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Optimal execution with multiplicative price impact and incomplete information on the return

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Abstract

We study an optimal liquidation problem with multiplicative price impact in which the trend of the asset price is an unobservable Bernoulli random variable. The investor aims at selling over an infinite time horizon a fixed amount of assets in order to maximise a net expected profit functional, and lump-sum as well as singularly continuous actions are allowed. Our mathematical modelling leads to a singular stochastic control problem featuring a finite-fuel constraint and partial observation. We provide a complete analysis of an equivalent three-dimensional degenerate problem under full information, whose state process is composed of the asset price dynamics, the amount of available assets in the portfolio, and the investor's belief about the true value of the asset's trend. Its value function and optimal execution rule are expressed in terms of the solution to a truly two-dimensional optimal stopping problem, whose associated belief-dependent free boundary b triggers the investor's optimal selling rule. The curve b is uniquely determined through a nonlinear integral equation, for which we derive a numerical solution through an application of the Monte Carlo method. This allows us to understand the value of information in our model as well as the sensitivity of the problem's solution with respect to the relevant model parameters.

Keywords Optimal execution problem · Multiplicative price impact · Singular stochastic control · Partial observation · Optimal stopping

Mathematics Subject Classification (2020) 93E20 · 93C41 · 49L20 · 91G80

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1 Introduction

In this paper, we consider an investor who possesses a fixed amount of assets and aims at selling them on the market. We assume that the investor faces the issue of causing an adverse price reaction, so that fast selling depresses the stock price while splitting the order over time may take too long. This problem – also known as the *optimal execution problem* in algorithmic trading – thus deals with the question of how to trade optimally in order to maximise a given profit, and therefore of how to determine the times as well as the sizes of the orders.

Dating back to the early works of Bertsimas and Lo [9], Almgren and Chriss [2] and Almgren [1], the study of optimal execution strategies has received much attention and resulted in a series of important contributions in various settings which, amongst other modelling features, can be distinguished with respect to the considered type of price impact: additive or multiplicative. A comprehensive discussion on the latter class of models can be found in Guo and Zervos [44], who also point out that models with multiplicative price impact seem to be more natural since they ensure prices to remain positive. Amongst those works dealing with multiplicative price impact, let us mention Bertsimas et al. [10] for a discrete-time framework, Forsyth et al. [37] for a continuous-time model à la Black–Scholes, and Guo and Zervos [44] and Becherer et al. [5] for settings involving singular stochastic controls.

A common feature in the literature is the assumption that the investor has full information on the trend of the asset. This, however, can be a strong requirement. As pointed out by Ekström and Lu [30], a statistical estimation of the drift is not an efficient procedure, and obtaining a reasonable precision would need decades or even centuries of data under the same market conditions, which is simply not feasible in reality (see also the discussion in Rogers [58, Sect. 4.2]). In some cases, such as initial public offerings, this price history does not even exist.

To account for this fact, we propose a model of optimal execution with multiplicative price impact in which the drift of the stock price dynamics is a random variable which is not directly observable by the investor. Through monitoring the evolution of the price on the market, the investor is able to update her belief regarding the drift value. However, that observation is noisy as the investor cannot perfectly distinguish whether price variations are caused by the drift or by the stochastic driver of the underlying dynamics. From a mathematical point of view, our model leads to a finite-fuel singular stochastic control problem under partial observation, and we investigate how the presence of incomplete information influences the selling strategy of the investor. In particular, we show that the flow of incoming information – through the observation of the asset's market price – has a direct effect on the optimal execution rule. Indeed, differently from the case of full information treated in Guo and Zervos [44], the decision to sell is no longer triggered by a constant critical price, but the execution threshold changes dynamically depending on the investor's current belief on the future trend of the asset. Our results show that the optimal execution strategy is in fact determined by a boundary that is increasing in the belief in the larger drift value, corresponding to the intuition that the decision maker chooses to delay selling assets if future prices are expected to increase.

In this regard, our work relates to the strand of economic and financial literature where questions of optimal decision-making under partial observation have been

considered; amongst a large number of contributions, we refer to the seminal papers on portfolio selection by Detemple [26] and Gennotte [42], to Veronesi [64] for an equilibrium model with uncertain dividend drift, to Sass and Haussmann [59] for a terminal-wealth portfolio optimisation problem, and to the more recent paper by Colaneri et al. [15] for an optimal liquidation problem with rate strategies and partial observation. Notably, the recent papers Drissi [29] and Bismuth et al. [11] incorporate Bayesian learning in a model of multi-asset optimal execution, although restricting the agent to absolutely continuous (regular) controls.

Furthermore, we contribute to those models dealing with problems of optimal stopping and singular stochastic control. To name just a few recent works, we mention Callegaro et al. [13] for public debt control, De Angelis [19] and Décamps and Villeneuve [25] for dividend payments, Décamps et al. [24] for investment timing, Ekström and Lu [30] as well as Ekström and Vaicenavicius [31] for asset liquidation, Federico et al. [33] for inventory management, Johnson and Peskir [45] for quickest detection, Gapeev [39] for the pricing problem of perpetual commodity equities, and Gapeev and Rodosthenous [40] for a zero-sum optimal stopping game associated with perpetual convertible bonds.

1.1 Our model, approach and overview of the mathematical analysis

We now discuss the mathematical modelling and analysis. Consider an investor holding a fixed amount y of assets in her portfolio. In the absence of investor actions, the stock price evolves according to a geometric Brownian motion $dS_t = \beta S_t dt + \sigma S_t dW_t$, where W is a standard Brownian motion and $\sigma > 0$ a constant volatility parameter. Furthermore, the price process exhibits a random future trend β , which is, however, unknown to the decision maker and is assumed to be a random variable, independent of the Brownian noise, taking two values $\beta_0 < \beta_1$ for some $\beta_0, \beta_1 \in \mathbb{R}$ and $\beta_0 < 0$.

The decision maker is able to sell the assets on the market over an infinite time horizon, and we denote by ξ_t the cumulative amount of assets liquidated up to time t . Consequently, the remaining assets in the portfolio follow the dynamics $Y_t^\xi = y - \xi_t$. Clearly, we must have $\xi_t \leq y$ at any time $t \geq 0$ (finite-fuel constraint) since no more than the initial amount of assets can be sold. As announced, we assume that the investor causes an adverse price reaction upon selling which, following Guo and Zervos [44], we assume to be of multiplicative type. Hence, the controlled asset price evolves as

$$dS_t^\xi = \beta S_t^\xi dt + \sigma S_t^\xi dW_t - \alpha S_t^\xi \circ d\xi_t, \quad S_{0-}^\xi = s > 0,$$

where $\alpha > 0$ denotes the parameter of price impact and the operator \circ is defined in (2.3) below so as to take care of the continuous and jump components of any admissible selling strategy ξ . As will become clear later, see (2.6), the multiplicative price impact structure allows expressing the asset price process as $S^\xi = \exp(X^\xi)$, where X^ξ is a linearly controlled drifted Brownian motion with volatility $\sigma > 0$ and drift value $\mu = \beta - \frac{1}{2}\sigma^2$.

The investor aims at maximising the total expected discounted reward upon selling, net of transaction costs; that is, she looks for

$$\sup_{\xi} \mathbb{E} \left[\int_0^{\infty} e^{-rt} (e^{X_t^{\xi}} - \kappa) \circ d\xi_t \right],$$

where the optimisation is taken over a suitable admissible class of selling strategies ξ and the investor discounts her future revenues with a strictly positive factor $r > 0$ that can be interpreted as her subjective impatience. This yields a finite-fuel singular stochastic control problem under partial observation.

By relying on classical filtering techniques (cf. Shiryaev [61, Sect. 4.2]), we begin by determining an equivalent Markovian problem – the so-called *separated problem* – under full information (see Fleming and Pardoux [35] as a classical reference on the separated problem). To this end, we introduce the process Π according to which the investor can update her belief regarding the true value of the drift. This is done by observing the evolution of the process X^0 (denoting the uncontrolled version of the process X^{ξ}), whose natural filtration $(\mathcal{F}_t^{X^0})$ models the overall information available up to time t . More precisely, after forming a prior $\pi := \mathbb{P}[\mu = \mu_1] \in (0, 1)$, the investor dynamically updates her belief upon the arrival of new information through observing the process X^{ξ} so that the belief process is given by $\Pi_t = \mathbb{P}[\mu = \mu_1 | \mathcal{F}_t^{X^{\xi}}]$. Notice that a value of Π close to 1 indicates a strong belief in the larger value of the drift, while Π close to 0 displays a strong belief in the lower value. Hence we expect the investor to change the liquidation strategy dynamically and base it not solely on the current price on the market, but also on the present belief at that time.

The separated problem turns out to be a *three-dimensional degenerate finite-fuel singular stochastic control problem*, so that obtaining explicit solutions through a traditional “guess-and-verify approach” is in general not feasible. We note that this approach would be applicable if we take $\beta_0 = -\beta_1$, which indeed allows a dimension reduction; see e.g. Décamps and Villeneuve [25]. In the present paper, however, we do not assume any relation between β_0 and β_1 other than $\beta_0 < \beta_1$.

In order to tame the multidimensional nature of the resulting optimal execution problem under full information, we then follow a direct approach which hinges on the study of a suitable optimal stopping problem, with value v , that we expect to be associated to the singular stochastic control problem. This method was studied and refined by many authors such as Beneš et al. [6], El Karoui and Karatzas [32] and Karatzas and Shreve [49], or De Angelis [19], De Angelis et al. [20, 21] and Guo and Tomecek [43] for more recent contributions. The optimal stopping problem, which involves the underlying two-dimensional diffusion (X^0, Π) taking values in $\mathbb{R} \times (0, 1)$, can be interpreted as an optimal selling problem and exhibits a structure similar to that of the problem treated by Décamps et al. [24] (see also Ekström and Lu [30] for a parabolic version). We then solve the optimal stopping problem by relying on techniques from free-boundary theory (as illustrated in the monograph by Peskir and Shiryaev [57, Chap. 3]) and first show that the optimal stopping rule is characterised through a belief-dependent free boundary $a(\pi)$ for $\pi \in (0, 1)$.

However, the coupled dynamics of the underlying processes X^0 and Π as well as the fact that they are driven by the same Brownian motion makes a further study of

the free boundary and the value function v not feasible. For that reason, we proceed by deriving two equivalent representations of the optimal stopping problem which allow a thorough analysis. First, via a change of measure, the state process (X^0, Π) is transformed into (X^0, Φ) taking values in $\mathbb{R} \times (0, \infty)$ and with decoupled dynamics. Here, the process Φ is the so-called “likelihood ratio”. Again, we can express the optimal stopping strategy in terms of a free boundary $\varphi \mapsto b(\varphi)$ which results from a simple transformation of the boundary $\pi \mapsto a(\pi)$. Second, we pass to yet another formulation by deriving the intrinsic parabolic formulation of the stopping problem in coordinates (X^0, Z) , in which the process Z now follows purely deterministic dynamics and takes values in \mathbb{R} . Even though the monotonicity result of the associated free boundary $z \mapsto c(z)$ is not trivial to derive and calls for a rigorous technical analysis, it is in this formulation that we are able to provide further regularity results for c and for the transformed optimal stopping value function \widehat{v} . In fact, borrowing arguments from De Angelis [19], suitably adapted to the present setting, we achieve a global regularity of \widehat{v} , namely $\widehat{v} \in C^1(\mathbb{R}^2)$. The latter result also allows proving that $\widehat{v}_{xx} \in L^\infty_{\text{loc}}(\mathbb{R}^2)$ and finally obtaining a nonlinear integral equation uniquely solved by the optimal stopping boundary c . It is worth mentioning that this characterisation can be translated back to both optimal stopping boundaries b and a and is thus tantamount to a complete specification of the optimal stopping rule in the original (x, π) -coordinates.

The thorough analysis developed for the optimal stopping problem is then exploited in order to identify an optimal execution strategy. In fact, the derived regularity results for \widehat{v} permit us to prove a verification theorem that identifies an optimal execution rule and shows that the optimal stopping value function v indeed coincides with a directional derivative of the separated problem’s value function V . Namely, we show that

$$V(x, y, \pi) := \frac{1}{\alpha} \int_{x-\alpha y}^x v(x', \pi) dx', \quad (x, y, \pi) \in \mathbb{R} \times (0, \infty) \times (0, 1).$$

Notice that if $\alpha \downarrow 0$, one finds $V(x, y, \pi) = yv(x, \pi)$, which is the value of the problem in which the investor has no market impact.

The optimal execution rule can be thought of as a “myopic one”. Indeed, it prescribes to sell assets as if the size of the investor’s portfolio were infinite, and to stop selling once the asset’s inventory is depleted (see also Karatzas [47] and El Karoui and Karatzas [32]). The optimal selling rule involves lump-sum executions (whenever the asset price is sufficiently large) that could eventually result in an immediate depletion of the portfolio (if the initial portfolio size is sufficiently small). However, for relatively large portfolios, an initial lump-sum selling is followed by a policy of oblique reflection type. This is triggered by the belief-dependent boundary $\varphi \mapsto b(\varphi)$ (equivalently, $\pi \mapsto a(\pi)$). Notably, given that all the transformations developed for the resolution of the optimal stopping problem are one-to-one and onto, the integral equation for the boundary $z \mapsto c(z)$ yields an integral equation for $\varphi \mapsto b(\varphi)$ and therefore a complete characterisation of the optimal execution rule. In order to provide insights about the sensitivity of the optimal decision mechanism of the investor with respect to the model parameters, we develop a recursive numerical scheme which relies on an application of the Monte Carlo method. Overall, we believe that the contributions of

this paper are the following. Even though the literature on optimal execution problems is extensive (to name just a few, see Almgren and Chriss [2], Almgren [1], Becherer et al. [5], Bertsimas and Lo [9], Bertsimas et al. [10], Colaneri et al. [15], Gatheral and Schied [41], Guo and Zervos [44], Moreau et al. [54], Schied and Schöneborn [60]), the combination of incomplete information on the future price trend while allowing lump-sum as well as singularly continuous executions constitutes a novelty. Furthermore, the present study on the optimal execution strategy complements as well as extends the literature on problems with a similar structure under full information. As a matter of fact, the derived optimal execution rule exhibits a broader structure and prescribes to take actions depending on the current belief on the future trend of the asset.

1.2 Our contributions

From a mathematical point of view, to the best of our knowledge, ours is the first work providing a complete characterisation of the value function and of the optimal control rule in a finite-fuel singular stochastic control problem under partial observation (which, in the present setting, is equivalent to a three-dimensional degenerate singular stochastic control problem). Furthermore, we believe that the optimal stopping (selling) problem studied as a device to characterise the optimal solution of the optimal execution problem is of interest on its own. By performing a thorough analysis on the regularity of (a transformed version of) its value function and free boundary, we are able to provide a complete characterisation of the optimal selling rule through a nonlinear integral equation, thus extending the results of the related model studied by Décamps et al. [24]. Notice that an integral equation for the free boundary has been obtained also in Ekström and Lu [30] and Ekström and Vaicenavicius [31], but in settings where the parabolic nature of the problem arises because of an explicit time-dependence. Finally, the probabilistic numerical approach developed for the resolution of the free boundary's integral equation allows understanding the dependence of the investor's optimal execution strategy on relevant model parameters such as volatility and trend. Moreover, based on the numerical evaluation of the boundary, we can compare the value of the control problem with partial information with that of an associated *average drift* problem under full information. This allows us to numerically evaluate the question on whether the introduction of uncertainty over the drift actually harms or benefits the investor.

1.3 Organisation of the paper

The rest of the paper is organised as follows. In Sect. 2, we present our setting and first preliminary results. In Sect. 3, we investigate the benchmark problem under full information, before we consider a corresponding optimal stopping problem and its optimal boundary in Sect. 4. In Sects. 5 and 6, we derive two equivalent formulations of the optimal stopping problem under partial observation which allow a more thorough study. Eventually, in Sect. 7, we return to the optimal control problem and characterise the optimal selling rule of the investor. A numerical study based on the derived integral equation of the execution boundary is then carried out in Sect. 8.

2 Setting and problem formulation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space rich enough to accommodate a standard one-dimensional Brownian motion $(W_t)_{t \geq 0}$ and an independent random variable β taking two values β_0 and β_1 . We denote by $\mathbb{F}^W := (\mathcal{F}_t^W)_{t \geq 0}$ the filtration generated by $(W_t)_{t \geq 0}$ and augmented by the \mathbb{P} -nullsets of \mathcal{F} . We assume that in the absence of any actions of the investor, the asset price on the stock market evolves stochastically according to a geometric Brownian motion

$$dS_t^0 = \beta S_t^0 dt + \sigma S_t^0 dW_t, \quad S_0^0 = s > 0, \tag{2.1}$$

where $\sigma > 0$ is a constant volatility. The investor holds a finite amount $y \geq 0$ of assets which she is able to sell. We identify the cumulative amount of assets sold up to time $t \geq 0$, which we denote by ξ_t , as the investor’s control variable. We denote by $\mathbb{F}^Z := (\mathcal{F}_t^Z)_{t \geq 0}$ the natural filtration of any process Z , augmented by the \mathbb{P} -nullsets of \mathcal{F} . The set of admissible execution strategies in this context is given by

$$\begin{aligned} \mathcal{A}(y) := \{ \xi : \Omega \times [0, \infty) \rightarrow \mathbb{R}_+ : (\xi_t)_{t \geq 0} \text{ is } \mathbb{F}^{S^0}\text{-adapted, increasing, càdlàg} \\ \text{and } \xi_{0-} = 0, \xi_t \leq y \text{ a.s.} \}, \end{aligned}$$

where the last condition naturally arises from the fact that the investor cannot sell more than the initial amount of assets. Moreover, the remaining assets in the portfolio evolve according to the dynamics

$$Y_t^\xi = y - \xi_t, \quad Y_{0-}^\xi = y \geq 0,$$

where we stress the dependence on the selling strategy ξ . Following Guo and Zervos [44], we assume in our model that the investor’s transactions on the market have a proportional impact on the asset price. More precisely, when selling a small amount $\epsilon > 0$ of assets at time t , the price exhibits a jump of size

$$\Delta S_t = S_t - S_{t-} = -\alpha \epsilon S_{t-},$$

for $\alpha > 0$ denoting the parameter of permanent price impact (see Almgren and Chriss [2], Almgren [1] for early works and Becherer et al. [4], Ferrari and Koch [34], Guo and Zervos [44] for more recent contributions). Hence a small transaction is such that $S_t = (1 - \alpha \epsilon) S_{t-} \simeq e^{-\alpha \epsilon} S_{t-}$, and by interpreting a lump-sum sale of $\Delta \xi_t$ shares as a sequence of N individual sales of size $\epsilon = \Delta \xi_t / N$, we have

$$S_t = e^{-\alpha N \epsilon} S_{t-} = e^{-\alpha \Delta \xi_t} S_{t-}$$

for N large enough. It follows that for any $\xi \in \mathcal{A}(y)$, we can model the controlled asset price process by

$$dS_t^\xi = \beta S_t^\xi dt + \sigma S_{t-}^\xi dW_t - \alpha S_t^\xi \circ d\xi_t, \quad S_{0-}^\xi = s, \tag{2.2}$$

where

$$\begin{aligned} \int_0^\cdot S_t^{\xi} \circ d\xi_t &:= \int_0^\cdot S_t^{\xi} d\xi_t^c + \sum_{t \leq \cdot : \Delta\xi_t \neq 0} \frac{1}{\alpha} S_{t-}^{\xi} (1 - e^{-\alpha \Delta\xi_t}) \\ &= \int_0^\cdot S_t^{\xi} d\xi_t^c + \sum_{t \leq \cdot : \Delta\xi_t \neq 0} S_{t-}^{\xi} \int_0^{\Delta\xi_t} e^{-\alpha u} du, \end{aligned} \tag{2.3}$$

ξ^c denotes the continuous part of the process ξ and $\Delta\xi_t := \xi_t - \xi_{t-}$. The solution to (2.2) can be explicitly determined via Itô’s formula and is given by

$$S_t^{\xi} = s \exp\left(\left(\beta - \frac{1}{2}\sigma^2\right)t + \sigma W_t - \alpha\xi_t\right) = S_t^0 \exp(-\alpha\xi_t), \tag{2.4}$$

where S^0 is the solution to (2.1) and we observe that the price impact of selling is additive with respect to the logarithm of the asset price.

We assume that the investor aims at maximising the total expected (discounted) profits, net of the total cost of selling, and thus seeks to solve

$$\begin{aligned} &\sup_{\xi \in \mathcal{A}(y)} \mathbb{E} \left[\int_0^\infty e^{-rt} (S_t^{\xi} - \kappa) \circ d\xi_t \right] \\ &= \sup_{\xi \in \mathcal{A}(y)} \mathbb{E} \left[\int_0^\infty e^{-rt} (S_t^{\xi} - \kappa) d\xi_t^c + \sum_{t: \Delta\xi_t \neq 0} e^{-rt} \int_0^{\Delta\xi_t} (S_{t-}^{\xi} e^{-\alpha u} - \kappa) du \right]. \end{aligned} \tag{2.5}$$

Here, $\kappa > 0$ is a proportional transaction cost which, thinking of S_t^{ξ} as the ask price of the stock at time t , can also be interpreted as a constant bid–ask spread. Notice that the structure of the expected net-profit functional in (2.5) can also be justified through stability results in the Skorokhod M_1 -topology in probability (see Becherer et al. [5]). Moreover, (2.5) has a finite value due to $\xi_t \leq y$ a.s. Thanks to (2.4), we have $S_t^{\xi} = \exp(X_t^{\xi})$, where

$$dX_t^{\xi} = \mu dt + \sigma dW_t - \alpha d\xi_t, \quad X_{0-}^{\xi} = x, \tag{2.6}$$

with $x := \ln s$ and $\mu := \beta - \frac{1}{2}\sigma^2$. In particular, the drift can take two values $\mu_i = \beta_i - \frac{1}{2}\sigma^2$, $i = 0, 1$. In the following, when needed, we let X^0 denote the solution to (2.6) with $\xi \equiv 0$, which is then an arithmetic Brownian motion. Furthermore, we state the following assumption.

Assumption 2.1 We have $\beta_1 > \beta_0$ and $\beta_0 < 0$, which implies $\mu_0 < 0$.

