

A Service of

ZBW

Leibniz-Informationszentrum Wirtschaft Leibniz Information Centre for Economics

Fülle, Markus J.; Herwartz, Helmut

Article — Published Version Predicting tail risks by a Markov switching MGARCH model with varying copula regimes

Journal of Forecasting

Provided in Cooperation with: John Wiley & Sons

Suggested Citation: Fülle, Markus J.; Herwartz, Helmut (2024) : Predicting tail risks by a Markov switching MGARCH model with varying copula regimes, Journal of Forecasting, ISSN 1099-131X, Wiley, Hoboken, NJ, Vol. 43, Iss. 6, pp. 2163-2186, https://doi.org/10.1002/for.3117

This Version is available at: https://hdl.handle.net/10419/306122

Standard-Nutzungsbedingungen:

Die Dokumente auf EconStor dürfen zu eigenen wissenschaftlichen Zwecken und zum Privatgebrauch gespeichert und kopiert werden.

Sie dürfen die Dokumente nicht für öffentliche oder kommerzielle Zwecke vervielfältigen, öffentlich ausstellen, öffentlich zugänglich machen, vertreiben oder anderweitig nutzen.

Sofern die Verfasser die Dokumente unter Open-Content-Lizenzen (insbesondere CC-Lizenzen) zur Verfügung gestellt haben sollten, gelten abweichend von diesen Nutzungsbedingungen die in der dort genannten Lizenz gewährten Nutzungsrechte.



http://creativecommons.org/licenses/by-nc/4.0/

Terms of use:

Documents in EconStor may be saved and copied for your personal and scholarly purposes.

You are not to copy documents for public or commercial purposes, to exhibit the documents publicly, to make them publicly available on the internet, or to distribute or otherwise use the documents in public.

If the documents have been made available under an Open Content Licence (especially Creative Commons Licences), you may exercise further usage rights as specified in the indicated licence.





WILEY

Predicting tail risks by a Markov switching MGARCH model with varying copula regimes

Markus J. Fülle | Helmut Herwartz

Chair of Econometrics, University of Goettingen, Goettingen, Germany

Correspondence

Markus J. Fülle, Chair of Econometrics, University of Goettingen, Humboldtallee 3, D-37073 Goettingen, Germany. Email: fuelle@uni-goettingen.de

Funding information Deutsche Forschungsgemeinschaft, Grant/Award Number: HE 2188/14-1

Abstract

To improve the dynamic assessment of risks of speculative assets, we apply a Markov switching MGARCH approach to portfolio risk forecasting. More specifically, we take advantage of the flexible Markov switching copula multivariate GARCH (MS-C-MGARCH) model of Fülle and Herwartz (2022). As an empirical illustration, we take the perspective of a risk-averse agent and employ the suggested model for assessments of future risks of portfolios composed of a high-yield equity index (S&P 500) and two safe-haven investment instruments (i.e., Gold and US Treasury Bond Futures). We follow recent suggestions to employ the expected shortfall as a prime assessment of tail risks. To accurately evaluate the merits of the new model, we back-test the risk forecasting for daily returns over 10 years for heterogeneous market environments including, for example, the COVID-19 pandemic. We find that the MS-C-MGARCH model outperforms benchmark volatility models (MGARCH, C-MGARCH) in predicting both value-at-risk and expected shortfall. The superiority of the MS-C-MGARCH model becomes stronger, when the share of comparably risky assets in the portfolio is relatively large.

KEYWORDS

copula, ES, forecasting, Markov switching, MGARCH, VaR

1 | INTRODUCTION

Since the financial crisis in 2007–2009 and its impacts on the stability of the financial system, banking regulators have increasingly kept an eye on the resilience of financial institutions. As a replacement of the so-called Basel II regularities, Basel III (The Basel Committee on Banking Supervision, 2011) was introduced to sharpen the capital requirements for financial institutions. To assess the risks and determine the required common equity according to Basel III, financial institutions have to use forecasting models that are capable of measuring risk such that the institution using it withstands certain stress tests. The Basel Committee on Banking Supervision (2011) requires financial institutions to use the value at risk (VaR) (Jorion, 1996) as a risk measure. However, if return distributions feature excess (left-hand side) tail probability mass, estimated VaRs might become inaccurate (Acerbi & Tasche, 2002; Bradley & Taqqu, 2003; Tasche, 2002). The expected shortfall (ES) has been developed to account for such tail risks. Specifically, it represents the expected loss encountered in the (lowest) quantiles of return distributions (Acerbi & Tasche, 2002; Bradley & Taqqu, 2003; Tasche, 2002). The Basel

This is an open access article under the terms of the Creative Commons Attribution-NonCommercial License, which permits use, distribution and reproduction in any medium, provided the original work is properly cited and is not used for commercial purposes. © 2024 The Authors. *Journal of Forecasting* published by John Wiley & Sons Ltd. Committee on Banking Supervision (2013) has proposed to use the ES for risk assessment.

Besides the choice of an unnecessarily weak risk measure, false distributional assumptions regarding a predictive model could result in inadequate and imprecise risk forecasts that might lead to overfulfillment or underfulfillment of capital requirements. Hence, a more accurate distributional model has the potential to mitigate the capital tied up due to Basel III requirements and/or improve the valuation of default risks. In this regard, this work aims to improve the risk forecasting accuracy by a sophisticated and flexible process for (co-)variance and higher order dependency estimation. The generalized autoregressive conditional heteroskedasticity (GARCH) models of Bollerslev (1986) and their multivariate extensions of Engle and Kroner (1995) and Engle (2002) have been widely applied to capture stylized facts of (vector-valued) speculative returns, for example, volatility clustering and contagion. However, studies with non-Gaussian distributed residuals (Aloui et al., 2013; Huang et al., 2009; Weiß, 2013), asymmetric conditional volatility (Brooks & Persand, 2003b), changing (volatility) regimes (Ardia et al., 2018; Haas et al., 2004), or conditional skewness and/or kurtosis (Bali et al., 2008) have highlighted the benefits of greater model-implied flexibility in risk forecasting. Going beyond the assumption of time-invariant distributions of model residuals, Liu et al. (2017), Wang et al. (2011), Chollete et al. (2009), and Ang and Chen (2002) have advocated the existence of changing structural dependence patterns among speculative return series. Taking advantage of flexible distributional models, we evaluate the predictive performance of the Markov switching Copula-MGARCH (MS-C-MGARCH) introduced by Fülle and Herwartz (2022) as a generalization of the C-MGARCH model of Lee and Long (2009),¹ since it promises to increase the risk forecasting performance compared with benchmark models. For the included MGARCH process, we consider the asymmetric BEKK model of Kroner and Ng (1998) to account for leverage effects of (co)variances in response to unfavorable market news (see, among others, Nelson, 1991; Bekaert & Wu, 2000; Baur, 2012). Moreover, Billio and Pelizzon (2000); Brooks et al. (2002), and Brooks and Persand (2003a) have advocated the merits of accounting for asymmetries when it comes to estimating hedging flows and the VaR. However, Brooks and Persand (2003b) and Berkowitz and O'Brien (2002) have argued that the use of MGARCH models is computationally very demanding. Against this background, we employ the recently developed R-package for the estimation of BEKK(1,1,1) models of Fülle et al. (2022) that allows fast and efficient estimation and inference for this model class.²

In order to protect firms or portfolios against losses, so-called safe-haven assets are often added to portfolios or firm's balances to cut down risk in periods of excess market volatility. According to Baur and Lucey (2010), Baur (2012), and Baur and McDermott (2010), the most prominent safe havens are gold and long-term treasury bonds (e.g., US Treasury Bond Futures). Accordingly, modeling such safe-haven dependencies may enhance the risk assessment, especially in times of economic turmoil when hedging becomes increasingly important. To investigate the predictive out-of-sample performance for such portfolios admix with safe havens, we consider a 30-year three-dimensional time series stacking returns of gold, the S&P 500 index, and 10-year US Treasury Bonds Futures. To compare the risk-forecasting performance of benchmark specifications and the MS-C-MGARCH models, we apply a 20-year rolling window on the selected time series and estimate the ES 1 day ahead by the parameters, which are estimated for each time using the returns from the previous 20 years. In order to evaluate risk predictions, Christoffersen (1998) derived coverage tests to compare VaR out-of-sample forecasting performance across models. Here, we use the most recent coverage tests for the ES of Du and Escanciano (2017), which are based on the benchmark tests of Kupiec (1995) and Christoffersen (1998) for VaR. We concentrate on the ES forecasting performance to compare the MS-C-MGARCH model with benchmark processes in terms of predicting tail risks. Going beyond the testing of unconditional and conditional coverage, we use the quantile (QL) and the Fissler and Ziegel (2016) loss (FZL) as two specific loss functions for VaR and ES statistics and employ the Diebold and Mariano (1995) test (DM) and the so-called model confidence set (MCS) of Hansen et al. (2011) for comparative model assessments. We find that the MS-C-MGARCH model outperforms the benchmark models of Engle and Kroner (1995), Kroner and Ng (1998), Grier et al. (2004), and Lee and Long (2009), when it comes to assessing (downside) risks for portfolios comprising a relatively large share of volatile, that is, relatively risky, asset classes.

The remainder of this work is organized as follows. In Section 2, we briefly introduce the asymmetric BEKK, C-MGARCH, and MS-C-MGARCH models. Section 3 outlines our approach to ES backtesting and the portfolio distribution functions for the MS-C-MGARCH model. In Section 4, we provide an empirical study in the format discussed above. Section 5 concludes. In the appendix, we provide the definition for copulas, model-implied portfolio distributions, and additional further graphical representations of empirical results.

2 | MODEL SETUP

In this section, we first sketch the asymmetric BEKK model of Kroner and Ng (1998) and subsequently outline the copula MGARCH (C-MGARCH) model of Lee and Long (2009) and its Markov switching generalization of Fülle and Herwartz (2022).

2.1 | Asymmetric BEKK

Let r_t define an *N*-dimensional time series with $t \in \{1,...,T\}$. The information available at time *t* is $\Omega_t = \{r_1,...,r_t\}$. Then, the asymmetric BEKK(1,1,1) model of Kroner and Ng (1998) reads as

$$r_t = \mu_t + e_t, \tag{1}$$

$$e_t = H_t^{1/2} \eta_t, \ \eta_t \sim N(0, I_N),$$
 (2)

$$H_{t} = QQ' + A'e_{t-1}e'_{t-1}A + B'\gamma_{t-1}\gamma'_{t-1}B + G'H_{t-1}G, \quad (3)$$

where $\mu_t = \mathbb{E}(r_t | \Omega_{t-1})$ is a conditional expectation, and the exponent 1/2 indicates a covariance matrix decomposition.³ Without loss of generality, we henceforth assume that an empirical analysis is based on centered return processes such that $\mu_t = \mu = 0$. The matrices Q, A, B and Gare $(N \times N)$ -dimensional coefficient matrices, where Q is lower triangular. Accordingly, the parameter vector of the asymmetric BEKK(1,1,1) model is $\phi = (\operatorname{vech}(Q)', \operatorname{vec}(A)', \operatorname{vec}(B)', \operatorname{vec}(G)')' \in \mathbb{R}^{\frac{N(N+1)}{2} + 2N^2}$, and $\Phi \subseteq \mathbb{R}^{\frac{N(N+1)}{2} + 3N^2}$ is the corresponding parameter space.

