

Fischer, Matthias J.

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Tailoring copula-based multivariate generalized hyperbolic secant distributions to financial return data: An empirical investigation

Matthias Fischer†

Institute of Statistics and Econometrics
University of Erlangen-Nürnberg,
Lange Gasse 20, D-90403 Nürnberg, Germany

†*Email:* Matthias.Fischer@wiso.uni-erlangen.de

Abstract

One of the crucial questions in risk management is how to aggregate individual risk into overall portfolio risk.

JEL classification:

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1 Introduction

2 Multivariate distributions: The copula approach

There are several possibilities to construct two-dimensional distribution families proposed in the literature, one of them being the so-called copula approach, where "copulas" play the central part. The axiomatic definition of such a (two dimensional) copula is as follows:

Definition 2.1 (2-Copula) *A two-dimensional copula (or 2-copula) is a function $C : [0, 1] \times [0, 1] \rightarrow [0, 1]$ which satisfies the following properties:*

1. C is 2-increasing, i.e. for $0 \leq u_1 \leq v_1 \leq 1$ and $0 \leq u_2 \leq v_2 \leq 1$ holds:

$$C(v_1, v_2) - C(v_1, u_2) - C(u_1, v_2) + C(u_1, u_2) \geq 0.$$

2. For all $u, v \in [0, 1]$: $C(u, 0) = C(0, v) = 0$ and $C(u, 1) = C(1, v) = u$.

Theorem 2.1 (Sklar) *Let F_1 and F_2 be two univariate distribution functions. Then $C(F_1(x_1), F_2(x_2))$ defines a bivariate probability distribution with margins F_1 and F_2 . On the other hand, let F be a 2-dimensional distribution function with margins F_1 and F_2 . Then F has the copula representation $F(x_1, x_2) = C(F_1(x_1), F_2(x_2))$. The copula C is unique if the margins are continuous.*

Following this fundamental theorem of Sklar (1957), the procedure to construct a multivariate distribution is fairly simple at first sight:

1. Choose a suitable copula C that can rebuild the dependence structure between financial returns.
2. Choose two suitable marginal distributions F_X and F_Y which rebuild the distributional stylized facts of log-returns.
3. The bivariate distribution function can then be obtained by means of

$$F_{X,Y}(x, y) = C(F_X(x), F_Y(y)) \tag{1}$$

or, in terms of a bivariate density function, by means of

$$f_{X,Y}(x, y) = c(F_X(x), F_Y(y)) \cdot f_X(x) \cdot f_Y(y) \quad \text{with} \quad c(x, y) = \frac{\partial^2 C(x, y)}{\partial x \partial y} \tag{2}$$

It is obvious that the main obstacle is not to construct the bivariate distribution (assuming that the margins and the copula are given) but to select the margins and – above all – the copulas knowing that there are dozens of proposals for a copula (see, for example, Joe (1997)) and several flexible multi-parametric distribution families. For that reason we postulate certain requirements on both the margins and the copulas which will be discussed in the next two sections.

3 The choice of the marginal distribution

A marginal distribution should essentially fulfil two requirements: Firstly and most important, the underlying distribution model is proclaimed to be flexible with respect to different combination of skewness and kurtosis. Secondly, we also proclaim certain "numerical" properties. In particular – according to equation (2) – a cumulative distribution function in closed form will avoid numerical integration methods and save computational time. In the literature, several flexible distribution families have been proposed up to now. All of them more or less seem to satisfy the first requirement. Take for example the generalized hyperbolic family (see Eberlein and Keller (1995) or Prause (1999)), the generalized logistic family (see McDonald (1992) or Fischer (2002)), the α -stable family (see Mittnik, Paolella and Rachev (1997)) or families of generalized t-distributions (see Theodossiou (1998) or Grottko (2001)). However, neither the generalized hyperbolic distribution nor the family of generalized logistic distribution, neither the α -stable family nor the generalized t-distributions provide a closed form for the cumulative distribution function. For that reason we put emphasize on a distribution to which not much attention was paid in the context of financial return data: The family of skewed generalized secant distribution which has its origin in the hyperbolic secant distribution which was studied first by Baten(1934) and Talacko (1956) and has density

$$f_{HS}(x) = \frac{1}{2 \cosh(\pi x/2)}$$

Since 1956, two generalization have been proposed which allow for a more flexible form concerning skewness and leptokurtosis: The first proposal was introduced by Morris (1982) in the context of natural exponential families (NEF) with quadratic variance function (i.e. the variance is a quadratic function of the mean). In this class with six members, one is generated by the hyperbolic secant distribution, the so-called NEF-GHS distribution with pdf

$$f(x; \lambda, \theta) = \frac{2^{\lambda-2}}{\pi \Gamma(\lambda)} \cdot \left| \Gamma \left(\frac{\lambda + ix}{2} \right) \right|^2 \cdot \exp \{ \theta x + \lambda \log(\cos(\theta)) \}$$

for $\lambda > 0$ and $|\theta| < \pi/2$. The NEF-GHS distribution allows for skewness and arbitrarily high excess kurtosis. Morris showed that this class is again reproductive, infinitely divisible with existing moment-generating function and existing moments. However, the corresponding cumulative distribution function doesn't admit a closed form.

Recently, Vaughan (2002) proposes a family of symmetric distributions — which he called *generalized secant hyperbolic (GSH) distribution* — with kurtosis ranging from 1.8 to infinity which includes both the hyperbolic secant and the logistic distribution and closely approximates the Student t-distribution with corresponding kurtosis. The probability density function of a GSH variate is given by

$$f_{GSH}(x; t) = c_1(t) \cdot \frac{\exp(c_2(t)x)}{\exp(2c_2(t)x) + 2a(t) \exp(c_2(t)x) + 1}, \quad x \in \mathbb{R} \quad (3)$$

with

$$\begin{aligned} a(t) &= \cos(t), & c_2(t) &= \sqrt{\frac{\pi^2 - t^2}{3}} & c_1(t) &= \frac{\sin(t)}{t} \cdot c_2(t), & \text{for } -\pi < t \leq 0, \\ a(t) &= \cosh(t), & c_2(t) &= \sqrt{\frac{\pi^2 + t^2}{3}} & c_1(t) &= \frac{\sinh(t)}{t} \cdot c_2(t), & \text{for } t > 0 \end{aligned}$$

The density from (3) is chosen so that X has zero mean and unit variance. The GSH distribution includes the logistic distribution ($t = 0$) and the hyperbolic secant distribution ($t = -\pi/2$) as special cases and the uniform distribution on $(-\sqrt{3}, \sqrt{3})$ as limiting case for $t \rightarrow \infty$. Vaughan derives the cumulative distribution function, depending on the parameter t , as

$$F_{GSH}(x; t) = \begin{cases} 1 + \frac{1}{t} \operatorname{arccot} \left(-\frac{\exp(c_2(t)x + \cos(t))}{\sin(t)} \right) & \text{for } t \in (-\pi, 0), \\ \frac{\exp(\pi x / \sqrt{3})}{1 + \exp(\pi x / \sqrt{3})} & \text{for } t = 0, \\ 1 - \frac{1}{t} \operatorname{arccoth} \left(\frac{\exp(c_2(t)x + \cosh(t))}{\sinh(t)} \right) & \text{for } t > 0. \end{cases} \quad (4)$$

the inverse distribution function

$$F_{GSH}^{-1}(u; t) = \begin{cases} \frac{1}{c_2(t)} \ln \left(\frac{\sin(tu)}{\sin(t(1-u))} \right) & \text{für } t \in (-\pi, 0), \\ \frac{\sqrt{3}}{\pi} \ln \left(\frac{u}{1-u} \right) & \text{für } t = 0, \\ \frac{1}{c_2(t)} \ln \left(\frac{\sinh(tu)}{\sinh(t(1-u))} \right) & \text{für } t > 0. \end{cases}$$

