We find the CD_{W+} test lacks power against network alternatives, in turn suggesting that Δ_{nT} need not diverge sufficiently fast under such alternatives. Our Monte Carlo experiments also show that CD_{W+} test tends to over-reject when $n \gg T$ and the errors are chi-squared distributed.

S2 Simulation results

This section provides Monte Carlo simulation results for the experiments discussed in Section 7 of the main paper. Tables S.1-S.4 report the results for the DGPs with serially independent errors. Tables S.5-S.12 report the results for the DGPs with serially correlated errors using variance adjustment. Tables S.13-S.20 report the results for the DGPs with serially correlated errors using ARDL adjustment. Figures S.1-S.4 report the simulated power functions for the DGPs with serially independent errors.

Table S.1: Size and power of tests of error cross-sectional dependence for the latent factor model with one factor ($m_0 = 1$) and serially independent non-Gaussian errors

$m = 1$		Power ($H_1 : \lambda = 0.25$)											
Tests	$n \setminus T$	Size ($H_0 : \lambda = 0$)						$\alpha = 1$					
		$\alpha = 1$			$\alpha = 2/3$			$\alpha = 1/2$			$\alpha = 2/3$		
		100	200	500	100	200	500	100	200	500	100	200	500
CD	100	64.5	86.4	97.7	5.3	8.9	23.1	6.1	6.6	8.0	23.5	37.1	57.5
	200	68.8	93.1	99.6	5.2	7.6	13.5	5.7	5.3	5.5	16.0	29.2	50.0
	500	69.1	94.0	100.0	4.8	5.8	9.8	5.8	6.0	5.4	11.1	23.8	46.5
	1000	69.4	95.4	100.0	4.9	5.0	6.9	6.0	5.0	5.1	10.0	19.1	43.8
CD^*	100	5.1	5.4	5.6	5.2	5.0	5.8	7.1	6.2	5.8	58.8	82.4	98.8
	200	3.8	4.4	4.4	5.8	5.4	5.1	5.5	5.3	5.0	58.4	81.4	99.3
	500	5.4	5.6	5.1	5.3	5.5	5.3	6.0	6.2	5.5	58.6	83.5	99.5
	1000	5.4	5.2	4.6	5.3	5.4	5.3	6.4	5.0	4.9	60.6	84.4	99.4
CD_{W+}	100	8.3	5.9	6.0	7.4	6.0	5.4	7.9	6.2	7.4	9.8	11.7	55.1
	200	8.6	5.8	6.0	9.4	7.5	5.6	9.7	6.2	5.3	10.8	11.7	66.9
	500	17.3	10.1	6.1	16.7	10.9	5.1	17.6	9.8	5.3	18.3	17.7	79.8
	1000	27.8	14.9	6.1	28.5	16.3	6.2	27.0	16.6	6.4	27.8	21.2	85.4
$m = 2$		Power ($H_1 : \lambda = 0.25$)											
Tests	$n \setminus T$	Size ($H_0 : \lambda = 0$)						$\alpha = 1$					
		$\alpha = 1$			$\alpha = 2/3$			$\alpha = 1/2$			$\alpha = 2/3$		
		100	200	500	100	200	500	100	200	500	100	200	500
CD	100	65.8	87.0	97.6	5.9	10.4	22.8	6.3	6.9	9.4	26.5	44.2	62.5
	200	68.2	93.2	99.6	5.2	6.9	13.9	5.2	5.1	5.9	18.5	32.6	55.7
	500	68.8	93.7	100.0	4.7	5.1	9.5	5.6	6.1	5.8	11.4	24.9	50.2
	1000	68.4	95.3	100.0	5.1	5.1	6.5	6.1	5.5	5.1	10.3	20.1	45.9
CD^*	100	6.2	5.9	7.2	5.9	5.5	5.9	7.1	6.7	5.8	57.5	81.7	98.3
	200	4.7	4.6	5.1	6.1	5.7	5.8	6.2	5.3	5.3	58.8	81.3	99.1
	500	5.4	5.7	5.6	5.4	5.8	5.2	6.3	6.5	5.4	58.4	82.6	99.5
	1000	5.5	5.3	4.4	5.9	5.9	5.5	6.2	5.5	4.7	60.8	83.8	99.2
CD_{W+}	100	6.4	6.4	6.1	6.5	5.8	5.6	7.0	5.6	8.1	7.8	10.9	37.6
	200	7.3	5.9	6.6	10.1	6.3	5.0	9.8	5.5	5.6	9.3	10.5	52.0
	500	15.8	8.3	5.8	13.8	9.0	5.9	15.1	8.4	5.9	16.8	15.0	73.3
	1000	27.1	15.0	6.1	27.4	15.9	6.6	27.6	16.4	6.7	26.4	21.5	81.3

Note: The DGP is given by (44) with $\beta_{11} = \beta_{12} = 0$ and contains a single latent factor with factor strength $\alpha = 1, 2/3, 1/2$. λ is the spatial autocorrelation coefficient of the error term. m_0 is the true number of factors and m is the number of selected PCs used to estimate factors. CD denotes the standard CD test statistic while CD^* denotes the bias-corrected CD test statistic. CD_{W+} denotes the power-enhanced randomized CD test statistic.

Table S.2: Size and power of tests of error cross-sectional dependence for the latent factor model with two factors ($m_0 = 2$) and serially independent non-Gaussian errors

$m = 2$		Power ($H_1 : \lambda = 0.25$)											
Tests	$n \setminus T$	Size ($H_0 : \lambda = 0$)						Power ($H_1 : \lambda = 0.25$)					
		$\alpha_1 = 1, \alpha_2 = 1$			$\alpha_1 = 1, \alpha_2 = 2/3$			$\alpha_1 = 2/3, \alpha_2 = 1/2$			$\alpha_1 = 1, \alpha_2 = 1$		
CD	100	99.9	100.0	100.0	98.4	99.9	100.0	9.9	17.3	43.1	99.1	99.9	100.0
	200	100.0	100.0	100.0	99.2	100.0	100.0	7.6	8.9	25.3	99.8	100.0	100.0
	500	100.0	100.0	100.0	99.5	100.0	100.0	7.4	5.3	10.8	100.0	100.0	100.0
	1000	100.0	100.0	100.0	99.7	100.0	100.0	7.6	5.1	7.9	100.0	100.0	100.0
CD^*	100	5.3	5.5	5.0	5.9	4.8	5.1	10.8	6.7	5.4	22.8	35.1	61.3
	200	5.4	4.7	4.1	6.5	5.1	5.8	9.0	5.9	5.5	20.8	33.8	62.6
	500	5.6	6.1	5.9	6.3	5.1	5.2	9.0	5.9	5.0	22.0	35.1	63.6
	1000	5.8	4.7	4.7	4.8	5.1	5.9	8.6	5.7	4.5	23.4	35.1	64.5
CD_{W+}	100	7.1	6.7	7.0	7.3	5.9	7.8	8.2	6.1	10.8	9.4	11.8	43.5
	200	9.1	7.0	5.9	8.8	6.5	5.1	9.6	6.8	5.7	10.6	13.1	58.1
	500	15.4	9.3	5.8	16.0	8.6	5.1	17.5	9.2	6.1	17.3	16.5	76.8
	1000	26.9	18.2	5.6	29.0	16.0	5.9	30.3	17.7	6.7	27.4	24.0	82.6
$m = 4$		Power ($H_1 : \lambda = 0.25$)											
Tests	$n \setminus T$	Size ($H_0 : \lambda = 0$)						Power ($H_1 : \lambda = 0.25$)					
		$\alpha_1 = 1, \alpha_2 = 1$			$\alpha_1 = 1, \alpha_2 = 2/3$			$\alpha_1 = 2/3, \alpha_2 = 1/2$			$\alpha_1 = 1, \alpha_2 = 1$		
CD	100	99.8	100.0	100.0	98.4	99.9	100.0	9.3	16.4	44.1	99.3	99.9	100.0
	200	100.0	100.0	100.0	99.4	100.0	100.0	6.3	8.5	27.1	99.8	100.0	100.0
	500	100.0	100.0	100.0	99.5	100.0	100.0	7.2	5.1	11.4	100.0	100.0	100.0
	1000	100.0	100.0	100.0	99.8	100.0	100.0	7.7	5.3	8.6	100.0	100.0	100.0
CD^*	100	7.4	8.4	14.0	7.5	6.2	8.0	10.5	8.8	8.0	28.5	43.3	75.0
	200	6.2	5.8	7.2	7.2	5.6	6.7	8.4	6.6	7.4	24.1	37.2	69.4
	500	7.0	6.1	6.9	6.3	5.3	5.5	9.1	5.9	5.1	24.2	37.5	65.5
	1000	5.9	4.7	4.4	5.4	5.1	6.2	9.2	5.4	5.0	24.8	37.6	65.1
CD_{W+}	100	6.5	7.8	26.9	6.6	6.9	10.7	6.3	6.6	9.7	7.8	10.1	51.2
	200	8.6	5.5	6.8	9.0	7.1	5.7	8.3	6.5	5.3	8.8	9.7	39.5
	500	15.5	8.7	5.1	14.9	9.0	5.8	14.8	9.5	6.2	15.9	13.7	63.9
	1000	29.8	14.1	7.0	27.1	16.1	6.1	28.1	14.7	5.5	27.5	20.5	76.4

Note: The DGP is given by (44) with $\beta_{11} = \beta_{12} = 0$ and contains two latent factors with factor strengths $(\alpha_1, \alpha_2) = [(1, 1), (1, 2/3), (2/3, 1/2)]$. λ is the spatial autocorrelation coefficient of the error term. m_0 is the true number of factors and m is the number of selected PCs used to estimate factors. CD denotes the standard CD test statistic while CD^* denotes the bias-corrected CD test statistic. CD_{W+} denotes the power-enhanced randomized CD test statistic.

Table S.3: Size and power of tests of error cross-sectional dependence for panel regression model with one latent factor ($m_0 = 1$) and serially independent non-Gaussian errors

$m = 1$		Power ($H_1 : \lambda = 0.25$)											
Tests	$n \setminus T$	Size ($H_0 : \lambda = 0$)						Power ($H_1 : \lambda = 0.25$)					
		$\alpha = 1$		$\alpha = 2/3$		$\alpha = 1/2$		$\alpha = 1$		$\alpha = 2/3$		$\alpha = 1/2$	
CD	100	65.7	87.9	98.4	6.1	9.3	20.3	6.9	6.6	9.0	26.1	38.0	56.0
	200	67.6	91.4	99.6	5.8	7.2	13.0	5.5	6.0	6.3	16.6	30.8	49.6
	500	67.4	94.9	99.9	5.3	5.0	7.4	6.9	5.8	4.9	12.0	22.3	45.6
	1000	68.7	95.3	100.0	6.3	4.9	7.0	6.7	6.3	5.0	10.0	19.3	44.1
CD^*	100	5.3	4.9	5.1	5.9	5.8	5.9	7.6	6.8	6.1	57.0	82.4	98.4
	200	6.0	5.4	5.1	6.2	6.1	5.3	6.3	6.8	5.5	58.5	83.0	99.2
	500	5.5	5.1	4.5	6.2	5.0	4.5	7.3	6.1	4.9	60.2	83.2	99.5
	1000	5.5	5.4	4.8	7.0	5.6	5.3	6.7	6.5	5.4	58.4	84.3	99.2
CD_{W+}	100	7.5	7.0	5.3	7.0	5.3	5.9	7.2	6.7	7.5	9.0	11.7	53.8
	200	9.0	7.6	4.5	8.6	7.8	5.1	8.8	7.0	5.9	10.3	13.4	64.9
	500	15.9	8.9	6.1	15.8	9.5	5.8	16.1	9.7	5.7	16.6	15.8	77.5
	1000	26.0	14.9	6.0	26.1	16.9	5.8	26.8	14.4	6.1	24.5	22.6	84.5
$m = 2$		Power ($H_1 : \lambda = 0.25$)											
Tests	$n \setminus T$	Size ($H_0 : \lambda = 0$)						Power ($H_1 : \lambda = 0.25$)					
		$\alpha = 1$		$\alpha = 2/3$		$\alpha = 1/2$		$\alpha = 1$		$\alpha = 2/3$		$\alpha = 1/2$	
CD	100	67.2	88.4	98.4	5.9	9.2	21.3	29.7	45.1	64.4	60.3	73.3	83.5
	200	67.3	90.6	99.8	5.5	6.9	13.0	18.1	35.1	55.5	72.9	88.9	98.7
	500	67.3	94.7	100.0	6.1	4.8	7.6	12.6	23.4	48.8	83.9	96.5	99.9
	1000	68.4	95.0	100.0	6.2	4.7	7.3	10.0	19.4	45.9	84.8	97.9	100.0
CD^*	100	6.1	5.9	6.5	6.2	6.3	6.4	56.1	81.2	98.0	84.9	97.8	100.0
	200	6.8	5.8	5.8	6.3	6.1	5.1	58.3	82.9	99.1	85.5	98.5	100.0
	500	5.7	5.0	5.1	6.8	5.1	4.1	59.6	83.5	99.6	89.4	98.7	100.0
	1000	5.5	5.4	5.0	7.2	5.7	5.6	57.8	84.5	99.2	87.3	99.0	100.0
CD_{W+}	100	6.5	6.4	7.1	6.3	5.8	6.6	8.9	10.0	38.2	8.1	8.6	40.8
	200	8.6	6.5	5.2	9.0	6.9	5.8	10.2	10.7	50.6	11.4	11.3	54.8
	500	14.1	9.5	5.0	15.5	10.0	5.6	15.8	13.4	70.7	15.8	15.4	72.8
	1000	26.0	15.3	6.9	24.0	14.5	6.4	25.0	20.8	79.3	24.1	21.7	81.0

Note: The DGP is given by (44) with β_{11} and β_{12} both generated from normal distribution, and contains a single latent factor with factor strength $\alpha = 1, 2/3, 1/2$. λ is the spatial autocorrelation coefficient of the error term. m_0 is the true number of factors and m is the number of selected PCs used to estimate factors. CD denotes the standard CD test statistic while CD^* denotes the bias-corrected CD test statistic. CD_{W+} denotes the power-enhanced randomized CD test statistic.

