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## A Better Cycle-Breaker for Swiss Democracy?

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# A Better Cycle-Breaker for Swiss Democracy?

## Abstract

When a counter-proposal is made to an initiative to change the Swiss constitution, the citizenry makes three binary majority choices: the initiative versus the status quo, the initiative versus the counter-proposal, and the status quo versus the counterproposal as a tie-breaker. If there is a cycle, the alternative that beats the status-quo wins. This system invites strategic voting, as exemplified by the 2010 case of the “Ausschaffungsinitiative”. We suggest to break cycles differently by choosing the middle alternative in case of a cycle, which will normally be the counter-proposal. More precisely, we show that there always exists a strong Nash equilibrium in which all citizens vote sincerely. Moreover, the outcome of all alternative strong equilibria with strategic voting is the same as if everybody votes sincerely. We also show that other common cycle-breaker rules cannot achieve the same result.

JEL-Codes: C720, D700, D720.

Keywords: Swiss democracy, three-way referendum, Condorcet Winner, manipulation, information sharing, initiative.

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# 1 Introduction

The Swiss democratic system has a procedure called “initiative”. Initiatives are proposals submitted by a group of citizens to change the constitution. At least 100’000 signatures are required to support an initiative before it can be voted on. The procedure also allows the parliament to make a counter-proposal<sup>1</sup>. If an initiative (say A) and a counter-proposal (say B) are made to change the status-quo (S), then the citizenry<sup>2</sup> has three binary choices:

A versus S,  
B versus S,  
A versus B.

If either A or B wins against S but not both simultaneously, this alternative will be implemented. If both A and B win against S, then the third binary choice between A and B (the so-called “tie-breaker”) will be used to determine the winner. Hence, the third binary choice is only relevant if both A and B win against the status-quo. If there is a cycle of the majority preferences, i.e., the citizenry expresses one of the two preferences

$A \succ S \succ B \succ A,$   
 $B \succ S \succ A \succ B,$

then the alternative which wins against the status-quo is chosen, i.e., A wins in the first case and B wins in the second case (see Schweizerische Eidgenossenschaft (1999)), where  $\succ$  denotes the preference relationship.

This voting scheme might invite strategic voting, as we discuss in Section 6. This is the starting point of our investigation. We build a simple model of a citizenry deciding by pairwise majority voting about three alternatives. We assume that the three alternatives can be ordered in the following sense: A is a proposal for a more drastic change of the status-quo than B. Typically, counter-proposals by the government are more moderate than initiatives (Linder and Mueller (2021), Ford and Kemokai (2014))<sup>3</sup>. Specifically, we

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<sup>1</sup>Schweizerische Eidgenossenschaft (1999).

<sup>2</sup>To approve either an initiative or a counter-proposal to change the status-quo, a so-called “double majority” support of citizens and cantons is required (see Gersbach (2009) and Gersbach (2024)). For a description of requirements, see also Schweizerische Eidgenossenschaft (1999).

<sup>3</sup>See also [https://de.wikipedia.org/wiki/Gegenentwurf\\_\(Schweiz\)](https://de.wikipedia.org/wiki/Gegenentwurf_(Schweiz)) (accessed on November 22, 2023).

order the three alternatives on a line and assume citizens have single-peaked preferences regarding these three alternatives. We look at voting outcomes where citizens cast their votes simultaneously. Moreover, we require that a voting equilibrium must be robust against deviations by groups of voters with the same preferences. It will turn out that the equilibria are robust against deviations from any coalition of citizens, and thus they are strong Nash equilibria in the sense of Aumann (1959).

Our main insights are as follows. First, we find states of nature in which strategic voting occurs and no equilibrium exists in which all groups vote sincerely for the most common cycle-breaking rules. The famous case of the “Ausschaffungsinitiative” in 2010<sup>4</sup> illustrates this case, as we describe in Section 6. Second, we suggest to break cycles differently. In particular, we show that the impact of strategic voting can be avoided if the middle alternative is chosen by default in case of a cycle. More precisely, we show that there always exists a strong Nash equilibrium in which everybody votes sincerely. Moreover, the outcome of any other strong Nash equilibrium is the same as if everybody voted sincerely and thus all equilibria are equivalent in this sense to the sincere one. Third, we show that other commonly known cycle-breaking rules cannot achieve the same result.

## 2 Relation to the literature

Our paper is related to two strands of the literature. On the one hand, since Arrow (1951) put the Condorcet Paradox in a general formal structure and established his Impossibility Theorem, a large literature has examined ways to select alternatives when agents take majority pairwise decisions with more than two alternatives and there is a cyclic outcome without any Condorcet Winner.

Some important rules resolve cycles by focusing on the *size* of majority victories. This size is, for instance, measured by the majority *margin* between candidates  $x$  and  $y$ , defined as the number of voters who prefer  $x$  to  $y$  minus the number of those who prefer  $y$  to  $x$ . The Ranked Pairs are an example of such a rule (Tideman (1987), Zavist and Tideman (1989)), together with River (Heitzig (2004)), and Beat Path (Schulze (2011), Schulze (2018)). Other examples are Weighted Covering (Dutta and Laslier (1999), Pérez-Fernández and

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<sup>4</sup>Schweizerische Eidgenossenschaft (2010) .

De Baets (2018)), and Split Cycle rules (Holliday and Pacuit (2021), Ding et al. (2022)). Alternative ways to handle cycles are to resort to other aggregation rules such as the Copeland Score (Rubinstein (1980)) and Borda Scores (Nitzan and Rubinstein (1981), Mihara (2017)). We contribute to this literature by exploring ways to break cycles when three alternatives can be ordered linearly.

On the other hand, we focus on one aspect of the Swiss democratic system. There is a very large literature exploring the procedures and outcomes of this system. Standard references are Linder et al. (2010)<sup>5</sup>, Kriesi et al. (2006, 2008), Feld and Kirchgässner (2000), and Frey (1994). Recent contributions on voting on initiatives in this system are, for instance, Leemann (2015), Leemann and Wasserfallen (2016), and Eichenberger and Schafer (2022), who examine how Swiss-type election procedures impact typical political features.

The paper is organized as follows. In Section 3, the model and working definitions are introduced. In Section 4, we show that strategy-proofness of a three-alternative pairwise majority vote may be achieved by selecting the middle alternative as a cycle-breaker when utility functions are single-peaked, and that the outcome of any strong Nash equilibrium is the same as if all voters vote sincere voting. In Section 5, alternative cycle-breakers are explored and we show that they are not strategy-proof. Finally, in Section 6, we show that the Swiss system is not strategy-proof and how this is exemplified through the Ausschaffungsinitiative. Section 7 concludes.

### 3 The model

A society collectively decides to choose a one-dimensional linear policy  $x \in \mathbb{R}$ . The society consists of a set of citizens with mass one, that is, it is modeled as a probability space. Each citizen has a unique *type*, defined as a point  $z \in \mathbb{R}$  in the same space as the policy  $x \in \mathbb{R}$ . The type of a citizen represents his/her preferred policy, i.e., that policy from which s/he derives maximal utility. It is convenient to refer to a citizen of type  $z$  as *citizen*  $z$ . Thus, the citizen type is modeled as a random variable  $Z \in \mathbb{R}$  of the probability space

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<sup>5</sup>A more detailed discussion about the Swiss System can be found in Linder and Mueller (2012), Kriesi (2005), and Klöti (2006).

of the population. We denote the distribution function of  $Z$  by  $F$ , that is,

$$F(z) = \mathbb{P}(Z \leq z),$$

and the *utility function* of citizens by  $u : \mathbb{R}^2 \rightarrow \mathbb{R}$ , where  $u(x, z)$  represents the utility derived by a citizen of type  $z$  from policy  $x$ . In particular, citizen  $z$ 's utility from policy  $x$  is given by

$$u_z : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto u(x, z).$$

Of course, citizens aim to maximize their own utility gain through their voting strategy. Throughout, we assume that  $u : \mathbb{R}^2 \rightarrow \mathbb{R}$  is a continuous function and that  $u_z(x)$  has a unique global maximum at  $x = z$ , for all types  $z \in \mathbb{R}$ . We adopt the convention of referring to the pair  $(F, u)$  of a distribution  $F$  together with a utility function  $u$  as a *state of nature*.

