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RESEARCH ARTICLE

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Generalized Ng–Kundu–Chan model of adaptive progressive Type-II censoring and related inference

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Abstract

The model of adaptive progressive Type-II censoring introduced by Ng et al. (2009) (referred to as Ng–Kundu–Chan model) is extended to allow switching from a given initial censoring plan $\mathscr{R}$ to any arbitrary given plan $\mathscr{S}$ of the same length. In this generalized model, the joint distribution of the failure times and the corresponding likelihood function is derived. It is illustrated that the computation of maximum likelihood and Bayesian estimates are along the same lines as for standard progressive Type-II censoring. However, the distributional properties of the estimators will usually be different since the censoring plan actually applied in the (generalized) Ng–Kundu–Chan model is random. As already mentioned in Cramer and Iliopoulos (2010), we directly show that the normalized spacings are independent and identically exponentially distributed. However, it turns out that the spacings themselves are generally dependent with mixtures of exponential distributions as marginals. These results are used to study linear estimators. Finally, we propose an algorithm for generating random numbers in the generalized Ng–Kundu–Chan model and present some simulation results. The results obtained also provide new findings in the original Ng–Kundu–Chan model; the corresponding implications are highlighted.

KEYWORDS

adaptive progressive censoring, exponential distribution, likelihood inference, linear inference, Ng–Kundu–Chan model, progressive Type-II censoring, spacings

1 | INTRODUCTION

In the last decade, progressively censored life testing has received great attention in the literature and various modifications to the standard model of progressive Type-II censoring have been proposed. In such a life test n items are put simultaneously on test. At the time of the first failure, a (prefixed) number of R_1 items is randomly selected and removed from the experiment. Then, the next (second) failure is observed where R_2 items are withdrawn. This process is continued till the m-th failure observation where the test is terminated. In particular, m failure times $X^{\mathscr{R}}_{1:m:n}, \ldots, X^{\mathscr{R}}_{m:m:n}$ out of n lifetimes are observed where $\mathscr{R} = (R_1, \ldots, R_m)$ denotes the censoring plan. Obviously, it satisfies the linear equation $n = m + \sum_{i=1}^{m} R_i$. More details on the model as well as recent reviews of results and references are provided by Balakrishnan and Cramer (2014); Balakrishnan and Cramer (2023). A schematic illustration of the standard model of progressive Type-II censoring is given in Figure 1.

In order to increase the flexibility of the censoring scheme, various modifications have been proposed. A data based adaption of the censoring plan $\mathscr{R}$ has been presented in Ng et al. (2009), where, given a time threshold T, the original censoring plan $\mathscr{R}$ is modified once the first failure time exceeds the threshold. Then, no further removals occur which means that the experimenter intends to terminate the experiment as soon as possible given the restriction that exactly m

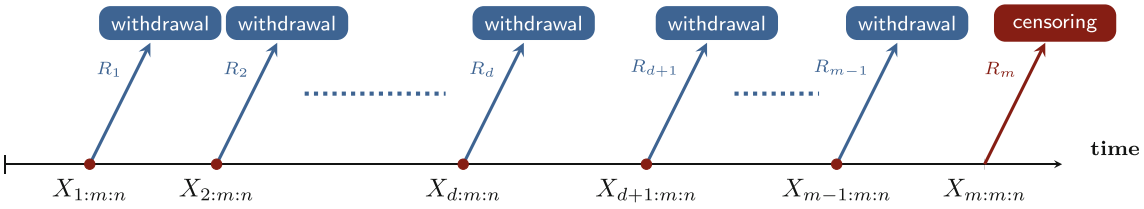

FIGURE 1 Schematic representation of original Type-II progressive censoring.

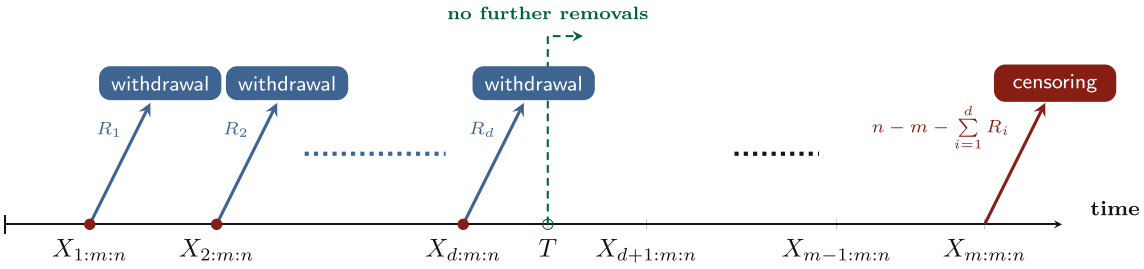

FIGURE 2 Schematic representation of the Ng–Kundu–Chan model of adaptive type-II progressive censoring.

measurements are made. Once the m-th failure is observed, the experiment is terminated. The censoring scheme can therefore be understood as a compromise between a mandatory termination at time T (as in Type-I censoring) and a desired number of m observations. In the following, this model will be referred to as the *Ng–Kundu–Chan model* (of adaptive progressive Type-II censoring). A schematic illustration of the Ng–Kundu–Chan model is provided in Figure 2.

Ng et al. (2009) presented inferential results for (one-parameter) exponentially distributed lifetimes. In particular, they obtained the conditional distribution of the maximum likelihood estimate (given $D = d$) and used this result to construct conditional confidence intervals for the scale parameter of the exponential distribution. Furthermore, approximate confidence intervals based on normal approximations as well as Bayesian credible intervals have been proposed. Using a conditional distribution construction, Cramer and Iliopoulos (2010) proposed a very general account to adaptive progressive Type-II censoring allowing a general study of such models. They pointed out that their approach includes, for example, standard progressive Type-II censoring, the Ng–Kundu–Chan model described above, as well as so-called progressive Type-II censoring with random removals (see, e.g., Yuen and Tse (1996)). They found that, for exponentially distributed lifetimes, the distribution of the maximum likelihood estimator in both the adaptive and non-adaptive model are the same. Therefore, inferential results obtained in the non-adaptive model can be directly applied in the adaptive model (for more details as well as statistical implications of this property, see Remark 3.2). An extension including also (adaptive) Type-I progressive censoring has been developed in Cramer and Iliopoulos (2015). Yan et al. (2021) proposed a modification of the Ng–Kundu–Chan model called *improved adaptive progressive Type-II censoring scheme* (see also Dutta and Kayal (2021)). This model is a Type-I censored version of the Ng–Kundu–Chan model where the life test is terminated at a second threshold $T^* > T$. Following the wording of Balakrishnan et al. (2023), this model can be considered as an hybrid censored version of the original data.