In addition, the moment-generating function and all moments exist, and the cumulative distribution is given in closed form. Unfortunately, this family does not allow for skewness. For this purpose, we introduce a skewness parameter by means of splitting the scale parameter according to Fernandez, Osiewalski and Steel (1995). In particular, this transformation preserves the closed form for the density, given by

$$\begin{aligned} f_{SGSH}(x; t, \gamma) &= \frac{2}{\gamma + \frac{1}{\gamma}} \{ f_{GSH}(x/\gamma) \cdot \mathbf{I}^-(x) + f_{GSH}(\gamma x) \cdot \mathbf{I}^+(x) \} \\ &= \frac{2c_1}{\gamma + \frac{1}{\gamma}} \cdot \left(\frac{\exp(c_2 x / \gamma) \cdot \mathbf{I}^-(x)}{\exp(2c_2 x / \gamma) + 2a \exp(c_2 x / \gamma) + 1} + \frac{\exp(c_2 \gamma x) \cdot \mathbf{I}^+(x)}{\exp(2c_2 \gamma x) + 2a \exp(c_2 \gamma x) + 1} \right), \end{aligned} \quad (5)$$

where is symmetric for $\gamma = 1$, skewed to the right for $\gamma > 1$ and skewed to the left for $0 < \gamma < 1$. The corresponding distribution will be termed as *skewed generalized secant hyperbolic distribution* in the sequel. The corresponding cumulative distribution function is given by

$$F_{SGSH}(x; t, \gamma) = \frac{2\gamma^2}{\gamma^2 + 1} \cdot \left(F_{GSH}(x/\gamma) \cdot \mathbf{I}^-(x) + \left(\frac{\gamma^2 - 1 + 2F_{GSH}(\gamma x)}{2\gamma^2} \right) \cdot \mathbf{I}^+(x) \right),$$

More details can be found in Fischer and Vaughan (2002), where the inverse cumulative distribution function and the moments are calculated. It is also shown that this family provides an excellent fit to financial return data.

4 The choice of the copula

Much more less explored than the choice of the marginal distribution is the selection problem of an "optimal" copula. Due to Durrleman et al. (2000) we assume that we have a finite subset of copulas $\mathcal{C}_0 \subset \mathcal{C}$. As we do not restrict our research to elliptical copulas, we also use certain distance measures between the empirical copula and some parametric copulas: Let $\{x_n^1, x_n^2\}_{n=1, \dots, N}$ be a bivariate sample of size N . The *empirical copula* dates back to Deheuvel (1979) and is defined by

$$C_{emp}(i^*, j^*) = \frac{1}{N} \sum_{n=1}^N \mathbf{1}_{\{x_n^1 \leq x_{(i)}^1, x_n^2 \leq x_{(j)}^2\}}, \quad (6)$$

where $i^* = i/N$ and $j^* = j/N$. To judge the goodness-of-fit we calculate several distance measures between the empirical copula C_{emp} and the (estimated) parametric copula C_{para} . Putting emphasize on the deviations in the tails, *Anderson-Darling statistic* is calculated

$$\mathcal{AD}_0 = \sqrt{N} \cdot \max_{1 \leq i, j \leq N} \frac{|C_{emp}(i^*, j^*) - C_{para}(i^*, j^*)|}{\sqrt{(C_{para}(i^*, j^*))(1 - C_{para}(i^*, j^*))}} \quad (7)$$

Taking also the second and third largest value into account, we also calculate \mathcal{AD}_1 and \mathcal{AD}_2 . As a global goodness-of-fit statistic which has still focus on the tails we also use the *integrated Anderson-Darling statistic* given by

$$\mathcal{IAD} = \frac{1}{N} \sum_{i=1}^N \sum_{j=1}^N \frac{(C_{emp}(i^*, j^*) - C_{para}(i^*, j^*))^2}{C_{para}(i^*, j^*)(1 - C_{para}(i^*, j^*))}. \quad (8)$$

In addition, we also calculate a distance based on the discrete \mathcal{L}^2 -norm by

$$\mathcal{L}_2 = \sqrt{\sum_{i=1}^N \sum_{j=1}^N (C_{emp}(i^*, j^*) - C_{para}(i^*, j^*))^2}. \quad (9)$$

We will essentially focus on three classes (which are not disjoint) of copulas

Elliptical copulas in general cannot be expressed in closed form. Moreover, only symmetric tail dependence can be captured. Examples are the Gaussian copula,

$$C_G = C_G(u, v, \rho) = \Phi_\rho(\Phi^{-1}(u), \Phi^{-1}(v))$$

the t-copula

$$C_t = C_t(u, v; \rho, \nu) = T_{\nu, \rho}(t_\nu^{-1}(u), t_\nu^{-1}(v)).$$

Up to There are two classes (which are not disjoint!) of copulas

1. *Elliptical copulas* associated to elliptical distributions are widely used in statistics and econometrics, especially in finance. Applications and limitations are discussed by Frahm, Junker and Szimayer (2002). Popular in finance are the multivariate Gaussian, the multivariate Student t and the sub-Gaussian α -stable distribution.

2. *Archimedean copulas* with generator φ which are defined by

$$C(u, v) = \varphi^{-1}(\varphi(u) + \varphi(v)), \quad (10)$$

where φ is a continuous, convex and strictly decreasing function from $[0, 1]$ to $[0, \infty]$ and $\varphi(1) = 0$, and φ^{-1} denotes its pseudo-inverse. Popular examples are the Frank copula with $\varphi(u) = -\ln[(e^{-\theta u} - 1)/(e^{-\theta} - 1)]$, the Clayton copula with $\varphi(t) = (t^{-\theta} - 1)/\theta$ or the Gumbel copula with $\varphi(t) = -\ln(t)^\theta$. If C is absolutely continuous, its density is given by

$$c(u, v) = -\frac{\varphi''(C(u, v))\varphi'(u)\varphi'(v)}{[\varphi(C'(u, v))]^3}. \quad (11)$$

Furthermore, for Archimedean copulas, tail dependence can be expressed in terms of the generators φ (see, for example, Embrechts, Lindskog and McNeil (2002)). Members of this class can have both symmetric and asymmetric tail dependence. It is easy constructing a new Archimedean copula by constructing a new generator via $\bar{\varphi} = g \circ \varphi$, where $g : [0, 1] \rightarrow [0, 1]$ is strictly monotone increasing, concave with $g(1) = 1$. In order to introduce tail dependence, Junker and May (2002) chose $g(x) = x^\delta$, $\delta \geq 1$. The so-called transformed Frank copula is given by

$$C(u, v) = -\frac{1}{\theta} \cdot \ln [1 + (e^{-\theta} - 1) \exp(-\{\varphi(u; \theta)^\delta + \varphi(v; \theta)^\delta\}^{1/\delta})]$$

with density given in (11), where

$$\begin{aligned} \bar{\varphi}'(t; \theta, \delta) &= \frac{\delta\theta}{1 - e^{\theta t}} \cdot \varphi(t, \theta)^{\delta-1} \\ \bar{\varphi}''(t; \theta, \delta) &= \frac{\delta\theta^2 e^{-\theta t}}{(e^{-\theta t} - 1)^2} \cdot \left(e^{-\theta t}(\delta - 1) + \varphi(t, \theta) \right) \cdot \varphi(t, \theta)^{\delta-2} \end{aligned}$$

Clearly, for $\delta = 1$ we have the Frank copula. Junker and May (2002) showed that C_{TF} has no lower tail dependence but is upper tails dependent with tail dependence parameter $\lambda_U = 2 - 2^{1/\delta}$.

