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A NOTE ON A NON-PARAMETRIC TAIL DEPENDENCE ESTIMATOR

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SUMMARY

We present a non-parametric tail dependence estimator which arises naturally from a specific regression model. Above that, this tail dependence estimator also results from a specific copula mixture.

Keywords and phrases: Upper tail dependence; nonparametric estimation; copula

1 Coefficients of Tail Dependence (TDC)

Let X and Y denote two random variables with joint distribution $F_{X,Y}(x, y)$ and continuous marginal distribution functions $F_X(x)$ and $F_Y(y)$. According to Sklar's (1960) fundamental theorem, there exists a unique decomposition

$$F_{X,Y}(x, y) = C(F_X(x), F_Y(y))$$

of the joint distribution into its marginal distribution functions and the so-called copula (function)

$$C(u, v) = P(U \leq u, V \leq v), \quad U \equiv F_X(X), \quad V \equiv F_Y(Y)$$

on $[0, 1]^2$ which comprises the information about the underlying dependence structure (For details on copulas we refer to Joe, 1997). The concept of tail dependence provides, roughly speaking, a measure for extreme co-movements in the lower and upper tail of $F_{X,Y}(x, y)$, respectively. The upper tail dependence coefficient (TDC) is usually defined by

$$\lambda_U \equiv \lim_{u \rightarrow 1^-} P(Y > F_Y^{-1}(u) | X > F_X^{-1}(u)) = \lim_{u \rightarrow 1^-} \frac{1 - 2u + C(u, u)}{1 - u} \in [0, 1]. \quad (1.1)$$

noting that λ_U is solely depending on $C(u, v)$ and not on the marginal distributions. Analogously, the lower TDC is defined as

$$\lambda_L \equiv \lim_{u \rightarrow 0^+} P(Y \leq F_Y^{-1}(u) | X \leq F_X^{-1}(u)) = \lim_{u \rightarrow 0^+} \frac{C(u, u)}{u}. \quad (1.2)$$

Coles et al. (1999) provide asymptotically equivalent versions of (1.1) and (1.2),

$$\lambda_L = 2 - \lim_{u \rightarrow 0^+} \frac{\log(1 - 2u + C(u, u))}{\log(1 - u)} \quad \text{and} \quad \lambda_U = 2 - \lim_{u \rightarrow 1^-} \frac{\log C(u, u)}{\log(u)}. \quad (1.3)$$

For reason of brevity, we focus on the upper TDC λ_U . Results on the lower TDC can be obtained in a similar manner.

2 Reviewing non-parametric TDC-estimators

For a given (bivariate) random sample of length n $(X_1, Y_1), \dots, (X_n, Y_n)$ from (X, Y) let

$$X_{(1)} \equiv \min\{X_1, \dots, X_n\} \leq \dots \leq X_{(n)} \equiv \max\{X_1, \dots, X_n\}$$

denote the corresponding order statistics. All of the relevant non-parametric TDC-estimator $\widehat{\lambda}_U$ of λ_U (See, e.g., Schmidt & Stadtmüller, 2006, Frahm, Junker & Schmidt, 2005 and Dobric & Schmid, 2005) rest upon the non-parametric copula estimator

$$C_n(i/n, j/n) = \frac{1}{n} \sum_{l=1}^n \mathbf{1}(X_l \leq X_{(i)}, Y_l \leq Y_{(j)}). \quad (2.1)$$

Plugging (2.1) into equation (1.1) and its asymptotically equivalent version (1.3), respectively, a first pair of estimators is obtained (via "simple replacement")

$$\widehat{\lambda}_U^{[1]} \equiv \frac{C_n((1 - k/n, 1] \times (1 - k/n, 1])}{1 - (1 - k/n)} \quad \text{and} \quad \widehat{\lambda}_U^{[2]} = 2 - \frac{\log C_n(1 - k/n, 1 - k/n)}{\log(1 - k/n)},$$

where $k \approx \sqrt{n}$ seems to be appropriate (cp. Dobric & Schmid, 2005, section 4). Secondly, Dobric & Schmid (2005) interpret equation (1.1) after suitable re-formulations as regression equation

$$C_n((1 - i/n, 1] \times (1 - i/n, 1]) = \lambda_U \cdot \frac{i}{n} + \varepsilon_i, \quad i = 1, \dots, k \quad (2.2)$$

which motivates $\widehat{\lambda}_U^{[3]}$ as OLS-estimator of equation (2.2). Thirdly, Dobric & Schmid (2005) propose to approximate the unknown copula $C(u, v)$ by the convex-combination $\widetilde{C}(u, v) \equiv \alpha \min\{u, v\} + (1 - \alpha)uv$ of the maximum (co-monotonicity) copula $\min\{u, v\}$ and the independence copula $C_\perp(u, v) = uv$ (i.e. copula **B11** in Joe, 1997). Noting that λ_U of \widetilde{C} is given by α , Dobric & Schmid (2005) introduce $\widehat{\lambda}_U^{[4]}$ which corresponds to that α which minimizes

$$F(\alpha) \equiv \sum_{i=1}^k \left(C_n \left(1 - \frac{i}{n}, 1 - \frac{i}{n} \right) - \widetilde{C} \left(1 - \frac{i}{n}, 1 - \frac{i}{n} \right) \right)^2.$$