The maximisation problem (2.5) can thus be rewritten in terms of (2.6) as

$$\sup_{\xi \in \mathcal{A}(y)} \mathbb{E} \left[\int_0^\infty e^{-rt} (e^{X_t^{\xi}} - \kappa) \circ d\xi_t \right]. \tag{2.7}$$

Notice that for a constant non-random drift coefficient, a close variant of this problem was considered and solved by Guo and Zervos [44], who also incorporate the option of buying shares of assets and the constraint that the whole inventory must be depleted at the terminal time. However, due to the presence of incomplete information on the drift of the asset, (2.7) is not of Markovian nature and thus requires a different analysis. In order to obtain an equivalent Markovian formulation of (2.7), we rely on classical results from filtering theory, dating back to the contribution of A.N. Shiryaev in the context of quickest detection models (see Shiryaev [62] for a survey). To this end, we introduce the *belief* process

$$\Pi_t := \mathbb{P}[\mu = \mu_1 | \mathcal{F}_t^{X^0}], \quad t \geq 0,$$

which reflects the conditional probability at time t that $\mu = \mu_1$ given the observations of the price process up to that time (indeed, $\mathbb{F}^{S^0} = \mathbb{F}^{X^0} = \mathbb{F}^{X^\xi}$, noting that ξ must be \mathbb{F}^{S^0} -adapted). According to this process, the investor is able to update the belief regarding the true value of the drift, based on the arrival of new information by observing the asset price evolution on the market. Notice that a large value of Π close to 1 implies a strong belief in the larger drift value μ_1 , while a low value of Π implies the contrary. It follows (see e.g. Shiryaev [61, Sect. 4.2]) that the dynamics of X^ξ , Π and Y^ξ can be written as

$$\begin{cases} dX_t^\xi = (\mu_1 \Pi_t + \mu_0(1 - \Pi_t))dt + \sigma d\bar{W}_t - \alpha d\xi_t, & X_{0-}^\xi = x \in \mathbb{R}, \\ d\Pi_t = \gamma \Pi_t(1 - \Pi_t)d\bar{W}_t, & \Pi_0 = \pi \in (0, 1), \\ Y_t^\xi = y - \xi_t, & Y_{0-}^\xi = y \geq 0, \end{cases} \quad (2.8)$$

where $\gamma = (\mu_1 - \mu_0)/\sigma$ is the *signal-to-noise ratio* and

$$d\bar{W}_t = \frac{dX_t^0}{\sigma} - \left(\frac{\mu_0}{\sigma} + \gamma \Pi_t \right) dt$$

denotes the *innovation process* which is an \mathbb{F}^{X^0} -Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$. Moreover, notice that the value $\pi := \mathbb{P}[\mu = \mu_1]$ reflects the initial subjective belief of the investor regarding the true value of the drift. We do not question the origin of this initial belief; this can be an instinctive decision or even the result of a constructive approach, for instance by observing the trends of similar assets over the past years. In the new formulation, the process (X^ξ, Y^ξ, Π) is an \mathbb{F}^{X^0} -adapted and time-homogeneous Markov process, as it is the unique strong solution to the system of stochastic differential equations in (2.8). Furthermore, we observe that the drift μ is replaced by its conditional estimate, and the process Π is a bounded martingale valued in $[0, 1]$ with $\Pi_\infty \in \{0, 1\}$ as all information will eventually get revealed. Denoting $\mathbb{E}_{x,y,\pi}[\cdot] = \mathbb{E}[\cdot | X_{0-}^\xi = x, Y_{0-}^\xi = y, \Pi_0 = \pi]$, we can thus reformulate the problem of incomplete information as a so-called *separated problem* (cf. Bensoussan [7, Chap. 7.1], and Fleming and Pardoux [35])

$$V(x, y, \pi) := \sup_{\xi \in \mathcal{A}(y)} J(x, y, \pi, \xi) \quad (2.9)$$

with

$$J(x, y, \pi, \xi) := \mathbb{E}_{x,y,\pi} \left[\int_0^\infty e^{-rt} (e^{X_t^\xi} - \kappa) d\xi_t^c + \sum_{t:\Delta\xi_t \neq 0} e^{-rt} \int_0^{\Delta\xi_t} (e^{X_{t-}^\xi - \alpha u} - \kappa) du \right],$$

for any $(x, y, \pi) \in \mathbb{R} \times (0, \infty) \times (0, 1)$. Notice indeed that $\Pi_t \in (0, 1)$ for all $t \geq 0$ a.s. if $\pi \in (0, 1)$, while $\Pi_t \equiv \pi_0$ for all $t \geq 0$ a.s. if $\pi_0 \in \{0, 1\}$. Problem (2.9) is equivalent to (2.5): They share the same value, and because of the uniqueness of the strong solution to (2.8), a control is optimal for (2.5) if and only if it is optimal for (2.9).

2.1 The Hamilton–Jacobi–Bellman equation

Problem (2.9) takes the form of a *three-dimensional singular stochastic control problem with finite-fuel constraint* (cf. Baldursson [3], Beneš et al. [6], El Karoui and Karatzas [32], Karatzas [46] and Karatzas et al. [48] for early contributions). We start our analysis by providing a heuristic derivation of the dynamic programming equation that we expect the value function V to satisfy. To this end, we notice that the investor is faced with two possible actions at the initial time. On the one hand, she could choose to wait for a short period of time Δt , not sell any fraction of the assets and then continue with an optimal execution strategy (supposing that one exists). Since this strategy is not necessarily optimal, we obtain

$$V(x, y, \pi) \geq \mathbb{E}_{x,y,\pi} [e^{-r\Delta t} V(X_{\Delta t}, y, \Pi_{\Delta t})], \quad (x, y, \pi) \in \mathbb{R} \times (0, \infty) \times (0, 1).$$

If we assume that the value function V has enough regularity, we can apply Itô’s formula, divide by Δt and invoke the mean value theorem to let $t \rightarrow 0$ and obtain

$$(\mathcal{L}_{X,\Pi} - r)V \leq 0.$$

Here, $\mathcal{L}_{X,\Pi}$ denotes the second-order differential operator acting on twice-continuously differentiable functions which is given by

$$\begin{aligned} \mathcal{L}_{X,\Pi} := & \frac{1}{2} \gamma^2 \pi^2 (1 - \pi)^2 \partial_{\pi\pi} \\ & + \frac{1}{2} \sigma^2 \partial_{xx} + (\pi \mu_1 + (1 - \pi) \mu_0) \partial_x + \sigma \gamma \pi (1 - \pi) \partial_{x\pi}. \end{aligned} \tag{2.10}$$

On the other hand, the investor can instantaneously sell an amount $\epsilon > 0$ of the assets and then proceed by following an optimal execution strategy. Again, this strategy is a priori suboptimal, and since this action is associated with the inequality

$$V(x, y, \pi) \geq V(x - \alpha\epsilon, y - \epsilon, \pi) + \frac{1}{\alpha} e^x (1 - e^{-\alpha\epsilon}) - \kappa\epsilon,$$

adding and subtracting $V(x - \alpha\epsilon, y, \pi)$ and dividing by ϵ yields

$$\frac{V(x, y, \pi) - V(x - \alpha\epsilon, y, \pi)}{\epsilon} \geq \frac{V(x - \alpha\epsilon, y - \epsilon, \pi) - V(x - \alpha\epsilon, y, \pi)}{\epsilon} + \frac{1}{\alpha} e^x \frac{(1 - e^{-\alpha\epsilon})}{\epsilon} - \kappa.$$

Hence by letting $\epsilon \downarrow 0$, we obtain

$$\alpha V_x(x, y, \pi) \geq -V_y(x, y, \pi) + e^x - \kappa.$$

Since only one of these actions should be optimal and given the Markovian setting of (2.9), we thus expect that the value function V should identify with an appropriate solution to the Hamilton–Jacobi–Bellman equation

$$\begin{aligned} \max\{(\mathcal{L}_{X,\Pi} - r)u, -\alpha u_x - u_y + e^x - \kappa\} &= 0, \\ (x, y, \pi) \in \mathbb{R} \times (0, \infty) \times (0, 1), \end{aligned} \tag{2.11}$$

with boundary condition $u(x, 0, \pi) = 0$ since $y = 0$ implies $\mathcal{A}(0) = \{\xi \equiv 0\}$ and $J(x, 0, \pi, 0) = 0$. It is worth noticing that the variable y plays the role of a parameter in (2.11), which is then a two-dimensional elliptic partial differential equation with a state-dependent directional derivative constraint, parametrised by $y > 0$. With reference to (2.11) and the reasoning above, we can introduce the *waiting region*

$$\begin{aligned} \mathbb{W}_1 := \{(x, y, \pi) \in \mathbb{R} \times (0, \infty) \times (0, 1) : (\mathcal{L}_{X,\Pi} - r)V &= 0, \\ &- \alpha V_x - V_y + e^x - \kappa < 0\} \end{aligned}$$

in which it is expected to be suboptimal to sell any assets, and the *selling/execution region*, where it should be profitable for the investor to sell a fraction of the assets, as

$$\begin{aligned} \mathbb{S}_1 := \{(x, y, \pi) \in \mathbb{R} \times (0, \infty) \times (0, 1) : (\mathcal{L}_{X,\Pi} - r)V &\leq 0, \\ &- \alpha V_x - V_y + e^x - \kappa = 0\}. \end{aligned}$$

Due to the multidimensional structure of the problem, a traditional *guess-and-verify approach* as seen for instance in Guo and Zervos [44] and Ferrari and Koch [34] is not effective. In fact, this would require the construction of an explicit solution to the second-order PDE with state dependent gradient constraint seen in (2.11) above, which is not feasible in general. Instead, we use a different approach and construct an optimal stopping problem connected to the stochastic control problem (2.9), which is then of a simpler structure. Before we do so and in order to get insights from a benchmark problem, we briefly discuss the problem under *full information*, i.e., where the drift coefficient is constant and equal to either μ_0 or μ_1 .

3 Benchmark problem under full information

Suppose that the initial subjective belief $\pi = \mathbb{P}[\mu = \mu_1]$ is such that $\pi \in \{0, 1\}$. Observe that there exists no uncertainty in the model other than the Brownian one and

the belief process Π will remain constant, as the investor is already certain at the initial time about the true value of the drift. Hence in this formulation, we are in the case of *full information*. The problem we address in this section has a similar structure to the ones studied by Guo and Zervos [44] as well as Koch [52, Chap. 2] and we therefore do not provide full details. Let us assume $\pi = 0$. We thus obtain $\Pi_t = 0$ for all $t \geq 0$, and the dynamics of X^ξ and Y^ξ then write as

$$\underline{X}_t^\xi = x + \mu_0 t + \sigma W_t - \alpha \xi_t, \quad Y_t^\xi = y - \xi_t. \tag{3.1}$$

We denote the corresponding value function as

$$V_0(x, y) := \sup_{\xi \in \mathcal{A}(y)} \mathbb{E}_{x,y} \left[\int_0^\infty e^{-rt} (e^{\underline{X}_t^\xi} - \kappa) \circ d\xi_t \right], \quad (x, y) \in \mathbb{R} \times (0, \infty), \tag{3.2}$$

where $\mathbb{E}_{x,y}[\cdot] = \mathbb{E}[\cdot | \underline{X}_{0-}^\xi = x, Y_{0-}^\xi = y]$. By similar arguments as in the case of incomplete information, we expect that V_0 should identify with an appropriate solution to the HJB equation

$$\max\{(\mathcal{L}_\underline{X} - r)w, -\alpha w_x - w_y + e^x - \kappa\} = 0 \quad \text{with } \mathcal{L}_\underline{X} = \frac{1}{2}\sigma^2 \partial_{xx} + \mu_0 \partial_x \tag{3.3}$$

and $w(x, 0) = 0$. Defining the associated waiting and selling regions as

$$\mathbb{W}^{\mu_0} := \{(x, y) \in \mathbb{R} \times [0, \infty) : (\mathcal{L}_\underline{X} - r)w = 0, -\alpha w_x - w_y + e^x - \kappa < 0\}, \tag{3.4}$$

$$\mathbb{S}^{\mu_0} := \{(x, y) \in \mathbb{R} \times [0, \infty) : (\mathcal{L}_\underline{X} - r)w \leq 0, -\alpha w_x - w_y + e^x - \kappa = 0\}, \tag{3.5}$$

we expect that the investor is only willing to sell shares of the asset when its price is sufficiently large. Hence we guess that for every $y \geq 0$, there exists a critical price $G(y)$ such that (3.4) and (3.5) rewrite as

$$\begin{aligned} \mathbb{W}^{\mu_0} &= \{(x, y) \in \mathbb{R} \times [0, \infty) : y > 0 \text{ and } x < G(y)\} \cup (\mathbb{R} \times \{0\}), \\ \mathbb{S}^{\mu_0} &= \{(x, y) \in \mathbb{R} \times [0, \infty) : y > 0 \text{ and } x \geq G(y)\}. \end{aligned}$$

Notice that the candidate value function should then satisfy $(\mathcal{L}_\underline{X} - r)w = 0$ on \mathbb{W}^{μ_0} . It is well known that the latter equation admits two fundamental strictly positive solutions; the only solution that remains bounded as $x \downarrow -\infty$ is then given by

$$w(x, y) = A(y)e^{nx}$$

for some functions $A : [0, \infty) \rightarrow \mathbb{R}$ and where n is the positive solution to the quadratic equation $(\sigma^2/2)n^2 + \mu_0 n - r = 0$. On the other hand, on \mathbb{S}^{μ_0} , we expect that the value function V_0 should instead satisfy

$$-\alpha w_x - w_y + e^x - \kappa = 0 \quad \text{and thus } -\alpha w_{xx} - w_{yx} + e^x = 0.$$

In order to derive the solutions for $A(y)$ and $G(y)$, we evaluate the two previous formulas at $x = G(y)$, require that $A(0) = 0$ and obtain

$$G(y) = \ln \frac{\kappa n}{n-1} =: x_0^*, \quad A(y) = \frac{\kappa}{\alpha n(n-1)} \left(\frac{\kappa n}{n-1} \right)^{-n} (1 - e^{-\alpha n y}). \tag{3.6}$$

Notice that the optimal execution threshold – determining the price at which the investor should sell – is independent of the current amount of assets in the portfolio. Moreover, the selling region is partitioned into

$$\begin{aligned} \mathbb{S}_1^{\mu_0} &:= \left\{ (x, y) \in \mathbb{R} \times (0, \infty) : x \geq x_0^*, y \leq \frac{x - x_0^*}{\alpha} \right\}, \\ \mathbb{S}_2^{\mu_0} &:= \left\{ (x, y) \in \mathbb{R} \times (0, \infty) : x \geq x_0^*, y > \frac{x - x_0^*}{\alpha} \right\}, \end{aligned}$$

and we expect that in $\mathbb{S}_1^{\mu_0}$, it is optimal to sell the complete amount of assets instantaneously, while in $\mathbb{S}_2^{\mu_0}$, the investor makes a lump-sum execution and then follows the strategy that keeps the process (X, Y) inside $\overline{\mathbb{W}}^{\mu_0}$ until all assets are sold. The candidate value function, according to our previous considerations, then takes the shape

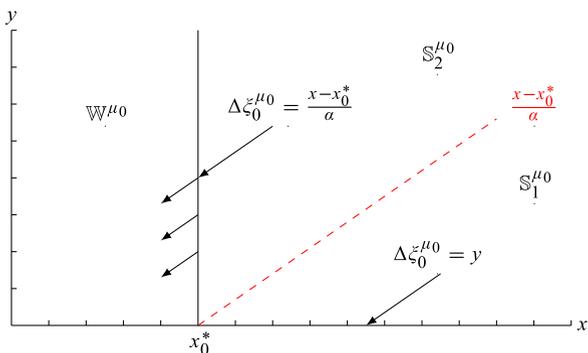
$$w(x, y) = \begin{cases} A(y)e^{nx}, & (x, y) \in \overline{\mathbb{W}}^{\mu_0}, \\ A(y - \frac{x-x_0^*}{\alpha})e^{nx_0^*} + \frac{1}{\alpha}(e^x - e^{x_0^*}) - \frac{\kappa}{\alpha}(x - x_0^*), & (x, y) \in \mathbb{S}_2^{\mu_0}, \\ \frac{1}{\alpha}e^x(1 - e^{-\alpha y}) - \kappa y, & (x, y) \in \mathbb{S}_1^{\mu_0}, \end{cases} \quad (3.7)$$

and via a verification theorem (cf. Guo and Zervos [44, Prop. 5.1], Koch [52, Prop. 2.4.1]), one can indeed show that this w is a $C^{2,1}$ solution to the HJB equation (3.3) and coincides with the value function V_0 of (3.2). Moreover, the process

$$\xi_t^{\mu_0} := y \wedge \sup_{0 \leq s \leq t} \frac{1}{\alpha}(x - x_0^* + \mu_0 s + \sigma W_s)^+, \quad t \geq 0, \xi_{0-}^{\mu_0} = 0, \quad (3.8)$$

belongs to $\mathcal{A}(y)$ and provides an optimal execution strategy for (3.2) (cf. Guo and Zervos [44, Prop. 5.1]; recall that here we are not assuming $\lim_{T \uparrow \infty} Y_T^\xi = 0$ as admissibility condition, see also Remark 7.6 below). Figure 1 sketches the optimal execution strategy (3.8) for problem (3.2) under full information. We observe that for an initial price x strictly larger than x_0^* , the investor immediately does a lump-sum execution. The latter can already deplete the whole portfolio, whenever $y \leq \frac{1}{\alpha}(x - x_0^*)$, or bring it to the level $(x_0^*, y - \frac{1}{\alpha}(x - x_0^*))$ otherwise. Afterwards, the optimal strategy prescribes to keep the state process (X, Y) inside the waiting region $\overline{\mathbb{W}}^{\mu_0}$ with

Fig. 1 Illustrative drawing of the optimal execution strategy (3.8) under full information



minimal effort, by reflecting it in the direction $(-\alpha, -1)$ according to a *Skorokhod reflection-type* policy (realised through the running supremum in (3.8)).

In light of our subsequent analysis, it is interesting to notice that the derivative $\alpha \partial_x V_0 + \partial_y V_0$ can be checked from (3.7) to identify with the value function of an optimal stopping problem. More precisely, for any $x \in \mathbb{R}$, one has

$$\alpha \partial_x V_0(x, y) + \partial_y V_0(x, y) =: v_0(x) = \sup_{\tau \geq 0} \mathbb{E}_x [e^{-r\tau} (e^{\underline{X}_\tau^0} - \kappa)], \tag{3.9}$$

where \underline{X}^0 denotes the solution to (3.1) with $\xi \equiv 0$, the optimisation is performed over all stopping times of the Brownian filtration and \mathbb{E}_x is the expectation under $\mathbb{P}_x[\cdot] = \mathbb{P}[\cdot | \underline{X}_0^0 = x]$. Moreover, the stopping time

$$\tau_0^*(x) := \inf\{t \geq 0 : \underline{X}_t^0 \geq x_0^*\} \quad \mathbb{P}_x\text{-a.s.}, x \in \mathbb{R}, \tag{3.10}$$

is optimal for (3.9). We can interpret (3.10) as the optimal time at which the investor should sell another unit of shares and notice that it in fact characterises the time at which the marginal expected profit $\alpha \partial_x V_0 + \partial_y V_0$ coincides with the marginal instantaneous net profit $e^x - \kappa$ from selling.

Remark 3.1 It is easily checked that the results we obtained for the case $\mu \equiv \mu_0$ can be replicated for the case $\mu \equiv \mu_1$. More precisely, considering the dynamics

$$\overline{X}_t^\xi = x + \mu_1 t + \sigma W_t - \alpha \xi_t, \quad t \geq 0, \tag{3.11}$$

and the value function

$$V_1(x, y) := \sup_{\xi \in \mathcal{A}(y)} \left[\int_0^\infty e^{-rt} (e^{\overline{X}_t^\xi} - \kappa) \circ d\xi_t \right], \quad (x, y) \in \mathbb{R} \times (0, \infty),$$

we can verify the existence of an optimal execution threshold x_1^* which triggers the selling strategy of the investor through the optimal control ξ^{μ_1} , which is of similar structure as (3.8) with μ_0 replaced by μ_1 . Furthermore, we have

$$\alpha \partial_x V_1 + \partial_y V_1 =: v_1(x) = \sup_{t \geq 0} \mathbb{E}_x [e^{-rt} (e^{\overline{X}_t^0} - \kappa)], \tag{3.12}$$

where \overline{X}^0 is the solution to (3.11) with $\xi \equiv 0$, and $\tau_1^*(x) := \inf\{t \geq 0 : \overline{X}_t^0 \geq x_1^*\}$ \mathbb{P}_x -a.s. is the optimal stopping time for problem (3.12).

4 A related optimal stopping problem

Motivated by the observed connection to an optimal stopping problem in the benchmark problem of Sect. 3 (see (3.12)), we pursue the following approach in the subsequent analysis: (i) We introduce and study an optimal stopping problem, with value v , that we expect to be associated to the singular stochastic control problem

(2.9); (ii) We provide a complete analysis of the optimal stopping problem which is achieved by studying two equivalent formulations of it (cf. Sects. 5 and 6). More precisely, we derive regularity results for the value function (cf. Proposition 6.9) as well as an integral equation for the free boundary (cf. Proposition 6.11); (iii) We verify the expected connection to the original problem of (2.9) by showing that (cf. Theorem 7.3)

$$V(x, y, \pi) = \frac{1}{\alpha} \int_{x-\alpha y}^x v(x', \pi) dx', \quad (x, y, \pi) \in \mathbb{R} \times (0, \infty) \times (0, 1),$$

and that the optimal execution strategy is triggered by the optimal stopping boundary studied in step (ii). In fact, as in the benchmark case, we can interpret the optimal stopping problem as the *marginal problem* in the sense that its value v coincides with the derivative of the value V of (2.9) in the direction of actions/execution and its optimal stopping strategy characterises the time at which it is optimal to sell a unit of assets.

We recall that $(X_t^0, \Pi_t)_{t \geq 0}$ is the two-dimensional strong Markov process solving

$$\begin{cases} dX_t^0 = (\mu_1 \Pi_t + \mu_0(1 - \Pi_t))dt + \sigma d\bar{W}_t, & X_0^0 = x, \\ d\Pi_t = \gamma \Pi_t(1 - \Pi_t)d\bar{W}_t, & \Pi_0 = \pi, \end{cases} \quad (4.1)$$

and **in the following**, in order to simplify notation, we write X instead of X^0 . For a stopping time τ of the filtration \mathbb{F}^X , we then define

$$\Psi(x, \pi, \tau) := \mathbb{E}_{x, \pi}[e^{-r\tau}(e^{X_\tau} - \kappa)], \quad (x, \pi) \in \mathbb{R} \times (0, 1),$$

and consider the optimal stopping problem

$$v(x, \pi) := \sup_{\tau} \Psi(x, \pi, \tau). \quad (4.2)$$

Above and in the following, $\mathbb{E}_{x, \pi}[\cdot] = \mathbb{E}[\cdot | X_0 = x, \Pi_0 = \pi]$. Also, denoting by $(X_t^{x, \pi}, \Pi_t^\pi)_{t \geq 0}$ the unique strong solution to (4.1), we often employ the equivalent notation $\mathbb{E}[f(X_t^{x, \pi}, \Pi_t^\pi)] = \mathbb{E}_{x, \pi}[f(X_t, \Pi_t)]$ for any integrable measurable function $f : \mathbb{R} \times [0, 1] \rightarrow \mathbb{R}$.

We make the next **standing assumption**.

Assumption 4.1 We assume that

$$r > \left(\mu_1 + \frac{1}{2}\sigma^2\right) \vee \left(\mu_1 + \frac{1}{2}\sigma^2 + \frac{(2\mu_1 + \sigma^2)(\mu_1 - \mu_0)}{\sigma^2}\right) \vee \left(\frac{\gamma}{2\sigma}|\mu_0 + \mu_1|\right).$$

Remark 4.2 (i) The different conditions we impose on the (subjective) discount factor r serve distinct purposes. Notice that the first condition is equivalent to imposing $r > \beta_1$ and guarantees wellposedness of problem (4.2).

(ii) Moreover, the forthcoming analysis (in particular Sect. 6) reveals that the other two terms are sufficient to ensure monotonicity of (a transformation of) the optimal

stopping boundary of problem (4.2) (cf. Propositions 6.3 and 6.5). This result is crucial when deriving the smooth-fit property and thus, by relying on arguments developed in De Angelis and Peskir [22], the global C^1 -regularity of (a transformation of) the value function v of (4.2). When r does not satisfy Assumption 4.1, the monotonicity of the (transformed version of the) boundary is not clear and one needs an alternative route to achieve the needed regularity of \widehat{v} . A possible approach could be to prove directly the (local) Lipschitz-regularity of the free boundary (cf. De Angelis and Stabile [23]) and then infer the C^1 -property of \widehat{v} from the continuity of the optimal stopping time with respect to the initial data. Since this is not straightforward to obtain in our formulation, we leave it for future research.

In the following, we derive some preliminary results for the optimal stopping problem (4.2) and its associated free boundary. Noticing that $(x, \pi) \mapsto X_t^{x, \pi}$ as well as $\pi \mapsto \Pi_t^\pi$ are continuous and nondecreasing due to classical comparison theorems for strong solutions to stochastic differential equations, the proof of the following lemma follows from standard arguments and is therefore skipped.

Lemma 4.3 *The value function v of (4.2) is such that*

- i) $x \mapsto v(x, \pi)$ is nondecreasing;
- ii) $\pi \mapsto v(x, \pi)$ is nondecreasing.

Furthermore, as $(x, \pi) \mapsto (X_t^{x, \pi}, \Pi_t^\pi)$ is continuous \mathbb{P} -a.s., Assumption 4.1 and standard estimates using the fact that Π is bounded allow us to invoke dominated convergence and obtain that

$$(x, \pi) \mapsto \mathbb{E}[e^{-r\tau}(e^{X_\tau^{x, \pi}} - \kappa)]$$

is continuous, and hence $(x, \pi) \mapsto v(x, \pi)$ is lower semicontinuous. As it is customary in optimal stopping theory, we introduce the continuation and stopping regions associated to v as

$$\mathcal{C}_1 := \{(x, \pi) \in \mathbb{R} \times (0, 1) : v(x, \pi) > e^x - \kappa\}, \quad (4.3)$$

$$\mathcal{S}_1 := \{(x, \pi) \in \mathbb{R} \times (0, 1) : v(x, \pi) = e^x - \kappa\}. \quad (4.4)$$

Then the continuation region \mathcal{C}_1 is an open set while the stopping region \mathcal{S}_1 in (4.4) is closed, and by Peskir and Shiryaev [57, Corollary 1.2.2.9], the stopping time

$$\tau^* = \tau^*(x, \pi) := \inf\{t \geq 0 : (X_t^{x, \pi}, \Pi_t^\pi) \in \mathcal{S}_1\}$$

is optimal whenever it is \mathbb{P} -a.s. finite; otherwise it is an optimal Markov time. We set

$$a(\pi) := \inf\{x \in \mathbb{R} : v(x, \pi) \leq e^x - \kappa\}, \quad (4.5)$$

with the convention $\inf \emptyset = \infty$, and state the following lemma.