Further adaptive extensions have been proposed in Bairamov and Parsi (2011) and Kinaci (2013) (see also Balakrishnan and Cramer (2014, Sec. 6.2.3)). Moreover, it has to be mentioned that the Ng–Kundu–Chan model has been studied under the label *adaptive Type-II progressive hybrid censoring* (see Balakrishnan et al. (2023), Chapter 13, as well as references cited therein).

In the present paper, we provide an extension of the Ng–Kundu–Chan model which allows to switch from the original censoring plan $\mathscr{R}$ to a *new* one $\mathscr{S}$. Thus, let $T \in \mathbb{R}$ be a threshold and let $\mathscr{R} = (R_1, \ldots, R_m)$ and $\mathscr{S} = (S_1, \ldots, S_m)$ be progressive censoring plans taken from the set

$$\mathfrak{C}_{m,n} = \left\{(r_1, \ldots, r_m) | r_{\bullet m} = n - m, r_j \in \{1, \ldots, m\}\right\}$$

of all admissible censoring plans of length m, where, for convenience, we use the notation $r_{\bullet m} = \sum_{j=1}^{m} r_j$. We also write $\gamma_j = \gamma_j(\mathscr{R})$ for short, $j \in \{1, \ldots, m\}$, which denotes the number of units in the experiment just before the j-th failure occurs when the original censoring plan $\mathscr{R}$ is employed. A schematic representation of the generalized Ng–Kundu–Chan model is provided in Figure 3.

Compared to the original Ng–Kundu–Chan model of adaptive progressive Type-II censoring (cf. Figure 2), the generalization adds further flexibility to the experimental design. The new censoring plan $\mathscr{S}$ can be chosen as any arbitrary admissible censoring plan of the same length as $\mathscr{R}$. Choosing $\mathscr{S}$ as right censoring, that is, $\mathscr{S} = (0, \ldots, 0, n - m)$, no further removals occur after the time threshold T until exactly m measurements are made and we get the original Ng–Kundu–Chan model as a special case.

Furthermore, this flexibility in the choice of $\mathscr{S}$ enables the experimenter to design the experiment according to the

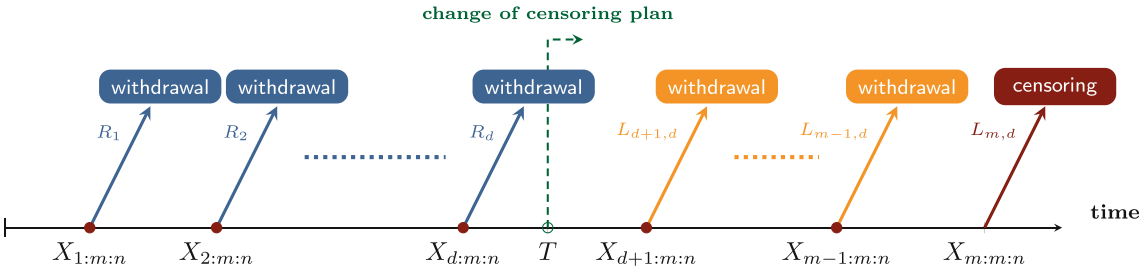

FIGURE 3 Schematic representation of the generalized Ng–Kundu–Chan model with adapted censoring plan $\mathscr{L}_d$.

given needs where the time threshold T can be considered as a control parameter. As mentioned above, choosing $\mathscr{S}$ as in Type-II right censoring, the experiment will be terminated as soon as possible with exactly m observed failure times. This shows that the Ng–Kundu–Chan model is designed to save time. However, the generalized Ng–Kundu–Chan model allows for the implementation of other requirements that may be imposed by management or external conditions. Recall that a popular argument for progressive Type II censoring is to use the censored objects for other tests. Therefore, the control parameter allows modelling of a changing demand for the available objects after time T.

The constraint of observing exactly m failures causes some technical difficulties in the construction of the adaptive censoring plan. In particular, the censoring plan $\mathscr{S}$ may not be applied in its original form in order to ensure the observation of m failures. Hence, an adaptation to an admissible censoring plan will possibly be necessary. A detailed construction of the adapted censoring plan $\mathscr{L}_d$ along with illustrative examples will be presented in Section 2.

The outline of the paper and its main contributions are as follows. In Section 2, we formally introduce the generalized Ng–Kundu–Chan model and discuss the aforementioned construction of the adaptive progressive Type-II censoring plan. Moreover, we derive the joint distribution of the failure times, followed by the respective likelihood function in Section 3. In the spirit of the representation in terms of adaptive progressive Type-II censoring in Cramer and Iliopoulos (2010), we explain that many results of a general kind can be directly obtained from this general account. However, as often, a more detailed analysis is possible when focusing on the generalized Ng–Kundu–Chan model. For this purpose, we assume (two-parameter) exponential distributed lifetimes in Section 3.1. In particular, in Sections 3.1.1 and 3.1.2, we pay attention to the role of (normalized) spacings and derive their distributions by the density transformation theorem. As pointed out in Cramer and Iliopoulos (2010), the normalized spacings are independent and identically exponentially distributed. However, it turns out that this is not true for the spacings themselves which are seen to be dependent in general. Furthermore, we show that the marginal distributions are given by mixtures of exponential distributions. This result will be used in Section 3.1.3 to study linear estimators based on adaptive progressive Type-II censored order statistics. In particular, we establish explicit expressions for the expectation of linear estimators. Note that the normalized spacings can not be used in linear inference since the normalizing factors are random (and depend on the failure times)! Furthermore, in Section 3.1.3, we study the existence of linear unbiased estimators and present conditions which ensure their existence. Then, in Section 3.2, we illustrate how our results can be applied to the Ng–Kundu–Chan model, particularly with respect to linear inference. Finally, we provide an algorithm to generate random numbers in the generalized Ng–Kundu–Chan model and present some simulation results for exponentially distributed lifetimes in Section 4.

At this point, it is worth mentioning that the results obtained in the generalized Ng–Kundu–Chan model also provide new findings in the original Ng–Kundu–Chan model. This particularly addresses the results on exact confidence intervals, the properties of (non-)normalized spacings, as well as the discussion of (best) linear inference. Note that the role of spacings and its implications on linear inference, to our knowledge, have not been discussed in the literature for adaptive progressive Type-II censoring so far. The corresponding implications for the original Ng–Kundu–Chan model are also highlighted.