Table S.4: Size and power of tests of error cross-sectional dependence for panel regression model with two latent factors ($m_0 = 2$) and serially independent non-Gaussian errors

$m = 2$		Size ($H_0 : \lambda = 0$)												Power ($H_1 : \lambda = 0.25$)																								
		$\alpha_1 = 1, \alpha_2 = 1$						$\alpha_1 = 1, \alpha_2 = 2/3$						$\alpha_1 = 2/3, \alpha_2 = 1/2$						$\alpha_1 = 1, \alpha_2 = 1$						$\alpha_1 = 1, \alpha_2 = 2/3$						$\alpha_1 = 2/3, \alpha_2 = 1/2$						
		100	200	500	1000	100.0	100.0	100	200	500	1000	8.6	13.7	39.7	98.9	100.0	100.0	98.9	100.0	100.0	91.4	99.1	100.0	91.4	99.1	100.0	92.6	99.6	100.0	93.0	99.9	100.0	83.7	97.7	100.0	83.7	97.7	100.0
Tests	n/T	100	200	500	1000	100.0	100.0	100.0	100.0	100.0	100.0	8.6	13.7	39.7	98.9	100.0	100.0	98.9	100.0	100.0	91.4 <td>99.1</td> <td>100.0</td> <td>91.4</td> <td>99.1</td> <td>100.0</td> <td>92.6</td> <td>99.6</td> <td>100.0</td> <td>93.0</td> <td>99.9</td> <td>100.0</td> <td>85.7</td> <td>97.5</td> <td>100.0</td> <td>85.7</td> <td>97.5</td> <td>100.0</td>	99.1	100.0	91.4	99.1	100.0	92.6	99.6	100.0	93.0	99.9	100.0	85.7	97.5	100.0	85.7	97.5	100.0
	CD	100	200	500	1000	100.0	100.0	100.0	100.0	100.0	100.0	6.6	7.4	22.9	99.7	100.0	100.0	99.7	100.0	100.0	91.4	99.1	100.0	91.4	99.1	100.0	92.6	99.6	100.0	93.0	99.9	100.0	83.0	95.9	99.9	83.0	95.9	99.9
	CD^*	100	200	500	1000	100.0	100.0	100.0	100.0	100.0	100.0	6.6	5.9	7.6	100.0	100.0	100.0	100.0	100.0	100.0	92.6	99.6	100.0	92.6	99.6	100.0	93.0	99.9	100.0	93.0	99.9	100.0	85.7	97.5	100.0	85.7	97.5	100.0
	CD_{W+}	100	200	500	1000	5.3	5.4	5.4	6.9	5.7	4.9	9.3	6.3	6.2	23.0	34.5	63.1	23.0	34.5	63.1	32.6	49.6	78.3	32.6	49.6	78.3	32.6	49.6	78.3	32.6	49.6	78.3	83.7	97.7	100.0	83.7	97.7	100.0
$m = 4$	n/T	100	200	500	1000	5.2	5.1	5.1	6.3	4.7	5.5	8.2	6.7	5.4	21.2	35.9	64.3	21.2	35.9	64.3	34.5	50.2	81.1	34.5	50.2	81.1	34.5	50.2	81.1	34.5	50.2	81.1	87.6	98.5	100.0	87.6	98.5	100.0
	CD	100	200	500	1000	5.9	4.7	5.8	5.9	5.3	4.8	9.2	5.9	5.9	21.9	37.6	64.9	21.9	37.6	64.9	36.2	53.4	83.9	36.2	53.4	83.9	36.2	53.4	83.9	36.2	53.4	83.9	88.4	99.0	100.0	88.4	99.0	100.0
	CD^*	100	200	500	1000	6.8	4.8	5.5	6.5	4.5	5.0	7.6	6.4	5.0	23.4	33.6	63.2	23.4	33.6	63.2	34.2	52.7	86.4	34.2	52.7	86.4	34.2	52.7	86.4	34.2	52.7	86.4	88.9	98.9	100.0	88.9	98.9	100.0
	CD_{W+}	100	200	500	1000	7.1	6.4	5.6	7.1	5.8	6.8	7.2	5.6	9.9	8.4	10.5	42.2	7.2	5.6	9.9	9.0	9.9	50.6	9.0	9.9	50.6	9.0	9.9	50.6	9.0	9.9	50.6	10.7	13.1	67.6	10.7	13.1	67.6
$m = 4$	n/T	100	200	500	1000	10.1	7.0	5.1	9.9	7.5	4.9	9.5	5.9	6.1	10.8	12.1	58.1	10.8	12.1	58.1	10.9	12.4	58.8	10.9	12.4	58.8	10.9	12.4	58.8	10.9	12.4	58.8	12.4	12.0	69.0	12.4	12.0	69.0
	CD	100	200	500	1000	16.2	10.4	6.1	15.8	9.0	5.4	16.3	9.8	5.8	16.5	15.2	74.4	16.5	15.2	74.4	16.0	14.2	74.8	16.0	14.2	74.8	16.0	14.2	74.8	16.0	14.2	74.8	17.7	17.6	79.8	17.7	17.6	79.8
	CD^*	100	200	500	1000	26.6	14.7	6.2	28.1	18.0	5.2	28.1	15.0	5.7	25.4	21.5	84.1	25.4	21.5	84.1	27.5	24.2	82.8	27.5	24.2	82.8	27.5	24.2	82.8	27.5	24.2	82.8	27.4	22.2	85.1	27.4	22.2	85.1
	CD_{W+}	100	200	500	1000	99.9	100.0	100.0	98.4	99.9	100.0	8.1	15.3	41.7	99.2	100.0	100.0	99.2	100.0	100.0	91.6	98.1	99.5	91.6	98.1	99.5	91.6	98.1	99.5	91.6	98.1	99.5	38.4	45.2	54.9	38.4	45.2	54.9
$m = 4$	n/T	100	200	500	1000	100.0	100.0	100.0	99.1	100.0	100.0	6.1	8.6	23.5	99.7	100.0	100.0	99.7	100.0	100.0	92.6	99.2	100.0	92.6	99.2	100.0	92.6	99.2	100.0	92.8	99.8	100.0	77.9	94.0	99.9	77.9	94.0	99.9
	CD	100	200	500	1000	100.0	100.0	100.0	99.5	100.0	100.0	7.4	5.6	11.6	100.0	100.0	100.0	100.0	100.0	100.0	92.8	99.8	100.0	92.8	99.8	100.0	92.8	99.8	100.0	93.0	99.9	100.0	83.2	97.2	100.0	83.2	97.2	100.0
	CD^*	100	200	500	1000	8.2	8.9	15.0	8.9	8.6	11.0	9.8	7.9	9.0	29.3	41.4	75.1	29.3	41.4	75.1	34.7	52.6	82.0	34.7	52.6	82.0	34.7	52.6	82.0	34.7	52.6	82.0	80.0	95.7	100.0	80.0	95.7	100.0
	CD_{W+}	100	200	500	1000	5.8	6.0	7.1	6.9	5.5	6.9	8.9	8.1	7.0	25.1	39.8	70.7	25.1	39.8	70.7	35.3	51.0	82.4	35.3	51.0	82.4	35.3	51.0	82.4	35.3	51.0	82.4	84.1	97.7	100.0	84.1	97.7	100.0
$m = 4$	n/T	100	200	500	1000	6.1	5.2	5.8	6.5	5.4	5.4	9.6	6.7	6.2	24.3	39.4	66.7	24.3	39.4	66.7	36.3	53.9	84.4	36.3	53.9	84.4	36.3	53.9	84.4	36.3	53.9	84.4	85.8	98.3	100.0	85.8	98.3	100.0
	CD	100	200	500	1000	7.0	4.8	5.6	7.0	5.0	5.4	7.9	6.6	5.1	25.1	35.2	64.8	25.1	35.2	64.8	35.4	53.2	86.7	35.4	53.2	86.7	35.4	53.2	86.7	35.4	53.2	86.7	86.7	98.7	100.0	86.7	98.7	100.0
	CD^*	100	200	500	1000	7.3	7.6	26.2	6.8	6.8	15.5	5.5	6.4	12.0	8.9	11.1	48.9	8.9	11.1	48.9	7.6	9.6	43.3	7.6	9.6	43.3	7.6	9.6	43.3	7.6	9.6	43.3	6.2	8.6	38.0	6.2	8.6	38.0
	CD_{W+}	100	200	500	1000	8.2	6.8	5.9	9.0	7.3	5.8	9.5	5.9	5.7	9.6	10.1	38.7	9.6	10.1	38.7	8.7	11.7	47.8	8.7	11.7	47.8	8.7	11.7	47.8	8.7	11.7	47.8	10.4	9.7	38.0	10.4	9.7	38.0
$m = 4$	n/T	100	200	500	1000	14.5	9.6	4.9	15.1	8.8	6.6	13.3	8.1	5.3	15.8	13.6	61.3	15.8	13.6	61.3	15.6	13.8	68.8	15.6	13.8	68.8	15.6	13.8	68.8	15.6	13.8	68.8	14.6	12.5	62.7	14.6	12.5	62.7
	CD	100	200	500	1000	26.3	13.2	6.9	25.2	14.4	6.3	24.7	16.0	5.6	25.1	19.8	75.1	25.1	19.8	75.1	26.4	21.2	77.8	26.4	21.2	77.8	26.4	21.2	77.8	26.4	21.2	77.8	23.4	21.5	77.5	23.4	21.5	77.5
	CD^*	100	200	500	1000	26.3	13.2	6.9	25.2	14.4	6.3	24.7	16.0	5.6	25.1	19.8	75.1	25.1	19.8	75.1	26.4	21.2	77.8	26.4	21.2	77.8	26.4	21.2	77.8	26.4	21.2	77.8	23.4	21.5	77.5	23.4	21.5	77.5
	CD_{W+}	100	200	500	1000	26.3	13.2	6.9	25.2	14.4	6.3	24.7	16.0	5.6	25.1	19.8	75.1	25.1	19.8	75.1	26.4	21.2	77.8	26.4	21.2	77.8	26.4	21.2	77.8	26.4	21.2	77.8	23.4	21.5	77.5	23.4	21.5	77.5

Note: The DGP is given by (44) with β_{11} and β_{22} both generated from normal distribution, and contains two latent factors with factor strengths $(\alpha_1, \alpha_2) = [(1, 1), (1, 2/3), (2/3, 1/2)]$. λ is the spatial autocorrelation coefficient of the error term. m_0 is the true number of factors and m is the number of selected PCs used to estimate factors. CD denotes the standard CD test statistic while CD^* denotes the bias-corrected CD test statistic. CD_{W+} denotes the power-enhanced randomized CD test statistic.

Table S.5: Size and power of variance adjusted tests of error cross-sectional dependence for the latent factor model with one factor ($m_0 = 1$) and serially correlated Gaussian errors

$m = 1$		Power ($H_1 : \lambda = 0.25$)											
Tests	$n \setminus T$	Size ($H_0 : \lambda = 0$)						$\alpha = 1$					
		$\alpha = 2/3$						$\alpha = 1/2$					
		100	200	500	100	200	500	100	200	500	100	200	500
Variance adjusted CD	100	41.0	71.4	93.8	5.3	6.6	12.9	6.7	6.5	5.9	12.8	26.1	44.1
	200	37.0	75.8	97.2	5.2	5.0	7.5	6.0	5.9	5.5	7.7	18.8	34.8
	500	33.5	76.6	98.6	6.7	4.8	5.7	6.5	5.4	5.6	5.3	14.4	31.9
	1000	28.8	76.6	99.6	10.1	5.9	4.5	6.9	6.6	4.2	4.4	10.4	27.4
Variance adjusted CD^*	100	5.6	5.3	5.6	6.2	6.2	6.3	6.8	6.7	4.9	40.9	61.7	89.3
	200	5.0	4.7	5.9	6.5	4.4	6.1	6.6	6.5	5.4	45.6	66.3	92.9
	500	6.3	5.0	5.4	7.2	5.5	4.5	6.8	5.8	5.6	46.2	63.5	93.5
	1000	6.7	5.2	5.1	11.4	6.3	4.0	6.9	6.4	4.4	49.9	66.7	94.2
Variance adjusted CD_{W+}	100	16.0	15.5	12.8	16.0	14.5	14.9	15.7	16.9	17.0	22.8	39.1	86.8
	200	20.2	25.3	19.8	22.8	26.1	20.6	20.5	26.8	22.4	29.1	55.8	98.6
	500	30.4	54.2	51.5	30.5	54.7	51.1	40.1	58.0	51.7	35.1	79.6	100.0
	1000	36.6	80.8	85.5	39.5	78.3	87.0	53.9	82.6	86.9	42.5	92.6	100.0
$m = 2$		Power ($H_1 : \lambda = 0.25$)											
Tests	$n \setminus T$	Size ($H_0 : \lambda = 0$)						$\alpha = 1$					
		$\alpha = 2/3$						$\alpha = 1/2$					
		100	200	500	100	200	500	100	200	500	100	200	500
Variance adjusted CD	100	41.7	72.2	93.3	4.2	5.6	12.8	6.1	6.3	6.1	15.5	28.1	50.0
	200	38.2	75.3	96.9	4.8	5.0	7.4	6.4	5.2	5.4	8.1	19.4	37.4
	500	34.1	77.0	98.7	7.3	4.6	5.6	6.8	5.2	5.6	5.1	15.5	33.9
	1000	29.5	77.0	99.6	11.2	5.8	3.9	7.0	6.5	4.4	4.5	10.4	28.5
Variance adjusted CD^*	100	4.9	4.8	5.7	5.9	6.7	5.8	7.0	7.4	6.2	41.1	62.3	88.8
	200	5.6	4.2	6.1	6.8	4.9	7.2	7.1	6.0	5.5	45.5	65.7	93.6
	500	5.6	5.5	5.3	8.2	5.1	5.1	7.0	5.3	5.8	45.2	63.4	93.9
	1000	6.9	5.7	5.2	11.7	7.7	4.1	7.1	6.7	4.3	50.2	67.3	94.4
Variance adjusted CD_{W+}	100	13.4	15.6	14.1	14.9	14.3	13.4	15.0	15.6	16.9	19.3	34.4	80.7
	200	18.3	22.9	20.2	19.9	25.2	18.3	17.4	27.0	21.8	26.3	47.6	97.9
	500	27.3	52.1	49.7	27.0	54.1	47.9	33.7	55.6	51.0	31.3	77.7	100.0
	1000	36.3	80.2	82.9	36.5	78.7	83.5	50.1	84.4	83.8	40.6	93.5	100.0

Note: The DGP is given by (44) with $\beta_{i1} = \beta_{i2} = 0$ and contains a single latent factor with factor strength $\alpha = 1, 2/3, 1/2$. λ is the spatial autocorrelation coefficient of the error term. m_0 is the true number of factors and m is the number of selected PCs used to estimate factors. CD denotes the standard CD test statistic while CD^* denotes the bias-corrected CD test statistic. CD_{W+} denotes the power-enhanced randomized CD test statistic.