Possibilities to construct new copulas, as for example

1. Rotation of copulas by means of

$$\bar{C}(u, v) = u + v - 1 + C(u, v)$$

2. It can be easily shown that any convex combination of two (or more generally of n) copulas C_1 and C_2

$$C(u, v) = \lambda C_1(u, v) + (1 - \lambda)C_2(u, v) \text{ for } \lambda \in [0, 1]$$

is again a copula. In order to allow both for lower and for upper tail dependence, Junker and May (2002) propose to convex-combine a Cook-Johnson (Clayton) copula and a survival copula of the same family:

$$\begin{aligned} C_{cCJ} &= \alpha \cdot \bar{C}_{JC}(u, v; \delta_1) + (1 - \alpha) \cdot C_{JC}(u, v; \delta_2) \\ &= \alpha \cdot (u + v - 1 + C_{JC}(1 - u, 1 - v; \delta_1)) + (1 - \alpha) \cdot C_{JC}(u, v; \delta_2), \end{aligned} \quad (12)$$

where $\alpha \in [0, 1]$. The convex-combined Cook-Johnson copula C_{cCJ} is lower tail dependent with parameter $\lambda_L = \alpha 2^{-1/\delta_1}$ and upper tail dependent with parameter $\lambda_U = (1 - \alpha) 2^{-1/\delta_2}$. In order to fulfil certain requirements (i.e. asymmetric tail dependence, whole range for lower tail dependence, any degree of Spearman's rank correlation coefficient and nesting the Gaussian copula), Fortin and Kuzmics (2002) convex-combined a t-copula/Gaussian copula with a Clayton copula, a rotated Gumbel and a rotated Joe copula.

3. Transforming copulas: Durrleman, Nikeghbali and Roncalli (2000) showed that

$$C_\gamma(u, v) = \gamma^{-1}(C(\gamma(u), \gamma(v)))$$

is a copula if $\gamma : [0, 1] \rightarrow [0, 1]$ is a concave bijection with $\gamma(0) = 0$ and $\gamma(1) = 1$, which is C^1 -diffeomorphism from $(0, 1)$ onto $(0, 1)$ and is twice differentiable on $(0, 1)$. Examples are

$$\gamma(x) = x^{1/\beta}, \quad \beta \geq 1, \quad \gamma(x) = \sin(\pi x/2), \quad \gamma(x) = \frac{\beta_1 x + \beta_2 x}{\beta_1 x + \beta_2}, \quad \beta_1, \beta_2 > 0.$$

4. Construction via Laplace transforms: Let F be the cumulative distribution function of a positive random variable X_F . Then the Laplace transform ψ of F is defined as

$$\psi(s) = \int_0^\infty e^{-sx} dF(x)$$

For every copula K we can derive another copula C via

$$C(u, v) = \psi \left[-\log K(e^{-p_1 \psi^{-1}(u)}, e^{-p_2 \psi^{-1}(v)}) + \nu_1 p_1 \psi^{-1}(u) + \nu_2 p_2 \psi^{-1}(v) \right]$$

with $\nu_i \geq 0$ and $p_i = \frac{1}{\nu_i + 1}$. In particular, setting $\nu_1 = \nu_2 = 0$ implying $p_1 = p_2 = 1$,

$$C(u, v) = \psi \left[-\log K(e^{-\psi^{-1}(u)}, e^{-\psi^{-1}(v)}) \right] \quad (13)$$

Note, that for the independence copula $K(u, v) = uv$ equation (13) reduces to (10). Several examples can be found in Joe (2000): Combining the Gumbel copula with Laplace transform LTB [LTC] we end up with the two-parameter copula family $BB1$ [$BB7$ or Joe-Clayton], Clayton copula in combination with LTA generates copula family $BB3$, Galambos copula with LTB generate copula family $BB4$, the independence copula with the two-parameter family LTL results in copula family $BB10$. Furthermore, let K denote a Frank copula and ψ the Laplace transform LTC . With $\check{x} = 1 - e^{-\delta(1-\bar{x}^\theta)}$, the corresponding copula (we call it $BBMF$) is

$$C = C(u, v; \theta, \delta) = 1 - \left(1 + \ln \left(1 - \frac{\check{u}\check{v}}{1 - \exp(-\delta)} \right) \frac{1}{\delta} \right)^{\frac{1}{\theta}}.$$

It can be shown that the corresponding copula density is given by

$$c(u, v; \theta, \delta) = \left(\frac{1 - C}{\check{u}\check{v}} \right)^{1-\theta} \frac{(1 - \check{u})(1 - \check{v})}{(1 - e^{-\delta} - \check{u}\check{v})^2} \left\{ \frac{\check{u}\check{v}(\theta - 1)}{(1 - C)^\theta} + \delta\theta(1 - e^{-\delta}) \right\}$$

Elliptical copulas:

$$c(u, v; \rho) = \frac{1}{\sqrt{\det(\Sigma)}} \cdot \exp(-0.5\zeta'(\Sigma^{-1} - \mathbf{I}_2)\zeta)$$

$$c(u, v; \rho, \nu) = \frac{1}{\sqrt{1 - \rho^2}} \cdot \frac{\Gamma((\nu + 2)/2)\Gamma(\nu/2)}{(\Gamma((\nu + 1)/2))^2} \cdot \frac{\left[\left(1 + \frac{\Phi^{-1}(u)^2}{\nu} \right) \left(1 + \frac{\Phi^{-1}(v)^2}{\nu} \right) \right]^{\frac{\nu+1}{2}}}{\left(1 + \frac{\zeta'\Sigma^{-1}\zeta}{\nu} \right)^{\frac{\nu+2}{2}}}$$

Independence or product copula

$$C_I = C_\Pi(u, v) = u \cdot v$$

The minimum copula

$$C_L = C_L(u, v) = \max(u + v - 1, 0)$$

The maximum copula

$$C_U = C_U(u, v) = \min(u, v)$$

Tail dependence: If C is a bivariate copula such that

$$\lim_{u \rightarrow 1} \frac{\overline{C}(u, u)}{1 - u} = \lim_{u \rightarrow 1} P(F_X(x) > 1 - u | F_Y(y) > 1 - u) = \lambda_U$$

exists, then C is said to be upper tail dependent if $\lambda_U \in (0, 1]$ and no upper tail dependent if $\lambda_U = 0$. Similarly, if

$$\lim_{u \rightarrow 0} \frac{C(u, u)}{u} = \lambda_L$$

exists, C has lower tail dependence if $\lambda_L \in (0, 1]$ and no lower tail dependence if $\lambda_L = 0$.

Tail dependence for a convex combination $C(u, v) = \sum_i \lambda_i C_i(u, v)$ mit $\sum_i \lambda_i = 1$. Tail dependence of the rotated copula: The lower (upper) tail dependence of C^r is the same as the upper (lower) tail dependence of C .