2 | MODEL DESCRIPTION AND DISTRIBUTION OF THE DATA

Let $\mathscr{L}_D$ be an adapted censoring plan where, after exceeding the threshold T, the original censoring plan $\mathscr{R}$ is adapted as follows. Let

$$D = \sum_{j=1}^{m} \mathbb{1}_{(-\infty,T]}\left(X^{\mathscr{R}}_{j:m:n}\right) \tag{2.1}$$

be the random variable counting the number or progressively Type-II censored order statistics $X^{\mathscr{R}}_{j:m:n}$ which do not exceed T. It is well-known that D has the following properties (see, e.g., Balakrishnan et al., 2023, eq. (7.4)), i.e. Lemma 2.1 follows from the identity $P(D = d) = P\left(X^{\mathscr{R}}_{d:m:n} \le T < X^{\mathscr{R}}_{d+1:m:n}\right)$ and the fact that the distribution of $\left(X^{\mathscr{R}}_{d:m:n}, X^{\mathscr{R}}_{d+1:m:n}\right)$ depends only on $\gamma_1, \ldots, \gamma_{d+1}$.

Lemma 2.1. *Let $F_{d:m:n}$ be the cumulative distribution function (cdf) of $X^{\mathscr{R}}_{d:m:n}$, $1 \le d \le m$. The probability mass function of D is given by*

$$P(D=d) = P_T(D=d) = \begin{cases} 1 - F_{1:m:n}(T), & d=0 \\ F_{d:m:n}(T) - F_{d+1:m:n}(T), & d=1,\ldots,m-1 \\ F_{m:m:n}(T), & d=m \end{cases}. \tag{2.2}$$

Furthermore, the probabilities $P(D=d)$ depend only on the first $d+1$ parameters $\gamma_1, \ldots, \gamma_{d+1}$ which are not affected by the change of the censoring plan.

The observed censoring plan $\mathscr{L}_d = (L_{1,d}, \ldots, L_{m,d})$ given $D=d$ for some $d \in \{0, \ldots, m\}$ is defined by

$$L_{i,d} = \begin{cases} R_i, & \text{if } i \le d \\ \varrho_{i,d}(R_{\bullet d}, S_{d+1}, \ldots, S_i), & \text{if } d+1 \le i \le m-1 \\ n - m - R_{\bullet d} - \sum_{j=d+1}^{m-1} L_{j,d}, & \text{if } i = m \end{cases}, \tag{2.3}$$

where $\varrho_{i,k}(R_{\bullet d}, S_{d+1}, \ldots, S_i)$ denotes for any $i \in \{d+1, \ldots, m\}$ a function of $R_{\bullet d} = \sum_{j=1}^{d} R_j$ and $S_{d+1}, \ldots, S_i$ leading to an admissible censoring plan with exactly m measurements. Of course, $\varrho_{i,d}(R_{\bullet d}, S_{d+1}, \ldots, S_i)$ can be chosen in many ways depending, for example, on whether the test time is to be extended, shortened or the number of withdrawals is to be kept constant. For convenience, let $R_0 = R_{\bullet 0} = 0$.

Note that in (2.3), for any $d \in \{0, \ldots, m\}$, conditions on the withdrawal numbers $\varrho_{i,d}(R_{\bullet d}, S_{d+1}, \ldots, S_i)$, $d+1 \le i \le m-1$, need to be imposed in order to ensure that the experiment is not terminated before m observations are made, that is, in order to ensure that exactly $\sum_{i=1}^{m} L_{i,d} = n - m$ items are withdrawn or censored. In the following, we present a strategy where withdrawals are essentially made according to the censoring plan $\mathcal{S}$.

2.1 | Construction of the adaptive censoring plan

By analogy with non-adaptive progressive Type-II censoring, we consider $\boldsymbol{\gamma}_d = (\gamma_{1,d}, \ldots, \gamma_{m,d})$, where $\gamma_{j,d}$ has to be interpreted as the number of units in the experiment immediately before the j-th failure occurs. In particular, for any $j \in \{d+2, \ldots, m\}$, we must ensure $\gamma_{j,d} \ge m - j + 1$ in order to guarantee that exactly m measurements are observed. For $1 \le j \le d+1$, the condition is satisfied by assumption since $\mathscr{R} \in \mathfrak{C}_{m,n}$ is a proper censoring plan. For $d+2 \le j \le m$, we define

$$\eta_{j,d} = n - j + 1 - R_{\bullet d} - \sum_{i=d+1}^{j-1} S_i. \tag{2.4}$$

Then, for $j = d+2$, we have the condition

$$\eta_{d+2,k} = n - d - 1 - R_{\bullet d} - S_{d+1} \ge m - d - 1 \iff n - m - R_{\bullet d} \ge S_{d+1}.$$

Thus, the number of withdrawn items in step $d+1$ can be maximally chosen as $L_{d+1,d} = (n - m - R_{\bullet d}) \wedge S_{d+1}$. This construction can be extended by iteration which leads to the conditions

$$\eta_{j,d} = n - j + 1 - R_{\bullet d} - \sum_{i=d+1}^{j-2} L_{i,d} - S_{j-1} \ge m - j + 1 \iff n - m - R_{\bullet d} - \sum_{i=d+1}^{j-2} L_{i,d} \ge S_{j-1}, \tag{2.5}$$

for $j \in \{d+2, \ldots, m\}$ so that

$$L_{j-1,d} = \left(n - m - R_{\bullet d} - \sum_{i=d+1}^{j-2} L_{i,d} \right) \wedge S_{j-1}, \quad j \in \{d+2, \ldots, m\}.$$

where $x \wedge y = \min\{x, y\}$, $x, y \in \mathbb{R}$. Similarly, we use the notation $x \vee y = \max\{x, y\}$. For $j = m$, everything left is removed. Consequently, we set

$$\gamma_{j,d} = \begin{cases} \gamma_j = n - j + 1 - R_{\bullet j-1}, & 1 \le j \le d+1 \\ \eta_{j,d} \vee (m - j + 1), & d+2 \le j \le m \end{cases}. \tag{2.6}$$

Then, a censoring plan (2.3) where withdrawals are essentially made according to the censoring plan $\mathcal{S}$ once the threshold T has been exceeded, is given by

$$L_{j,d} = \gamma_{j,d} - \gamma_{j+1,d} - 1, \quad 1 \le j \le m, \tag{2.7}$$

where $\gamma_{m+1,d} = 0$ for convenience. In particular, $\gamma_{1,d} = n$ and $\gamma_{m,d} = L_{m,d} + 1$.