Unlike the symmetric model specification of Engle and Kroner (1995), the asymmetric model in (3) allows for so-called leverage effects. Specifically, the model additionally allows jointly negative returns to exert a positive incremental influence on the conditional heteroscedasticity by setting

$$\gamma_t = e_t \mathbb{I}(e_t < N_0), \tag{4}$$

where $\mathbb{I}(\cdot)$ is the indicator function returning 1 if the condition inside the parentheses is fulfilled and 0 if not. Moreover, the relation $<_l$ for two *l*-dimensional vectors $x = (x_1, ..., x_l)'$ and $y = (y_1, ..., y_l)'$ is defined as

$$x < ly \Leftrightarrow x_j < y_j \forall j \in \{1, ..., l\}.$$

The asymmetric BEKK nests the symmetric BEKK model of Engle and Kroner (1995) in the case of B = 0. According to Grier et al. (2004), the asymmetric effects in the BEKK model might be chosen in a more flexible way by selecting a combination of positive and negative returns that is supposed to have the strongest effects on the conditional covariance. Covariance stationarity and uniqueness (identification) of the asymmetric BEKK (1,1,1) hold, respectively, if the following two conditions are fulfilled (Engle & Kroner, 1995):

$$\max\{|Eig(A \otimes A + \mathbb{E}(\mathbb{I}(e_t <_N 0))B \otimes B + G \otimes G)|\} < 1 \text{ and} \\ \operatorname{diag}(Q), \ a_{11}, \ b_{11}, \\ g_{11} > 0,$$
(5)

where a_{11}, b_{11} and g_{11} denote the first diagonal element of *A*, *B* and *G*, respectively, and the diag(\cdot) operator collects the diagonal elements of a square matrix. The expected occurrence of asymmetric effects, $\mathbb{E}(\mathbb{I}(e_t < N0))$, is estimated by the empirical mean of the return residuals, that

is,
$$1/T \sum_{t=1}^{r} \mathbb{I}(e_t < N0)$$
.

The log-likelihood of the asymmetric BEKK(1,1,1) model for a return series $(e_t)_{t=1,..,T}$ is given by

$$\mathcal{L}(\phi) = \sum_{t=1}^{T} l_t(\phi) = \sum_{t=1}^{T} \left(\frac{N}{2} \log(2\pi) - \frac{1}{2} \log(\det H_t) - \frac{1}{2} e'_t H_t^{-1} e_t \right),$$

where $H_t = H_t(\phi)$, and the (quasi) maximum likelihood ((Q)ML) estimator is $\hat{\phi}_{QML} = \arg\max_{\phi \in \Phi} \mathcal{L}(\phi)$. Asymptotic normality and consistency of $\hat{\phi}_{QML}$ may be shown analogously to Comte and Lieberman (2003) and Hafner and Preminger (2009). They prove that the QML estimation of MGARCH models is asymptotically consistent and normal under some regularity conditions.

2.2 | C-MGARCH

For their Copula MGARCH (C-MGARCH) model, Lee and Long (2009) modified the residuals in (2) in the following way:

$$\eta_t = \Sigma^{-1/2} \xi_t, \tag{6}$$

where ξ_t follows a Copula distribution with covariance matrix $E[\xi_t \xi'_t | \Omega_{t-1}] = \Sigma$.⁴ Accordingly,

$$E[\eta_t \eta'_t | \Omega_{t-1}] = I_N, E[e_t | \Omega_{t-1}] = 0, \text{ and } E[e_t e'_t | \Omega_{t-1}] = H_t.$$

To specify dependence characteristics for each bivariate relation in a flexible manner, we consider a so-called vine copula. For instance, in the case of N = 3, the CDF of a vine copula distribution is defined as

$$F(\xi_{1,t},\xi_{2,t},\xi_{3,t};\theta) = \int_{-\infty}^{\xi_{3,t}} C_{12|3}\Big(F_{1|3}\big(\xi_{1,t}|z;\theta\big), \qquad (7)$$
$$F_{2|3}\big(\xi_{2,t}|z;\theta\big);\theta\Big)f_{3}(z)dz,$$

which stacks three potentially different copulas and its respective parameters. With regard to Joe (1996a) and Nelsen (2006), the number of copulas increases by N, when the dimension of the series e_t is augmented from N to N+1. To address a potential curse of dimensionality, one may restrict (economically) less relevant pair-wise copulas to the independence copula. As an alternative and more restrictive approach, one might formalize an N-dimensional Archimedean copula with only one parameter to impose a uniform dependence structure over all pair-wise series. Although the choice of a particular copula function is essential for the implied dependence structure, the full specification of the distributional model requires the choice of marginal distributions as well. In this study, we assume throughout that marginal distributions are standard Gaussian. However, it is worth stressing that non-trivial convolutions of dependent Gaussian marginals $\xi_{i,t}$ —as considered within the C-MGARCH model-do not necessarily translate to normally distributed marginals of e_t (Joe, 1996b). Moreover, the time-varying convolutions triggered by decompositions of H_t lead to time-varying marginal distributions. Complementing the analysis of Lee and Long (2009), Fülle and Herwartz (2022) discuss distributional features of the model-implied transformations, $e_t = H_t^{1/2} \eta_t =$ $H_t^{1/2} \Sigma^{-1/2} \xi_t$ conditional on the conventions that the matrices $H_t^{1/2}$ and $\Sigma^{1/2}$ refer to the eigenvalue decomposition of the MGARCH covariance H_t and the lower triangular Cholesky factor of Σ , respectively. In particular, their findings show that, first, an upper (lower) tail dependence in ξ_t is rotated by 180° and results in a lower (upper) tail dependence of e_t . Second, probability mass is shifted such that the joint occurrence of extreme returns is more likely than under of independence of the random variables in ξ_t .

With the distribution defined in (7), the log-likelihood of the C-MGARCH model is

$$\mathcal{L}(\theta) = \sum_{t=1}^{T} \log \left(f\left(\Sigma^{1/2} H_t^{-1/2} e_t; \theta\right) \right) + \log \left(\det \left(\Sigma^{1/2} H_t^{-1/2} \right) \right), \tag{8}$$

where f is the PDF implied by the distribution in (7) and the likelihood is determined by using the transformation theorem for densities. For a more detailed discussion of the C-MGARCH model, we refer the reader to Lee and Long (2009).

2.3 | MS-C-MGARCH

Noticing that, among others, Ang and Chen (2002) found that the dependencies between financial assets are changing over time, and Fülle and Herwartz (2022) have suggested recently a Markov switching generalization of the copula MGARCH model of Lee and Long (2009). The MS-C-MGARCH model accounts for *k* different (hidden) states $s_t \in \{1,...,K\}$ such that shocks $\xi_t(s_t = k)$ originate from different copula distributions. In analogy to Fülle and Herwartz (2022), we define $\Sigma_k := \Sigma(s_t = k)$, $e_t(k) := e_t(s_t = k)$ and $\xi_t(k) := \xi_t(s_t = k)$. More formally, the model reads as

$$\begin{aligned} e_t(k) &= H_t^{1/2} \Sigma_k^{-1/2} \xi_t(k), \quad \text{Var}[\xi_t(k) | \Omega_{t-1}] \\ &= \Sigma_k \text{ and } \xi_t(k) | \Omega_{t-1} \sim F_k(\xi_{1,t}, ..., \xi_{N,t}; \theta_k) \\ &= C_k(F_{k,1}(\xi_{1,t}; \theta_{k,1}), ..., F_{k,N}(\xi_{N,t}; \theta_{k,N}); \theta_k), \end{aligned}$$

where F_k is the CDF of a copula distribution with copula C_k and parameter θ_k . The corresponding marginal distributions are $F_{k,j}$, j = 1, 2, ..., N, with parameters $\theta_{k,j}$. The hidden regimes are modeled by the Hamiltonian filter (Hamilton, 1989, 2016) and a Markov Chain with transition matrix M. In analogy to the state invariant case (i.e., the C-MGARCH model in Section 2.2), the PDF of $e_t(k)$ is

$$g_k(e_t(k)) = f_k \left(\Sigma_k^{1/2} H_t^{-1/2} e_t \right) \cdot \det \left(\Sigma_k^{1/2} H_t^{-1/2} \right).$$

The probability of being in (the hidden) state *i* given Ω_t is

$$P(s_{t}=i|\Omega_{t}) = \frac{P(s_{t}=i|\Omega_{t-1})f_{i}(e_{t} \mid \Omega_{t-1})}{\sum_{j=1}^{K} P(s_{t}=j|\Omega_{t-1})f_{j}(e_{t} \mid \Omega_{t-1})}.$$
 (9)

Since the states follow a Markov process, conditional on Ω_{t-1} the respective probability is

$$P(s_{t} = i | \Omega_{t-1}) = p_{ii} P(s_{t-1} = i | \Omega_{t-1}) + \sum_{j=1, j \neq i}^{K} p_{ij} P(s_{t-1} = j | \Omega_{t-1}), \quad (10)$$

where p_{ij} is the probability of being in state *i* in time *t* under the condition that the state of the previous period t-1 was *j*, that is, $p_{ij} = P(s_t = i | s_{t-1} = j)$. The log-likelihood of the Markov switching model is

į

$$\begin{aligned} \mathcal{L}(\kappa) &= \sum_{t=1}^{T} l_t(\kappa) \\ &= \sum_{t=1}^{T} \log \left(\sum_{i=1}^{K} \left(f_i \left(\Sigma_i^{1/2} H_t^{-1/2} e_t \right) \right. \right. \\ &\left. \cdot \det \left(\Sigma_i^{1/2} H_t^{-1/2} \right) \cdot P(s_t = i | \Omega_{t-1}) \right) \right), \end{aligned} \tag{11}$$

with parameter vector $\kappa = (\phi, \psi) = (\phi, \theta_1, ..., \theta_K, \theta_{1,1}, ..., \theta_{1,N}, ..., \theta_{K,N}, \widetilde{\text{vec}}(M)')'$, where $\widetilde{\text{vec}}(M)$ is a N(N-1)-dimensional vector stacking the free parameters of the transition matrix M. The respective parameter space is $\mathcal{K} = \Phi \cup \Psi$. The PDFs f_k for state k in (11) are determined as in (7) with copula density c_k . The ML estimator is $\widehat{\kappa}_{\text{ML}} = \arg \max_{\kappa \in K} \mathcal{L}(\kappa)$.

For the estimation of the MS-C-MGARCH model, we proceed in analogy to Lee and Long (2009) and suggest a two-step estimation procedure, where the MGARCH part of the model is estimated first by QML estimation, that is, $\hat{\phi}_{QML}$, to obtain an assessment of the covariance process. The second step is to estimate the remaining parameters conditional on the results from the first step (i.e., $\hat{\psi}_{CML}$). For further details and a discussion of consistency and asymptotic normality, we refer the reader to Fülle and Herwartz (2022). Hereinafter, we denote the two-step estimator as $\hat{\kappa}_{2SE} = (\hat{\phi}_{QML}, \hat{\psi}_{CML})$.

3 | ESTIMATING AND BACKTESTING RISKS

In this section, we first define VaR and ES and derive the distribution of a portfolio of (dependent) returns. Then, the backtesting procedure for ES of Du and Escanciano (2017) is briefly sketched along with further loss statistics.

3.1 | Risk measures

The VaR has become a prominent measure to estimate the risks of, for instance, financial assets and portfolios. It quantifies the loss of an asset or portfolio Z that is not exceeded with probability $1 - \alpha$, that is,

$$\operatorname{VaR}_{\alpha}(Z) = F_Z^{-1}(\alpha) = \inf\{z \in \mathbb{R} : F_Z(z) > \alpha\}, \qquad (12)$$

where F_Z is the (estimated) CDF of Z. Two main critics have been raised against this measure. First, it is nonsubadditive, which means that the VaR of compound assets is not less than the sum of the particular VaRs of those assets (see also Acerbi & Tasche, 2002; Tasche, 2002). Second, the VaR does only depict the minimal loss that is not exceeded with probability $1 - \alpha$. This especially leads to potential underestimation of risks by the VaR, when the true distribution of the asset is heavy tailed (i.e., non-normal). More specifically, the expected loss in the α quantile could be sizeably larger than the respective VaR. The ES has been designed to mitigate the described underestimation of risks featuring the VaR. According to Acerbi and Tasche (2002), the ES is defined as

$$ES_{\alpha} = \mathbb{E}(Z \mid Z \leq VaR_{\alpha}(Z)) = \frac{1}{\alpha} \int_{0}^{\alpha} VaR_{\alpha}(Z)$$
$$= \frac{1}{\alpha} \int_{-\infty}^{VaR_{\alpha}} zf_{Z}(z)dz, \qquad (13)$$

where f_Z is the PDF of portfolio Z.

