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On the Uniqueness of the Mixed Equilibrium in the Tullock Contest

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On the Uniqueness of the Mixed Equilibrium in the Tullock Contest

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Abstract. It is shown that the two-player Tullock contest admits precisely one equilibrium in randomized strategies.

Keywords. Tullock contest, mixed equilibrium, sequence spaces, Cauchy matrices, Dirichlet series

JEL codes. C72 Noncooperative games; D72 Political Processes: Rent-Seeking, Lobbying, Elections, Legislatures, and Voting Behavior

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1 Preliminaries

1.1 Introduction

In [Tullock's \(1980\)](#) model of political conflict, contestants compete in efforts to obtain a reward that cannot be easily allocated otherwise. Suppose there are two contestants, A and B, with respective valuations of winning given by $V_A \geq V_B > 0$. When contestants choose efforts $x_A \geq 0$ and $x_B \geq 0$, contestant i 's payoff is given by

$$u_i(x_A, x_B) = \frac{x_i^R}{x_A^R + x_B^R} V_i - x_i,$$

where $R \geq 0$ is an exogenous parameter, and the ratio is interpreted as $\frac{1}{2}$ if otherwise undefined. While that model has been the workhorse of contest theory for several decades ([Beviá and Corchón, 2024](#)), the equilibrium analysis remained incomplete. Specifically, prior work did not address the question whether the equilibrium for $R > 2$ is unique. In this paper, we establish the uniqueness of the mixed equilibrium in the two-player Tullock contest.

1.2 Statement of the main result

Suppose that each contestant $i \in \{A, B\}$ chooses a probability distribution over the interval $[0, V_i]$.¹ Then, payoffs are bounded, and a *mixed-strategy Nash equilibrium* (MSNE) may be defined as usual ([Dasgupta and Maskin, 1986](#)). The main result of the present paper is the following.

Theorem 1. *The two-player Tullock contest has a unique MSNE, for any $V_A \geq V_B > 0$ and $R > 2$.*

Proof. See [Section 3](#). □

¹This assumption is reasonable because bids that exceed the contestant's valuation are strictly dominated by the zero bid.

There is a sense in which Theorem 1 completes the equilibrium analysis of the two-player case. Indeed, for $R \in [0, 1 + (V_B/V_A)^R]$, it was known that a unique equilibrium in pure strategies exists (Nti, 1999). Moreover, for $R \in (1 + (V_B/V_A)^R, 2]$, a semi-mixed equilibrium, in which contestant A plays a pure strategy while contestant B randomizes between a positive and a zero effort, was known to exist (Wang, 2010). Finally, for $R \in [0, 2]$, there are no other MSNE, i.e., the equilibrium is always unique (Ewerhart, 2017b; Feng and Lu, 2017). Thus, by covering the remaining case $R > 2$, Theorem 1 indeed rounds up the equilibrium analysis of the two-player Tullock contest.

1.3 Summary of the proof

The proof of Theorem 1 makes use of sequence spaces, Cauchy matrices, and Dirichlet series. Background information on these mathematical tools is provided in an [Appendix](#). The proof has two steps. First, we rewrite the condition of complete rent dissipation as an operator equation in the Banach space of absolutely summable sequences. The operator is represented by an infinite symmetric Cauchy matrix with finite supremum norm. Second, we show that the infinite Cauchy matrix is positive definite. For this, we rewrite the infinite Cauchy matrix as an integral over a parameterized matrix that decomposes naturally as the outer product of two identical infinite vectors. This approach reduces the problem of uniqueness of the mixed equilibrium to the question if the coefficients of a Dirichlet series are uniquely determined. As the representation is indeed unique (Hardy and Riesz, 1915), this completes the proof.

1.4 Overview

The remainder of the paper is structured as follows. Section 2 reviews prior work. Section 3 presents the proof of Theorem 1. The Appendix provides the necessary background on the mathematical tools employed in the proof.

2 Prior work on the case $R > 2$

We are aware of four papers that made progress on the research question addressed in the present paper.