Note that, given d, once $\gamma_{j_0,d} = \eta_{j_0,d} \vee (m - j_0 + 1) = m - j_0 + 1$ for a certain $j_0 \in \{d+2, \ldots, m\}$, it follows $L_{j,d} = 0$ for all $j \in \{j_0, \ldots, m\}$ in the sequel so that there is no withdrawal or censoring anymore, as illustrated in Example 2.4 (see Table 2, $d \in \{1, 2\}$).

Remark 2.2. The max operator in (2.6) ensures that we observe exactly m measurements. If the censoring plans satisfy the condition

$$\gamma_j = \gamma_j(\mathscr{R}) \le \gamma_j(\mathcal{S}), \quad j = 1, \ldots, m-1, \tag{2.8}$$

then the operator can be dropped in (2.6). Condition 2.8 means that, if we run the life test with the censoring plan $\mathscr{R}$, we will have at most the same number of items in the life test as under $\mathcal{S}$ at any censoring time. In this sense, progressive censoring according to the censoring plan $\mathscr{R}$ will be performed earlier than using $\mathcal{S}$. Moreover, in this case, the resulting censoring plan is constructed as a concatenation of the original ones (except for the last component) as

$$(R_1, \ldots, R_d, S_{d+1}, \ldots, S_{m-1}, n - m - R_{\bullet d} - S_{\bullet m-1} + S_{\bullet d})$$

as desired. In the following, censoring plans satisfying this condition for $d \in \{0, \ldots, m\}$ are called regular paired.

In general, see for example, Example 2.4, the effectively applied plan looks like

$$(R_1, \ldots, R_d, S_{d+1}, \ldots, S_{d'}, S^*_{d'+1}, 0, \ldots, 0)$$

with $S^*_{d'+1} \in \{0, \dots, S_{d'+1} - 1\}$. Here, $d' + 1 \geq d+1$ denotes the number where not enough items would be available to ensure m measurements when continuing with the removals according to $\mathcal{S}$.

Furthermore, we note the following simple but very useful fact which follows directly from (2.6).

Lemma 2.3. *Given $d \in \{0, \dots, m\}$, the first d censoring numbers of the adaptive censoring plan $\mathscr{L}_d$ are equal to those of $\mathscr{R}$, that is,*

$$L_{j,d} = R_j, \quad j = 1, \dots, d.$$

Moreover, this implies that $\gamma_{j,d} = \gamma_j$, $j = 1, \dots, d+1$.

Furthermore, for $j \in \{d+2, \dots, m\}$,

$$\gamma_{j,d} = \gamma_j \Leftrightarrow \sum_{i=d+1}^{j-1} R_i = \sum_{i=d+1}^{j-1} S_i.$$

Example 2.4. For illustration, consider the two censoring plans $\mathscr{R} = (1,2,3,5,1,4)$ and $\mathcal{S} = (6,3,0,1,4,2)$ with $n = 22$ and $m = 6$. For $d \in \{0, \dots, 6\}$, the respective observed censoring plans $\mathscr{L}_d$ are given in Tables 1 and 2. Note that, in order to illustrate the change in the censoring plan, neither the threshold T nor the failure times $X^{\mathscr{R}}_{1:m:n}, \dots, X^{\mathscr{R}}_{m:m:n}$ need to be specified; according to Equation (2.1), any relevant information is already included in the realization $d \in \{0, \dots, 6\}$ of the random variable D, that is, in the observed number of failures before the threshold T. In Table 1, we start with the censoring plan $\mathscr{R}$ and switch to $\mathcal{S}$ after the threshold T. In Table 2, we apply the censoring plans in reversed order. Notice that both censoring plans are employed as planned in the first situation presented in Table 1. In the situation of Table 2, the censoring plan $\mathscr{R}$ can not be completely realized as designed for $d \in \{1,2,3,4\}$ since not enough items remain in the experiment after the change of the censoring plan at time T. This applies to the cases $D = d \in \{1,2\}$ in Table 2. Notice that in these situations no progressive censoring occurs after $X_{d+1:m:n}$, which is indicated by zeros in the adapted censoring plan. Of course, this comment does not apply to the last withdrawal time where the experiment is terminated. Further examples of the second phase's censoring plans are presented in Section 4.

Remark 2.5.

(i) In the Ng–Kundu–Chan model introduced by Ng et al. (2009), we have in (2.3) $S_j = 0$, $1 \leq j \leq m-1$. This choice of the alternate censoring plan $\mathcal{S}$ emphasises the idea that the life testing experiment should be terminated as soon as possible once the test duration exceeds the prefixed time threshold T. Then, (2.6) may be written as $\gamma_{j,d} = n - j + 1 - R_{\bullet \min(d,j-1)}$, $1 \leq j \leq m$. (In Ng et al. (2009), the max-operator is erroneously written instead of min-operator as upper limit in the sum in $R_{\bullet \min(d,j-1)} = \sum_{i=1}^{\min(d,j-1)} R_i$). Note that $\sum_{i=1}^{m} L_{i,d} = n - m$ is always satisfied as long as $\mathscr{R}$ remains an admissible censoring plan.

(ii) As mentioned above, adaptive versions of progressive Type-II censoring have also been proposed in Bairamov and Parsi (2011) and Kinaci (2013), see also Balakrishnan and Cramer (2014, Sec. 6.2.3). In Bairamov and Parsi (2011), an extension regarding the number of possible adaptions of the censoring plan is discussed. According to the notation in (2.3), $m-1$ thresholds $0 \leq T_1 < T_2 < \cdots < T_{m-1}$ are considered, where R_j units are removed from the experiment when the j-th failure occurs before time T_j

TABLE 1 Adaptive censoring plans with initial plan $\mathscr{R}$ and $\mathcal{S}$ as desired plan after the change time T.

Change number	Adapted censoring plan	Vector of γ_d Minimum requirement: (6,5,4,3,2,1)					
d	$\mathscr{L}_d$	$\gamma_{1,d}$	$\gamma_{2,d}$	$\gamma_{3,d}$	$\gamma_{4,d}$	$\gamma_{5,d}$	$\gamma_{6,d}$
0	$(6,3,0,1,4,2) = \mathcal{S}$	22	15	11	10	8	3
1	$(1,3,0,1,4,7)$	22	20	16	15	13	8
2	$(1,2,0,1,4,8)$	22	20	17	16	14	9
3	$(1,2,3,1,4,5)$	22	20	17	13	11	6
4	$(1,2,3,5,4,1)$	22	20	17	13	7	2
5	$(1,2,3,5,1,4) = \mathscr{R}$	22	20	17	13	7	5
6	$(1,2,3,5,1,4) = \mathscr{R}$	22	20	17	13	7	5

Note: The parts of the adapted censoring plans are marked in the respective color of the original censoring plans.