3.2 | Distribution of portfolio returns

Let z_t denote the return of a weighted portfolio comprising *N* assets; the returns of which are collected in e_t . The portfolio weights $\omega = (\omega_1, \omega_2, ..., \omega_N)$ are assumed to be positive (i.e., no short selling) and constant over time, where $\omega_i \in [0, 1]$ and $\sum_{i=1}^{N} \omega_i = 1$. The portfolio returns obtain as

$$z_t = \omega e_t = \omega H_t^{1/2} \Sigma_k^{-1/2} \xi_t(k).$$
 (14)

Accordingly, z_t is a weighted, time-varying, and statedependent sum of the stochastic model residuals $\xi_t(k)$. Let $\nu_t(k) = (\nu_{1,t}(k), \nu_{2,t}(k), ..., \nu_{N,t}(k))$ denote the vector of those (time- and state-varying) weights. Accordingly, z_t reads as

$$\begin{split} & \mathcal{I}_t = \nu_t(k)\xi_t(k) \\ & = \nu_{1,t}(k)\xi_{1,t}(k) + \nu_{2,t}(k)\xi_{2,t}(k) + ... + \nu_{N,t}(k)\xi_{N,t}(k) \\ & = \sum_{i=1}^K \ P(s_t = i|\Omega_{t-1})\nu_t(s_t = i)\xi_t(s_t = i), \end{split}$$

where $\nu_t(s_t) = \omega H_t^{1/2} \Sigma^{-1/2}(s_t)$. The PDF of the portfolio can be obtained by applying the convolution formula for densities. Accordingly, let $f_{s_t}(\eta_{1,t}, \eta_{2,t}, ..., \eta_{N,t})$ denote the PDF assigned to state s_t . Then, the density of the portfolio returns z_t is

$$f_{z_t}(z) = \sum_{i=1}^{K} P(s_t = i | \Omega_{t-1}) \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_i \left(\frac{z}{\nu_{1,t}(s_t = i)} - \frac{\eta_2 \nu_{2,t}(s_t = i)}{\nu_{1,t}(s_t = i)} - \frac{\eta_N \nu_{N,t}(s_t = i)}{\nu_{1,t}(s_t = i)}, \eta_2, ..., \eta_N \right)$$

$$d\eta_2 d\eta_3 \cdots d\eta_N,$$

where $\nu_{1,t}(s_t = i) \neq 0$. However, the complexity of the PDF and CDF can be mitigated, if at least one of the state-dependent weights $\nu_t(s_t)$ is equal to zero.⁵ For the CDF in the case of a C-MGARCH and MS-C-MGARCH model, we refer to Appendix B.

3.3 | Backtesting

Backtesting helps to assess the accuracy of risk models and to compare alternative model specifications. For backtesting, we closely follow the recent literature using a rolling window approach. More specifically, we divide the sample $(e_t)_{t=1}^T$ into an in-sample part of length T_{in} and an out-of-sample part of length $T - T_{in}$ (i.e., the remainder of the series e_t). A more detailed outline of the rolling window design will be given below in Section 4.3.

The backtesting statistics for the ES of Du and Escanciano (2017) build upon the backtesting models of Kupiec (1995) and Christoffersen (1998) that depart from the definition of a hit occurring in time *t* (Christoffersen, 1998) for VaR level of α as

$$h_t(\alpha) = \mathbb{I}(z_t < VaR_t(\alpha)|\Omega_{t-1}), \ t = T_{in} + 1, ..., T.$$
 (15)

Based on this hit function, the cumulative violation of the forecast to test the ES forecasting accuracy is given by

$$\mathcal{H}_t = \frac{1}{\alpha} \int_0^\alpha h_t(u) du = \frac{1}{\alpha} (\alpha - u_t) \mathbb{I}(u_t \le \alpha), \qquad (16)$$

where $u_t = F_z(z_t; \hat{\kappa}_{2SE} | \Omega_{t-1}^L)$ is the quantile of the observed returns under the estimated parameter vector, that is, $\hat{\kappa}_{2SE}$. Assuming that $F_z(z_t; \hat{\phi}_{QML} | \Omega_{t-1}^L)$ is the true CDF of the observed returns z_t , the cumulative violation \mathcal{H}_t is uniformly distributed on the interval $[0, \alpha]$ (Berkowitz, 2001; Hong & Lee, 2003; Rosenblatt, 1952). Then, the expected value of \mathcal{H}_t is $\alpha/2$. Following Du and Escanciano (2017), the hypothesis

$$H_0: \mathbb{E}(\mathcal{H}_t) = \alpha/2 \tag{17}$$

is used to test the forecasting accuracy of the ES.⁶ The corresponding *t*-statistic is given by

$$U_{ES} = \frac{\sqrt{T - T_{in}} \left(\overline{\mathcal{H}}_t - \alpha/2\right)}{\sqrt{\alpha(1/3 - \alpha/4)}},$$
(18)

where $\overline{\mathcal{H}}_t = 1/(T - T_{in}) \sum_{t=T_{in}+1}^{T} \mathcal{H}_t$ is the sample mean. Here, we base on Corollary 1 of Du and Escanciano (2017), since we use 1-day-ahead forecasts and assume a sufficiently large in-sample period. As a result, the *t*-statistic is asymptotically Gaussian, that is, $U_{ES} \sim \mathcal{N}(0, 1)$.

Besides opting for average accuracy of the risk forecasts, one would like to avoid systematic (i.e., dynamic) ES violations, as this can lead to an increased risk of default due to depleted capital after a first violation. Therefore, we test for independence similarly to Kupiec (1995). The hypothesis for conditional coverage, that is, independence of the ES forecasts on the previous periods' forecasts, is

$$H_0: \mathbb{E}(\mathcal{H}_t - \alpha/2 \mid \Omega_{t-1}) = 0.$$
(19)

The test statistic is then given by

$$C_{ES}(m) = (T - T_{in}) \sum_{j=1}^{m} \widehat{\rho}_j^2,$$
 (20)

where $\hat{\rho}_j = \hat{\gamma}_j / \hat{\gamma}_0$, $\hat{\gamma}_j = \frac{1}{(T - T_{in} - j)} \sum_{t = T_{in} + 1 + j}^T \left[(\mathcal{H}_t - \alpha/2) (\mathcal{H}_{t-j} - \alpha/$

 $\alpha/2$], and the number of lagged autocorrelations subjected to testing is *m*. With similar arguments as for the unconditional coverage test, and under Corollary 2 of Du and Escanciano (2017), the test statistic is asymptotically χ_m^2 distributed under the null hypothesis in (19).

In order to evaluate at which time instances one model outperforms another model in terms of risk forecasting, we largely follow Ardia et al. (2018) and Patton et al. (2019) and employ two specific loss functions. First, to assess the VaR forecasting accuracy, the so-called quantile loss (QL) is

$$QL_t^{\alpha} = (\alpha - \mathbb{I}(z_t < VaR_t(\alpha)))(z_t - VaR_t(\alpha)), \qquad (21)$$

which is minimized by the most accurate VaR measure retrieved from a set of alternative prediction models. Second, to test alternative measures for VaR and ES jointly, the combined loss function of Fissler and Ziegel (2016) is

$$FZL_{t}^{\alpha} = \frac{1}{\alpha ES_{t}(\alpha)} \mathbb{I}(z_{t} < VaR_{t}(\alpha))(z_{t} - VaR_{t}(\alpha)) + \frac{VaR_{t}(\alpha)}{ES_{t}(\alpha)} + \log(-ES_{t}(\alpha)) - 1.$$
(22)

4 | RISK FORECASTING FOR A PORTFOLIO WITH SAFE HAVENS

In this section, we outline the estimation results of the MS-C-MGARCH model and compare them with the

outcomes of benchmark models. Then, we estimate 1-day-ahead VaR and ES forecasts using rolling windows and compare backtesting results obtained from the diagnostics encountered in Section 3.

4.1 | Data and model specification

For the assessment and comparison of different conditional volatility processes with respect to risk forecasting, we consider a three-dimensional time series of centered, daily log-returns of gold, S&P 500 index returns, and yields of 10-year US Treasury Bond Futures as depicted in Figure 1. Corresponding quotes of closing prices have been drawn from Refinitiv. The sample covers the period from October 1, 1991, until September 30, 2021, that is, 7346 observations. Since copula-MGARCH models have the potential to assign considerable probability mass to the joint occurrence of returns that are relatively large in absolute value, we refrain from any trimming of the return series and removing outlying observations on the basis of the (ex-post) unconditional distribution. For the empirical analysis, the sample information is split into two parts. First, to compare the in-sample fit of alternative models and to infer about dependencies among the three asset classes, we consider a period of 20 years,

that is, October 1, 1991, to September 30, 2011, that comprises $T_{in} = 4893$ return observations. Second, the remaining 2453 observations (October 1, 2011, until September 30, 2021) are used to evaluate the ex-ante forecasting performance. Throughout, we compare the MS-C-MGARCH model with benchmark models such as the C-MGARCH model of Lee and Long (2009) and the (asymmetric) BEKK model in terms of goodness-of-fit and predictive ES performance. The following three statistics are considered to compare the goodness-of-fit of the alternative models:

Log-likelihood value: LLV = $\mathcal{L}(\hat{\kappa})$, Akaike information criterion: AIC = $-2 \cdot \text{LLV} + 2 \cdot J$, Bayesian information criterion: BIC = $-2 \cdot \text{LLV} + J \cdot \ln(T)$,

where J is the number of model parameters.

To account for patterns of upper and lower tail dependence, we consider four alternative copulas that have been widely applied in financial econometrics (see, e.g., Diks et al., 2010; Rodriguez, 2007), namely, Gumbel (G), Clayton (C), Gumbel survival (GS), and Clayton survival (CS). Hence, there are $4^3 = 64$ possible combinations of these copulas to form copula-distributed shocks by means of regular D-vine copulas. To reduce the



FIGURE 1 2000 random draws from a Gumbel copula with $\theta = 8$ and Gaussian marginals, that is, the sample has an upper tail dependence. Arrows indicate the transformation obtained by reverting the correlation of the sample. Here, we assume an upper tail dependence between η_1 and η_2 . Hence, the dependence is rotated (by changing the sign of the correlation) into a dependence between high negative values of η_2 and high positive values η_1 (and vice versa if η_1 and η_2 are exchanged). number of specification alternatives, we take account of empirical findings on the (asymmetric) dependencies between safe havens and stock markets and impose some restrictions on the set of copula combinations to be evaluated. Baur and Lucey (2010), Baur (2012), and Fülle and Herwartz (2022) showed that stock markets are negatively correlated with gold or treasury bonds in periods of economic downturns. For this reason, we concentrate on copulas that capture tail dependence characterizing the relationship between a safe haven and stocks. More specifically, there should be a dependence between highly positive safe haven and highly negative stock returns in times of economic turmoil and vice versa for periods of economic recovery. Figure 2 displays the changing tail dependence profile of random variables η_1 and η_2 . Under negative return correlation (i.e., a 90° clockwise rotation of a bivariate return vector), an initially upper tail dependence is transformed into a joint dependence between large positive values of η_1 and large negative values of η_2 . Furthermore, Fülle and Herwartz (2022) showed that the underlying dependence structure is non-trivially influenced by the transformations characterizing (MS)-C-MGARCH models. As described in Section 2.2, the tail dependence pattern is rotated by 180° inside the model, such that an upper (lower) tail dependence of ξ_t ends up in a lower (upper) tail dependence of the process e_t .

