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## Article

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## Missing data estimation based on the chaining technique in survey sampling

Narendra Singh Thakur<sup>1</sup>, Diwakar Shukla<sup>2</sup>

### ABSTRACT

Sample surveys are often affected by missing observations and non-response caused by the respondents' refusal or unwillingness to provide the requested information or due to their memory failure. In order to substitute the missing data, a procedure called imputation is applied, which uses the available data as a tool for the replacement of the missing values. Two auxiliary variables create a chain which is used to substitute the missing part of the sample. The aim of the paper is to present the application of the Chain-type factor estimator as a means of source imputation for the non-response units in an incomplete sample. The proposed strategies were found to be more efficient and bias-controllable than similar estimation procedures described in the relevant literature. These techniques could also be made nearly unbiased in relation to other selected parametric values. The findings are supported by a numerical study involving the use of a dataset, proving that the proposed techniques outperform other similar ones.

**Key words:** estimation, missing data, chaining, imputation, bias, mean squared error (MSE), factor type (F-T), chain type estimator, double sampling.

Mathematical Subject Code: 62D05

### 1. Introduction

In sample surveys, the auxiliary information is used to improve efficiency of the estimate [see, Cochran (2005), Sukhatme et al. (1984)]. The use of a ratio estimator is preferred when the population mean of auxiliary variate is known. However, when it is unknown then it is not possible to apply the ratio estimator directly and the concept of two-phase sampling is applied to get a sample-based estimate of population mean. Sometimes information on one more auxiliary variable highly correlated to earlier

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auxiliary variate is available and easy to access at a lesser cost. This additional information could be intelligently utilized for obtaining efficient estimates. Chaining is one such technique, used by Chand (1975), Sukhatme and Chand (1977), which has a mechanism of combining wisely two auxiliary variates. Kiregyera (1980, 1984) proposed some chain type ratio and regression estimators whereas Singh et al. (1994) developed a class of chain type estimators under a double sample scheme. Al-Jararha and Ahmed (2002) discussed the class of chain type estimators for population variance using double a sampling scheme. Some other useful contributions are Kumar and Bahl (2006), Pradhan (2005), Rao and Sitter (1995), Sharma and Tailor (2010), Shukla (2002), Singh and Espejo (2007), Singh et al. (2009), Singh et al. (1993), Srivastava and Jhaji (1980), etc.

The use of auxiliary information in the estimation of population values of the study variate is a common phenomenon in sampling theory of surveys. Auxiliary information is successfully utilized either at the planning stage or at the design stage or at the information stage to arrive at improved estimator compared to those not utilizing auxiliary information. The use of ratio and product strategies in survey sampling solely depends upon the knowledge of population mean  $\bar{X} = N^{-1} \sum_{i=1}^N X_i$  of the auxiliary character  $X$ . In many situations of practical importance, the population mean  $\bar{X}$  is unknown before the start of a survey. In such a situation, the usual thing to do is to estimate it by the sample mean  $\bar{x}_m = m^{-1} \sum_{i=1}^m x_i$  based on a preliminary sample of size  $m$  of which  $n$  is a sub-sample ( $n < m$ ). If the population mean  $\bar{Z} = N^{-1} \sum_{i=1}^N Z_i$  of another auxiliary variate  $Z$ , closely related to auxiliary variate  $X$  but compared to  $X$  remotely related to study variate  $Y$  is known, it is advisable to estimate  $\bar{X}$  by  $\bar{X} = \bar{x}_m \frac{\bar{Z}}{\bar{z}_m}$ , where  $\bar{z}_m = m^{-1} \sum_{i=1}^m z_i$ , which would provide better estimate of  $\bar{X}$  than  $\bar{x}_m$  to the terms of order  $o(n^{-1})$  if  $\rho_{XZ} \frac{C_X}{C_Z} > 0.5$  [see, Choudhury and Singh (2012)]. The symbol  $\rho_{XZ}$  is the coefficient of correlation between  $X$  and  $Z$  and  $C_X, C_Z$  are the coefficient of variation of  $X$  and  $Z$  respectively. Chand (1975) and Sukhatme and Chand (1977) proposed a technique of chaining of the available information on auxiliary characteristics with the main characteristic. Kiregyera (1980, 1984), Singh et al. (2006) also proposed some chain type ratio and regression estimators based on two auxiliary variables. Using prior information on parameters of auxiliary variate some useful contributions are Shukla et al. (1991), Bose (1943), Kadilar and Cingi (2003), Srivastava et al. (1990), Srivenkataramana (1980), etc.

According to Hietjan and Basu (1996), incompleteness in the form of missingness, censoring or grouping, is a troubling feature of several data sets. A key question is what one needs to assume to justify ignoring the incompleteness mechanism. Rubin (1976) addressed this question for Bayes/likelihood and frequentist inferences. Little and Rubin (1987) recognized for some time that failure to account for the stochastic nature of incompleteness can spoil inferences.

In brief, Rubin (1976) defined three key concepts: missing at random (MAR), observed at random (OAR) and Parameter Distinctness (PD). The data are MAR if the probability of the observed missingness pattern, given the observed and unobserved data, does not depend on the values of the unobserved data. The data are OAR if, for every possible value of the missing data, the probability of the observed missingness pattern, given the observed and unobserved data, does not depend on the values of the observed data. PD holds if there are no a priori ties between the parameters of the missingness model and those of the data model. For Bayesian inference this means that the parameters of the data model and missingness model are a priori independent. For direct likelihood inference it means that knowledge of one parameter's value does not place any constraints on the other parameter's value. Ignoring the missingness mechanism is justified for Bayes/likelihood inference if MAR and PD hold. The combination of MAR and OAR is called missing completely at random (MCAR). In what follows missing completely at random (MCAR) by Hietjan and Basu (1996) is used in this article. Some useful contributions available in the literature are Weeks (1999), Shukla et al. (2009), Seaman et al. (2013), Bhaskaran and Smeeth (2014), Pandey et al. (2015), Pandey et al. (2016), Doretti et al. (2018), etc. This manuscript presents the use of Chain-Type estimator as an imputation source for dealing with missing observations to estimate the population mean.

### 1.1. Some existing imputation strategies

A simple random sample  $S$  without replacement (SRSWOR), of size  $n$  is drawn from population  $\Omega = \{1, 2, \dots, N\}$  with  $Y_i$  as  $i^{th}$  unit of variable  $Y$  under study. Let  $\bar{Y} = N^{-1} \sum_{i=1}^N Y_i$  be the mean of a finite population under estimation. The sample  $S$  of  $n$  units contains  $r$  responding units ( $r < n$ ) forming a sub-space  $R$  and  $(n - r)$  non-responding with the sub-space  $(n - r)$  having symbol  $R^C$  in the space  $S$ . The sub-spaces  $R$  and  $R^C$  are disjoint and  $R \cup R^C = S$ . The variable  $Y$  is of main interest and  $X$  is auxiliary correlated with  $Y$ . For every unit  $i \in R$ , the value  $y_i$  is observed available. For units  $i \in R^C$ , the  $y_i$  values are missing and imputed values are to be derived. The  $i^{th}$  value  $x_i$  of  $X$  could be used as a source of imputation for  $y_i$ ,  $i \in R^C$ . This is to assume for sample  $S$ , the data  $x_s = \{x_i : i \in S\}$  is known and available completely. Responding units have missing data only for the study variable  $Y$ . Under this two variable set-up, some well-known imputation methods available in the literature are:

### 1.1.1. Ratio method of imputation

For  $y_i$  and  $x_i$ , define  $y_{\bullet i}$  as

$$y_{\bullet i} = \begin{cases} y_i & \text{if } i \in R \\ \hat{b}x_i & \text{if } i \in R^C \end{cases} \quad (1.1)$$

Where  $\hat{b} = \frac{\sum_{i \in R} y_i}{\sum_{i \in R} x_i} = \frac{\bar{y}_r}{\bar{x}_r}$

Using the above, the imputation-based estimator is:

$$\bar{y}_S = \frac{1}{n} \sum_{i \in S} y_{\bullet i} = \frac{1}{n} \left[ \sum_{i \in R} y_i + \hat{b} \sum_{i \in R^C} x_i \right] = \bar{y}_r \left( \frac{\bar{x}_n}{\bar{x}_r} \right) = \bar{y}_{RAT} \quad (1.2)$$

Where  $\bar{y}_r = \frac{1}{r} \sum_{i \in R} y_i$ ,  $\bar{x}_r = \frac{1}{r} \sum_{i \in R} x_i$  and  $\bar{x}_n = \frac{1}{n} \sum_{i \in S} x_i$

### 1.1.2. Mean method of imputation

For  $y_i$  define  $y_{\bullet i}$  as

$$y_{\bullet i} = \begin{cases} y_i & \text{if } i \in R \\ \bar{y}_r & \text{if } i \in R^C \end{cases} \quad (1.3)$$

Using the above, the imputation-based estimator of population mean  $\bar{Y}$  is:

$$\bar{y}_m = \frac{1}{r} \sum_{i \in R} y_i = \bar{y}_r \quad (1.4)$$

### 1.1.3. Compromised method of imputation

Singh and Horn (2000) proposed a compromised imputation procedure:

$$y_{\bullet i} = \begin{cases} (an/r)y_i + (1-\alpha)\hat{b}x_i & \text{if } i \in R \\ (1-\alpha)\hat{b}x_i & \text{if } i \in R^C \end{cases} \quad (1.5)$$

Where  $\alpha$  is a suitably chosen constant, such that the resultant variance of the estimator is minimum. The imputation-based estimator, for this case, is

$$\bar{y}_{COMP} = \left[ \alpha \bar{y}_r + (1-\alpha) \bar{y}_r \frac{\bar{x}_n}{\bar{x}_r} \right] \quad (1.6)$$

### 1.1.4. Ahmed methods of imputation

For the case where  $y_{ji}$  denotes the  $i^{\text{th}}$  available observation for the  $j^{\text{th}}$  imputation method Ahmed et al. (2006) suggested:

$$(A) \quad y_{li} = \begin{cases} y_i & \text{if } i \in R \\ \frac{1}{(n-r)} \left[ n \bar{y}_r \left( \frac{\bar{X}}{\bar{x}_n} \right)^{\beta_1} - r \bar{y}_r \right] & \text{if } i \in R^C \end{cases} \quad (1.7)$$

Under this, the point estimator is:

$$t_1 = \bar{y}_r \left( \frac{\bar{X}}{\bar{x}_n} \right)^{\beta_1} \quad (1.8)$$

$$(B) \quad y_{2i} = \begin{cases} y_i & \text{if } i \in R \\ \frac{1}{(n-r)} \left[ n \bar{y}_r \left( \frac{\bar{x}_n}{\bar{x}_r} \right)^{\beta_2} - r \bar{y}_r \right] & \text{if } i \in R^C \end{cases} \quad (1.9)$$

The point estimator is under this set-up:

$$t_2 = \bar{y}_r \left( \frac{\bar{x}_n}{\bar{x}_r} \right)^{\beta_2} \quad (1.10)$$

$$(C) \quad y_{3i} = \begin{cases} y_i & \text{if } i \in R \\ \frac{1}{(n-r)} \left[ n \bar{y}_r \left( \frac{\bar{X}}{\bar{x}_n} \right)^{\beta_3} - r \bar{y}_r \right] & \text{if } i \in R^C \end{cases} \quad (1.11)$$

The point estimator is:

$$t_3 = \bar{y}_r \left( \frac{\bar{X}}{\bar{x}_r} \right)^{\beta_3} \quad (1.12)$$

Terms  $\beta_1$ ,  $\beta_2$  and  $\beta_3$  are suitably chosen constants, so as to keep the variance of the resultant estimator minimum. As special cases, when

$$\beta_3 = 1, \quad t_{Ratio} = \bar{y}_r \left( \frac{\bar{X}}{\bar{x}_r} \right) \quad (1.13)$$

$$\text{and } \beta_3 = -1, \quad t_{Product} = \bar{y}_r \left( \frac{\bar{x}_r}{\bar{X}} \right) \quad (1.14)$$

The last one (1.14) is natural analogue of the ratio estimator called the product estimator used when an auxiliary variate  $X$  has negative correlation with  $Y$ .

### 1.1.5. Factor type methods of imputation

Shukla and Thakur (2008) suggested factor-type imputation procedures as:

$$(D) \quad (y_{FT1})_i = \begin{cases} y_i & \text{if } i \in R \\ \frac{\bar{y}_r}{(n-r)} [n \phi_1(k) - r] & \text{if } i \in R^C \end{cases} \quad (1.15)$$

$$(E) \quad (y_{FT2})_i = \begin{cases} y_i & \text{if } i \in R \\ \frac{\bar{y}_r}{(n-r)} [n \phi_2(k) - r] & \text{if } i \in R^C \end{cases} \quad (1.16)$$

$$(F) \quad (y_{FT3})_i = \begin{cases} y_i & \text{if } i \in R \\ \frac{\bar{y}_r}{(n-r)} [n\phi_3(k) - r] & \text{if } i \in R^C \end{cases} \quad (1.17)$$

$$\text{Where } \phi_1(k) = \frac{(A+C)\bar{X} + fB\bar{x}_n}{(A+fB)\bar{X} + C\bar{x}_n}, \phi_2(k) = \frac{(A+C)\bar{x}_n + fB\bar{x}_r}{(A+fB)\bar{x}_n + C\bar{x}_r}, \phi_3(k) = \frac{(A+C)\bar{X} + fB\bar{x}_r}{(A+fB)\bar{X} + C\bar{x}_r},$$

$$A = (k-1)(k-2), B = (k-1)(k-4), C = (k-2)(k-3)(k-4), f = \frac{n}{N}, 0 < k < \infty$$

Under (1.15), (1.16) and (1.17) point estimators are:

$$\left. \begin{aligned} T_{FT1} &= \bar{y}_r \phi_1(k) \\ T_{FT2} &= \bar{y}_r \phi_2(k) \\ T_{FT3} &= \bar{y}_r \phi_3(k) \end{aligned} \right\} \quad (1.18)$$

As special cases, when  $k=1, \beta_l=1$  then  $T_{FTl} = t_l$  when  $k=2, \beta_l=-1$  then  $T_{FTl} = t_l$   
when  $k=4, \beta_l=0$  then  $T_{FTl} = t_l = \bar{y}_r$ ;  $(l=1,2,3)$

## 2. Proposed imputation strategies

Consider a double sampling set-up with three variables  $Y$ ,  $X$  and  $Z$  where  $Y$  is the main variable and  $X, Z$  are auxiliary variates. The correlation between  $X$  and  $Z$  is higher than other two. A specific way of combining  $X$  and  $Z$  is “chaining”, which generates chain-type estimators in double sampling, and several authors have used this [see Singh and Singh (1991), Singh et al. (1994)] to get a series of alternative estimators for estimating population mean. Singh and Shukla (1987) discussed a family of factor-type ratio estimator for estimating population mean. In one more contribution, Singh and Shukla (1993) derived efficient factor-type estimator for estimating the same population parameter. Using the above contributions Singh et al. (1994) developed a factor-type-chain estimator, whose application as an imputation tool is the main source of motivation in this article.