TABLE 2 Adaptive censoring plans with initial plan $\mathcal{S}$ and $\mathcal{R}$ as desired plan after the change time T.

Change number	Adapted censoring plan	Vector of γ_d Minimum requirement: (6, 5, 4, 3, 2, 1)					
d	$\mathcal{L}_d$	$\gamma_{1,d}$	$\gamma_{2,d}$	$\gamma_{3,d}$	$\gamma_{4,d}$	$\gamma_{5,d}$	$\gamma_{6,d}$
0	$(1,2,3,5,1,4) = \mathcal{R}$	22	20	17	13	7	5
1	$(6,2,3,5,0,0)$	22	15	12	8	2	1
2	$(6,3,3,4,0,0)$	22	15	11	7	2	1
3	$(6,3,0,5,1,1)$	22	15	11	10	4	2
4	$(6,3,0,1,1,5)$	22	15	11	10	8	6
5	$(6,3,0,1,4,2) = \mathcal{S}$	22	15	11	10	8	3
6	$(6,3,0,1,4,2) = \mathcal{S}$	22	15	11	10	8	3

Note: The parts of the adapted censoring plans are marked in the respective color of the original censoring plans.

and S_j units otherwise, whereby $R_j \leq S_j$, $1 \leq j \leq m-1$. It is worth noticing that the restriction $R_j \leq S_j$ might be quite counter intuitive to an experimentor's aim to reduce the time on test. Also, the model presented in Bairamov and Parsi (2011) cannot be considered as a generalization of the Ng–Kundu–Chan model, having $S_j = 0$, $1 \leq j \leq m-1$, in the latter. In the model introduced in the present paper, it is $T_j = T$, $1 \leq j \leq m-1$, and we detach ourselves from the assumption that $R_j \leq S_j$.

(iii) Condition (2.8) holds for example when the inequalities $S_j \leq R_j$, $1 \leq j \leq m-1$, are satisfied so that exactly m measurements are guaranteed in the considered life test. Note that this condition is trivially fulfilled for any admissible censoring plan $\mathcal{R}$ in the Ng–Kundu–Chan model since $S_j = 0$, $1 \leq j \leq m-1$.

However, in general, dropping the maximum operator in (2.6) may lead to a violation of condition (2.5), that is,

$$\gamma_{j,d} < m - j + 1 \quad \text{for some } j > d+1.$$

Therefore, if sampling were carried out as intended according to the switch from $\mathcal{R}$ to $\mathcal{S}$ after time T, there might be too few units available to ensure a total of m observations. The experiment would therefore end before the m-th failure occurs, with the sample size being at most m. However, by design, at least one observation (the minimum of the data) would be assured.

Another extension would be to consider censoring plans of different lengths and thus different numbers of intended observations, that is, $\mathcal{R}$ would have the length m_1 and $\mathcal{S}$ would have the length m_2.

Both modifications would lead to models with a random sample size. Such extensions will be subject of future research.

2.2 | Joint distribution

Let $X^{\mathcal{L}_D}_{1:m:n}, \ldots, X^{\mathcal{L}_D}_{m:m:n}$ be the adaptive progressively Type-II censored data with proper censoring plan as in (2.3) (for convenience, let $X^{\mathcal{L}_D}_{0:m:n} = -\infty$). It is worth noticing that the censoring plan $\mathcal{L}_D$ is random in general. Moreover, after exceeding the threshold T, the observation of the subsequent failure time $X^{\mathcal{L}_D}_{k+1:m:n}$ does only depend on $R_{\bullet k}$ of the initially applied censoring plan $\mathcal{R} \in \mathfrak{C}_{m,n}$.

In order to derive the joint cumulative distribution function of $X^{\mathcal{L}_D}_{1:m:n}, \ldots, X^{\mathcal{L}_D}_{m:m:n}$, for $t_1, \ldots, t_m \in \mathbb{R}$, by similarity with the derivations in Cramer and Balakrishnan (2013) (see also Balakrishnan et al. (2023)), we get

$$\begin{aligned}
&P\left(X^{\mathcal{L}_D}_{j:m:n} \leq t_j, 1 \leq j \leq m\right) \\
&= \sum_{k=0}^{m} P\left(X^{\mathcal{L}_D}_{j:m:n} \leq t_j, 1 \leq j \leq m, D = k\right) \\
&= \sum_{k=0}^{m-1} P\left(X^{\mathcal{L}_k}_{j:m:n} \leq t_j, 1 \leq j \leq m, X^{\mathcal{L}_k}_{k:m:n} \leq T < X^{\mathcal{L}_k}_{k+1:m:n}\right) \\
&\quad + P\left(X^{\mathcal{R}}_{j:m:n} \leq t_j, 1 \leq j \leq m, X^{\mathcal{R}}_{m:m:n} \leq T\right).
\end{aligned}$$

Then,

$$\begin{aligned}
&P\left(X^{\mathcal{R}}_{j:m:n} \leq t_j, 1 \leq j \leq m, X^{\mathcal{R}}_{m:m:n} \leq T\right) \\
&\quad = F^{\mathcal{R}}_{1,\ldots,m:m:n}(t_1, \ldots, t_{m-1}, t_m \wedge T).
\end{aligned}$$

For $k \in \{0, \ldots, m-1\}$, we get

$$\begin{aligned}
&P\Big(X^{\mathcal{L}_k}_{j:m:n} \leq t_j, j \in \{1, \ldots, m\} \setminus \\
&\qquad \{k\}, X^{\mathcal{L}_k}_{k:m:n} \leq t_k \wedge T, T < X^{\mathcal{L}_k}_{k+1:m:n}\Big) \\
&= P\Big(X^{\mathcal{L}_k}_{j:m:n} \leq t_j, j \in \{1, \ldots, m\} \setminus \\
&\qquad \{k\}, X^{\mathcal{L}_k}_{k:m:n} \leq t_k \wedge T, T < X^{\mathcal{L}_k}_{k+1:m:n}, T \leq \min_{k+1 \leq j \leq m} t_j\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{1}_{[T,\infty)}\left(\min_{k+1\le j\le m} t_j\right)\Big[P\Big(X^{\mathscr{L}_k}_{j:m:n} \le t_j, j\in\{1,\ldots,m\}\setminus \\
&\quad \{k\}, X^{\mathscr{L}_k}_{k:m:n} \le t_k \wedge T\Big) \\
&\quad - P\Big(X^{\mathscr{L}_k}_{j:m:n} \le t_j, j\in\{1,\ldots,m\}\setminus \\
&\quad \{k,k+1\}, X^{\mathscr{L}_k}_{k:m:n} \le t_k \wedge T, X^{\mathscr{L}_k}_{k+1:m:n} \le T\Big)\Big] \\
&= \mathbb{1}_{[T,\infty)}\left(\min_{k+1\le j\le m} t_j\right) \\
&\Big[F^{\mathscr{L}_k}_{1,\ldots,m:m:n}(t_1,\ldots,t_{k-1},t_k\wedge T,t_{k+1},\ldots,t_m) \\
&\quad -F^{\mathscr{L}_k}_{1,\ldots,m:m:n}(t_1,\ldots,t_{k-1},t_k\wedge T,T,t_{k+2},\ldots,t_m)\Big].
\end{aligned}
$$