Lemma 4.4 *It holds that*

$$\mathcal{C}_1 = \{(x, \pi) \in \mathbb{R} \times (0, 1) : x < a(\pi)\},$$

$$\mathcal{S}_1 = \{(x, \pi) \in \mathbb{R} \times (0, 1) : x \geq a(\pi)\}.$$

Proof Recalling (2.10), an application of Dynkin’s formula yields

$$\begin{aligned}
 u(x, \pi) &:= v(x, \pi) - (e^x - \kappa) \\
 &= \sup_{\tau} \mathbb{E}_{x, \pi} \left[\int_0^{\tau} e^{-rt} \left(e^{X_t} \left(\mu_1 \Pi_t + (1 - \Pi_t) \mu_0 + \frac{1}{2} \sigma^2 - r \right) + r\kappa \right) dt \right].
 \end{aligned}$$

For $x_1 < x_2$ and τ^* optimal for $v(x_2, \pi)$, we have by Assumption 4.1 that

$$\begin{aligned}
 u(x_1, \pi) - u(x_2, \pi) &\geq \mathbb{E} \left[\int_0^{\tau^*} e^{-rt} (e^{X_t^{x_1, \pi}} - e^{X_t^{x_2, \pi}}) \right. \\
 &\quad \times \left. \left(\mu_1 \Pi_t^{\pi} + (1 - \Pi_t^{\pi}) \mu_0 + \frac{1}{2} \sigma^2 - r \right) dt \right] \\
 &\geq \mathbb{E} \left[\int_0^{\tau^*} e^{-rt} (e^{X_t^{x_2, \pi}} - e^{X_t^{x_1, \pi}}) \left(r - \mu_1 - \frac{1}{2} \sigma^2 \right) dt \right] \geq 0.
 \end{aligned}$$

For $(x_1, \pi) \in \mathcal{S}_1$ and $x_2 > x_1$, we thus obtain $0 \leq u(x_2, \pi) \leq u(x_1, \pi) = 0$ so that $(x_2, \pi) \in \mathcal{S}_1$. □

The free boundary $a(\pi)$ thus splits $\mathbb{R} \times (0, 1)$ into the continuation and stopping region. In the following lemma, we derive some preliminary properties.

Lemma 4.5 *One has the following properties:*

- i) $\pi \mapsto a(\pi)$ is nondecreasing on $(0, 1)$.
- ii) $\pi \mapsto a(\pi)$ is left-continuous on $(0, 1)$.
- iii) There exist constants such that $x_0^* \leq a(\pi) \leq x_1^*$ for all $\pi \in (0, 1)$.

Proof i) If $\pi_2 > \pi_1$ and $(x, \pi_2) \in \mathcal{S}_1$, we have $x \geq a(\pi_2)$ and $v(x, \pi_2) = e^x - \kappa$. Since $\pi \mapsto v(x, \pi)$ is nondecreasing, $v(x, \pi_1) \leq v(x, \pi_2) = e^x - \kappa$, which together with $v(x, \pi_1) \geq (e^x - \kappa)$ gives $(x, \pi_1) \in \mathcal{S}_1$. Therefore $a(\pi_2) \geq a(\pi_1)$.

ii) For a sequence $(\pi_n)_{n \in \mathbb{N}}$ with $\pi_n \uparrow \pi$, the sequence $(a(\pi_n))$ is increasing by i) and $a(\pi_n) \leq a(\pi)$. So $\lim_{n \rightarrow \infty} a(\pi_n) =: a(\pi^-)$ exists and $a(\pi^-) \leq a(\pi)$. Because $v(a(\pi_n), \pi_n) = e^{a(\pi_n)} - \kappa$ for all $n \in \mathbb{N}$, lower semicontinuity of $(x, \pi) \mapsto v(x, \pi)$ yields $v(a(\pi^-), \pi) = e^{a(\pi^-)} - \kappa$. Hence $a(\pi) \leq a(\pi^-)$ and $\lim_{n \rightarrow \infty} a(\pi_n) = a(\pi)$.

iii) Recall from (3.9) and (3.12) the value functions v_0 and v_1 in the optimal stopping problems with full information when either $\mu \equiv \mu_0$ or $\mu \equiv \mu_1$. The associated continuation regions are given by

$$\begin{aligned}
 \{x \in \mathbb{R} : x \geq x_1^*\} &= \{x \in \mathbb{R} : v_1(x) \leq e^x - \kappa\}, \\
 \{x \in \mathbb{R} : x \geq x_0^*\} &= \{x \in \mathbb{R} : v_0(x) \leq e^x - \kappa\},
 \end{aligned}$$

where x_0^* and x_1^* are the optimal execution thresholds (cf. (3.6) and Remark 3.1). Recalling $\mu_0 < \mu_1$ and $\Pi_t \in (0, 1)$ for $\pi \in (0, 1)$, we have $\underline{X}_t^0 \leq X_t \leq \overline{X}_t^0$ \mathbb{P} -a.s. for any $t \geq 0$ by classical comparison arguments, where \underline{X}^0 and \overline{X}^0 denote the solutions

to (3.1) and (3.11) with $\xi \equiv 0$. Thus $v_0(x) \leq v(\pi, x) \leq v_1(x)$, which implies

$$\begin{aligned} \{x \in \mathbb{R} : v_1(x) \leq e^x - \kappa\} &\subseteq \{(x, \pi) \in \mathbb{R} \times (0, 1) : v(x, \pi) \leq e^x - \kappa\} \\ &\subseteq \{x \in \mathbb{R} : v_0(x) \leq e^x - \kappa\}, \end{aligned}$$

and the latter combined with (4.5) allows concluding that $x_0^* \leq a(\pi) \leq x_1^*$. □

5 Decoupling change of measure and a new optimal selling problem

We notice that the underlying dynamics in (4.1) are coupled. In order to derive further results about the properties of the optimal stopping problem (4.2) and its associated free boundary, it is useful to address the problem under a different probability measure. With reference to related contributions (cf. De Angelis [19], Ekström and Lu [30], Johnson and Peskir [45] and Shiryaev [62] and references therein), we introduce the so-called *likelihood ratio process* via

$$\Phi_t := \frac{\Pi_t}{1 - \Pi_t}, \quad t \geq 0.$$

Through an application of Itô’s formula, we can derive its associated dynamics given by

$$d\Phi_t = \gamma \Phi_t (\gamma \Pi_t dt + d\bar{W}_t), \quad \Phi_0 = \varphi := \frac{\pi}{1 - \pi},$$

and we aim to remove its dependence on the process Π through a change of measure. For a fixed $T > 0$, we define the measure $\mathbb{Q}_T \approx \mathbb{P}$ on (Ω, \mathcal{F}_T) via the Radon–Nikodým derivative

$$\eta_T := \frac{d\mathbb{Q}_T}{d\mathbb{P}} := \exp\left(-\int_0^T \gamma \Pi_s d\bar{W}_s - \frac{1}{2} \int_0^T \gamma^2 \Pi_s^2 ds\right) \tag{5.1}$$

and notice that the process

$$dB_t = d\bar{W}_t + \gamma \Pi_t dt$$

is a Brownian motion under \mathbb{Q}_T on $[0, T]$. Rewriting the state process (X, Φ) under \mathbb{Q}_T then yields

$$\begin{cases} dX_t = \mu_0 dt + \sigma dB_t, & t \in (0, T], X_0 = x, \\ d\Phi_t = \gamma \Phi_t dB_t, & t \in (0, T], \Phi_0 = \varphi, \end{cases} \tag{5.2}$$

and we notice that the processes decouple under this formulation. In the following, when needed, we write $\mathbb{E}_{x, \varphi}^{\mathbb{Q}_T}$ to denote the expectation under \mathbb{Q}_T conditionally on $X_0 = x, \Phi_0 = \varphi$. To rewrite (4.2) in terms of the new variables (X, Φ) , we introduce

$$\Theta_t := \frac{1 + \Phi_t}{1 + \varphi}, \quad t \in [0, T],$$

and by an application of Itô’s formula, it can be verified that Θ admits the representation

$$\Theta_t = \exp \left(\int_0^t \gamma \Pi_s d\overline{W}_s + \frac{1}{2} \int_0^t \gamma^2 \Pi_s^2 ds \right) = \frac{1}{\eta_t}, \quad t \in [0, T]. \tag{5.3}$$

Upon using (5.1) and (5.3), we find that

$$\begin{aligned} & \mathbb{E}_{x,\pi} [e^{-r(\tau \wedge T)} (e^{X_{\tau \wedge T}} - \kappa)] \\ &= \mathbb{E}_{x,\pi} [e^{-r(\tau \wedge T)} (e^{X_{\tau \wedge T}} - \kappa) \eta_{\tau \wedge T} \Theta_{\tau \wedge T}] \\ &= \mathbb{E}_{x,\varphi}^{\mathbb{Q}_T} \left[e^{-r(\tau \wedge T)} (e^{X_{\tau \wedge T}} - \kappa) \frac{1 + \Phi_{\tau \wedge T}}{1 + \varphi} \right] \\ &= (1 + \varphi)^{-1} \mathbb{E}_{x,\varphi}^{\mathbb{Q}_T} [e^{-r(\tau \wedge T)} (e^{X_{\tau \wedge T}} - \kappa) (1 + \Phi_{\tau \wedge T})] \end{aligned} \tag{5.4}$$

for any stopping time τ and $(x, \varphi) \in \mathbb{R} \times (0, \infty)$. With regard to (5.4), we introduce the stopping problems

$$\begin{aligned} v(x, \pi; T) &:= \sup_{\tau} \mathbb{E}_{x,\pi} [e^{-r(\tau \wedge T)} (e^{X_{\tau \wedge T}} - \kappa)], \\ v^{\mathbb{Q}_T}(x, \varphi; T) &:= \sup_{\tau} \mathbb{E}_{x,\varphi}^{\mathbb{Q}_T} [e^{-r(\tau \wedge T)} (e^{X_{\tau \wedge T}} - \kappa) (1 + \Phi_{\tau \wedge T})], \end{aligned}$$

and notice that (5.4) implies $v^{\mathbb{Q}_T}(x, \varphi; T) = (1 + \varphi)v(x, \varphi/(1 + \varphi); T)$ for fixed $T > 0$. However, since the measure \mathbb{Q}_T changes with T , passing to the limit $T \rightarrow \infty$ in (5.4) requires a bit of care. To this end, we define a probability space $(\tilde{\Omega}, \tilde{\mathbb{F}}, \tilde{\mathbb{Q}})$ with a Brownian motion \tilde{B} and a filtration $\tilde{\mathbb{F}} = (\tilde{\mathcal{F}}_t)_{t \geq 0}$. Moreover, we let $(\tilde{X}, \tilde{\Phi})$ be the strong solution to the stochastic differential equation (5.2) driven by the Brownian motion \tilde{B} instead of B . Let $\tilde{\mathbb{E}}_{x,\varphi}[\cdot]$ denote the expectation under $\tilde{\mathbb{Q}}$ and define the stopping problems

$$\begin{aligned} \bar{v}(x, \varphi; T) &:= \sup_{\tau} \tilde{\mathbb{E}}_{x,\varphi} [e^{-r(\tau \wedge T)} (e^{\tilde{X}_{\tau \wedge T}} - \kappa) (1 + \tilde{\Phi}_{\tau \wedge T})], \\ \bar{v}(x, \varphi) &:= \sup_{\tau} \tilde{\mathbb{E}}_{x,\varphi} [e^{-r\tau} (e^{\tilde{X}_{\tau}} - \kappa) (1 + \tilde{\Phi}_{\tau})]. \end{aligned}$$

Due to the equivalence in law of the process $(\tilde{X}_t, \tilde{\Phi}_t, \tilde{B}_t)_{t \geq 0}$ under $\tilde{\mathbb{Q}}$ and the process $(X_t, \Phi_t, B_t)_{t \geq 0}$ under \mathbb{Q}_T , both on $[0, T]$, we have $v^{\mathbb{Q}_T}(x, \varphi; T) = \bar{v}(x, \varphi; T)$. Moreover, using Fatou’s lemma and simple comparison arguments, one can show that

$$\lim_{T \rightarrow \infty} v(x, \pi; T) = v(x, \pi), \quad \lim_{T \rightarrow \infty} \bar{v}(x, \varphi; T) = \bar{v}(x, \varphi).$$

Hence we finally obtain

$$\begin{aligned} \bar{v}(x, \varphi) &= \lim_{T \rightarrow \infty} \bar{v}(x, \varphi; T) = \lim_{T \rightarrow \infty} v^{\mathbb{Q}_T}(x, \varphi; T) \\ &= (1 + \varphi) \lim_{T \rightarrow \infty} v(x, \varphi/(1 + \varphi); T) = (1 + \varphi)v(x, \varphi/(1 + \varphi)). \end{aligned} \tag{5.5}$$

For the sake of clarity – and with a slight abuse of notation –, we simply write from now on $(\Omega, \mathbb{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{Q}, \mathbb{E}^{\mathbb{Q}}, X, \Phi, B)$ instead of $(\tilde{\Omega}, \tilde{\mathbb{F}}, (\tilde{\mathcal{F}}_t)_{t \geq 0}, \tilde{\mathbb{Q}}, \tilde{\mathbb{E}}, \tilde{X}, \tilde{\Phi}, \tilde{B})$. Henceforth, we thus study the optimal stopping problem

$$\bar{v}(x, \varphi) = \sup_{\tau} \mathbb{E}_{x, \varphi}^{\mathbb{Q}} [e^{-r\tau} (e^{X_{\tau}} - \kappa)(1 + \Phi_{\tau})]. \tag{5.6}$$

In the sequel, we often write $\mathbb{E}_{x, \varphi}^{\mathbb{Q}} [f(X_t, \Phi_t)] = \mathbb{E}^{\mathbb{Q}} [f(X_t^x, \Phi_t^{\varphi})]$, where $(X_t^x, \Phi_t^{\varphi})_{t \geq 0}$ is the unique strong solution to (5.2). The continuation and stopping regions associated to this problem are then given by

$$C_2 := \{(x, \varphi) \in \mathbb{R} \times (0, \infty) : \bar{v}(x, \varphi) > (e^x - \kappa)(1 + \varphi)\}, \tag{5.7}$$

$$S_2 := \{(x, \varphi) \in \mathbb{R} \times (0, \infty) : \bar{v}(x, \varphi) = (e^x - \kappa)(1 + \varphi)\}. \tag{5.8}$$

Given the lower semicontinuity of v and (5.5), we find that $(x, \varphi) \mapsto \bar{v}(x, \varphi)$ is lower semicontinuous as well. Hence the stopping region S_2 of (5.8) is a closed set, while the continuation region C_2 of (5.7) is open. Also,

$$\tau^* := \tau^*(x, \varphi) := \inf\{t \geq 0 : (X_t^x, \Phi_t^{\varphi}) \in S_2\}$$

is optimal by Peskir and Shiryaev [57, Chap. 1] whenever it is \mathbb{Q} -a.s. finite. Furthermore, we define

$$b(\varphi) := \inf\{x \in \mathbb{R} : \bar{v}(x, \varphi) \leq (e^x - \kappa)(1 + \varphi)\}, \tag{5.9}$$

with $\inf \emptyset = \infty$. In the following lemma, we derive some preliminary properties of the value function (5.6). In light of the relation (5.5), we notice that some of the following results are a direct consequence of Lemma 4.3.

Lemma 5.1 *The value function \bar{v} of (5.6) has the following properties:*

- i) $0 \leq \bar{v}(x, \varphi) \leq K_1 e^x (1 + \varphi)$ for all $(x, \varphi) \in \mathbb{R} \times (0, \infty)$ and some $K_1 > 0$.
- ii) $x \mapsto \bar{v}(x, \varphi)$ is nondecreasing.
- iii) $\varphi \mapsto \bar{v}(x, \varphi)$ is nondecreasing.
- iv) $(x, \varphi) \mapsto \bar{v}(x, \varphi)$ is locally Lipschitz over $\mathbb{R} \times (0, \infty)$.
- v) $\varphi \mapsto \bar{v}(x, \varphi)$ and $x \mapsto \bar{v}(x, \varphi)$ are convex.

Proof ii) follows from Lemma 4.3 i) upon using (5.5).

i) For the lower bound, note that $\{(x, \varphi) \in \mathbb{R} \times (0, \infty) : e^x - \kappa < 0\} \subseteq C_2$. Hence, since $\Phi^{\varphi} \geq 0$ a.s., we have $\bar{v}(x, \varphi) \geq 0$ for all $(x, \varphi) \in \mathbb{R} \times (0, \infty)$. For the upper bound, we observe that for any stopping time τ , setting $\pi = \varphi / (1 + \varphi)$ gives

$$\begin{aligned} \mathbb{E}_{x, \varphi}^{\mathbb{Q}} [e^{-r\tau} (e^{X_{\tau}} - \kappa)(1 + \Phi_{\tau})] &= (1 + \varphi) \mathbb{E}_{x, \pi} [e^{-r\tau} (e^{X_{\tau}} - \kappa)] \\ &\leq (1 + \varphi) \mathbb{E} [e^{-r\tau} e^{x + \mu_1 \tau + \sigma W_{\tau}}] \leq K_1 e^x (1 + \varphi), \end{aligned}$$

where the last inequality follows from standard estimates upon using Assumption 4.1.

iii) Let $\varphi, \varphi' \in (0, \infty)$ with $\varphi' > \varphi$ and notice that $\Phi_t^\varphi = \varphi e^{-\frac{1}{2}\gamma^2 t + \gamma B_t}$. For $x \in \mathbb{R}$ and $\tau^* := \tau^*(x, \varphi)$ optimal for $\bar{v}(x, \varphi)$, we have

$$\begin{aligned} \bar{v}(x, \varphi') - \bar{v}(x, \varphi) &\geq \mathbb{E}_{x, \varphi'}^{\mathbb{Q}}[e^{-r\tau^*} (e^{X_{\tau^*}} - \kappa)(1 + \Phi_{\tau^*})] \\ &\quad - \mathbb{E}_{x, \varphi}^{\mathbb{Q}}[e^{-r\tau^*} (e^{X_{\tau^*}} - \kappa)(1 + \Phi_{\tau^*})] \\ &= \mathbb{E}^{\mathbb{Q}}[e^{-r\tau^*} (e^{X_{\tau^*}^x} - \kappa)(\varphi' - \varphi)e^{-\frac{1}{2}\gamma^2 \tau^* + \gamma B_{\tau^*}}] \geq 0, \end{aligned}$$

where the last inequality exploits that $\{(x, \varphi) \in \mathbb{R} \times (0, \infty) : e^x - \kappa < 0\} \subseteq \mathcal{C}_2$.

iv) Let $x, x' \in \mathbb{R}$, $\pi \in (0, 1)$ and $\varphi, \varphi' \in (0, \infty)$. Recall v from (4.2). Again, standard estimates yield

$$\begin{aligned} |v(x, \pi) - v(x', \pi)| &\leq K_1 |e^x - e^{x'}|, \\ |\bar{v}(x, \varphi) - \bar{v}(x, \varphi')| &\leq K_2 e^x |\varphi - \varphi'| \end{aligned}$$

for some $K_1, K_2 > 0$. Hence using (5.5), we obtain

$$\begin{aligned} |\bar{v}(x, \varphi) - \bar{v}(x', \varphi')| &\leq |\bar{v}(x, \varphi) - \bar{v}(x', \varphi)| + |\bar{v}(x', \varphi) - \bar{v}(x', \varphi')| \\ &\leq K_1(1 + \varphi) |e^x - e^{x'}| + K_2 e^{x'} |\varphi - \varphi'|, \end{aligned} \tag{5.10}$$

and the local Lipschitz property follows.

v) We first prove convexity with respect to $\varphi \in (0, \infty)$. For $\varphi_1, \varphi_2 \in (0, \infty)$, $x \in \mathbb{R}$ and $\lambda \in (0, 1)$, we set $\bar{\varphi} := \lambda\varphi_1 + (1 - \lambda)\varphi_2$ and obtain

$$\begin{aligned} \bar{v}(x, \bar{\varphi}) &= \sup_{\tau} \mathbb{E}_{x, \bar{\varphi}}^{\mathbb{Q}}[e^{-r\tau} (e^{X_{\tau}} - \kappa)(1 + \bar{\varphi} e^{-\frac{1}{2}\gamma^2 \tau + \gamma B_{\tau}})] \\ &\leq \sup_{\tau} \mathbb{E}^{\mathbb{Q}}[e^{-r\tau} (e^{X_{\tau}^x} - \kappa)\lambda(1 + \varphi_1 e^{-\frac{1}{2}\gamma^2 \tau + \gamma B_{\tau}})] \\ &\quad + \sup_{\tau} \mathbb{E}^{\mathbb{Q}}[e^{-r\tau} (e^{X_{\tau}^x} - \kappa)(1 - \lambda)(1 + \varphi_2 e^{-\frac{1}{2}\gamma^2 \tau + \gamma B_{\tau}})] \\ &= \lambda\bar{v}(x, \varphi_1) + (1 - \lambda)\bar{v}(x, \varphi_2). \end{aligned}$$

Analogously, upon exploiting the convexity of $x \mapsto e^x$, one can prove the convexity of $x \mapsto \bar{v}(x, \varphi)$. □

Lemma 5.2 *The continuation and stopping region regions in (5.7) and (5.8) satisfy*

$$\mathcal{C}_2 = \{(x, \varphi) \in \mathbb{R} \times (0, \infty) : x < b(\varphi)\}, \quad \mathcal{S}_2 = \{(x, \varphi) \in \mathbb{R} \times (0, \infty) : x \geq b(\varphi)\}.$$

Proof We proceed similarly to Lemma 4.4. We first notice that the second-order differential operator associated with the four-dimensional process (X, Φ) is given by

$$\mathcal{L}_{X, \Phi} f = \mu_0 \partial_x f + \frac{1}{2} \sigma^2 \partial_{xx} f + \frac{1}{2} \gamma^2 \varphi^2 \partial_{\varphi\varphi} f + \gamma \varphi \sigma \partial_{x\varphi} f \tag{5.11}$$

for $f \in C^2(\mathbb{R} \times (0, \infty))$, and we apply Dynkin’s formula to obtain

$$\begin{aligned} \bar{u}(x, \varphi) &:= \bar{v}(x, \varphi) - (e^x - \kappa)(1 + \varphi) \\ &= \sup_{\tau} \mathbb{E}_{x, \varphi}^{\mathbb{Q}} \left[\int_0^{\tau} e^{-rt} \left(e^{X_t} \left(\mu_0 + \frac{1}{2} \sigma^2 - r \right) + r\kappa \right. \right. \\ &\quad \left. \left. + \Phi_t \left(e^{X_t} \left(\mu_1 + \frac{1}{2} \sigma^2 - r \right) + r\kappa \right) \right) dt \right]. \end{aligned} \tag{5.12}$$

For $x_2 > x_1$ and $\tau^* := \tau^*(x_2, \varphi)$ optimal for $\bar{v}(x_2, \varphi)$, we have

$$\begin{aligned} \bar{u}(x_1, \varphi) - \bar{u}(x_2, \varphi) &\geq \mathbb{E}^{\mathbb{Q}} \left[\int_0^{\tau^*} e^{-rt} \left((e^{X_t^{x_2}} - e^{X_t^{x_1}}) \left(r - \mu_0 - \frac{1}{2} \sigma^2 \right) \right. \right. \\ &\quad \left. \left. + \Phi_t (e^{X_t^{x_2}} - e^{X_t^{x_1}}) \left(r - \mu_1 + \frac{1}{2} \sigma^2 \right) \right) dt \right] \geq 0, \end{aligned}$$

where the last inequality follows from $X^{x_2} \geq X^{x_1}$ \mathbb{Q} -a.s. and Assumption 4.1. Hence for $(x_1, \varphi) \in \mathcal{S}_2$ and $x_2 > x_1$, we obtain $0 \leq \bar{u}(x_2, \varphi) \leq \bar{u}(x_1, \varphi) = 0$ and the claim follows. □

It is interesting to notice that there exists a one-to-one correspondence between the continuation regions \mathcal{C}_1 and \mathcal{C}_2 of (4.3) and (5.7) as well as the stopping regions \mathcal{S}_1 and \mathcal{S}_2 of (4.4) and (5.8). Indeed, introducing the diffeomorphism

$$\begin{aligned} T &:= (T_1, T_2) : \mathbb{R} \times (0, 1) \rightarrow \mathbb{R} \times (0, \infty), \\ (T_1(x, \pi), T_2(x, \pi)) &:= \left(x, \frac{\pi}{1 - \pi} \right), \end{aligned} \tag{5.13}$$

with inverse

$$T^{-1}(x, \varphi) := \left(x, \frac{\varphi}{1 + \varphi} \right), \quad (x, \varphi) \in \mathbb{R} \times (0, \infty),$$

one has

$$\mathcal{C}_2 = T(\mathcal{C}_1) \quad \text{as well as} \quad \mathcal{S}_2 = T(\mathcal{S}_1).$$

Furthermore, upon using Lemma 4.4 and Lemma 5.2, we find that

$$b(\varphi) = a \left(\frac{\varphi}{1 + \varphi} \right). \tag{5.14}$$

Due to this explicit relationship between the optimal stopping boundaries, we obtain some first results on b thanks to Lemma 4.5.

