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# Article

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Statistics in Transition new series (SiTns)

## **Provided in Cooperation with:** Polish Statistical Association

*Suggested Citation:* Jiratampradab, Arisa; Supapakorn, Thidaporn; Suntornchost, Jiraphan (2022) : Comparison of confidence intervals for variance components in an unbalanced one-way random effects model, Statistics in Transition new series (SiTns), ISSN 2450-0291, Sciendo, Warsaw, Vol. 23, Iss. 4, pp. 149-160, https://doi.org/10.2478/stattrans-2022-0047

This Version is available at: https://hdl.handle.net/10419/301894

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*STATISTICS IN TRANSITION new series, December 2022 Vol. 23, No. 4, pp. 149–160, DOI 10.2478/stattrans-2022-0047* Received – 23.12.2021; accepted – 22.08.2022



# Comparison of confidence intervals for variance components in an unbalanced one-way random effects model

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# ABSTRACT

The purpose of this paper is to study and compare the methods for constructing confidence intervals for variance components in an unbalanced one-way random effects model. The methods are based on a classical exact, generalised pivotal quantity, a fiducial inference and a fiducial generalised pivotal quantity. The comparison of criteria involves the empirical coverage probability that maintains at the nominal confidence level of 0.95 and the shortest average length of the confidence interval. The simulation results show that the method based on the generalised pivotal quantity and the fiducial inference perform very well in terms of both the empirical coverage probability and the average length of the confidence interval. The classical exact method performs well in some situations, while the fiducial generalised pivotal quantity performs well in a very unbalanced design. Therefore, the method based on the generalised pivotal quantity is recommended for all situations.

**Key words:** variance components, unbalanced one-way random effects model, pivotal quantity, fiducial inference, coverage probability.

### 1. Introduction

The one-way random effects model is studied in many applications, such as medical treatment, animal breeding studies, agricultural genetics and industrial process management, etc. The variance components of this model are used to consider the different sources of variation. For example, radiotherapy doses for cancer treatment are determined by process variation due to difference in area of organs of individual patients and diagnosis of individual physician (Demetrashvili et al., 2016). Thus, the inferences for variance components in the model is of interest. Consider the one-way random effects model

$$y_{ij} = \mu + a_i + e_{ij}, \ i = 1, \dots, g, \ j = 1, \dots, n_i,$$
 (1)

where  $y_{ij}$  is the random observation,  $\mu$  is the overall mean. The random group effects  $a_i$  and the random errors  $e_{ij}$  are mutually independent random variables, and distributed

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as  $N(0, \sigma_a^2)$  and  $N(0, \sigma_e^2)$ , respectively. In addition, let  $n = \sum_{i=1}^g n_i$  denote the number of the total observations. When the number of observations  $n_i$  of each group is equal, model (1) is called balanced model. Otherwise, it is called unbalanced model. The source of variation is known as the variance components, namely,  $\sigma_a^2$  and  $\sigma_e^2$ . In general,  $\sigma_a^2$  is called between-group variance component, and  $\sigma_e^2$  is called within-group variance component. The proportion of the between-group variance component and the total variation can be written in the form  $\rho = \sigma_a^2/(\sigma_a^2 + \sigma_e^2)$ , which measures the importance of one effect related to the other effect.

One very important property of an estimator is minimal sufficient statistics. A closedform function of the minimal sufficient statistics is available in balanced random model. However, these functions are unavailable in unbalanced random model as described by Searle et al. (2006). Furthermore, solving the closed-form functions of the minimal sufficient statistics in the unbalanced case is computationally complicated for estimation of the variance components. There are several works in the literature that studied inferences for variance components in unbalanced model, such as Wald (1940), Thomas and Hultquist (1978), Park and Burdick (2003), and Arendacká (2005) which are based on a pivotal quantity approach. Ting et al. (1990) and Hartung and Knapp (2000) studied that by the classical exact method. Li and Li (2007) and Lidong et al. (2008) used the idea of a fiducial generalized confidence interval for variance components. Liu et al. (2016) proposed the concept of the fiducial generalized pivotal quantity for constructing the confidence interval for variance components in unbalanced model.

