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# A NOTE ON A NON-PARAMETRIC TAIL DEPENDENCE ESTIMATOR

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## SUMMARY

We present a non-parametric tail dependence estimator which arises naturally from a specific regression model. Above that, this tail dependence estimator also results from a specific copula mixture.

*Keywords and phrases:* Upper tail dependence; nonparametric estimation; copula

## 1 Coefficients of Tail Dependence (TDC)

Let  $X$  and  $Y$  denote two random variables with joint distribution  $F_{X,Y}(x,y)$  and continuous marginal distribution functions  $F_X(x)$  and  $F_Y(y)$ . According to Sklar's (1960) fundamental theorem, there exists a unique decomposition

$$F_{X,Y}(x,y) = C(F_X(x), F_Y(y))$$

of the joint distribution into its marginal distribution functions and the so-called copula (function)

$$C(u,v) = P(U \leq u, V \leq v), \quad U \equiv F_X(X), \quad V \equiv F_Y(Y)$$

on  $[0,1]^2$  which comprises the information about the underlying dependence structure (For details on copulas we refer to Joe, 1997). The concept of tail dependence provides, roughly speaking, a measure for extreme co-movements in the lower and upper tail of  $F_{X,Y}(x,y)$ , respectively. The upper tail dependence coefficient (TDC) is usually defined by

$$\lambda_U \equiv \lim_{u \rightarrow 1^-} P(Y > F_Y^{-1}(u) | X > F_X^{-1}(u)) = \lim_{u \rightarrow 1^-} \frac{1 - 2u + C(u,u)}{1-u} \in [0,1]. \quad (1.1)$$

noting that  $\lambda_U$  is solely depending on  $C(u,v)$  and not on the marginal distributions. Analogously, the lower TDC is defined as

$$\lambda_L \equiv \lim_{u \rightarrow 0^+} P(Y \leq F_Y^{-1}(u) | X \leq F_X^{-1}(u)) = \lim_{u \rightarrow 0^+} \frac{C(u,u)}{u}. \quad (1.2)$$

Coles et al. (1999) provide asymptotically equivalent versions of (1.1) and (1.2),

$$\lambda_L = 2 - \lim_{u \rightarrow 0^+} \frac{\log(1 - 2u + C(u,u))}{\log(1-u)} \quad \text{and} \quad \lambda_U = 2 - \lim_{u \rightarrow 1^-} \frac{\log C(u,u)}{\log(u)}. \quad (1.3)$$

For reason of brevity, we focus on the upper TDC  $\lambda_U$ . Results on the lower TDC can be obtained in a similar manner.

## 2 Reviewing non-parametric TDC-estimators

For a given (bivariate) random sample of length  $n$   $(X_1, Y_1), \dots, (X_n, Y_n)$  from  $(X, Y)$  let

$$X_{(1)} \equiv \min\{X_1, \dots, X_n\} \leq \dots \leq X_{(n)} \equiv \max\{X_1, \dots, X_n\}$$

denote the corresponding order statistics. All of the relevant non-parametric TDC-estimator  $\hat{\lambda}_U$  of  $\lambda_U$  (See, e.g., Schmidt & Stadtmüller, 2006, Frahm, Junker & Schmidt, 2005 and Dobric & Schmid, 2005) rest upon the non-parametric copula estimator

$$C_n(i/n, j/n) = \frac{1}{n} \sum_{l=1}^n \mathbf{1}(X_l \leq X_{(i)}, Y_l \leq Y_{(j)}). \quad (2.1)$$

Plugging (2.1) into equation (1.1) and its asymptotically equivalent version (1.3), respectively, a first pair of estimators is obtained (via "simple replacement")

$$\hat{\lambda}_U^{[1]} \equiv \frac{C_n((1 - k/n, 1] \times (1 - k/n, 1])}{1 - (1 - k/n)} \quad \text{and} \quad \hat{\lambda}_U^{[2]} = 2 - \frac{\log C_n(1 - k/n, 1 - k/n)}{\log(1 - k/n)},$$

where  $k \approx \sqrt{n}$  seems to be appropriate (cp. Dobric & Schmid, 2005, section 4). Secondly, Dobric & Schmid (2005) interpret equation (1.1) after suitable re-formulations as regression equation

$$C_n((1 - i/n, 1] \times (1 - i/n, 1]) = \lambda_U \cdot \frac{i}{n} + \varepsilon_i, \quad i = 1, \dots, k \quad (2.2)$$

which motivates  $\hat{\lambda}_U^{[3]}$  as OLS-estimator of equation (2.2). Thirdly, Dobric & Schmid (2005) propose to approximate the unknown copula  $C(u, v)$  by the convex-combination  $\tilde{C}(u, v) \equiv \alpha \min\{u, v\} + (1 - \alpha)uv$  of the maximum (co-monotonicity) copula  $\min\{u, v\}$  and the independence copula  $C_{\perp}(u, v) = uv$  (i.e. copula **B11** in Joe, 1997). Noting that  $\lambda_U$  of  $\tilde{C}$  is given by  $\alpha$ , Dobric & Schmid (2005) introduce  $\hat{\lambda}_U^{[4]}$  which corresponds to that  $\alpha$  which minimizes

$$F(\alpha) \equiv \sum_{i=1}^k \left( C_n \left( 1 - \frac{i}{n}, 1 - \frac{i}{n} \right) - \tilde{C} \left( 1 - \frac{i}{n}, 1 - \frac{i}{n} \right) \right)^2.$$

The following tables classifies the above-mentioned estimators  $\hat{\lambda}_U^{[i]}, i = 1, \dots, 4$ . By the end

Underlying formula ↓	Underlying method		
	Simple replacement	Regression approach	Approximation
(1.1)	$\hat{\lambda}_U^{(1)}$ ✓	$\hat{\lambda}_U^{(3)}$ ✓	$\hat{\lambda}_U^{(4)}$ ✓
(1.3)	$\hat{\lambda}_U^{(2)}$ ✓	$\hat{\lambda}_U^{(5)}$	$\hat{\lambda}_U^{(6)}$

Table 1: A classification scheme of non-parametric TDC estimators.

of this work we present an TDC-estimator  $\lambda_U^{(6)}$  which coincides with the TDC-estimator  $\lambda_U^{(5)}$  which arises from a regression equation derived from TDC-formula (1.3).