Assuming homogeneous valuations and $R > 2$, Baye et al. (1994) proved existence of a MSNE with complete rent dissipation. Further, working with inequalities, they provided an upper bound on the equilibrium payoffs in a discrete version of the contest where efforts are chosen from an equidistant grid. That upper bound tends to zero as the grid becomes finer and finer.

Alcalde and Dahm (2010) defined an *all-pay auction equilibrium* of a probabilistic contest as a MSNE in which bids, winning probabilities, and payoffs coincide in expectation with the respective values of the corresponding all-pay auction. To a given MSNE in the two-player contest with homogeneous valuations, they associated an all-pay auction equilibrium in the contest with heterogeneous valuations.²

Ewerhart (2017a) studied the equilibrium set of probabilistic contests. It was shown, in particular, that any MSNE of the two-player Tullock contest with $R > 2$ is an all-pay auction equilibrium. Therefore, uniqueness of the MSNE in the special case where $V_A = V_B > 0$ implies uniqueness for general valuations

²Their findings also imply that Theorem 1 does not extend to more than two players. Specifically, suppose that $N \geq 3$ contestants compete for a homogeneously valued prize and $R > 2$. Then, for any pair of contestants, an all-pay auction equilibrium exists in which these two contestants are active while all other contestants exert no effort (Alcalde and Dahm, 2010, proof of Thm. 3.2). Thus, there are multiple equilibria in this case.

$V_A \geq V_B > 0$. Further, given that a homogeneous prize in the Tullock contest may always be normalized to unity without loss of generality, this shows that it suffices to verify the uniqueness claim in Theorem 1 in the special case where $V_A = V_B = 1$ and $R > 2$.

With a focus on this special case, [Ewerhart \(2015\)](#) further explored the nature of the MSNE. In the proof of Theorem 1, we will make use of the following result.

Lemma 1 ([Ewerhart, 2015](#)). *Suppose that $V_A = V_B = 1$ and $R > 2$. Then, there is a strictly declining sequence of positive bid levels $\{y_k\}_{k=1}^{\infty}$ with $\lim_{k \rightarrow \infty} y_k = 0$ that jointly nest the support of any equilibrium strategy. Further, for any $k \in \{1, 2, \dots\}$, we have the condition of complete rent dissipation at y_k ,*

$$y_k = \sum_{l=1}^{\infty} \frac{q_l y_k^R}{y_k^R + y_l^R},$$

where q_l denotes the probability attached by the equilibrium strategy to the bid level y_l .

Proof. See [Ewerhart \(2015\)](#). □

Thus, under the assumptions of Lemma 1, any equilibrium strategy concentrates all probability weight on countably many positive bid levels that are contained in a known set. In particular, any such strategy may be represented by a probability distribution $\{q_k\}_{k=1}^{\infty}$.

3 Proof of Theorem 1

Consider the infinite symmetric matrix

$$Y = \left\{ \frac{y_k^R y_l^R}{y_k^R + y_l^R} \right\}_{k,l=1}^{\infty},$$

where the sequence $\{y_k\}_{k=1}^\infty$ is defined via Lemma 1. Given that $\{y_k\}_{k=1}^\infty$ is strictly declining, the entries of Y are bounded by

$$\frac{y_k^R y_l^R}{y_k^R + y_l^R} \leq \frac{y_1^R}{2}.$$

The following lemma says that, if probability distributions $q^* = \{q_k^*\}_{k=1}^\infty$ and $q^{**} = \{q_k^{**}\}_{k=1}^\infty$ each represent an equilibrium strategy, then the difference $q^* - q^{**} \in \ell^1(\mathbb{R})$ is in the null space of Y .