Accordingly, the joint returns of gold and stocks should be modeled by a copula featuring upper tail dependence (here denoted with $C_{12|3}$), since these assets are negatively correlated in times of crisis (for more details, see Fülle & Herwartz, 2022). Moreover, the returns of the S&P 500 and US Treasury Bond Futures are modeled by the copula $C_{23}(\eta_2,\eta_3)$. Setting for this combination $\eta_1 =$ η_2 and $\eta_2 = \eta_3$ in Figure 2 represents a corresponding structure of upper tail dependence between S&P 500 returns and Treasury Bond yields. Accordingly, a 90° rotation of the (upper-tail) dependence structure of these two assets leads to a dependence profile that is untypical for crisis times (i.e., a dependence between large positive stock returns and [in absolute value] large negative bond returns). Consequently, using a copula with lower-tail dependence incorporates a dependence that is more reasonable in crisis times.

In light of these considerations, we restrict the space of examined copula combinations and use a copula with lower tail dependence for $C_{12|3}$ and a copula with upper tail dependence for C_{23} for crisis periods and vice versa for economic recovery periods. For the relationship between the two safe havens, both upper and lower tail dependencies are considered.⁷ Table 1 depicts all combinations of copulas used in our comparative risk analysis.



FIGURE 2 Daily log-returns of gold, S&P 500, and US Treasury Bond Futures (T = 7316).

TABLE 1 Used copula combinations to form D-vine copulas.

Economic state/copulas	$C_{12 3}$	<i>C</i> ₁₃	C ₂₃
Crisis	GS/C	GS/C	G/CS
Crisis	GS/C	G/CS	G/CS
Recovery	G/CS	GS/C	GS/C
Recovery	G/CS	G/CS	GS/C

Note: C_{ij} stands for the copula to model the dependence between $e_{i,t}$ and $e_{j,t}$.

TABLE 2 Estimation results and information criteria for asymmetric BEKK(1,1,1) models (with symmetric benchmark in the last row).

Asyr	nmetri	es	Goodness-	of-fit criteria	
e ₁	<i>e</i> ₂	<i>e</i> ₃	LLV	AIC	BIC
_	_	_	49,906.73	-99,747.46	-99,533.11
_	+	+	49,840.96	-99,615.93	-99,401.58
_	+	_	49,887.10	-99,708.21	-99,493.86
_	_	+	49,964.57	-99,863.15	-99,648.79
+	-	_	49,840.96	-99,615.93	-99,401.58
+	+	+	49,882.06	-99,698.12	-99,483.76
+	+	_	49,840.96	-99,615.93	-99,401.58
+	_	+	49,967.23	-99,868.46	-99,654.11
Sym.	BEKK		49,840.96	-99,633.93	-99,478.04

Note: The labels "–" and "+" indicate an accounting for asymmetric conditional volatility patterns for negative and positive returns, respectively (see Section 2.1). Largest LLV and smallest AIC and BIC statistics are in boldface.

4.2 | In-sample estimation

First, we examine eventually asymmetric patterns for the BEKK model. For this purpose, we consider all possible combinations of positive and negative returns in the series. Respective estimation results are documented in Table 2. The highest likelihood value is obtained, if the model is specified such that negative stock (i.e., S&P 500) returns joint with positive gold and bond returns exert an additional effect on conditional volatility. This coincides with the findings of Baur and Lucey (2010), Baur (2012), and Fülle and Herwartz (2022), since it implies that in times of crisis, safe-haven patterns in the return series are associated with higher volatility. Henceforth, we condition on this selection when considering empirical asymmetric BEKK models. Estimated conditional volatilities using the R-Package BEKKS of Fülle et al. (2022) are shown in Figure C1 in Appendix C.

Similar to Fülle and Herwartz (2022), we proceed in three steps for generalizing the asymmetric MGARCH model specification. Regarding the first step, results

shown in Table 3 indicate that the consideration of state invariant copula distributions, that is, the C-MGARCH model, improves model fit in terms of both selection criteria. As a second step and going beyond state invariant models, we set up two regime MS-C-MGARCH models, which we denote as 2MS-C-MGARCH. According to estimation results documented in Table 4, the consideration of 2MS-C-MGARCH processes further improves the model fit in terms of information criteria. The copula combinations signifying flight-to-safety effects in the presence of economic crisis lead to the best fitting model. Figure C2 in Appendix C shows the implied smoothed regime probabilities for the case of the preferred model comprising two regimes, with the second regime capturing dependence patterns that are typical for crisis periods with copula combination (GS, G, G). Smoothed regime probabilities allow for the interpretation that the copula regimes are more likely to occur in times of economic crisis. With the described evidence in favor of a 2MS-C-MGARCH model, a natural interest arises in testing a combination of three regimes. Hence, as a third step of model augmentation, we consider 3MS-C-MGARCH models featuring a Gaussian regime for non-crisis times and two copula regimes (one for crisis periods and one for recovery). Corresponding estimation results are displayed in Table 5 and Figure C3. The best-fitting copula combination features a crisis regime with copulas (GS, G, G) and a recovery regime with copulas (G, GS, GS). This copula structure further underpins the safe-haven characteristics of gold and US Treasury Bonds, since the crisis and recovery regime are more likely to occur in the context of major economic or political pressures (see Figure C3 in Appendix C). Moreover, the information criteria show that the best asymmetric three-regime model leads to a considerable increase of the log-likelihood value (LLV = 50446.88).

By implication of the best-fitting copula combination, we find that gold and bond returns are subject to lower and upper tail dependence in times of crisis and recovery, respectively. Aligning with the economic rationale of safe havens, this evidence hints more at comovements and less at transitions of such safe-haven instruments in times of extreme events. Moreover, we find a stronger relationship to exist between bonds and stocks than between gold and stocks. According to the best-fitting 3MS-C-MGARCH model, we find evidence for periods of strong upper ($\lambda_U = 0.941$) and lower ($\lambda_L = 0.965$) tail dependence between gold and stocks. Similarly, we see periods of strong upper ($\lambda_U = 0.972$) and lower ($\lambda_L = 0.959$) tail dependence between bonds and stocks.⁸ Hence, one may conclude that in times of extreme stock market turmoil, bonds are more often chosen as a safe haven than gold because of stronger tail dependence in crisis periods. In

Copula	as		Estimat	es		Goodness-	of-fit criteria	
$C_{12 3}$	C ₁₃	<i>C</i> ₂₃	$\widehat{\theta}_{12}$	$\widehat{ heta}_{13}$	$\widehat{\theta}_{23}$	LLV	AIC	BIC
G	GS	GS	1.013	1.005	1.018	49,997.33	-99,922.65	-99,688.81
			(0.008)	(0.011)	(0.009)			
G	G	GS	1.018	1.033	1.018	50,000.11	-99,928.22	-99,694.38
			(0.010)	(0.023)	(0.009)			
GS	G	G	1.026	1.015	1.135	50,012.88	-99,953.76	-99,719.92
			(0.012)	(0.013)	(0.081)			
С	С	G	0.032	0.026	1.149	50,014.64	-99,957.29	-99,723.45
			(0.014)	(0.018)	(0.049)			

TABLE 3 Estimation results and goodness-of-fit criteria for alternative C-MGARCH models with an asymmetric BEKK(1,1,1) model to formalize conditional (co)variances.

Note: The asymmetric patterns are chosen to follow the best-fitting model as highlighted in Table 2.

TABLE 4 Estimation results and goodness-of-fit criteria for alternative two-state MS-C-MGARCH models.

Copulas	5		Estimate	s				Goodness-o	of-fit criteria	
$C_{12 3}$	<i>C</i> ₁₃	C ₂₃	\widehat{p}_{11}	\widehat{p}_{22}	$\widehat{ heta}_{12}$	$\widehat{ heta}_{13}$	$\widehat{ heta}_{23}$	LLV	AIC	BIC
G	GS	GS	0.782	0.000	12.883	30.493	30.032	50,336.52	-100,597.00	-100,350.20
			(0.088)	(0.354)	(7.977)	(1.523)	(3.264)			
G	G	GS	0.827	0.047	1.523	9.638	31.500	50,334.57	-100,593.10	-100,346.30
			(0.101)	(2.106)	(1.801)	(0.686)	(0.494)			
GS	G	G	0.757	0.139	26.058	26.032	31.499	50,393.51	-100,711.00	-100,464.20
			(0.183)	(0.709)	(0.767)	(0.054)	(0.852)			
GS	GS	G	0.674	0.221	1.061	31.499	6.384	50,343.08	$-100,\!610.20$	-100,363.30
			(0.557)	(1.169)	(0.906)	(1.084)	(10.238)			

Note: Estimated standard errors are calculated using the distribution for the two-stage ML estimator outlined in Fülle and Herwartz (2022). The conditional (co)variances align with the best-performing asymmetric BEKK(1,1,1) model as highlighted in Table 2.

conjunction with the results from Section 4.1 and Table 1 on model implied tail dependence patterns, we find crisis and recovery regimes for which a slump (boom) in stocks is likely to coincide with a boom (slump) in bonds and gold given negative correlation between stocks and the two safe havens. Moreover, extreme gold and bond returns are estimated to be highly synchronized during both market phases. This finding aligns with the literature on safe-haven characteristics of gold and bonds (Baur & Lucey, 2010; Baur & McDermott, 2010; Fülle & Herwartz, 2022). Unlike the flexible MS-C-MGARCH approach, the restrictive single regime C-MGARCH model fails to detect the described higher order comovements. By implication of copula parameter estimates, we barely find tail dependencies between gold and stocks $(\lambda_L = 0.000)$ or between bonds and stocks $(\lambda_U = 0.172)$ for the best-fitting C-MGARCH model.

In sum, we find evidence for better fitting MS-C-MGARCH models compared with benchmark models. Regarding two particular steps of model generalization from MGARCH to C-MGARCH and from C-MGARCH to MS-C-MGARCH, the latter is of particular effectiveness. In fact, log-likelihood differences for the second generalization step are by a factor of about eight larger than those for the first step. This is remarkable even after accounting for the implied enlargement of the parameter space. To examine model-implied forecasting performance, we next take a look at rolling window, 1-dayahead predictive exercises.

4.3 | Out-of-sample analysis

We next analyze the performance of alternative models and benchmark specifications in ex-ante VaR and ES prediction. Regarding the considered copula selections, we mainly build upon those models that provided the most effective in-sample estimates. Unreported further results for all model specifications are available upon request. More specifically, the crisis and recovery regimes featuring (GS, G, G) and (G, GS, GS) vine copulas are used hereinafter for C-MGARCH, 2MS-C-MGARCH, and

	Estimat	se											Informati	on criteria	
Copulas	\widehat{p}_{11}	\widehat{p}_{12}	\widehat{p}_{21}	\widehat{p}_{22}	\widehat{p}_{31}	\widehat{p}_{32}	$\widehat{\theta}_{12}^1$	$\widehat{\theta}_{13}^1$	$\widehat{\theta}_{23}^1$	$\widehat{\theta}^2_{12}$	$\widehat{\theta}^2_{13}$	$\widehat{ heta}^2_{23}$	LLV	AIC	BIC
G/GS/GS	0.565	0.232	0.624	0.001	0.405	0.087	12.114	17.885	17.357	20.408	19.342	24.884	50,446.9	-100,803.8	-100,511.5
& GS/G/G	(1.532)	(1.238)	(1.591)	(1.190)	(1.583)	(0.047)	(28.617)	(13.942)	(14.671)	(1.006)	(0.029)	(1.093)			
G/G/GS	0.847	0.007	0.612	0.204	0.000	0.486	9.097	13.306	14.539	22.773	17.328	21.299	50,387.4	-100,684.8	-100,392.5
& GS/GS/G	(0.022)	(0.048)	(4.998)	(0.279)	(0.164)	(060.0)	(2.509)	(8.929)	(9.559)	(25.387)	(0.435)	(0.178)			
Note: For further	information	see Table 4.													

Estimation results and goodness-of-fit criteria for alternative three state MS-C-MGARCH models.