The aim of this paper is to compare five methods which are applicable to confidence intervals for between-group variance component in unbalanced one-way random effects model. These five methods are as follows: the Ting and others (TG) method (Ting et al., 1990), the Hartung-Knapp (HK) method (Hartung and Knapp, 2000), the Park-Burdick (PB) method (Park and Burdick, 2003), the Li-Li (LL) method (Li and Li, 2007), and the Liu-Xu-Hannig (LXH) method (Liu et al., 2016).

The paper is organized as follows. Section 2 describes the model and notation. Section 3 presents the methods for constructing a confidence interval for  $\sigma_a^2$ . Section 4 shows the results of a simulation study and compare the performance of the methods. Section 5 provides previously published data example. In the final Section 6, a conclusion is given.

### 2. Model and notation

A matrix formulation of the model (1) is given by

$$\mathbf{Y} = \mathbf{1}_n \boldsymbol{\mu} + \mathbf{Z} \mathbf{A} + \mathbf{E},\tag{2}$$

where  $\mathbf{Y} = (\mathbf{Y}'_1, \dots, \mathbf{Y}'_g)'$  with  $\mathbf{Y}_i = (Y_{i1}, \dots, Y_{in_i})'$  for  $i = 1, \dots, g$ ,  $\mathbf{1}_n = (\mathbf{1}'_{n_1}, \dots, \mathbf{1}'_{n_g})'$  with  $\mathbf{1}_{n_i}$  is a  $n_i \times 1$  vector of ones, and  $n = \sum_{i=1}^g n_i$ . The matrix  $\mathbf{Z} = \text{diag}(\mathbf{1}_{n_1}, \dots, \mathbf{1}_{n_g})$  is known as incidence matrix of size  $n \times g$ . The random group effects vector  $\mathbf{A} = (A_1, \dots, A_g)'$  is distributed as  $N(\mathbf{0}_g, \sigma_a^2 \mathbf{I}_g)$  and the random errors vector  $\mathbf{E} = (\mathbf{E}'_1, \dots, \mathbf{E}'_g)'$  with  $\mathbf{E}_i = (E_{i1}, \dots, E_{in_i})'$  is distributed as  $N(\mathbf{0}_n, \sigma_e^2 \mathbf{I}_n)$ , where  $\mathbf{0}_c$  is a  $c \times 1$  vector of zeros, and  $\mathbf{I}_c$  is a  $c \times c$  identity matrix. The random vectors  $\mathbf{A}$  and  $\mathbf{E}$  are mutually independent. De-

note  $r_1 = \operatorname{rank}(\mathbf{X}) - \operatorname{rank}(\mathbf{1}_n)$  and  $r_2 = n - \operatorname{rank}(\mathbf{X})$ , where  $\mathbf{X} = (\mathbf{1}_n, \mathbf{Z}\mathbf{Z}')$  is the horizontal concatenation of matrices  $\mathbf{1}_n$  and  $\mathbf{Z}\mathbf{Z}'$ . Under model (2), the distribution function of  $\mathbf{Y}$  is  $\mathbf{Y} \sim N(\mu \mathbf{1}_n, \sigma_a^2 \mathbf{Z}\mathbf{Z}' + \sigma_e^2 \mathbf{I}_n)$ , then  $\mathbf{H}'\mathbf{Y} \sim N(\mathbf{0}, \sigma_a^2 \mathbf{W} + \sigma_e^2 \mathbf{I})$ , where  $\mathbf{H}$  is matrix whose columns span the space orthogonal to the space spanned by the column vector of ones (Burch, 2011),  $\mathbf{W}$  is the part of the variance-covariance matrix associated with  $\sigma_a^2$ ,  $\mathbf{0}$  is vector of zeros, and  $\mathbf{I}$  is identity matrix. The quadratic form is denoted by  $T = \mathbf{Y}'\mathbf{B}\mathbf{Y}$ , where  $\mathbf{B}$  is an appropriately chosen symmetric matrix of constants called the matrix of the quadratic form (Milliken and Johnson, 2009).