We are interested in the conditions that the utility functions  $u_z$  must additionally satisfy to ensure that the procedure is strategy-proof. The fact that we must ask additional conditions from the utility functions to obtain a manipulation-proof procedure follows directly from the classic Gibbard-Satterthwaite Theorem: if the members of the electorate have arbitrary non-single-peaked preferences, binary majority voting with three alternatives may be manipulated whatever the cycle-breaker is. It turns out that this condition is “single-peakedness”, defined as follows:

**Definition 1 (Single-peaked utility functions)** *The utility function  $u_z$  is single-peaked if it is strictly increasing for  $x \leq z$  and strictly decreasing for  $z \leq x$ , i.e.,  $u_z(x) < u_z(y)$  for all  $x < y < z$  or  $z < y < x$ .*

We note that no assumptions are made on the distribution  $F$  of the citizen type, which in particular includes both a discrete and continuous setting. Nonetheless, we will in general think of it as continuous to obtain a description of the game in terms of the citizen type distribution rather than in terms of specific citizens.

We also assume that, prior to the voting round, each citizen is informed of his/her own private type  $z$  as well as of the state of nature, i.e., of the global citizen type distribution and of the shape of the utility functions of others (or, equivalently, of the true preferences of the other types). Thus, the state of nature  $(F, u)$  is public knowledge.

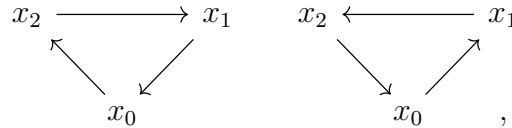


We will denote the three policy alternatives  $x_0, x_1, x_2 \in \mathbb{R}$  such that  $x_0 < x_1 < x_2$ , that is, the three competing policy proposals are labelled following the natural policy order<sup>6</sup>. Each voter is asked to submit his/her pairwise preferences over the three alternatives  $x_0, x_1, x_2$ , i.e., their preference between  $x_0$  and  $x_1$ , between  $x_1$  and  $x_2$ , and between  $x_0$  and  $x_2$ . We propose the following procedure to decide the final outcome: if a majority pairwise prefers an alternative  $x_i$  for some  $i = 0, 1, 2$  over both other alternatives, i.e.,  $x_i$  is the Condorcet Winner, then  $x_i$  is the outcome of the voting process. If, on the contrary, the outcome produces a cycle, then one of the alternatives is declared to be the winner and this alternative is called the *default alternative*. In particular, this is a case of the procedures that we call *Condorcet-consistent*:

**Definition 2 (Condorcet-consistent voting procedure)** *A Condorcet-consistent voting procedure is one where the outcome is the Condorcet Winner whenever it exists.*

Different Condorcet-consistent voting procedures may differ on the procedure to break cyclic outcomes with no Condorcet Winner. This can be done by pre-selecting a default alternative as described above or through some other device.

The outcome of such a three-way pairwise voting can be easily represented via diagrams where an arrow  $x \rightarrow y$  means that the proposal  $y$  wins against the proposal  $x$  in that particular binary choice. For example, the two possible cyclic or non-Condorcet outcomes are:



which we call the “clockwise” and “counter-clockwise” cycles, respectively. In this paper, we study which cycle-breaking mechanisms provide a strategy-proof procedure, focusing on those that select a default alternative as the outcome of the cyclic results. We also consider procedures where the default outcome of each of the two cycles is different.

To make the notion of “strategy-proofness” precise, we first define sincere voting and our notion of voting equilibrium.

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<sup>6</sup>Following the notation from the introduction for the alternatives of a Swiss-type procedure, with a proposal  $A$ , a counter-proposal  $B$ , and a status-quo  $S$ , with this new notation we may assume that  $A \equiv x_2$ ,  $B \equiv x_1$ , and  $S \equiv x_0$ .

**Definition 3 (Sincere voting)** *Given a voting procedure, sincere voting is the set of strategies where all voters submit all pairwise votes according to their true preferences.*

This means that a voter who expects the most utility in alternative  $x_i$ , followed by  $x_j$ , and expects the least utility from  $x_k$ , would accordingly vote for  $x_i$  against  $x_j$  and against  $x_k$ , and similarly would vote for  $x_j$  over  $x_k$ . Reciprocally, by the *manipulation* of a voting procedure we refer to any strategy deviating from sincere voting.

We next describe the game and the equilibrium concept. The game is a simple static game in which citizens vote simultaneously and the winning alternative is selected by a particular procedure. As we said, there is no information sharing between citizens and the only public information is the state of nature. We will examine different selection procedures in the following sections.

We look for Nash equilibria in which citizens of the same type (i.e., with the same preferences) vote in the same way as a best response. We require that an equilibrium be robust to deviations of groups of voters with the same preferences, and even to entire coalitions of groups of citizens. Hence, we require that equilibria are strong Nash equilibria in the sense of Aumann (1959). Henceforth, we simply call such strong equilibria *voting equilibria*. For concreteness, we recall here the definition of strong Nash equilibria:

**Definition 4 (Strong Nash equilibrium)** *A Nash equilibrium is strong if there exists no subset of players that could simultaneously deviate in a way that strictly benefits all the members of the subset, assuming the strategies of all non-members remain the same.*

In our context, this means that there exists no subset of groups of voters, i.e., a coalition, with a new strategy for each member of the coalition, such that all members of the coalition obtain strictly more utility if they all follow the new strategy, assuming the others vote in the same way as before.

We note that the assumption that all citizens with the same preferences vote in the same way is no loss of generality. Intuitively, if within a group with common preferences there is a splitting of the votes between the two alternatives of a given pairwise comparison, the same outcome is obtained by jointly voting for the winning alternative, assuming the others' voting strategies are fixed. For formal proofs that deviations of groups of voters

with the same preferences are also Nash equilibria if deviations by individual voters are considered, see Tsakas and Xeferis (2023). Finally, to simplify the exposition, we assume that exact ties regarding voting outcomes do not occur. This avoids the necessity to deal with ties.

Thus, the aim of the paper is to establish the conditions under which manipulation is never optimal. This is captured by the following definition:

**Definition 5 (Strategy-proof voting procedure)** *A voting procedure is strategy-proof if sincere voting is a voting equilibrium, i.e., if each group of voters with common preferences finds it optimal to vote sincerely, assuming all others do so, and even if coalitions are allowed.*

As we already advanced, we establish in Section 4.2 that the voting procedure is strategy-proof for single-peaked utility functions if the alternative  $x_1$  is the default alternative. We also establish in Section 4.3 that, under the same hypotheses, the outcome of every strong Nash equilibrium coincides with the outcome of sincere voting. Furthermore, in Section 5 we see that other common voting schemes are not strategy-proof. We conclude in Section 6 by providing a real example of the manipulation described in our proofs when the procedure is not Condorcet-consistent, the so-called “Ausschaffungsinitiative” that was put to a vote in Switzerland in 2010.

## 4 Middle alternative as cycle-breaker

### 4.1 The starting point

In this section, we examine voting equilibria when the middle alternative is the default alternative if there is a cycle, that is, when the middle alternative  $x_1$  is the cycle-breaker. In principle, there are six ways to rank-order the alternatives  $\{x_0, x_1, x_2\}$  by some preference order  $\succ$  (where  $x_i \succ x_j$  if  $x_i$  is preferred to  $x_j$ ):

- Group  $a$ :  $x_2 \succ x_1 \succ x_0$ ,
- Group  $b$ :  $x_1 \succ x_2 \succ x_0$ ,
- Group  $c$ :  $x_1 \succ x_0 \succ x_2$ ,

- Group  $d$ :  $x_0 \succ x_1 \succ x_2$ ,
- Group  $e$ :  $x_2 \succ x_0 \succ x_1$ ,
- Group  $f$ :  $x_0 \succ x_2 \succ x_1$ .