TABLE 5

WILEY 2173

3MS-C-MGARCH models.9 In line with the recent literature on VaR and ES forecasting (see, e.g., Kratz et al., 2018; Patton et al., 2019), we adopt a rolling window approach with a window length of 20 years (in total 4893 daily observations). For each time instance t of the out-of-sample period $t \in \{T_{in} + 1, ..., T\}$, 1-day-ahead VaR and ES forecasts ($\widehat{VaR}_t(\alpha)$ and $\widehat{ES}_t(\alpha)$) are determined in an iterative manner on the basis of updated OML esticondition upon sampled mates that returns $\{r_{t-T_{in}},...,r_{t-1}\}$. Subsequently, we investigate for three alternative VaR coverage levels $\alpha = 0.01, 0.025$ and $\alpha =$ 0.05 and two portfolio compositions (i.e., an equal weight portfolio and a 0.15, 0.70, and 0.15 portfolio with the highest weight assigned to equity) out-of-sample risk predictions with regard to several criteria. First, we implement ES backtests of Du and Escanciano (2017) outlined in Section 3 (see Equations 18 and 20). Second, we perform pairwise Diebold-Mariano tests (DM; Diebold & Mariano, 1995) with regard to the two loss functions QL and FZL as given in (21) and (22), respectively. Third, we consider MCS in the vein of Hansen et al. (2011) for joint model comparisons of QL and FZL statistics.

Overall, the in-sample results on relative model performance discussed in Section 4.2 are well reflected in the ES forecasting performance of alternative models. As documented in Table 6, MS-C-MGARCH models tend to result in higher p-values of diagnostic tests for unconditional and conditional levels of the ES. Accordingly, the flexible regime switching models are of particular convenience for all three VaR coverage levels (1%, 2.5%, 5%)and both portfolio compositions. For instance, regarding risk predictions for portfolios with a higher share of equity ($\omega = (0.15, 0.70, 0.15)$) and the 1% coverage level, the null hypothesis of unconditional coverage is throughout rejected except for the asymmetric 2MS-C-MGARCH model (with a p-value of 0.1436). Although diagnostic results are not that clear for remaining cases, testing for conditional or unconditional coverage provides the largest *p*-values for regime switching models generally.

The findings for the ES diagnostics are underpinned by results for the loss functions QL and FZL. For instance, as depicted in Figure 4 (for portfolio weights of $\omega = (0.15, 0.70, 0.15)$) and Figure C4 in Appendix C (equal weight portfolio), loss differentials between the bestfitting benchmark model and the best-fitting MS-C-MGARCH model indicate a lead of the latter. Although the loss differentials between the two models are less pronounced in periods of economic prosperity, cumulative losses show that the risk forecasting is especially improved by the best-performing 3MS-C-MGARCH model during times of economic stress such as the outburst of the trade conflict between China and the United States and the COVID 19 pandemic and their

TABLE 6 ES-forecasting results for rolling windows using 20 previous years as in-sample.

		Hit rate	$\mathbf{s}\widehat{\mathcal{H}}_t$		U _{ES} p-va	alues		<i>C_{ES}</i> (5) <i>p</i>	-values	
Model	Copula	1%	2.5%	5%	1%	2.5%	5%	1%	2.5%	5%
Portfolio weights: ω	=(1/3,1/3,1/3)									
BEKK		.0090	.0141	.0227	.0006	.3880	.3630	.4199	.4294	.4408
C-BEKK	G/G/GS	.0129	.0208	.0317	.0000	.0000	.0091	.4223	.4332	.4459
C-BEKK	GS/G/G	.0129	.0208	.0317	.0000	.0000	.0091	.4216	.4313	.4467
2MS-C-BEKK	G/GS/GS	.0080	.0135	.0225	.0099	.6014	.3202	.4219	.4361	.4481
2MS-C-BEKK	GS/G/G	.0077	.0133	.0221	.0192	.6615	.2622	.4224	.4369	.4550
3MS-C-BEKK	G/GS/GS & GS/G/G	.0093	.0159	.0258	.0002	.0621	.7507	.4236	.4326	.4402
Portfolio weights: ω	= (0.15,0.7,0.15)									
BEKK		.0102	.0160	.0236	.0000	.0567	.5894	.4207	.4305	.4355
C-BEKK	G/G/GS	.0156	.0214	.0286	.0000	.0000	.1592	.4228	.4306	.4358
C-BEKK	GS/G/G	.0153	.0210	.0284	.0000	.0000	.1872	.4207	.4297	.4368
2MS-C-BEKK	G/GS/GS	.0159	.0166	.0238	.0000	.0248	.6280	.4189	.4350	.4400
2MS-C-BEKK	GS/G/G	.0067	.0116	.0195	.1436	.6401	.0313	.4230	.4377	.4427
3MS-C-BEKK	G/GS/GS & GS/G/G	.0074	.0128	.0211	.0387	.8624	.1309	.4214	.4306	.4399

Note: Largest p-values are highlighted in boldface.

2174

IFY



FIGURE 3 Estimated 1-day-ahead 1% ES for a portfolio with weights $\omega = (0.15, 0.7, 0.15)$. The blue line displays the best-performing C-MGARCH model (i.e., the benchmark model, GS/G/G). The red line displays results for the 3MS-C-MGARCH model (G/GS/GS & GS/G/G) featuring the best predictive performance (see results shown in Table 7).

aftermaths. These insights are underpinned by Figure 3 highlighting that the Markov switching model with three regimes provides comparably more accurate risk forecasts in comparison with time-invariant benchmark models. Especially in the wake of the European debt crisis and

during the COVID crash in March 2020, MS-C-MGARCH models appear to signify more realistic risk levels, since the two most extreme downturns in 2013 are only covered by the ES forecasts of the more flexible 3MS-C-MGARCH model. During the COVID crash, the best-

FÜLLE and HERWARTZ



FIGURE 4 Model comparisons by means of cumulative loss differentials (Ardia et al., 2018). The upper and lower panels show, respectively, QL and FZL differentials between the best-performing benchmark and the best-performing MS-C-MGARCH model from VaR and ES with nominal coverage of $\alpha = 1\%$. The underlying portfolio has weights of $\omega = (0.15, 0.70, 0.15)$. Corresponding results for equal-weight portfolios are shown in Appendix C.

		VaR cover	age α				
		1%		2.5%		5%	
Model	Copulas	QL	FZL	QL	FZL	QL	FZL
Portfolio weights: $\omega = (1/2)$	3,1/3,1/3)						
BEKK		0.198e-3	-3.692	0.366e-3	-4.132	0.591e-3	-4.396
Asym. BEKK		0.201e-3	-3.645	0.372e-3	-4.102	0.598e-3	-4.377
C-BEKK		0.205e-3	-3.727	0.369e-3	-4.174	0.587e-3	-4.434
Asym. C-BEKK		0.211e-3	-3.674	0.376e-3	-4.142	0.595e-3	-4.417
2MS-C-BEKK	G/G/GS	0.201e-3	-3.789	0.372e-3	-4.163	0.592e-3	-4.368
Asym. 2MS-C-BEKK	G/G/GS	0.199e-3	-3.839	0.372e-3	-4.179	0.599e-3	-4.386
3MS-C-BEKK	G/GS/GS & GS/G/G	0.199e-3	-3.737	0.365e-3	-4.150	0.586e-3	-4.404
Asym. 3MS-C-BEKK	G/GS/GS & GS/G/G	0.204e-3	-3.728	0.373e-3	-4.126	0.596e-3	-4.376
Portfolio weights: $\omega = (0.1)$	5,0.70,0.15)						
BEKK		0.342e-3	-2.631	0.602e-3	-3.397	0.940e-3	-3.808
Asym. BEKK		0.333e-3	-2.753	0.595e-3	-3.445	0.934e-3	-3.836
C-BEKK		0.416e-3	-2.184	0.650e-3	-3.332	0.960e-3	-3.842
Asym. C-BEKK		0.404e-3	-2.251	0.639e-3	-3.362	0.954e-3	-3.855
2MS-C-BEKK	G/G/GS	0.343e-3	-2.702	0.606e-3	-3.429	0.944e-3	-3.828
Asym. 2MS-C-BEKK	G/G/GS	0.315e-3	-3.090	0.586e-3	-3.572	0.937e-3	-3.882
3MS-C-BEKK	G/GS/GS & GS/G/G	0.327e-3	-2.957	0.597e-3	-3.506	0.938e-3	-3.859
Asym. 3MS-C-BEKK	G/GS/GS & GS/G/G	0.318e-3	-3.092	0.588e-3	-3.565	0.933e-3	-3.871

TABLE 7 Average QL and FZL statistics for equal-weight portfolios (upper panel) and portfolios with a higher share of equity (lower).

Note: The smallest mean losses are highlighted in boldface. We apply the MCS test of Hansen et al. (2011) to all models listed in the leftmost column. The tests are performed with a confidence level of 5% using the R-package MCS of Catania and Bernardi (2017). Numbers in *italic* indicate that the respective model is excluded from the set of superior models.

²¹⁷⁶ WILEY-

performing benchmark model overestimates actual risks, whereas the best regime switching model results in an eventually more realistic risk assessment, since realized returns deviate less from ES forecasts.

Adding to the graphical display of selected QL and FZL statistics in Figure 4, Table 7 documents average losses for all competing models. Although we document results for four MS and four time-invariant benchmark models, minimum losses are found more often for MS-C-MGARCH models for both QL and FZL statistics. This performance lead is especially pronounced, if the portfolio can be considered more risky (i.e., portfolio weights imply a higher share of equity). In this case, benchmark models fail throughout to yield minimum loss statistics. To infer on the relative performance of MS-C-MGARCH models over benchmark models by means of explicit tests, we perform pairwise DM and overall MCS tests for OL and FZL statistics. Results are documented in Table 8 and indicate that VaR and ES forecasts of MS-C-MGARCH models often outperform benchmark models, especially in the case of more risky portfolios that comprise relatively large shares of equity.

Although DM tests only allow for pairwise comparisons of model-implied losses, the MCS method of Hansen et al. (2011) enables a joint assessment of all models listed in Table 7. This test procedure consists of iterative exclusions of models for which forecast-implied losses are pairwise outperformed by all remaining models (inside the MCS) with 5% significance. Although results for the equalweight portfolios do not signify a superior performance of MS-C-MGARCH models, the picture changes when portfolios comprise a relatively large share of risky assets (i.e., equity). Especially for the most extreme events (i.e., a VaR coverage level of $\alpha = 1\%$), the MCS mainly consists of the new flexible MS-C-MGARCH models. More specifically, when we are looking at the FZL, the outperformance of the benchmark approaches by the new model class is evident. The best-performing model with regard to average FZL and 1% coverage level is the asymmetric 3MS-C-MGARCH model with the so-called "recovery" and "crisis" regimes.

In sum, the introduction of regime-specific higher order dependencies of speculative returns improves both ex-post and ex-ante performance of model-implied risk measures. The suggested Markov switching approach promises particular performance leads for portfolios comprising relatively large shares of risky assets as, for instance, equity. Regarding the overall performance of the considered benchmark and Markov switching specifications, it is important to notice that (almost) all models considered in this study build upon a flexible asymmetric MGARCH specification in BEKK form that allows for rather general spillover effects among returns. Hence, the performance differentials examined in this study can be

TABLE 8 Diebold–Mariano tests for QL and FZL statistics of equal-weight portfolios (upper panel) and portfolios with an extended share of equity $\omega = (0.15, 0.7, 0.15)$ (lower panel).