Lemma 5.3 *The boundary $b(\varphi)$ of (5.9) has the following properties:*

- i) $\varphi \mapsto b(\varphi)$ is nondecreasing on $(0, \infty)$.
- ii) $\varphi \mapsto b(\varphi)$ is left-continuous.
- iii) b is bounded by $x_0^* \leq b(\varphi) \leq x_1^*$ for all $\varphi \in (0, \infty)$, with x_0^* and x_1^* as in Lemma 4.5.

The relationship (5.14) and the transformation (5.13) allow us to translate back our results from this section – as well as from the following section – to the initial optimal stopping problem (4.2). Moreover, (5.14) turns out to be valuable in the proof of Lemma 5.3, since proving the monotonicity result i) as well as the boundedness iii) is not straightforward without exploiting the relation between b and a and the results of Lemma 4.5.

6 A parabolic formulation

Observe that the dynamics of the processes X and Φ in (5.2) are driven by the same Brownian motion. In order to account for this degeneracy, we pass to yet another formulation of the optimal stopping problem. To this end, we rely on a transformation that reveals the true parabolic nature of the generator $\mathcal{L}_{X, \Phi}$ as in (5.11), i.e., that poses it in its canonical form (cf. Strauss [63, Chap. 1.6]). Define

$$\begin{aligned} \bar{T} &:= (\bar{T}_1, \bar{T}_2) : \mathbb{R} \times (0, \infty) \rightarrow \mathbb{R}^2, \\ (\bar{T}_1(x, \varphi), \bar{T}_2(x, \varphi)) &:= \left(x, \frac{\sigma}{\gamma} \ln \varphi - x \right), \end{aligned} \tag{6.1}$$

which is a diffeomorphism with inverse given by

$$\bar{T}^{-1}(x, z) := \left(x, e^{\frac{\gamma}{\sigma}(x+z)} \right), \quad (x, z) \in \mathbb{R}^2.$$

With regard to the transformation (6.1), we introduce the process

$$Z_t = \frac{\sigma}{\gamma} \ln \Phi_t - X_t, \quad t \geq 0, \tag{6.2}$$

and an application of Itô’s formula reveals that its dynamics are given by

$$dZ_t = -\frac{1}{2}(\mu_1 + \mu_0)dt, \quad Z_0 = z := \frac{\sigma}{\gamma} \ln \varphi - x. \tag{6.3}$$

Furthermore, we define the transformed version of the value function \bar{v} of (5.6) via

$$\widehat{v}(x, z) := \bar{v}(x, e^{\frac{\gamma}{\sigma}(x+z)}) = \sup_{\tau} \mathbb{E}_{x,z}^{\mathbb{Q}}[e^{-r\tau}(e^{X_\tau} - \kappa)(1 + e^{\frac{\gamma}{\sigma}(X_\tau + Z_\tau)})] \tag{6.4}$$

for $(x, z) \in \mathbb{R}^2$ and where now $\mathbb{E}_{x,z}^{\mathbb{Q}}[\cdot] = \mathbb{E}^{\mathbb{Q}}[\cdot | X_0 = x, Z_0 = z]$. In light of this explicit relationship between the value functions \bar{v} and \widehat{v} , we obtain the following result from Lemma 5.1.

Lemma 6.1 *The value function \widehat{v} of (6.4) is locally Lipschitz-continuous on \mathbb{R}^2 .*

The associated continuation and stopping regions are given by

$$C_3 := \{(x, z) \in \mathbb{R}^2 : \widehat{v}(x, z) > (e^x - \kappa)(1 + e^{\frac{z}{\sigma}(x+z)})\}, \tag{6.5}$$

$$S_3 := \{(x, z) \in \mathbb{R}^2 : \widehat{v}(x, z) = (e^x - \kappa)(1 + e^{\frac{z}{\sigma}(x+z)})\}, \tag{6.6}$$

where C_3 is open and S_3 is closed. Furthermore, the global diffeomorphism (6.1) implies that $C_3 = \overline{T(C_2)}$ as well as $S_3 = \overline{T(S_2)}$, with C_2 and S_2 as in (5.7) and (5.8). Notice that the second-order infinitesimal generator associated to the process (X, Z) is now such that

$$\mathcal{L}_{X,Z}f = \mu_0 \partial_x f + \frac{1}{2} \sigma^2 \partial_{xx} f - \frac{1}{2} (\mu_1 + \mu_0) \partial_z f, \quad \forall f \in C^{2,1}(\mathbb{R}^2). \tag{6.7}$$

We can rely on standard arguments from classical PDE theory as well as optimal stopping theory (see e.g. Karatzas and Shreve [50, Theorem 2.7.7]) and obtain the following lemma.

Lemma 6.2 *The value function \widehat{v} of (5.5) is the unique classical $C^{2,1}$ -solution to the boundary value problem*

$$(\mathcal{L}_{X,Z} - r)w = 0 \quad \text{in } \mathcal{R} \quad \text{and} \quad w|_{\partial \mathcal{R}} = \widehat{v}|_{\partial \mathcal{R}},$$

for $\mathcal{L}_{X,Z}$ as in (6.7) and any open set \mathcal{R} such that its closure is contained in the continuation region C_3 of (6.5). In particular, $\widehat{v} \in C^{2,1}(C_3)$.

In the following, we aim at investigating the geometry of the state space in the coordinates (X, Z) . To this end, we define the generalised inverse of the nondecreasing boundary b by

$$b^{-1}(x) := \inf\{\varphi \in (0, \infty) : b(\varphi) > x\} \tag{6.8}$$

so that the continuation region C_2 of (5.7) is rewritten as

$$C_2 = \{(x, \varphi) \in \mathbb{R} \times (0, \infty) : b^{-1}(x) < \varphi\}.$$

Since $\varphi \mapsto b(\varphi)$ is nondecreasing by Lemma 5.3, we observe that

$$\begin{aligned} (x, z) \in C_3 &\iff (x, e^{\frac{z}{\sigma}(x+z)}) \in C_2 \\ &\iff e^{\frac{z}{\sigma}(x+z)} > b^{-1}(x) \iff z > \frac{\sigma}{\gamma} \log b^{-1}(x) - x, \end{aligned}$$

and by setting

$$c^{-1}(x) := \frac{\sigma}{\gamma} \log b^{-1}(x) - x, \tag{6.9}$$

we can rewrite (6.5) and (6.6) as

$$\mathcal{C}_3 = \{(x, z) \in \mathbb{R}^2 : z > c^{-1}(x)\}, \quad \mathcal{S}_3 = \{(x, z) \in \mathbb{R}^2 : z \leq c^{-1}(x)\}. \tag{6.10}$$

In contrast to the optimal stopping problems in the formulations (4.2) and (5.6), deriving the monotonicity of the boundary $x \mapsto c^{-1}(x)$ is not straightforward. Moreover – and differently to related contributions such as Federico et al. [33] –, we cannot translate it back to the monotonicity of the boundary b of (5.9) since the generalised inverse b^{-1} is nondecreasing as well, and this does not imply monotonicity of $x \mapsto c^{-1}(x)$. Therefore we follow and adapt arguments presented in De Angelis [19, Sect. 4.4] who studies separately the two cases in which the deterministic process Z in (6.3) is either increasing ($\mu_0 + \mu_1 \geq 0$) or decreasing ($\mu_0 + \mu_1 < 0$).

For the following analysis, it is useful to define

$$\widehat{u}(x, z) := \widehat{v}(x, z) - (e^x - \kappa)(1 + e^{\frac{\gamma}{\sigma}(x+z)}) \tag{6.11}$$

as well as

$$\begin{aligned} g(x, z) &:= (\mathcal{L}_{X,Z} - r)((e^x - \kappa)(1 + e^{\frac{\gamma}{\sigma}(x+z)})) \\ &= e^x \left(\frac{1}{2}\sigma^2 + \mu_0 - r \right) + r\kappa + e^{\frac{\gamma}{\sigma}(x+z)} \left(e^x \left(\frac{1}{2}\sigma^2 + \mu_1 - r \right) + r\kappa \right), \end{aligned} \tag{6.12}$$

and we observe that an application of Dynkin’s formula implies that

$$\widehat{u}(x, z) = \sup_{\tau} \mathbb{E}_{x,z}^{\mathbb{Q}} \left[\int_0^{\tau} e^{-rt} g(X_t, Z_t) dt \right], \quad (x, z) \in \mathbb{R}^2. \tag{6.13}$$

In the following Propositions 6.3 and 6.5, we establish the existence of a monotone boundary $c : \mathbb{R} \rightarrow \mathbb{R}$ that splits the state space into continuation and stopping region. As we verify later in Remark 6.6, the function c^{-1} in (6.9) is indeed the right-continuous inverse of this function.

Proposition 6.3 *If $\mu_0 + \mu_1 \geq 0$, there exists a nondecreasing function $c : \mathbb{R} \rightarrow \mathbb{R}$ such that the continuation region \mathcal{C}_3 of (6.5) is rewritten as*

$$\mathcal{C}_3 = \{(x, z) \in \mathbb{R}^2 : x < c(z)\}. \tag{6.14}$$

Proof Let $(x_0, z_0) \in \mathcal{S}_3$, $x_1 > x_0$ and notice that (6.10) implies $(-\infty, z_0] \times \{x_0\} \subseteq \mathcal{S}_3$. Furthermore, we have $x_0 > x_0^*$, and since the process Z is decreasing, we observe that the process (X^{x_1}, Z^{z_0}) crosses the half-line $(-\infty, z_0] \times \{x_0\}$ before reaching the level x_0^* . Hence we have $\mathbb{Q}_{x_1, z_0}[\tau^* < \tau_{x_0^*}] = 1$, where $\tau_{x_0^*} := \inf\{t \geq 0 : X_t^{x_1} = x_0^*\}$ and $\mathbb{Q}_{x_1, z_0}[\cdot] = \mathbb{Q}[\cdot | X_0 = x_1, Z_0 = z_0]$. Moreover, it can be verified that the second condition of Assumption 4.1 implies $x_0^* > \tilde{x}$, with the latter given by

$$\tilde{x} := \log \frac{r\kappa}{r - \frac{1}{2}\sigma^2 - \mu_1}. \tag{6.15}$$

Consequently, we have $\exp(X_s^{x_1})(r - \frac{1}{2}\sigma^2 - \mu_1) > r\kappa$ for all $s \in [0, \tau^*)$, and (6.12) and (6.13) imply $\widehat{u}(x_1, z_0) \leq 0$ for all $x_1 > x_0$ and therefore $\{z_0\} \times [x_0, \infty) \subseteq \mathcal{S}_3$. We can thus define

$$c(z) := \inf\{x \in \mathbb{R} : (x, z) \in \mathcal{S}_3\} \tag{6.16}$$

and observe that (6.10) implies that $z \mapsto c(z)$ is nondecreasing. □

In order to establish the same result in the case when $\mu_0 + \mu_1 < 0$, we first state the following lemma.

Lemma 6.4 *We have*

$$\widehat{v}_z(x, z) = \mathbb{E}_{x,z}^{\mathbb{Q}} \left[\frac{\gamma}{\sigma} e^{-r\tau^*} (e^{X_{\tau^*}} - \kappa) e^{\frac{\gamma}{\sigma}(X_{\tau^*} + Z_{\tau^*})} \mathbb{1}_{\{\tau^* < \infty\}} \right] \tag{6.17}$$

for all $(x, z) \in \mathbb{R}^2 \setminus \partial\mathcal{C}_3$ and $\tau^* := \tau^*(x, z)$.

Proof For $(x, z) \in \mathcal{S}_3$, the claim follows immediately since $\mathbb{Q}_{x,z}[\tau^* = 0] = 1$. Hence we let $(x, z) \in \mathcal{C}_3$, and for $\epsilon > 0$, we obtain

$$\begin{aligned} \widehat{v}(x, z + \epsilon) - \widehat{v}(x, z) &\geq \mathbb{E}^{\mathbb{Q}} \left[e^{-r(\tau^* \wedge t)} (\widehat{v}(X_{\tau^* \wedge t}^x, Z_{\tau^* \wedge t}^{z+\epsilon}) - \widehat{v}(X_{\tau^* \wedge t}^x, Z_{\tau^* \wedge t}^z)) \right] \\ &\geq \mathbb{E}^{\mathbb{Q}} \left[e^{-r\tau^*} (e^{X_{\tau^*}^x} - \kappa) e^{\frac{\gamma}{\sigma} X_{\tau^*}^x} (e^{\frac{\gamma}{\sigma} Z_{\tau^*}^{z+\epsilon}} - e^{\frac{\gamma}{\sigma} Z_{\tau^*}^z}) \mathbb{1}_{\{\tau^* < t\}} \right] \\ &\quad + \mathbb{E}^{\mathbb{Q}} \left[e^{-rt} (\widehat{v}(X_t^x, Z_t^{z+\epsilon}) - \widehat{v}(X_t^x, Z_t^z)) \mathbb{1}_{\{\tau^* > t\}} \right], \end{aligned} \tag{6.18}$$

where the first inequality follows from the supermartingale property of the process $(e^{-r(\tau \wedge t)} \widehat{v}(X_{\tau \wedge t}^x, Z_{\tau \wedge t}^{z+\epsilon}))$ and the martingale property of $(e^{-r(\tau^* \wedge t)} \widehat{v}(X_{\tau^* \wedge t}^x, Z_{\tau^* \wedge t}^z))$ for $\tau^* := \tau^*(x, z)$. Upon employing a change of measure as in Sect. 5, we find

$$\begin{aligned} \mathbb{E}_{x,z}^{\mathbb{Q}} [e^{-rt} |\widehat{v}(X_t, Z_t)|] &\leq \mathbb{E}_{x,z}^{\mathbb{Q}} [e^{-rt} |\overline{v}(x, e^{\frac{\gamma}{\sigma}(X_t + Z_t)})|] \\ &\leq K_1 \mathbb{E}_{x, \exp(\frac{\gamma}{\sigma}(x+z))}^{\mathbb{Q}} [e^{-rt} e^{X_t} (1 + \Phi_t)] \\ &= K_1 (1 + e^{\frac{\gamma}{\sigma}(x+z)}) \mathbb{E}_{x,\pi} [e^{-rt} e^{X_t}], \end{aligned}$$

where $\pi = e^{\frac{\gamma}{\sigma}(x+z)} / (1 + e^{\frac{\gamma}{\sigma}(x+z)})$. It is then easy to verify that Assumption 4.1 implies

$$\lim_{t \rightarrow \infty} \mathbb{E}_{x,z}^{\mathbb{Q}} [e^{-rt} \widehat{v}(X_t, Z_t)] = 0,$$

and hence applying dominated convergence in (6.18) as $t \rightarrow \infty$ yields

$$\begin{aligned} \widehat{v}(x, z + \epsilon) - \widehat{v}(x, z) &\geq \mathbb{E}^{\mathbb{Q}} [e^{-r\tau^*} (e^{X_{\tau^*}^x} - \kappa) e^{\frac{\gamma}{\sigma} X_{\tau^*}^x} (e^{\frac{\gamma}{\sigma} Z_{\tau^*}^{z+\epsilon}} - e^{\frac{\gamma}{\sigma} Z_{\tau^*}^z}) \mathbb{1}_{\{\tau^* < \infty\}}]. \end{aligned} \tag{6.19}$$

Similar arguments show

$$\begin{aligned} & \widehat{v}(x, z) - \widehat{v}(x, z - \epsilon) \\ & \leq \mathbb{E}^{\mathbb{Q}}[e^{-r\tau^*} (e^{X_{\tau^*}^x} - \kappa) e^{\frac{\gamma}{\sigma} X_{\tau^*}^x} (e^{\frac{\gamma}{\sigma} Z_{\tau^*}^{z+\epsilon}} - e^{\frac{\gamma}{\sigma} Z_{\tau^*}^z}) \mathbb{1}_{\{\tau^* < \infty\}}], \end{aligned} \tag{6.20}$$

and since $\widehat{v} \in C^{2,1}(\mathcal{C}_3)$ by Lemma 6.2, dividing (6.19) and (6.20) by ϵ and letting $\epsilon \downarrow 0$ yields the desired result. \square

Proposition 6.5 *If $\mu_0 + \mu_1 < 0$, there exists a nondecreasing function $c : \mathbb{R} \rightarrow \mathbb{R}$ (the same as in Proposition 6.3) such that the continuation region of (6.5) can be written as*

$$\mathcal{C}_3 = \{(x, z) \in \mathbb{R}^2 : x < c(z)\}.$$

Proof Let $(x, z) \in \mathbb{R}^2$. Notice that $x < x_0^*$ implies $(x', z) \in \mathcal{C}_3$ for all $x' < x$ and $z \in \mathbb{R}$ because of Lemma 4.5 and since the transformations T_1 and \overline{T}_1 of (5.13) and (6.1), respectively, are the identity; hence $\{(x, z) : x < x_0^*\} \subseteq \mathcal{C}_3$. We can thus focus on the case $x \geq x_0^*$ and distinguish two possibilities:

- i) $\widehat{u}_x(x, z) \leq 0$ for all $x \in (x_0^*, \infty)$ such that $(x, z) \in \mathcal{C}_3$;
- ii) there exists $x_0 \in (x_0^*, \infty)$ such that $(x_0, z) \in \mathcal{C}_3$ and $\widehat{u}_x(x_0, z) > 0$.

In case i), the map $x \mapsto \widehat{u}(x, z)$ is decreasing for $x \in (x_0^*, \infty)$ and $(x, z) \in \mathcal{C}_3$. Hence for any $(x, z) \in \mathcal{C}_3$, we obtain $(-\infty, x] \times \{z\} \subseteq \mathcal{C}_3$ and the claim follows in the same spirit as in Proposition 6.3. In case ii), we establish a contradiction. As a first step, we show that ii) implies $[x_0, \infty) \times \{z\} \subseteq \mathcal{C}_3$ which will then lead to a contradiction. We start by noticing that Lemma 6.2 and (6.13) imply

$$(\mathcal{L}_{X,Z} - r)\widehat{u}(x_0, z) = -g(x_0, z) \tag{6.21}$$

for (x_0, z) as in ii). As $\mu_0 < 0$ and $\widehat{u}_x(x_0, z) > 0$, we have $\mu_0 \widehat{u}(x_0, z) < 0$ and thus

$$\begin{aligned} \frac{1}{2}\sigma^2 \widehat{u}_{xx}(x_0, z) &= r\widehat{u}(x_0, z) - \mu_0 \widehat{u}_x(x_0, z) + \frac{1}{2}(\mu_0 + \mu_1)\widehat{u}_z(x_0, z) - g(x_0, z) \\ &> r\widehat{u}(x_0, z) + \frac{1}{2}(\mu_0 + \mu_1)\widehat{u}_z(x_0, z) - g(x_0, z). \end{aligned} \tag{6.22}$$

Next, we notice that we can rewrite (6.17) as

$$\widehat{v}_z(x, z) = \frac{\gamma}{\sigma} (\widehat{v}(x, z) - \mathbb{E}^{\mathbb{Q}}[e^{-r\tau^*} (e^{X_{\tau^*}^x} - \kappa)]), \tag{6.23}$$

and since

$$\begin{aligned} \widehat{v}_z(x, z) &= \widehat{u}_z(x, z) + \frac{\gamma}{\sigma} (e^x - \kappa) e^{\frac{\gamma}{\sigma}(x+z)}, \\ \widehat{v}(x, z) &= \widehat{u}(x, z) + (e^x - \kappa) (1 + e^{\frac{\gamma}{\sigma}(x+z)}), \end{aligned}$$

(6.23) gives

$$\widehat{u}_z(x, z) + \frac{\gamma}{\sigma}(e^x - \kappa)e^{\frac{\gamma}{\sigma}(x+z)} = \frac{\gamma}{\sigma}(\widehat{u}(x, z) + (e^x - \kappa)(1 + e^{\frac{\gamma}{\sigma}(x+z)}) - \mathbb{E}^{\mathbb{Q}}[e^{-r\tau^*}(e^{X_{\tau^*}^x} - \kappa)]),$$

which is equivalent to

$$\widehat{u}_z(x, z) = \frac{\gamma}{\sigma}\widehat{u}(x, z) + \frac{\gamma}{\sigma}(e^x - \kappa - \mathbb{E}^{\mathbb{Q}}[e^{-r\tau^*}(e^{X_{\tau^*}^x} - \kappa)]).$$

We can thus plug this last equality into (6.22) and obtain

$$\begin{aligned} & \frac{1}{2}\sigma^2\widehat{u}_{xx}(x_0, z) \\ & > r\widehat{u}(x_0, z) + \frac{1}{2}(\mu_0 + \mu_1)\left(\frac{\gamma}{\sigma}\widehat{u}(x_0, z) + \frac{\gamma}{\sigma}(e^{x_0} - \kappa - \mathbb{E}^{\mathbb{Q}}[e^{-r\tau^*}(e^{X_{\tau^*}^{x_0}} - \kappa)])\right) \\ & \quad - g(x_0, z) \\ & = \left(r + \frac{1}{2}(\mu_0 + \mu_1)\frac{\gamma}{\sigma}\right)(\widehat{u}(x_0, z) + e^{x_0} - \kappa) \\ & \quad - \frac{1}{2}(\mu_0 + \mu_1)\frac{\gamma}{\sigma}\mathbb{E}^{\mathbb{Q}}[e^{-r\tau^*}(e^{X_{\tau^*}^{x_0}} - \kappa)] - g(x_0, z) > 0, \end{aligned}$$

where the last inequality follows from $r > \frac{\gamma}{2\sigma}|\mu_0 + \mu_1|$ in Assumption 4.1 upon noticing that $x_0 > x_0^*$. We deduce that $\widehat{u}_x(\cdot, z)$ increases in a right neighbourhood of x_0 , and repeating the arguments for every $x > x_0$ yields $\widehat{u}_x(\cdot, z) > 0$ on $[x_0, \infty)$. It follows that $\widehat{u}(\cdot, z)$ is increasing on $[x_0, \infty)$ so that $[x_0, \infty) \times \{z\} \subseteq \mathcal{C}_3$, and combining the latter with (6.10), we have $\mathcal{A} := [x_0, \infty) \times [z_0, \infty) \subseteq \mathcal{C}_3$. However, this leads to a contradiction. To see this, let $(x, z) \in \mathcal{A}$ and $\tau_{x_0} := \inf\{t > 0 : X_t^x \leq x_0\}$. Since $t \mapsto Z_t^z$ is increasing, the only possibility for the process (X^x, Z^z) to exit \mathcal{A} , and thus eventually the continuation region, is by passing through the horizontal line $[x_0, \infty) \times \{z_0\}$. We thus have $\tau_{x_0} \leq \tau^*$ $\mathbb{Q}_{x,z}$ -a.s. and moreover, since $\mu_0 < 0$, the stopping time τ_{x_0} is finite a.s. Upon using Lemma 5.1 i) and (6.4), it follows that

$$\begin{aligned} (e^x - \kappa)(1 + e^{\frac{\gamma}{\sigma}(x+z)}) &< \widehat{v}(x, z) \\ &= \mathbb{E}_{x,z}^{\mathbb{Q}}[e^{-r\tau_{x_0}}\widehat{v}(X_{\tau_{x_0}}, Z_{\tau_{x_0}})] \\ &= \mathbb{E}_{x,z}^{\mathbb{Q}}\left[e^{-r\tau_{x_0}}\widehat{v}\left(x_0, z - \frac{1}{2}(\mu_0 + \mu_1)\tau_{x_0}\right)\right] \\ &\leq K_1 e^{x_0} \mathbb{E}_{x,z}^{\mathbb{Q}}[e^{-r\tau_{x_0}}] \\ &\quad + K_1 e^{\frac{\gamma}{\sigma}(x_0+z)} e^{x_0} \mathbb{E}_{x,z}^{\mathbb{Q}}[e^{-(r - \frac{1}{2}\frac{\gamma}{\sigma}|\mu_0 + \mu_1|)\tau_{x_0}}]. \end{aligned}$$

Let now $\widehat{r} := r - \frac{\gamma}{2\sigma}|\mu_0 + \mu_1| > 0$, where the inequality is from Assumption 4.1, and denote by ϕ_r (resp. $\phi_{\widehat{r}}$) the strictly decreasing solution to $\frac{1}{2}\sigma^2 f_{xx} + \mu_0 f_x - qf = 0$

for $q \in \{r, \widehat{r}\}$. Then by results on hitting times for one-dimensional diffusions (see e.g. Borodin and Salminen [12, Chap. II.10]), the above inequality is equivalent to

$$(e^x - \kappa)(1 + e^{\frac{y}{\sigma}(x+z)}) \leq K_1 e^{x_0} \frac{\phi_r(x)}{\phi_r(x_0)} + K_1 e^{x_0} e^{\frac{y}{\sigma}(x_0+z)} \frac{\phi_{\widehat{r}}(x)}{\phi_{\widehat{r}}(x_0)}, \tag{6.24}$$

which thus holds true for all $(x, z) \in \mathcal{A}$. Since \mathcal{A} is right-connected, we can let $x \rightarrow \infty$ and notice that $(e^x - \kappa)(1 + e^{\frac{y}{\sigma}(x+z)}) \rightarrow \infty$, while the right-hand side of (6.24) decreases to 0 due to the decreasing property of $x \rightarrow \phi_q(x)$ for $q > 0$. We thus obtain a contradiction, which concludes our proof. \square

Remark 6.6 Due to the implied geometry of the state space, as observed in Propositions 6.3 and 6.5, we notice that the function $x \mapsto c^{-1}(x)$ is nondecreasing as well. Moreover, we notice that

$$z > c^{-1}(x) \iff c(z) > x,$$

and hence the function c^{-1} is the right-continuous inverse of c and thus admits the representation

$$c^{-1}(x) = \inf\{z \in \mathbb{R} : c(z) > x\}. \tag{6.25}$$

In light of the connection (6.9) between c^{-1} and b^{-1} (the generalised inverse of the boundary b), (6.25) allows us to translate back our results to the formulation of Sect. 5 and then – through the representation (5.14) – to the original setting of Sect. 4.