The following tables classifies the above-mentioned estimators $\widehat{\lambda}_U^{[i]}$, $i = 1, \dots, 4$. By the end

| Underlying formula ↓ | Underlying method | | |
|----------------------|--|--|--|
| | Simple replacement | Regression approach | Approximation |
| (1.1) | $\widehat{\lambda}_U^{(1)} \checkmark$ | $\widehat{\lambda}_U^{(3)} \checkmark$ | $\widehat{\lambda}_U^{(4)} \checkmark$ |
| (1.3) | $\widehat{\lambda}_U^{(2)} \checkmark$ | $\widehat{\lambda}_U^{(5)}$ | $\widehat{\lambda}_U^{(6)}$ |

Table 1: A classification scheme of non-parametric TDC estimators.

of this work we present an TDC-estimator $\lambda_U^{(6)}$ which coincides with the TDC-estimator $\lambda_U^{(5)}$ which arises from a regression equation derived from TDC-formula (1.3).

3 Derivation of a new non-parametric TDC-estimator

Instead of considering the arithmetic mean of the independence copula C_{\perp} and the comonotonicity copula C_U , we now focus on the geometric mean of C_{\perp} and C_U (i.e. copula family **B12** in Joe, 1997), that is

$$C^*(u, v) = (\min\{u, v\})^{\delta} \cdot (uv)^{1-\delta}, \quad \delta \in [0, 1]. \quad (3.1)$$

We first prove that δ corresponds to the lower TDC of \bar{C} : Using equation (1.3),

$$\lambda_U = 2 - \lim_{u \rightarrow 1^-} \frac{\log C^*(u, u)}{\log(u)} = 2 - \lim_{u \rightarrow 1^-} \frac{\log(u^{2-\delta})}{\log(u)} = \delta.$$

In accordance to Dobric & Schmid (2005), approximating the (unknown) copula $C(u, v)$ by $C^*(u, v)$, an estimator for λ_U is given by

$$\widehat{\lambda}_U^{(6)} = \operatorname{argmin}_{\lambda \in [0, 1]} \sum_{i=1}^k \left(C_n \left(1 - \frac{i}{n}, 1 - \frac{i}{n} \right) - \left(1 - \frac{i}{n} \right)^{2-\lambda} \right)^2. \quad (3.2)$$

We next show that $\widehat{\lambda}_U^{(6)}$ equals $\widehat{\lambda}_U^{(5)}$, a new estimator which results from an LS-estimator of the equation

$$\log C_n \left(1 - \frac{i}{n}, 1 - \frac{i}{n} \right) = (2 - \lambda_U) \cdot \log \left(\frac{i}{n} \right) + \varepsilon_i, \quad i = 1, \dots, k \quad (3.3)$$

which itself follows from the Coles et al. (1999) formula (1.3) in combination with (2.1).

Lemma 3.1. *The tail dependence estimator $\widehat{\lambda}_U^{(6)}$ and $\widehat{\lambda}_U^{(5)}$ are asymptotically equivalent.*

Proof: We first observe that the LS estimator can be represented as

$$\widehat{\lambda}_U^{(5)} = \operatorname{argmin}_{\lambda \in [0, 1]} \sum_{i=1}^k \left(\log C_n \left(1 - \frac{i}{n}, 1 - \frac{i}{n} \right) - (2 - \lambda) \cdot \log \left(1 - \frac{i}{n} \right) \right)^2.$$

Now, using the relationship $\log(y^d) \approx 1 - y^d$ for $y \approx 1$ and $d \in [0, 1]$,

$$\begin{aligned} \widehat{\lambda}_U^{(5)} &= \operatorname{argmin}_{\lambda \in [0, 1]} \sum_{i=1}^k \left(\log C_n \left(1 - \frac{i}{n}, 1 - \frac{i}{n} \right) - \log \left(1 - \frac{i}{n} \right)^{2-\lambda} \right)^2 \\ &\approx \operatorname{argmin}_{\lambda \in [0, 1]} \sum_{i=1}^k \left(1 - C_n \left(1 - \frac{i}{n}, 1 - \frac{i}{n} \right) - 1 + \left(1 - \frac{i}{n} \right)^{2-\lambda} \right)^2 \\ &= \operatorname{argmin}_{\lambda \in [0, 1]} \sum_{i=1}^k \left(C_n \left(1 - \frac{i}{n}, 1 - \frac{i}{n} \right) - \left(1 - \frac{i}{n} \right)^{2-\lambda} \right)^2 = \widehat{\lambda}_U^{(6)}. \quad \square \end{aligned}$$

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