Graybill (1976) described the properties of quadratic forms for estimation of the variance components. The independently quadratic forms, denoted by  $T_1, \ldots, T_d, T_{d+1}$ , are minimal sufficient statistics for  $(\sigma_a^2, \sigma_e^2)$  under multivariate normal distribution of **Y**. Burch (2011) showed that the sum of squares due to between groups  $SS_a$  and the sum of squares due to within groups  $SS_e$  can be expressed as quadratic forms

 $\mathbf{Y}'(\mathbf{X}(\mathbf{X}'\mathbf{X})^{-}\mathbf{X}' - \mathbf{1}_{n}(\mathbf{1}'_{n}\mathbf{1}_{n})^{-}\mathbf{1}'_{n})\mathbf{Y} = T_{1} + \dots + T_{d}$  and  $\mathbf{Y}'(\mathbf{I}_{n} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-}\mathbf{X}')\mathbf{Y} = T_{d+1}$ , respectively. The mean square for between groups and the mean square for within groups are denoted by  $MS_{a} = SS_{a}/r_{1}$  and  $MS_{e} = SS_{e}/r_{2}$ , respectively. Furthermore,  $MS_{a}$  and  $MS_{e}$  are independent, and  $SS_{e}/\sigma_{e}^{2}$  has a chi-squared distribution with  $r_{2}$  degrees of freedom.

# **3.** Approximate confidence intervals for $\sigma_a^2$

Several existing methods for constructing the confidence interval for  $\sigma_a^2$  are reviewed in this section.

#### 3.1. The TG method

Ting et al. (1990) suggested the method for constructing the confidence interval for the variance components in random effect model applying results provided by Howe (1974) and using cross-product terms in Ting et al. (1989). Let  $\mathbf{W}_{\text{TG}} = \mathbf{H}_{\text{TG}}\mathbf{Z}\mathbf{Z}'\mathbf{H}_{\text{TG}}$ , where  $\mathbf{H}_{\text{TG}}$  is a  $n \times n$  matrix such that  $\mathbf{H}_{\text{TG}} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-}\mathbf{X}' - \mathbf{1}_{n}(\mathbf{1}'_{n}\mathbf{1}_{n})^{-}\mathbf{1}'_{n}$ . Let  $\lambda_{1} > \cdots > \lambda_{d} > 0$  be the distinct positive eigenvalues of  $\mathbf{W}_{\text{TG}}$  having multiplicities  $s_{1}, \ldots, s_{d}$ . Define  $SS_{a} = \mathbf{Y}'\mathbf{H}'_{\text{TG}}\mathbf{W}_{\text{TG}}\mathbf{H}_{\text{TG}}\mathbf{Y}$ .

The approximate  $100(1-\alpha)\%$  confidence interval for  $\sigma_a^2$  is derived by

$$[MS_a - \frac{1}{b}MS_e - (G_1^2MS_a^2 + \frac{1}{b^2}C_2^2MS_e^2 + \frac{1}{b}G_{12}MS_aMS_e)^{1/2},$$
  
$$MS_a - \frac{1}{b}MS_e + (C_1^2MS_a^2 + \frac{1}{b^2}G_2^2MS_e^2 + \frac{1}{b}C_{12}MS_aMS_e)^{1/2}],$$

where  $b = r_1 (\sum_{\ell=1}^d s_\ell / \lambda_\ell)^{-1}$ ,  $G_1 = 1 - 1/F_{1-\alpha,(r_1,\infty)}$ ,  $C_2 = 1/F_{\alpha,(r_2,\infty)} - 1$ ,  $G_{12} = [(F_{1-\alpha,(r_1,r_2)} - 1)^2 - G_1^2 F_{1-\alpha,(r_1,r_2)}^2 - C_2^2]/F_{1-\alpha,(r_1,r_2)}$ ,  $C_1 = 1/F_{\alpha,(r_1,\infty)} - 1$ ,  $G_2 = 1 - 1/F_{1-\alpha,(r_2,\infty)}$ ,  $C_{12} = [(1 - F_{\alpha,(r_1,r_2)})^2 - C_1^2 F_{\alpha,(r_1,r_2)}^2 - G_2^2]/F_{\alpha,(r_1,r_2)}$ .