### 2.1. Preliminaries

Typically, in double sampling, the population mean  $\bar{X}$  of variable  $X$  is unknown. Hence, let  $S'$  be the preliminary sample drawn from  $\Omega = \{1, 2, \dots, N\}$  by SRSWOR containing  $m$  units with mean  $\bar{x}_m, \bar{z}_m$  of  $X$  and  $Z$ . This implies  $x_{s'} = \{x_j : j \in S'\}$ ,  $z_{s'} = \{z_j : j \in S'\}$  are known data and at this stage data linked with variable  $Y$  are not known. A sub-sample  $S$  of  $n$  units ( $n < m$ ) is drawn from  $S'$  by SRSWOR having  $r$  responding units ( $r < n$ ) forming subspace  $R$ , having  $(n-r)$  non-responding units with the sub-space  $R^C$ . Also, in  $S$ ,  $y_R = \{y_i, i \in R\}$ ,  $x_S = \{x_i, i \in S\}$ ,  $z_S = \{z_i, i \in S\}$  are available,

whereas  $y_{R^C} = \{y_i, i \in R^C\}$  is missing and needs to be estimated by an appropriate imputation technique. As discussed in previous section the sub-spaces  $R$  and  $R^C$  are disjoint and  $R \cup R^C = S$ .

Let us consider Ahmed et al. (2006) point estimator from equation (1.10)  $t_2$  with  $\beta_2 = 1$  :

$$t_2^* = \bar{y}_r \frac{\bar{x}_n}{\bar{x}_r} \quad (*)$$

The term  $\bar{x}_n$  could be improved by Chaining Technique as suggested by Chand (1975), Sukhatme and Chand (1977), Singh and Singh (1991) as:

$$t_2^{**} = \bar{y}_r \frac{\bar{x}_m}{\bar{x}_r} \frac{\bar{Z}}{\bar{z}_m} \quad (\text{With } \bar{z}_m \text{ and } \bar{Z} \text{ known}) \quad (**)$$

Motivated from the above discussion, some proposed imputation strategies using Singh et al. (1994) are:

$$(G) \quad (y_{C1})_i = \begin{cases} y_i & \text{if } i \in R \\ \frac{1}{(n-r)} [n\psi_1(k) - r\bar{y}_r] & \text{if } i \in R^C \end{cases} \quad (2.1)$$

$$(H) \quad (y_{C2})_i = \begin{cases} y_i & \text{if } i \in R \\ \frac{1}{(n-r)} [n\psi_2(k) - r\bar{y}_r] & \text{if } i \in R^C \end{cases} \quad (2.2)$$

$$(I) \quad (y_{C3})_i = \begin{cases} y_i & \text{if } i \in R \\ \frac{1}{(n-r)} [n\psi_3(k) - r\bar{y}_r] & \text{if } i \in R^C \end{cases} \quad (2.3)$$

Where 
$$\psi_1(k) = \bar{y}_r \frac{\bar{x}_m}{\bar{x}_r} \frac{(A+C)\bar{Z} + fB\bar{z}_m}{(A+fB)\bar{Z} + C\bar{z}_m} \quad (2.4)$$

$$\psi_2(k) = \bar{y}_r \frac{\bar{x}_m}{\bar{x}_r} \frac{(A+C)\bar{z}_m + fB\bar{z}_r}{(A+fB)\bar{z}_m + C\bar{z}_r} \quad (2.5)$$

$$\psi_3(k) = \bar{y}_r \frac{\bar{x}_m}{\bar{x}_r} \frac{(A+C)\bar{Z} + fB\bar{z}_r}{(A+fB)\bar{Z} + C\bar{z}_r} \quad (2.6)$$

Where  $A = (k-1)(k-2)$ ;  $B = (k-1)(k-4)$ ;  $C = (k-2)(k-3)(k-4)$  and  $0 < k < \infty$ , is a constant. Also,  $\bar{y}_r = \frac{1}{r} \sum_{i \in R} y_i$ ,  $\bar{x}_r = \frac{1}{r} \sum_{i \in R} x_i$ ,  $\bar{z}_r = \frac{1}{r} \sum_{i \in R} z_i$ ,  $\bar{x}_m = \frac{1}{m} \sum_{i \in S'} x_i$ ,  $\bar{z}_m = \frac{1}{m} \sum_{i \in S'} z_i$ ,

$$\bar{Z} = \frac{1}{N} \sum_{i \in \Omega} Z_i.$$



Under strategies (2.1), (2.2) and (2.3) the point estimators of population mean of study variable  $\bar{Y}$  are like (2.4), (2.5) and (2.6) respectively.

## 2.2. Special Cases:

(i) At  $k=1$ ;  $A=0, B=0, C=-6$

$$\psi_1(1) = \bar{y}_r \frac{\bar{x}_m}{\bar{x}_r} \frac{\bar{Z}}{\bar{z}_m}; \quad \psi_2(1) = \bar{y}_r \frac{\bar{x}_m}{\bar{x}_r} \frac{\bar{z}_m}{\bar{z}_r}; \quad \psi_3(1) = \bar{y}_r \frac{\bar{x}_m}{\bar{x}_r} \frac{\bar{Z}}{\bar{z}_r} \quad (2.7)$$

(ii) At  $k=2$ ;  $A=0, B=-2, C=0$

$$\psi_1(2) = \bar{y}_r \frac{\bar{x}_m}{\bar{x}_r} \frac{\bar{z}_m}{\bar{Z}}; \quad \psi_2(2) = \bar{y}_r \frac{\bar{x}_m}{\bar{x}_r} \frac{\bar{z}_r}{\bar{z}_m}; \quad \psi_3(2) = \bar{y}_r \frac{\bar{x}_m}{\bar{x}_r} \frac{\bar{z}_r}{\bar{Z}} \quad (2.8)$$

(iii) At  $k=3$ ;  $A=2, B=-2, C=0$

$$\psi_1(3) = \bar{y}_r \frac{\bar{x}_m}{\bar{x}_r} \frac{\bar{Z} - f\bar{z}_m}{(1-f)\bar{Z}}; \quad \psi_2(3) = \bar{y}_r \frac{\bar{x}_m}{\bar{x}_r} \frac{\bar{z}_m - f\bar{z}_r}{(1-f)\bar{z}_m}; \quad \psi_3(3) = \bar{y}_r \frac{\bar{x}_m}{\bar{x}_r} \frac{\bar{Z} - f\bar{z}_r}{(1-f)\bar{Z}} \quad (2.9)$$

(iv) At  $k=4$ ;  $A=6, B=0, C=0$

$$\psi_1(4) = \bar{y}_r \frac{\bar{x}_m}{\bar{x}_r}; \quad \psi_2(4) = \bar{y}_r \frac{\bar{x}_m}{\bar{x}_r}; \quad \psi_3(4) = \bar{y}_r \frac{\bar{x}_m}{\bar{x}_r} \quad (2.10)$$

## 3. Properties of the estimators under proposed strategies

Let  $B(\cdot)$  and  $M(\cdot)$  be the bias and mean squared error (MSE) of the estimators under a given sampling design respectively. Let the large sample approximations as  $n \rightarrow N$  be:  $\bar{y}_r = \bar{Y}(1 + \delta_1)$ ;  $\bar{x}_r = \bar{X}(1 + \delta_2)$ ;  $\bar{x}_m = \bar{X}(1 + \delta_3)$ ;  $\bar{z}_r = \bar{Z}(1 + \delta_4)$  and  $\bar{z}_m = \bar{Z}(1 + \delta_5)$

Here,  $|\delta_i| < 1$ ;  $i = 1, 2, 3, 4, 5$ .