Table S.6: Size and power of variance adjusted tests of error cross-sectional dependence for the latent factor model with two factors ($m_0 = 2$) and serially correlated Gaussian errors

$m = 2$		Size ($H_0 : \lambda = 0$)												Power ($H_1 : \lambda = 0.25$)											
Tests	$n \setminus T$	$\alpha_1 = 1, \alpha_2 = 1$						$\alpha_1 = 2/3, \alpha_2 = 1/2$						$\alpha_1 = 1, \alpha_2 = 1$						$\alpha_1 = 1, \alpha_2 = 2/3$					
		100	200	500	100	200	500	100	200	500	100	200	500	100	200	500	100	200	500	100	200	500	100	200	500
Variance adjusted CD	100	98.2	100.0	100.0	87.5	98.7	100.0	8.4	6.4	22.4	92.9	99.1	100.0	67.6	90.6	97.8	54.0	61.0	69.8						
	200	99.0	100.0	100.0	88.7	99.8	100.0	9.0	5.2	9.8	95.4	99.7	100.0	66.6	94.6	99.4	64.8	77.8	94.0						
	500	99.2	100.0	100.0	88.2	100.0	100.0	12.2	8.2	7.2	95.1	100.0	100.0	63.4	96.8	100.0	80.8	90.8	99.2						
	1000	98.0	100.0	100.0	84.6	99.9	100.0	13.4	8.4	4.8	92.6	100.0	100.0	52.4	95.4	100.0	81.6	93.0	99.8						
Variance adjusted CD^*	100	5.0	5.0	4.5	5.9	5.4	4.4	12.2	5.8	6.0	17.4	22.0	40.3	27.0	34.8	57.8	71.6	91.4	99.6						
	200	5.9	4.2	4.8	7.1	5.7	4.2	10.8	6.6	6.8	19.2	24.1	42.1	27.8	35.8	64.6	75.0	91.0	99.8						
	500	8.4	4.8	4.3	9.0	6.2	4.5	14.2	9.6	6.6	25.3	27.1	45.3	30.0	41.4	71.2	83.8	93.0	99.8						
	1000	13.7	4.7	5.1	12.5	7.7	5.6	14.4	11.6	6.6	33.6	28.4	46.4	42.6	41.2	65.8	83.6	94.8	100.0						
Variance adjusted CD_{W+}	100	17.2	16.5	14.1	16.8	18.7	20.9	16.8	17.8	23.6	23.3	37.8	85.9	21.2	33.4	76.6	26.0	43.6	95.4						
	200	20.9	29.2	24.2	19.4	27.6	24.6	28.2	30.6	26.2	28.0	53.0	98.3	21.2	39.2	90.8	35.4	58.2	99.6						
	500	32.4	54.9	51.7	29.1	55.4	52.7	45.4	63.2	55.4	38.1	79.6	100.0	27.6	51.6	99.4	50.4	80.8	100.0						
	1000	39.4	80.9	85.9	38.5	81.3	87.7	59.8	87.6	88.0	43.8	94.0	100.0	36.4	79.0	100.0	60.2	96.4	100.0						

$m = 4$		Size ($H_0 : \lambda = 0$)												Power ($H_1 : \lambda = 0.25$)											
Tests	$n \setminus T$	$\alpha_1 = 1, \alpha_2 = 1$						$\alpha_1 = 2/3, \alpha_2 = 1/2$						$\alpha_1 = 1, \alpha_2 = 1$						$\alpha_1 = 2/3, \alpha_2 = 1/2$					
		100	200	500	100	200	500	100	200	500	100	200	500	100	200	500	100	200	500	100	200	500	100	200	500
Variance adjusted CD	100	98.7	100.0	100.0	88.7	98.8	100.0	5.8	6.2	20.4	95.4	99.7	99.9	73.2	94.0	99.4	44.4	46.6	46.2						
	200	99.5	100.0	100.0	91.2	99.8	100.0	6.6	5.2	12.2	96.3	99.9	100.0	69.4	95.2	99.6	59.8	69.6	87.6						
	500	99.2	100.0	100.0	88.4	100.0	100.0	11.4	7.6	7.6	95.5	100.0	100.0	61.4	97.6	100.0	77.4	87.6	98.4						
	1000	98.1	100.0	100.0	83.5	99.7	100.0	12.2	8.2	5.8	92.4	100.0	100.0	51.6	96.6	100.0	78.8	91.8	99.8						
Variance adjusted CD^*	100	5.6	6.8	9.3	7.8	7.0	6.6	9.2	6.2	8.8	20.7	27.5	51.6	28.4	43.4	65.8	71.0	89.4	99.8						
	200	5.5	5.8	5.7	7.1	6.9	5.1	10.2	7.2	8.4	20.3	26.9	48.1	29.4	38.2	67.2	73.8	90.6	100.0						
	500	8.9	5.4	4.1	9.3	6.6	5.4	13.0	10.8	6.4	25.9	28.0	46.6	34.0	43.2	71.6	82.6	92.8	99.6						
	1000	14.3	4.6	5.1	13.5	7.6	5.5	14.2	10.8	7.2	34.9	29.1	47.9	42.2	43.6	66.0	81.0	93.4	100.0						
Variance adjusted CD_{W+}	100	13.4	17.7	34.6	13.2	16.7	32.5	14.0	20.6	25.0	17.4	30.5	82.4	18.2	30.2	80.6	18.2	34.2	80.8						
	200	18.8	26.3	23.4	20.6	24.0	25.8	22.8	25.6	22.4	22.7	45.5	93.6	23.6	43.0	95.6	27.6	45.2	95.4						
	500	29.6	53.6	48.4	28.9	52.6	49.9	32.6	54.4	53.4	33.9	76.1	100.0	34.0	77.0	99.8	35.6	77.4	100.0						
	1000	38.9	79.3	84.4	40.9	79.7	82.9	45.6	83.0	81.2	43.0	92.6	100.0	45.6	91.4	100.0	49.8	93.0	100.0						

Note: The DGP is given by (44) with $\beta_{11} = \beta_{12} = 0$, and contains two latent factors with factor strengths $(\alpha_1, \alpha_2) = [(1, 1), (1, 2/3), (2/3, 1/2)]$. λ is the spatial autocorrelation coefficient of the error term. m_0 is the true number of factors and m is the number of selected PCs used to estimate factors. CD denotes the standard CD test statistic while CD^* denotes the bias-corrected CD test statistic. CD_{W+} denotes the power-enhanced randomized CD test statistic.

Table S.7: Size and power of variance adjusted tests of error cross-sectional dependence for panel regression model with one latent factor ($m_0 = 1$) and serially correlated Gaussian errors

$m = 1$		Power ($H_1 : \lambda = 0.25$)											
Tests	$n \setminus T$	Size ($H_0 : \lambda = 0$)						$\alpha = 1$					
		$\alpha = 2/3$						$\alpha = 1/2$					
Variance adjusted CD	100	100	200	500	100	200	500	100	200	500	100	200	500
	100	44.1	73.2	94.5	8.0	5.6	11.5	6.4	5.6	6.6	14.5	24.7	43.0
	200	39.6	74.0	97.0	5.2	5.4	8.3	6.1	7.1	4.6	9.3	18.9	36.3
	500	32.9	77.8	96.3	8.9	5.1	5.0	8.7	6.0	4.9	5.3	13.4	29.4
	1000	27.7	76.2	77.2	8.4	6.4	5.3	7.3	5.3	4.0	5.0	9.3	27.9
Variance adjusted CD^*	100	5.0	4.1	5.4	8.6	5.9	5.8	7.0	6.2	5.4	40.8	62.1	88.3
	200	5.3	6.6	5.0	5.9	6.5	5.8	6.4	7.0	5.1	42.5	64.7	91.8
	500	5.7	5.1	4.9	9.5	6.3	4.8	8.8	6.7	5.0	48.1	65.1	93.8
	1000	8.9	4.0	4.2	9.2	6.8	5.1	7.3	5.2	4.2	51.7	66.8	93.5
	1000	14.4	16.1	12.5	14.7	15.8	15.0	13.6	16.2	16.1	19.6	35.6	88.5
Variance adjusted CD_{W+}	100	16.1	25.2	20.6	18.2	25.1	22.8	20.5	26.9	19.8	22.6	50.7	99.4
	200	26.6	50.1	50.7	26.8	48.8	52.6	26.0	54.1	50.8	31.0	75.4	100.0
	500	29.0	75.5	76.0	34.1	74.3	85.6	34.4	78.2	82.1	35.2	90.0	100.0
	1000												
	1000												

$m = 2$		Power ($H_1 : \lambda = 0.25$)											
Tests	$n \setminus T$	Size ($H_0 : \lambda = 0$)						$\alpha = 1$					
		$\alpha = 2/3$						$\alpha = 1/2$					
Variance adjusted CD	100	43.4	72.9	93.8	7.2	5.7	11.3	6.0	5.9	7.3	15.6	28.3	51.4
	200	40.1	73.7	96.9	4.7	5.8	8.0	5.4	6.8	4.7	10.4	20.3	38.6
	500	34.4	77.7	96.3	7.5	5.0	5.1	7.8	5.5	5.2	5.1	14.0	31.5
	1000	27.1	75.2	75.9	7.1	5.3	4.8	6.0	5.8	3.8	5.5	10.1	29.5
	1000	5.5	4.4	5.7	8.4	6.5	6.6	6.9	7.1	7.0	40.4	63.3	90.4
Variance adjusted CD^*	100	5.2	6.0	5.8	6.0	6.9	5.3	6.0	6.9	5.3	41.4	64.5	92.4
	200	6.0	5.3	5.1	8.3	6.4	5.7	7.8	5.7	5.3	47.7	65.4	94.6
	500	9.1	4.3	4.4	7.5	6.0	5.6	6.2	5.8	4.1	51.2	66.5	93.7
	1000												
	1000	11.6	15.2	14.8	12.5	15.7	15.5	14.5	14.7	14.7	16.4	30.5	80.4
Variance adjusted CD_{W+}	100	19.3	22.3	19.7	18.3	21.9	20.8	15.3	23.8	22.6	21.8	44.1	97.3
	200	23.8	46.9	50.5	25.7	47.0	46.7	26.9	50.6	52.3	28.7	71.3	100.0
	500	30.4	75.9	76.0	31.2	74.4	83.5	32.0	74.4	83.8	32.8	91.6	100.0
	1000												
	1000												

Note: The DGP is given by (44) with β_{11} and β_{12} both generated from normal distribution, and contains a single latent factor with factor strength $\alpha = 1, 2/3, 1/2$. λ is the spatial autocorrelation coefficient of the error term. m_0 is the true number of factors and m is the number of selected PCs used to estimate factors. CD denotes the standard CD test statistic while CD^* denotes the bias-corrected CD test statistic. CD_{W+} denotes the power-enhanced randomized CD test statistic.

Table S.8: Size and power of variance adjusted tests of error cross-sectional dependence for panel regression model with two latent factors ($m_0 = 2$) and serially correlated Gaussian errors

$m = 2$		Power ($H_1 : \lambda = 0.25$)											
Tests	$n \setminus T$	Size ($H_0 : \lambda = 0$)						$\alpha_1 = 1, \alpha_2 = 1$					
		$\alpha_1 = 1, \alpha_2 = 1$			$\alpha_1 = 2/3, \alpha_2 = 1/2$			$\alpha_1 = 1, \alpha_2 = 1$			$\alpha_1 = 1, \alpha_2 = 2/3$		
		100	200	500	100	200	500	100	200	500	100	200	500
Variance adjusted CD	100	98.6	100.0	100.0	89.0	98.6	100.0	6.3	8.1	18.1	93.2	99.5	99.9
	200	99.6	100.0	100.0	91.0	99.5	100.0	8.6	6.2	9.9	94.6	100.0	100.0
	500	99.2	100.0	100.0	88.3	99.9	100.0	7.5	6.1	6.2	95.3	100.0	100.0
	1000	97.4	100.0	100.0	84.5	99.8	100.0	11.3	7.3	3.5	91.2	100.0	100.0
Variance adjusted CD^*	100	4.4	4.4	5.6	6.4	4.9	4.2	9.0	9.0	6.0	18.0	26.0	37.6
	200	6.0	3.2	4.8	6.2	5.6	4.3	10.3	7.2	6.0	19.1	22.0	43.8
	500	8.4	5.2	4.2	7.6	5.4	5.2	9.0	8.6	6.0	25.2	26.3	44.3
	1000	15.0	4.8	5.6	13.6	6.6	5.7	12.5	8.9	5.3	35.6	27.4	45.4
Variance adjusted CD_{W+}	100	14.6	14.6	15.6	15.7	15.8	16.6	17.3	19.2	22.8	19.0	36.1	86.3
	200	19.2	23.6	23.8	18.9	25.1	22.8	23.8	28.6	25.9	24.4	50.3	98.3
	500	28.6	52.8	48.2	26.8	50.8	48.7	38.2	53.3	50.1	33.2	76.8	100.0
	1000	36.4	76.4	84.0	33.0	74.3	84.7	49.0	81.4	83.9	40.0	89.9	100.0

$m = 4$		Power ($H_1 : \lambda = 0.25$)											
Tests	$n \setminus T$	Size ($H_0 : \lambda = 0$)						$\alpha_1 = 1, \alpha_2 = 1$					
		$\alpha_1 = 1, \alpha_2 = 1$			$\alpha_1 = 2/3, \alpha_2 = 1/2$			$\alpha_1 = 1, \alpha_2 = 1$			$\alpha_1 = 1, \alpha_2 = 2/3$		
		100	200	500	100	200	500	100	200	500	100	200	500
Variance adjusted CD	100	98.6	100.0	100.0	90.3	98.7	100.0	6.8	8.2	18.0	94.3	99.5	99.9
	200	99.0	100.0	100.0	92.2	99.6	100.0	5.4	4.7	12.2	96.5	100.0	100.0
	500	99.6	100.0	100.0	89.7	99.8	100.0	6.1	5.7	5.7	94.8	100.0	100.0
	1000	96.6	100.0	100.0	84.1	99.8	100.0	10.0	7.3	3.6	91.2	100.0	100.0
Variance adjusted CD^*	100	6.0	5.8	8.0	6.6	7.4	8.0	9.9	9.5	7.8	20.0	30.3	49.9
	200	6.6	3.6	5.2	7.4	6.5	5.6	7.8	7.5	6.3	19.8	24.2	49.0
	500	9.4	5.6	4.4	7.8	5.5	5.1	7.7	7.6	5.8	26.2	27.6	46.6
	1000	16.0	4.6	5.4	12.4	6.5	5.4	10.2	9.3	5.5	35.5	27.9	47.8
Variance adjusted CD_{W+}	100	13.8	17.2	31.8	10.8	16.0	30.2	13.5	17.9	23.8	16.2	26.3	81.2
	200	13.2	22.6	23.8	15.4	18.4	24.6	18.6	24.0	22.9	19.5	40.6	94.6
	500	27.0	46.8	46.8	24.5	46.3	49.6	28.5	50.4	49.5	30.3	70.1	99.9
	1000	32.6	75.2	82.4	33.3	73.5	81.0	38.5	78.9	82.4	36.9	88.4	100.0

Note: The DGP is given by (44) with β_{11} and β_{22} both generated from normal distribution, and contains two latent factors with factor strengths $(\alpha_1, \alpha_2) = [(1, 1), (1, 2/3), (2/3, 1/2)]$. λ is the spatial autocorrelation coefficient of the error term. m_0 is the true number of factors and m is the number of selected PCs used to estimate factors. CD denotes the standard CD test statistic while CD^* denotes the bias-corrected CD test statistic. CD_{W+} denotes the power-enhanced randomized CD test statistic.