Note that  $F_{\alpha,(r_1,r_2)}$  and  $F_{1-\alpha,(r_1,r_2)}$  are the  $\alpha$  and  $1-\alpha$  quantiles of the *F*-distribution with degrees of freedom  $r_1$  and  $r_2$ , respectively. Furthermore,  $F_{\alpha,(r_1,\infty)} = \chi^2_{\alpha,r_1}/r_1$ ,  $F_{1-\alpha,(r_1,\infty)} = \chi^2_{1-\alpha,r_1}/r_1$ ,  $F_{\alpha,(r_2,\infty)} = \chi^2_{\alpha,r_2}/r_2$ , and  $F_{1-\alpha,(r_2,\infty)} = \chi^2_{1-\alpha,r_2}/r_2$  (Milliken and Johnson, 2009).

#### 3.2. The HK method

Hartung and Knapp (2000) developed the method for constructing the confidence interval for the between-group variance component using the concept of Wald (1940). The sufficient statistics of the HK method are defined as  $T_{\text{HK}_1}, \ldots, T_{\text{HK}_g}$ , where  $T_{\text{HK}_i} = (\mathbf{1}'_{n_i} \mathbf{1}_{n_i})^{-1} \mathbf{1}'_{n_i} \mathbf{Y}_i$ ,  $i = 1, \ldots, g$ .

The approximate  $100(1-\alpha)\%$  confidence interval for  $\sigma_a^2$  is derived by

$$[MS_eR_1, MS_eR_2]$$

Note that  $R_1$  and  $R_2$  are the root of the equations as follows:

$$f(R_1) = \frac{\sum_{i=1}^{g} w_i (T_{\mathrm{HK}_i} - \sum_{i=1}^{g} w_i T_{\mathrm{HK}_i} / \sum_{i=1}^{g} w_i)^2}{r_1 M S_e} \sim F_{1-\alpha/2,(r_1,r_2)} \text{ and}$$
$$f(R_2) = \frac{\sum_{i=1}^{g} v_i (T_{\mathrm{HK}_i} - \sum_{i=1}^{g} v_i T_{\mathrm{HK}_i} / \sum_{i=1}^{g} v_i)^2}{r_1 M S_e} \sim F_{\alpha/2,(r_1,r_2)},$$

where  $w_i = n_i / (1 + n_i R_1)$  and  $v_i = n_i / (1 + n_i R_2)$ .

#### 3.3. The PB method

Park and Burdick (2003) proposed the generalized pivotal quantity for constructing the confidence interval for the between-group variance component using results provided by Olsen et al. (1976). Let  $\mathbf{W}_{PB} = \mathbf{H}_{PB}\mathbf{Z}\mathbf{Z}'\mathbf{H}_{PB}$ , where  $\mathbf{H}_{PB}$  is a  $n \times n$  matrix such that  $\mathbf{H}_{PB} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-}\mathbf{X}' - \mathbf{1}_{n}(\mathbf{1}'_{n}\mathbf{1}_{n})^{-}\mathbf{1}'_{n}$ . Let  $\lambda_{1} > \cdots > \lambda_{d} > 0$  be the distinct positive eigenvalues of  $\mathbf{W}_{PB}$  having multiplicities  $s_{1}, \ldots, s_{d}$ . Let  $\mathbf{P}_{PB} = [\mathbf{P}_{PB_{1}}, \ldots, \mathbf{P}_{PB_{d}}]$  be  $n \times n$  orthogonal matrix such that  $\mathbf{P}'_{PB}\mathbf{W}_{PB}\mathbf{P}_{PB} = \text{diag}(\lambda_{1}\mathbf{1}'_{s_{1}}, \ldots, \lambda_{d}\mathbf{1}'_{s_{d}})$ , where  $\mathbf{P}_{PB_{\ell}}$ ,  $\ell = 1, \ldots, d$  corresponding to  $\lambda_{\ell}$  is of dimension  $n \times s_{\ell}$ .