Using the concept of two-phase sampling, following Rao and Sitter (1995) and using the mechanism of MCAR [Heitjan and Basu (1996)], for given  $r$ ,  $n$  and  $m$ , we have:

$$\begin{aligned} E(\delta_i) &= 0; \quad i = 1, 2, 3, 4, 5; \quad E(\delta_1^2) = M_1 C_Y^2; \quad E(\delta_2^2) = M_1 C_X^2; \quad E(\delta_3^2) = M_2 C_X^2; \quad E(\delta_4^2) = M_1 C_Z^2; \\ E(\delta_5^2) &= M_2 C_Z^2; \quad E(\delta_1 \delta_2) = M_1 \rho_{YX} C_Y C_X; \quad E(\delta_1 \delta_3) = M_2 \rho_{YX} C_Y C_X; \quad E(\delta_1 \delta_4) = M_1 \rho_{YZ} C_Y C_Z; \\ E(\delta_1 \delta_5) &= M_2 \rho_{YZ} C_Y C_Z; \quad E(\delta_2 \delta_3) = M_2 C_X^2; \quad E(\delta_2 \delta_4) = M_1 \rho_{XZ} C_X C_Z; \quad E(\delta_2 \delta_5) = M_2 \rho_{XZ} C_X C_Z; \\ E(\delta_3 \delta_4) &= M_2 \rho_{XZ} C_X C_Z; \quad E(\delta_3 \delta_5) = M_2 \rho_{XZ} C_X C_Z; \quad E(\delta_4 \delta_5) = M_2 C_Z^2 \end{aligned}$$

and  $M_1 = \frac{1}{r} - \frac{1}{N}$ ;  $M_2 = \frac{1}{m} - \frac{1}{N}$ ;  $M_3 = M_1 - M_2 = \frac{1}{r} - \frac{1}{m}$ .

**Remark 3.1:** Define the symbols

$$\begin{aligned} \phi_1 &= \frac{fB}{A + fB + C}; \quad \phi_2 = \frac{C}{A + fB + C}; \quad \phi_3 = \frac{A + C}{A + fB + C}; \quad \phi_4 = \frac{A + fB}{A + fB + C}; \quad (\phi_1 + \phi_3) = (\phi_2 + \phi_4) = 1 \\ \phi &= (\phi_1 - \phi_2) = -(\phi_3 - \phi_4); \quad K_{YX} = \rho_{YX} \frac{C_Y}{C_X}; \quad K_{YZ} = \rho_{YZ} \frac{C_Y}{C_Z}; \quad K_{XZ} = \rho_{XZ} \frac{C_X}{C_Z} \end{aligned}$$

**Theorem 3.1:**

[a<sub>1</sub>] The estimator  $\psi_1(k)$  in terms of  $\delta_i$ ;  $i=1,2,3,4,5$  up to the first order of approximation is:

$$\psi_1(k) = \bar{Y} \left[ 1 + \delta_1 - \delta_2 + \delta_3 - \delta_1\delta_2 + \delta_1\delta_3 - \delta_2\delta_3 + \delta_2^2 + \phi(\delta_5 + \delta_1\delta_5 - \delta_2\delta_5 + \delta_3\delta_5 - \phi_2\delta_5^2) \right] \quad (3.1)$$

[a<sub>2</sub>] Bias of  $\psi_1(k)$ :

$$B[\psi_1(k)] = \bar{Y} \left[ M_3 C_X^2 (1 - K_{YX}) - \phi M_2 C_Z^2 (\phi_2 - K_{YZ}) \right] \quad (3.2)$$

[a<sub>3</sub>] Mean squared error of  $\psi_1(k)$ :

$$M[\psi_1(k)] = \bar{Y}^2 \left[ M_1 C_Y^2 + M_3 C_X^2 (1 - 2K_{YX}) - \phi M_2 C_Z^2 (\phi + 2K_{YZ}) \right] \quad (3.3)$$

[a<sub>4</sub>] Minimum MSE of the estimator  $\psi_1(k)$  is when  $\phi = -K_{YZ}$  holds and the expression is:

$$M[\psi_1(k)]_{\min} = \bar{Y}^2 \left[ M_1 C_Y^2 + M_3 C_X^2 (1 - 2K_{YX}) + M_2 K_{YZ}^2 C_Z^2 \right] \quad (3.4)$$

**Proof:**

$$\begin{aligned} [a_1] \quad \psi_1(k) &= \bar{y}_r \frac{\bar{x}_m}{\bar{x}_r} \left[ \frac{(A+C)\bar{Z} + fB\bar{z}_m}{(A+fB)\bar{Z} + C\bar{z}_m} \right] \\ &= \bar{Y} (1 + \delta_1) (1 + \delta_2)^{-1} (1 + \delta_3) (1 + \phi_1\delta_5) (1 + \phi_2\delta_5^2)^{-1} \\ &= \bar{Y} \left[ 1 + \delta_1 - \delta_2 + \delta_3 - \delta_1\delta_2 + \delta_1\delta_3 - \delta_2\delta_3 + \delta_2^2 + \phi(\delta_5 + \delta_1\delta_5 - \delta_2\delta_5 + \delta_3\delta_5 - \phi_2\delta_5^2) \right] \end{aligned}$$

$$[a_2] \quad B[\psi_1(k)] = E[\psi_1(k) - \bar{Y}] = [E[\psi_1(k)] - \bar{Y}]$$

Using (3.1) and taking expectation both sides

$$\begin{aligned} E[\psi_1(k)] &= \bar{Y} E \left[ 1 + \delta_1 - \delta_2 + \delta_3 - \delta_1\delta_2 + \delta_1\delta_3 - \delta_2\delta_3 + \delta_2^2 + \phi(\delta_5 + \delta_1\delta_5 - \delta_2\delta_5 + \delta_3\delta_5 - \phi_2\delta_5^2) \right] \\ &= \bar{Y} \left[ 1 + M_3 C_X^2 (1 - K_{YX}) - \phi M_2 C_Z^2 (\phi_2 - K_{YZ}) \right] \\ B[\psi_1(k)] &= \bar{Y} \left[ M_3 C_X^2 (1 - K_{YX}) - \phi M_2 C_Z^2 (\phi_2 - K_{YZ}) \right] \end{aligned}$$

$$\begin{aligned} [a_3] \quad M[\psi_1(k)] &= E[\psi_1(k) - \bar{Y}]^2 \\ &= E \left[ \bar{Y} \left\{ 1 + \delta_1 - \delta_2 + \delta_3 - \delta_1\delta_2 + \delta_1\delta_3 - \delta_2\delta_3 + \delta_2^2 + \phi(\delta_5 + \delta_1\delta_5 - \delta_2\delta_5 + \delta_3\delta_5 - \phi_2\delta_5^2) \right\} - \bar{Y} \right]^2 \\ &\quad \text{[Using (3.1)]} \\ &= \bar{Y}^2 \left[ M_1 C_Y^2 + M_3 C_X^2 (1 - 2K_{YX}) - \phi M_2 C_Z^2 (\phi + 2K_{YZ}) \right] \end{aligned}$$