Table S.9: Size and power of variance adjusted tests of error cross-sectional dependence for the latent factor model with one factor ($m_0 = 1$) and serially correlated non-Gaussian errors

$m = 1$																			
Tests	$n \setminus T$	Size ($H_o : \lambda = 0$)						Power ($H_1 : \lambda = 0.25$)											
		$\alpha = 1$			$\alpha = 2/3$			$\alpha = 1/2$			$\alpha = 1$			$\alpha = 2/3$			$\alpha = 1/2$		
		100	200	500	100	200	500	100	200	500	100	200	500	100	200	500	100	200	500
Variance adjusted CD	100	41.6	70.9	93.4	4.6	7.0	10.6	6.4	5.8	6.3	14.1	26.1	41.6	59.7	76.6	94.1	66.4	87.0	97.4
	200	38.4	73.2	96.2	6.0	5.7	7.1	6.3	5.6	5.2	9.3	16.4	34.6	67.9	85.3	98.8	72.2	91.8	99.8
	500	36.3	76.5	98.8	7.2	6.4	6.3	7.3	7.2	5.5	5.4	12.1	31.0	74.7	89.6	100.0	76.0	93.8	100.0
	1000	29.4	75.6	99.5	8.9	6.6	4.5	6.8	6.7	4.0	5.3	9.5	26.8	78.2	92.5	99.9	78.6	91.8	100.0
Variance adjusted CD^*	100	5.4	5.0	3.3	5.8	5.7	4.2	6.7	5.5	5.3	41.0	61.3	91.7	70.4	89.8	99.9	72.1	93.1	99.9
	200	4.5	6.2	6.2	6.9	6.6	5.7	6.6	6.3	6.7	42.6	66.0	92.9	74.7	92.3	99.8	74.2	94.3	99.9
	500	5.1	5.2	4.7	8.4	6.8	4.6	7.5	7.2	5.6	44.1	66.1	93.8	76.8	93.1	100.0	76.9	94.1	100.0
	1000	7.6	5.9	4.0	9.3	6.9	4.1	6.8	6.8	4.2	48.9	67.9	93.8	79.5	94.8	100.0	79.1	92.0	100.0
Variance adjusted CD_{W+}	100	25.3	19.5	14.1	30.1	23.4	15.6	28.2	23.0	20.1	37.7	47.5	89.8	42.2	55.2	94.5	42.6	58.4	96.2
	200	57.4	50.8	26.6	57.1	52.4	29.1	60.2	56.4	30.1	66.3	81.6	99.4	64.6	81.2	99.9	69.7	85.2	99.6
	500	95.6	96.5	75.9	95.0	95.3	77.8	95.7	96.3	79.8	96.4	99.5	100.0	96.4	99.5	100.0	97.4	99.9	100.0
	1000	100.0	100.0	99.3	99.9	100.0	99.5	99.9	100.0	99.6	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
$m = 2$																			
Tests	$n \setminus T$	Size ($H_o : \lambda = 0$)						Power ($H_1 : \lambda = 0.25$)											
		$\alpha = 1$			$\alpha = 2/3$			$\alpha = 1/2$			$\alpha = 1$			$\alpha = 2/3$			$\alpha = 1/2$		
		100	200	500	100	200	500	100	200	500	100	200	500	100	200	500	100	200	500
Variance adjusted CD	100	42.9	71.5	93.4	6.0	6.4	10.7	5.3	5.2	7.2	14.1	29.8	46.6	53.4	66.7	82.2	60.4	78.0	88.5
	200	38.2	74.2	96.6	6.3	5.0	7.9	6.1	6.0	6.2	9.0	18.1	40.0	63.9	81.4	97.0	69.5	90.2	99.4
	500	35.9	76.2	98.8	7.6	6.0	6.3	7.9	7.3	6.0	5.4	11.7	33.6	73.0	88.3	99.8	75.8	92.7	100.0
	1000	28.4	75.2	99.4	9.3	6.4	4.2	7.1	6.8	4.1	4.9	9.1	27.4	77.6	91.8	99.9	76.9	91.6	100.0
Variance adjusted CD^*	100	5.5	5.4	4.2	7.2	6.2	5.2	6.5	6.0	6.4	40.5	62.9	92.4	69.9	89.7	99.7	72.1	93.3	99.9
	200	4.7	6.3	6.2	7.0	6.4	5.6	6.9	7.1	7.6	42.8	65.6	92.7	75.5	91.8	99.6	73.7	93.6	99.8
	500	5.6	5.2	5.2	9.2	7.1	4.8	8.3	7.4	5.7	45.0	65.7	94.3	77.9	92.3	100.0	76.7	93.8	100.0
	1000	8.9	6.0	4.5	9.9	7.0	4.3	7.2	6.8	4.3	50.2	67.6	93.5	80.5	93.9	100.0	77.4	91.8	100.0
Variance adjusted CD_{W+}	100	21.2	17.9	17.3	20.9	21.0	16.2	25.2	21.1	19.8	30.0	39.0	81.3	31.8	45.3	84.7	35.0	44.4	88.8
	200	47.0	43.0	26.0	48.4	44.3	27.7	48.3	46.9	29.6	56.7	72.8	98.5	57.1	73.2	99.2	58.2	75.6	98.9
	500	93.1	92.9	75.0	93.0	92.6	72.4	94.1	94.5	75.2	95.0	98.9	100.0	95.1	98.6	100.0	94.9	99.2	100.0
	1000	99.8	99.9	99.2	100.0	100.0	99.1	99.9	100.0	99.6	99.9	100.0	100.0	99.9	100.0	100.0	99.9	100.0	100.0

Note: The DGP is given by (44) with $\beta_{11} = \beta_{12} = 0$ and contains a single latent factor with factor strength $\alpha = 1, 2/3, 1/2$. λ is the spatial autocorrelation coefficient of the error term. m_0 is the true number of factors and m is the number of selected PCs used to estimate factors. CD denotes the standard CD test statistic while CD^* denotes the bias-corrected CD test statistic. CD_{W+} denotes the power-enhanced randomized CD test statistic.

Table S.10: Size and power of variance adjusted tests of error cross-sectional dependence for the latent factor model with two factors ($m_0 = 2$) and serially correlated non-Gaussian errors

$m = 2$		Power ($H_1 : \lambda = 0.25$)											
Tests	$n \setminus T$	Size ($H_0 : \lambda = 0$)						Power ($H_1 : \lambda = 0.25$)					
		$\alpha_1 = 1, \alpha_2 = 1$		$\alpha_1 = 1, \alpha_2 = 2/3$		$\alpha_1 = 2/3, \alpha_2 = 1/2$		$\alpha_1 = 1, \alpha_2 = 1$		$\alpha_1 = 1, \alpha_2 = 2/3$		$\alpha_1 = 2/3, \alpha_2 = 1/2$	
Variance adjusted CD	100	98.6	100.0	100.0	89.0	98.6	100.0	6.3	8.1	18.1	93.2	99.5	99.9
	200	99.6	100.0	100.0	91.0	99.5	100.0	8.6	6.2	9.9	94.6	100.0	100.0
	500	99.2	100.0	100.0	88.3	99.9	100.0	7.5	6.1	6.2	95.3	100.0	100.0
	1000	97.4	100.0	100.0	84.5	99.8	100.0	11.3	7.3	3.5	91.2	100.0	100.0
Variance adjusted CD^*	100	4.4	4.4	5.6	6.4	4.9	4.2	9.0	9.0	6.0	18.0	26.0	37.6
	200	6.0	3.2	4.8	6.2	5.6	4.3	10.3	7.2	6.0	19.1	22.0	43.8
	500	8.4	5.2	4.2	7.6	5.4	5.2	9.0	8.6	6.0	25.2	26.3	44.3
	1000	15.0	4.8	5.6	13.6	6.6	5.7	12.5	8.9	5.3	35.6	27.4	45.4
Variance adjusted CD_{W+}	100	14.6	14.6	15.6	15.7	15.8	16.6	17.3	19.2	22.8	19.0	36.1	86.3
	200	19.2	23.6	23.8	18.9	25.1	22.8	23.8	28.6	25.9	24.4	50.3	98.3
	500	28.6	52.8	48.2	26.8	50.8	48.7	38.2	53.3	50.1	33.2	76.8	100.0
	1000	36.4	76.4	84.0	33.0	74.3	84.7	49.0	81.4	83.9	40.0	89.9	100.0

$m = 4$		Power ($H_1 : \lambda = 0.25$)											
Tests	$n \setminus T$	Size ($H_0 : \lambda = 0$)						Power ($H_1 : \lambda = 0.25$)					
		$\alpha_1 = 1, \alpha_2 = 1$		$\alpha_1 = 1, \alpha_2 = 2/3$		$\alpha_1 = 2/3, \alpha_2 = 1/2$		$\alpha_1 = 1, \alpha_2 = 1$		$\alpha_1 = 1, \alpha_2 = 2/3$		$\alpha_1 = 2/3, \alpha_2 = 1/2$	
Variance adjusted CD	100	98.6	100.0	100.0	90.3	98.7	100.0	6.8	8.2	18.0	94.3	99.5	99.9
	200	99.0	100.0	100.0	92.2	99.6	100.0	5.4	4.7	12.2	96.5	100.0	100.0
	500	99.6	100.0	100.0	89.7	99.8	100.0	6.1	5.7	5.7	94.8	100.0	100.0
	1000	96.6	100.0	100.0	84.1	99.8	100.0	10.0	7.3	3.6	91.2	100.0	100.0
Variance adjusted CD^*	100	6.0	5.8	8.0	6.6	7.4	8.0	9.9	9.5	7.8	20.0	30.3	49.9
	200	6.6	3.6	5.2	7.4	6.5	5.6	7.8	7.5	6.3	19.8	24.2	49.0
	500	9.4	5.6	4.4	7.8	5.5	5.1	7.7	7.6	5.8	26.2	27.6	46.6
	1000	16.0	4.6	5.4	12.4	6.5	5.4	10.2	9.3	5.5	35.5	27.9	46.5
Variance adjusted CD_{W+}	100	13.8	17.2	31.8	10.8	16.0	30.2	13.5	17.9	23.8	16.2	26.3	81.2
	200	13.2	22.6	23.8	15.4	18.4	24.6	18.6	24.0	22.9	19.5	40.6	94.6
	500	27.0	46.8	46.8	24.5	46.3	49.6	28.5	50.4	49.5	30.3	70.1	99.9
	1000	32.6	75.2	82.4	33.3	73.5	81.0	38.5	78.9	82.4	36.9	88.4	100.0

Note: The DGP is given by (44) with $\beta_{11} = \beta_{12} = 0$, and contains two latent factors with factor strengths $(\alpha_1, \alpha_2) = [(1, 1), (1, 2/3), (2/3, 1/2)]$. λ is the spatial autocorrelation coefficient of the error term. m_0 is the true number of factors and m is the number of selected PCs used to estimate factors. CD denotes the standard CD test statistic while CD^* denotes the bias-corrected CD test statistic. CD_{W+} denotes the power-enhanced randomized CD test statistic.