			BEKK		Asym. B	BEKK	C-BEKK	:	Asym. C-BEKK	
α	Model	Copulas	QL	FZL	QL	FZL	QL	FZL	QL	FZL
Portfolio	weights: $\omega = (1/3)$	3,1/3,1/3)								
1%	2MS-C-BEKK	G/G/GS	0.678	-2.686	-1.438	-2.571	-1.674	-5.005	-2.830	-5.520
	3MS-C-BEKK	G/GS/GS & GS/G/G	3.401	-1.395	2.576	-1.498	-0.687	-3.738	-2.739	-6.085
2.5%	2MS-C-BEKK	G/G/GS	3.627	-2.958	-0.006	-2.658	0.716	-4.387	-1.026	-4.956
	3MS-C-BEKK	G/GS/GS & GS/G/G	3.669	-2.372	0.821	-1.888	1.749	-0.650	-1.328	-2.321
5%	2MS-C-BEKK	G/G/GS	5.157	-8.746	0.762	-6.728	2.982	-2.072	0.838	-2.937
	3MS-C-BEKK	G/GS/GS & GS/G/G	2.912	-4.895	-1.426	-3.849	3.962	-0.502	0.367	-1.358
Portfolio	weights: $\omega = (0.1)$	5,0.70,0.15)								
1%	2MS-C-BEKK	G/G/GS	-4.274	-7.427	-3.846	-7.532	-6.984	-11.204	-6.746	-11.245
	3MS-C-BEKK	G/GS/GS & GS/G/G	-4.386	-7.148	-4.229	-7.444	-7.134	-11.108	-6.925	-11.294
2.5%	2MS-C-BEKK	G/G/GS	-2.963	-8.579	-2.733	-9.093	-5.323	-12.659	-5.034	-12.856
	3MS-C-BEKK	G/GS/GS & GS/G/G	-2.983	-7.896	-2.354	-8.531	-5.582	-12.321	-5.358	-12.758
5%	2MS-C-BEKK	G/G/GS	-0.584	-9.083	1.262	-10.468	-2.339	-11.008	-2.052	-10.734
	3MS-C-BEKK	G/GS/GS & GS/G/G	-1.449	-7.792	-0.599	-10.069	-3.067	-8.159	-2.952	-7.195

Note: Results are shown for alternative coverage levels α . In-sample best-performing MS-C-MGARCH models (see Tables 4 and 5) with two and three regimes are ordered row-wise, whereas non-MS benchmark models (symmetric and asymmetric) are ordered columnwise. Bold entries indicate a non-zero performance difference with 1% significance. Negative (positive) DM statistics are in favor of MS-C-MGARCH (benchmark) models.

reasonably attributed to higher order (and non-trivial) dependence structures as, for instance, the rare cases when large returns of either size occur jointly within the constituents of a portfolio under scrutiny. Hence, the relative performance of the flexible Markov switching models could be subject to time variation and confined in particular to episodes of market turmoil.

5 | CONCLUSION

In accordance with Basel III regulations, risk managers will likely focus on ES as a more accurate measure for tail risks. Especially for very rare events such as the COVID crash or the Ukrainian crisis a predictive tool for tail risks is of particular interest. We have estimated and tested the ES forecasting performance of the regime switching copula MGARCH model of Fülle and Herwartz (2022). To analyze the risks for different portfolios combining a stock index and two major safe-haven assets, we deployed a rolling window ex-ante analysis. From these exercises, three results are of particular importance. First, the MS-C-MGARCH model improves the risk forecasting of speculative assets and safe havens compared with benchmark models for alternative nominal VaR coverage levels. Second, we find that choosing the origin of leveraged (co)variances properly enhances the model fit and risk forecasting performance. In this framework, accounting for additional (conditional) volatility induced by jointly positive returns of safe havens and negative returns of stock indices has led to the most favorable results and underpins the consideration of gold and US Treasury Bonds as safe havens. The patterns of the best fitting copula combinations for two- and three-regime MS-C-MGARCH models further stress the role of proper model selection, since copula distributions are of particular relevance in times of crisis and recovery. Third, we find that the more flexible model with two and three regimes enhances the forecasting accuracy of US speculative asset returns. More specifically, accounting for a setup featuring one regime for recovery and one for crisis yields the best data-fitting and risk-predicting model in our empirical framework.

A topic for future research is to investigate whether the MGARCH models help to evaluate and forecast the risks in a striking environment, for instance, under financial market stress as invoked by the current war in Ukraine. In the wake of such a ground-breaking crisis, extreme comovements and counter movements can be expected. Against this background, the MS-C-MGARCH model might help to capture such capital movements and improve risk forecasting in states when it is of crucial interest. Throughout, the analysis in this work relied on time invariance of marginal distributions prior to standardization. For future research, it appears tempting to systematically address the scope of extending Markov switching patterns to induce time variation of marginal distributions. As potential distributional frameworks, one might consider heavy tailed and/or skewed distributions in the vein of Bali et al. (2008).

ACKNOWLEDGMENTS

We gratefully acknowledge the financial support from the Deutsche Forschungsgemeinschaft (HE 2188/14-1). We are also grateful for the helpful comments from Danielle Barna, Huynh Tuan Duy Bui, and two anonymous reviewers. Open Access funding enabled and organized by Projekt DEAL.

CONFLICT OF INTEREST STATEMENT

The authors declare that there is no conflict of interest.

DATA AVAILABILITY STATEMENT

The data that support the findings of this study are available in None at https://finance.yahoo.com. These data were derived from the following resources available in the public domain: Yahoo Finance Data, https://finance. yahoo.com, and Gold Data may be found on Quandl, https://demo.quandl.com/.

ENDNOTES

- ¹ It is worth to notice that in the present context, the definition of copula models and/or Markov switching is independent of the specification for time-varying volatilities or correlations. Hence, instead of our focus on MGARCH covariance models, one could also consider constant or dynamic correlation models (CCC or DCC) to formalize conditional heteroskedasticity (Bollerslev, 1990; Engle, 2002). We utilize MGARCH models in BEKK form, since they better capture spillovers in comparison with DCC. Although this has to be compromised by system dimensionality (in comparison with CCC or DCC), however, Markov switching might suffer from a curse of dimensionality, if it is intended to cover the marginal volatility processes.
- 2 The package employs analytical derivatives of the BEKK(1,1,1) log-likelihood as suggested by Hafner and Herwartz (2008), which are implemented in C++ to enable fast nonlinear optimization. For instance, subjecting a five-dimensional time series comprising about 5000 observations obtains parameter estimates within less than 1 min.
- ³ Among the possible decompositions of a positive definite symmetric matrix *D* into LL' = D is, for example, the Cholesky decomposition, where *L* is a lower triangular matrix. The symmetric (eigenvalue) decomposition with $D = D^{1/2}D^{1/2}$ is $D^{1/2} = \Gamma \Lambda^{\frac{1}{2}} \Gamma'$, where Γ denotes the matrix of eigenvectors and Λ is a diagonal matrix containing the eigenvalues of *D*.
- ⁴ See the textbook treatments of Joe (1996a) or Nelsen (2006) and Appendix A for definitions of copulas and their probability densities.

2178 WILEY-

⁵ In case of independently distributed residuals η_t , the PDF of the portfolio (state) is given by integrating over the weighted product of the marginal PDFs. For example, in case of normal residuals

(here: $\mathcal{N}(0,1)$), $z_t \sim \mathcal{N}\left(0, \sum_{j=1}^{3} \nu_{j,t}\right)$.

- ⁶ In addition to the null hypothesis of a correctly predicted ES, the test conditions implicitly on the correctness of the estimated CDF.
- ⁷ One may observe jointly positive returns in times of downturns. However, there might be also a transfer from a (considerably weaker) safe haven to another in crisis times.
- ⁸ The notations λ_U and λ_L indicate, respectively, upper and lower tail dependence. For an explicit definition of these moments, see Appendix A. For a detailed investigation of the of the copula-specific transmission of higher order (tail) dependencies from latent random variables ξ_t to observables e_t , see Fülle and Herwartz (2022).
- ⁹ We restrict the analysis to only one vine copula setup for crisis and recovery to enable a sound performance comparison across model classes C-MGARCH and MS-C-MGARCH.

REFERENCES

- Acerbi, C., & Tasche, D. (2002). Expected shortfall: A natural coherent alternative to value at risk. *Economic Notes*, 31(2), 379–388. https://doi.org/10.1111/1468-0300.00091
- Aloui, R., Ben Aïssa, M. S., & Nguyen, D. K. (2013). Conditional dependence structure between oil prices and exchange rates: A copula-GARCH approach. *Journal of International Money and Finance*, 32, 719–738. https://doi.org/10.1016/j.jimonfin.2012. 06.006
- Ang, A., & Chen, J. (2002). Asymmetric correlations of equity portfolios. *Journal of Financial Economics*, 63(3), 443–494. https:// doi.org/10.1016/S0304-405X(02)00068-5
- Ardia, D., Bluteau, K., Boudt, K., & Catania, L. (2018). Forecasting risk with Markov-switching GARCH models: A large-scale performance study. *International Journal of Forecasting*, 34(4), 733–747. https://doi.org/10.1016/j.ijforecast.2018.05.004
- Bali, T. G., Mo, H., & Tang, Y. (2008). The role of autoregressive conditional skewness and kurtosis in the estimation of conditional var. *Journal of Banking and Finance*, 32(2), 269–282. https://doi.org/10.1016/j.jbankfin.2007.03.009
- Baur, D. G. (2012). Asymmetric volatility in the gold market. The Journal of Alternative Investments, 14(4), 26–38. https://doi.org/ 10.3905/jai.2012.2012.1.016
- Baur, D. G., & Lucey, B. M. (2010). Is gold a hedge or a safe haven? An analysis of stocks, bonds and gold. *Financial Review*, 45(2), 217–229. https://doi.org/10.1111/j.1540-6288.2010.00244.x
- Baur, D. G., & McDermott, T. K. (2010). Is gold a safe haven? International evidence. *Journal of Banking and Finance*, 34(8), 1886–1898. https://doi.org/10.1016/j.jbankfin.2009.12.008
- Bekaert, G., & Wu, G. (2000). Asymmetric volatility and risk in equity markets. *Review of Financial Studies*, 13(1), 1–42. https://doi.org/10.1093/rfs/13.1.1
- Berkowitz, J. (2001). Testing density forecasts, with applications to risk management. *Journal of Business and Economic Statistics*, 19(4), 465–474. https://doi.org/10.1198/07350010152596718

- Berkowitz, J., & O'Brien, J. (2002). How accurate are value-at-risk models at commercial banks? *The Journal of Finance*, *57*, 1093– 1111. https://doi.org/10.1111/1540-6261.00455
- Billio, M., & Pelizzon, L. (2000). Value-at-risk: A multivariate switching regime approach. *Journal of Empirical Finance*, 7(5), 531–554. https://doi.org/10.1016/S0927-5398(00)00022-0
- Bollerslev, T. (1986). Generalized autoregressive conditional heteroskedasticity. Journal of Econometrics, 31(3), 307–327. https:// doi.org/10.1016/0304-4076(86)90063-1
- Bollerslev, T. (1990). Modelling the coherence in short-run nominal exchange rates: A multivariate generalized arch model. *The Review of Economics and Statistics*, 72(3), 498–505. https://doi. org/10.2307/2109358
- Bradley, B. O., & Taqqu, M. S. (2003). Financial risk and heavy tails. In S. T. Rachev (Ed.), Handbook of heavy tailed distributions in finance. Vol. 1 of handbooks in finance (pp. 35–103). North-Holland, Amsterdam. https://doi.org/10.1016/B978-044450896-6.50004-2
- Brooks, C., Henry, Ó. T., & Persand, G. (2002). The effect of asymmetries on optimal hedge ratios. *The Journal of Business*, 75(2), 333–352. https://doi.org/10.1086/338484
- Brooks, C., & Persand, G. (2003a). The effect of asymmetries on stock index return value-at-risk estimates. *The Journal of Risk Finance*, 4(2), 29–42. https://doi.org/10.1108/eb022959
- Brooks, C., & Persand, G. (2003b). Volatility forecasting for risk management. *Journal of Forecasting*, 22(1), 1–22. https://doi.org/10.1002/for.841
- Catania, L., & Bernardi, M. (2017). MCS: Model confidence set procedure. R package version 0.1.3.
- Cherubini, U., Luciano, E., & Vecchiato, W. (2004). Copula methods in finance. John Wiley & Sons. https://doi.org/10.1002/ 9781118673331
- Chollete, L., Heinen, A., & Valdesogo, A. (2009). Modeling international financial returns with a multivariate regime-switching copula. *Journal of Financial Econometrics*, 7(4), 437–480. https://doi.org/10.1093/jjfinec/nbp014
- Christoffersen, P. F. (1998). Evaluating interval forecasts. International Economic Review, 39(4), 841–862. https://doi.org/10. 2307/2527341
- Comte, F., & Lieberman, O. (2003). Asymptotic theory for multivariate GARCH processes. Journal of Multivariate Analysis, 84(1), 61–84. https://doi.org/10.1016/s0047-259x(02)00009-x
- Diebold, F., & Mariano, R. (1995). Comparing predictive accuracy. Journal of Business & Economic Statistics, 13(3), 253–263. https://doi.org/10.1198/073500102753410444
- Diks, C., Panchenko, V., & van Dijk, D. (2010). Out-of-sample comparison of copula specifications in multivariate density forecasts. *Journal of Economic Dynamics and Control*, 34(9), 1596–1609. https://doi.org/10.1016/j.jedc.2010.06.021
- Du, Z., & Escanciano, J. C. (2017). Backtesting expected shortfall: Accounting for tail risk. *Management Science*, 63(4), 940–958. https://doi.org/10.1287/mnsc.2015.2342
- Engle, R. F. (2002). Dynamic conditional correlation: A simple class of multivariate generalized autoregressive conditional heteroskedasticity models. *Journal of Business and Economic Statistics*, 20(3), 339–350. https://doi.org/10.1198/07350010-2288618487