Therefore, we find the following expression

$$
\begin{aligned}
&P\left(X^{\mathscr{L}_D}_{j:m:n} \le t_j, 1\le j\le m\right) \\
&= \sum_{k=0}^{m} P\left(X^{\mathscr{L}_D}_{j:m:n} \le t_j, 1\le j\le m, D=k\right) \\
&= \sum_{k=0}^{m} \mathbb{1}_{[T,\infty)}\left(\min_{k+1\le j\le m} t_j\right) \\
&\quad F^{\mathscr{L}_k}_{1,\ldots,m:m:n}(t_1,\ldots,t_{k-1},t_k\wedge T,t_{k+1},\ldots,t_m) \\
&\quad - \sum_{k=0}^{m-1} \mathbb{1}_{[T,\infty)}\left(\min_{k+1\le j\le m} t_j\right) \\
&\quad F^{\mathscr{L}_k}_{1,\ldots,m:m:n}(t_1,\ldots,t_{k-1},t_k\wedge T,T,t_{k+2},\ldots,t_m). \qquad (2.9)
\end{aligned}
$$

Thus, differentiating the cumulative distribution function w.r.t. $t_1,\ldots,t_m$, we get the density function (with $t_0 = -\infty, t_{m+1} = \infty$)

$$
f^{\mathscr{L}_D}_{1,\ldots,m:m:n}(t_1,\ldots,t_m) = \sum_{k=0}^{m} \mathbb{1}_{(t_k,t_{k+1}]}(T) f^{\mathscr{L}_k}_{1,\ldots,m:m:n}(t_1,\ldots,t_m),
$$

$$
t_1 < \cdots < t_m. \qquad (2.10)
$$

Obviously, the joint density function in the generalized adaptive model with random censoring plan $\mathscr{L}_D$ is defined step-wise in terms of the joint density functions of non-adaptive progressively Type II censored order statistics with deterministic censoring plans $\mathscr{L}_k, 0\le k\le m$. The result can be found for the standard Ng–Kundu–Chan model by Ng et al. (2009) in Balakrishnan et al. (2023, Remark 13.4). In the following, we will use the identity (2.10) for the likelihood to point out that many results in statistical inference in the generalized Ng–Kundu–Chan can be directly obtained from the corresponding results under progressive Type-II censoring (see also comments in Cramer and Iliopoulos (2010)).

3 | INFERENCE IN THE GENERALIZED NG–KUNDU–CHAN MODEL

In the following, we assume that the items under study have a life time distribution with an absolutely continuous cumulative distribution function F_θ with density function f_θ that depends depends on a finite-dimensional parameter $\theta\in\Theta$. Given observed failure times $x_1 < \cdots < x_m$, we get from (2.10) the likelihood function

$$
\begin{aligned}
\mathcal{L}(\theta|x_1,\ldots,x_m) &= \sum_{k=0}^{m} \mathbb{1}_{(x_k,x_{k+1}]}(T) f^{\mathscr{L}_k}_{1,\ldots,m:m:n;\theta}(x_1,\ldots,x_m) \\
&= f^{\mathscr{L}_d}_{1,\ldots,m:m:n;\theta}(x_1,\ldots,x_m) \qquad (3.1)
\end{aligned}
$$

with $d = \sum_{j=1}^{m} \mathbb{1}_{(-\infty,T]}(x_j)$ being obtained directly from the data. Although the censoring plan is random in the adaptive model, according to (3.1), the likelihood function in the adaptive model equals that in the non-adaptive model when the observed censoring plan is used. Consequently, assuming that $X^{\mathscr{L}_D}_{1:m:n},\ldots,X^{\mathscr{L}_D}_{m:m:n}$ are based on a cumulative distribution function F_θ with density function f_θ with some unknown finite-dimensional parameter $\theta\in\Theta$, the likelihood function (3.1) is given by

$$
\begin{aligned}
&\mathcal{L}(\theta|x_1,\ldots,x_m) \\
&= \prod_{j=1}^{m}\left(\gamma_{j,d} f_\theta(x_j)\,(1-F_\theta(x_j))^{L_{j,d}}\right) \\
&= c(\mathscr{L}_d)\cdot\left(\prod_{j=1}^{m} f_\theta(x_j)\right)\cdot\left(\prod_{j=1}^{d}\left(1-F_\theta(x_j)\right)^{R_j}\right) \\
&\quad \cdot\left(\prod_{j=d+1}^{m-1}\left(1-F_\theta(x_j)\right)^{L_{j,d}}\right)(1-F_\theta(x_m))^{\gamma_{m,d}-1}, \qquad (3.2)
\end{aligned}
$$

where $L_{j,d}$ is defined as in (2.7) and $c(\mathscr{L}_d) = \prod_{j=1}^{m}\gamma_{j,d}$. Thus, maximization of $\mathcal{L}(\theta|x_1,\ldots,x_m)$ w.r.t. to θ in the in the generalized Ng–Kundu–Chan model yields the same solution as in the non-adaptive model. Consequently, results on maximum likelihood inference regarding existence, uniqueness of maximum likelihood estimates and their explicit expressions or, respectively, approximation methods can be directly derived from the non-adaptive model of progressive Type II censored order statistics. For a comprehensive summary, we refer to Balakrishnan and Cramer (2014, Chapter 12).