As we explained in the previous section, we may assume that all citizens within a group with common preferences will coordinate and act as a single player. Thus, the setting is a voting game of 6 players  $a$  to  $f$  where, depending on the state of nature  $(F, u)$ , each player is assigned a different population mass given by  $F$ . In this setting and being aware of the other players' preferences and masses, each player votes to maximize his/her utility. In the following we use the convention of also denoting by  $a, b, c, d, e$ , or  $f$ , respectively, the mass associated with each of these groups.

However, groups  $e$  and  $f$  are inconsistent with the assumption about single-peaked utility functions  $u_z$ : if utility functions  $u_z$  are single-peaked for almost all  $z \in Z$  (according to the distribution  $F$ ), then groups  $e$  and  $f$  have mass zero, i.e.,  $e = f = 0$ . The converse is also true for functions  $u_z$  with unique global maximum at  $z$ : if for all  $x_0 < x_1 < x_2$  the groups  $e$  and  $f$  do not occur, in the sense that the sets of individuals with such preferences have mass zero, then the utility functions  $u_z$  are single-peaked for almost all  $z$ , i.e., up to subsets of  $F$ -mass zero. To see it, we observe the contra-positive statement: if there exists a non-zero mass of  $z$ 's such that  $u_z(x)$  does not satisfy the single-peaked assumption, then there exist  $x_0 < x_1 < x_2$  such that these citizens fall into group  $e$ , if  $u_z$  is not strictly increasing for  $x \leq z$ , or into group  $f$ , if  $u_z$  is not strictly decreasing for  $z \leq x$ .

Hence, under the assumption of single-peaked utility functions, we can restrict our attention to groups  $a, b, c$  and  $d$ . We study this setting in the next two sections.

## 4.2 Sincere voting equilibria

We emphasize again that all citizens which share policy preferences can and will coordinate their votes, so that, for single-peaked utility functions, we have four players:  $a, b, c$ , and  $d$ . To verify strategy-proofness we have to identify profitable deviations from sincere voting by each of the groups, assuming that the remaining three groups vote sincerely.

**Proposition 1** *Assume that utility functions are single-peaked and that the voting proce-*

*dure is Condorcet-consistent. Then, if the default alternative is the middle alternative  $x_1$ , there is a voting equilibrium in which all citizens vote sincerely.*

**Proof.** According to our definition of voting equilibria given in Section 3, we have to prove that sincere voting is a strong Nash equilibrium. That is, it is optimal if the other groups vote sincerely, and this remains the case even if we allow the formation of coalitions.

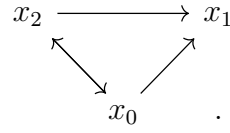
The outline of the argument is as follows: assuming there are no coalitions among groups and that no group is a majority, the outcome of a sincere procedure is always the middle alternative  $x_1$ . Thus, on the one hand groups  $b$  and  $c$ , whose most preferred outcome is  $x_1$ , have no incentive to deviate. On the other hand, setting  $x_1$  as cycle-breaker prevents any scenario where either group  $a$  or  $d$  could, by themselves, force a change of outcome, and thus sincere voting is individually an optimal strategy. To conclude, we rule out coordinated deviation by the members of a coalition observing that no coalition could induce a strict utility gain for all its members at the same time.

We begin with the case where coalitions are not allowed, arguing for a general setting where groups  $e$  and  $f$  need not be empty, to then particularize for single-peaked  $u_z$ , since this will be helpful later on. We note the following: for the sincere strategy, i.e., assuming all groups vote according to their true preferences, the following pairwise voting outcomes will take place if and only if the corresponding groups are a majority (denoting by  $a + b$  the joint total mass of groups  $a$  and  $b$ , and so on):

- $x_1 \rightarrow x_2 \iff a + e + f > 1/2$ ,
- $x_1 \rightarrow x_0 \iff d + e + f > 1/2$ ,
- $x_0 \rightarrow x_2 \iff a + b + e > 1/2$ .

If either of the 6 groups  $a$  to  $f$  is a majority, then, since the procedure is Condorcet-consistent, by voting sincerely they obtain their most desired outcome regardless of the votes of the other groups. Thus, we may assume that none of the individual groups is a majority. Similarly, if the combined mass of a group combination with a common most preferred outcome is a majority, then this majority enforces that outcome by all voting sincerely, so we may also assume that not to be the case. This means that neither  $a + e$ , which prefer  $x_2$ , nor  $d + f$ , which prefer  $x_0$ , nor  $b + c$ , which prefer  $x_1$ , are jointly a majority.

Assume now that  $u_z$  are single-peaked, so that  $e = f = 0$ . Then groups  $a$  and  $d$  are not a majority and both  $x_2$  and  $x_0$  necessarily lose against  $x_1$  under sincere voting, with outcome:



Thus, if all vote sincerely and no proposal is preferred by a majority, the outcome is the middle alternative  $x_1$ . In particular, neither  $b$  nor  $c$  has an incentive to deviate, assuming the other groups vote sincerely (deviation either makes no difference or strictly reduces their utility).

But groups  $a$  and  $d$  are not a majority, so that manipulation by only one of them could never benefit them. To see this, we first take group  $a$ , with preferences  $x_2 \succ x_1 \succ x_0$ . Since they are a minority, the pairwise vote  $x_1 \leftrightarrow x_2$  will always be lost, since all the others vote sincerely against  $x_2$ . Thus,  $x_2$  cannot be the Condorcet Winner, regardless of group  $a$ 's vote. What is more, manipulation from  $a$  at  $x_0 \leftrightarrow x_2$  or  $x_0 \leftrightarrow x_1$  could only either make no difference or produce a strictly worse outcome for them. This is due to the fact that their true vote is already cast trying to prevent that  $x_0$  wins, given the default alternative  $x_1$ . Thus, either group  $a$ 's strategy is irrelevant for the outcome, or it is strictly optimal for  $a$  to vote sincerely. The argument for group  $d$  is symmetric, and thus sincere voting is optimal for all and in particular a Nash equilibrium if no coalitions are allowed.

Consider now what potential coalitions may be formed. For a coalition to be preferred to sincere voting the outcome obtained from the joint coalition strategy must be strictly better for all members of the coalition than the sincere outcome. Thus, a coalition will not contain both  $a$  and  $d$ , since there is no change in the procedure's outcome that would result in a strict utility gain for both  $a$  and  $d$  at the same time. Furthermore, consider a coalition containing  $a$  but not  $d$  (or vice versa). The only case where all members of such a coalition stand to gain something compared to individual play is if  $x_0$  is the outcome with no coalition. But as we have just proven, under sincere voting this only occurs if group  $d$  is a majority, in which case not even a coalition could change the outcome if  $d$  votes sincerely (which is of course optimal for  $d$ ). Finally, a coalition between  $b$  and  $c$  is also superfluous since, both individually and as a coalition, voting for  $x_1$  over  $x_0$  and over  $x_2$  is optimal, while the third comparison vote between  $x_0$  and  $x_2$  does not matter if the other

groups vote sincerely. Thus, sincere voting is robust also against deviation from coalitions of groups, which completes the proof.  $\square$

### 4.3 Equilibria with strategic voting

While we have just shown in Section 4.2 that voting sincerely is a voting equilibrium, in this section we examine other possible strong Nash equilibria. We show that, for single-peaked utility functions and a Condorcet-consistent procedure with default middle alternative, all other possible strong Nash equilibria lead to the same outcome as sincere voting. Strong Nash equilibria rule out implausible Nash equilibria in which groups of citizens are artificially set to vote for their least desired alternative, even if their vote has no influence over the outcome, which as we know may occur in voting games. Alternatively, one may deal with such equilibria by iterative elimination of weakly-dominated strategies for all groups of citizens.