Lemma 2. *Let $q^* = \{q_k^*\}_{k=1}^\infty$ and $q^{**} = \{q_k^{**}\}_{k=1}^\infty$ be probability distributions each representing an equilibrium. Then, for any $k \in \{1, 2, \dots\}$, we have*

$$\sum_{l=1}^{\infty} \frac{y_k^R y_l^R}{y_k^R + y_l^R} (q_l^* - q_l^{**}) = 0.$$

Proof. From Lemma 1, we see that

$$y_k = \sum_{l=1}^{\infty} \frac{q_l^* y_k^R}{y_k^R + y_l^R}.$$

Hence, exploiting that q^* is a probability distribution,

$$\begin{aligned} 1 - y_k &= \sum_{l=1}^{\infty} q_l^* \left(1 - \frac{y_k^R}{y_k^R + y_l^R} \right) \\ &= \sum_{l=1}^{\infty} \frac{q_l^* y_l^R}{y_k^R + y_l^R}. \end{aligned}$$

Multiplying through with y_k^R yields

$$\sum_{l=1}^{\infty} \frac{q_l^* y_k^R y_l^R}{y_k^R + y_l^R} = (1 - y_k) y_k^R.$$

Subtracting the analogous relationship in with q^* is replaced by q^{**} yields the claim. \square

We note now that

$$Y = \left\{ \frac{1}{y_k^{-R} + y_l^{-R}} \right\}_{k,l=1}^{\infty}.$$

The following result, which might be of independent interest, therefore completes the proof of Theorem 1.

Lemma 3. *Let $C = \{1/(\lambda_k + \lambda_l)\}_{k,l=1}^\infty$ be an infinite symmetric Cauchy matrix with $0 < \lambda_1 < \lambda_2 < \dots$ and $\lim_{k \rightarrow \infty} \lambda_k = \infty$. Then, $C\alpha = 0$ implies $\alpha = 0$, for any $\alpha \in \ell^1(\mathbb{R})$.*

Proof. We note that

$$C = \int_0^\infty B(s) ds,$$

where

$$B(s) = \{\exp(-s(\lambda_k + \lambda_l))\}_{k,l=1}^\infty.$$

Moreover, $B(s) = b(s)b(s)^T$, with

$$b(s) = \{\exp(-s\lambda_k)\}_{k=1}^\infty \in \ell^\infty(\mathbb{R}),$$

for any $s \geq 0$. Take now some $\alpha \in \ell^1(\mathbb{R})$. Then,

$$\begin{aligned} \alpha^T B(s) \alpha &= \sum_{k,l=1}^\infty \exp(-s(\lambda_k + \lambda_l)) \alpha_k \alpha_l \\ &= \left(\sum_{k=1}^\infty \exp(-s\lambda_k) \alpha_k \right)^2 \\ &= |b(s)^T \alpha|^2. \end{aligned}$$

Noting that

$$b(s)^T \alpha = \sum_{k=1}^\infty \alpha_k \exp(-s\lambda_k) \leq \|\alpha\|_1 \exp(-s\lambda_1)$$

holds for any $s \geq 0$, Lebesgue's dominated convergence theorem implies

$$\int_0^\infty |b(s)^T \alpha|^2 ds = \alpha^T C \alpha.$$

Hence, if $C\alpha = 0$, then the Dirichlet series $\sum_{k=1}^\infty \alpha_k \exp(-s\lambda_k)$ vanishes almost everywhere on the positive real axis. By Lemma A.1 in the [Appendix](#), this shows that the coefficients α_k all vanish. Hence, $\alpha = 0$, which proves the lemma. \square

A Appendix: Mathematical tools

This section provides background on the mathematical tools used in the proof of Theorem 1. Specifically, we discuss sequence spaces (Subsection A.1), Cauchy matrices (Subsection A.2), and Dirichlet series (Subsection A.3).

A.1 Sequence spaces

Let $\ell^1(\mathbb{R}) = \{x = (x_1, x_2, \dots) : \|x\|_1 < \infty\}$ denote the Banach space of absolutely summable sequences, where $\|x\|_1 = \sum_{k=1}^{\infty} |x_k|$. Further, let $\ell^\infty(\mathbb{R}) = \{x = (x_1, x_2, \dots) : \|x\|_\infty < \infty\}$ denote the Banach space of bounded sequences, where $\|x\|_\infty = \sup_{k \in \{1, 2, \dots\}} |x_k|$. Then, the product $x^T b \equiv b^T x \equiv \sum_{k=1}^{\infty} x_k b_k \in \mathbb{R}$ converges absolutely for any $x \in \ell^1(\mathbb{R})$ and $b \in \ell^\infty(\mathbb{R})$ (Aliprantis and Border, 1994, Ch. 16).