[a<sub>4</sub>] To obtain minimum MSE, let

$$\begin{aligned} \frac{d}{d\phi} M[\psi_1(k)] &= 0 \Rightarrow \bar{Y}^2 \left[ M_2 C_Z^2 (2\phi + 2K_{YZ}) \right] = 0 \Rightarrow \phi = -K_{YZ} \\ M[\psi_1(k)]_{\min} &= \bar{Y}^2 \left[ M_1 C_Y^2 + M_3 C_X^2 (1 - 2K_{YX}) + M_2 K_{YZ}^2 C_Z^2 \right] \end{aligned}$$

based...

**Theorem 3.2:**

[a<sub>5</sub>] The estimator  $\psi_2(k)$  in terms of  $\delta_i$ ;  $i=1,2,3,4,5$  up to the first order of approximation is:

$$\psi_2(k) = \bar{Y} \left[ 1 + \delta_1 - \delta_2 + \delta_3 - \delta_1\delta_2 + \delta_1\delta_3 - \delta_2\delta_3 + \delta_2^2 + \phi(\delta_4 - \delta_5 + \delta_1\delta_4 - \delta_1\delta_5 - \delta_2\delta_4 + \delta_2\delta_5 + \delta_3\delta_4 - \delta_3\delta_5 + (\phi_2 - \phi_4)\delta_4\delta_5 - \phi_2\delta_4^2 + \phi_4\delta_5^2) \right] \quad (3.5)$$

[a<sub>6</sub>] Bias of the estimator  $\psi_2(k)$ :

$$B[\psi_2(k)] = \bar{Y} M_3 [C_X^2 (1 - K_{YX}) - \phi C_Z^2 (\phi_2 - K_{YZ} + K_{XZ})] \quad (3.6)$$

[a<sub>7</sub>] Mean squared error of  $\psi_2(k)$ :

$$M[\psi_2(k)] = \bar{Y}^2 [M_1 C_Y^2 + M_3 \{C_X^2 (1 - 2K_{YX}) + \phi C_Z^2 (\phi + 2K_{YZ} - 2K_{XZ})\}] \quad (3.7)$$

[a<sub>8</sub>] Minimum MSE of  $\psi_2(k)$  is at  $\phi = (-K_{YZ} + K_{XZ})$ :

$$M[\psi_2(k)]_{\min} = \bar{Y}^2 [M_1 C_Y^2 + M_3 \{ (1 - 2K_{YX}) C_X^2 - (K_{YZ} - K_{XZ})^2 C_Z^2 \}] \quad (3.8)$$

**Proof:**

$$\begin{aligned} [a_5] \quad \psi_2(k) &= \bar{y}_r \left( \frac{\bar{x}_m}{\bar{x}_r} \right) \left[ \frac{(A+C)\bar{z}_m + fB\bar{z}_r}{(A+fB)\bar{z}_m + C\bar{z}_r} \right] \\ &= \bar{Y} (1 + \delta_1) (1 + \delta_2)^{-1} (1 + \delta_3) (1 + \phi_1\delta_4 + \phi_3\delta_5) (1 + \phi_2\delta_4 + \phi_4\delta_5)^{-1} \\ &= \bar{Y} \left[ 1 + \delta_1 - \delta_2 + \delta_3 - \delta_1\delta_2 + \delta_1\delta_3 - \delta_2\delta_3 + \delta_2^2 + \phi \{ \delta_4 - \delta_5 + \delta_1\delta_4 - \delta_1\delta_5 \right. \\ &\quad \left. - \delta_2\delta_4 + \delta_2\delta_5 + \delta_3\delta_4 - \delta_3\delta_5 + (\phi_2 - \phi_4)\delta_4\delta_5 - \phi_2\delta_4^2 + \phi_4\delta_5^2 \} \right] \end{aligned}$$

$$[a_6] \quad B[\psi_2(k)] = E[\psi_2(k) - \bar{Y}] = E[\psi_2(k)] - \bar{Y}$$

Using (3.5) and taking the expectation both sides,

$$\begin{aligned} E[\psi_2(k)] &= \bar{Y} E \left[ 1 - \delta_1\delta_2 + \delta_1\delta_3 - \delta_2\delta_3 + \delta_2^2 + \phi \{ \delta_1\delta_4 - \delta_1\delta_5 - \delta_2\delta_4 + \delta_2\delta_5 + \delta_3\delta_4 - \delta_3\delta_5 \right. \\ &\quad \left. + (\phi_2 - \phi_4)\delta_4\delta_5 - \phi_2\delta_4^2 + \phi_4\delta_5^2 \} \right] \\ &= \bar{Y} \left[ 1 + M_3 \{ C_X^2 (1 - K_{YX}) - \phi C_Z^2 (\phi_2 - K_{YZ} + K_{XZ}) \} \right] \end{aligned}$$

$$\begin{aligned} B[\psi_2(k)] &= E[\psi_2(k)] - \bar{Y} \\ &= M_3 \bar{Y} [C_X^2 (1 - K_{YX}) - \phi C_Z^2 (\phi_2 - K_{YZ} + K_{XZ})] \end{aligned}$$

$$\begin{aligned} [a_7] \quad M[\psi_2(k)] &= E[\psi_2(k) - \bar{Y}]^2 \\ &= \bar{Y}^2 E \left[ (\delta_1 - \delta_2 + \delta_3 + \phi(\delta_4 - \delta_5))^2 \right] \quad [\text{Using (3.5)}] \end{aligned}$$

$$M[\psi_2(k)] = \bar{Y}^2 [M_1 C_Y^2 + M_3 \{ C_X^2 (1 - 2K_{YX}) + \phi C_Z^2 (\phi + 2K_{YZ} - 2K_{XZ}) \}]$$

[a<sub>8</sub>] To obtain minimum MSE, let

$$\frac{d}{d\phi} M[\psi_2(k)] = 0 \Rightarrow \phi = K_{XZ} - K_{YZ}$$

and substitution provides

$$M[\psi_2(k)]_{\min} = \bar{Y}^2 \left[ M_1 C_Y^2 + M_3 \left\{ (1 - 2K_{YX}) C_X^2 - (K_{YZ} - K_{XZ})^2 C_Z^2 \right\} \right]$$