**Proposition 2** *Assume utility functions are single-peaked and the voting procedure is Condorcet-consistent. Then, if the default alternative is the middle alternative  $x_1$ , the outcome of all voting equilibria coincides with the outcome of sincere voting.*

**Proof.** Consider a given strong Nash equilibrium. If either group holds a majority then, as argued before, the optimal strategy of the majority group is to vote sincerely, while the other groups' strategies are irrelevant. In particular, the outcome is the same as if all groups vote sincerely.

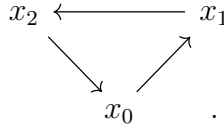
Suppose then that no group is a majority. As in Proposition 1, the outcome of a sincere procedure is guaranteed to be the middle alternative  $x_1$ . In this way, we must only rule out a polar outcome  $x_2$  or  $x_0$ . We argue by contradiction starting with  $x_2$ , and assume that the outcome of the strong Nash equilibrium under consideration is  $x_2$ . By assumption, group  $a$  is not a majority, so that the other groups  $b + c + d$  must be jointly a majority. But then  $b + c + d$  may form a coalition and coordinate to vote for  $x_1$  over  $x_2$  and force  $x_1$  to be the Condorcet Winner. Since this new strategy provides a strict utility gain for all  $b, c$  and  $d$ , this equilibrium could not be strong, and thus the assumption that the outcome is  $x_2$  must be false. By the symmetric argument, it cannot be either  $x_0$ , and thus it must be  $x_1$ , as desired.  $\square$

We emphasize that this result does not imply that the outcome of all Nash equilibria is the same as sincere voting if we don't restrict ourselves to strong equilibria. A trivial counter-example is one where no group is a majority and the strategies of all groups are all equal: to vote for a particular polar alternative, either  $x_2$  or  $x_0$ . No group can change the outcome by itself and thus the strategy is individually optimal, but of course a coalition would prevent this, as argued in the proof above. Similarly, Proposition 2 does not imply that sincere voting is always optimal:

**Example 1** Consider a Condorcet-consistent procedure with default middle alternative and single-peaked utility functions, where the group mass distribution is as follows:

$$a = 30\%, \quad b = 40\%, \quad c = 0\%, \quad d = 30\%.$$

Suppose that  $a$  votes sincerely,  $b$  votes sincerely except at  $x_2 \leftrightarrow x_0$ , voting for  $x_0$  instead, and  $d$  votes sincerely except at  $x_2 \leftrightarrow x_1$ , voting for  $x_2$  instead. We obtain a counter-clockwise cycle and the outcome is  $x_1$ :



This is a Nash equilibria, since group  $a$  cannot change the outcome of  $x_2 \leftrightarrow x_0$  against both  $b$  and  $d$ , and  $x_1$  is its next most preferred outcome . Similarly,  $d$  cannot improve the result, for it cannot modify the outcome of  $x_1 \leftrightarrow x_0$  against  $a$  and  $b$ . Finally,  $b$  gets its most desired outcome, and thus all groups are playing optimally. Since the outcome is the middle alternative this is in fact a strong equilibrium, arguing as before.

In this example sincere voting is however *strictly* worse for  $b$ : if they voted sincerely also at  $x_2 \leftrightarrow x_0$ , voting for  $x_2$  together with  $a$ , then their combined votes would turn  $x_2$  into the Condorcet Winner. It is thus not true that sincere voting is *always* optimal, not even for strong Nash equilibria. However, if groups can coordinate, Proposition 2 states that coalitions ensure that the outcome is the same as if all voted sincerely, making manipulation pointless.



## 5 Alternative cycle-breakers and main result

In this section we explore alternative rules to select the outcome of the three-alternative voting procedure and study whether they entail strategy-proof voting procedures.

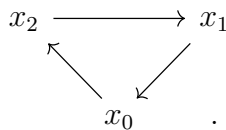
### 5.1 Polar alternatives as cycle-breakers

We start with the case where polar alternatives, i.e., alternatives differing from the middle one  $x_1$ , are the default alternative. We establish that such voting schemes are not strategy-proof and, as in the previous section, we discover a non-sincere Nash equilibrium in the process.

**Proposition 3** *There exist states of nature with single-peaked utility functions where a Condorcet-consistent procedure with a default alternative differing from the middle alternative  $x_1$  is not strategy-proof.*

**Proof.** As at the start of the proof of Proposition 1, we may assume that no group is a majority, so that  $x_1$  is the Condorcet Winner under sincere voting. As we saw, there is no incentive to manipulate if the default alternative is  $x_1$ . To find a scenario where manipulation may occur if the default alternative is not  $x_1$ , we reverse-engineer a situation where a group stands to benefit from such a manipulation versus the sincere case, keeping track of all the requirements for successfully changing the outcome. It turns out that these conditions are not mutually exclusive and that one may find multiple states of nature satisfying all of them.

Assume instead that the default is  $x_2$ . Then, group  $a$  could have an incentive to manipulate and generate a cyclic outcome. Since it cannot enforce a change in the voting result  $x_2 \rightarrow x_1$  if the others vote sincerely, as it is the losing minority of that pairwise comparison, its only hope is to induce a clockwise cyclic outcome:



This means group  $a$  has incentives to manipulate the voting  $x_0 \leftrightarrow x_1$  and to vote  $x_1 \rightarrow x_0$ , together with the sincere preferences of group  $d$ . It will do so only if  $x_2$  wins against  $x_0$ , i.e., if  $a + b$  is a majority, assuming the other groups vote sincerely. For this outcome to

succeed, we thus require that neither  $a$  nor  $b + c$  nor  $d$  are majorities, but  $a + d$  and  $a + b$  are each a majority:

$$a < \frac{1}{2}, \quad b + c < \frac{1}{2}, \quad d < \frac{1}{2}, \quad a + b > \frac{1}{2}, \quad a + d > \frac{1}{2}.$$

Thus, for all states of nature satisfying the above conditions and if the outcome of the clockwise cycle is  $x_2$ , sincere voting is not a Nash equilibrium, since it is strictly suboptimal for group  $a$  to vote sincerely. In particular, such a procedure is not strategy-proof.  $\square$

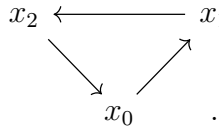
**Example 2** A concrete example of the above manipulation conditions is a state of nature where the population is divided according to

$$a = 40\%, \quad b = c = 15\%, \quad d = 30\%.$$

Clearly, there are infinitely many such states of nature, such as any small perturbation of the values of  $a, b, c$  and  $d$  given above that preserve the relation  $a + b + c + d = 100\%$ .

Actually, continuing the argument of the proof we may compute the optimal adversarial strategies of the other groups and find an actual Nash equilibrium with manipulation. If group  $a$  manipulates in  $x_1$  against  $x_0$ , group  $d$  can reply by also manipulating and voting  $x_0 \rightarrow x_1$  against its true preferences. Since  $a$  is a minority, this implies that the final outcome of this pairwise vote is  $x_0 \rightarrow x_1$ , breaking the cycle desired by  $a$  and making  $x_1$  become again the Condorcet Winner and thus the final outcome. This is, of course, optimal for  $b$  and  $c$ , and as we have seen also for  $d$ , since its preferred outcome  $x_0$  had no chance to be chosen if  $a + b$  is a majority and both  $a$  and  $b$  are voting  $x_0 \rightarrow x_1$ . Finally,  $a$  cannot modify the outcome now in any way, and thus we have a Nash equilibrium. Since the outcome is  $x_1$  it is, as before, a strong Nash equilibrium.