Let $\mathcal{M}^\infty(\mathbb{R})$ denote the space of infinite matrices $A = \{a_{k,l}\}_{k,l=1}^{\infty}$ (Cooke, 1950) with finite supremum norm, i.e., matrices for which $\sup_{k,l} |a_{k,l}| < \infty$. Then, for any $A \in \mathcal{M}^\infty(\mathbb{R})$ and $x \in \ell^1(\mathbb{R})$, we may define $Ax \in \ell^\infty(\mathbb{R})$ component-wise via $(Ax)_k = \sum_{l=1}^{\infty} a_{k,l} x_l \in \mathbb{R}$. In particular, for any $x, \hat{x} \in \ell^1(\mathbb{R})$, we have $x^T A \hat{x} \equiv (Ax)^T \hat{x} = x^T (A \hat{x}) = \sum_{k,l=1}^{\infty} a_{k,l} x_k \hat{x}_l \in \mathbb{R}$. The *null space* of $A \in \mathcal{M}^\infty(\mathbb{R})$ is the set of $x \in \ell^1(\mathbb{R})$ such that $Ax = 0$.

An infinite matrix $A = \{a_{k,l}\}_{k,l=1}^{\infty}$ is called *symmetric* if $a_{k,l} = a_{l,k}$ for all $k, l \in \{1, 2, \dots\}$. An example is the *outer product* $bb^T = \{b_k b_l\}_{k,l=1}^{\infty} \in \mathcal{M}^\infty(\mathbb{R})$, where $b \in \ell^\infty(\mathbb{R})$. We say that a symmetric infinite matrix $A \in \mathcal{M}^\infty(\mathbb{R})$ is *positive semi-definite* if $x^T A x \geq 0$ holds for all $x \in \ell^1(\mathbb{R})$. If, in addition, $x^T A x = 0$ implies $x = 0$, then we say that A is *positive definite*.

A.2 Cauchy matrices

Given positive parameters c_1, \dots, c_n , for some finite $n \geq 1$, the matrix

$$C_n = \left\{ \frac{1}{c_k + c_l} \right\}_{k,l=1}^n$$

is called a symmetric Cauchy matrix. If the parameters c_1, \dots, c_n are, in addition, pairwise distinct, then C_n is positive definite (Fiedler, 2010, Thm. A).

Infinite Cauchy matrices may be defined by letting $n = \infty$. As noted by Schur (1911, p. 18), infinite Cauchy matrices with pairwise different entries are positive semi-definite. Lemma 3 in the body of the paper provides conditions under which such matrices are positive definite.

A.3 Dirichlet series

An infinite series of the form

$$f(s) = \sum_{k=1}^{\infty} \alpha_k \exp(-s\lambda_k), \tag{A.1}$$

for a sequence of coefficients $\{\alpha_k\}_{k=1}^{\infty}$ and a *frequency* $\lambda = \{\lambda_k\}_{k=1}^{\infty}$, where $\lim_{k \rightarrow \infty} \lambda_k = \infty$, is called a (*general*) *Dirichlet series*. In the special case where $\lambda_k = k$, the series in (A.1) reduces to a power series in the variable $z = \exp(-s)$. Our interest, however, lies in the case of a general frequency.

Lemma A.1 (Hardy and Riesz, 1915). *Suppose that the series in (A.1) is convergent for $s = 0$, and that for some $\delta > 0$, we have $f(s) = 0$ for infinitely many $s \geq \delta$. Then $\alpha_k = 0$ for all $k \in \{1, 2, \dots\}$.*

Proof. See Hardy and Riesz (1915, Thm. 6). □

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