**Theorem 3.3:**

[a<sub>9</sub>] The estimator  $\psi_3(k)$  in terms of  $\delta_i$ ;  $i=1,2,3,4,5$  up to the first order of approximation could be expressed as:

$$\psi_3(k) = \bar{Y} [1 + \delta_1 - \delta_2 + \delta_3 - \delta_1 \delta_2 + \delta_1 \delta_3 - \delta_2 \delta_3 + \delta_2^2 + \phi (\delta_4 + \delta_1 \delta_4 - \delta_2 \delta_4 + \delta_3 \delta_4 - \phi_2 \delta_4^2)] \quad (3.9)$$

[a<sub>10</sub>] Bias of  $\psi_3(k)$  :

$$B[\psi_3(k)] = \bar{Y} [M_3 C_X^2 (1 - K_{YX}) + \phi C_Z^2 (M_1 K_{YZ} - M_3 K_{XZ} - M_1 \phi_2)] \quad (3.10)$$

[a<sub>11</sub>] Mean squared error of  $\psi_3(k)$  :

$$M[\psi_3(k)] = \bar{Y}^2 [M_1 C_Y^2 + M_3 C_X^2 (1 - 2K_{YX}) + \phi C_Z^2 (\phi M_1 + 2M_1 K_{YZ} - 2M_3 K_{XZ})] \quad (3.11)$$

[a<sub>12</sub>] Minimum MSE of  $\psi_3(k)$  is when  $\phi = M_1^{-1} (M_3 K_{XZ} - M_1 K_{YZ})$  and the expression is:

$$M[\psi_3(k)]_{\min} = \bar{Y}^2 [M_1 C_Y^2 + M_3 C_X^2 (1 - 2K_{YX}) - (M_3 K_{XZ} - M_1 K_{YZ})^2 M_1^{-1} C_Z^2] \quad (3.12)$$

**Proof:**

$$\begin{aligned} [a_9] \quad \psi_3(k) &= \bar{y}_r \left( \frac{\bar{x}_m}{\bar{x}_r} \right) \left[ \frac{(A+C)\bar{Z} + fB\bar{z}_r}{(A+fB)\bar{Z} + C\bar{z}_r} \right] = \bar{Y} (1 + \delta_1) (1 + \delta_2)^{-1} (1 + \delta_3) (1 + \phi_1 \delta_4) (1 + \phi_2 \delta_4)^{-1} \\ &= \bar{Y} (1 + \delta_1) (1 - \delta_2 + \delta_2^2 - \delta_3^3 + \dots) (1 + \delta_3) (1 + \phi_1 \delta_4) (1 - \phi_2 \delta_4 + \phi_2^2 \delta_4^2 - \phi_2^2 \delta_4^3 + \dots) \\ &= \bar{Y} [1 + \delta_1 - \delta_2 + \delta_3 - \delta_1 \delta_2 + \delta_1 \delta_3 - \delta_2 \delta_3 + \delta_2^2 + \phi (\delta_4 + \delta_1 \delta_4 - \delta_2 \delta_4 + \delta_3 \delta_4 - \phi_2 \delta_4^2)] \end{aligned}$$

$$\begin{aligned} [a_{10}] \quad B[\psi_3(k)] &= E[\psi_3(k) - \bar{Y}] \\ &= \bar{Y} E[\delta_1 - \delta_2 + \delta_3 - \delta_1 \delta_2 + \delta_1 \delta_3 - \delta_2 \delta_3 + \delta_2^2 + \phi (\delta_4 + \delta_1 \delta_4 - \delta_2 \delta_4 + \delta_3 \delta_4 - \phi_2 \delta_4^2)] \\ &= \bar{Y} [M_3 C_X^2 (1 - K_{YX}) + \phi C_Z^2 (M_1 K_{YZ} - M_3 K_{XZ} - M_1 \phi_2)] \end{aligned}$$

$$\begin{aligned} [a_{11}] \quad M[\psi_3(k)] &= E[\psi_3(k) - \bar{Y}]^2 = \bar{Y}^2 E[\delta_1 - \delta_2 + \delta_3 + \phi \delta_4]^2 \\ &= \bar{Y}^2 [M_1 C_Y^2 + M_3 C_X^2 - 2M_3 \rho_{YX} C_Y C_X + \phi^2 M_1 C_Z^2 \\ &\quad + 2\phi (M_1 \rho_{YZ} C_Y C_Z - M_3 \rho_{XZ} C_X C_Z)] \\ &= \bar{Y}^2 [M_1 C_Y^2 + M_3 C_X^2 (1 - 2K_{YX}) + \phi C_Z^2 (\phi M_1 + 2M_1 K_{YZ} - 2M_3 K_{XZ})] \end{aligned}$$

[a<sub>12</sub>] To obtain minimum MSE, let

$$\frac{d}{d\phi} M[\psi_3(k)] = 0 \Rightarrow \phi = M_1^{-1} (M_3 K_{XZ} - M_1 K_{YZ})$$

and substitution provides

$$M[\psi_3(k)]_{\min} = \bar{Y}^2 [M_1 C_Y^2 + M_3 C_X^2 (1 - 2K_{YX}) - (M_3 K_{XZ} - M_1 K_{YZ})^2 M_1^{-1} C_Z^2]$$

#### 4. Comparison of the estimators under proposed imputation strategies

$$[b_1] \quad D_1 = M[\psi_1(k)]_{\min} - M[\psi_2(k)]_{\min} \\ = \bar{Y}^2 C_Z^2 [M_3(K_{YZ} - K_{XZ})^2 - M_2 K_{YZ}^2] \quad (4.1)$$

$\psi_2(k)$  is better over  $\psi_1(k)$  if  $D_1 > 0$

$$\Rightarrow \frac{K_{YZ} - K_{XZ}}{K_{YZ}} > \sqrt{\frac{M_2}{M_3}} \Rightarrow F_1 > F_2 \quad (\text{let})$$

$$[b_2] \quad D_2 = M[\psi_1(k)]_{\min} - M[\psi_3(k)]_{\min} \\ = \bar{Y}^2 C_Z^2 [(M_3 K_{XZ} - M_1 K_{YZ})^2 M_1^{-1} - M_2 K_{YZ}^2] \quad (4.2)$$

$\psi_3(k)$  is better over  $\psi_1(k)$  if  $D_2 > 0$

$$\Rightarrow \frac{K_{XZ}}{K_{YZ}} > \frac{M_1 + \sqrt{M_1 M_2}}{M_3} \Rightarrow F_3 > F_4 \quad (\text{let})$$

$$[b_3] \quad D_3 = M[\psi_2(k)]_{\min} - M[\psi_3(k)]_{\min} \\ = \bar{Y}^2 C_Z^2 [(M_3 K_{XZ} - M_1 K_{YZ})^2 M_1^{-1} - M_3 (K_{YZ} - K_{XZ})^2] \quad (4.3)$$

$\psi_3(k)$  is better than  $\psi_2(k)$  if  $D_3 > 0$

$$\Rightarrow \frac{K_{XZ}}{K_{YZ}} > \frac{M_1 + \sqrt{M_1 M_3}}{M_3 + \sqrt{M_1 M_3}} \Rightarrow F_3 > F_5 \quad (\text{let})$$