Similarly, if  $x_0$  were the default alternative, group  $d$  would have an incentive to manipulate. By symmetry, its desired outcome would be the counter-clockwise cyclic outcome



The conditions for group  $d$  to manipulate (by voting  $x_1 \rightarrow x_2$ ) are:  $a, b + c$  and  $d$  are not majorities,  $a + d$  and  $c + d$  are majorities. Analogously, an alternative strong Nash

equilibrium may consist in  $b$  and  $c$  voting sincerely,  $a$  manipulating at  $x_2 \rightarrow x_1$ , and  $d$  manipulating at  $x_1 \rightarrow x_2$ , with  $x_1$  the Condorcet Winner.

**Remark 1** We have produced an example of manipulation when the default outcome is  $x_2$ , so that group  $a$  has an incentive to induce the clockwise cycle. However, the result is also valid if the default of the other cycle, the counter-clockwise cycle, is  $x_1$  instead of also  $x_2$ : the same strategies would again comprise a Nash equilibrium, since this modification does not allow an outcome different of  $x_1$ .

Similarly, the symmetric argument is valid if the default outcome of the counter-clockwise cycle is  $x_0$  and the default of the clockwise cycle is either  $x_1$  or  $x_0$ . Thus, we have actually proven a slightly stronger result:

**Corollary 1** *There exist states of nature with single-peaked utility functions where a Condorcet-consistent procedure is not strategy-proof if the default outcome of both clockwise and counter-clockwise cyclic outcomes is not  $x_1$ .*

## 5.2 The main result

Combining Corollary 1 with Proposition 1, we obtain a characterisation of manipulation-proof procedures:

**Theorem 1** *A Condorcet-consistent voting procedure with a default alternative for each cyclic outcome is strategy-proof for all states of nature with single-peaked utility functions if and only if the default is the middle alternative  $x_1$  for both cycles.*

In light of our previous remark about the Gibbard-Satterthwaite Theorem and the fact that manipulation is always possible for arbitrary utility functions, we see that single-peakedness is also a necessary condition, not only sufficient. Thus, we may strengthen the theorem to state that a Condorcet-consistent voting procedure with a default alternative is strategy-proof if and only if the default is the middle alternative  $x_1$  and we only have single-peaked utility functions.

This result supports our proposal that the middle alternative should be used as a cycle-breaker instead of the current system. As the parliament is expected to produce a more moderate version of the initiative, it is expected that the counter-proposal will typically be

the cycle-breaker. In the remainder of this section, we explore two common non-Condorcet-consistent voting procedures and prove that they are not strategy-proof.

In the remainder of this section we explore two common non-Condorcet-consistent set of rules and show that they as well are susceptible to manipulation.

### 5.3 Greatest vote share

The first non-Condorcet-consistent set of rules for a three-alternative voting procedure is to choose the proposal with the *greatest share of votes* supporting it over the total votes in which that alternative has participated. This algorithm generalizes immediately to any number of alternatives, and we introduce it for this more general case.

Consider  $n$  ordered proposed policy alternatives,  $x_0 < x_1 < \dots < x_{n-1}$ . As before, a population with single-peaked utility functions votes, each citizen casting a pairwise preference between every unordered pair of alternatives. In particular, every citizen casts  $\binom{n}{2} = n(n-1)/2$  votes, one for every pair  $\{i, j\}$ ,  $i \neq j$ . We denote by  $x_{ij}$  the fraction of votes in favor of  $j$  against  $i$  in the pairwise comparison  $x_i \leftrightarrow x_j$ , so that  $x_{ji}$  is the fraction in favor of  $i$  against  $j$  and clearly

$$x_{ij} + x_{ji} = 1.$$

We assume that all citizens vote for all comparisons and that there are no invalid votes. The share of votes in favor of policy  $x_i$  over the total votes where  $x_i$  participated, denoted by  $y_i$ , is given by:

$$y_i := \frac{1}{n-1} \sum_{j:j \neq i} x_{ji}.$$

As a convenient notation, we denote by  $\hat{y}_i := \sum_{j:j \neq i} x_{ji}$  the non-normalized vote share. One sees that

$$\sum_i y_i = \frac{1}{n-1} \sum_{i,j:j \neq i} x_{ji} = \frac{1}{n-1} \sum_{i,j:j < i} (x_{ji} + x_{ij}) = \frac{1}{n-1} \binom{n}{2} = \frac{n}{2}.$$

In particular, this means that there always exists at least one index  $i$  such that  $y_i \geq 1/2$ . Under this “greatest vote share procedure”, the outcome of an  $n$ -alternative voting is the

alternative  $x_i$  with the greatest vote share  $y_i$ , that is, such that:

$$y_i = \max_j \{y_j\}.$$

Clearly, there is always at least one such alternative, and if multiple indices attain the maximum value, we can decide randomly as tie-breaker. We assume, however, that this does not happen and that with probability 1 there is a unique index  $i$  such that  $y_i = \max_j \{y_j\}$ .

The outcome thus produced does not necessarily select the Condorcet Winner, if it exists. In fact, we now give a counterexample where the Condorcet Winner does not have the maximum vote share, even under the stronger assumptions of sincere voting and assuming that utility functions are not only single-peaked but *symmetric*, that is, for every type  $z \in \mathbb{R}$ , we additionally have that:

$$u_z(x) = u_z(y), \quad \forall x, y \text{ s.t. } |z - x| = |z - y|;$$

i.e.,  $u_z$  is symmetric around its maximum at  $z$ . This assumption ensures that a citizen of type  $z$  will always prefer the alternative that is closest to  $z$  in policy space, i.e., the alternative  $x_i$  such that  $|z - x_i| = \min_j |z - x_j|$ . In particular, we can easily compute the sincere support fractions  $x_{ij}$  as the total mass of the types  $z$  that lie closer to  $x_j$  than to  $x_i$ , for e.g.  $x_i < x_j$ :

$$x_{ij} = \mathbb{P}_z[z : |z - x_j| < |z - x_i|] = \mathbb{P}_z\left[\left(\frac{x_i + x_j}{2}, \infty\right)\right].$$

**Example 3** Let us consider a three-option greatest vote share procedure where the three alternatives satisfy  $x_0 < x_1 < 0 < x_2$ , with  $x_0 = 2x_1$  and  $x_2 < |x_1|$ . We consider furthermore a state of nature  $(F, u)$  where utility functions are symmetric and single-peaked, and where  $F$  is an absolutely continuous distribution of the voter type  $Z$  with nowhere-zero density function  $f > 0$  and such that  $\hat{z} = 0$ , where  $\hat{z}$  is the median voter type, i.e.,  $F$  satisfies  $\mathbb{P}_z[(-\infty, 0)] = \mathbb{P}_z[(0, \infty)] = 1/2$ . For an arbitrary fixed  $\varepsilon$  satisfying  $0 < \varepsilon < 1/4$ , we also assume that  $\mathbb{P}_z[(-\infty, x_0)] < \varepsilon$ ,  $\mathbb{P}_z[(x_1, 0)] < \varepsilon$ ,  $\mathbb{P}_z[(x_2, \infty)] < \varepsilon$ . Thus,  $\mathbb{P}_z[(x_0, x_1)] > 1/2 - 2\varepsilon$  and  $\mathbb{P}_z[(0, x_2)] > 1/2 - \varepsilon$ . We may also assume that  $\mathbb{P}_z[(x_0, (x_1 + x_0)/2)] = \varepsilon$ .

Since  $x_2 < |x_1|$ ,  $x_2$  lies closer to 0 than  $x_1$  and is the Condorcet Winner, in virtue of the symmetry of the utility functions. To see this, one may argue using the fact that a majority of the population ( $\mathbb{P}_z[(0, \infty)] = 1/2$ ) is distributed closer to  $x_2$  than to either  $x_1$  or  $x_0$ , since  $|x_2| < |x_1| < |x_0|$ , and then use the fact that policies that lie closer to  $z$  produce strictly more utility for citizen  $z$ .