- Engle, R. F., & Kroner, K. F. (1995). Multivariate simultaneous generalized ARCH. *Econometric Theory*, 11(1), 122–150. https:// doi.org/10.1017/S0266466600009063
- Fissler, T., & Ziegel, J. F. (2016). Higher order elicitability and Osband's principle. *The Annals of Statistics*, 44(4), 1680–1707. https://doi.org/10.1214/16-AOS1439
- Fülle, M. J., & Herwartz, H. (2022). Is gold always a safe-haven? Evidence from a novel Markov-switching multivariate GARCH model with copula-distributed innovations. https://doi.org/10. 2139/ssrn.3947734
- Fülle, M. J., Lange, A., Hafner, C. M., & Herwartz, H. (2022). BEKKs: Multivariate conditional volatility modelling and forecasting. R package version 1.4.1.
- Grier, K. B., Henry, Ó. T., Olekalns, N., & Shields, K. (2004). The asymmetric effects of uncertainty on inflation and output growth. *Journal of Applied Econometrics*, 19(5), 551–565. https://doi.org/10.1002/jae.763
- Haas, M., Mittnik, S., & Paolella, M. S. (2004). A new approach to Markov-switching GARCH models. *Journal of Financial Econometrics*, 2(4), 493–530. https://doi.org/10.1093/jjfinec/ nbh020
- Hafner, C. M., & Herwartz, H. (2008). Analytical quasi maximum likelihood inference in multivariate volatility models. *Metrika*, 67(2), 219–239. https://doi.org/10.1007/s00184-007-0130-y
- Hafner, C. M., & Preminger, A. (2009). On asymptotic theory for multivariate GARCH models. *Journal of Multivariate Analysis*, 100(9), 2044–2054. https://doi.org/10.1016/j.jmva.2009.03.011
- Hamilton, J. D. (1989). A new approach to the economic analysis of nonstationary time series and the business cycle. *Econometrica*, 57(2), 357. https://doi.org/10.2307/1912559
- Hamilton, J. D. (2016). *Regime switching models* (pp. 1–7). Palgrave Macmillan UK. https://doi.org/10.1057/978-1-349-95121-5_ 2459-1
- Hansen, P. R., Lunde, A., & Nason, J. M. (2011). The model confidence set. *Econometrica*, 79(2), 453–497. https://doi.org/10. 3982/ECTA5771
- Hong, Y., & Lee, T.-H. (2003). Diagnostic checking for the adequacy of nonlinear time series models. *Econometric Theory*, 19(6), 1065–1121. https://doi.org/10.1017/S0266466603196089
- Huang, J.-J., Lee, K.-J., Liang, H., & Lin, W.-F. (2009). Estimating value at risk of portfolio by conditional copula-GARCH method. *Insurance: Mathematics & Economics*, 45(3), 315–324. https://doi.org/10.1016/j.insmatheco.2009.09.009
- Joe, H. (1996a). Families of *m*-variate distributions with given margins and m(m-1)/2 bivariate dependence parameters. Institute of Mathematical Statistics Lecture Notes Monograph Series. (pp. 120–141). Institute of Mathematical Statistics. https://doi.org/10.1214/lnms/1215452614
- Joe, H. (1996b). Time series models with univariate margins in the convolution-closed infinitely divisible class. *Journal of Applied Probability*, 33(3), 664–677. https://doi.org/10.2307/3215348
- Jorion, P. (1996). Risk2: Measuring the risk in value at risk. Financial Analysts Journal, 52(6), 47–56. https://doi.org/10.2469/faj. v52.n6.2039

- Kim, C. J. (1994). Dynamic linear models with Markov-switching. Journal of Econometrics, 60(1–2), 1–22. https://doi.org/10.1016/ 0304-4076(94)90036-1
- Kratz, M., Lok, Y. H., & McNeil, A. J. (2018). Multinomial VaR backtests: A simple implicit approach to backtesting expected shortfall. *Journal of Banking and Finance*, 88, 393–407. https:// doi.org/10.1016/j.jbankfin.2018.01.002
- Kroner, K. F., & Ng, V. K. (1998). Modeling asymmetric comovements of asset returns. *Review of Financial Studies*, 11(4), 817– 844. https://doi.org/10.1093/rfs/11.4.817
- Kupiec, P. H. (1995). Techniques for verifying the accuracy of risk measurement models. *The Journal of Derivatives*, 3(2), 73–84. https://doi.org/10.3905/jod.1995.407942
- Lee, T.-H., & Long, X. (2009). Copula-based multivariate GARCH model with uncorrelated dependent errors. *Journal of Econometrics*, 150(2), 207–218. https://doi.org/10.1016/j.jeconom. 2008.12.008
- Liu, B.-Y., Ji, Q., & Fan, Y. (2017). A new time-varying optimal copula model identifying the dependence across markets. *Quantitative Finance*, 17(3), 437–453. https://doi.org/10.1080/ 14697688.2016.1205208
- Nelsen, R. B. (2006). An introduction to copulas. Springer. https:// doi.org/10.1007/0-387-28678-0
- Nelson, D. B. (1991). Conditional heteroskedasticity in asset returns: A new approach. *Econometrica*, 59(2), 347–370. https://doi.org/10.2307/2938260
- Patton, A. J., Ziegel, J. F., & Chen, R. (2019). Dynamic semiparametric models for expected shortfall (and value-at-risk). *Journal* of *Econometrics*, 211(2), 388–413. https://doi.org/10.1016/j. jeconom.2018.10.008
- Rodriguez, J. C. (2007). Measuring financial contagion: A copula approach. *Journal of Empirical Finance*, 14(3), 401–423. https:// doi.org/10.1016/j.jempfin.2006.07.002
- Rosenblatt, M. (1952). Remarks on a multivariate transformation. The Annals of Mathematical Statistics, 23(3), 470–472. https:// doi.org/10.1214/aoms/1177729394
- Tasche, D. (2002). Expected shortfall and beyond. *Journal of Bank*ing and Finance, 26(7), 1519–1533. https://doi.org/10.1016/ S0378-4266(02)00272-8
- The Basel Committee on Banking Supervision. (2011). Basel III: A global regulatory framework for more resilient banks and banking systems.
- The Basel Committee on Banking Supervision. (2013). Consultative document, fundamental review of the trading book: A revised market risk framework.
- Wang, K., Chen, Y. H., & Huang, S. W. (2011). The dynamic dependence between the Chinese market and other international stock markets: A time-varying copula approach. *International Review* of Economics and Finance, 20(4), 654–664. https://doi.org/10. 1016/j.iref.2010.12.003
- Weiß, G. N. F. (2013). Copula-GARCH versus dynamic conditional correlation: An empirical study on VaR and ES forecasting accuracy. *Review of Quantitative Finance and Accounting*, 41(2), 179–202. https://doi.org/10.1007/s11156-012-0311-2

AUTHOR BIOGRAPHIES

Markus J. Fülle has completed his PhD in applied statistics and empirical methods in Göttingen. His research focuses on time series econometrics, spatial pattern recognition, and copula modeling with applications in financial economics. He now works in the finance industry assessing financial risks as a data scientist.

Helmut Herwartz is Professor of Econometrics at Georg-August University of Goettingen. His research interests cover robust and bootstrap-based inference, applied financial and macroeconomic analysis, and structural vector autoregressive modeling. He has published in major econometric journals, for instance, the Journal of Econometrics, Journal of Business and Economics Statistics, Econometric Reviews, and Journal of Money, Credit and Banking.

How to cite this article: Fülle, M. J., & Herwartz, H. (2024). Predicting tail risks by a Markov switching MGARCH model with varying copula regimes. *Journal of Forecasting*, *43*(6), 2163–2186. https://doi.org/10.1002/for.3117

APPENDIX A: COPULAS

This section originates broadly from Joe (1996a) and Nelsen (2006).

Definition 1. A bivariate function $C: [0,1]^2 \rightarrow [0,1]$ is defined as a copula if the conditions

1.
$$C(u_1, u_2) = 0$$
 for $u_1 = 0 \lor u_2 = 0$.
2. $\sum_{i=1}^{2} \sum_{j=1}^{2} (-1) C(u_{1,i}, u_{2,j}) \ge 0 \forall (u_{1,i}, u_{2,j}) \in [0,1]^2$ with $u_{1,1} < u_{1,2}$ and $u_{2,1} < u_{2,2}$.
3. $C(u_1, 1) = u_1$ and $C(1, u_2) = u_2 \forall u_1, u_2 \in [0,1]$ are fulfilled.

Definition 2. We define upper and lower tail dependence (TD), that is, the comovement in the extreme quantiles, of two random variables X_1 , X_2 as

$$\lambda_{U} \coloneqq \lim_{u \to 1^{-}} \mathbb{P}\left\{X_{1} > F_{1}^{-1}(u) | X_{2} > F_{2}^{-1}(u)\right\} = \lim_{u \uparrow 1} \frac{1 - 2u + C(u, u)}{1 - u}$$
$$\lambda_{L} \coloneqq \lim_{u \to 0^{+}} \mathbb{P}\left\{X_{1} \le F_{1}^{-1}(u) | X_{2} \le F_{2}^{-1}(u)\right\} = \lim_{u \downarrow 0} \frac{C(u, u)}{u},$$

where F_i^{-1} are the inverse cumulative distribution functions (CDF) of X_i .

Definition 3. The survival copula C_S of a copula *C* is defined as

$$C_{\rm S}(u_1, u_2; \theta) = u_1 + u_2 - 1 + C(1 - u_1, 1 - u_2; \theta)$$

Consequently, it is a copula rotated by 180°.

Definition 4. The Gumbel copula is defined as

$$C_{Gumbel}(u_1, u_2; \theta) = \exp\left\{-\left((-\ln(u_1))^{\theta} + (-\ln(u_2))^{\theta}\right)^{\frac{1}{\theta}}\right\}$$

$$c_{Gumbel}(u_1, u_2; \theta) = \frac{C_{Gumbel}(u_1, u_2; \theta)(\ln(u_1)\ln(u_2))^{\theta-1}\left(\left((-\ln(u_1))^{\theta} + (-\ln(u_2))^{\theta}\right)^{\frac{1}{\theta}} + \theta - 1\right)}{u_1 u_2 \left((-\ln(u_1))^{\theta} + (-\ln(u_2))^{\theta}\right)^{2-\frac{1}{\theta}}}$$

where $\theta \ge 1$ must hold. The lower tail dependence is zero, where the upper tail dependence is $\lambda_U = 2 - 2^{\frac{1}{\theta}}$.