Bayesian inference in the generalized Ng–Kundu–Chan model can be conducted by similarity with progressively Type-II censored data. A general discussion of Bayesian inference under progressive Type-II censoring can be found in Chapter 15 of Balakrishnan and Cramer (2014). Generally, given the prior distribution with density function $\pi_{\mathbf{a}}(\theta)$ and hyperparameters $\mathbf{a}$, the posterior density function $\pi^*_{\mathbf{a}}(\cdot|x_1,\ldots,x_m)$ is given by

$$
\pi^*_{\mathbf{a}}(\theta|x_1,\ldots,x_m) = \frac{\pi_{\mathbf{a}}(\theta)\mathcal{L}(\theta|x_1,\ldots,x_m)}{\int \mathcal{L}(\xi|x_1,\ldots,x_m)\pi_{\mathbf{a}}(\xi)\,d\xi}. \qquad (3.3)
$$

Clearly, the likelihood function in (3.1) is that of a progressively Type-II censored sample with the adapted censoring plan. Therefore, all the Bayesian inferential results can directly be applied to construct Bayesian estimates etc. in the generalized Ng–Kundu–Chan model. As a matter of fact, we get the same form of the estimators (like e.g., the posterior mean). But, as for the likelihood estimator, one has to take into account that the censoring plan is not fixed in

advance but observed during the life test. Respective results can be found in, for example, Kundu (2008) (Weibull distribution), Ali Mousa and Al-Sagheer (2006), Wu et al. (2006) (Rayleigh distribution), Ali Mousa and Jaheen (2002) (Burr distribution), and Ali Mousa (2001) (Pareto distribution). Further references are given in Balakrishnan and Cramer (2014); Balakrishnan and Cramer (2023).

3.1 | Exponential distribution

Let the adaptive progressively Type-II censored order statistics $X^{\mathscr{L}_D}_{1:m:n}, \ldots, X^{\mathscr{L}_D}_{m:m:n}$ be based on a two-parameter exponential distribution $\mathrm{Exp}(\mu, \vartheta)$ having cumulative distribution function

$$F_{\mu,\vartheta}(t) = 1 - \exp(-(t-\mu)/\vartheta), \quad t > \mu,$$

and density function $f_{\mu,\vartheta}(t) = 1/\vartheta \ \exp(-(t-\mu)/\vartheta)$, $t > \mu$, with distribution parameters $\mu \in \mathbb{R}$ and $\vartheta > 0$. If $\mu = 0$, then we write F_ϑ, f_ϑ for short.

Then, given observations $\mu < x_1 < \cdots < x_m$ with $D = d$, the likelihood function is given by

$$\mathcal{L}(\mu, \vartheta | x_1, \ldots, x_m) = c(\mathscr{L}_d)\vartheta^{-m} \exp\left(-\frac{1}{\vartheta}\left(\sum_{j=1}^{m}(L_{j,d}+1)(x_j - \mu)\right)\right)\mathbb{1}_{(\mu,\infty)}(x_1). \tag{3.4}$$

By similarity with standard progressive Type-II censoring (see, e.g., Balakrishnan and Cramer, 2014, Sec. 12.1), it is maximized by the maximum likelihood estimators (MLEs)

$$\widehat{\mu} = X^{\mathscr{R}}_{1:m:n} = X^{\mathscr{L}_D}_{1:m:n} \text{ and } \widehat{\vartheta}_2 = \frac{1}{m}\left(\sum_{j=2}^{m}(L_{j,D}+1)\,(X^{\mathscr{L}_D}_{j:m:n} - X^{\mathscr{L}_D}_{1:m:n})\right). \tag{3.5}$$

If μ is supposed known, one gets the maximum likelihood estimator for ϑ as

$$\widehat{\vartheta}_1 = \frac{1}{m}\left(\sum_{j=1}^{m}(L_{j,D}+1)\,X^{\mathscr{L}_D}_{j:m:n}\right) = \frac{1}{m}\mathrm{TTT}, \tag{3.6}$$

where $\mathrm{TTT} = \sum_{j=1}^{m}(L_{j,D}+1)\,X^{\mathscr{L}_D}_{j:m:n}$ denotes the total time on test in the generalized Ng–Kundu–Chan model.

Note that $D = \sum_{j=1}^{m}\mathbb{1}_{(-\infty,T]}(X^{\mathscr{R}}_{j:m:n})$ and the effectively applied censoring plan $\mathscr{L}_D$ in (3.6) and (3.5) are random so that the MLE is a weighted version of the failure times with random weights. Note that the form of the MLE equals that under non-adaptive progressive Type-II censoring when interpreting the effectively applied censoring plan $\mathscr{L}_D$ as non-random. However, as pointed out in Cramer and Iliopoulos (2010), the distribution of the MLE under adaptive progressive Type-II censoring equals that of the MLE under non-adaptive censoring. In order to illustrate this result, we derive the distribution of $\widehat{\vartheta}_i$, $i = 1, 2$, directly by studying the distribution of the (normalized) spacings under adaptive progressive Type-II censoring. In particular, the following representations of the scale MLEs in terms of the spacings

$$W_j^{\mathscr{L}_D} = X^{\mathscr{L}_D}_{j:m:n} - X^{\mathscr{L}_D}_{j-1:m:n}, \quad 1 \le j \le m, \quad X^{\mathscr{L}_D}_{0:m:n} = \mu \tag{3.7}$$

are obviously true:

$$\widehat{\vartheta}_1 = \frac{1}{m}\sum_{j=1}^{m}\gamma_{j,D}W_j^{\mathscr{L}_D}, \qquad \widehat{\vartheta}_2 = \frac{1}{m}\sum_{j=2}^{m}\gamma_{j,D}W_j^{\mathscr{L}_D}. \tag{3.8}$$

3.1.1 | Normalized spacings

To derive the distribution of the maximum likelihood estimators $\widehat{\vartheta}_i$, $i = 1, 2$, we rewrite (3.6) and (3.5) in terms of normalized spacings of the adaptive progressively Type-II censored order statistics $X^{\mathscr{L}_D}_{1:m:n}, \ldots, X^{\mathscr{L}_D}_{m:m:n}$. We show subsequently for exponentially distributed lifetimes that the normalized spacings are independent and identically exponentially distributed random variables. To prove these properties, we will apply the density transformation theorem as well as the law of total probability by conditioning on $\{D = d\}$, $d \in \{0, \ldots, m\}$. Thus, let

$$S_j^{\mathscr{L}_D} = \gamma_{j,D}W_j^{\mathscr{L}_D}, \quad 1 \le j \le m, \tag{3.9}$$

be the normalized spacing of the sample $X^{\mathscr{L}_D}_{1:m:n}, \ldots, X^{\mathscr{L}_D}_{m:m:n}$, where $X^{\mathscr{L}_D}_{0:m:n} = \mu$. In (3.9), the normalizing factors $\gamma_{1,D}, \ldots, \gamma_{m,D}$ are random turning to fixed values $\gamma_{1,d}, \ldots, \gamma_{m,d}$ as defined in (2.6), once the normalized spacings $S_1^{\mathscr{L}_D}, \ldots, S_m^{\mathscr{L}_D}$ are considered conditionally on the number of observed failures $D = d$ before time T. The joint probability mass function of the random normalizing factors is given in Balakrishnan et al. (2023, Sec. 13.2).