If we now compute the sincere vote shares, we see that:

- Since  $\frac{x_1 + x_2}{2} \in (x_1, 0)$ , we have that  $\mathbb{P}_z[((x_1 + x_2)/2, 0)] < \mathbb{P}_z[(x_1, 0)] < \varepsilon$  and thus

$$x_{12} = \mathbb{P}_z\left[\left(\frac{x_1 + x_2}{2}, \infty\right)\right] = \mathbb{P}_z\left[\left(\frac{x_1 + x_2}{2}, 0\right)\right] + \mathbb{P}_z[(0, \infty)] < \varepsilon + \frac{1}{2}.$$

- Similarly, we have  $\frac{x_0 + x_2}{2} \in (x_1, 0)$  and thus, for the same reason, also  $x_{02} < \frac{1}{2} + \varepsilon$ .
- Finally, we see that

$$x_{10} = \mathbb{P}_z\left[\left(-\infty, \frac{x_1 + x_0}{2}\right)\right] = \mathbb{P}_z[(-\infty, x_0)] + \mathbb{P}_z\left[\left(x_0, \frac{x_1 + x_0}{2}\right)\right] < 2\varepsilon.$$

With this we may bound the vote shares of  $x_2$  and  $x_1$  as follows:

- $y_2 = \frac{x_{12} + x_{02}}{2} < \frac{1}{2} + \varepsilon$ ;
- $y_1 = \frac{x_{21} + x_{01}}{2} = \frac{1 - x_{12} + 1 - x_{10}}{2} > \frac{1/2 - \varepsilon + 1 - 2\varepsilon}{2} = \frac{3}{4} - \frac{3\varepsilon}{2}.$

For  $\varepsilon < 1/10$ , we see that  $y_2 < y_1$  but that  $x_2$  is the Condorcet Winner. Note that the construction is very flexible and that infinitely many similar settings exist. For example, the hypothesis that  $\hat{z} = 0$  may be weakened to  $\hat{z} \in (-\varepsilon, \varepsilon)$  with minor changes in our argument.

We now show how manipulation can occur with this procedure, even for single-peaked utility functions.

**Example 4** Again in a three-option procedure, we use the same notation of the previous section for groups with common preference rankings. Consider a state of nature where neither  $a$ ,  $b + c$ , nor  $d$  are majorities. In particular,  $x_1$  is the Condorcet Winner. The vote shares after sincere voting are (recall that  $\hat{y}_i := \sum_{j:j \neq i} x_{ji}$ ):

- $\widehat{y}_2 = x_{02} + x_{12} = (a + b) + (a) = 2a + b$ ;
- $\widehat{y}_1 = x_{01} + x_{21} = (a + b + c) + (b + c + d) = a + 2b + 2c + d$ ;
- $\widehat{y}_0 = x_{10} + x_{20} = (d) + (c + d) = c + 2d$ .

Assuming that

$$\widehat{y}_1 = a + 2b + 2c + d > 2a + b = \widehat{y}_2, \quad \widehat{y}_1 = a + 2b + 2c + d > c + 2d = \widehat{y}_0,$$

then  $x_1$  is the outcome of a sincere procedure.

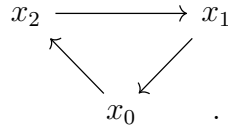
We show now the conditions under which  $a$  has an incentive to misreport its true preference  $x_0 \rightarrow x_1$ . Suppose  $a$  instead votes  $x_1 \rightarrow x_0$ ; then, assuming the other votes are sincere, the new vote shares are:

- $\widehat{y}_2' = x_{02} + x_{12} = (a + b) + (a) = 2a + b$ ;
- $\widehat{y}_1' = x_{01} + x_{21} = (b + c) + (b + c + d) = 2b + 2c + d$ ;
- $\widehat{y}_0' = x_{10} + x_{20} = (a + d) + (c + d) = a + c + 2d$ .

In this manner,  $a$  has an incentive to manipulate if

$$\widehat{y}_2' = 2a + b > 2b + 2c + d = \widehat{y}_1', \quad \widehat{y}_2' = 2a + b > a + c + 2d = \widehat{y}_0'.$$

If  $a + d > 1/2$ , then this manipulation would change the outcome  $x_0 \rightarrow x_1$ , and in particular, induce the cycle:



A particular example of the above conditions, i.e., where neither  $a$ ,  $b + c$ , nor  $d$  are majorities, and where:

$$a + 2b + 2c + d > 2a + b,$$

$$a + 2b + 2c + d > c + 2d,$$

$$2a + b > 2b + 2c + d,$$

$$2a + b > a + c + 2d,$$

is given by:

$$a = 40\%, \quad b = 30\%, \quad c = 10\%, \quad d = 20\%.$$

This example also shows that manipulation is possible even in the following Condorcet-consistent variation of the greatest vote share procedure: if there is a Condorcet Winner, that alternative is the outcome; if there is a cyclic outcome, the option with the highest vote share is the outcome. To see it, we note that with this variation the same manipulation by  $a$  modifies the sincere outcome, where  $x_1$  is the Condorcet Winner and has the greatest vote share, to a cyclic outcome where  $x_2$  has the greatest vote share. Thus, even if we make this procedure be Condorcet-consistent,  $a$  would still have incentives to manipulate.

## 5.4 Borda count

We next consider a three-alternative voting procedure with proposals  $x_0 < x_1 < x_2$  where each voter submits a vote consisting of a ranking of the three alternatives:  $x_i \succ x_j \succ x_k$ . Then, for each alternative  $x_i$ , the following count is made: a vote as highest-ranking alternative is worth 2 points, a vote as intermediate-ranking alternative is worth 1 point, and vote as least-ranking alternative is worth no points. The outcome of the *Borda count* procedure is the alternative with the highest total count of points. We again assume that exact draws occur with zero probability.

Clearly, the procedure is invariant if the values 0, 1, and 2 points are modified by a monotone affine transformation, i.e., we add or subtract to 0, 1, and 2 a common amount of points or we multiply the points received by any positive factor. The outcome might, however, be different after non-affine transformations, i.e., if the ratio of the difference between points awarded to each ranking position varies.

In the notation of groups of the previous section and assuming that every voter reports his/her preference ranking sincerely, the total count of points of each alternative is:

- For  $x_2$  :  $2a + b$ .
- For  $x_1$  :  $a + 2b + 2c + d$ .
- For  $x_0$  :  $c + 2d$ .



It is interesting that under sincere voting, Borda count follows the exact same rules as the greatest vote share procedure under sincere voting, since the values whose maximum decides the outcome are the same as the vote shares  $\hat{y}_i$  of the previous subsection. However, in Borda count the only preferences that may be expressed by voting are monotone or non-cyclic, in the sense that it is not possible to vote for  $x_i \succ x_j$ ,  $x_j \succ x_k$ , and also for  $x_k \succ x_i$ , unlike for the greatest vote share procedure. Thus, the two procedures provide the same outcome if all voters submit non-cyclic votes, but not necessarily otherwise.

Again, Borda count is not strategy-proof either. Before exemplifying this, we make the following observation:

**Lemma 1** *The outcome of a sincere Borda count procedure is:*

$$\begin{cases} x_2, & \text{if } a > \frac{1+c}{2}, \\ x_0, & \text{if } d > \frac{1+b}{2}, \\ x_1, & \text{otherwise.} \end{cases}$$

*The three conditions are exclusive and exhaustive with probability one.*

**Proof.** We assume that voters vote sincerely and compute the conditions under which the outcome is the middle alternative  $x_1$ . According to the previous observations, this is equivalent to:

$$a + 2b + 2c + d > 2a + b, \quad a + 2b + 2c + d > c + 2d.$$

Since  $a + b + c + d = 1$ , each equation can be expressed equivalently as the following two conditions:

$$\frac{1+c}{2} > a, \quad \frac{1+b}{2} > d.$$

Another observation is that the event that none of the two conditions are satisfied has zero probability since, if neither is satisfied, then:

$$1 \leq 1 + \frac{b+c}{2} = \frac{1+b}{2} + \frac{1+c}{2} \leq a + d \leq 1 \iff a + d = 1,$$

which, combined with  $a \geq 1/2$  and  $d \geq 1/2$ —since  $a \geq (1+c)/2$ ,  $d \geq (1+b)/2$ —is equivalent to  $a = d = 1/2$ ,  $b = c = 0$ . Since this is an exact draw, it occurs with probability 0.