The minimal sufficient statistics of the PB method are defined as  $T_{PB_1}, \ldots, T_{PB_d}$ , where  $T_{PB_\ell} = \mathbf{Y}' \mathbf{H}'_{PB} \mathbf{P}_{PB_\ell} (\mathbf{P}'_{PB_\ell} \mathbf{P}_{PB_\ell})^- \mathbf{P}'_{PB_\ell} \mathbf{H}_{PB} \mathbf{Y}, \ell = 1, \ldots, d$ . Lamotte (1976) showed that  $SS_a = \sum_{\ell=1}^{d} T_{PB_\ell}$ , where  $T_{PB_\ell}/(\lambda_\ell \sigma_a^2 + \sigma_e^2), \ell = 1, \ldots, d$  has the chi-squared distribution with  $s_\ell$  degrees of freedom. The function of the generalized pivotal quantity is defined by R as the solution for  $\sigma_a^2$  in the non-linear equation given by

$$U = \sum_{\ell=1}^{d} \frac{T_{\text{PB}_{\ell}}}{\lambda_{\ell} R + r_2 M S_e / K},\tag{3}$$

where  $U \sim \chi_{r_1}^2$  and  $K \sim \chi_{r_2}^2$ .

The approximate  $100(1-\alpha)\%$  confidence interval for  $\sigma_a^2$  is derived by

$$[\max(0, R_{\alpha/2}), \max(0, R_{1-\alpha/2})],$$

where  $R_{\alpha/2}$  and  $R_{1-\alpha/2}$  are the  $\alpha/2$  and  $1-\alpha/2$  quantiles of the distribution of *R* in equation (3), respectively. Note that the solutions of  $R_{\alpha/2}$  and  $R_{1-\alpha/2}$  are based on pivotal quantities.

#### 3.4. The LL method

Li and Li (2007) presented the concept of the fiducial inference for constructing the confidence interval for the between-group variance component in random effect model applying results provided by Li and Li (2005). Let  $\mathbf{W}_{LL} = \mathbf{H}_{LL}(\mathbf{Z}'\mathbf{Z})^{-}\mathbf{H}'_{LL}$ , where  $\mathbf{H}_{LL}$  is a  $(g-1) \times g$  matrix such that  $\mathbf{H}_{LL}\mathbf{H}'_{LL} = \mathbf{I}_{g-1}$  and  $\mathbf{H}'_{LL}\mathbf{H}_{LL} = \mathbf{I}_g$ . Let  $\lambda_1 > \cdots > \lambda_d \ge 0$  be the distinct eigenvalues of  $\mathbf{W}_{LL}$  having multiplicities  $s_1, \ldots, s_d$ . Let  $\mathbf{P}_{LL} = [\mathbf{P}_{LL_1}, \ldots, \mathbf{P}_{LL_d}]$  be  $(g-1) \times (g-1)$  orthogonal matrix such that  $\mathbf{P}'_{LL}\mathbf{W}_{LL}\mathbf{P}_{LL} = \text{diag}(\lambda_1\mathbf{1}'_{s_1}, \ldots, \lambda_d\mathbf{1}'_{s_d})$ , where  $\mathbf{P}_{LL_\ell}, \ell = 1, \ldots, d$  corresponding to  $\lambda_\ell$  is of dimension  $(g-1) \times s_\ell$ .

The sufficient statistics of the LL method are defined as  $T_{LL} = P_{LL}H_{LL}(\mathbf{Z}'\mathbf{Z})^{-}\mathbf{Z}'\mathbf{Y}$ . The function of the fiducial inference is given by

$$R = \frac{\mathbf{T}_{LL}' \mathbf{T}_{LL} - \mathbf{Q}' \mathbf{C} \mathbf{Q} S S_e / K}{\mathbf{Q}' \mathbf{Q}},\tag{4}$$

where  $\mathbf{C} = \mathbf{P}_{LL}\mathbf{W}_{LL}\mathbf{P}'_{LL}$ ,  $\mathbf{Q} \sim N(\mathbf{0}, \mathbf{I}_{r_1})$ , and  $K \sim \chi^2_{r_2}$ .