#### 5. Empirical study

For numerical study consider the data as attached in Appendix A, which is a generated artificial population of size  $N = 200$  containing values of main variable  $Y$  and auxiliary variables  $X, Z$ . Parameters of this population are given below:

$$\bar{Y} = 42.485; \bar{X} = 18.515; \bar{Z} = 20.52; S_Y^2 = 199.0598; S_X^2 = 48.5375; S_Z^2 = 45.7684;$$

$$\rho_{YX} = 0.8734; \rho_{YZ} = 0.8667; \rho_{XZ} = 0.9943; C_Y = 0.3287; C_X = 0.3755; C_Z = 0.3296;$$

$$K_{YZ} = 0.8643; K_{XZ} = 1.1326; K_{YX} = 0.7645$$

Reddy (1978) proved that  $K_{YX}$ ,  $K_{YZ}$ ,  $K_{XZ}$  are ratio values and bear very small change over a span of time. It could be easily guessed or assumed to be known a priori. Using preliminary large sample of size  $m = 80$  and sub-random sample of size  $n = 30$  with the number of responding units  $r = 22$  and  $f = 0.15$  by SRSWOR. The optimum values of constants of different estimators at their optimal condition are  $\alpha = 0.2354$ ,  $\beta_1 = \beta_2 = \beta_3 = 0.7646$ ,  $k_1' = 1.5206$ ,  $k_2' = 2.4505$ ,  $k_3' = 8.9456$  for compromised, Ahmed's methods and Factor Type F-T Estimators of imputation respectively. By simplifying optimum conditions of proposed estimators for minimum MSE, the cubic equations provide the values of constants  $k$  as shown in Table 5.1.

**Table 5.1.** Optimum  $k$ -values for minimum MSE of proposed estimators

Estimators	Condition for Optimum MSE	Three optimum Values of $k$ on one condition		
$\psi_1(k)$	$\phi = -K_{YZ}$	$k_1 = 1.3137$	$k_2 = 2.5180$	$k_3 = 13.5979$
$\psi_2(k)$	$\phi = K_{XZ} - K_{YZ}$	$k_4 = 1.9321$	$k_5 = \text{-----}$	$k_6 = \text{-----}$
$\psi_3(k)$	$\phi = M_1^{-1}(M_3K_{XZ} - M_1K_{YZ})$	$k_7 = 1.8759$	$k_8 = 3.2154$	$k_9 = 4.0919$

**Note:**  $k_5, k_6$  do not exist because the solution of cubic equations provided no real roots.

The formula for efficiency measurement is  $e(\hat{T}) = \frac{MSE(\bar{y}_r)}{MSE(\hat{T})}$ , where  $\hat{T}$  is any estimator under consideration. The steps followed for the simulation procedure are:

**Step 1:** Draw a preliminary random sample  $S'$  of size  $m = 80$  from the population of size 200.

**Step 2:** Again draw a random sub-sample of size  $n = 30$  from  $S'$  drawn in step 1.

**Step 3:** Drop away 8 units randomly from each sample corresponding to variable  $Y$ .

**Step 4:** Compute and impute the dropped units of  $Y$  with the help of existing and proposed imputation methods.

**Step 5:** Obtain the estimates of the population mean for existing and proposed imputation methods.

**Step 6:** Repeat the above steps (1 to 5) 50,000 times, which provides multiple sample based estimates  $\hat{T}_1, \hat{T}_2, \hat{T}_3, \dots, \hat{T}_{50,000}$ .

**Step 7:** The bias of  $\hat{T}$  is obtained by  $B(\hat{T}) = \frac{1}{50000} \sum_{i=1}^{50000} (\hat{T}_i - \bar{Y})$ .

**Step 8:** The MSE of  $\hat{T}$  is obtained by  $MSE(\hat{T}) = \frac{1}{50000} \sum_{i=1}^{50000} (\hat{T}_i - \bar{Y})^2$ .

Following the above procedure bias and MSE of the existing and proposed estimators are computed based on 50,000 repeated samples drawn by SRSWOR from population of  $N = 200$ . These computations and efficiencies with respect to  $\bar{y}_r$  are given in Tables 5.2 and 5.3 respectively.

**Table 5.2.** Bias and MSE of existing estimators

Estimators	Optimum Value	Bias	MSE	Efficiency
$\bar{y}_r$	-----	-0.3123	9.7252	1
$\bar{y}_{RAT}$	-----	-0.0996	7.8457	1.2395
$\bar{y}_{COMP}$	$\alpha = 0.2354$	-0.0809	6.9649	1.3963
$t_1$	$\beta_1 = 0.7646$	-0.3983	5.8967	1.6492

based...

Table 5.2. Bias and MSE of existing estimators (cont.)

Estimators	Optimum Value	Bias	MSE	Efficiency
$t_2$	$\beta_2 = 0.7646$	-0.1871	7.6655	1.2686
$t_3$	$\beta_3 = 0.7646$	-0.2151	3.2967	2.9499
$T_{FT1}$	$k_1' = 1.5206$	-0.3878	4.8327	2.0123
	$k_2' = 2.4505$	-0.3736	5.1655	1.8827
	$k_3' = 8.9456$	-0.3961	4.9454	1.9665
$T_{FT2}$	$k_1' = 1.5206$	-0.1071	6.3071	1.5419
	$k_2' = 2.4505$	-0.0329	6.1072	1.5924
	$k_3' = 8.9456$	-0.0980	6.0561	1.6058
$T_{FT3}$	$k_1' = 1.5206$	-0.1826	1.8399	5.2857
	$k_2' = 2.4505$	-0.1944	2.2685	4.2870
	$k_3' = 8.9456$	-0.1818	1.9894	4.8885

5.1. Numerical computation of proposed estimators

From Section 4.0 we get computational values of conditions on the population given in Appendix A.  $F_1 = \frac{K_{YZ} - K_{XZ}}{K_{YZ}} = -0.3104$ ;  $F_2 = \sqrt{\frac{M_2}{M_3}} = 0.4774$ ;  $F_3 = \frac{K_{XZ}}{K_{YZ}} = 1.3104$ ;  $F_4 = \frac{M_1 + \sqrt{M_1 M_2}}{M_3} = 1.7570$  and  $F_5 = \frac{M_1 + \sqrt{M_1 M_3}}{M_3 + \sqrt{M_1 M_3}} = 1.1082$

Since  $F_1 < F_2$  holds,  $\psi_1(k)$  is better than  $\psi_2(k)$  for this data set. Again,  $F_3 < F_4$ , which implies  $\psi_1(k)$  is better than  $\psi_3(k)$  for the data set, and  $F_3 > F_5$ , which implies  $\psi_3(k)$  is better than  $\psi_2(k)$  for this data set. Overall  $\psi_1(k)$  is the best estimator.