Table S.11: Size and power of variance adjusted tests of error cross-sectional dependence for panel regression model with one latent factor ($m_0 = 1$) and serially correlated non-Gaussian errors

$m = 1$		Power ($H_1 : \lambda = 0.25$)											
Tests	$n \setminus T$	Size ($H_0 : \lambda = 0$)						$\alpha = 1$					
		$\alpha = 2/3$						$\alpha = 1/2$					
Variance adjusted CD	100	100	200	500	100	200	500	100	200	500	100	200	500
	200	40.0	70.6	93.3	6.1	7.0	13.3	7.2	6.0	7.3	15.9	24.7	44.2
	500	38.4	71.7	96.5	6.4	5.3	7.9	6.2	5.1	5.0	7.4	16.5	36.7
	1000	32.6	75.5	99.2	8.0	6.3	5.9	9.4	7.0	3.8	6.3	13.4	29.9
	1000	28.3	69.0	69.0	8.3	4.8	4.5	7.1	5.4	4.8	5.5	9.1	27.1
Variance adjusted CD^*	100	5.6	5.8	4.2	7.6	6.5	5.5	8.0	7.0	6.4	42.8	62.4	89.9
	200	4.6	5.1	6.1	8.1	5.7	5.6	6.7	5.6	5.4	43.1	64.9	92.6
	500	5.8	5.2	5.4	8.5	7.2	4.7	9.7	7.3	3.6	46.2	67.6	93.3
	1000	7.2	5.9	5.9	9.2	5.5	5.1	7.2	5.6	5.9	51.0	65.7	95.4
	1000	22.5	17.1	13.5	24.5	22.2	19.1	21.7	21.9	17.8	33.0	43.8	90.2
Variance adjusted CD_{W+}	100	48.2	44.6	27.9	51.9	48.0	26.4	52.7	48.0	28.8	61.0	74.4	99.3
	200	93.1	93.5	73.9	91.7	94.7	76.6	91.7	94.2	77.4	94.1	98.5	100.0
	500	98.1	100.0	100.0	99.7	100.0	99.5	99.9	100.0	99.5	99.9	100.0	100.0
	1000												
	1000												
$m = 2$		Power ($H_1 : \lambda = 0.25$)											
Tests	$n \setminus T$	Size ($H_0 : \lambda = 0$)						$\alpha = 1$					
		$\alpha = 2/3$						$\alpha = 1/2$					
Variance adjusted CD	100	100	200	500	100	200	500	100	200	500	100	200	500
	200	41.2	70.6	93.1	4.4	5.5	13.6	6.8	5.6	7.3	15.5	27.2	50.2
	500	39.2	72.4	97.0	5.7	4.6	6.2	5.2	3.9	5.3	8.7	17.9	38.0
	1000	32.0	75.6	99.3	6.7	6.1	5.9	7.4	6.4	3.3	6.3	12.8	31.1
	1000	28.9	69.1	69.1	7.2	5.0	4.5	7.0	6.1	5.2	5.2	9.5	28.7
Variance adjusted CD^*	100	5.4	4.9	5.2	6.7	6.7	6.8	8.6	7.2	7.0	40.5	61.4	89.5
	200	4.6	4.2	6.7	6.8	4.2	5.6	6.3	4.8	6.3	41.4	64.8	92.7
	500	6.1	5.2	4.7	7.6	6.9	4.9	7.8	6.8	4.0	48.5	66.5	93.6
	1000	8.1	6.3	6.3	8.0	5.3	4.4	7.1	6.0	5.5	51.0	65.4	95.2
	1000	21.0	16.9	14.7	17.4	18.9	16.2	19.1	17.8	17.8	25.6	36.7	80.4
Variance adjusted CD_{W+}	100	42.2	40.4	26.5	45.7	40.8	25.7	44.4	39.2	25.1	50.7	64.5	98.2
	200	88.5	91.7	72.7	89.2	92.3	74.3	86.6	92.0	73.7	89.8	98.2	100.0
	500	98.0	100.0	100.0	99.7	100.0	99.3	99.4	100.0	99.0	99.3	100.0	100.0
	1000												
	1000												

Note: The DGP is given by (44) with β_{11} and β_{12} both generated from normal distribution, and contains a single latent factor with factor strength $\alpha = 1, 2/3, 1/2$. λ is the spatial autocorrelation coefficient of the error term. m_0 is the true number of factors and m is the number of selected PCs used to estimate factors. CD denotes the standard CD test statistic while CD^* denotes the bias-corrected CD test statistic. CD_{W+} denotes the power-enhanced randomized CD test statistic.

Table S.12: Size and power of variance adjusted tests of error cross-sectional dependence for panel regression model with two latent factors ($m_0 = 2$) and serially correlated non-Gaussian errors

$m = 2$		Power ($H_1 : \lambda = 0.25$)											
Tests	$n \setminus T$	Size ($H_0 : \lambda = 0$)						$\alpha_1 = 1, \alpha_2 = 1$					
		$\alpha_1 = 1, \alpha_2 = 1$		$\alpha_1 = 1, \alpha_2 = 2/3$		$\alpha_1 = 2/3, \alpha_2 = 1/2$		$\alpha_1 = 1, \alpha_2 = 1$		$\alpha_1 = 1, \alpha_2 = 2/3$		$\alpha_1 = 2/3, \alpha_2 = 1/2$	
Variance adjusted CD	100	97.9	100.0	100.0	87.6	99.0	100.0	8.2	6.7	20.8	91.5	99.2	100.0
	200	98.4	100.0	100.0	88.1	100.0	100.0	7.4	6.0	11.6	93.2	99.9	100.0
	500	98.4	100.0	100.0	88.2	100.0	100.0	10.7	7.5	5.6	93.9	100.0	100.0
	1000	98.1	100.0	100.0	82.3	100.0	100.0	11.0	8.7	5.0	92.7	100.0	100.0
Variance adjusted CD^*	100	6.3	4.5	5.2	6.5	5.3	4.3	9.8	7.7	6.7	16.6	21.9	36.8
	200	5.9	4.1	5.1	7.3	4.9	4.1	10.0	8.2	5.6	20.7	24.9	40.4
	500	8.9	6.1	4.4	9.0	5.4	5.1	11.9	9.6	5.2	25.7	26.8	44.5
	1000	14.2	4.9	5.4	13.5	6.9	5.6	12.0	10.5	6.2	33.5	29.1	44.1
Variance adjusted CD_{W+}	100	21.9	19.4	14.0	22.1	19.2	18.9	25.6	24.6	25.0	28.2	40.6	83.1
	200	45.7	43.7	24.3	46.9	43.3	28.2	54.7	50.6	30.1	54.7	72.3	98.5
	500	90.1	94.7	74.3	89.1	93.4	74.0	93.2	95.8	77.5	92.0	98.1	100.0
	1000	99.9	100.0	100.0	99.7	100.0	100.0	99.7	100.0	100.0	99.5	100.0	100.0

$m = 4$		Power ($H_1 : \lambda = 0.25$)											
Tests	$n \setminus T$	Size ($H_0 : \lambda = 0$)						$\alpha_1 = 1, \alpha_2 = 1$					
		$\alpha_1 = 1, \alpha_2 = 1$		$\alpha_1 = 1, \alpha_2 = 2/3$		$\alpha_1 = 2/3, \alpha_2 = 1/2$		$\alpha_1 = 1, \alpha_2 = 1$		$\alpha_1 = 1, \alpha_2 = 2/3$		$\alpha_1 = 2/3, \alpha_2 = 1/2$	
Variance adjusted CD	100	97.8	100.0	100.0	88.9	99.3	100.0	6.8	6.2	20.9	93.3	99.4	100.0
	200	98.6	100.0	100.0	89.2	100.0	100.0	6.8	5.6	11.3	94.8	99.9	100.0
	500	98.4	100.0	100.0	87.9	100.0	100.0	8.5	6.8	5.1	94.3	100.0	100.0
	1000	97.5	100.0	100.0	81.1	99.8	100.0	10.1	7.9	5.1	91.5	100.0	100.0
Variance adjusted CD^*	100	6.7	6.7	7.1	7.4	7.1	8.0	9.1	9.0	8.8	22.9	29.4	47.0
	200	7.7	5.7	5.7	6.9	6.1	5.2	9.1	7.8	4.9	22.3	28.0	46.0
	500	9.0	7.0	5.2	9.9	6.0	6.5	10.5	8.8	6.4	26.8	28.6	45.4
	1000	16.0	6.5	5.2	15.1	6.6	5.3	11.4	9.2	6.8	36.1	29.0	44.5
Variance adjusted CD_{W+}	100	18.6	17.3	33.9	14.4	16.6	28.9	16.7	18.8	24.6	23.1	31.5	82.3
	200	35.2	36.8	23.9	34.9	35.0	26.5	38.3	37.4	26.3	39.4	57.4	95.8
	500	83.6	88.1	69.0	84.6	89.7	70.5	84.8	92.4	71.6	87.3	96.6	100.0
	1000	99.4	99.9	100.0	99.6	100.0	100.0	99.4	100.0	100.0	99.3	100.0	100.0

Note: The DGP is given by (44) with β_{11} and β_{22} both generated from normal distribution, and contains two latent factors with factor strengths $(\alpha_1, \alpha_2) = [(1, 1), (1, 2/3), (2/3, 1/2)]$. λ is the spatial autocorrelation coefficient of the error term. m_0 is the true number of factors and m is the number of selected PCs used to estimate factors. CD denotes the standard CD test statistic while CD^* denotes the bias-corrected CD test statistic. CD_{W+} denotes the power-enhanced randomized CD test statistic.

Table S.13: Size and power of ARDL adjusted tests of error cross-sectional dependence for the latent factor model with one factor ($m_0 = 1$) and serially correlated Gaussian errors

$m = 2$		Power ($H_1 : \lambda = 0.25$)											
Tests	$n \setminus T$	Size ($H_0 : \lambda = 0$)						$\alpha = 1$					
		$\alpha = 1$						$\alpha = 2/3$					
		100	200	500	100	200	500	100	200	500	100	200	500
ARDL adjusted CD	100	68.9	91.3	98.7	6.5	9.0	20.8	6.2	5.7	9.4	54.3	70.5	82.6
	200	70.6	94.4	99.8	5.1	7.0	14.3	5.7	5.5	5.7	69.3	88.8	98.1
	500	71.2	96.5	100.0	4.5	5.1	7.4	7.3	6.0	5.1	79.9	95.9	100.0
	1000	73.0	97.2	100.0	5.9	5.4	5.6	7.2	5.1	5.2	81.7	96.7	100.0
ARDL adjusted CD^*	100	5.0	5.3	5.9	6.8	6.3	5.8	7.4	6.9	7.1	57.1	83.9	99.1
	200	5.5	5.4	5.4	5.8	5.9	4.7	6.3	6.3	5.5	59.1	84.9	99.4
	500	5.0	5.5	5.3	5.5	5.3	5.5	7.4	6.7	5.2	60.7	85.5	99.5
	1000	5.4	4.5	4.6	6.1	6.1	4.9	7.7	5.3	5.1	61.0	85.6	99.8
ARDL adjusted CD_{W+}	100	5.6	5.4	7.2	5.5	5.6	5.2	6.5	5.3	7.5	7.0	6.5	28.9
	200	5.0	6.1	5.4	4.4	5.3	5.7	4.2	5.0	5.5	6.0	6.5	36.2
	500	6.3	5.1	4.8	5.4	4.6	5.1	4.3	5.2	5.7	5.5	6.3	45.7
	1000	4.7	6.1	5.0	5.6	4.6	4.3	4.3	5.5	5.5	4.4	5.7	47.0
$m = 4$		Power ($H_1 : \lambda = 0.25$)											
Tests	$n \setminus T$	Size ($H_0 : \lambda = 0$)						$\alpha = 1$					
		$\alpha = 1$						$\alpha = 2/3$					
		100	200	500	100	200	500	100	200	500	100	200	500
ARDL adjusted CD	100	68.9	91.6	98.6	6.2	9.2	22.6	6.9	5.8	10.2	33.7	56.9	78.1
	200	71.5	94.0	99.7	5.4	7.1	13.9	5.5	5.4	6.0	22.3	42.7	69.3
	500	70.8	96.9	100.0	5.0	5.8	7.6	7.3	5.9	4.8	14.7	30.8	60.9
	1000	72.4	97.5	100.0	5.9	5.9	6.0	6.9	5.3	5.7	10.8	24.7	54.6
ARDL adjusted CD^*	100	7.0	7.1	12.0	7.9	8.1	10.3	8.9	9.7	12.0	58.0	84.7	99.5
	200	6.4	5.8	6.6	6.4	6.5	5.7	6.7	6.8	6.8	57.5	84.2	99.3
	500	5.4	5.9	5.2	5.3	5.3	5.5	7.9	6.7	5.2	58.6	85.2	99.6
	1000	4.9	4.9	4.7	6.3	6.1	5.3	7.3	5.3	5.6	59.9	85.0	99.8
ARDL adjusted CD_{W+}	100	5.9	5.9	8.2	5.5	6.1	7.4	5.7	6.2	7.8	6.8	7.0	20.5
	200	5.7	5.1	5.2	4.8	5.1	4.8	5.0	5.7	5.1	5.9	6.1	20.2
	500	4.7	4.5	5.7	5.7	5.4	4.7	5.4	5.9	5.9	5.1	4.9	31.8
	1000	6.7	4.9	4.5	5.7	4.4	4.6	6.5	5.1	5.5	5.8	5.8	40.2

Note: The DGP is given by (44) with $\beta_{11} = \beta_{12} = 0$ and contains a single latent factor with factor strength $\alpha = 1, 2/3, 1/2$. λ is the spatial autocorrelation coefficient of the error term. m_0 is the true number of factors and m is the number of selected PCs used to estimate factors. CD denotes the standard CD test statistic while CD^* denotes the bias-corrected CD test statistic. CD_{W+} denotes the power-enhanced randomized CD test statistic.