6.1 Regularity of the value function and of the optimal stopping boundary

We have established the existence of a nondecreasing boundary $z \mapsto c(z)$ such that \mathbb{R}^2 is split into the continuation region \mathcal{C}_3 of (6.5) and the stopping region \mathcal{S}_3 of (6.6). In the following, we derive some further properties of the optimal stopping boundary and of the value function \widehat{v} of (6.4). We first state the following result, which will be helpful in the forthcoming analysis.

Lemma 6.7 *We have $\widehat{u}_z(x, z) \geq 0$ for $(x, z) \in \mathcal{C}_3$.*

Proof Due to (6.4) and (6.1), \bar{v} of (5.6) satisfies $\bar{v}(x, \varphi) = \widehat{v}(x, \frac{\sigma}{\gamma} \ln \varphi - x)$ for all $(x, \varphi) \in \mathbb{R} \times (0, \infty)$. Since $\widehat{v}_z \in C^0(\mathcal{C}_3)$ by Lemma 6.2, we then also have $\bar{v}_\varphi \in C^0(\mathcal{C}_2)$. Furthermore, $\varphi \mapsto \bar{v}(x, \varphi)$ is convex on $(0, \infty)$ by Lemma 5.1 iv) and thus so is $\varphi \mapsto \bar{u}(x, \varphi)$ of (5.12). Then for $(x, \varphi) \in \mathcal{C}_2$ and $\varphi' = b^{-1}(x)$ so that $(x, \varphi') \in \partial \mathcal{C}_2$, we obtain, as $\bar{u}_\varphi \in C^0(\mathcal{C}_2)$ as well,

$$0 \leq \bar{u}(x, \varphi) = \bar{u}(x, \varphi) - \bar{u}(x, b^{-1}(x)) \leq \bar{u}_\varphi(x, \varphi)(\varphi - b^{-1}(x)),$$

and $\varphi > b^{-1}(x)$ implies $\bar{u}_\varphi(x, \varphi) \geq 0$ for $(x, \varphi) \in \mathcal{C}_2$. In light of the relation (6.4), we then obtain $\widehat{u}_z(x, z) \geq 0$ on \mathcal{C}_3 . \square

Proposition 6.8 *The optimal stopping boundary $c(z)$ is such that $x_0^* \leq c(z) \leq x_1^*$ for all $z \in \mathbb{R}$ and with x_0^* and x_1^* as in Lemma 4.5. Furthermore, we have $c \in C(\mathbb{R})$.*

Proof The first assertion follows from Lemma 5.3 iii) and by noticing that the transformation \bar{T}_1 of (6.1) is the identity. We derive the continuity of $z \mapsto c(z)$ in two steps.

1) Left-continuity. Let $z_0 \in \mathbb{R}$ and $z_n \uparrow z_0$ as $n \rightarrow \infty$. Since $z \mapsto c(z)$ is nondecreasing and \mathcal{S}_3 is closed, we obtain $\lim_{n \rightarrow \infty} (c(z_n), z_n) = (c(z_0-), z_0) \in \mathcal{S}_3$, where $c(z_0-)$ denotes the left limit of c at z_0 . The definition of c in (6.16) implies that $c(z_0-) \geq c(z_0)$; but since c is nondecreasing, we must have $c(z_0-) = c(z_0)$ and the claim follows.

2) Right-continuity. We argue by contradiction and assume there exists $z_0 \in \mathbb{R}$ with $c(z_0) < c(z_0+)$. Using techniques developed in De Angelis [18], we take $c(z_0) < x_1 < x_2 < c(z_0+)$ and a nonnegative function $\phi \in C_c^\infty(x_1, x_2)$ such that $\int_{x_1}^{x_2} \phi(x) dx = 1$. Recalling (6.21), we have

$$\mathcal{L}_{X,Z} \widehat{u}(x, z) - r\widehat{u}(x, z) = -g(x, z) \tag{6.26}$$

for $(x, z) \in (x_1, x_2) \times (z_0, \infty)$. In the following, it is helpful to treat the cases i) $\mu_0 + \mu_1 \geq 0$ and ii) $\mu_0 + \mu_1 < 0$ separately. Let us start with i) and recall that $\widehat{u}_z(x, z) \geq 0$ for x and z as above, due to Lemma 6.7. Integration by parts yields

$$\begin{aligned} 0 &\geq -\frac{1}{2}(\mu_0 + \mu_1) \int_{x_1}^{x_2} \widehat{u}_z(x, z) \phi(x) dx \\ &= \int_{x_1}^{x_2} \left(r\widehat{u}(x, z) - \mu_0 \widehat{u}_x(x, z) - \frac{1}{2} \sigma^2 \widehat{u}_{xx}(x, z) - g(x, z) \right) \phi(x) dx \\ &= \int_{x_1}^{x_2} \left(r\widehat{u}(x, z) \phi(x) + \mu_0 \widehat{u}(x, z) \phi'(x) - \frac{1}{2} \sigma^2 \widehat{u}(x, z) \phi''(x) - g(x, z) \phi(x) \right) dx. \end{aligned}$$

Hence using dominated convergence as $z \downarrow z_0$ and $\widehat{u}(x, z_0) = 0$, we get

$$0 \geq - \int_{x_1}^{x_2} g(x, z) \phi(x) dx > 0,$$

where the latter inequality follows from $x_1, x_2 \geq x_0^*$ and Assumption 4.1, which implies $x > \tilde{x}$ for all $x \in [x_1, x_2]$ and \tilde{x} as in (6.15). We thus obtain a contradiction and $c(z_0) = c(z_0+)$.

In case ii), we rely on classical results of internal regularity of PDEs (cf. Friedman [38, Theorem 3.10]) which allow taking derivatives in (6.26) with respect to x and have $\widehat{u}_x \in C^{2,1}(\mathcal{C}_3)$ solving

$$(\mathcal{L}_{X,Z} - r)\widehat{u}_x(x, z) = -g_x(x, z), \quad (x, z) \in (x_1, x_2) \times (z_0, \infty).$$

Then we obtain for $z > z_0$ that

$$\int_{x_1}^{x_2} ((\mathcal{L}_{X,Z} - r)\widehat{u}_x(x, z) + g_x(x, z)) \phi(x) dx = 0. \tag{6.27}$$

Let $F_\phi(z) := \int_{x_1}^{x_2} \widehat{u}_{xz}(x, z)\phi(x)dx$. Integration by parts allows rewriting (6.27) as

$$\begin{aligned} & \frac{1}{2}|\mu_0 + \mu_1|F_\phi(z) \\ &= \int_{x_1}^{x_2} \left(r\widehat{u}_x(x, z) - \frac{1}{2}\sigma^2\widehat{u}_{xx}(x, z) - \mu_0\widehat{u}_{xx}(x, z) - g_x(x, z) \right)\phi(x)dx \\ &= \int_{x_1}^{x_2} \left(-r\widehat{u}(x, z)\phi'(x) + \frac{1}{2}\sigma^2\widehat{u}(x, z)\phi'''(x) \right. \\ & \quad \left. - \mu_0\widehat{u}(x, z)\phi''(x) - g_x(x, z)\phi(x) \right)dx, \end{aligned}$$

and using dominated convergence as $z \downarrow z_0$ as well as $\widehat{u}(x, z_0) = 0$ results in

$$F_\phi(z_0+) = \frac{2}{|\mu_0 + \mu_1|} \int_{x_1}^{x_2} -g_x(x, z_0)\phi(x)dx \geq p_0 > 0$$

for some p_0 , where the first inequality again follows from Assumption 4.1. Thus there exists $\epsilon > 0$ such that $F_\phi(z) \geq p_0/2$ for all $z \in (z_0, z_0 + \epsilon)$, and we finally obtain

$$\begin{aligned} \frac{1}{2}p_0\epsilon &\leq \int_{z_0}^{z_0+\epsilon} F_\phi(z)dz = \int_{z_0}^{z_0+\epsilon} \int_{x_1}^{x_2} \widehat{u}_{xz}(x, z)\phi(x)dx dz \\ &= - \int_{x_1}^{x_2} \int_{z_0}^{z_0+\epsilon} \widehat{u}_z(x, z)\phi'(x)dz dx \\ &= - \int_{x_1}^{x_2} (\widehat{u}(x, z_0 + \epsilon) - \widehat{u}(x, z_0))\phi'(x)dx \\ &= \int_{x_1}^{x_2} \widehat{u}_x(x, z_0 + \epsilon)\phi(x)dx \leq 0, \end{aligned}$$

where we used $\widehat{u}(x, z_0) = 0$ as well as $\widehat{u}_x(x, z) \leq 0$ for $x \in [x_1, x_2]$ and $z > z_0$ (cf. Proposition 6.5). Hence $c(z) = c(z+)$ for all $z \in \mathbb{R}$ and together with 1), we conclude that $z \mapsto c(z)$ is continuous. □

In the next result, we derive the regularity of the value function. Its proof can be found in Appendix A.

Proposition 6.9 *The value function \widehat{v} of (6.4) has $\widehat{v} \in C^1(\mathbb{R}^2)$ and $\widehat{v}_{xx} \in L^\infty_{loc}(\mathbb{R}^2)$.*

In light of Proposition 6.9, we are able to derive an integral equation for the free boundary c . Let us first recall that by standard arguments based on the strong Markov property and Proposition 6.9, the value function \widehat{v} and the free boundary c solve the

free-boundary problem

$$\begin{cases} (\mathcal{L}_{X,Z} - r)\widehat{v}(x, z) \leq 0, & (x, z) \in \mathbb{R}^2, \\ (\mathcal{L}_{X,Z} - r)\widehat{v}(x, z) = 0, & x < c(z), z \in \mathbb{R}, \\ \widehat{v}(x, z) \geq (e^x - \kappa)(1 + e^{\frac{\gamma}{\sigma}(x+z)}), & (x, z) \in \mathbb{R}^2, \\ \widehat{v}(x, z) = (e^x - \kappa)(1 + e^{\frac{\gamma}{\sigma}(x+z)}), & x \geq c(z), z \in \mathbb{R}, \\ \widehat{v}_x(x, z) = e^x(1 + e^{\frac{\gamma}{\sigma}(x+z)}) + \frac{\gamma}{\sigma}(e^x - \kappa)e^{\frac{\gamma}{\sigma}(x+z)}, & x = c(z), z \in \mathbb{R}, \\ \widehat{v}_z(x, z) = \frac{\gamma}{\sigma}(e^x - \kappa)e^{\frac{\gamma}{\sigma}(x+z)}, & x = c(z), z \in \mathbb{R}. \end{cases} \tag{6.28}$$

In the next result, with a suitable application of Itô’s lemma, we derive a probabilistic representation of the value function \widehat{v} . Its proof is postponed to Appendix B.

Proposition 6.10 *Recall the free boundary c of (6.16) and the function g of (6.12). For any $(x, z) \in \mathbb{R}^2$, the value function \widehat{v} can be written as*

$$\widehat{v}(x, z) = \mathbb{E}_{x,z}^{\mathbb{Q}} \left[- \int_0^\infty e^{-rs} g(X_s, Z_s) \mathbb{1}_{\{X_s \geq c(Z_s)\}} ds \right]. \tag{6.29}$$

Denote now by

$$G(w; m, v) := \frac{1}{\sqrt{2\pi v^2}} e^{-\frac{(w-m)^2}{2v^2}}, \quad w \in \mathbb{R}, m \in \mathbb{R}, v > 0, \tag{6.30}$$

the density function of a Gaussian random variable with mean m and variance v^2 . Then from Proposition 6.10, we obtain the following result.

Proposition 6.11 *Let*

$$\mathcal{M} := \{f : \mathbb{R} \mapsto \mathbb{R} : f \text{ is nondecreasing, continuous and } x_0^* \leq f(z) \leq x_1^*\}.$$

Then the free boundary c of (6.16) is the unique solution in \mathcal{M} to the integral equation

$$\begin{aligned} & (e^{c(z)} - \kappa)(1 + e^{\frac{\gamma}{\sigma}(c(z)+z)}) \\ &= \int_0^\infty e^{-rs} \left(\int_{\mathbb{R}} -g(w, Z_s) G(w; c(z) + \mu_0 s, \sigma \sqrt{s}) \mathbb{1}_{\{w \geq c(z)\}} dw \right) ds \end{aligned} \tag{6.31}$$

with g as in (6.12) and G as in (6.30).

Proof We take $x = c(z)$ in Proposition 6.10. Using the continuity of the value function, we find for $z \in \mathbb{R}$ that

$$(e^{c(z)} - \kappa)(1 + e^{\frac{\gamma}{\sigma}(c(z)+z)}) = \mathbb{E}^{\mathbb{Q}} \left[- \int_0^\infty e^{-rs} g(X_s^{c(z)}, Z_s^z) \mathbb{1}_{\{X_s^{c(z)} \geq c(Z_s^z)\}} ds \right]. \tag{6.32}$$

Because Z^z is deterministic and $X_s^{c(z)}$ is Gaussian under \mathbb{Q} with mean $c(z) + \mu_0 s$ and variance $\sigma^2 s$, we can reformulate (6.32) as (6.31) upon using (6.30). To show unique-

ness, one can employ a four-step-approach exploiting the superharmonic characterisation of \widehat{v} as originally developed in Peskir [55, Theorem 3.1]. Since the present setting does not exhibit additional challenges, we omit the details for the sake of brevity. \square

Remark 6.12 As it turns out, the integral equation (6.31) allows us to derive an integral equation for the boundary b^{-1} of (6.8) as well. Indeed, taking $z = c^{-1}(x)$ in (6.31) and using (6.9) yields

$$\begin{aligned} &(e^x - \kappa)(1 + b^{-1}(x)) \\ &= \mathbb{E}^{\mathbb{Q}} \left[- \int_0^\infty e^{-rs} g \left(X_s^x, \frac{\sigma}{\gamma} \ln \Phi_s^{b^{-1}(x)} - X_s^x \right) \mathbb{1}_{\{\Phi_s^{b^{-1}(x)} \leq b^{-1}(X_s^x)\}} ds \right], \quad x \in \mathbb{R}. \end{aligned}$$

In particular, it follows from the latter that

$$\begin{aligned} b^{-1}(x) = \frac{1}{e^x - \kappa} \mathbb{E}^{\mathbb{Q}} \left[- \int_0^\infty e^{-rs} g \left(X_s^x, \frac{\sigma}{\gamma} \ln \Phi_s^{b^{-1}(x)} - X_s^x \right) \right. \\ \left. \times \mathbb{1}_{\{\Phi_s^{b^{-1}(x)} \leq b^{-1}(X_s^x)\}} ds \right] - 1, \quad x \in \mathbb{R}. \end{aligned} \tag{6.33}$$

Notice that the domain of b^{-1} is given by the interval $[x_0^*, x_1^*]$ (cf. Lemma 5.3), and hence we do not encounter any problems when dividing by $e^x - \kappa$ since Assumption 4.1 guarantees that $e^x - \kappa > 0$ for $x \geq x_0^*$.

7 Solution of the optimal execution problem

In this section, we finally return to the optimal execution problem of Sect. 4 and provide its solution. Before we do so, it is helpful to transform the singular stochastic control problem (2.9) by arguing as for the optimal stopping problems in Sects. 5 and 6, respectively. Since the arguments are in the same spirit of those developed in Sect. 5, the details are omitted (see also Federico et al. [33, Sect. 4]). First, we make a change of measure as in Sect. 5, and for \mathbb{Q} as introduced there, we let

$$dX_t^\xi = \mu_0 dt + \sigma dB_t - \alpha d\xi_t, \quad X_{0-}^\xi = x,$$

denote the dynamics of the controlled process X^ξ under \mathbb{Q} . Conditionally on $X_{0-}^\xi = x$, $Y_{0-}^\xi = y$ and $\Phi_0 = \varphi$, we then introduce the transformed optimal control problem

$$\begin{aligned} \overline{V}(x, y, \varphi) := \sup_{\xi \in \mathcal{A}(y)} \mathbb{E}_{x, y, \varphi}^{\mathbb{Q}} \left[\int_0^\infty e^{-rt} (e^{X_t^\xi} - \kappa)(1 + \Phi_t) \circ d\xi_t \right], \\ (x, y, \varphi) \in \mathbb{R} \times (0, \infty) \times (0, \infty), \end{aligned} \tag{7.1}$$

and observe that $\bar{V}(x, y, \varphi) = (1 + \varphi)V(x, y, \frac{\varphi}{1+\varphi})$. Furthermore, we set

$$Z_t^\xi := \frac{\sigma}{\gamma} \log \Phi_t - X_t^\xi, \quad z := \frac{\sigma}{\gamma} \log \varphi - x,$$

for any $(x, \varphi) \in \mathbb{R} \times (0, \infty)$. An application of Itô’s formula then gives the dynamics

$$dZ_t^\xi = -\frac{1}{2}(\mu_0 + \mu_1)dt + \alpha d\xi_t, \quad Z_{0-}^\xi = z. \tag{7.2}$$

Finally, analogously to (6.4), we define, for $(x, y, z) \in \mathcal{O} := \mathbb{R} \times (0, \infty) \times \mathbb{R}$,

$$\begin{aligned} \widehat{V}(x, y, z) &:= \bar{V}(x, y, e^{\frac{\gamma}{\sigma}(x+z)}) \\ &= \sup_{\xi \in \mathcal{A}(y)} \mathbb{E}_{x, y, z}^{\mathbb{Q}} \left[\int_0^\infty e^{-rt} (e^{X_t^\xi} - \kappa) (1 + e^{\frac{\gamma}{\sigma}(X_t^\xi + Z_t^\xi)}) \circ d\xi_t \right], \end{aligned} \tag{7.3}$$

where $\mathbb{E}_{x, y, z}^{\mathbb{Q}}$ is the expectation conditionally on $X_{0-}^\xi = x, Y_{0-}^\xi = y$ and $Z_{0-}^\xi = z$.

In the following, we introduce a candidate for the value function V of (2.9) and – through the explicit relationships between the value functions v, \bar{v} and \widehat{v} – also for the value functions \bar{V} and \widehat{V} of (7.1) and (7.3). To this end, we set

$$U(x, y, \pi) := \frac{1}{\alpha} \int_{x-\alpha y}^x v(x', \pi) dx', \tag{7.4}$$

where v denotes the value function of (4.2). Upon using the explicit relationship (5.5) of v and \bar{v} , it follows that

$$\begin{aligned} \bar{U}(x, y, \varphi) &:= (1 + \varphi)U\left(x, y, \frac{\varphi}{1 + \varphi}\right) = (1 + \varphi) \frac{1}{\alpha} \int_{x-\alpha y}^x v\left(x', \frac{\varphi}{1 + \varphi}\right) dx' \\ &= \frac{1}{\alpha} \int_{x-\alpha y}^x \bar{v}(x', \varphi) dx', \end{aligned} \tag{7.5}$$

which gives a candidate for the value function \bar{V} of (7.1). Furthermore, we let $\widehat{U}(x, y, z) := \bar{U}(x, y, e^{\frac{\gamma}{\sigma}(x+z)})$ and exploit the relationship (6.4) to derive

$$\widehat{U}(x, y, z) = \frac{1}{\alpha} \int_{x-\alpha y}^x \widehat{v}(x', x + z - x') dx' = \frac{1}{\alpha} \int_z^{z+\alpha y} \widehat{v}(x + z - q, q) dq, \tag{7.6}$$

where the last equality follows from a change of variables. With regard to Proposition 6.9, we can state the following result; its proof is based on direct computations.

Lemma 7.1 *The function \widehat{U} of (7.6) has $\widehat{U} \in C^1(\mathcal{O})$. Moreover, $\widehat{U}_{xy}, \widehat{U}_{yz} \in C(\mathcal{O})$ as well as $\widehat{U}_{xx}, \widehat{U}_{xz} \in L^\infty_{\text{loc}}(\mathcal{O})$.*

Proof Notice that (7.6) gives

$$\begin{aligned} \widehat{U}_x(x, y, z) &= \frac{1}{\alpha} \int_z^{z+\alpha y} \widehat{v}_x(x+z-q, q) dq, \\ \widehat{U}_y(x, y, z) &= \widehat{v}(x-\alpha y, z+\alpha y), \end{aligned} \tag{7.7}$$

$$\begin{aligned} \widehat{U}_z(x, y, z) &= \frac{1}{\alpha} \int_z^{z+\alpha y} \widehat{v}_x(x+z-q, q) dq + \frac{1}{\alpha} (\widehat{v}(x-\alpha y, z+\alpha y) - \widehat{v}(x, z)) \\ &= \frac{1}{\alpha} \int_z^{z+\alpha y} \widehat{v}_z(x+z-q, q) dq, \end{aligned} \tag{7.8}$$

so that $\widehat{U}_x, \widehat{U}_y$ and \widehat{U}_z are continuous due to Proposition 6.9. Moreover, Proposition 6.9 also implies that

$$\widehat{U}_{xx}(x, y, z) = \frac{1}{\alpha} \int_z^{z+\alpha y} \widehat{v}_{xx}(x+z-q, q) dq \tag{7.9}$$

is locally bounded, and the mixed derivatives

$$\widehat{U}_{xy}(x, y, z) = \widehat{v}_x(x-\alpha y, z+\alpha y), \quad \widehat{U}_{yz}(x, y, z) = \widehat{v}_z(x-\alpha y, z+\alpha y)$$

are continuous, while

$$\widehat{U}_{xz}(x, y, z) = \frac{1}{\alpha} \int_z^{z+\alpha y} \widehat{v}_{xx}(x+z-q, q) dq + \frac{1}{\alpha} (\widehat{v}_x(x-\alpha y, z+\alpha y) - \widehat{v}_x(x, z))$$

is locally bounded. Furthermore, it is easy to see that $\widehat{U}_{yx} = \widehat{U}_{xy}, \widehat{U}_{xz} = \widehat{U}_{zx}$ and $\widehat{U}_{yz} = \widehat{U}_{zy}$. \square

The proof of the next result follows from (7.7), (7.8) and direct computations.