Table 5.3. Bias and MSE of proposed chain type estimators

Estimator	<i>k-optimum</i>	Bias	MSE	Efficiency
$\psi_1(k)$	$k_1 = 1.3137$	-0.0030	1.9169	5.0734
	$k_2 = 2.5180$	0.0215	1.9328	5.0317
	$k_3 = 13.5979$	-0.0038	1.9409	5.0106
$\psi_2(k)$	$k_4 = 1.9321$	0.3534	9.0303	1.0769
	$k_5 = \text{-----}$	—	—	-----
	$k_6 = \text{-----}$	—	—	-----
$\psi_3(k)$	$k_7 = 1.8759$	0.6036	8.6779	1.1206
	$k_8 = 3.2154$	0.6215	8.6360	1.1261
	$k_9 = 4.0919$	0.5992	8.6621	1.1227

## 6. Almost unbiased imputation based chain type estimator

By expression (3.2), (3.6) and (3.10), bias of  $\psi_i(k)$ ;  $i = 1, 2, 3$  could be made zero up to the first order of approximation. This provides three equations:

$$M_3 C_X^2 (1 - K_{YX}) - \phi M_2 C_Z^2 (\phi_2 - K_{YZ}) = 0 \quad (6.1)$$

$$C_X^2 (1 - K_{YX}) - \phi C_Z^2 (\phi_2 - K_{YZ} + K_{XZ}) = 0 \quad (6.2)$$

$$\text{and} \quad M_3 C_X^2 (1 - K_{YX}) + \phi C_Z^2 (M_1 K_{YZ} - M_3 K_{XZ} - M_1 \phi_2) = 0 \quad (6.3)$$

These equations are cubic or more function of  $k$ -values to provide multiple values of  $k$  on which bias is zero. The best choice is to have lowest mean squared error. So, the proposed estimators bear property of reducing  $MSE$  along with being almost unbiased also. Many similar estimators existing in the literature do not control both bias and  $MSE$  at their optimal level but the proposed estimators have this property. For equation (6.1), we get two real values  $k_1'' = 0.3829$  and  $k_2'' = 6.5038$  and from (6.2) and (6.3) all values are imaginary, viz. there are no real roots. These results are obtained using the data set on which the empirical study was performed. The term almost unbiased is used because biases of proposed estimates  $\psi_i(k)$  are obtained only up to the first order of approximation. The bias  $B[\psi_2(k)] = 0$  holds approximately not completely, therefore mentioned almost unbiased.

**Table 6.1.** Almost unbiased comparison of chain type estimators

$k$ -values	$\psi_1(k)$		$\psi_2(k)$		$\psi_3(k)$	
	Bias	MSE	Bias	MSE	Bias	MSE
$k_1'' = 0.3829$	0.0005	4.4522	0.0002	15.4062	0.0002	14.4033
$k_2'' = 6.5038$	0.0004	2.4831	0.0001	7.4559	0.0011	6.4898

## 7. Discussion and conclusions

In the present article some imputation procedures and their estimators of population mean are suggested and the expression of their bias, mean squared error and minimum mean squared error have been derived under large sample approximations up to the first order. An empirical study has been done over a data set and the bias and mean squared error have been calculated. Among the existing and proposed estimators, under Chain-based imputation strategies, i.e.  $\psi_i(k)$ ; ( $i = 1, 2, 3$ ), the estimator  $\psi_1(k)$  is found best. The general perception regarding imputation of missing data is that it increases the bias of the estimate when  $MSE$  is optimized. In contrary, a key feature of  $\psi_i(k)$ ; ( $i = 1, 2, 3$ ) is that there are many values of the parameter  $k$  on which  $MSE$  is optimal. One can choose the value with the lowest bias. Therefore, suggested strategies are bias-controlled at the optimum level of  $MSE$ . Apart from this, estimators are almost unbiased also over multiple choices of  $k$ -values. The



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best selection is to have the lowest *MSE* by proposed strategies one can get almost unbiased estimator with lowest possible *MSE*. Thus, the suggested Chain-based imputation strategies  $\psi_i(k)$ ; ( $i = 1, 2, 3$ ) are useful and have advantage over other similar procedures.

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## Appendix

### A. Population (N = 200)

$Y_i$	45	50	39	60	42	38	28	42	38	35
$X_i$	15	20	23	35	18	12	8	15	17	13
$Z_i$	16	22	26	37	19	14	11	17	18	15
$Y_i$	40	55	45	36	40	58	56	62	58	46
$X_i$	29	35	20	14	18	25	28	21	19	18
$Z_i$	30	37	23	15	19	27	30	22	21	21
$Y_i$	36	43	68	70	50	56	45	32	30	38
$X_i$	15	20	38	42	23	25	18	11	09	17
$Z_i$	18	22	39	44	25	26	19	13	12	20
$Y_i$	35	41	45	65	30	28	32	38	61	58
$X_i$	13	15	18	25	09	08	11	13	23	21
$Z_i$	16	17	19	27	12	10	13	14	24	23
$Y_i$	65	62	68	85	40	32	60	57	47	55
$X_i$	27	25	30	45	15	12	22	19	17	21
$Z_i$	28	26	33	46	17	15	23	20	19	23
$Y_i$	67	70	60	40	35	30	25	38	23	55
$X_i$	25	30	27	21	15	17	09	15	11	21
$Z_i$	26	32	30	23	17	18	12	18	14	24
$Y_i$	50	69	53	55	71	74	55	39	43	45
$X_i$	15	23	29	30	33	31	17	14	17	19
$Z_i$	17	24	30	33	35	32	19	16	19	21
$Y_i$	61	72	65	39	43	57	37	71	71	70
$X_i$	25	31	30	19	21	23	15	30	32	29
$Z_i$	27	33	32	21	23	25	17	32	33	32
$Y_i$	73	63	67	47	53	51	54	57	59	39
$X_i$	28	23	23	17	19	17	18	21	23	20
$Z_i$	30	25	24	20	22	20	21	23	26	22
$Y_i$	23	25	35	30	38	60	60	40	47	30
$X_i$	07	09	15	11	13	25	27	15	17	11
$Z_i$	10	11	18	14	14	26	29	18	20	14
$Y_i$	57	54	60	51	26	32	30	45	55	54
$X_i$	31	23	25	17	09	11	13	19	25	27
$Z_i$	32	25	27	19	12	13	14	20	27	28
$Y_i$	33	33	20	25	28	40	33	38	41	33
$X_i$	13	11	07	09	13	15	13	17	15	13
$Z_i$	16	14	9	10	14	17	14	20	17	15
$Y_i$	30	35	20	18	20	27	23	42	37	45
$X_i$	11	15	08	07	09	13	12	25	21	22
$Z_i$	13	18	11	8	12	16	14	26	24	23

$Y_i$	37	37	37	34	41	35	39	45	24	27
$X_i$	15	16	17	13	20	15	21	25	11	13
$Z_i$	16	18	19	16	22	18	23	26	14	14
$Y_i$	23	20	26	26	40	56	41	47	43	33
$X_i$	09	08	11	12	15	25	15	25	21	15
$Z_i$	11	10	14	15	17	26	17	27	22	17
$Y_i$	37	27	21	23	24	21	39	33	25	35
$X_i$	17	13	11	11	09	08	15	17	11	19
$Z_i$	19	16	13	12	12	11	17	20	13	20
$Y_i$	45	40	31	20	40	50	45	35	30	35
$X_i$	21	23	15	11	20	25	23	17	16	18
$Z_i$	22	25	18	13	21	27	26	19	17	19
$Y_i$	32	27	30	33	31	47	43	35	30	40
$X_i$	15	13	14	17	15	25	23	17	16	19
$Z_i$	17	16	16	14	17	28	25	18	18	22
$Y_i$	35	35	46	39	35	30	31	53	63	41
$X_i$	19	19	23	15	17	13	19	25	35	21
$Z_i$	22	21	24	17	20	15	22	26	36	23
$Y_i$	52	43	39	37	20	23	35	39	45	37
$X_i$	25	19	18	17	11	09	15	17	19	19
$Z_i$	26	20	20	19	13	12	17	18	21	22