Table 1: Archimedean copulas

| Name | $\varphi(u)$ | $\varphi(u)^{-1}$ |
|------|---|-----------------------|
| $B0$ | $-\ln(u)$ | $\exp(-u)$ |
| $B3$ | $-\log\left(\frac{\exp(-\alpha u)-1}{\exp(-\alpha)-1}\right)$ | |
| $B4$ | $u^{-\alpha} - 1$ | $(u + 1)^{-1/\alpha}$ |
| $B6$ | $(-\ln(u))^\alpha$ | $\exp(-u^{1/\alpha})$ |
| $B5$ | | |

5 Application to bivariate financial return data

Throughout this work we focus on the bivariate analysis

Table 2: Parameterized copulas

| name | Abr. | formula | parameter |
|--------------|-----------|----------------------|-----------|
| Minimum | <i>BL</i> | $\max(u + v - 1, 0)$ | – |
| Maximum | <i>BU</i> | $\min(u, v)$ | – |
| Independence | <i>B0</i> | $u \cdot v$ | – |

Bivariate one-parameter families

| | | | |
|-----------------|------------|--|-------------------|
| Normal | <i>B1</i> | $\Phi_\rho(\Phi^{-1}(u), \Phi^{-1}(v))$ | $ \rho \leq 1$ |
| Plackett | <i>B2</i> | $\frac{1}{2\eta}(1 + \eta(u + v) - \sqrt{(1 + \eta(u + v))^2 - 4\delta\eta uv})$ | $\delta \geq 0$ |
| Frank | <i>B3</i> | $-\frac{1}{\delta} \log\left(1 - \frac{(1-e^{-\delta u})(1-e^{-\delta v})}{1-e^{-\delta}}\right)$ | $\delta \geq 0$ |
| Clayton | <i>B4</i> | $(u^{-\delta} + v^{-\delta} - 1)^{-\frac{1}{\delta}}$ | $\delta \geq 0$ |
| Joe | <i>B5</i> | $1 - (\bar{u}^\delta + \bar{v}^\delta - \bar{u}^\delta \bar{v}^\delta)^{1/\delta}$ | $\delta \geq 1$ |
| Gumbel | <i>B6</i> | $\exp(-(\tilde{u}^\delta + \tilde{v}^\delta)^{1/\delta})$ | $\delta \geq 1$ |
| Galambos | <i>B7</i> | $uv \exp((\tilde{u}^{-\delta} + \tilde{v}^{-\delta})^{-1/\delta})$ | $\delta \geq 0$ |
| Hüsler-Reiss | <i>B8</i> | $\exp(-\tilde{u}\Phi[\frac{1}{\delta} + \frac{\delta}{2} \ln(\frac{\tilde{u}}{\tilde{v}})] - \tilde{v}\Phi[\frac{1}{\delta} + \frac{\delta}{2} \ln(\frac{\tilde{v}}{\tilde{u}})])$ | $\delta \geq 0$ |
| Rotated Clayton | <i>B4'</i> | $u + v - 1 + (\bar{u}^{-\delta} + \bar{v}^{-\delta} - 1)^{-\frac{1}{\delta}}$ | $\delta \geq 0$ |
| Rotated Joe | <i>B5'</i> | $u + v - (u^\delta + v^\delta - u^\delta v^\delta)^{1/\delta}$ | $\delta \geq 1$ |
| Rotated Gumbel | <i>B6'</i> | $u + v - 1 + \exp(-(\tilde{u}^\delta + \tilde{v}^\delta)^{1/\delta})$ | $\delta \geq 1$ |
| Ali-Mikhail-Haq | <i>B12</i> | $(uv)/(1 - \delta \cdot \bar{u}\bar{v})$ | $ \delta \leq 1$ |

Bivariate two-parameter families

| | | | |
|---------------|-------------|---|-----------------------------|
| | <i>BB1</i> | $(1 + [\hat{u}^\delta + \hat{v}^\delta]^{1/\delta})^{-1/\theta}$ | $\theta \geq 1, \delta > 0$ |
| | <i>BB2</i> | $\exp\left(-\left[\frac{1}{\delta} \log(\exp(\delta\tilde{u}^\theta) + \exp(\delta\tilde{v}^\theta) - 1)\right]^{1/\theta}\right)$ | $\theta > 0, \delta \geq 1$ |
| | <i>BB4</i> | $(\hat{u} + \hat{v} + 1 - [\hat{u}^{-\delta} + \hat{v}^{-\delta}]^{-\frac{1}{\delta}})^{-\frac{1}{\theta}}$ | $\theta \geq 0, \delta > 0$ |
| Joe-Clayton | <i>BB7</i> | $1 - \left(1 - [(1 - \bar{u}^\kappa)^{-\delta} + (1 - \bar{v}^\kappa)^{-\delta} - 1]^{-\frac{1}{\delta}}\right)^{\frac{1}{\kappa}}$ | $\kappa \geq 1, \delta > 0$ |
| Transf. Frank | <i>BB10</i> | $-\theta^{-1} \cdot \ln\left[1 + \frac{\exp(-\{\varphi(u;\theta)^\delta + \varphi(v;\theta)^\delta\}^{1/\delta})}{(e^{-\theta} - 1)^{-1}}\right]$ | $\theta > 0, \delta \geq 0$ |
| t | <i>BB11</i> | $T_{\nu,\rho}(t_\nu^{-1}(u), t_\nu^{-1}(v))$ | $ \rho \leq 1, \nu > 0$ |
| MF | <i>BBMF</i> | $1 - \left(1 + \ln\left(1 - \frac{\hat{u}\hat{v}}{1 - \exp(-\delta)}\right)\right)^{\frac{1}{\theta}}$ | $\theta > 1, \delta \geq 1$ |

Bivariate three-parameter families

| | | | |
|-------------------|------------|---|--|
| CC Clayton | <i>BC1</i> | $\alpha C_{B4}(u, v; \delta_1) + (1 - \alpha)\overline{C}_{B4}(u, v; \delta_2)$ | $\delta_i > 0, \alpha \in [0, 1]$ |
| CC Joe | <i>BC2</i> | $\alpha C_{B5}(u, v; \delta_1) + (1 - \alpha)\overline{C}_{B5}(u, v; \delta_2)$ | $\delta_i \geq 1, \alpha \in [0, 1]$ |
| CC Normal-Clayton | <i>BC3</i> | $\alpha C_{B4}(u, v; \delta) + (1 - \alpha)C_{B1}(u, v; \rho)$ | $\delta > 0, \rho \leq 1, \alpha \in [0, 1]$ |

Note, that $\bar{u} = 1 - u$, $\hat{u} = u^{-\theta} - 1$, $\tilde{u} = -\log(u)$, \overline{C} Survival copula.

Table 3: Tail dependence and limiting cases

| name | τ^L | τ^U | Dependence |
|------------------------------------|---|-----------------------------------|---|
| Minimum | | | C_L |
| Maximum | | | C_U |
| Independence | | | C_I |
| Bivariate one-parameter families | | | |
| Normal ($B1$) | 0 | 0 | $C_L (\rho = -1), C_I (\rho = 0), C_U (\rho = 1)$ |
| Plackett ($B2$) | | | $C_L (\delta \rightarrow 0), C_I (\delta \rightarrow 1), C_U (\delta \rightarrow \infty)$ |
| Frank ($B3$) | 0 | 0 | $C_I (\delta \rightarrow 0), C_U (\delta \rightarrow \infty)$ |
| Clayton ($B4$) | $2^{-1/\delta}$ | 0 | $C_I (\delta \rightarrow 0), C_U (\delta \rightarrow \infty)$ |
| Joe ($B5$) | 0 | $2 - 2^{1/\delta}$ | $C_I (\delta = 1), C_U (\delta \rightarrow \infty)$ |
| Gumbel ($B6$) | 0 | $2 - 2^{1/\delta}$ | $C_I (\delta = 1), C_U (\delta \rightarrow \infty)$ |
| Galambos ($B7$) | | | $C_I (\delta \rightarrow 0), C_U (\delta \rightarrow \infty)$ |
| Hüsler-Reiss ($B8$) | | | $C_I (\delta \rightarrow 0), C_U (\delta \rightarrow \infty)$ |
| Rotated Clayton ($B4'$) | 0 | $2^{-1/\delta}$ | |
| Rotated Joe ($B5'$) | $2 - 2^{1/\delta}$ | 0 | |
| Rotated Gumbel ($B6$) | $2 - 2^{1/\delta}$ | 0 | |
| Ali-Mikhail-Haq | | | |
| Bivariate two-parameter families | | | |
| $BB1$ | $2^{-1/(\delta\theta)}$ | $2 - 2^{1/\delta}$ | $B4 (\theta = 1), B6 (\delta \rightarrow 0)$ |
| $BB3$ | 1 | $2 - 2^{1/\theta}$ | $B4 (\theta = 1), B6 (\delta \rightarrow 0)$ |
| $BB4$ | $(2 - 2^{1/\delta})^{-1/\theta}$ | $2^{-1/\delta}$ | $B7 (\theta \rightarrow 0), B4 (\delta \rightarrow 0)$ |
| $BB7$ | $2^{-1/\delta}$ | $2 - 2^{-1/\kappa}$ | $B4 (\kappa = 1), B5 (\delta \rightarrow 0)$ |
| $BB10$ | 0 | $2 - 2^{1/\delta}$ | $B3 (\delta = 1)$ |
| $BB11$ | $2 \left(1 - t_{\nu+1} \left(\frac{\sqrt{\nu+1}\sqrt{1-\rho}}{\sqrt{1+\rho}} \right) \right)$ | | $C_L (\rho = -1), C_I (\rho = 0), C_U (\rho = 1)$ |
| Bivariate three-parameter families | | | |
| CC Clayton | $\alpha 2^{-1/\delta_1}$ | $(1 - \alpha) 2^{-1/\delta_2}$ | $B4 (\alpha = 1), B4' (\alpha = 0)$ |
| CC Joe | $\alpha (2 - 2^{1/\delta})$ | $(1 - \alpha) (2 - 2^{1/\delta})$ | $B5 (\alpha = 1), B5' (\alpha = 0)$ |
| CC Normal-Clayton | $\alpha 2^{-1/\delta}$ | 0 | $B4 (\alpha = 1), B1 (\alpha = 0)$ |