### 3 Derivation of a new non-parametric TDC-estimator

Instead of considering the arithmetic mean of the independence copula  $C_{\perp}$  and the co-monotonicity copula  $C_U$ , we now focus on the geometric mean of  $C_{\perp}$  and  $C_U$  (i.e. copula family **B12** in Joe, 1997), that is

$$C^*(u, v) = (\min\{u, v\})^\delta \cdot (uv)^{1-\delta}, \quad \delta \in [0, 1]. \quad (3.1)$$

We first proof that  $\delta$  corresponds to the lower TDC of  $\bar{C}$ : Using equation (1.3),

$$\lambda_U = 2 - \lim_{u \rightarrow 1^-} \frac{\log C^*(u, u)}{\log(u)} = 2 - \lim_{u \rightarrow 1^-} \frac{\log(u^{2-\delta})}{\log(u)} = \delta.$$

In accordance to Dobric & Schmid (2005), approximating the (unknown) copula  $C(u, v)$  by  $C^*(u, v)$ , an estimator for  $\lambda_U$  is given by

$$\hat{\lambda}_U^{(6)} = \operatorname{argmin}_{\lambda \in [0, 1]} \sum_{i=1}^k \left( C_n \left( 1 - \frac{i}{n}, 1 - \frac{i}{n} \right) - \left( 1 - \frac{i}{n} \right)^{2-\lambda} \right)^2. \quad (3.2)$$

We next show that  $\hat{\lambda}_U^{(6)}$  equals  $\hat{\lambda}_U^{(5)}$ , a new estimator which results from an LS-estimaton of the equation

$$\log C_n \left( 1 - \frac{i}{n}, 1 - \frac{i}{n} \right) = (2 - \lambda_U) \cdot \log \left( \frac{i}{n} \right) + \varepsilon_i, \quad i = 1, \dots, k \quad (3.3)$$

which itself follows from the Coles et al. (1999) formula (1.3) in combination with (2.1).

**Lemma 3.1.** *The tail dependence estimator  $\hat{\lambda}_U^{(6)}$  and  $\hat{\lambda}_U^{(5)}$  are asymptotically equivalent.*

*Proof.* We first observe that the LS estimator can be represented as

$$\hat{\lambda}_U^{(5)} = \operatorname{argmin}_{\lambda \in [0, 1]} \sum_{i=1}^k \left( \log C_n \left( 1 - \frac{i}{n}, 1 - \frac{i}{n} \right) - (2 - \lambda) \cdot \log \left( 1 - \frac{i}{n} \right) \right)^2.$$

Now, using the relationship  $\log(y^d) \approx 1 - y^d$  for  $y \approx 1$  and  $d \in [0, 1]$ ,

$$\begin{aligned} \hat{\lambda}_U^{(5)} &= \operatorname{argmin}_{\lambda \in [0, 1]} \sum_{i=1}^k \left( \log C_n \left( 1 - \frac{i}{n}, 1 - \frac{i}{n} \right) - \log \left( 1 - \frac{i}{n} \right)^{2-\lambda} \right)^2. \\ &\approx \operatorname{argmin}_{\lambda \in [0, 1]} \sum_{i=1}^k \left( 1 - C_n \left( 1 - \frac{i}{n}, 1 - \frac{i}{n} \right) - 1 + \left( 1 - \frac{i}{n} \right)^{2-\lambda} \right)^2 \\ &= \operatorname{argmin}_{\lambda \in [0, 1]} \sum_{i=1}^k \left( C_n \left( 1 - \frac{i}{n}, 1 - \frac{i}{n} \right) - \left( 1 - \frac{i}{n} \right)^{2-\lambda} \right)^2 = \hat{\lambda}^{(6)}. \quad \square \end{aligned}$$

## References

- [1] Coles, S.; Heffernan, J.; Tawn, J. Dependence measures for extreme value analyses. *Extremes*, **1999**, 2, 339-365.
- [2] Dobric, J.; Schmid, F. Nonparametric Estimation of the Lower Tail Dependence l in Bivariate Copulas. *Journal of Applied Statistics*, **2005**, 32(4), 387-407 .
- [3] Frahm, G.; Junker, M.; Schmidt, R. Estimating the tail dependence coefficient. *Insurance: Mathematics and Economics*, **2005**, 37, 80-100.
- [4] Joe, H. *Multivariate Models and Dependence Concepts*. Chapman & Hall, London, **1997**.
- [5] Schmidt, R.; Stadtmüller, U. Non-parametric estimation of tail dependence. *Scandinavian Journal of Statistics*, **2006**, 33, 307-335.
- [6] Sklar, A. Fonctions de répartitions à  $n$  dimensions et leurs marges. *Publications de l'Institut de Statistique de l'Université de Paris*, **1959**, 8(1), 229-231.