To conclude, we observe that if  $a > (1 + c)/2$ , then  $x_2$  has more total points than  $x_1$ , and then the condition for  $d$  must be false, i.e.,  $d < (1 + b)/2$ . Thus,  $x_1$  has more points than  $x_0$ , and  $x_2$  is the final outcome. The symmetric argument for  $d$  provides the remaining case.  $\square$

Actually, given the observation that, under sincere voting, Borda count coincides with selecting the alternative with the greatest vote share, this result is also true for a sincere greatest vote share procedure. We also see that Borda count makes it more demanding for the extreme proposals to win under sincere voting, since now a share greater than the absolute majority is required. However, it is not strategy-proof:

**Example 5** Assume that

$$\frac{1 + c}{2} > a, \quad \frac{1 + b}{2} > d,$$

so that  $x_1$  is the sincere outcome of the Borda count procedure. Let us explore the conditions under which  $a$  is tempted to manipulate. The only ranking it may submit that could change the outcome without hurting its own interests is to report  $x_2 \succ x_0 \succ x_1$ . This would give total points, assuming the other groups vote sincerely:

- For  $x_2$  :  $2a + b$ .
- For  $x_1$  :  $2b + 2c + d$ .
- For  $x_0$  :  $a + c + 2d$ .

Thus,  $x_2$  wins if

$$2a > b + 2c + d, \quad a + b > c + 2d.$$

A simple scenario where  $a$  will try to manipulate in this fashion is the case where  $c = d = 0$  and  $a < b$ . The above conditions reduce to  $2a > b$ , and this is satisfied, for example, for

$$a = 40\%, \quad b = 60\%, \quad c = 0\%, \quad d = 0\%.$$

In fact, under these conditions,  $b$  would also manipulate and submit the ranking  $x_1 \succ x_0 \succ x_2$ . This way, the total points of  $x_2$  are  $2a$ , those of  $x_1$  are  $2b$ , and those of  $x_0$  are  $a + b$ , making  $x_1$  be the outcome, regardless of the strategy of  $a$ . Since all submitted

preferences are non-cyclic, this is another example of manipulation for the greatest vote share procedure.

## 5.5 A general approach

Studying equilibria under more general voting procedures that may not be Condorcet-consistent is more intricate, since a priori the procedure could be extremely complex. Actually, any voting procedure can be described exclusively via a function that takes as input the total voting share in favor of one of the two alternatives of each pairwise comparison. If we denote by  $x_{01}$  the vote share in favour of  $x_1$  at  $x_0 \leftrightarrow x_1$ , by  $x_{02}$  the vote share in favour of  $x_2$  at  $x_0 \leftrightarrow x_1$ , and by  $x_{12}$  the vote share in favour of  $x_2$  at  $x_1 \leftrightarrow x_2$ , then a procedure is a map  $f$  such that

$$f : [0, 1]^3 \longrightarrow \{x_0, x_1, x_2\} : (x_{01}, x_{02}, x_{12}) \longmapsto f(x_{01}, x_{02}, x_{12}).$$

If the map  $f$  is arbitrary, it is difficult to understand the structure of eventual Nash equilibria. Even in the Condorcet-consistent case, where  $f$  outputs  $x_j$  whenever  $x_{ij} > 1/2$  and  $x_{kj} > 1/2$  for  $\{i, j, k\} = \{1, 2, 3\}$  (introducing  $x_{10} := 1 - x_{01}$ , and similarly for the other cases in the obvious way), there is still much freedom of choice in  $f$ .

Finally, we note the following. To obtain the map sending a state of nature (together with a set of strategies for each group) to the final outcome of the procedure, we would evaluate such an  $f$  at the result of the map that assigns a group distribution<sup>7</sup> together with a set of group strategies to the voting shares  $x_{ij}$ . A thorough examination of this general approach is left to future research.

## 6 The Swiss example

In this section we exemplify the manipulation situation used to prove Proposition 3 with a concrete historical example that took place in the Swiss democratic system in 2010.

Switzerland provides a particular framework for a three-way vote scenario. Any citizen may submit a proposal for a constitutional change, referred to as an “initiative”. The

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<sup>7</sup>The space of group distributions is given by the points  $(a, b, c, d)$  in the 4-simplex for single-peaked utility functions, or the points  $(a, b, c, d, e, f)$  in the 6-simplex for the general case.

only requirement is to collect a specified number of supporting signatures within a given time-frame<sup>8</sup>. If the initiator obtains these signatures, the matter will be put to vote to the Swiss people<sup>9</sup>. Typically, there is a binary vote between the status-quo and the initiative. The initiative is only approved if it obtains a double majority that consists of a majority of the popular vote, the so-called “Volksmehr”, and a majority of the cantons, the so-called “Ständemehr”<sup>10</sup>. Nevertheless, in certain exceptional instances, the Swiss parliament may introduce a third option, known as a counter-proposal, to the voting process. This counter-proposal typically corresponds to a middle ground between the existing status-quo and the proposed initiative. The rationale behind this third option is to address situations where the proposed change, though deemed favorable by the parliament, may be too radical in certain aspects. Consequently, this dynamic entails a procedure with a three-way vote.<sup>11</sup> In the following we discuss the issues arising from the popular vote. A similar discussion can be held analogously for the “Ständemehr”.

## 6.1 The Swiss decision-making rule

In contrast to the Condorcet-consistent voting procedure with the default middle alternative as cycle-breaker described in Sections 3 and 4, Switzerland has adopted a specific approach, as explained in the introduction. The determination of the winner hinges on prevailing over the status-quo. The tie-breaking vote between the initiative and the third alternative counts only if both of these new options succeed against the status-quo<sup>12</sup>.

In formal terms, let  $x_0$  represent the status-quo, while  $x_1$  and  $x_2$  denote the alternative and the initiative, respectively. As described above,  $x_1$  is the counter-proposal introduced by the parliament aiming to balance the potentially too radical proposal  $x_2$  by giving a less radical, intermediate option.

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<sup>8</sup>The number of required signatures lies at 100'000 within a time-frame of 18 months. See Schweizerische Eidgenossenschaft (2023c) for more information.

<sup>9</sup>See Schweizerische Eidgenossenschaft (1999).

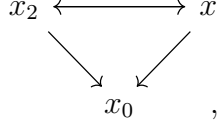
<sup>10</sup>See Schweizerische Eidgenossenschaft (2023b)

<sup>11</sup>It is noteworthy that the parliament has introduced a counter-proposal in only three instances since the introduction of the current scheme in the year 1987 (Bundesamt für Statistik (BFS) (2023)).

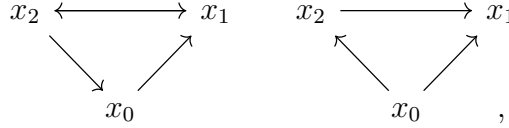
<sup>12</sup>If both alternatives obtain the necessary majorities against the status-quo, the winner of the tie-breaker vote is decided by a majority in both the popular vote and the majority of cantons. If both options fail to receive the double majority, then the option with the bigger sum of shares wins the tie-breaker. For more information, see Schweizerische Eidgenossenschaft (2023a) and Schweizerische Eidgenossenschaft (2023c).