| | AD_0 | AD_1 | AD_2 | IAD | KS | \mathcal{LL} | AIC | χ^2 |
|---------------------|--------|--------|--------|--------|--------|----------------|----------|----------|
| Independence copula | | | | | | | | |
| Norm | 0.619 | 0.607 | 0.604 | 51.495 | 85.129 | -3699.439 | 7408.929 | |
| SH | 0.619 | 0.607 | 0.604 | 51.495 | 85.129 | -3660.336 | 7330.722 | |
| t | 0.619 | 0.607 | 0.604 | 51.495 | 85.129 | -3652.784 | 7319.663 | |
| GSH | 0.619 | 0.607 | 0.604 | 51.495 | 85.129 | -3653.702 | 7325.556 | |
| SGSH | 0.619 | 0.607 | 0.604 | 51.495 | 85.129 | -3643.008 | 7304.169 | |
| Gaussian copula | | | | | | | | |
| Norm | 0.262 | 0.262 | 0.262 | 0.307 | 6.087 | -3290.909 | 6593.889 | |
| SH | 0.262 | 0.262 | 0.262 | 0.254 | 4.839 | -3231.147 | 6474.364 | |
| t | 0.262 | 0.262 | 0.262 | 0.275 | 5.476 | -3225.345 | 6466.894 | |
| GSH | 0.262 | 0.262 | 0.262 | 0.277 | 5.435 | -3227.928 | 6476.042 | |
| SGSH | 0.262 | 0.262 | 0.262 | 0.273 | 5.450 | -3216.788 | 6453.788 | |
| t-copula | | | | | | | | |
| Norm | 0.064 | 0.061 | 0.058 | 0.096 | 4.500 | -3287.724 | 6589.542 | |
| SH | 0.062 | 0.059 | 0.056 | 0.056 | 4.186 | -3228.954 | 6472.003 | |
| t | | | | | | | | |
| GSH | | | | | | | | |
| SGSH | | | | | | | | |
| Frank copula | | | | | | | | |
| Norm | 0.262 | 0.220 | 0.216 | 0.416 | 8.679 | -3274.776 | 6561.622 | |
| SH | 0.262 | 0.232 | 0.230 | 0.289 | 5.742 | -3254.022 | 6520.114 | |
| t | 0.262 | 0.229 | 0.227 | 0.301 | 6.230 | -3240.971 | 6498.063 | |
| GSH | 0.262 | 0.228 | 0.226 | 0.305 | 6.378 | -3241.456 | 6503.098 | |
| SGSH | 0.262 | 0.229 | 0.226 | 0.303 | 6.301 | -3232.565 | 6485.316 | |
| Plackett copula | | | | | | | | |
| Norm | 0.198 | 0.161 | 0.157 | 0.167 | 5.124 | -3281.934 | 6575.938 | |
| SH | 0.211 | 0.171 | 0.169 | 0.214 | 5.749 | -3252.308 | 6516.687 | |
| t | 0.210 | 0.171 | 0.168 | 0.210 | 5.677 | -3243.102 | 6502.326 | |
| GSH | 0.209 | 0.170 | 0.168 | 0.205 | 5.601 | -3243.676 | 6507.537 | |
| SGSH | 0.210 | 0.171 | 0.168 | 0.209 | 5.664 | -3235.040 | 6490.265 | |

Table 4: Goodness-of-fit for different copulas I

| | AD_0 | AD_1 | AD_2 | IAD | KS | \mathcal{LL} | AIC | χ^2 |
|------------------------|--------|--------|--------|-------|--------|----------------|----------|----------|
| Ali-Mikhail-Haq copula | | | | | | | | |
| Norm | 0.147 | 0.146 | 0.145 | 4.221 | 33.410 | -3381.853 | 6775.777 | |
| SH | 0.146 | 0.145 | 0.144 | 4.115 | 33.028 | -3326.394 | 6664.858 | |
| t | 0.146 | 0.145 | 0.144 | 4.115 | 33.029 | -3321.289 | 6658.700 | |
| GSH | 0.146 | 0.145 | 0.144 | 4.138 | 33.114 | -3322.734 | 6665.653 | |
| SGSH | 0.146 | 0.145 | 0.145 | 4.177 | 33.253 | -3315.525 | 6651.236 | |
| Clayton copula | | | | | | | | |
| Norm | 0.103 | 0.102 | 0.102 | 1.681 | 20.278 | -3374.627 | 6761.324 | |
| SH | 0.101 | 0.101 | 0.100 | 1.628 | 19.807 | -3292.300 | 6596.670 | |
| t | 0.102 | 0.101 | 0.101 | 1.644 | 19.953 | -3289.384 | 6594.889 | |
| GSH | 0.101 | 0.101 | 0.101 | 1.641 | 19.921 | -3291.867 | 6603.919 | |
| SGSH | 0.097 | 0.097 | 0.097 | 1.548 | 19.036 | -3290.960 | 6602.107 | |
| Rotated Clayton copula | | | | | | | | |
| Norm | 0.437 | 0.430 | 0.426 | 9.117 | 33.560 | -3469.762 | 6951.595 | |
| SH | 0.407 | 0.401 | 0.396 | 5.814 | 25.412 | -3346.311 | 6704.692 | |
| t | 0.400 | 0.394 | 0.389 | 5.163 | 23.545 | -3332.888 | 6681.898 | |
| GSH | 0.408 | 0.402 | 0.398 | 5.917 | 25.697 | -3344.691 | 6709.567 | |
| SGSH | 0.376 | 0.371 | 0.366 | 3.400 | 17.871 | -3309.397 | 6638.980 | |
| Gumbel copula | | | | | | | | |
| Norm | 0.159 | 0.154 | 0.150 | 0.869 | 11.387 | -3377.305 | 6766.680 | |
| SH | 0.154 | 0.150 | 0.146 | 0.730 | 10.317 | -3280.451 | 6572.972 | |
| t | 0.153 | 0.149 | 0.145 | 0.693 | 10.011 | -3273.972 | 6564.066 | |
| GSH | 0.155 | 0.150 | 0.146 | 0.736 | 10.365 | -3280.408 | 6581.001 | |
| SGSH | 0.137 | 0.133 | 0.129 | 0.351 | 6.767 | -3244.594 | 6509.375 | |
| Joe-Clayton copula | | | | | | | | |
| Norm | 0.092 | 0.091 | 0.090 | 1.124 | 16.924 | -3347.234 | 6708.563 | |
| SH | 0.080 | 0.079 | 0.078 | 0.579 | 11.583 | -3247.610 | 6509.314 | |
| t | 0.079 | 0.079 | 0.078 | 0.518 | 10.917 | -3241.816 | 6500.384 | |
| GSH | 0.080 | 0.079 | 0.078 | 0.591 | 11.727 | -3247.610 | 6517.443 | |
| SGSH | 0.074 | 0.073 | 0.072 | 0.486 | 10.854 | -3241.532 | 6505.287 | |