Table S.14: Size and power of ARDL adjusted tests of error cross-sectional dependence for the latent factor model with two factors ($m_0 = 2$) and serially correlated Gaussian errors

$m = 4$		Power ($H_1 : \lambda = 0.25$)											
Tests	$n \setminus T$	Size ($H_0 : \lambda = 0$)						$\alpha_1 = 1, \alpha_2 = 1$					
		$\alpha_1 = 1, \alpha_2 = 2/3$						$\alpha_1 = 2/3, \alpha_2 = 1/2$					
		100	200	500	100	200	500	100	200	500	100	200	500
ARDL adjusted CD	100	100.0	100.0	100.0	98.9	99.9	100.0	7.4	11.9	38.0	99.6	100.0	100.0
	200	100.0	100.0	100.0	99.5	100.0	100.0	6.9	8.8	21.8	99.8	100.0	100.0
	500	100.0	100.0	100.0	99.9	100.0	100.0	7.0	5.7	10.7	99.9	100.0	100.0
	1000	100.0	100.0	100.0	99.8	100.0	100.0	7.1	5.5	7.5	100.0	100.0	100.0
ARDL adjusted CD^*	100	8.0	9.3	14.5	7.8	7.2	10.5	10.2	9.9	9.9	31.8	53.7	87.9
	200	5.7	7.2	7.9	6.6	6.2	7.1	8.1	7.6	6.3	30.2	50.0	86.0
	500	6.1	6.1	6.5	7.0	6.2	5.0	8.6	6.1	4.7	29.7	49.5	83.5
	1000	5.5	6.0	5.9	6.3	6.1	5.0	8.2	7.2	6.4	28.3	48.1	81.8
ARDL adjusted CD_{W+}	100	6.3	6.9	8.5	5.4	6.2	10.2	5.8	5.6	9.2	6.1	8.7	21.1
	200	5.3	5.9	6.5	5.5	6.0	5.1	6.0	5.5	6.3	5.5	7.0	25.3
	500	5.4	4.0	4.4	5.3	4.7	4.4	5.9	4.7	6.2	5.6	5.1	36.1
	1000	5.4	5.3	5.0	5.1	5.9	5.1	5.2	5.4	6.3	5.6	5.1	40.7

$m = 6$		Power ($H_1 : \lambda = 0.25$)											
Tests	$n \setminus T$	Size ($H_0 : \lambda = 0$)						$\alpha_1 = 1, \alpha_2 = 1$					
		$\alpha_1 = 1, \alpha_2 = 2/3$						$\alpha_1 = 2/3, \alpha_2 = 1/2$					
		100	200	500	100	200	500	100	200	500	100	200	500
ARDL adjusted CD	100	100.0	100.0	100.0	98.8	99.9	100.0	7.4	12.4	38.6	99.7	100.0	100.0
	200	100.0	100.0	100.0	99.6	100.0	100.0	6.5	8.9	22.4	100.0	100.0	100.0
	500	100.0	100.0	100.0	99.9	100.0	100.0	6.5	5.1	10.5	100.0	100.0	100.0
	1000	100.0	100.0	100.0	99.8	100.0	100.0	7.1	5.5	7.7	100.0	100.0	100.0
ARDL adjusted CD^*	100	11.4	15.6	30.3	9.8	12.5	24.1	12.3	13.2	17.3	37.0	62.5	93.1
	200	7.5	9.1	12.7	6.9	7.2	10.5	8.8	7.9	8.6	32.1	53.4	89.8
	500	6.6	6.6	7.4	7.1	6.8	5.6	8.3	5.9	5.6	29.8	50.2	84.6
	1000	6.0	6.4	6.1	6.8	5.8	4.9	8.2	7.0	6.4	28.3	47.0	81.6
ARDL adjusted CD_{W+}	100	5.5	7.4	16.2	6.1	6.9	16.5	5.5	6.3	10.4	5.9	8.4	26.2
	200	5.5	5.5	6.5	5.9	5.5	6.7	5.4	5.7	6.3	6.2	6.5	17.8
	500	5.6	4.9	5.1	6.2	4.7	5.1	6.3	6.6	4.7	5.1	6.3	27.9
	1000	5.2	5.4	5.3	5.4	5.0	4.8	6.3	5.5	5.7	5.5	6.0	35.9

Note: The DGP is given by (44) with $\beta_{i1} = \beta_{i2} = 0$, and contains two latent factors with factor strengths $(\alpha_1, \alpha_2) = [(1, 1), (1, 2/3), (2/3, 1/2)]$. λ is the spatial autocorrelation coefficient of the error term. m_0 is the true number of factors and m is the number of selected PCs used to estimate factors. CD denotes the standard CD test statistic while CD^* denotes the bias-corrected CD test statistic. CD_{W+} denotes the power-enhanced randomized CD test statistic.

Table S.15: Size and power of ARDL adjusted tests of error cross-sectional dependence for panel regression model with one latent factor ($m_0 = 1$) and serially correlated Gaussian errors

$m = 2$		Power ($H_1 : \lambda = 0.25$)											
Tests	$n \setminus T$	Size ($H_0 : \lambda = 0$)						$\alpha = 1$					
		$\alpha = 1$			$\alpha = 2/3$			$\alpha = 1/2$			$\alpha = 1$		
ARDL adjusted CD	100	69.1	91.0	98.9	7.0	9.2	22.1	8.1	7.4	9.4	29.5	49.6	67.5
	200	69.6	93.5	99.9	7.0	6.6	11.9	7.4	6.8	5.6	18.5	34.7	60.2
	500	67.2	96.4	100.0	7.1	4.6	7.9	8.0	6.6	5.1	11.4	25.4	53.4
	1000	69.0	96.8	100.0	7.6	6.0	6.2	8.0	6.7	5.1	10.5	22.2	51.2
ARDL adjusted CD^*	100	5.7	6.1	7.8	7.9	7.6	6.9	9.4	8.6	7.4	55.8	81.6	98.9
	200	5.1	5.7	5.1	8.3	6.7	5.3	8.4	8.3	5.4	58.7	84.3	99.6
	500	6.5	6.2	5.2	8.6	6.5	5.8	8.0	7.4	5.7	61.8	85.6	99.6
	1000	6.2	4.7	4.6	8.3	6.0	6.4	8.2	7.0	5.9	60.1	85.3	99.5
ARDL adjusted CD_{W+}	100	5.4	6.4	6.4	5.3	5.9	5.9	5.6	5.5	6.3	5.4	8.0	29.1
	200	6.5	5.9	5.3	6.6	6.4	5.0	5.3	5.7	5.5	7.0	7.4	34.7
	500	5.8	5.6	4.7	5.4	6.2	5.1	5.1	5.4	4.4	6.0	5.8	44.3
	1000	6.7	5.6	4.9	5.9	5.3	5.8	6.7	6.0	5.2	6.5	5.5	42.0
$m = 4$		Power ($H_1 : \lambda = 0.25$)											
Tests	$n \setminus T$	Size ($H_0 : \lambda = 0$)						$\alpha = 1$					
		$\alpha = 1$			$\alpha = 2/3$			$\alpha = 1/2$			$\alpha = 1$		
ARDL adjusted CD	100	69.6	90.3	99.2	6.7	10.4	23.7	7.1	7.3	10.8	38.9	60.2	80.9
	200	69.5	93.8	100.0	6.7	6.9	11.6	7.1	5.9	6.8	23.4	42.0	70.1
	500	67.9	96.0	100.0	6.3	4.7	7.8	6.9	6.6	5.1	14.1	28.0	58.8
	1000	68.5	96.6	100.0	7.0	5.7	6.1	8.0	6.5	5.8	12.2	24.8	54.7
ARDL adjusted CD^*	100	6.7	8.3	12.0	9.0	10.3	11.9	9.3	10.5	12.6	55.9	83.0	99.4
	200	5.4	6.6	6.6	8.1	7.6	6.4	8.6	8.8	8.1	57.4	84.1	99.5
	500	6.6	7.0	5.4	7.1	6.2	6.1	7.6	7.6	6.2	58.9	84.8	99.7
	1000	6.6	5.4	5.1	7.6	6.1	6.5	8.2	7.1	6.2	59.1	84.2	99.6
ARDL adjusted CD_{W+}	100	7.0	6.5	9.5	5.8	4.2	7.3	6.6	6.6	7.6	7.1	8.3	20.5
	200	5.3	5.9	4.9	6.5	5.5	5.5	6.3	5.7	5.4	5.7	6.6	21.6
	500	5.5	5.1	5.3	4.8	5.7	4.9	6.4	5.5	5.1	5.6	6.1	30.0
	1000	6.0	5.6	5.5	5.8	5.4	6.3	5.6	5.0	5.5	6.3	6.5	39.7

Note: The DGP is given by (44) with β_{11} and β_{12} both generated from normal distribution, and contains a single latent factor with factor strength $\alpha = 1, 2/3, 1/2$. λ is the spatial autocorrelation coefficient of the error term. m_0 is the true number of factors and m is the number of selected PCs used to estimate factors. CD denotes the standard CD test statistic while CD^* denotes the bias-corrected CD test statistic. CD_{W+} denotes the power-enhanced randomized CD test statistic.

Table S.16: Size and power of ARDL adjusted tests of error cross-sectional dependence for panel regression model with two latent factors ($m_0 = 2$) and serially correlated Gaussian errors

$m = 4$		Power ($H_1 : \lambda = 0.25$)											
Tests	$n \setminus T$	Size ($H_0 : \lambda = 0$)						Power ($H_1 : \lambda = 0.25$)					
		$\alpha_1 = 1, \alpha_2 = 1$		$\alpha_1 = 1, \alpha_2 = 2/3$		$\alpha_1 = 2/3, \alpha_2 = 1/2$		$\alpha_1 = 1, \alpha_2 = 1$		$\alpha_1 = 1, \alpha_2 = 2/3$		$\alpha_1 = 2/3, \alpha_2 = 1/2$	
ARDL adjusted CD	100	100.0	100.0	100.0	100.0	98.4	100.0	8.8	12.0	37.8	99.4	100.0	100.0
	200	100.0	100.0	100.0	100.0	99.3	100.0	7.7	8.4	19.6	99.8	100.0	100.0
	500	100.0	100.0	100.0	100.0	99.6	100.0	6.8	6.1	8.3	99.9	100.0	100.0
	1000	100.0	100.0	100.0	100.0	100.0	100.0	8.3	6.0	6.9	100.0	100.0	100.0
ARDL adjusted CD^*	100	7.7	7.7	11.9	7.8	8.8	11.6	11.9	11.4	12.4	29.3	51.8	84.9
	200	6.1	7.0	6.0	6.6	6.3	6.1	11.6	9.3	6.7	29.3	48.8	81.9
	500	7.2	5.7	6.0	7.4	5.7	4.7	10.1	8.6	6.6	29.6	41.9	78.6
	1000	5.9	5.8	5.2	7.3	5.6	5.5	9.0	8.4	6.5	27.1	41.6	76.6
ARDL adjusted CD_{W+}	100	6.4	6.8	10.2	5.7	6.9	9.4	5.7	6.3	9.1	8.3	9.1	25.2
	200	5.8	5.9	6.0	5.1	5.9	5.3	5.8	4.7	5.5	6.2	6.8	24.3
	500	5.9	5.8	5.5	6.4	5.5	5.6	5.8	5.7	5.5	5.9	7.0	34.3
	1000	5.9	5.2	4.9	6.5	5.6	6.2	5.5	5.6	5.7	6.2	6.1	38.7

$m = 6$		Power ($H_1 : \lambda = 0.25$)											
Tests	$n \setminus T$	Size ($H_0 : \lambda = 0$)						Power ($H_1 : \lambda = 0.25$)					
		$\alpha_1 = 1, \alpha_2 = 1$		$\alpha_1 = 1, \alpha_2 = 2/3$		$\alpha_1 = 2/3, \alpha_2 = 1/2$		$\alpha_1 = 1, \alpha_2 = 1$		$\alpha_1 = 1, \alpha_2 = 2/3$		$\alpha_1 = 2/3, \alpha_2 = 1/2$	
ARDL adjusted CD	100	100.0	100.0	100.0	100.0	98.5	100.0	9.0	13.0	38.7	99.8	100.0	100.0
	200	100.0	100.0	100.0	100.0	99.4	100.0	8.3	8.5	20.2	99.8	100.0	100.0
	500	100.0	100.0	100.0	100.0	99.7	100.0	7.3	5.7	8.4	100.0	100.0	100.0
	1000	100.0	100.0	100.0	100.0	100.0	100.0	6.9	6.1	7.0	100.0	100.0	100.0
ARDL adjusted CD^*	100	10.1	13.8	28.7	10.5	14.2	26.9	13.1	15.3	19.1	32.0	59.4	91.6
	200	7.4	8.0	9.7	8.0	8.0	9.0	11.4	10.4	9.3	30.1	50.9	86.9
	500	7.8	5.7	6.8	8.0	6.0	5.4	9.5	8.7	7.3	29.3	42.4	79.4
	1000	6.3	6.2	5.1	8.3	6.5	5.6	8.4	7.7	6.4	27.7	42.4	77.8
ARDL adjusted CD_{W+}	100	6.3	6.8	22.5	6.9	7.2	16.9	5.5	7.1	10.7	6.9	8.6	32.1
	200	5.9	5.9	5.9	6.5	5.5	6.4	6.6	5.3	5.6	6.2	7.0	16.8
	500	7.4	5.8	5.5	6.7	5.8	5.7	5.7	4.9	4.2	6.9	6.3	28.4
	1000	5.9	5.9	5.2	6.4	6.3	5.3	6.1	5.8	5.3	6.1	6.5	33.1

Note: The DGP is given by (44) with β_{11} and β_{22} both generated from normal distribution, and contains two latent factors with factor strengths $(\alpha_1, \alpha_2) = [(1, 1), (1, 2/3), (2/3, 1/2)]$. λ is the spatial autocorrelation coefficient of the error term. m_0 is the true number of factors and m is the number of selected PCs used to estimate factors. CD denotes the standard CD test statistic while CD^* denotes the bias-corrected CD test statistic. CD_{W+} denotes the power-enhanced randomized CD test statistic.

Table S.17: Size and power of ARDL adjusted tests of error cross-sectional dependence for the latent factor model with one factor ($m_0 = 1$) and serially correlated non-Gaussian errors

$m = 2$

Tests	$n \setminus T$	Size ($H_0 : \lambda = 0$)						Power ($H_1 : \lambda = 0.25$)					
		$\alpha = 1$		$\alpha = 2/3$		$\alpha = 1/2$		$\alpha = 1$		$\alpha = 2/3$		$\alpha = 1/2$	
		100	200	500	100	200	500	100	200	500	100	200	500
ARDL adjusted CD	100	62.2	86.3	97.2	6.7	9.7	23.6	6.6	6.4	10.0	28.5	46.4	67.0
	200	66.2	91.6	96.4	4.9	7.0	14.2	5.1	5.4	6.9	19.2	34.4	58.8
	500	66.0	88.8	97.3	5.6	5.8	9.5	7.3	5.9	5.9	12.2	27.9	52.8
	1000	63.1	96.8	100.0	6.4	5.7	7.4	7.2	5.2	5.0	9.6	21.7	50.4
ARDL adjusted CD^*	100	5.5	5.9	5.7	7.5	6.2	7.6	8.6	7.6	6.9	58.8	84.5	99.1
	200	5.5	5.4	6.3	5.6	5.5	5.7	6.1	6.3	6.2	59.6	86.1	99.3
	500	5.8	5.3	5.8	6.4	5.5	5.6	7.6	6.6	5.8	61.7	85.5	99.4
	1000	5.5	6.2	5.4	7.0	6.0	4.8	7.4	5.6	5.8	62.3	85.8	99.5
ARDL adjusted CD_{W+}	100	6.5	7.0	6.4	6.2	5.1	5.7	5.9	5.7	7.3	7.9	10.9	33.0
	200	8.5	6.5	6.1	8.5	6.5	5.4	8.1	5.5	5.1	10.4	10.7	47.1
	500	13.5	10.9	6.9	15.9	8.1	5.3	14.1	7.8	5.0	16.9	16.0	67.8
	1000	24.3	13.9	5.9	27.4	14.2	7.0	26.0	16.4	6.5	25.2	19.9	81.7