Under the current Swiss voting system, the winning conditions for each alternative can be defined as follows:

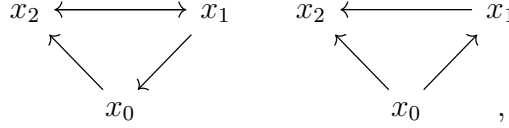
- $x_0$  wins  $\iff x_1 \rightarrow x_0$  and  $x_2 \rightarrow x_0$ , i.e.:



- $x_1$  wins  $\iff (x_0 \rightarrow x_1 \text{ and } x_2 \rightarrow x_0)$  or  $(x_0 \rightarrow x_2 \rightarrow x_1)$ , i.e.:



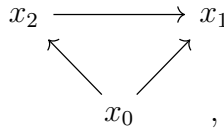
- $x_2$  wins  $\iff (x_0 \rightarrow x_2 \text{ and } x_1 \rightarrow x_0)$  or  $(x_0 \rightarrow x_1 \rightarrow x_2)$ , i.e.:



where the notation  $c \rightarrow b \rightarrow a$  means that  $a$  won both votes over  $b$  and  $c$ , and that  $b$  won the vote over  $c$ , as before. We note that the procedure is Condorcet-consistent, since it selects the Condorcet Winner whenever it exists, and that the cycle-breaking rules are summarized by selecting  $x_1$  for a counter-clockwise cycle, and  $x_2$  for a clockwise cycle.

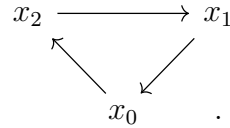
## 6.2 Non-strategy-proofness

The lack of strategy-proofness of this procedure is manifest, as Remark 1 after Proposition 3 states. Consider a scenario where sincere voting implies the collective pairwise voting outcome  $x_0 \rightarrow x_2 \rightarrow x_1$ ,



leading to the selection of  $x_1$ . However, group  $a$ , with preferences  $x_2 \succ x_1 \succ x_0$ , will have an incentive to engage in strategic voting. In this strategic approach,  $a$  would sincerely support  $x_2$  against  $x_0$ , and  $x_2$  against  $x_1$ , but would “dishonestly” support  $x_0$  against  $x_1$ , despite its actual preference for  $x_1$  over  $x_0$ . The strategic intent is to ensure that  $x_1$  loses against  $x_0$ . If  $x_2$  wins against  $x_0$ , it becomes the selected alternative, regardless of the

outcome in the tie-breaker question involving  $x_2$  and  $x_1$ . In effect, this would induce the cycle:



Therefore, if such a cycle can be induced through strategic voting, the polar alternative  $x_2$  becomes the default alternative and serves as the cycle-breaker.

As we saw in Remark 1, this scenario shows that the current Swiss procedure is not strategy-proof. As a real-life example for strategic voting, we consider the *Ausschaffungsinitiative* put forth in the year 2010 by the Swiss right-wing party SVP. This initiative sought to make the deportation of criminal foreigners easier or even automatic<sup>13</sup>. However, the majority of the Swiss parliament disagreed with the proposal and introduced an alternative that set higher hurdles for deportation, but still made it easier than in the status quo. Notably, there was an obvious recommendation of the SVP for strategic voting. It recommended to accept only the proposal while rejecting the alternative<sup>14</sup>. Conversely, other parties were divided, with some supporting the alternative and others favoring the status-quo.

Under the assumption of single-peakedness, the voting recommendation of SVP is a clear case of manipulation within our theoretical framework, as it supports the status-quo over the alternative, against true preferences.

In this particular instance, the official recommendation of SVP to vote in a particular way proved successful and was necessary in order to win. The proposal won against the status-quo, while the alternative failed to surpass the status-quo. Consequently, the proposal was accepted. Intriguingly, the proposal lost in the inconsequential vote against the alternative, which indicates that strategic voting was needed for the proposal to be implemented<sup>15</sup>. We present the number of votes for the respective choices in Figure 1.

<sup>13</sup>See Schweizerische Eidgenossenschaft (2010)

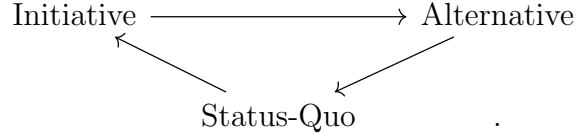
<sup>14</sup>We note that “manipulation” is the common term used in the literature to describe strategic voting, i.e., voting that does not reflect the preferences of voters. There is of course, no unlawful conduct with regard to such voting behaviour.

<sup>15</sup>See Bundeskanzlei (2023) for precise results.

Option 1	Number of votes	Option 2	Number of votes
Status-Quo	1'243'942	Initiative	1'397'923
Status-Quo	1'407'830	Alternative	1'189'269
Initiative	1'252'761	Alternative	1'271'365

Figure 1: Results of the vote on the Ausschaffungsinitiative.

The results in Figure 1 can be represented by the following cycle:



This outcome was facilitated by the left-wing parties *refraining from strategic voting recommendations* and withholding support for both the proposal and the alternative<sup>16</sup>. Thus, the final outcome was the polar outcome  $x_2$ , contradicting Proposition 2, since the main hypothesis of this result, that of having game-theoretical optimal play by all voters in the form of a Nash equilibrium, was not satisfied. This is perhaps a reminder of the difficulty of translating theoretical results to real-world settings. However, the strategic approach of SVP was indeed the game-theoretic optimal strategy described in the proof of Proposition 3, and afforded them success in this case.

This case stresses the critical importance of strategy-proofness in decision-making procedures. A non-strategy-proof process as the one above has tangible consequences in the real world. The ability of participants to vote strategically introduces a level of unpredictability and potential distortion of the outcome.

### 6.3 Historical development of the counter-proposal

While the current system of a three-way vote with its cycle-breaker rules is not strategy-proof, it is important to put it into historical context. The current system was introduced in 1987. The previous system was not a three-way vote but two separate votes between the initiative and status-quo on the one hand, and the alternative and status-quo on the other. The undesirable case of both proposals winning against the status-quo was prevented by introducing a rule that each voter is only allowed to accept one of the two proposals. If a

<sup>16</sup>See Année Politique Suisse (2010a) and Année Politique Suisse (2010b).

voter accepted both proposals, his/her vote would be invalid. S/he had to leave the other voting slip blank or accept the status-quo if s/he wanted to vote for the other proposal. Blank votes increased the total number of votes and therefore also the absolute majority needed to win against the status-quo<sup>17</sup>. This rule ensured that only one alternative could win against the status-quo<sup>18</sup>.

It is easy to see that this system heavily favors the status-quo. For example, if the counter-proposal and the initiative are very similar, the votes split up evenly between them, which, in turn, makes it impossible to obtain a majority for either alternative. The extra rule preventing from voting twice against the status-quo *inherently makes it impossible* for anyone favoring the most extreme alternative to vote sincerely, as s/he cannot vote for the middle alternative over the status-quo. While the current system is not optimal in terms of strategy-proofness, it is still a clear improvement over the previous system which made sincere voting impossible.

## 7 Conclusion

We have advanced a simple, concrete proposal how to make the Swiss democratic system strategy-proof. We have backed up our arguments with a real example where the current rules were exploited as described in our proofs. Furthermore, we have proved that our suggestion is robust, even against attempts of *several* parties to vote against their preferences. Finally, we have shown how other common three-way voting procedures also suffer from the same problem as the system currently used in Switzerland. This advocates an update of the current Swiss procedure for three-way referenda. Although the procedural details have to be designed carefully, as much of the existing literature covers, strategy-proofness appears to be crucial to ensure that voting outcomes reflect the will of the people in the best possible way.

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<sup>17</sup>See Schweizerische Eidgenossenschaft (2024).

<sup>18</sup>See Schweizerische Eidgenossenschaft (2018).



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