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A NOTE ON A NON-PARAMETRIC TAIL DEPENDENCE ESTIMATOR

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SUMMARY
We present a non-parametric tail dependence estimator which arises naturally from a specific regression model. Above that, this tail dependence estimator also results from a specific copula mixture.

Keywords and phrases: Upper tail dependence; nonparametric estimation; copula

1 Coefficients of Tail Dependence (TDC)

Let $X$ and $Y$ denote two random variables with joint distribution $F_{X,Y}(x, y)$ and continuous marginal distribution functions $F_X(x)$ and $F_Y(y)$. According to Sklar’s (1960) fundamental theorem, there exists a unique decomposition

$$F_{X,Y}(x, y) = C(F_X(x), F_Y(y))$$

of the joint distribution into its marginal distribution functions and the so-called copula (function)

$$C(u, v) = P(U \leq u, V \leq v), \quad U \equiv F_X(X), \quad V \equiv F_Y(Y)$$

on $[0, 1]^2$ which comprises the information about the underlying dependence structure (For details on copulas we refer to Joe, 1997). The concept of tail dependence provides, roughly speaking, a measure for extreme co-movements in the lower and upper tail of $F_{X,Y}(x, y)$, respectively. The upper tail dependence coefficient (TDC) is usually defined by

$$\lambda_U \equiv \lim_{u \to 1^-} P(Y > F_Y^{-1}(u) | X > F_X^{-1}(u)) = \lim_{u \to 1^-} \frac{1 - 2u + C(u, u)}{1 - u} \in [0, 1]. \quad (1.1)$$

noting that $\lambda_U$ is solely depending on $C(u, v)$ and not on the marginal distributions. Analogously, the lower TDC is defined as

$$\lambda_L \equiv \lim_{u \to 0^+} P(Y \leq F_Y^{-1}(u) | X \leq F_X^{-1}(u)) = \lim_{u \to 0^+} C(u, u). \quad (1.2)$$

Coles et al. (1999) provide asymptotically equivalent versions of (1.1) and (1.2),

$$\lambda_L = 2 - \lim_{u \to 0^+} \frac{\log(1 - 2u + C(u, u))}{\log(1 - u)} \quad \text{and} \quad \lambda_U = 2 - \lim_{u \to 1^-} \frac{\log C(u, u)}{\log(u)}. \quad (1.3)$$

For reason of brevity, we focus on the upper TDC $\lambda_U$. Results on the lower TDC can be obtained in a similar manner.
2 Reviewing non-parametric TDC-estimators

For a given (bivariate) random sample of length \(n\) \((X_1, Y_1), \ldots, (X_n, Y_n)\) from \((X, Y)\) let

\[X(1) = \min\{X_1, \ldots, X_n\} \leq \ldots \leq X(n) = \max\{X_1, \ldots, X_n\}\]

denote the corresponding order statistics. All of the relevant non-parametric TDC-estimator \(\hat{\lambda}_U\) of \(\lambda_U\) (See, e.g., Schmidt & Stadtmüller, 2006, Frahm, Junker & Schmidt, 2005 and Dobric & Schmid, 2005) rest upon the non-parametric copula estimator

\[
C_n(i/n, j/n) = \frac{1}{n} \sum_{l=1}^{n} 1(X_l \leq X(i), Y_l \leq Y(j)).
\]  

(2.1)

Plugging (2.1) into equation (1.1) and its asymptotically equivalent version (1.3), respectively, a first pair of estimators is obtained (via "simple replacement")

\[
\hat{\lambda}_U^{[1]} = \frac{C_n((1-k/n, 1) \times (1-k/n, 1))}{1 - (1-k/n)} \quad \text{and} \quad \hat{\lambda}_U^{[2]} = 2 - \frac{\log C_n(1-k/n, 1-k/n)}{\log(1-k/n)},
\]

where \(k \approx \sqrt{n}\) seems to be appropriate (cp. Dobric & Schmid, 2005, section 4). Secondly, Dobric & Schmid (2005) interpret equation (1.1) after suitable re-formulations as regression equation

\[
C_n((1-i/n, 1) \times (1-i/n, 1)) = \lambda_U \cdot \frac{i}{n} + \varepsilon_i, \quad i = 1, \ldots, k
\]  