$m = 4$

Tests	$n \setminus T$	Size ($H_0 : \lambda = 0$)						Power ($H_1 : \lambda = 0.25$)					
		$\alpha = 1$		$\alpha = 2/3$		$\alpha = 1/2$		$\alpha = 1$		$\alpha = 2/3$		$\alpha = 1/2$	
		100	200	500	100	200	500	100	200	500	100	200	500
ARDL adjusted CD	100	63.6	86.2	97.5	6.7	9.5	24.6	7.2	6.7	10.7	35.2	60.2	79.8
	200	66.5	90.9	96.3	5.4	7.6	14.5	4.8	5.5	6.7	20.8	41.9	67.9
	500	64.4	88.4	97.3	6.1	5.6	9.2	6.9	6.3	5.8	13.8	29.5	58.7
	1000	62.0	96.8	100.0	6.5	5.4	7.8	7.0	5.7	5.8	10.1	23.5	53.8
ARDL adjusted CD^*	100	7.6	7.0	10.7	9.8	8.5	11.3	9.0	8.3	8.6	58.0	83.8	99.7
	200	7.2	6.6	7.1	6.5	6.3	7.1	6.1	6.4	6.6	59.9	86.1	99.1
	500	6.4	5.3	5.7	7.0	6.4	5.6	7.2	7.0	5.7	61.3	85.6	99.6
	1000	6.5	6.1	5.3	7.0	5.9	5.1	7.4	6.2	6.0	61.6	85.6	99.5
ARDL adjusted CD_{W+}	100	6.7	6.1	7.4	6.7	7.0	7.4	6.3	5.8	8.0	8.2	8.3	22.0
	200	6.8	5.9	6.3	6.3	6.3	6.9	7.0	6.1	5.3	7.6	7.5	28.2
	500	13.5	10.4	6.1	14.1	8.4	6.4	14.0	9.3	5.7	15.4	13.4	52.7
	1000	22.8	13.4	5.3	24.3	14.9	6.3	23.3	14.5	6.2	24.0	18.6	73.0

Note: The DGP is given by (44) with $\beta_{11} = \beta_{12} = 0$ and contains a single latent factor with factor strength $\alpha = 1, 2/3, 1/2$. λ is the spatial autocorrelation coefficient of the error term. m_0 is the true number of factors and m is the number of selected PCs used to estimate factors. CD denotes the standard CD test statistic while CD^* denotes the bias-corrected CD test statistic. CD_{W+} denotes the power-enhanced randomized CD test statistic.

Table S.18: Size and power of ARDL adjusted tests of error cross-sectional dependence for the latent factor model with two factors ($m_0 = 2$) and serially correlated non-Gaussian errors

$m = 4$		Power ($H_1 : \lambda = 0.25$)											
Tests	$n \setminus T$	Size ($H_0 : \lambda = 0$)						$\alpha_1 = 1, \alpha_2 = 1$					
		$\alpha_1 = 1, \alpha_2 = 2/3$						$\alpha_1 = 2/3, \alpha_2 = 1/2$					
		100	200	500	100	200	500	100	200	500	100	200	500
ARDL adjusted CD	100	100.0	100.0	100.0	98.6	100.0	100.0	8.3	14.6	39.4	99.6	100.0	100.0
	200	100.0	100.0	100.0	99.4	100.0	100.0	5.9	8.6	24.1	99.8	100.0	100.0
	500	100.0	100.0	100.0	99.7	100.0	100.0	7.5	5.2	11.5	99.9	100.0	100.0
	1000	100.0	100.0	100.0	99.7	100.0	100.0	7.4	6.1	7.7	100.0	100.0	100.0
ARDL adjusted CD^*	100	9.0	8.9	14.1	8.0	8.9	12.1	10.8	9.0	10.4	32.8	54.9	87.1
	200	6.5	7.1	8.0	6.4	6.9	7.1	8.2	7.4	7.0	29.8	49.2	86.8
	500	6.4	6.1	6.2	6.6	6.0	5.6	9.6	7.9	6.5	31.3	53.1	85.0
	1000	6.2	5.7	5.1	7.1	6.2	4.6	8.9	6.5	4.8	30.5	46.8	82.2
ARDL adjusted CD_{W+}	100	6.9	7.4	8.1	6.0	6.4	12.1	6.1	6.4	10.4	7.9	9.7	25.0
	200	8.1	6.3	5.2	7.8	6.9	6.0	7.8	6.7	6.4	9.7	9.9	31.8
	500	13.0	7.6	5.2	15.0	8.6	5.7	14.7	8.4	5.5	13.6	11.9	59.4
	1000	25.8	13.7	6.0	26.9	13.0	6.2	24.1	14.6	6.3	24.2	19.9	74.9

$m = 6$		Power ($H_1 : \lambda = 0.25$)											
Tests	$n \setminus T$	Size ($H_0 : \lambda = 0$)						$\alpha_1 = 1, \alpha_2 = 1$					
		$\alpha_1 = 1, \alpha_2 = 2/3$						$\alpha_1 = 2/3, \alpha_2 = 1/2$					
		100	200	500	100	200	500	100	200	500	100	200	500
ARDL adjusted CD	100	99.9	100.0	100.0	98.5	100.0	100.0	8.2	15.0	39.4	99.5	100.0	100.0
	200	100.0	100.0	100.0	99.2	100.0	100.0	5.8	8.5	24.3	99.9	100.0	100.0
	500	100.0	100.0	100.0	99.5	100.0	100.0	7.6	5.7	10.9	100.0	100.0	100.0
	1000	100.0	100.0	100.0	99.6	100.0	100.0	8.0	5.9	7.4	100.0	100.0	100.0
ARDL adjusted CD^*	100	13.1	14.5	30.0	11.9	14.3	25.1	12.7	12.2	17.2	38.6	61.7	93.3
	200	7.6	8.4	11.8	7.9	8.1	10.4	8.8	9.0	9.1	32.9	54.6	90.5
	500	6.8	6.8	6.7	6.9	5.9	6.0	10.0	8.3	6.5	31.9	52.8	84.4
	1000	7.1	6.2	5.7	8.3	7.0	4.9	9.3	6.6	4.9	33.0	47.4	82.4
ARDL adjusted CD_{W+}	100	6.0	6.9	15.6	6.4	6.3	17.1	6.8	6.1	10.6	6.6	7.6	27.7
	200	7.7	6.3	7.1	8.0	5.7	5.5	7.4	5.6	5.7	7.7	8.6	19.1
	500	12.1	8.2	5.8	13.9	8.4	6.3	12.9	8.4	5.8	14.2	11.9	44.1
	1000	24.0	12.8	5.9	26.5	14.4	5.5	23.2	13.5	6.9	24.2	18.2	67.7

Note: The DGP is given by (44) with $\beta_{i2} = 0$, and contains two latent factors with factor strengths $(\alpha_1, \alpha_2) = [(1, 1), (1, 2/3), (2/3, 1/2)]$. λ is the spatial autocorrelation coefficient of the error term. m_0 is the true number of factors and m is the number of selected PCs used to estimate factors. CD denotes the standard CD test statistic while CD^* denotes the bias-corrected CD test statistic. CD_{W+} denotes the power-enhanced randomized CD test statistic.

Table S.19: Size and power of ARDL adjusted tests of error cross-sectional dependence for panel regression model with one latent factor ($m_0 = 1$) and serially correlated non-Gaussian errors

$m = 2$		Power ($H_1 : \lambda = 0.25$)											
Tests	$n \setminus T$	Size ($H_0 : \lambda = 0$)						Power ($H_1 : \lambda = 0.25$)					
		$\alpha = 1$		$\alpha = 2/3$		$\alpha = 1/2$		$\alpha = 1$		$\alpha = 2/3$		$\alpha = 1/2$	
		100	200	500	100	200	500	100	200	500	100	200	500
ARDL adjusted CD	100	66.7	90.9	98.9	7.2	10.1	20.4	7.9	7.0	9.9	28.5	49.2	68.2
	200	68.7	93.2	99.9	7.5	5.8	13.6	8.6	5.5	6.4	18.7	36.4	58.7
	500	68.1	96.0	100.0	8.2	7.7	8.9	8.0	5.6	5.8	11.1	25.0	53.0
	1000	66.6	95.9	100.0	7.2	5.6	5.2	7.9	6.9	4.6	10.9	21.6	51.7
ARDL adjusted CD^*	100	5.8	6.3	7.4	8.6	8.2	6.8	8.8	7.2	7.4	57.8	84.2	99.5
	200	6.8	6.7	5.5	9.4	6.9	7.0	9.9	7.2	6.5	60.0	85.2	99.3
	500	6.1	5.5	5.1	9.5	7.5	6.5	8.5	6.7	6.1	60.3	84.7	99.6
	1000	6.2	5.7	5.0	8.5	6.8	5.1	8.2	7.4	4.8	60.6	85.7	99.6
ARDL adjusted CD_{W+}	100	7.6	6.3	6.2	6.9	6.4	7.0	6.7	6.2	6.4	8.4	9.9	32.9
	200	9.0	5.9	5.1	8.9	6.0	6.0	9.2	6.7	5.7	9.8	9.9	46.7
	500	15.0	9.1	5.3	13.2	9.3	5.5	12.8	9.7	5.3	15.0	14.1	67.8
	1000	25.9	14.0	6.2	23.8	15.6	5.9	22.8	14.2	5.8	24.1	19.4	79.9
$m = 4$		Power ($H_1 : \lambda = 0.25$)											
Tests	$n \setminus T$	Size ($H_0 : \lambda = 0$)						Power ($H_1 : \lambda = 0.25$)					
		$\alpha = 1$		$\alpha = 2/3$		$\alpha = 1/2$		$\alpha = 1$		$\alpha = 2/3$		$\alpha = 1/2$	
		100	200	500	100	200	500	100	200	500	100	200	500
ARDL adjusted CD	100	68.1	91.0	98.7	6.9	10.2	22.4	7.8	7.2	10.4	37.8	61.2	80.5
	200	67.9	93.2	99.8	6.7	6.6	13.0	8.4	6.0	7.4	23.0	43.1	69.7
	500	67.8	95.8	100.0	8.0	7.0	9.0	7.4	6.7	5.7	12.8	29.3	57.8
	1000	66.1	95.9	100.0	6.6	5.6	5.0	7.2	6.6	4.7	11.1	23.0	55.2
ARDL adjusted CD^*	100	7.6	8.0	11.8	10.0	9.8	10.3	10.7	10.0	12.9	56.4	84.6	99.7
	200	7.8	7.2	6.6	8.4	7.1	8.5	9.8	8.1	8.6	57.4	84.8	99.4
	500	6.8	5.6	5.8	9.1	7.7	6.9	7.9	7.8	6.5	59.9	84.4	99.5
	1000	6.9	5.9	5.5	7.6	6.5	5.9	7.7	7.6	5.3	60.1	85.8	99.7
ARDL adjusted CD_{W+}	100	7.0	6.7	7.4	6.3	6.1	7.8	6.6	6.6	8.3	7.8	7.6	21.8
	200	7.3	5.9	5.9	8.3	5.8	6.3	7.9	5.8	6.0	8.6	8.5	27.2
	500	13.1	9.8	5.1	14.4	7.8	6.2	11.9	8.7	5.5	14.4	13.6	53.8
	1000	23.9	14.3	6.5	24.1	11.9	5.7	22.6	13.3	6.0	23.0	19.3	71.8

Note: The DGP is given by (44) with β_{11} and β_{12} both generated from normal distribution, and contains a single latent factor with factor strength $\alpha = 1, 2/3, 1/2$. λ is the spatial autocorrelation coefficient of the error term. m_0 is the true number of factors and m is the number of selected PCs used to estimate factors. CD denotes the standard CD test statistic while CD^* denotes the bias-corrected CD test statistic. CD_{W+} denotes the power-enhanced randomized CD test statistic.

Table S.20: Size and power of ARDL adjusted tests of error cross-sectional dependence for panel regression model with two latent factors ($m_0 = 2$) and serially correlated non-Gaussian errors