Table 5: Goodness-of-fit for different copulas II

| | AD_0 | AD_1 | AD_2 | IAD | KS | \mathcal{LL} | AIC | χ^2 |
|---------------------|--------|--------|--------|-------|--------|----------------|----------|----------|
| Joe copula | | | | | | | | |
| Norm | 0.441 | 0.435 | 0.430 | 8.979 | 32.426 | -3485.862 | 6983.794 | |
| SH | 0.420 | 0.414 | 0.409 | 6.597 | 26.779 | -3362.337 | 6736.745 | |
| t | 0.413 | 0.408 | 0.403 | 5.938 | 25.047 | -3348.603 | 6713.328 | |
| GSH | 0.422 | 0.416 | 0.411 | 6.755 | 27.183 | -3360.708 | 6741.603 | |
| SGSH | 0.386 | 0.382 | 0.376 | 3.854 | 18.937 | -3321.045 | 6662.275 | |
| Hüsler-Reiss copula | | | | | | | | |
| Norm | 0.216 | 0.210 | 0.205 | 3.626 | 24.455 | -3463.404 | 6938.879 | |
| SH | 0.174 | 0.169 | 0.165 | 1.317 | 14.330 | -3300.558 | 6613.186 | |
| t | 0.164 | 0.159 | 0.155 | 0.934 | 11.847 | -3283.977 | 6584.075 | |
| GSH | 0.174 | 0.170 | 0.165 | 1.329 | 14.403 | -3300.306 | 6620.797 | |
| SGSH | 0.146 | 0.142 | 0.137 | 0.455 | 7.845 | -3256.091 | 6532.368 | |
| Galambos copula | | | | | | | | |
| Norm | 0.161 | 0.157 | 0.152 | 0.899 | 11.602 | -3393.960 | 6799.990 | |
| SH | 0.158 | 0.153 | 0.149 | 0.780 | 10.707 | -3283.701 | 6579.474 | |
| t | 0.155 | 0.151 | 0.147 | 0.718 | 10.207 | -3275.393 | 6566.907 | |
| GSH | 0.158 | 0.153 | 0.149 | 0.789 | 10.776 | -3283.606 | 6587.397 | |
| SGSH | 0.138 | 0.134 | 0.130 | 0.350 | 6.748 | -3245.811 | 6511.808 | |
| BB3 copula | | | | | | | | |
| Norm | 0.096 | 0.095 | 0.094 | 1.206 | 17.381 | -3357.775 | 6729.644 | |
| SH | 0.073 | 0.073 | 0.072 | 0.434 | 10.078 | -3247.201 | 6508.497 | |
| t | 0.076 | 0.076 | 0.075 | 0.399 | 9.477 | -3238.928 | 6496.007 | |
| GSH | 0.073 | 0.072 | 0.072 | 0.499 | 10.945 | -3247.855 | 6517.934 | |
| SGSH | 0.057 | 0.055 | 0.054 | 0.302 | 8.001 | -3234.625 | 6491.474 | |
| BB1 copula | | | | | | | | |
| Norm | 0.071 | 0.070 | 0.069 | 0.397 | 9.555 | -3320.232 | 6654.558 | |
| SH | 0.072 | 0.071 | 0.070 | 0.290 | 7.589 | -3233.869 | 6481.833 | |
| t | 0.070 | 0.070 | 0.069 | 0.285 | 7.605 | -3229.056 | 6476.265 | |
| GSH | 0.071 | 0.071 | 0.070 | 0.298 | 7.775 | -3233.525 | 6489.274 | |
| SGSH | 0.054 | 0.053 | 0.053 | 0.186 | 6.470 | -3225.909 | 6474.041 | |

Table 6: Goodness-of-fit for different copulas III

| | AD_0 | AD_1 | AD_2 | IAD | KS | \mathcal{LL} | AIC | χ^2 |
|--------------------------------|--------|--------|--------|-------|-------|----------------|----------|----------|
| Transformed Frank copula | | | | | | | | |
| Norm | 0.219 | 0.196 | 0.195 | 0.325 | 7.588 | -3267.640 | 6549.375 | |
| SH | 0.174 | 0.169 | 0.169 | 0.149 | 3.881 | -3241.531 | 6497.157 | |
| t | 0.185 | 0.182 | 0.181 | 0.175 | 4.430 | -3231.596 | 6481.344 | |
| GSH | 0.190 | 0.189 | 0.185 | 0.187 | 4.683 | -3232.691 | 6487.604 | |
| SGSH | 0.172 | 0.168 | 0.167 | 0.151 | 4.117 | -3221.541 | 6465.306 | |
| Convex-combined Clayton copula | | | | | | | | |
| Norm | 0.063 | 0.063 | 0.062 | 0.258 | 6.856 | -3294.796 | 6605.713 | |
| SH | 0.049 | 0.048 | 0.047 | 0.186 | 6.645 | -3241.571 | 6499.263 | |
| t | 0.049 | 0.048 | 0.047 | 0.191 | 6.756 | -3235.988 | 6492.161 | |
| GSH | 0.048 | 0.048 | 0.048 | 0.186 | 6.634 | -3240.200 | 6504.664 | |
| SGSH | 0.047 | 0.046 | 0.045 | 0.181 | 6.345 | -3233.567 | 6495.295 | |
| BBMF copula | | | | | | | | |
| Norm | 0.262 | 0.218 | 0.214 | 0.391 | 8.334 | -3269.644 | 6553.383 | |
| SH | 0.262 | 0.221 | 0.218 | 0.239 | 5.389 | -3246.859 | 6507.812 | |
| t | 0.262 | 0.222 | 0.219 | 0.264 | 5.872 | -3235.361 | 6488.874 | |
| GSH | 0.262 | 0.223 | 0.220 | 0.274 | 6.059 | -3236.273 | 6494.770 | |
| SGSH | 0.262 | 0.222 | 0.219 | 0.265 | 5.906 | -3228.640 | 6479.503 | |
| Convex-combined Joe copula | | | | | | | | |
| Norm | 0.062 | 0.061 | 0.061 | 0.252 | 6.839 | -3300.149 | 6616.419 | |
| SH | 0.049 | 0.048 | 0.048 | 0.201 | 6.869 | -3247.827 | 6511.775 | |
| t | 0.049 | 0.048 | 0.048 | 0.204 | 6.924 | -3242.680 | 6505.546 | |
| GSH | 0.049 | 0.048 | 0.048 | 0.197 | 6.782 | -3246.696 | 6517.657 | |
| SGSH | 0.057 | 0.057 | 0.057 | 0.162 | 5.656 | -3239.774 | 6503.812 | |