(2.2)

which motives \(\hat{\lambda}_U^{[3]}\) as OLS-estimator of equation (2.2). Thirdly, Dobric & Schmid (2005) propose to approximate the unknown copula \(\tilde{C}(u, v)\) by the convex-combination \(\tilde{C}(u, v) = \alpha \min\{u, v\} + (1-\alpha)uv\) of the maximum (co-monotonicity) copula \(\min\{u, v\}\) and the independence copula \(C_\perp(u, v) = uv\) (i.e. copula \(B_{11}\) in Joe, 1997). Noting that \(\lambda_U\) of \(\tilde{C}\) is given by \(\alpha\), Dobric & Schmid (2005) introduce \(\hat{\lambda}_U^{[4]}\) which corresponds to that \(\alpha\) which minimizes

\[
F(\alpha) = \sum_{i=1}^{k} \left( C_n \left( \frac{1-i}{n}, 1 - \frac{i}{n} \right) - \tilde{C} \left( \frac{1-i}{n}, 1 - \frac{i}{n} \right) \right)^2.
\]

The following tables classifies the above-mentioned estimators \(\hat{\lambda}_U^{[i]}, i = 1, \ldots, 4\). By the end of this work we present an TDC-estimator \(\lambda_U^{(6)}\) which coincides with the TDC-estimator \(\hat{\lambda}_U^{(5)}\) which arises from a regression equation derived from TDC-formula (1.3).

<table>
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<th>Underlying method</th>
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Table 1: A classification scheme of non-parametric TDC estimators.
3 Derivation of a new non-parametric TDC-estimator

Instead of considering the arithmetic mean of the independence copula $C_\perp$ and the co-monotonicity copula $C_U$, we now focus on the geometric mean of $C_\perp$ and $C_U$ (i.e. copula family B12 in Joe, 1997), that is

$$C^*(u, v) = (\min\{u, v\})^{\delta} \cdot (uv)^{1-\delta}, \quad \delta \in [0, 1].$$  \hspace{1cm} (3.1)

We first proof that $\delta$ corresponds to the lower TDC of $C$: Using equation (1.3),

$$\lambda_U = 2 - \lim_{u \to 1-} \frac{\log C^*(u, u)}{\log(u)} = 2 - \lim_{u \to 1-} \frac{\log(u^{2-\delta})}{\log(u)} = \delta. \hspace{1cm} (3.2)$$

In accordance to Dobric & Schmid (2005), approximating the (unknown) copula $C(u, v)$ by $C^*(u, v)$, an estimator for $\lambda_U$ is given by

$$\hat{\lambda}_U^{(6)} = \arg\min_{\lambda \in [0, 1]} \sum_{i=1}^{k} \left( C_n\left(1 - \frac{i}{n}, 1 - \frac{i}{n}\right) - \left(1 - \frac{i}{n}\right)^{2-\lambda}\right)^2. \hspace{1cm} (3.3)$$

We next show that $\hat{\lambda}_U^{(6)}$ equals $\hat{\lambda}_U^{(5)}$, a new estimator which results from an LS-estimator of the equation

$$\log C_n\left(1 - \frac{i}{n}, 1 - \frac{i}{n}\right) = (2 - \lambda_U) \cdot \log \left(\frac{i}{n}\right) + \varepsilon_i, \quad i = 1, \ldots, k$$

which itself follows from the Coles et al. (1999) formula (1.3) in combination with (2.1).

**Lemma 3.1.** The tail dependence estimator $\hat{\lambda}_U^{(6)}$ and $\hat{\lambda}_U^{(5)}$ are asymptotically equivalent.

**Proof:** We first observe that the LS estimator can be represented as

$$\hat{\lambda}_U^{(5)} = \arg\min_{\lambda \in [0, 1]} \sum_{i=1}^{k} \left( \log C_n\left(1 - \frac{i}{n}, 1 - \frac{i}{n}\right) - (2 - \lambda) \cdot \log \left(1 - \frac{i}{n}\right) \right)^2. \hspace{1cm} (3.4)$$

Now, using the relationship $\log(y^d) \approx 1 - y^d$ for $y \approx 1$ and $d \in [0, 1]$,

$$\hat{\lambda}_U^{(5)} \approx \arg\min_{\lambda \in [0, 1]} \sum_{i=1}^{k} \left( 1 - C_n\left(1 - \frac{i}{n}, 1 - \frac{i}{n}\right) - 1 + \left(1 - \frac{i}{n}\right)^{2-\lambda} \right)^2$$

$$\approx \arg\min_{\lambda \in [0, 1]} \sum_{i=1}^{k} \left( C_n\left(1 - \frac{i}{n}, 1 - \frac{i}{n}\right) - \left(1 - \frac{i}{n}\right)^{2-\lambda} \right)^2 = \hat{\lambda}_U^{(6)}. \hspace{1cm} \square$$
References