$m = 4$		Size ($H_0 : \lambda = 0$)												Power ($H_1 : \lambda = 0.25$)											
Tests	$n \setminus T$	$\alpha_1 = 1, \alpha_2 = 1$						$\alpha_1 = 2/3, \alpha_2 = 1/2$						$\alpha_1 = 1, \alpha_2 = 1$						$\alpha_1 = 1, \alpha_2 = 2/3$					
		100	200	500	100	200	500	100	200	500	100	200	500	100	200	500	100	200	500	100	200	500	100	200	500
ARDL adjusted CD	100	99.9	100.0	100.0	98.4	100.0	100.0	8.2	12.3	38.5	99.4	100.0	100.0	91.5	98.6	100.0	91.5	98.6	100.0	36.2	43.2	50.4	36.2	43.2	50.4
	200	100.0	100.0	100.0	99.3	100.0	100.0	8.3	7.0	21.5	99.6	100.0	100.0	92.2	99.7	100.0	92.2	99.7	100.0	59.8	75.4	85.4	59.8	75.4	85.4
	500	100.0	100.0	100.0	99.6	100.0	100.0	7.5	7.0	10.7	99.9	100.0	100.0	91.5	99.5	100.0	91.5	99.5	100.0	77.3	94.8	99.8	77.3	94.8	99.8
	1000	100.0	100.0	100.0	99.6	100.0	100.0	9.7	6.1	7.1	99.9	100.0	100.0	91.5	99.9	100.0	91.5	99.9	100.0	82.1	97.9	100.0	82.1	97.9	100.0
ARDL adjusted CD^*	100	8.4	8.4	12.0	8.3	9.4	9.6	11.2	12.1	9.3	33.3	50.8	83.4	40.5	64.4	92.1	40.5	64.4	92.1	79.4	96.5	100.0	79.4	96.5	100.0
	200	6.4	6.8	6.5	7.5	7.8	6.3	11.5	9.2	8.1	29.1	46.2	80.7	39.8	60.7	91.8	39.8	60.7	91.8	82.6	98.4	100.0	82.6	98.4	100.0
	500	7.0	6.7	5.8	6.6	5.6	5.4	10.1	9.2	8.2	30.3	45.0	80.1	43.5	61.9	91.9	43.5	61.9	91.9	85.9	98.3	100.0	85.9	98.3	100.0
	1000	6.4	5.0	5.8	7.5	6.6	5.8	10.5	9.0	7.2	29.3	44.2	78.7	42.4	64.2	90.6	42.4	64.2	90.6	86.6	99.2	100.0	86.6	99.2	100.0
ARDL adjusted CD_{W+}	100	7.2	7.2	10.3	7.3	6.6	10.2	6.1	6.4	9.9	7.9	9.4	27.9	7.6	8.6	29.4	7.6	8.6	29.4	7.7	10.2	28.1	7.7	10.2	28.1
	200	8.5	6.2	6.3	7.6	7.0	5.9	7.6	5.9	5.9	9.8	9.3	32.8	10.0	9.5	32.0	10.0	9.5	32.0	8.0	9.0	34.0	8.0	9.0	34.0
	500	12.1	8.8	5.6	12.2	8.2	5.8	12.8	9.0	4.8	13.9	13.2	59.5	14.7	12.5	57.6	14.7	12.5	57.6	14.5	14.0	60.7	14.5	14.0	60.7
	1000	24.6	15.6	5.9	25.0	12.8	6.6	23.5	12.6	6.5	23.7	22.2	73.1	24.2	19.1	73.0	24.2	19.1	73.0	23.3	18.9	75.6	23.3	18.9	75.6
$m = 6$		Size ($H_0 : \lambda = 0$)												Power ($H_1 : \lambda = 0.25$)											
Tests	$n \setminus T$	$\alpha_1 = 1, \alpha_2 = 1$						$\alpha_1 = 2/3, \alpha_2 = 1/2$						$\alpha_1 = 1, \alpha_2 = 1$						$\alpha_1 = 1, \alpha_2 = 2/3$					
		100	200	500	100	200	500	100	200	500	100	200	500	100	200	500	100	200	500	100	200	500	100	200	500
ARDL adjusted CD	100	100.0	100.0	100.0	97.9	100.0	100.0	8.1	13.2	40.7	99.6	100.0	100.0	93.9	99.3	100.0	93.9	99.3	100.0	24.7	32.6	50.5	24.7	32.6	50.5
	200	100.0	100.0	100.0	99.1	100.0	100.0	7.6	7.7	22.0	99.9	100.0	100.0	92.9	99.7	100.0	92.9	99.7	100.0	49.2	61.7	71.7	49.2	61.7	71.7
	500	100.0	100.0	100.0	99.4	100.0	100.0	8.6	6.5	11.3	99.9	100.0	100.0	92.5	99.6	100.0	92.5	99.6	100.0	70.8	91.6	99.2	70.8	91.6	99.2
	1000	100.0	100.0	100.0	99.5	100.0	100.0	8.9	6.1	6.8	99.9	100.0	100.0	91.4	99.9	100.0	91.4	99.9	100.0	76.6	96.6	100.0	76.6	96.6	100.0
ARDL adjusted CD^*	100	12.0	14.0	29.5	11.0	14.9	25.8	12.3	16.3	16.7	39.0	59.8	91.6	44.9	70.5	96.1	44.9	70.5	96.1	74.4	95.5	99.9	74.4	95.5	99.9
	200	7.3	8.4	9.6	8.8	8.6	9.2	11.5	10.3	9.5	32.5	49.6	85.1	42.4	64.2	94.0	42.4	64.2	94.0	78.1	97.1	100.0	78.1	97.1	100.0
	500	7.7	7.1	6.5	6.9	6.4	5.5	10.3	9.5	8.3	31.9	46.9	80.5	44.4	62.5	92.7	44.4	62.5	92.7	82.5	98.0	100.0	82.5	98.0	100.0
	1000	7.0	5.3	6.5	7.9	6.7	5.9	9.8	8.3	7.1	30.5	45.1	79.5	43.5	63.5	91.1	43.5	63.5	91.1	83.5	98.4	100.0	83.5	98.4	100.0
ARDL adjusted CD_{W+}	100	6.5	7.4	21.7	7.0	7.1	17.3	7.1	7.7	10.4	6.9	9.4	34.3	7.1	8.0	32.4	7.1	8.0	32.4	8.0	9.1	22.7	8.0	9.1	22.7
	200	6.9	5.8	6.8	7.6	6.4	7.0	7.4	6.3	6.2	8.0	9.2	21.9	8.0	7.8	20.1	8.0	7.8	20.1	8.6	7.7	19.6	8.6	7.7	19.6
	500	12.9	7.5	6.3	13.2	8.3	5.7	12.6	9.2	6.1	14.5	11.8	45.6	12.6	10.8	46.9	12.6	10.8	46.9	13.7	12.4	44.9	13.7	12.4	44.9
	1000	23.5	14.9	5.6	24.8	12.7	6.3	23.4	12.5	6.2	23.5	19.7	66.5	24.6	18.1	65.0	24.6	18.1	65.0	22.5	18.5	64.7	22.5	18.5	64.7

Note: The DGP is given by (44) with β_{11} and β_{22} both generated from normal distribution, and contains two latent factors with factor strengths $(\alpha_1, \alpha_2) = [(1, 1), (1, 2/3), (2/3, 1/2)]$. λ is the spatial autocorrelation coefficient of the error term. m_0 is the true number of factors and m is the number of selected PCs used to estimate factors. CD denotes the standard CD test statistic while CD^* denotes the bias-corrected CD test statistic. CD_{W+} denotes the power-enhanced randomized CD test statistic.

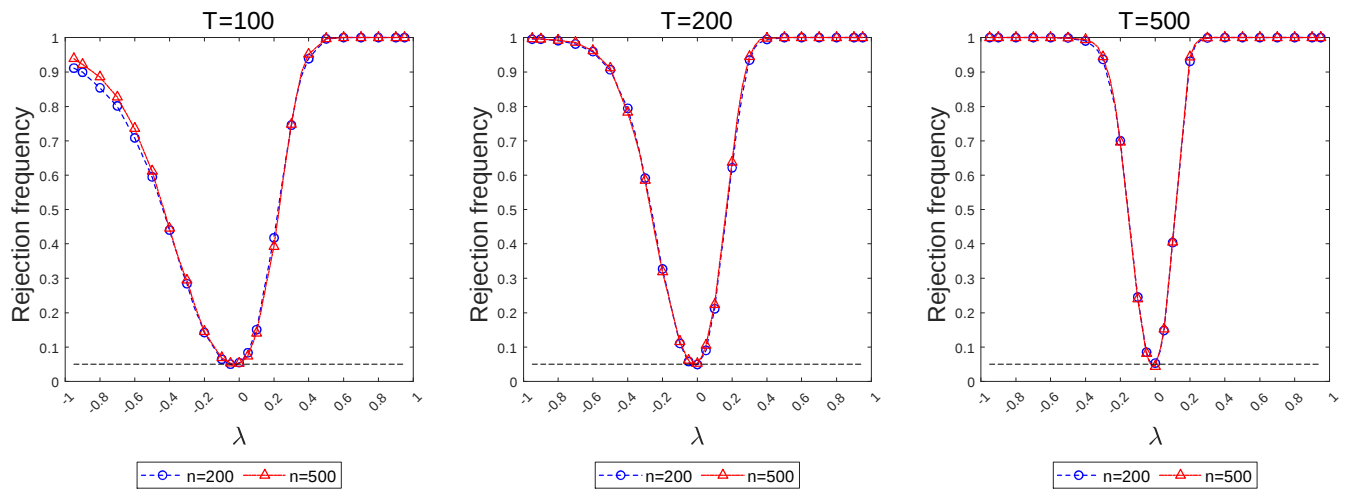


Figure S.1: Empirical power functions of the CD^* test against spatial alternatives for the pure latent factor model with one latent strong factor and serially independent Gaussian errors, for different sample sizes.

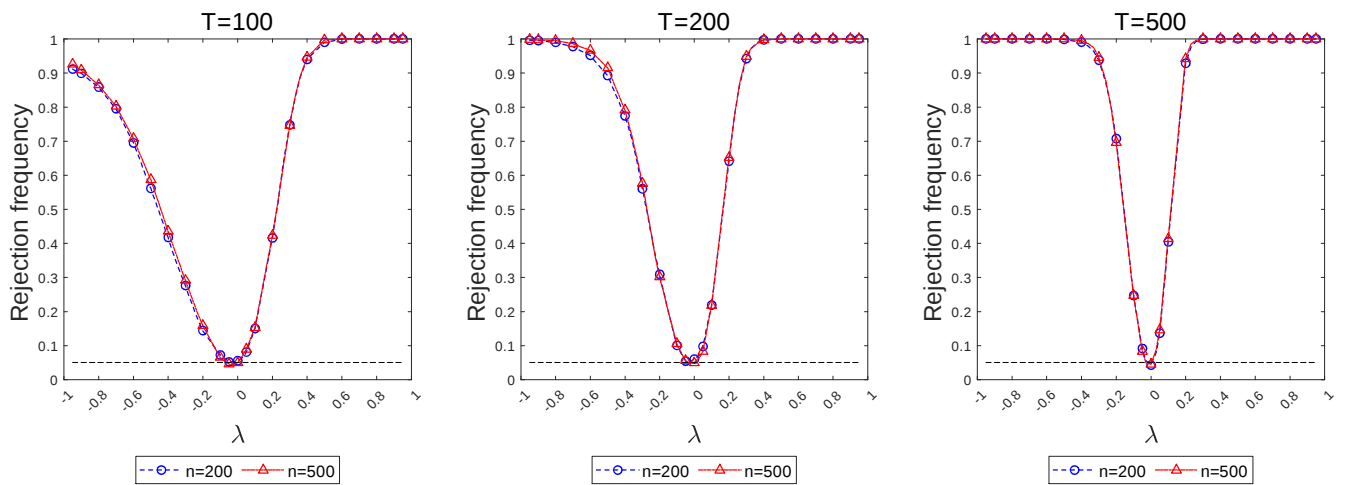


Figure S.2: Empirical power functions of the CD^* test against spatial alternatives for panel regression models with one latent strong factor and serially independent Gaussian errors, for different sample sizes.

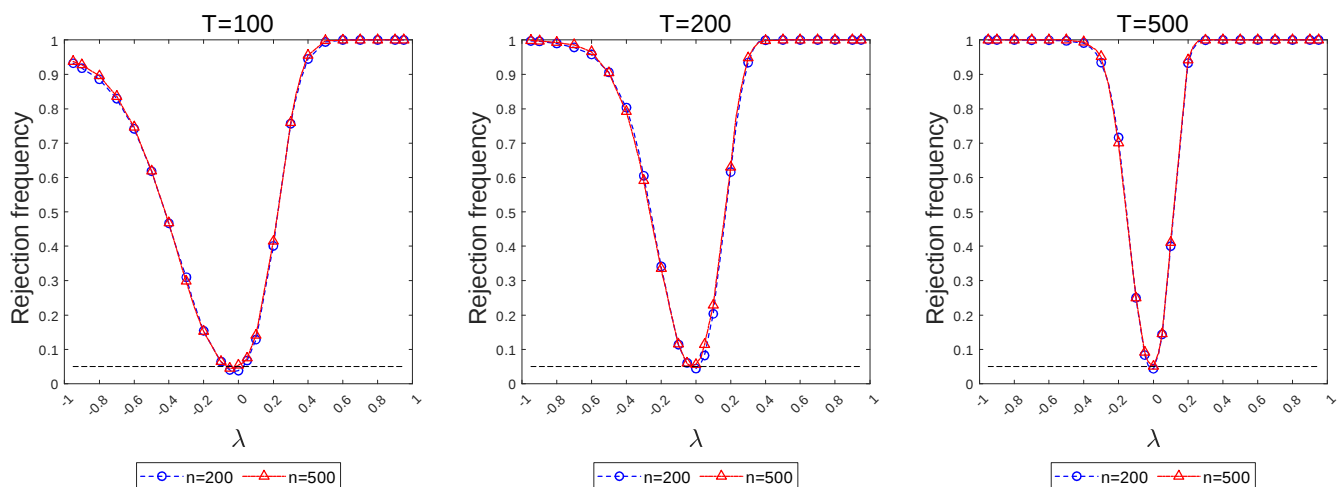


Figure S.3: Empirical power functions of the CD^* test against spatial alternatives for the pure latent factor model with one latent strong factor and serially independent non-Gaussian errors, for different sample sizes.

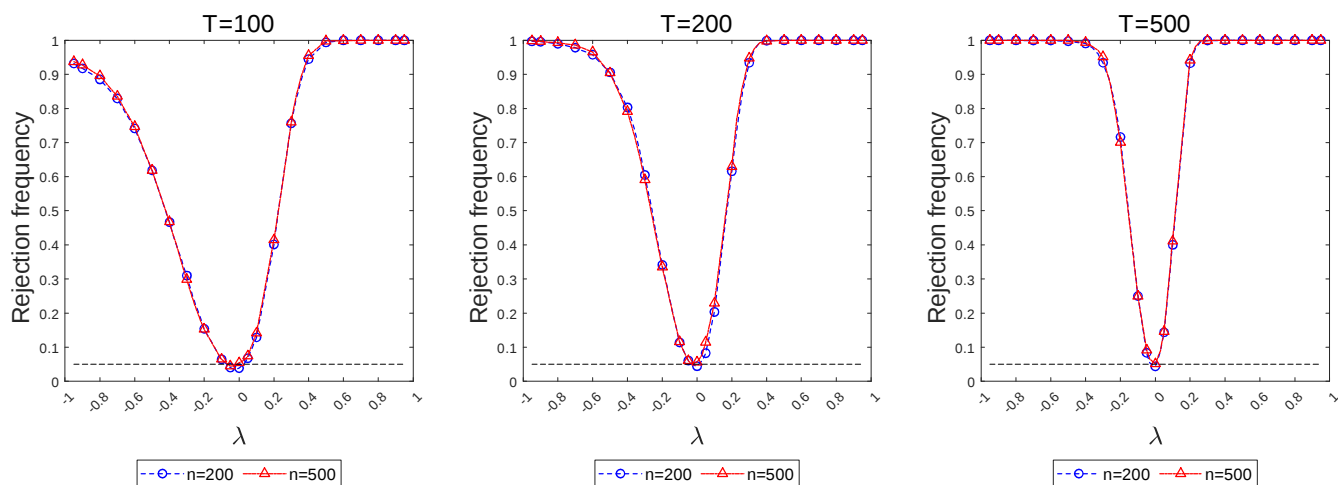


Figure S.4: Empirical power functions of the CD^* test against spatial alternatives for panel regression models with one latent strong factor and serially independent non-Gaussian errors, for different sample sizes.

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