The approximate  $100(1 - \alpha)\%$  confidence interval for  $\sigma_a^2$  is derived by

$$[\max(0, R_{\alpha/2}), \max(0, R_{1-\alpha/2})],$$

where  $R_{\alpha/2}$  and  $R_{1-\alpha/2}$  are the  $\alpha/2$  and  $1-\alpha/2$  quantiles of the distribution of *R* in equation (4), respectively. Note that the solutions of  $R_{\alpha/2}$  and  $R_{1-\alpha/2}$  are based on pivotal quantities.

#### 3.5. The LXH method

Liu et al. (2016) proposed the least squares idea of the fiducial generalized pi-votal quantity for constructing the confidence interval for the variance components in random effect model. Let  $\mathbf{W}_{\text{LXH}} = \mathbf{H}'_{\text{LXH}} \mathbf{Z}\mathbf{Z}'\mathbf{H}_{\text{LXH}}$ , where  $\mathbf{H}_{\text{LXH}}$  is a  $n \times (n-1)$  matrix such that  $\mathbf{H}_{\text{LXH}}\mathbf{H}'_{\text{LXH}} = \mathbf{I}_n - n^{-1}\mathbf{1}_n\mathbf{1}'_n$  and  $\mathbf{H}'_{\text{LXH}}\mathbf{H}_{\text{LXH}} = \mathbf{I}_{n-1}$ . Let  $\lambda_1 > \cdots > \lambda_d \ge 0$  be the distinct eigenvalues of  $\mathbf{W}_{\text{LXH}}$  having multiplicities  $s_1, \ldots, s_d$ . Let  $\mathbf{P}_{\text{LXH}} = [\mathbf{P}_{\text{LXH}_1}, \ldots, \mathbf{P}_{\text{LXH}_d}]$  be  $(n-1) \times (n-1)$  orthogonal matrix such that  $\mathbf{P}'_{\text{LXH}}\mathbf{W}_{\text{LXH}}\mathbf{P}_{\text{LXH}} = \text{diag}(\lambda_1\mathbf{1}'_{s_1}, \ldots, \lambda_d\mathbf{1}'_{s_d})$ , where  $\mathbf{P}_{\text{LXH}_\ell}$ ,  $\ell = 1, \ldots, d$  corresponding to  $\lambda_\ell$  is of dimension  $(n-1) \times s_\ell$ .

The minimal sufficient statistics of the LXH method are defined as  $T_{\text{LXH}_1}, \ldots, T_{\text{LXH}_d}$ , where  $T_{\text{LXH}_{\ell}} = \mathbf{Y}' \mathbf{H}_{\text{LXH}} \mathbf{P}_{\text{LXH}_{\ell}} \mathbf{P}'_{\text{LXH}_{\ell}} \mathbf{H}'_{\text{LXH}} \mathbf{Y}$ ,  $\ell = 1, \ldots, d$ . The variables  $U_{\ell}$ ,  $\ell = 1, \ldots, d$ are mutually independent and  $U_{\ell} = T_{\text{LXH}_{\ell}} / (\lambda_{\ell} \sigma_a^2 + \sigma_e^2)$ ,  $\ell = 1, \ldots, d$  has the chi-squared distribution with  $s_{\ell}$  degrees of freedom. The function of the least squares fiducial inference is given by

$$R = \frac{\sum_{\ell=1}^{d} U_{\ell}^{2} \sum_{\ell=1}^{d} \lambda_{\ell} T_{\text{LXH}_{\ell}} U_{\ell} - \sum_{\ell=1}^{d} \lambda_{\ell} U_{\ell}^{2} \sum_{\ell=1}^{d} T_{\text{LXH}_{\ell}} U_{\ell}}{\sum_{\ell=1}^{d} U_{\ell}^{2} \sum_{\ell=1}^{d} \lambda_{\ell}^{2} U_{\ell}^{2} - (\sum_{\ell=1}^{d} \lambda_{\ell} U_{\ell}^{2})^{2}}.$$
(5)

The approximate  $100(1-\alpha)\%$  confidence interval for  $\sigma_a^2$  is derived by

$$[\max(0, R_{\alpha/2}), \max(0, R_{1-\alpha/2})]$$

where  $R_{\alpha/2}$  and  $R_{1-\alpha/2}$  are the  $\alpha/2$  and  $1-\alpha/2$  quantiles of the distribution of *R* in equation (5), respectively. Note that the solutions of  $R_{\alpha/2}$  and  $R_{1-\alpha/2}$  are based on pivotal quantities.