Table 7: Goodness-of-fit for different copulas IV

| Copula | Parameters of the margins | | | | Copula parameter | | | \mathcal{LL} |
|-------------|---------------------------|------------------|---------------|------------------|------------------|--------|--------|----------------|
| | $\hat{\mu}_1$ | $\hat{\delta}_1$ | $\hat{\mu}_2$ | $\hat{\delta}_2$ | cp_1 | cp_2 | cp_3 | |
| <i>B0</i> | -0.014 | 1.056 | -0.014 | 1.229 | 0.000 | – | – | -3699.4 |
| <i>B1</i> | -0.021 | 1.054 | -0.025 | 1.222 | 0.708 | – | – | -3290.9 |
| <i>B2</i> | -0.015 | 1.084 | -0.019 | 1.257 | 12.574 | – | – | -3281.9 |
| <i>B3</i> | -0.016 | 1.091 | -0.020 | 1.271 | 6.748 | – | – | -3274.8 |
| <i>B4</i> | 0.022 | 1.123 | 0.027 | 1.285 | 1.591 | – | – | -3374.6 |
| <i>B5</i> | -0.042 | 1.085 | -0.052 | 1.278 | 1.897 | – | – | -3485.9 |
| <i>B4'</i> | -0.056 | 1.056 | -0.067 | 1.242 | 0.933 | – | – | -3469.8 |
| <i>B6</i> | 0.011 | 1.091 | 0.014 | 1.280 | 1.855 | – | – | -3377.3 |
| <i>B6'</i> | -0.028 | 1.101 | -0.027 | 1.264 | 2.038 | – | – | -3300.9 |
| <i>B7</i> | 0.015 | 1.099 | 0.018 | 1.291 | 1.134 | – | – | -3394.0 |
| <i>B8</i> | 0.009 | 1.101 | 0.016 | 1.306 | 1.316 | – | – | -3463.4 |
| <i>B12</i> | 0.076 | 1.029 | 0.089 | 1.184 | 0.992 | – | – | -3381.9 |
| <i>BB1</i> | -0.004 | 1.093 | 0.003 | 1.267 | 0.728 | 1.434 | – | -3320.2 |
| <i>BB3</i> | -0.008 | 1.072 | 0.001 | 1.246 | 1.506 | 0.280 | – | -3357.8 |
| <i>BB4</i> | 0.021 | 1.119 | 0.026 | 1.280 | 1.577 | 0.088 | – | -3374.3 |
| <i>BB7</i> | -0.005 | 1.088 | 0.004 | 1.257 | 1.338 | 1.301 | – | -3347.2 |
| <i>BB10</i> | -0.024 | 1.078 | -0.029 | 1.258 | 1.089 | 5.862 | – | -3267.6 |
| <i>BB11</i> | -0.012 | 1.063 | -0.013 | 1.232 | 0.718 | 18.523 | – | -3287.7 |
| <i>BBMF</i> | -0.021 | 1.082 | -0.027 | 1.261 | 1.056 | 6.488 | – | -3269.6 |
| <i>BC1</i> | -0.008 | 1.108 | 0.006 | 1.289 | 2.435 | 2.414 | 0.600 | -3294.8 |
| <i>BC2</i> | -0.008 | 1.113 | 0.006 | 1.293 | 3.227 | 3.286 | 0.407 | -3300.1 |

Table 8: Normal margins: Estimated parameters

| Copula | Parameters of the margins | | | | | | Copula parameter | | | \mathcal{LL} |
|-------------|---------------------------|------------------|---------------|---------------|------------------|---------------|------------------|--------|--------|----------------|
| | $\hat{\mu}_1$ | $\hat{\delta}_1$ | $\hat{\nu}_1$ | $\hat{\mu}_2$ | $\hat{\delta}_2$ | $\hat{\nu}_2$ | cp_1 | cp_2 | cp_3 | |
| <i>B0</i> | -0.036 | 0.89 | 6.91 | -0.053 | 1.04 | 7.05 | 0.00 | – | – | -3652.8 |
| <i>B1</i> | -0.061 | 0.87 | 5.89 | -0.077 | 1.01 | 5.96 | 0.72 | – | – | -3225.4 |
| <i>B2</i> | -0.047 | 0.91 | 7.28 | -0.060 | 1.06 | 7.64 | 11.19 | – | – | -3243.1 |
| <i>B3</i> | -0.042 | 0.93 | 8.09 | -0.054 | 1.09 | 8.12 | 6.22 | – | – | -3241.0 |
| <i>B4</i> | 0.026 | 0.87 | 4.35 | 0.023 | 1.02 | 5.02 | 1.62 | – | – | -3289.4 |
| <i>B4'</i> | -0.141 | 0.80 | 3.70 | -0.172 | 0.93 | 3.67 | 1.28 | – | – | -3332.9 |
| <i>B5</i> | -0.144 | 0.80 | 3.54 | -0.175 | 0.93 | 3.52 | 2.14 | – | – | -3348.6 |
| <i>B6</i> | -0.094 | 0.84 | 4.37 | -0.116 | 0.97 | 4.29 | 1.89 | – | – | -3274.0 |
| <i>B6'</i> | -0.031 | 0.88 | 5.22 | -0.038 | 1.03 | 5.84 | 2.02 | – | – | -3233.9 |
| <i>B7</i> | -0.101 | 0.83 | 4.19 | -0.124 | 0.96 | 4.11 | 1.17 | – | – | -3275.4 |
| <i>B8</i> | -0.113 | 0.81 | 3.82 | -0.140 | 0.94 | 3.67 | 1.62 | – | – | -3284.0 |
| <i>B12</i> | 0.045 | 0.84 | 5.65 | 0.047 | 0.98 | 6.36 | 1.00 | – | – | -3321.3 |
| <i>BB1</i> | -0.049 | 0.92 | 7.79 | -0.061 | 1.07 | 7.73 | 1.20 | 5.51 | – | -3229.1 |
| <i>BB3</i> | -0.060 | 0.82 | 4.03 | -0.072 | 0.96 | 4.21 | 1.66 | 0.31 | – | -3238.9 |
| <i>BB4</i> | 0.026 | 0.87 | 4.35 | 0.023 | 1.02 | 5.02 | 1.62 | 0.00 | – | -3289.4 |
| <i>BB7</i> | -0.050 | 0.83 | 4.11 | -0.062 | 0.97 | 4.32 | 1.66 | 1.27 | – | -3241.1 |
| <i>BB10</i> | -0.057 | 0.91 | 7.46 | -0.070 | 1.06 | 7.43 | 1.21 | 4.48 | – | -3231.6 |
| <i>BB11</i> | -0.059 | 0.87 | 5.78 | -0.073 | 1.01 | 5.90 | 0.72 | 19.53 | – | -3223.6 |
| <i>BBMF</i> | -0.049 | 0.92 | 7.79 | -0.061 | 1.07 | 7.73 | 1.19 | 5.51 | – | -3235.4 |
| <i>BC1</i> | -0.039 | 0.87 | 5.18 | -0.045 | 1.00 | 5.35 | 2.10 | 1.77 | 0.57 | -3236.0 |
| <i>BC2</i> | -0.040 | 0.88 | 5.15 | -0.046 | 1.00 | 5.24 | 2.62 | 2.95 | 0.43 | -3242.7 |