Corollary 7.2 *One has*

$$\begin{aligned} \alpha \widehat{U}_x(x, y, z) - \alpha \widehat{U}_z(x, y, z) + \widehat{U}_y(x, y, z) &= \widehat{v}(x, z) \\ &\geq (e^x - \kappa)(1 + e^{\frac{y}{\sigma}(x+z)}) \end{aligned} \tag{7.10}$$

so that

$$\begin{aligned} \mathbb{W}_3 := \{(x, y, z) \in \mathcal{O} : \alpha \widehat{U}_x(x, y, z) - \alpha \widehat{U}_z(x, y, z) + \widehat{U}_y(x, y, z) \\ > (e^x - \kappa)(1 + e^{\frac{y}{\sigma}(x+z)})\} = \mathcal{C}_3 \end{aligned} \tag{7.11}$$

with C_3 as in (6.5). Furthermore, $\widehat{U}_x - \widehat{U}_z = \overline{U}_x$ as well as $\widehat{U}_y = \overline{U}_y$, and we have

$$\begin{aligned} \mathbb{W}_2 := \{ (x, y, \varphi) \in \mathbb{R} \times (0, \infty) \times (0, \infty) : \alpha \overline{U}_x(x, y, \varphi) + \overline{U}_y(x, y, \varphi) \\ > (e^x - \kappa)(1 + \varphi) \} = C_2 \end{aligned} \tag{7.12}$$

with C_2 from (5.7).

7.1 Construction of the optimal control for the state space process (X, Y, Φ)

Recall b from (5.9), which is nondecreasing and left-continuous by Lemma 5.3. Then we define for any $(x, y, \varphi) \in \mathbb{R} \times (0, \infty) \times (0, \infty)$ the admissible control strategy

$$\widehat{\xi}_t := y \wedge \sup_{0 \leq s \leq t} \frac{1}{\alpha} (x - b(\Phi_s^\varphi) + \mu_0 s + \sigma B_s)^+, \quad t \geq 0, \widehat{\xi}_{0-} = 0, \tag{7.13}$$

according to which the investor should only execute a lump-sum amount of shares whenever X_{t-} is strictly inside the selling region and hence strictly above the boundary $b(\Phi_t)$. More precisely, if $y \leq \frac{1}{\alpha}(x - b(\varphi))$, it is optimal to sell the complete amount of shares instantaneously, while for $y > \frac{1}{\alpha}(x - b(\varphi))$, the system is brought immediately to the level $(X_0, Y_0, \Phi_0) = (b(\varphi), y - \frac{1}{\alpha}(x - b(\varphi)), \varphi)$. Afterwards, the strategy (7.13) prescribes to take action whenever X_t approaches the boundary $b(\Phi_t)$ from below and the process (X, Y) is obliquely reflected at the belief-dependent boundary $b(\Phi)$ in the direction $(-\alpha, -1)$. Hence X_t is kept inside the interval $(-\infty, b(\Phi_t)]$ with “minimal effort”. These actions are so-called *Skorokhod reflection-type policies* and caused by the continuous part $\widehat{\xi}^c$ of the control $\widehat{\xi}$. Notice that the nondecreasing process $\widehat{\xi}$ and the induced random measure $d\widehat{\xi}$ on $[0, \infty)$ are such that (recall (7.12))

$$\begin{cases} (X_t^{\widehat{\xi}}, Y_t^{\widehat{\xi}}, \Phi_t) \in \overline{\mathbb{W}}_2 & \mathbb{Q} \otimes dt\text{-a.s.}, \\ d\widehat{\xi} \text{ has support on } \{t \geq 0 : (X_{t-}^{\widehat{\xi}}, Y_{t-}^{\widehat{\xi}}, \Phi_t) \notin \mathbb{W}_2\}, \\ \widehat{\xi}_t \leq y, & t \geq 0. \end{cases}$$

Furthermore, due to (7.2), (7.3) and Corollary 7.2, we can express the control $\widehat{\xi}$ equivalently in terms of the state-process $(X^{\widehat{\xi}}, Y^{\widehat{\xi}}, Z^{\widehat{\xi}})$ by (cf. (7.11))

$$\begin{cases} (X_t^{\widehat{\xi}}, Y_t^{\widehat{\xi}}, Z_t^{\widehat{\xi}}) \in \overline{\mathbb{W}}_3 & \mathbb{Q} \otimes dt\text{-a.s.}, \\ d\widehat{\xi} \text{ has support on } \{t \geq 0 : (X_{t-}^{\widehat{\xi}}, Y_{t-}^{\widehat{\xi}}, Z_{t-}^{\widehat{\xi}}) \notin \mathbb{W}_3\}, \\ \widehat{\xi}_t \leq y, & t \geq 0. \end{cases} \tag{7.14}$$

In the following, we prove that $\widehat{\xi}$ is in fact an optimal control for problem (7.3) and that $\widehat{U} = \widehat{V}$. As an immediate consequence, we obtain $\overline{U} = \overline{V}$ and $U = V$.

Theorem 7.3 *Let $(x, y, z) \in \mathbb{R} \times [0, \infty) \times \mathbb{R}$ and $\widehat{U}(x, y, z)$ as in (7.6). Then $\widehat{U}(x, y, z) = \widehat{V}(x, y, z)$, and $\widehat{\xi}$ in (7.13) is optimal for the singular control problem (7.3).*

Proof First of all, for $y = 0$, we have $\widehat{U}(x, 0, z) = 0 = \widehat{V}(x, 0, z)$. Hence in the following, we assume $(x, y, z) \in \mathcal{O}$.

1) We prove $\widehat{U} \geq \widehat{V}$. Take an arbitrary control $\xi \in \mathcal{A}(y)$ and for $R > 0$ and $N \in \mathbb{N}$, define $\tau_{R,N} := \inf\{s \geq 0 : |(X_s^\xi, Z_s^\xi)| > R\} \wedge N$. Due to Lemma 7.1, we can proceed as in Fleming and Soner [36, Theorem 8.4.1] to obtain (after performing an approximation of \widehat{U} via mollifiers and taking limits)

$$\begin{aligned} & \mathbb{E}_{x,y,z}^{\mathbb{Q}} [e^{-r\tau_{R,N}} \widehat{U}(X_{\tau_{R,N}}^\xi, Y_{\tau_{R,N}}^\xi, Z_{\tau_{R,N}}^\xi) - \widehat{U}(x, y, z)] \\ &= \mathbb{E}_{x,y,z}^{\mathbb{Q}} \left[\int_0^{\tau_{R,N}} e^{-rs} (\mathcal{L}_{X,Z} - r) \widehat{U}(X_s^\xi, Y_s^\xi, Z_s^\xi) ds \right. \\ &\quad + \sigma \int_0^{\tau_{R,N}} e^{-rs} \widehat{U}_x(X_{\tau_{R,N}}^\xi, Y_{\tau_{R,N}}^\xi, Z_{\tau_{R,N}}^\xi) dB_s \\ &\quad + \sum_{0 \leq s \leq \tau_{R,N}} e^{-rs} (\widehat{U}(X_s^\xi, Y_s^\xi, Z_s^\xi) - \widehat{U}(X_{s-}^\xi, Y_{s-}^\xi, Z_{s-}^\xi)) \\ &\quad \left. + \int_0^{\tau_{R,N}} e^{-rs} (-\alpha \widehat{U}_x(X_s^\xi, Y_s^\xi, Z_s^\xi) - \widehat{U}_y(X_s^\xi, Y_s^\xi, Z_s^\xi) \right. \\ &\quad \left. + \alpha \widehat{U}_z(X_s^\xi, Y_s^\xi, Z_s^\xi)) d\xi_s^c \right]. \tag{7.15} \end{aligned}$$

Denote by $M_{R,N}$ the dB -integral in (7.15). Notice that

$$\begin{aligned} & \widehat{U}(X_s^\xi, Y_s^\xi, Z_s^\xi) - \widehat{U}(X_{s-}^\xi, Y_{s-}^\xi, Z_{s-}^\xi) \\ &= \widehat{U}(X_{s-}^\xi - \alpha \Delta \xi_s, Y_{s-}^\xi - \Delta \xi_s, Z_{s-}^\xi + \alpha \Delta \xi_s) - \widehat{U}(X_{s-}^\xi, Y_{s-}^\xi, Z_{s-}^\xi) \\ &= \int_0^{\Delta \xi_s} \frac{\partial \widehat{U}(X_{s-}^\xi - \alpha u, Y_{s-}^\xi - u, Z_{s-}^\xi + \alpha u)}{\partial u} du \\ &= \int_0^{\Delta \xi_s} (-\alpha \widehat{U}_x - \widehat{U}_y + \alpha \widehat{U}_z)(X_{s-}^\xi - \alpha u, Y_{s-}^\xi - u, Z_{s-}^\xi + \alpha u) du. \tag{7.16} \end{aligned}$$

Hence combining (7.15) and (7.16) and adding the term

$$\begin{aligned} & \mathbb{E}_{x,y,z}^{\mathbb{Q}} \left[\int_0^{\tau_{R,N}} e^{-rs} (e^{X_s^\xi} - \kappa) (1 + e^{\frac{\gamma}{\sigma}(X_s^\xi + Z_s^\xi)}) d\xi_s^c \right. \\ &\quad \left. + \sum_{0 \leq s \leq \tau_{R,N}} e^{-rs} \int_0^{\Delta \xi_s} (e^{X_{s-}^\xi - \alpha u} - \kappa) (1 + e^{\frac{\gamma}{\sigma}(X_{s-}^\xi + Z_{s-}^\xi)}) du \right] \end{aligned}$$

on both sides yields

$$\begin{aligned}
 & \mathbb{E}_{x,y,z}^{\mathbb{Q}} \left[\int_0^{\tau_{R,N}} e^{-rs} (e^{X_s^\xi} - \kappa) (1 + e^{\frac{\gamma}{\sigma}(X_s^\xi + Z_s^\xi)}) d\xi_s^c \right. \\
 & \quad \left. + \sum_{0 \leq s \leq \tau_{R,N}} e^{-rs} \int_0^{\Delta \xi_s} (e^{X_{s-}^\xi - \alpha u} - \kappa) (1 + e^{\frac{\gamma}{\sigma}(X_{s-}^\xi + Z_{s-}^\xi)}) du - \widehat{U}(x, y, z) \right] \\
 &= \mathbb{E}_{x,y,z}^{\mathbb{Q}} \left[\int_0^{\tau_{R,N}} e^{-rs} (\mathcal{L}_{X,Z} - r) \widehat{U}(X_s^\xi, Y_s^\xi, Z_s^\xi) ds \right. \\
 & \quad + M_{R,N} - e^{-r\tau_{R,N}} \widehat{U}(X_{\tau_{R,N}}^\xi, Y_{\tau_{R,N}}^\xi, Z_{\tau_{R,N}}^\xi) \\
 & \quad + \sum_{0 \leq s \leq \tau_{R,N}} e^{-rs} \\
 & \quad \quad \times \int_0^{\Delta \xi_s} ((-\alpha \widehat{U}_x - \widehat{U}_y + \alpha \widehat{U}_z)(X_{s-}^\xi - \alpha u, Y_{s-}^\xi - u, Z_{s-}^\xi + \alpha u) \\
 & \quad \quad + (e^{X_{s-}^\xi - \alpha u} - \kappa) (1 + e^{\frac{\gamma}{\sigma}(X_{s-}^\xi + Z_{s-}^\xi)})) du \\
 & \quad + \int_0^{\tau_{R,N}} e^{-rs} ((-\alpha \widehat{U}_x - \widehat{U}_y + \alpha \widehat{U}_z)(X_s^\xi, Y_s^\xi, Z_s^\xi) \\
 & \quad \quad + (e^{X_s^\xi} - \kappa) (1 + e^{\frac{\gamma}{\sigma}(X_s^\xi + Z_s^\xi)})) d\xi_s^c \left. \right]. \tag{7.17}
 \end{aligned}$$

We observe that (7.7)–(7.9) imply

$$(\mathcal{L}_{X,Z} - r) \widehat{U}(x, y, z) = \frac{1}{\alpha} \int_{x-\alpha y}^x (\mathcal{L}_{X,Z} - r) \widehat{v}(x', x + z - x') dx' \leq 0, \tag{7.18}$$

where the last inequality follows from the supermartingale property of the process $(e^{-rt} \widehat{v}(X_t, Z_t))$, combined with the regularity obtained in Proposition 6.9. Hence due to (7.10), $\widehat{U} \geq 0$ and $\mathbb{E}_{x,y,z}^{\mathbb{Q}}[M_{R,N}] = 0$, (7.17) is written as

$$\begin{aligned}
 & \widehat{U}(x, y, z) \\
 & \geq \mathbb{E}_{x,y,z}^{\mathbb{Q}} \left[\int_0^{\tau_{R,N}} e^{-rs} (e^{X_s^\xi} - \kappa) (1 + e^{\frac{\gamma}{\sigma}(X_s^\xi + Z_s^\xi)}) d\xi_s^c \right. \\
 & \quad \left. + \sum_{0 \leq s \leq \tau_{R,N}} e^{-rs} \int_0^{\Delta \xi_s} (e^{X_{s-}^\xi - \alpha u} - \kappa) (1 + e^{\frac{\gamma}{\sigma}(X_{s-}^\xi + Z_{s-}^\xi)}) du \right]. \tag{7.19}
 \end{aligned}$$

Taking limits as $R \uparrow \infty$ as well as $N \uparrow \infty$ and invoking the dominated convergence theorem thanks to Assumption 4.1, we obtain

$$\begin{aligned}
 & \widehat{U}(x, y, z) \geq \mathbb{E}_{x,y,z}^{\mathbb{Q}} \left[\int_0^\infty e^{-rs} (e^{X_s^\xi} - \kappa) (1 + e^{\frac{\gamma}{\sigma}(X_s^\xi + Z_s^\xi)}) d\xi_s^c \right. \\
 & \quad \left. + \sum_{s: \Delta \xi_s \neq 0} e^{-rs} \int_0^{\Delta \xi_s} (e^{X_{s-}^\xi - \alpha u} - \kappa) (1 + e^{\frac{\gamma}{\sigma}(X_{s-}^\xi + Z_{s-}^\xi)}) du \right] \\
 & = J(x, y, z, \xi).
 \end{aligned}$$

Since ξ was arbitrary, we have

$$\widehat{U}(x, y, z) \geq \widehat{V}(x, y, z) \tag{7.20}$$

for all $(x, y, z) \in \mathcal{O}$, that is, $\widehat{U} \geq \widehat{V}$ on \mathcal{O} .

2) We prove that $\widehat{U} \leq \widehat{V}$. Let $\widehat{\xi}$ satisfy the conditions in (7.14) and define $\widehat{\tau}_{R,N} = \inf\{t \geq 0 : |(X_t^{\widehat{\xi}}, Z_t^{\widehat{\xi}})| > R\} \wedge N$ for $R > 0$ and $N \in \mathbb{N}$. Notice that the properties of $\widehat{\xi}$ imply equalities in (7.10) and (7.18), where the equality in (7.18) follows from the monotonicity of c , and we can deduce that $(x', x + z - x') \in \mathbb{W}_3$ for $(x, y, z) \in \mathbb{W}_3$ and $x' \leq x$. The same arguments as in the first part of the proof yield

$$\begin{aligned} \widehat{U}(x, y, z) &= \mathbb{E}_{x,y,z}^{\mathbb{Q}} [e^{-r\widehat{\tau}_{R,N}} \widehat{U}(X_{\widehat{\tau}_{R,N}}^{\widehat{\xi}}, Y_{\widehat{\tau}_{R,N}}^{\widehat{\xi}}, Z_{\widehat{\tau}_{R,N}}^{\widehat{\xi}})] \\ &+ \mathbb{E}_{x,y,z}^{\mathbb{Q}} \left[\int_0^{\widehat{\tau}_{R,N}} e^{-rs} (e^{X_s^{\widehat{\xi}}} - \kappa) (1 + e^{\frac{\gamma}{\sigma}(X_s^{\widehat{\xi}} + Z_s^{\widehat{\xi}})}) d\widehat{\xi}_s^c \right. \\ &\quad \left. + \sum_{0 \leq s \leq \widehat{\tau}_{R,N}} e^{-rs} \int_0^{\Delta \widehat{\xi}_s} (e^{X_{s-}^{\widehat{\xi}} - \alpha u} - \kappa) (1 + e^{\frac{\gamma}{\sigma}(X_{s-}^{\widehat{\xi}} + Z_{s-}^{\widehat{\xi}})}) du \right]. \end{aligned}$$

It is thus left to prove that

$$\lim_{N \uparrow \infty} \lim_{R \uparrow \infty} \mathbb{E}_{x,y,z}^{\mathbb{Q}} [e^{-r\widehat{\tau}_{R,N}} \widehat{U}(X_{\widehat{\tau}_{R,N}}^{\widehat{\xi}}, Y_{\widehat{\tau}_{R,N}}^{\widehat{\xi}}, Z_{\widehat{\tau}_{R,N}}^{\widehat{\xi}})] = 0, \tag{7.21}$$

because combining this with (7.19) implies that $J(x, y, z, \widehat{\xi}) = \widehat{U}(x, y, z)$ and therefore $\widehat{V}(x, y, z) \geq \widehat{U}(x, y, z)$ for all $(x, y, z) \in \mathcal{O}$. Combining the latter with (7.20) yields $\widehat{U} = \widehat{V}$ on \mathcal{O} .

In order to prove (7.21), we notice that Lemma 5.1 i), (6.4) and (7.6) imply that

$$\begin{aligned} \widehat{U}(x, y, z) &\leq \frac{1}{\alpha} \int_z^{z+\alpha y} K_1 e^{x+z-q} (1 + e^{\frac{\gamma}{\sigma}(x+z-q+q)}) dq \\ &= \frac{1}{\alpha} K_1 e^x (1 + e^{\frac{\gamma}{\sigma}(x+z)})(1 - e^{-\alpha y}), \end{aligned}$$

and since $y \mapsto \widehat{U}(x, y, z)$ is increasing, we obtain

$$\begin{aligned} 0 &\leq e^{-r\widehat{\tau}_{R,N}} \widehat{U}(X_{\widehat{\tau}_{R,N}}^{\widehat{\xi}}, Y_{\widehat{\tau}_{R,N}}^{\widehat{\xi}}, Z_{\widehat{\tau}_{R,N}}^{\widehat{\xi}}) \\ &\leq e^{-r\widehat{\tau}_{R,N}} \widehat{U}(X_{\widehat{\tau}_{R,N}}^{\widehat{\xi}}, y, Z_{\widehat{\tau}_{R,N}}^{\widehat{\xi}}) \\ &\leq \frac{K_1}{\alpha} (1 - e^{-\alpha y}) e^{-r\widehat{\tau}_{R,N}} e^{X_{\widehat{\tau}_{R,N}}^0} (1 + e^{\frac{\gamma}{\sigma}(X_{\widehat{\tau}_{R,N}}^0 + Z_{\widehat{\tau}_{R,N}}^0)}), \end{aligned}$$

where we used that $X_t^\xi \leq X_t^0$ as well as $X_t^\xi + Z_t^\xi = X_t^0 + Z_t^0$ a.s. Hence taking expectations yields

$$\begin{aligned} 0 &\leq \mathbb{E}_{x,y,z}^{\mathbb{Q}} [e^{-r\widehat{\tau}_{R,N}} \widehat{U}(X_{\widehat{\tau}_{R,N}}^\xi, Y_{\widehat{\tau}_{R,N}}^\xi, Z_{\widehat{\tau}_{R,N}}^\xi)] \\ &\leq \frac{K_1}{\alpha} (1 - e^{-\alpha y}) \mathbb{E}_{x,y,z}^{\mathbb{Q}} [e^{-r\widehat{\tau}_{R,N}} e^{X_{\widehat{\tau}_{R,N}}^0} (1 + e^{\frac{y}{\sigma}(X_{\widehat{\tau}_{R,N}}^0 + Z_{\widehat{\tau}_{R,N}}^0)})] \\ &= \frac{K_1}{\alpha} (1 - e^{-\alpha y}) (1 + e^{\frac{y}{\sigma}(x+z)}) \mathbb{E}_{x,y,\pi} [e^{-r\widehat{\tau}_{R,N}} e^{X_{\widehat{\tau}_{R,N}}^0}] \end{aligned}$$

with $\pi := e^{\frac{y}{\sigma}(x+z)} / (1 + e^{\frac{y}{\sigma}(x+z)})$, where the last equality follows from a change of measure as in Sect. 5. Upon using Assumption 4.1, it is easy to check that (7.21) holds true, thus completing the proof. \square

Remark 7.4 We can use the transformation (6.1) from (x, z) - to (x, φ) -coordinates in order to show that $\widehat{\xi}$ is an optimal control for problem (7.1) as well. Indeed, recall (6.2) and the equality $\widehat{V}(x, y, z) = \widehat{V}(x, y, \frac{\sigma}{\gamma} \ln \varphi - x) = \overline{V}(x, y, \varphi)$ to conclude

$$\begin{aligned} \overline{V}(x, y, \varphi) &= \mathbb{E}_{x,y,\varphi}^{\mathbb{Q}} \left[\int_0^\infty e^{-rs} (e^{X_s^\xi} - \kappa) (1 + \Phi_s) d\widehat{\xi}_s^c \right. \\ &\quad \left. + \sum_{s:\Delta\widehat{\xi}_s \neq 0} e^{-rs} \int_0^{\Delta\widehat{\xi}_s} (e^{X_s^\xi - \alpha u} - \kappa) (1 + \Phi_s) du \right]. \end{aligned} \tag{7.22}$$

Furthermore, (7.22) and (7.5) imply that we have $U(x, y, \pi) = V(x, y, \pi)$ for all $(x, y, \pi) \in \mathbb{R} \times (0, \infty) \times (0, 1)$.

Remark 7.5 Letting $\tilde{\tau}(x, y, \varphi) := \inf\{t \geq 0 : x + \mu_0 t + \sigma B_t \geq b(\Phi_t^\varphi)\}$, the optimal execution strategy $\tilde{\xi}$ as in (7.13) converges as $\alpha \downarrow 0$ to the execution strategy

$$\tilde{\xi}_t = \begin{cases} 0 & \text{for } t < \tilde{\tau}(x, y, \varphi), \\ y & \text{for } t \geq \tilde{\tau}(x, y, \varphi), \end{cases}$$

which prescribes to sell the total amount of shares instantaneously when the process X reaches the optimal execution boundary $b(\Phi)$. It is interesting to notice that the optimal strategy and the value function are robust with respect to the parameter α . Indeed, by L'Hôpital's rule, we see from (7.5) that $\lim_{\alpha \downarrow 0} U = y\overline{v}(x, \varphi)$ for $(x, y, \varphi) \in \mathbb{R} \times (0, \infty) \times (0, \infty)$, and it is easy to show via a verification theorem that $y\overline{v}(x, \varphi)$ and $\tilde{\xi}$ are the value function and the optimal execution rule in the problem with no market impact.

Remark 7.6 Let $\widehat{\sigma} := \inf\{t \geq 0 : Y_t^\xi = 0\}$ denote the time at which the portfolio is fully depleted. Imposing the constraint that the investor has to sell all assets until the terminal time (cf. Guo and Zervos [44]), we notice that for $y \leq \frac{1}{\alpha}(x - b(\varphi))$, the control strategy $\widehat{\xi}$ of (7.13) still defines an optimal control as the complete amount

of shares is sold immediately at time $t = 0$. However, for $y > \frac{1}{\alpha}(x - b(\varphi))$, simple calculations yield

$$\lim_{T \uparrow \infty} \mathbb{Q}[\widehat{\sigma} > T] \geq 1 - \exp\left(\frac{2\mu_0}{\sigma^2}(\alpha y + x_0^* - x)\right),$$

and we notice that for increasing y and decreasing x , the probability increases that the investor does not sell the entire amount of shares until the terminal time. Hence if we restrict the admissible strategies to all $\xi \in \mathcal{A}(y)$ such that $\lim_{T \rightarrow \infty} Y_T^\xi = 0$, the control strategy $\widehat{\xi}$ of (7.13) does not provide an admissible execution strategy. In this case, arguing as in Guo and Zervos [44, Proposition 5.1], we can use $\widehat{\xi}$ to construct a sequence of ϵ -optimal strategies.