Definition 5. The Clayton copula is defined as

$$C_{Clayton}(u_1, u_2; \theta) = (u_1^{-\theta} + u_2^{-\theta} - 1)^{-\frac{1}{\theta}}$$
$$c_{Clayton}(u_1, u_2; \theta) = \frac{(1+\theta)(u_1^{-\theta} + u_2^{-\theta} - 1)^{-\frac{1}{\theta} - 2}}{(u_1 u_2)^{\theta + 1}}$$

where $\theta > 0$ must hold. Moreover, the copula is not continuous for $u_1 = 0 \lor u_2 = 0$. The upper tail dependence is zero, where the lower tail dependence is $\lambda_L = 2^{-\frac{1}{\theta}}$.

A.1 | Vine copulas

Besides generalizing the copulas for higher dimensions, for instance, described in Joe (1996a) and Lee and Long (2009). The so-called regular D-vine copulas provide a more flexible framework to model dependencies in higher dimensions. For N = 3, the D-vine copula is defined as

$$\begin{split} F(\xi_{1,t},\xi_{2,t},\xi_{3,t};\theta_1,\theta_2,\theta_3,\theta_{12},\theta_{13},\theta_{23}) &= \\ \int_{-\infty}^{\xi_{3,t}} C_{12|3}(F_{1|3}(\xi_{1,t}|z;\theta_{13}),F_{2|3}(\xi_{2,t}|z;\theta_{23});\theta_{12})f_3(z,\theta_3)dz, \\ F_{ij}(\xi_{i,t},\xi_{j,t};\theta_i,\theta_j\theta_{ij}) &= C_{ij}(F_i(\xi_{i,t};\theta_i),F_j(\xi_{j,t};\theta_j),\theta_{ij}), \end{split}$$

with $F_{i|j}(\eta_1,\eta_2) = C_{i|j}(F_1(\eta_1),F_2(\eta_2))$, conditional copula $C_{i|j}(u,v) = \frac{\partial C_{i|(u,v)}}{\partial v}$, C_{ij} being a two-dimensional copula, and F_i marginal CDFs. Hence, the PDF reads as

$$\begin{split} f_{123}(\xi_{1,t},\xi_{2,t},\xi_{3,t};\theta_1,\theta_2,\theta_3,\theta_{12},\theta_{13},\theta_{23}) &= \\ c_{12|3}(F_{1|3}(\xi_{1,t}|z;\theta_{13,t}),F_{2|3}(\xi_{2,t}|z;\theta_{23});\theta_{12})f_{13}(\xi_{1,t},\xi_{2,t};\theta_{12})f_{23}(\xi_{2,t},\xi_{3,t};\theta_{23})f_2(\xi_{2,t};\theta_2) &= \\ c_{12|3}(F_{1|3}(\xi_{1,t}|z;\theta_{13,t}),F_{2|3}(\xi_{2,t}|z;\theta_{23});\theta_{12})c_{13}(F_1(\xi_{1,t};\theta_1),F_3(\xi_{3,t};\theta_2);\theta_{13}) \times \\ c_{23}(F_2(\xi_{2,t};\theta_2),F_3(\xi_{3,t};\theta_3);\theta_{23})\prod_{i=1}^3 f_i(\xi_{i,t};\theta_i) \end{split}$$

The copulas for N > 3 can be derived by iterating that scheme. Joe (1996a) gives a detailed view.

APPENDIX B: PORTFOLIO DISTRIBUTIONS

For notational simplicity, we drop the copula parameters hereinafter. According to Cherubini et al. (2004), the distribution of the portfolio in the case of a single regime is given by

$$F_{z_{t}}(z) = \begin{cases} F_{1}\left(\frac{z}{\nu_{1,t}}\right) & \text{if } \nu_{2,t} = \nu_{3,t} = 0 \\ F_{2}\left(\frac{z}{\nu_{2,t}}\right) & \text{if } \nu_{1,t} = \nu_{2,t} = 0 \\ \int_{-\infty}^{\infty} F_{1|2,3}\left(\frac{z}{\nu_{1,t}} - \eta_{2,t}\frac{\nu_{2,t}}{\nu_{1,t}}, \eta_{2,t}\right) d\eta_{2,t} & \text{if } \nu_{3,t} = 0 \\ \int_{-\infty}^{\infty} F_{1|3}\left(\frac{z}{\nu_{1,t}} - \eta_{3,t}\frac{\nu_{3,t}}{\nu_{1,t}}, \eta_{3,t}\right) d\eta_{3,t} & \text{if } \nu_{2,t} = 0 \\ \int_{-\infty}^{\infty} F_{2|3}\left(\frac{z}{\nu_{2,t}} - \frac{\nu_{3,t}}{\nu_{2,t}}, \eta_{3,t}\right) d\eta_{3,t} & \text{if } \nu_{1,t} = 0 \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F_{1|2,3}\left(F_{1|3}\left(\frac{z}{\nu_{1,t}} - \eta_{3,t}\frac{\nu_{2,t}}{\nu_{1,t}}, \eta_{2,t} - \frac{\nu_{3,t}}{\nu_{1,t}}, \eta_{3,t}, \eta_{3,t}\right), F_{2|3}(\eta_{2,t}, \eta_{3,t})\right) \\ \cdot f_{23}(F_{2}(\eta_{2,t}), F_{3}(\eta_{3,t})) d\eta_{2,t} d\eta_{3,t} & \text{else} \end{cases}$$

and for the regime switching model with two regimes (for more than two regimes, this distribution can be derived by iterating that scheme):

$$F_{z_{t}}(z) = \begin{cases} \mathcal{P}_{1}\Phi_{\nu_{t}}(z) + \mathcal{P}_{2}F_{1}\left(\frac{z}{\nu_{1,t}(s_{t}=2)}\right) & \text{if } \nu_{2,t}(s_{t}=2) = \nu_{3,t}(s_{t}=2) = 0\\ \mathcal{P}_{1}\Phi_{\nu_{t}}(z) + \mathcal{P}_{2}F_{2}\left(\frac{z}{\nu_{2,t}(s_{t}=2)}\right) & \text{if } \nu_{1,t}(s_{t}=2) = \nu_{3,t}(s_{t}=2) = 0\\ \mathcal{P}_{1}\Phi_{\nu_{t}}(z) + \mathcal{P}_{2}F_{3}\left(\frac{z}{\nu_{3,t}(s_{t}=2)}\right) & \text{if } \nu_{1,t}(s_{t}=2) = \nu_{2,t}(s_{t}=2) = 0\\ \mathcal{P}_{1}\Phi_{\nu_{t}}(z) + \mathcal{P}_{2}\int_{-\infty}^{\infty}F_{1|2,3}\left(\frac{z}{\nu_{1,t}(s_{t}=2)} - \frac{\nu_{2,t}(s_{t}=2)}{\nu_{1,t}(s_{t}=2)}, \eta_{2,t}\right)d\eta_{2,t} & \text{if } \nu_{3,t}(s_{t}=2) = 0\\ \mathcal{P}_{1}\Phi_{\nu_{t}}(z) + \mathcal{P}_{2}\int_{-\infty}^{\infty}F_{1|3}\left(\frac{z}{\nu_{2,t}(s_{t}=2)} - \frac{\nu_{3,t}(s_{t}=2)}{\nu_{1,t}(s_{t}=2)}, \eta_{3,t}\right)d\eta_{3,t} & \text{if } \nu_{2,t}(s_{t}=2) = 0\\ \mathcal{P}_{1}\Phi_{\nu_{t}}(z) + \mathcal{P}_{2}\int_{-\infty}^{\infty}F_{2|3}\left(\frac{z}{\nu_{2,t}(s_{t}=2)} - \frac{\nu_{3,t}(s_{t}=2)}{\nu_{2,t}(s_{t}=2)}, \eta_{3,t}\right)d\eta_{3,t} & \text{if } \nu_{1,t}(s_{t}=2) = 0\\ \mathcal{P}_{1}\Phi_{\nu_{t}}(z) + \mathcal{P}_{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F_{1|2,3}(F_{1|3}\left(\frac{z}{\nu_{1,t}(s_{t}=2)} - \frac{\nu_{2,t}(s_{t}=2)}{\nu_{1,t}(s_{t}=2)}, \eta_{3,t}\right)d\eta_{3,t} & \text{if } \nu_{1,t}(s_{t}=2) = 0\\ \mathcal{P}_{1}\Phi_{\nu_{t}}(z) + \mathcal{P}_{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F_{1|2,3}(F_{1|3}\left(\frac{z}{\nu_{1,t}(s_{t}=2)} - \frac{\nu_{2,t}(s_{t}=2)}{\nu_{1,t}(s_{t}=2)}, \eta_{3,t}\right)d\eta_{3,t} & \text{if } \nu_{1,t}(s_{t}=2) = 0\\ \mathcal{P}_{1}\Phi_{\nu_{t}}(z) + \mathcal{P}_{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F_{1|2,3}(F_{1|3}\left(\frac{z}{\nu_{1,t}(s_{t}=2)} - \frac{\nu_{2,t}(s_{t}=2)}{\nu_{2,t}(s_{t}=2)}, \eta_{3,t}\right)d\eta_{3,t} & \text{if } \nu_{1,t}(s_{t}=2) = 0\\ \mathcal{P}_{1}\Phi_{\nu_{t}}(z) + \mathcal{P}_{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F_{1|2,3}(F_{1|3}\left(\frac{z}{\nu_{1,t}(s_{t}=2)} - \frac{\nu_{2,t}(s_{t}=2)}{\nu_{1,t}(s_{t}=2)}, \eta_{2,t}-\frac{\nu_{3,t}(s_{t}=2)}{\nu_{1,t}(s_{t}=2)}, \eta_{3,t}, \eta_{3,t}\right),\\ F_{2|3}(\eta_{2,t},\eta_{3,t})) \cdot f_{23}(F_{2}(\eta_{2,t}), F_{3}(\eta_{3,t}))d\eta_{2,t}d\eta_{3,t} & \text{else} \end{cases}$$

with $\mathcal{P}_i = P(s_t = i | \Omega_{t-1})$ and $\Phi_{\nu_t}(z)$ is the CDF of a univariate normal distribution with standard deviation $\sum_{l=1}^{3} \nu_{t,l}(s_t = 1)$.

0

0

0

WILEY-

APPENDIX C: ADDITIONAL GRAPHS AND TABLES



FIGURE C1 Estimated conditional standard deviations and correlations implied by the asymmetric BEKK(1,1,1) model with asymmetric affects set to (+, -, +). Estimated by the R-Package BEKKs of Fülle et al. (2022).



FIGURE C2 Estimated smoothed probabilities of being in the independence regime $P(s_t = 1 | \Omega_{T_m})$ by the algorithm of Kim (1994). Upper/lower panel shows the probabilities when the second regime is set to the best fitting crisis (GS, G, G)/recovery (G, GS, GS) regime. The grey-shaded areas indicate important economic and or political crisis events. I, Japan Crisis; II, Black Wednesday; III, Mexican peso crisis; IV, Asian financial crisis; V, Bursting of the Dotcom Bubble; VI, Financial Crisis; VII, European Sovereign Debt Crisis.

Date



FIGURE C3 From top to bottom: estimated smoothed probabilities of being in the independence, crisis, and recovery regime. For further information, see Figure C2.



FIGURE C4 Cumulative loss differentials (Ardia et al., 2018). The upper and lower panels show QL and FZL statistics from VaR and ES with nominal coverage of $\alpha = 1\%$. The underlying portfolio is equally weighted. For further information, see Figure 4.