### 4. Simulation study

In this section, a comparison of the methods for constructing the confidence interval for  $\sigma_a^2$  with the methods described in Section 3 is studied by the Monte Carlo simulation. Without loss of generality, it is assumed that  $\mu = 0$  in model (2). The values chosen for  $(\sigma_a^2, \sigma_e^2)$  are (0.001, 0.999), (0.1, 0.9), (0.2, 0.8), (0.3, 0.7), (0.4, 0.6), (0.5, 0.5), (0.6, 0.4), (0.7, 0.3), (0.8, 0.2), (0.9, 0.1), and (0.999, 0.001). The ratio of va-riance components,  $\rho = \sigma_a^2/(\sigma_a^2 + \sigma_e^2)$  varies from small to large. The nominal confidence level of 0.95 is considered. The simulation study is based on 5,000 iterations for each setting of the values  $(\sigma_a^2, \sigma_e^2)$  and the sample size pattern  $(n_i, i = 1, \dots, g)$ .

The criteria for analysing the performance of the methods are the empirical coverage probability that maintains at the nominal confidence level, and the shortest average length of the confidence interval. The empirical coverage prob-ability is firstly considered, and the average length of the confidence interval is later compared. The degree of imbalance is  $\Phi = (g/\sum_{i=1}^{g} n_i)(g/\sum_{i=1}^{g} 1/n_i)$ , which is used to measure imbalance in one-way model (Ahrens and Pincus, 1981). Note that  $0 < \Phi \le 1$  is equal to 1 if and only if the model is balanced, and  $\Phi$  is close to 0 when the model is very unbalanced. The coverage probability of confidence interval for  $\sigma_a^2$  depends on the degree of imbalance and the design  $(n_1, \ldots, n_g)$ . The simulation patterns are shown in Table 1.

Pattern	Φ	g				n <sub>i</sub>						
1	0.044	3	1	1	100							
2	0.570	3	3	7	20							
3	0.818	3	5	10	15							
4	0.068	6	1	1	1	1	1	100				
5	0.700	6	5	10	15	20	25	30				
6	0.957	6	6	6	8	8	10	10				
7	0.525	10	1	1	4	4	6	6	8	8	10	10
8	0.835	10	3	3	4	5	6	6	8	8	10	10

 Table 1. Unbalanced patterns used in simulations

The simulation results are represented in the boxplots of Figures 1 and 2. The empirical coverage probabilities of the confidence interval for  $\sigma_a^2$  with the number of groups g = 3, 6, and 10, where  $\rho < 0.5$  and  $\rho \ge 0.5$ , are shown in Figure 1. The relative difference of the average length of the confidence interval for  $\sigma_a^2$  with the number of groups g = 3, 6, and 10, where  $\rho < 0.5$  and  $\rho \ge 0.5$  is shown in Figure 2. The relative length is defined as  $(L_{\rm M} - L_{\rm PB})/L_{\rm PB}$ , where  $L_{\rm M}$  denotes the average interval length of competing methods and

 $L_{\text{PB}}$  denotes the average interval length of the PB method. Clearly, the positive value of the relative length implies that  $L_{\text{PB}}$  is shorter than  $L_{\text{M}}$ . On the contrary, the negative value of the relative length implies that  $L_{\text{M}}$  is shorter than  $L_{\text{PB}}$ . Moreover, the relative length equal to 0 implies that  $L_{\text{M}}$  and  $L_{\text{PB}}$  are equal.