8 Numerical study

In this section, we perform a comparative statics analysis on the optimal execution boundaries a and b of (4.5) and (5.9), respectively, and also investigate the *value of information* in our model by comparing the value function V of (2.9) to the value of an *average drift problem*.

8.1 Comparative statics analysis

Based on the integral equation (6.31), we implement a recursive numerical scheme which relies on an application of the Monte Carlo method. Let ζ denote an auxiliary exponentially distributed random variable with parameter r that is independent of the Brownian motion B . Recalling that (6.31) can be reformulated as (6.33), we notice that the latter takes the shape of a fixed point problem

$$b^{-1}(x) = \Gamma(b^{-1}(x), x; b^{-1}), \tag{8.1}$$

for $x \in \mathbb{R}$ and b^{-1} being the generalised inverse of b as in (6.8). Here, the operator Γ is defined via

$$\Gamma(\varphi, x; f) := \frac{1}{e^x - \kappa} \frac{1}{r} \mathbb{E}^{\mathbb{Q}} \left[-g\left(X_\zeta^x, \frac{\sigma}{\gamma} \ln \Phi_\zeta^\varphi - X_\zeta^x\right) \mathbb{1}_{\{\Phi_\zeta^\varphi \leq f(X_\zeta^x)\}} \right] - 1 \tag{8.2}$$

for $(x, \varphi) \in \mathbb{R} \times (0, \infty)$ and a function $f : \mathbb{R} \rightarrow (0, \infty)$. By employing techniques seen in Christensen and Salminen [14], Dammann and Ferrari [17] and Detemple and Kitapbayev [27], we aim to solve (8.1) via an iterative scheme. To this end, we let

$$(b^{-1})^{[n]}(x) = \Gamma((b^{-1})^{[n-1]}(x), x; (b^{-1})^{[n-1]}), \quad x \in \mathbb{R}, n \geq 1, \tag{8.3}$$

define a sequence of boundaries and – for a given boundary $(b^{-1})^{[k]}$ – we estimate the expectation in (8.2) by

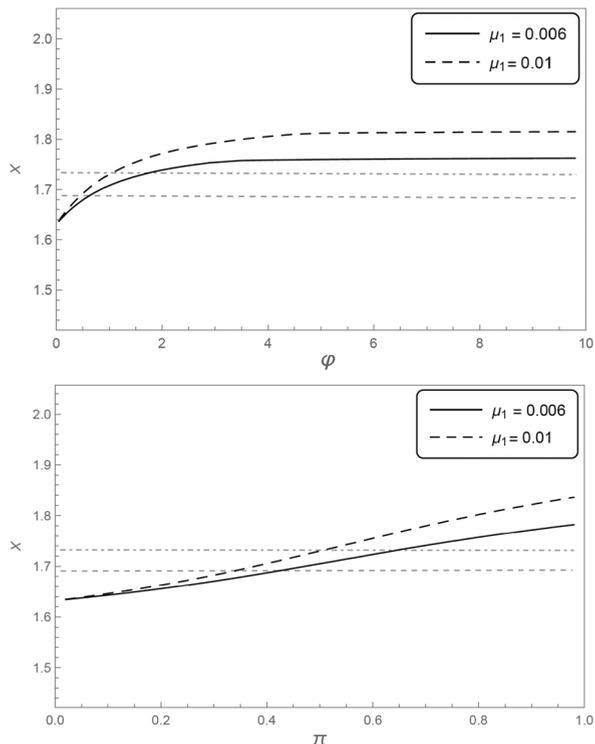
$$-\frac{1}{N} \sum_{i=1}^N g\left(X_{\xi_i}^{i,x}, \frac{\sigma}{\gamma} \ln \Phi_{\xi_i}^{i,(b^{-1})^{[k]}(x)} - X_{\xi_i}^{i,x}\right) \mathbb{1}_{\{\Phi_{\xi_i}^{i,(b^{-1})^{[k]}(x)} \leq (b^{-1})^{[k]}(X_{\xi_i}^{i,x})\}},$$

where N denotes the number of realisations of the exponential random variable. The initial boundary $(b^{-1})^{[0]}$ can be chosen as a simple exponential function with $(b^{-1})^{[0]}(x_0^*) = 0$ and $(b^{-1})^{[0]}(x) \rightarrow \infty$ for $x \uparrow x_1^*$ with x_0^* and x_1^* as in (3.6) and Remark 3.1, respectively. The numerical scheme (8.3) is then iterated on a grid over the interval $[x_0^*, x_1^*]$ until the variation of the boundary points between steps falls below a predetermined level. Finally, we calculate b from its generalised inverse b^{-1} and transform the resulting boundary according to the explicit relationship (5.14). We can thus study the sensitivity of $b(\varphi)$ as well as $a(\pi)$ with respect to some of the model parameters. Furthermore, we can compare the belief-dependent boundaries to the strategy of a pre-committed agent, who – after forming an initial belief $\pi = \mathbb{P}[\mu = \mu_1]$ – restrains from updating her belief and thus acts as if the drift value was constant and equal to $\mu_1\pi + \mu_0(1 - \pi)$. The resulting strategy is then triggered by a constant execution threshold which has a similar structure as the one derived in Sect. 3. Consequently, such an agent cannot react to price movements on the market and is thus not able to decrease or increase the target price at which she would like to sell the asset.

8.1.1 Sensitivity with respect to the drift

In Fig. 2, we can observe the sensitivity of the optimal execution boundaries with respect to one of the possible drift values. Since an increase in μ_1 implies higher

Fig. 2 The optimal execution boundaries $b(\varphi)$ and $a(\pi)$ as well as the pre-committed strategies for different values of μ_1 and parameters $r = 0.07$, $\mu_0 = -0.01$, $\sigma = 0.17$, $\kappa = 3$, $\pi = 0.6$

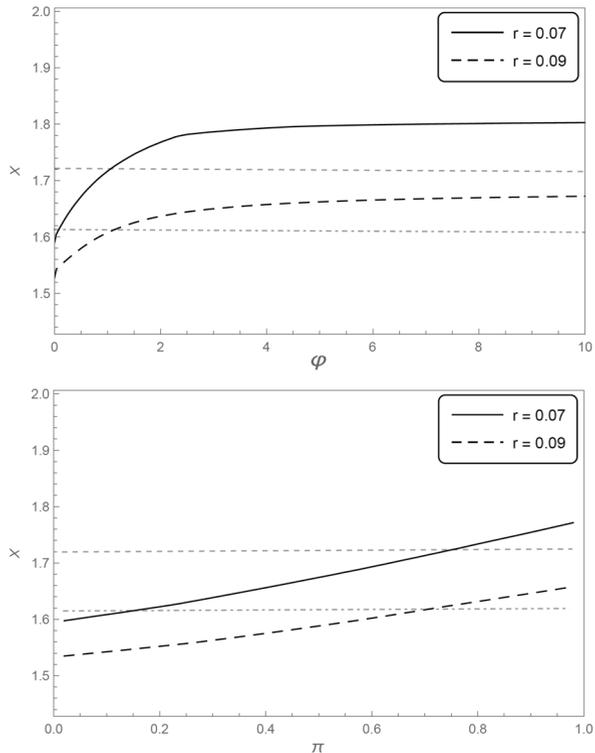


expected prices on the market, the investor delays her decision to sell a fraction of her shares and waits for higher prices to evolve. This effect is strongest for higher values of π , which reflect a stronger belief in the drift μ_1 . On the other hand, we notice that the lower bound x_0^* remains untouched by a change in μ_1 , since it results from the case of full information when $\mu = \mu_0$. Consequently, for a strong belief in the drift value μ_0 , the investor does not significantly change her execution strategy.

8.1.2 Sensitivity with respect to the discount rate

Figure 3 shows the effect on the boundaries a and b of a change in r ; the latter can be interpreted as the subjective impatience of the investor. For a larger value of r , the investor gets more impatient and discounts future revenues more heavily. Consequently, the investor is willing to liquidate her assets earlier, which is realised by decreasing the target price she aims at achieving on the market. This clear effect can be observed for every value of belief $\pi \in [0, 1]$.

Fig. 3 The optimal execution boundaries $b(\varphi)$ and $a(\pi)$ as well as the pre-committed strategies for different values of r and for the parameters $\mu_0 = -0.01, \mu_1 = 0.007, \sigma = 0.17, \kappa = 3, \pi = 0.6$

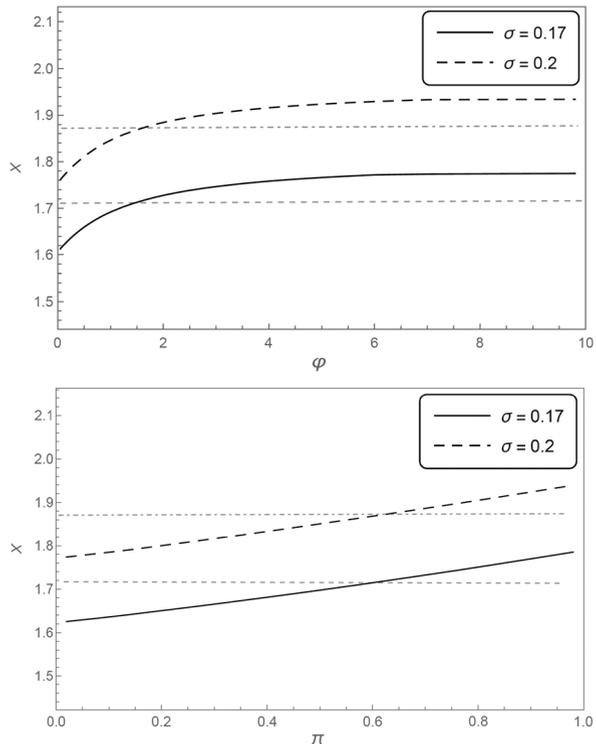


8.1.3 Sensitivity with respect to the volatility

The sensitivity of the optimal execution boundaries a and b on the volatility of the underlying asset is more delicate. As pointed out by Décamps et al. [24] who consider an optimal stopping problem of a structure similar to the one in (4.2), the effect of an increase in volatility is ambiguous and cannot always be predicted with the help of standard real option models (see for example Dixit and Pindyck [28, Chap. 6], McDonald and Siegel [53]). In general, one expects a value function increasing with rising volatility, as this increases the spread of possible future values of the asset and thus the maximal possible profit, while the maximal possible loss remains unchanged. The investor exploits this upside potential by delaying her liquidation decision and increasing the target price she aims at realising on the market. This effect, widely known and referred to as the “real option effect” in Décamps et al. [24], can be observed in the benchmark case (3.2) as well as in the problem (2.9) under partial information, as Fig. 4 reveals.

However, this effect need not be robust. To understand how an increase in volatility might indeed harm the investor, we recall the dynamics of the belief process Π given by (2.8). In particular, we observe that increasing volatility lowers the signal-to-noise ratio $\gamma = (\mu_1 - \mu_0)/\sigma$ (determining the variance of the process Π) and thus the *efficiency of learning*. The latter effect is in contrast to the mentioned real option effect, and the sensitivity of the value function with respect to an increase in volatility

Fig. 4 The optimal execution boundaries $b(\varphi)$ and $a(\pi)$ as well as the pre-committed strategies for different values of σ and for the parameters $r = 0.07, \mu_0 = -0.01, \mu_1 = 0.007, \kappa = 3, \pi = 0.6$



“depends on which of the real option and the inefficient learning effect dominates” (Décamps et al. [24, p. 487]). The overall impact of a change in volatility thus clearly depends on the parameter constellation of the model; however, a division of the parameter space is not straightforward. For a broader discussion on this subject, we refer to Décamps et al. [24, Sect. 6.2],

8.2 The value of information

Here, we want to address the question whether incomplete information about the drift actually harms or benefits the investor. To this end, we introduce the “average drift problem”, whose value is denoted by $V^A(x, y)$ and modelled as in (3.2), but with constant and known drift $\mu_1\pi + \mu_0(1 - \pi)$, i.e., the average of μ with respect to the prior Bernoulli distribution. We then investigate the preference of an investor faced with the decision of choosing between two portfolios containing assets with either an unknown drift coefficient, or with a constant and known *average* drift. An analytical attempt to answer this question is presented in Décamps et al. [24], although the derived result does not hold true in general, as pointed out by Klein [51]. Here, we are able to analyse this question with numerical methods based on the numerical evaluation of the optimal execution boundary (cf. Sect. 8.1) and the representation (6.29) of the optimal stopping value function \widehat{v} . In order to accomplish that, we plug in the numerical evaluation of b^{-1} into (6.29) and we transform the result according to (5.13). This yields the value function v of (4.2), which can be finally integrated via (7.4) to obtain a numerical approximation of the control problem’s value function V .

In general, the results derived in Décamps et al. [24] and Klein [51] suggest that the overall impact of introducing uncertainty over the drift is governed by two separate effects: the introduction of uncertainty in general and the impact of learning. If learning is efficient, which is achieved by – for example – specifying a small volatility coefficient σ , the latter effect seems to outweighs the former and the investor indeed prefers the problem with only incomplete information on the return. We observe this overall effect in Fig. 5.

In their model, Décamps et al. [24] give an analytical proof to this observation in an optimal stopping environment, although restricting the possible drift values to 0 and 1. For small values of σ , depending on the other parameters in the model, this result seems to hold true in our more generalised framework. Nevertheless, this effect cannot be expected to be robust over the whole parameter space. In an example where the parameter values are aligned such that $\beta_0 + \beta_1 = \sigma^2$ (and thus $\mu_0 = -\mu_1$ in our model), Klein [51] obtains an explicit solution to the optimal stopping problem and shows how the introduction of uncertainty might harm the decision maker. This effect appears to have the peculiarity of being, at least in some cases, dependent on the initial value of the price process, as it determines the distance to the target price at which the investor is willing to execute. We can observe an example of this in Fig. 6. In particular, if the asset price is close to the target value under the current belief and learning is inefficient, the investor will not choose a portfolio with drift uncertainty. This is due to the fact that the downside risk outweighs the upside potential, which could only be achieved if learning is efficient. On the other hand, we observe that for low prices the upside potential might still dominate and the investor is willing to choose the uncertain environment, even if learning is inefficient.

Fig. 5 The value function V of (2.9) and the average drift value function V^A as functions of x and π , respectively. The parameters of the model have been specified as $r = 0.15$, $\sigma = 0.15$, $\mu_0 = -0.012$, $\mu_1 = 0.01$, $\kappa = 1$, $\pi = 0.3$, $x = -0.1$

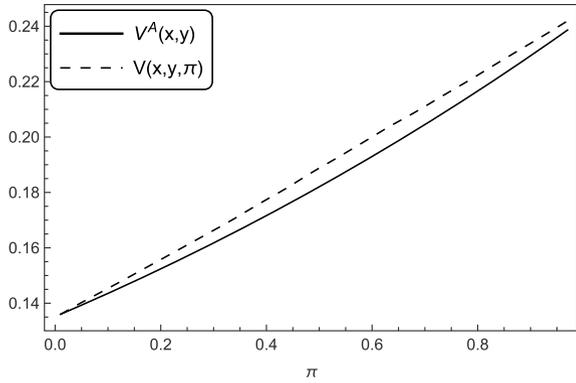
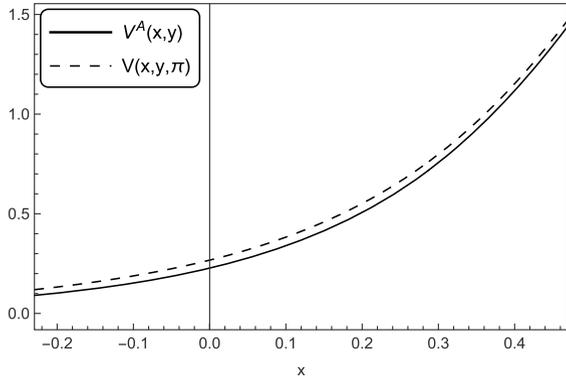
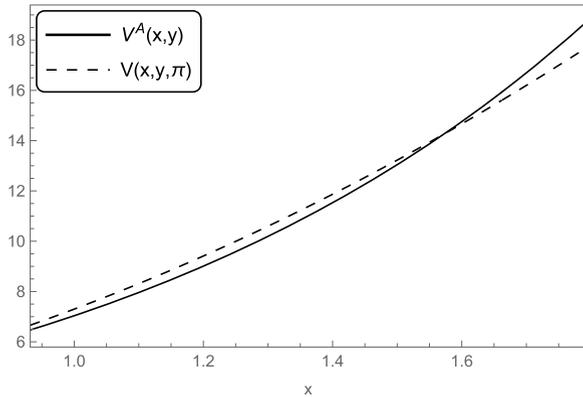


Fig. 6 The value function V of (2.9) and the average drift value function V^A as functions of x . The parameters of the model have been specified as $r = 0.2$, $\sigma = 0.5$, $\mu_0 = -0.012$, $\mu_1 = 0.01$, $\kappa = 1$, $\pi = 0.5$



Appendix A: Proof of Proposition 6.9

The proof follows the lines of De Angelis [19, Sect. 4], suitably adapted to the present setting, and is obtained through a series of intermediate results. Fix $(x, z) \in \mathbb{R}^2$, set

$$\begin{aligned} \sigma_* &:= \sigma_*(x, z) := \inf\{t \geq 0 : (X_t^x, Z_t^z) \in \mathcal{S}_3\}, \\ \widehat{\sigma}_* &:= \widehat{\sigma}_*(x, z) := \inf\{t \geq 0 : (X_t^x, Z_t^z) \in \text{int}(\mathcal{S}_3)\}, \end{aligned}$$

and observe that $\sigma_* = \tau_*$ \mathbb{Q} -a.s. on $\mathbb{R}^2 \setminus \partial\mathcal{C}_3$ due to the continuity of paths. It is crucial to show that this equality also holds for the boundary points $(x_0, z_0) \in \partial\mathcal{C}_3$. As it turns out, the cases i) $\mu_0 + \mu_1 \geq 0$ and ii) $\mu_0 + \mu_1 < 0$ should be treated in different fashions, and ii) exhibits some more technical difficulties than i). We start with case i), in which the needed result follows upon using the law of the iterated logarithm.

Proposition A.1 *Assume that $\mu_0 + \mu_1 \geq 0$. Let $(x_n, z_n)_{n \in \mathbb{N}} \subseteq \mathcal{C}_3$ be a sequence with $(x_n, z_n) \rightarrow (x_0, z_0) \in \partial\mathcal{C}_3$ such that $x_0 = c(z_0)$. We then have $\tau^*(x_n, z_n) \downarrow 0$ as well as $\widehat{\sigma}_*(x_n, z_n) \downarrow 0$, \mathbb{Q} -a.s.*

Proof Fix $\omega \in \Omega$ and assume that $\limsup_{n \rightarrow \infty} \tau^*(x_n, z_n)(\omega) =: \delta > 0$. Hence there exists a subsequence (still labelled by (x_n, z_n)) such that

$$X_t^{x_n}(\omega) < c(Z_t^{z_n}), \quad \forall n \in \mathbb{N}, \forall t \in [0, \delta/2], \tag{A.1}$$

which is equivalent to

$$x_n + \mu_0 t + \sigma B_t(\omega) < c\left(z_n - \frac{1}{2}(\mu_0 + \mu_1)t\right), \quad \forall n \in \mathbb{N}, \forall t \in [0, \delta/2].$$

Using that $z \mapsto c(z)$ is continuous, we let $n \rightarrow \infty$ and obtain for $t \in [0, \delta/2]$ that

$$\sigma B_t(\omega) \leq c\left(z_0 - \frac{1}{2}(\mu_0 + \mu_1)t\right) - x_0 - \mu_0 t \leq c(z_0) - x_0 - \mu_0 t = -\mu_0 t, \tag{A.2}$$

where the last inequality follows from $\mu_0 + \mu_1 \geq 0$ and Proposition 6.3. On the other hand, by the law of the iterated logarithm, there exists for each $\epsilon > 0$ a sequence $t_n \downarrow 0$ such that

$$B_{t_n} \geq (1 - \epsilon)\sqrt{2t_n \log(\log(1/t_n))}, \quad \forall n \in \mathbb{N}. \tag{A.3}$$

Combining (A.2) and (A.3) implies

$$\frac{1}{t} \sigma (1 - \epsilon) \sqrt{2t \log(\log(1/t))} \leq -\mu_0;$$

but since $\sqrt{2t \log(\log(1/t))}/t \rightarrow \infty$ for $t \downarrow 0$, (A.1) can only happen on a \mathbb{Q} -nullset. Thus we have $\tau^*(x_n, z_n) \downarrow 0$, and by replacing “ $<$ ” in (A.1) by “ \leq ”, we obtain $\widehat{\sigma}_*(x_n, z_n) \downarrow 0$ as well. □

Notice that the proof of Proposition A.1 cannot be replicated for case ii) in which $\mu_0 + \mu_1 < 0$ since the last inequality in (A.2) no longer applies. To prove the same result for case ii), we have to take a longer route. The reason lies in the fact that the process (X, Z) is moving towards the right in the state space and hence – keeping in mind that the continuation region \mathcal{C}_3 of (6.14) lies below the increasing boundary c – could possibly evade from the stopping set. In the following, we show that this is not the case by adapting the procedure in De Angelis [19, Sect. 4]. As a first step, we state the following result, whose proof follows the lines of Cox and Peskir [16, Corollary 8] and is thus omitted for the sake of brevity.

Lemma A.2 *If $\mu_0 + \mu_1 < 0$ and $r > \frac{\gamma}{2\sigma}|\mu_0 + \mu_1|$, then $\mathbb{Q}[\sigma_* = \widehat{\sigma}_*] = 1$.*

In the next step, we aim at proving *regularity* of the boundary points for the stopping set \mathcal{S}_3 in the sense of diffusions, that is, for $(x, z) \in \partial\mathcal{C}_3$, we have

$$\mathbb{Q}_{x,z}[\sigma_* > 0] = 0. \tag{A.4}$$

It is clear from Blumenthal’s 0–1 law that $\mathbb{Q}_{x,z}[\sigma_* > 0] = 1$ if (A.4) does not hold. Due to the geometry of the problem already mentioned, which implies that the process (X, Z) could evade from the stopping set when $\mu_0 + \mu_1 < 0$, proving (A.4) is not straightforward since we cannot apply an argument similar to the one for Proposition A.1. Instead, we establish (A.4) in two steps and begin by showing that the classical smooth-fit property, i.e., continuity of $\widehat{v}_x(\cdot, z)$, holds at the free boundary.