Table 9: $t(\nu)$ margins: Estimated parameters

| Copula | Parameters of the margins | | | | | | Copula parameter | | | \mathcal{LL} |
|-------------|---------------------------|------------------|-------------|---------------|------------------|-------------|------------------|--------|--------|----------------|
| | $\hat{\mu}_1$ | $\hat{\delta}_1$ | \hat{t}_1 | $\hat{\mu}_2$ | $\hat{\delta}_2$ | \hat{t}_2 | cp_1 | cp_2 | cp_3 | |
| <i>B0</i> | -0.039 | 1.05 | -0.73 | -0.050 | 1.22 | 0.66 | 0.00 | – | – | -3653.7 |
| <i>B1</i> | -0.062 | 1.06 | -1.12 | -0.076 | 1.23 | -0.82 | 0.71 | – | – | -3227.9 |
| <i>B2</i> | -0.047 | 1.07 | -0.45 | -0.059 | 1.23 | 0.98 | 11.27 | – | – | -3243.7 |
| <i>B3</i> | -0.041 | 1.07 | 0.78 | -0.051 | 1.24 | 1.27 | 6.25 | – | – | -3241.5 |
| <i>B4</i> | 0.020 | 1.13 | -1.57 | 0.021 | 1.29 | -1.32 | 1.62 | – | – | -3291.9 |
| <i>B4'</i> | -0.130 | 1.08 | -1.57 | -0.158 | 1.26 | -1.57 | 1.19 | – | – | -3344.7 |
| <i>B5</i> | -0.130 | 1.09 | -1.57 | -0.159 | 1.28 | -1.57 | 2.06 | – | – | -3360.7 |
| <i>B6</i> | -0.090 | 1.09 | -1.57 | -0.110 | 1.28 | -1.57 | 1.88 | – | – | -3280.4 |
| <i>B6'</i> | -0.035 | 1.10 | -1.39 | -0.040 | 1.25 | -0.92 | 2.02 | – | – | -3235.7 |
| <i>B7</i> | -0.097 | 1.09 | -1.57 | -0.118 | 1.27 | -1.57 | 1.15 | – | – | -3283.6 |
| <i>B8</i> | -0.10 | 1.08 | -1.57 | -0.123 | 1.28 | -1.57 | 1.55 | – | – | -3300.3 |
| <i>B12</i> | -0.042 | 1.02 | -1.20 | 0.049 | 1.17 | -0.52 | 0.99 | – | – | -3322.7 |
| <i>BB1</i> | -0.049 | 1.10 | -1.56 | -0.057 | 1.26 | -1.36 | 0.70 | 1.50 | – | -3233.5 |
| <i>BB3</i> | -0.060 | 1.09 | -1.57 | -0.068 | 1.26 | -1.47 | 1.64 | 0.29 | – | -3247.4 |
| <i>BB4</i> | 0.018 | 1.13 | -1.57 | 0.017 | 1.27 | -1.19 | 1.58 | 0.00 | – | -3292.2 |
| <i>BB7</i> | -0.047 | 1.10 | -1.57 | -0.055 | 1.27 | -1.57 | 1.61 | 1.28 | – | -3247.6 |
| <i>BB10</i> | -0.054 | 1.06 | 0.55 | -0.065 | 1.23 | 1.06 | 1.18 | 4.70 | – | -3232.7 |
| <i>BB11</i> | -0.060 | 1.06 | -1.15 | -0.072 | 1.23 | -0.86 | 0.72 | 20.10 | 0.00 | -3226.1 |
| <i>BBMF</i> | -0.060 | 1.09 | -1.57 | -0.068 | 1.26 | -1.47 | 1.64 | 0.29 | 0.00 | -3236.3 |
| <i>BC1</i> | -0.040 | 1.10 | -1.34 | -0.044 | 1.25 | -1.08 | 2.13 | 1.77 | 0.58 | -3240.2 |
| <i>BC2</i> | -0.041 | 1.10 | -1.35 | -0.044 | 1.26 | -1.14 | 2.62 | 2.98 | 0.42 | -3246.7 |

Table 10: *GSH* margins: Estimated parameters

| Copula | Parameters of the margins | | | | | | | | Copula parameter | | | \mathcal{LL} |
|-------------|---------------------------|------------------|-------------|---------------|-------------|------------------|-------------|-------------|------------------|--------|--------|----------------|
| | $\hat{\mu}_1$ | $\hat{\delta}_1$ | \hat{t}_1 | $\hat{\mu}_2$ | λ_1 | $\hat{\delta}_2$ | \hat{t}_2 | λ_2 | cp_1 | cp_2 | cp_3 | |
| <i>B0</i> | -0.19 | 1.05 | -0.85 | 0.89 | -0.30 | 1.20 | 0.83 | 0.86 | 0.00 | – | – | -3643.0 |
| <i>B1</i> | -0.19 | 1.05 | -1.10 | 0.89 | -0.29 | 1.20 | -0.44 | 0.86 | 0.72 | – | – | -3216.8 |
| <i>B2</i> | -0.19 | 1.06 | -0.54 | 0.90 | -0.29 | 1.20 | 1.14 | 0.86 | 11.21 | – | – | -3235.0 |
| <i>B3</i> | -0.19 | 1.07 | 0.69 | 0.90 | -0.29 | 1.22 | 1.39 | 0.87 | 6.23 | – | – | -3232.6 |
| <i>B4</i> | 0.08 | 1.13 | -1.57 | 1.05 | 0.06 | 1.30 | -1.37 | 1.03 | 1.69 | – | – | -3291.0 |
| <i>B4'</i> | -0.39 | 1.08 | -1.57 | 0.80 | -0.50 | 1.24 | -1.57 | 0.78 | 1.55 | – | – | -3309.4 |
| <i>B5</i> | -0.39 | 1.06 | -1.57 | 0.80 | -0.46 | 1.23 | -1.57 | 0.80 | 2.34 | – | – | -3323.3 |
| <i>B6</i> | -0.34 | 1.07 | -1.57 | 0.80 | -0.48 | 1.23 | -1.38 | 0.77 | 2.01 | – | – | -3244.6 |
| <i>B6'</i> | -0.02 | 1.10 | -1.36 | 1.01 | -0.07 | 1.25 | -0.93 | 0.98 | 2.02 | – | – | -3235.4 |
| <i>B7</i> | -0.34 | 1.08 | -1.57 | 0.80 | -0.48 | 1.23 | -1.42 | 0.77 | 1.30 | – | – | -3245.8 |
| <i>B8</i> | -0.35 | 1.07 | -1.56 | 0.79 | -0.49 | 1.23 | -1.57 | 0.76 | 1.75 | – | – | -3256.2 |
| <i>B12</i> | -0.07 | 1.02 | -1.25 | 0.91 | -0.13 | 1.17 | -0.48 | 0.89 | 0.99 | – | – | -3315.5 |
| <i>BB1</i> | -0.20 | 1.08 | -1.54 | 0.89 | -0.29 | 1.23 | -1.26 | 0.86 | 0.44 | 1.64 | – | -3225.9 |
| <i>BB3</i> | -0.21 | 1.06 | -1.57 | 0.88 | -0.32 | 1.22 | -1.48 | 0.84 | 1.70 | 0.16 | – | -3236.5 |
| <i>BB4</i> | 0.09 | 1.13 | -1.57 | 1.05 | 0.06 | 1.30 | -1.34 | 1.02 | 1.70 | 0.03 | – | -3291.0 |
| <i>BB7</i> | -0.18 | 1.08 | -1.57 | 0.90 | -0.27 | 1.25 | -1.56 | 0.87 | 1.80 | 1.03 | – | -3241.5 |
| <i>BB10</i> | -0.22 | 1.05 | -0.53 | 0.88 | -0.33 | 1.21 | 0.96 | 0.84 | 1.29 | 3.97 | – | -3221.5 |
| <i>BB11</i> | -0.19 | 1.06 | -1.15 | 0.89 | -0.29 | 1.20 | -0.52 | 0.86 | 0.72 | 19.59 | – | -3214.9 |
| <i>BBMF</i> | -0.18 | 1.06 | 0.58 | 0.91 | -0.28 | 1.21 | 1.29 | 0.87 | 1.19 | 5.54 | – | -3228.6 |
| <i>BC1</i> | -0.17 | 1.09 | -1.36 | 0.91 | -0.28 | 1.22 | -0.99 | 0.86 | 2.04 | 1.89 | 0.49 | -3233.6 |
| <i>BC2</i> | -0.17 | 1.09 | -1.37 | 0.91 | -0.29 | 1.23 | -1.05 | 0.86 | 2.73 | 2.91 | 0.51 | -3239.8 |

Table 11: *SGSH* margins: Estimated parameters

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