Regarding the empirical coverage probabilities, from Figure 1, the PB procedure maintains the nominal confidence level for all situations. The LL procedure provides a larger than the nominal confidence level for all situations. The TG procedure maintains the nominal confidence level for all situations except for g = 10. However, the TG procedure provides a smaller than the nominal confidence level when  $\rho < 0.5$  for g = 10. The HK procedure mostly maintains the nominal confidence level when  $\rho < 0.5$  and it provides a larger than the nominal confidence level when  $\rho < 0.5$  and it provides a larger than the nominal confidence level when  $\rho < 0.5$  and it provides a smaller than the nominal confidence level when  $\rho < 0.5$  and it provides a larger than the nominal confidence level when  $\rho > 0.5$  for g = 3. The HK procedure provides a smaller than the nominal confidence level of  $\rho = 6$  and 10. The LXH procedure provides a larger than the nominal confidence level for all  $\rho$  for g = 3. The LXH procedure provides a smaller than the nominal confidence level for all  $\rho$  for g = 6 and 10 except in a very unbalanced design (pattern 4), that is, the LXH procedure maintains the nominal confidence level in a very unbalanced design.

Comparing the average length of the confidence interval, Figure 2 clearly indicates that the average lengths of the TG, LL, and PB intervals behave very similar. The average length of the LXH interval is the shortest. For the number of groups g = 6 and 10, the average length of the HK interval is shorter than the average length of the PB interval when  $\rho < 0.5$ . Conversely, the average length of the PB interval is shorter than the average length of the HK interval when  $\rho \ge 0.5$ .

#### 5. Application

The numerical example from Brownlee (1965) is a study of the effects of environmental conditions on the measure of the ratio of electromagnetic and electrostatic units of electricity. The data set is shown in Table 2. Model (1) is used to des-cribe this data set, that is, g = 5,  $n_i = (11, 8, 6, 24, 15)$ , and  $\Phi = 0.796$ . Furthermore,  $a_i$  denote the random group effects of the environmental conditions and assume  $a_i \sim N(0, \sigma_a^2)$ ,  $e_{ij}$  represent the random effect of the *j*th measure of electricity on the *i*th environmental condition and assume  $e_{ij} \sim N(0, \sigma_e^2)$ . Independence among  $a_i$  and  $e_{ij}$  is also assumed. The five confidence intervals for  $\sigma_a^2$  based on the five methods in Section 3 are presented in Table 3. Table 3 shows that the PB method provides the shortest confidence interval for this data set.



Figure 1: The empirical coverage probabilities of 95% confidence interval for  $\sigma_a^2$ 



Figure 2: Relative difference of the average length of 95% confidence interval for  $\sigma_a^2$ 

Groups	Observations											
1	62	64	62	62	65	64	65	62	62	63	64	
2	65	64	63	62	65	63	64	63				
3	65	64	67	62	65	62						
4	62	66	64	64	63	62	64	64	66	64	66	63
	65	63	63	63	61	56	64	64	65	64	64	65
5	66	65	65	66	67	66	69	70	68	69	63	65
	64	65	64									

Table 2. The ratio of the electromagnetic to electrostatic units of electricity

Table 3. Nominally 95% confidence interval for the data

Method	TG	НК	PB	LL	LXH
confidence interval	(0, 10.901)	(0, 11.311)	(0, 9.595)	(0, 10.836)	(0, 9.728)

### 6. Conclusion

This article studies the methods for constructing 95% confidence intervals for variance components in an unbalanced one-way random effects model. Simulation studies indicate that the TG procedure maintains the nominal confidence level for all situations except for the number of group g = 10, which is liberal when  $\rho$  is small. The HK procedure is conservative when  $\rho$  is large. On the contrary, when  $\rho$  is small, the HK procedure mostly maintains the nominal confidence level for the number of groups g = 6 and 10. The PB procedure maintains the nominal confidence level for all situations. The LL procedure is conservative for all situations. The LL procedure is conservative for all situations. The LL procedure does not adequately maintain the nominal confidence level level. All of the average lengths of the confidence intervals behave similarly, but the average length of the LXH interval always has the shortest. Notice that the relative length values of the LXH method is negative.

In summary, the PB and LXH methods are recommended for the number of group g = 3. The PB and LL methods are recommended for the number of groups g = 6 and 10. The TG and HK methods are useful when  $\rho$  is large. Furthermore, the LXH method is preferred in a very unbalanced design.

### Acknowledgements

The authors thank the reviewers for their valuable comments and suggestions which substantially improved the quality of the article. This work is supported by the Science Achievement Scholarship of Thailand (SAST).

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