Lemma A.3 *Assume that $\mu_0 + \mu_1 < 0$ and $r > \frac{\gamma}{2\sigma}|\mu_0 + \mu_1|$. For \widehat{v} of (6.4), we have $\widehat{v}_x(\cdot, z) \in C(\mathbb{R})$ or, equivalently, $\widehat{u}_x(\cdot, z) \in C(\mathbb{R})$ for \widehat{u} of (6.11).*

Proof From (6.21), we obtain

$$\frac{1}{2}\sigma^2\widehat{u}_{xx}(x, z) = r\widehat{u}(x, z) - \mu_0\widehat{u}_x(x, z) + \frac{1}{2}(\mu_0 + \mu_1)\widehat{u}_z(x, z) - g(x, z)$$

for $(x, z) \in \mathcal{C}_3$, and due to (5.10) (which implies an analogous result for \widehat{v}), we deduce that for a bounded set B , \widehat{u}_{xx} is bounded on the closure of $B \cap \mathcal{C}_3$. Moreover, we recall that $\widehat{u}_x \leq 0$ in \mathcal{C}_3 as verified in the proof of Proposition 6.5. Aiming for a contradiction, we now assume that for $(x_0, z_0) \in \partial\mathcal{C}_3$ so that $x_0 = c(z_0)$, we have

$$\widehat{u}_x(x_0-, z_0) < -\delta_0 \tag{A.5}$$

for some $\delta_0 > 0$. We take a bounded rectangular neighbourhood B of (x_0, z_0) and define the stopping time $\tau_B := \inf\{t > 0 : (X_t, Z_t) \notin B\}$. Notice that

$$\widehat{u}(x_0, z_0) \geq \mathbb{E}_{x_0, z_0}^{\mathbb{Q}} \left[e^{-r(\tau_B \wedge t)} \widehat{u}(X_{\tau_B \wedge t}, Z_{\tau_B \wedge t}) + \int_0^{\tau_B \wedge t} e^{-rs} g(X_s, Z_s) ds \right] \tag{A.6}$$

from the supermartingale property of $(e^{-rt}\widehat{v}(X_t, Z_t))$. By Lemma 6.7 and because $t \mapsto Z_{\tau_B \wedge t}$ is increasing, we have $\widehat{u}(X_{\tau_B \wedge t}^{x_0}, Z_{\tau_B \wedge t}^{z_0}) \geq \widehat{u}(X_{\tau_B \wedge t}^{x_0}, z_0)$ \mathbb{Q} -a.s. Moreover, since the integrand on the right-hand side of (A.6) is bounded on B , we obtain

$$\widehat{u}(x_0, z_0) \geq \mathbb{E}_{x_0, z_0}^{\mathbb{Q}} [e^{-r(\tau_B \wedge t)} \widehat{u}(X_{\tau_B \wedge t}, z_0) - c_B(\tau_B \wedge t)], \tag{A.7}$$

where c_B is a constant depending on B . Due to the previously discussed local bound-ness of \widehat{u}_{xx} , we can apply the Itô–Tanaka formula to the first term in the expectation of (A.7). Let $\mathcal{L}_X := \frac{1}{2}\sigma^2\partial_{xx} + \mu_0\partial_x$ and denote the local time of X at x_0 by L^{x_0} . Moreover, noticing that $\widehat{u}_{xx}(\cdot, z_0) = 0$ for $x > x_0$, we obtain

$$\begin{aligned} & \mathbb{E}_{x_0, z_0}^{\mathbb{Q}} [e^{-r(\tau_B \wedge t)} \widehat{u}(X_{\tau_B \wedge t}, z_0)] \\ &= \widehat{u}(x_0, z_0) + \mathbb{E}_{x_0, z_0}^{\mathbb{Q}} \left[\int_0^{\tau_B \wedge t} e^{-rs} (\mathcal{L}_X - r) \widehat{u}(X_s, z_0) \mathbb{1}_{\{X_s \neq x_0\}} ds \right] \\ & \quad - \mathbb{E}_{x_0, z_0}^{\mathbb{Q}} \left[\int_0^{\tau_B \wedge t} e^{-rs} \widehat{u}_x(x_0-, z_0) dL_s^{x_0} \right], \end{aligned}$$

and combining this with (A.7) as well as noticing that $(\mathcal{L}_X - r)\widehat{u}(X_s, Z_s)$ is bounded on B , we find upon using the assumption (A.5) that

$$\begin{aligned} 0 &\geq \mathbb{E}_{x_0, z_0}^{\mathbb{Q}} \left[\int_0^{\tau_B \wedge t} e^{-rs} (\mathcal{L}_X - r) \widehat{u}(X_s, z_0) \mathbb{1}_{\{X_s \neq x_0\}} ds - c_B(\tau_B \wedge t) \right] \\ & \quad - \mathbb{E}_{x_0, z_0}^{\mathbb{Q}} \left[\int_0^{\tau_B \wedge t} e^{-rs} \widehat{u}_x(x_0-, z_0) dL_s^{x_0} \right] \\ &\geq \delta_0 e^{-rt} \mathbb{E}_{x_0, z_0}^{\mathbb{Q}} [L_{\tau_B \wedge t}^{x_0}] - c_B \mathbb{E}_{x_0, z_0}^{\mathbb{Q}} [\tau_B \wedge t]. \end{aligned}$$

This implies $c_B \mathbb{E}_{x_0, z_0}^{\mathbb{Q}} [\tau_B \wedge t] \geq \delta_0 e^{-rt} \mathbb{E}_{x_0, z_0}^{\mathbb{Q}} [L_{\tau_B \wedge t}^{x_0}]$, and because $\mathbb{E}_{x_0, z_0}^{\mathbb{Q}} [\tau_B \wedge t] \approx t$ while $\mathbb{E}_{x_0, z_0}^{\mathbb{Q}} [L_{\tau_B \wedge t}^{x_0}] \approx \sqrt{t}$ (see e.g. Peskir [56, Lemma 15]), we obtain the desired contradiction. Hence $\widehat{u}_x(\cdot, z) \in C(\mathbb{R})$. \square

We can now state the regularity of the boundary points.

Proposition A.4 *Assume that $\mu_0 + \mu_1 < 0$ and $r > \frac{\gamma}{2\sigma} |\mu_0 + \mu_1|$. Then all points $(x, z) \in \partial\mathcal{C}_3$ are regular, i.e., we have $\mathbb{Q}_{x, z}[\sigma_* > 0] = 0$.*

Proof We argue by contradiction and show that if $\mathbb{Q}_{x_0, z_0}[\sigma_* > 0] = 1$ for a boundary point $(x_0, z_0) \in \partial\mathcal{C}_3$, it follows that $\widehat{u}_x(x_0-, z_0) < 0$ which contradicts Lemma A.3. First, we establish an upper bound for \widehat{u}_x . Fix $(x, z) \in \mathcal{C}_3$ with $x > \tilde{x}$, with the latter given by (6.15). Define the stopping time $\tau_\epsilon := \tau_\epsilon(x) := \inf\{t \geq 0 : X_t^x = \tilde{x} + \epsilon\}$ and observe that by the strong Markov property, we have

$$\widehat{u}(x, z) = \sup_{\tau} \mathbb{E}_{x, z}^{\mathbb{Q}} \left[e^{-r\tau_\epsilon} \widehat{u}(\tilde{x} + \epsilon, Z_{\tau_\epsilon}) \mathbb{1}_{\{t > \tau_\epsilon\}} + \int_0^{\tau_\epsilon \wedge t} e^{-rt} g(X_t, Z_t) dt \right]. \tag{A.8}$$

Moreover, we let $\tilde{\tau} := \tilde{\tau}(x) := \inf\{t > 0 : X_t^x = \tilde{x}\}$, and for

$$\tau' := \tau^*(x, z) = \inf\{t > 0 : (X_t^x, Z_t^z) \in S_3\},$$

we obtain

$$\widehat{u}(x - \epsilon, z) \geq \mathbb{E}^{\mathbb{Q}}_{x-\epsilon, z} \left[e^{-r\tilde{\tau}(x-\epsilon)} \widehat{u}(\tilde{x}, Z_{\tilde{\tau}(x-\epsilon)}^z) \mathbb{1}_{\{\tau' > \tilde{\tau}(x-\epsilon)\}} + \int_0^{\tau' \wedge \tilde{\tau}(x-\epsilon)} e^{-rt} g(X_t, Z_t) dt \right]. \tag{A.9}$$

Notice that $\tau_\epsilon(x) = \tilde{\tau}(x - \epsilon)$ and τ' is optimal for (A.8). Hence subtracting (A.9) from (A.8) yields

$$\widehat{u}(x, z) - \widehat{u}(x - \epsilon, z) \leq \mathbb{E}^{\mathbb{Q}} \left[e^{-r\tau_\epsilon} (\widehat{u}(\tilde{x} + \epsilon, Z_{\tau_\epsilon}^z) - \widehat{u}(\tilde{x}, Z_{\tau_\epsilon}^z)) \mathbb{1}_{\{\tau' > \tau_\epsilon\}} + \mathbb{E}^{\mathbb{Q}} \left[\int_0^{\tau_\epsilon \wedge \tau'} e^{-rt} (g(X_t^x, Z_t^z) - g(X_t^{x-\epsilon}, Z_t^z)) dt \right] \right].$$

Since $(\tilde{x} + \epsilon, Z_{\tau_\epsilon}^z) \in \mathcal{C}_3$ on $\{\tau' > \tau_\epsilon\}$ and $\widehat{u}_x \leq 0$ in \mathcal{C}_3 (see Proposition 6.5), we must have

$$\widehat{u}(\tilde{x}, Z_{\tau_\epsilon}^z) \geq \widehat{u}(\tilde{x} + \epsilon, Z_{\tau_\epsilon}^z),$$

and so we obtain

$$\widehat{u}(x, z) - \widehat{u}(x - \epsilon, z) \leq \mathbb{E}^{\mathbb{Q}} \left[\int_0^{\tau_\epsilon \wedge \tau'} e^{-rt} (g(X_t^x, Z_t^z) - g(X_t^{x-\epsilon}, Z_t^z)) dt \right].$$

If we now divide by $\epsilon > 0$ and let $\epsilon \downarrow 0$, we obtain (since $\tau_\epsilon \downarrow \tilde{\tau}$ and $\tau' = \tau^*(x, z)$)

$$\widehat{u}_x(x, z) \leq \mathbb{E}^{\mathbb{Q}} \left[\int_0^{\tilde{\tau} \wedge \tau'} e^{-rt} g_x(X_t^x, Z_t) dt \right].$$

In the next step, we assume by contradiction that there exists $(x_0, z_0) \in \partial\mathcal{C}_3$ with $\mathbb{Q}_{x_0, z_0}[\sigma_* > 0] = 1$ and take an increasing sequence $x_n \uparrow x_0$ such that $x_n > \tilde{x}$ for all $n \in \mathbb{N}$, which is possible due to Assumption 4.1. Let $\tau_n := \tau^*(x_n, z_n)$ and notice that $\tau_n = \sigma_n := \sigma_*(x_n, z_0)$ for all $n \in \mathbb{N}$ due to continuity of paths. Furthermore, σ_n decreases in n and $\sigma_n \geq \sigma_* := \sigma_*(x_0, z_0)$ since $x \mapsto X_t^x$ is increasing. Set $\tilde{\tau}^n := \tilde{\tau}(x_n)$ and notice that $\tilde{\tau}^n \uparrow \tilde{\tau}$. Moreover, we let $\sigma^\infty := \lim_{n \rightarrow \infty} \sigma_n$ and have

$$\sigma^\infty \wedge \tilde{\tau} = \lim_{n \rightarrow \infty} (\sigma_n \wedge \tilde{\tau}^n) \geq \sigma_* \wedge \tilde{\tau} \quad \mathbb{Q}\text{-a.s.}$$

We then obtain

$$\begin{aligned} \widehat{u}_x(x_0-, z_0) &= \lim_{n \rightarrow \infty} \widehat{u}_x(x_n, z_0) \leq \lim_{n \rightarrow \infty} \mathbb{E}^{\mathbb{Q}} \left[\int_0^{\tilde{\tau} \wedge \sigma_n} e^{-rt} g_x(X_t^{x_n}, Z_t^{z_0}) dt \right] \\ &= \mathbb{E}^{\mathbb{Q}} \left[\int_0^{\tilde{\tau} \wedge \sigma^\infty} e^{-rt} g_x(X_t^{x_0}, Z_t^{z_0}) dt \right] < 0, \end{aligned}$$

where we used $x_0 > \tilde{x}$ as well as $\tilde{\tau} \wedge \sigma^\infty > 0$ thanks to our assumption that $\mathbb{Q}_{x_0, z_0}[\sigma^\infty \geq \sigma_* > 0] = 1$. As this contradicts Lemma A.3, the claim follows. \square

As a corollary of Lemma A.2 and Proposition A.4, we obtain

Corollary A.5 *Assume that $\mu_0 + \mu_1 < 0$ and $r > \frac{\gamma}{2\sigma}|\mu_0 + \mu_1|$. Then for all $(x, z) \in \mathbb{R}^2$, we have*

$$\mathbb{Q}_{x,z}[\tau^* = \sigma_* = \widehat{\sigma}_*] = 1.$$

This result allows us to state a continuity result for the optimal stopping time with respect to the initial data.

Lemma A.6 *Assume that $\mu_0 + \mu_1 < 0$ and $r > \frac{\gamma}{2\sigma}|\mu_0 + \mu_1|$. Then we have*

$$\lim_{n \rightarrow \infty} \tau^*(x_n, z_n) = \tau^*(x, z)$$

for any point $(x, z) \in \mathbb{R}^2$ and for any sequence $(x_n, z_n) \rightarrow (x, z)$. In particular, if $(x, z) \in \partial\mathcal{C}_3$, the limit is zero.

Proof Let $(x, z) \in \mathbb{R}^2$ and denote $\tau_n := \tau^*(x_n, z_n)$ as well as $\tau := \tau^*(x, z)$ for simplicity. In order to show lower semicontinuity, we fix $\omega \in \Omega$ outside of a nullset. For $\tau(\omega) = 0$, we are finished and thus assume $\tau(\omega) > \delta > 0$. Due to Proposition 6.8, there exists $k_{\delta,\omega} > 0$ such that $c(Z_t(\omega)) - X_t(\omega) > k_{\delta,\omega}$ for all $t \in [0, \delta]$. The map $(t, x, z) \mapsto c(Z_t^z(\omega)) - X_t^x(\omega)$ is uniformly continuous on any compact $[0, \delta] \times K$; hence we can find $N_\omega \geq 1$ such that for all $n \geq N_\omega$ and $t \in [0, \delta]$, $c(Z_t^{z_n}(\omega)) - X_t^{x_n}(\omega) > k_{\delta,\omega}$, and therefore $\liminf_{n \rightarrow \infty} \tau_n(\omega) \geq \delta$. Since ω and δ were arbitrary, we obtain $\liminf_{n \rightarrow \infty} \tau_n \geq \tau$ \mathbb{Q} -a.s. and thus lower semicontinuity. By employing similar arguments, we can show $\limsup_{n \rightarrow \infty} \widehat{\sigma}_n \leq \widehat{\sigma}$ \mathbb{Q} -a.s. and the claim thus follows together with Corollary A.5. \square

Before we finally state the proof of Proposition 6.9, we give a probabilistic representation of v_x by arguments similar to those in the proof of Lemma 6.4.

Lemma A.7 *For all $(x, z) \in \mathbb{R}^2 \setminus \partial\mathcal{C}_3$, we have*

$$\begin{aligned} & \widehat{v}_x(x, z) \\ &= \mathbb{E}_{x,z}^{\mathbb{Q}} \left[e^{-r\tau^*} \left(e^{X_{\tau^*}} \left(1 + e^{\frac{\gamma}{\sigma}(X_{\tau^*} + Z_{\tau^*})} \right) + \frac{\gamma}{\sigma} (e^{X_{\tau^*}} - \kappa) e^{\frac{\gamma}{\sigma}(X_{\tau^*} + Z_{\tau^*})} \right) \mathbb{1}_{\{\tau^* < \infty\}} \right]. \end{aligned}$$

We are now ready to prove Proposition 6.9.

Proof of Proposition 6.9 The first statement trivially holds true for $(x, z) \in \text{int}(\mathcal{S}_3)$ and $(x, z) \in \mathcal{C}_3$, due to Lemma 6.2. It thus remains to prove that $\nabla_{x,z} \widehat{v}$ is continuous across the boundary $\partial\mathcal{C}_3$. Let $(x_0, z_0) \in \partial\mathcal{C}_3$ and take a sequence $(x_n, z_n) \rightarrow (x_0, z_0)$ with $\tau_n := \tau^*(x_n, z_n)$. For a fixed $t > 0$, we notice that $(X_t, Z_t) \in \mathcal{C}_3$ on $\{\tau_n > t\}$ and

thus, upon using the tower and the Markov property, we obtain

$$\begin{aligned} &\widehat{v}_x(x_n, z_n) \\ &= \mathbb{E}_{x_n, z_n}^{\mathbb{Q}} \left[e^{-r\tau_n} \left(e^{X_{\tau_n}} \left(1 + e^{\frac{\gamma}{\sigma}(X_{\tau_n} + Z_{\tau_n})} \right) + \frac{\gamma}{\sigma} (e^{X_{\tau_n}} - \kappa) e^{\frac{\gamma}{\sigma}(X_{\tau_n} + Z_{\tau_n})} \right) \mathbb{1}_{\{\tau_n \leq t\}} \right] \\ &\quad + \mathbb{E}_{x_n, z_n}^{\mathbb{Q}} [e^{-rt} \widehat{v}_x(X_t, Z_t) \mathbb{1}_{\{\tau_n > t\}}]. \end{aligned}$$

Due to Assumption 4.1, we can invoke dominated convergence as well as Lemma A.6 to obtain

$$\begin{aligned} \lim_{n \rightarrow \infty} \widehat{v}_x(x_n, z_n) &= e^{x_0} (1 + e^{\frac{\gamma}{\sigma}(x_0 + z_0)}) + \frac{\gamma}{\sigma} (e^{x_0} - \kappa) e^{\frac{\gamma}{\sigma}(x_0 + z_0)} \\ &= \frac{\partial}{\partial x} \left((e^x - \kappa) (1 + e^{\frac{\gamma}{\sigma}(x+z)}) \right) \Big|_{(x_0, z_0)} \end{aligned}$$

and hence the continuity of \widehat{v}_x across the optimal boundary. The continuity of \widehat{v}_z across the free boundary follows similarly. For the last claim, we observe that Lemma 6.2 implies

$$\frac{1}{2} \sigma^2 \widehat{v}_{xx}(x, z) = r \widehat{v}(x, z) - \mu_0 \widehat{v}_x(x, z) + \frac{1}{2} (\mu_0 + \mu_1) \widehat{v}_z(x, z) \tag{A.10}$$

for all $(x, z) \in \mathcal{C}_3$. But the right-hand side of (A.10) only involves functions which are continuous on \mathbb{R}^2 ; hence we deduce that \widehat{v}_{xx} admits a continuous extension to $\overline{\mathcal{C}}_3$ and is therefore bounded there. It follows that $\widehat{v}_x(\cdot, z)$ is locally Lipschitz-continuous on $\overline{\mathcal{C}}_3$, with a Lipschitz constant $K(z)$ that is locally bounded on \mathbb{R} . Because $\widehat{v}_x(\cdot, z)$ is infinitely many times continuously differentiable in the stopping region \mathcal{S}_3 (and hence locally bounded there as well), we conclude that $\widehat{v}_{xx} \in L^\infty_{\text{loc}}(\mathbb{R}^2)$. \square

Appendix B: Proof of Proposition 6.10

Let $R > 0$ and define $\tau_R := \inf\{t \geq 0 : |X_t| \geq R \text{ or } |Z_t| \geq R\}$. Since $\widehat{v} \in C^1(\mathbb{R}^2)$ and $\widehat{v}_{xx} \in L^\infty_{\text{loc}}(\mathbb{R}^2)$, we can apply a weak version of Itô’s lemma (see e.g. Bensoussan and Lions [8, Lemma 2.8.1 and Theorem 2.8.5]) up to the stopping time $\tau_R \wedge T$ for some $T > 0$, which results in

$$\begin{aligned} \widehat{v}(x, z) &= \mathbb{E}_{x, z}^{\mathbb{Q}} \left[e^{-r(\tau_R \wedge T)} \widehat{v}(X_{\tau_R \wedge T}, Z_{\tau_R \wedge T}) \right. \\ &\quad \left. - \int_0^{\tau_R \wedge T} e^{-rs} (\mathcal{L}_{X, Z} - r) \widehat{v}(X_s, Z_s) ds \right]. \end{aligned} \tag{B.1}$$

The right-hand-side of (B.1) is well defined because Z is deterministic, X has an absolutely continuous transition density and $\mathcal{L}_{X, Z} \widehat{v}$ is defined up to a set of zero Lebesgue measure. Since \widehat{v} solves the free-boundary problem (6.28), we have

$$\begin{aligned} (\mathcal{L}_{X, Z} - r) \widehat{v}(x, z) &= (\mathcal{L}_{X, Z} - r) \widehat{v}(x, z) \mathbb{1}_{\{x < c(z)\}} + (\mathcal{L}_{X, Z} - r) \widehat{v}(x, z) \mathbb{1}_{\{x \geq c(z)\}} \\ &= g(x, z) \mathbb{1}_{\{x \geq c(z)\}} \end{aligned}$$

for almost all $(x, z) \in \mathbb{R}^2$. Using again that the transition density of X is absolutely continuous with respect to Lebesgue measure, (B.1) becomes

$$\widehat{v}(x, z) = \mathbb{E}_{x,z}^{\mathbb{Q}} \left[e^{-r(\tau_R \wedge T)} \widehat{v}(X_{\tau_R \wedge T}, Z_{\tau_R \wedge T}) - \int_0^{\tau_R \wedge T} e^{-rs} g(X_s, Z_s) \mathbb{1}_{\{x \geq c(z)\}} ds \right].$$

Now upon employing a change of measure as in Sect. 5, we obtain

$$\begin{aligned} & \mathbb{E}_{x,z}^{\mathbb{Q}} [e^{-r(\tau_R \wedge T)} |\widehat{v}(X_{\tau_R \wedge T}, Z_{\tau_R \wedge T})|] \\ &= \mathbb{E}_{x,z}^{\mathbb{Q}} [e^{-r(\tau_R \wedge T)} |\bar{v}(X_{\tau_R \wedge T}, e^{\frac{\gamma}{\sigma}(X_{\tau_R \wedge T} + Z_{\tau_R \wedge T})})|] \\ &\leq K_1 \mathbb{E}_{x, \exp(\frac{\gamma}{\sigma}(x+z))}^{\mathbb{Q}} [e^{-r(\tau_R \wedge T)} e^{X_{\tau_R \wedge T}} (1 + \Phi_{\tau_R \wedge T})] \\ &= K_1 (1 + e^{\frac{\gamma}{\sigma}(x+z)}) \mathbb{E}_{x,\pi} [e^{-r(\tau_R \wedge T)} e^{X_{\tau_R \wedge T}}], \end{aligned} \tag{B.2}$$

where $\pi = e^{\frac{\gamma}{\sigma}(x+z)} / (1 + e^{\frac{\gamma}{\sigma}(x+z)})$. Due to Assumption 4.1, it is easy to verify that taking limits in (B.2) yields

$$\lim_{T \uparrow \infty} \lim_{R \uparrow \infty} \mathbb{E}_{x,z}^{\mathbb{Q}} [e^{-r(\tau_R \wedge T)} \widehat{v}(X_{\tau_R \wedge T}, Z_{\tau_R \wedge T})] = 0. \tag{B.3}$$

Furthermore,

$$\begin{aligned} & \mathbb{E}_{x,z}^{\mathbb{Q}} \left[\int_0^{\tau_R \wedge T} e^{-rs} g(X_s, Z_s) \mathbb{1}_{\{x \geq c(z)\}} ds \right] \\ &\leq \mathbb{E}_{x,z}^{\mathbb{Q}} \left[\int_0^{\infty} e^{-rs} |g(X_s, Z_s)| ds \right] \\ &\leq \mathbb{E}_{x, \exp(\frac{\gamma}{\sigma}(x+z))}^{\mathbb{Q}} \left[\int_0^{\infty} e^{-rs} \left(e^{X_s} \left(r - \frac{1}{2} \sigma^2 - \mu_0 \right) \right. \right. \\ &\qquad \qquad \qquad \left. \left. + rk + \Phi_s \left(e^{X_s} \left(r - \frac{1}{2} \sigma^2 - \mu_1 \right) + rk \right) \right) ds \right] \\ &\leq \mathbb{E}_{x, \exp(\frac{\gamma}{\sigma}(x+z))}^{\mathbb{Q}} \left[\int_0^{\infty} e^{-rs} \left(e^{X_s} \left(r - \frac{1}{2} \sigma^2 - \mu_0 \right) + rk \right) ds \right] \\ &\quad + (1 + e^{\frac{\gamma}{\sigma}(x+z)}) \mathbb{E}_{x,\pi} \left[\int_0^{\infty} e^{-rs} \left(e^{X_s} \left(r - \frac{1}{2} \sigma^2 - \mu_1 \right) + rk \right) ds \right] < \infty, \end{aligned} \tag{B.4}$$

where $\pi = e^{\frac{\gamma}{\sigma}(x+z)} / (1 + e^{\frac{\gamma}{\sigma}(x+z)})$ and the last inequality follows again from Assumption 4.1. Hence given the finiteness of the expectation in (B.4), we can apply dominated convergence to interchange expectation and limits as $R \uparrow \infty$ and $T \uparrow \infty$. Combining this with (B.3) gives (6.29), which completes our proof. \square

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