For convenience, define the random vectors $\mathbf{W}^{\mathscr{L}_D} = (W_1^{\mathscr{L}_D}, \ldots, W_m^{\mathscr{L}_D})'$, $\mathbf{S}^{\mathscr{L}_D} = (S_1^{\mathscr{L}_D}, \ldots, S_m^{\mathscr{L}_D})'$, and $\mathbf{X}^{\mathscr{L}_D} = (X^{\mathscr{L}_D}_{1:m:n}, \ldots, X^{\mathscr{L}_D}_{m:m:n})'$ (for $a \in \mathbb{R}^m$, a' denotes the transpose of a). Then, given $D = d$, we can write $\mathbf{W}^{\mathscr{L}_d} = A \cdot (\mathbf{X}^{\mathscr{L}_d} - \mu\boldsymbol{e}_1)$ and $\mathbf{S}^{\mathscr{L}_d} = M_d \cdot (\mathbf{X}^{\mathscr{L}_d} - \mu\boldsymbol{e}_1)$ with matrices A and M_d being defined by

$$A = \begin{pmatrix} 1 & 0 & \ldots & \ldots & 0 \\ -1 & 1 & \ldots & \ldots & 0 \\ 0 & -1 & 1 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & -1 & 1 \end{pmatrix},$$

$$M_d = \begin{pmatrix} \gamma_{1,d} & 0 & \ldots & \ldots & 0 \\ -\gamma_{2,d} & \gamma_{2,d} & \ldots & \ldots & 0 \\ 0 & -\gamma_{3,d} & \gamma_{3,d} & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & -\gamma_{m,d} & \gamma_{m,d} \end{pmatrix}, \tag{3.10}$$

respectively, and vector $\boldsymbol{e}_1' = (1, 0, \ldots, 0) \in \mathbb{R}^m$.

From (2.10) and (3.2) we get

$$f^{\mathbf{X}^{\mathscr{L}_D}|D=d}_{\mu,\vartheta}(\boldsymbol{t}) = \frac{1}{P_{\mu,\vartheta}(D=d)} f^{\mathscr{L}_d}_{1,\ldots,m:m:n;\mu,\vartheta}(\boldsymbol{t})\mathbb{1}_{(t_d,t_{d+1}]}(T)$$

$$= \frac{c(\mathscr{L}_d)}{P_{\mu,\vartheta}(D=d)}\vartheta^{-m}\exp\left(-\frac{1}{\vartheta}\sum_{j=1}^{m}(L_{j,d}+1)\,(t_j-\mu)\right)$$
$$\mathbb{1}_{(t_d,t_{d+1}]}(T), \quad \mu < t_1 < \cdots < t_m.$$

By density transformation and proceeding as in Balakrishnan and Cramer (2014, Sec. 2.3.1), it follows that the conditional joint density function of $\mathbf{S}^{\mathscr{L}_D} = (S_1^{\mathscr{L}_D}, \ldots, S_m^{\mathscr{L}_D})'$ conditionally on $D = d$ is given by

$$f_{\mu,\vartheta}^{\mathbf{S}^{\mathscr{L}_D}|D=d}(\boldsymbol{t}) = \frac{1}{P_{\mu,\vartheta}(D=d)}\vartheta^{-m}\exp\left(-\frac{1}{\vartheta}\sum_{j=1}^{m}t_j\right)\mathbb{1}_{(\nu_{d,d}(\boldsymbol{t}),\nu_{d,d+1}(\boldsymbol{t})]}(T)$$
$$= \frac{1}{P_{\mu,\vartheta}(D=d)}\left(\prod_{j=1}^{m}f_\vartheta(t_j)\right)\mathbb{1}_{(\nu_{d,d}(\boldsymbol{t}),\nu_{d,d+1}(\boldsymbol{t})]}(T), \quad 0 < t_1, \ldots, t_m, \tag{3.11}$$

where

$$\nu_{d,j}(\boldsymbol{t}) = \left(M_d^{-1}\boldsymbol{t}\right)_j = \sum_{i=1}^{j} t_i/\gamma_{i,d},\ j \in \{1, \ldots, m\},$$
$$\nu_{d,0}(\boldsymbol{t}) = 0, \nu_{d,m+1}(\boldsymbol{t}) = \infty, \tag{3.12}$$

for $d \in \{0, \ldots, m\}$. Notice that

$$\nu_{d,d}(\boldsymbol{t}) < \nu_{d,d+1}(\boldsymbol{t}), \quad d \in \{0, \ldots, m\},$$

as $\mathscr{R}$ is assumed to be an admissible censoring plan in $\mathfrak{C}_{m,n}$ and $t_1, \ldots, t_m > 0$. In detail, we have

(i) $\nu_{0,0}(\boldsymbol{t}) = 0 < \nu_{0,1}(\boldsymbol{t}) = t_1/\gamma_{1,0}$ with $\gamma_{1,0} = n - R_{\bullet 0} = n$ for $d = 0$;
(ii) $\nu_{m,m}(\boldsymbol{t}) = \sum_{i=1}^{m} t_i/\gamma_{i,m} < \nu_{m+1,m}(\boldsymbol{t}) = \infty$ with $\gamma_{i,m} = n - i + 1 - R_{\bullet i-1} > 0$ for any $i \in \{1, \ldots, m\}$ for $d = m$;
(iii) $0 < \nu_{d,d}(\boldsymbol{t}) < \nu_{d,d+1}(\boldsymbol{t}) = \nu_{d,d}(\boldsymbol{t}) + t_{d+1}/\gamma_{d+1,d} < \infty$ with $\gamma_{d+1,d} = n - d - R_{\bullet d} = n - d - \sum_{i=1}^{d} R_i > 0$.

In particular, the numbers $\nu_{d,d}(\boldsymbol{t})$, $\nu_{d,d+1}(\boldsymbol{t})$ depend for any $d \in \{0, \ldots, m\}$ only on the initial censoring plan $\mathscr{R} \in \mathfrak{C}_{m,n}$.

Using the law of total probability, we conclude that the (unconditional) joint density function of $\mathbf{S}^{\mathscr{L}_D} = (S_1^{\mathscr{L}_D}, \ldots, S_m^{\mathscr{L}_D})'$ is given by

$$f_{\mu,\vartheta}^{\mathbf{S}^{\mathscr{L}_D}}(\boldsymbol{t}) = \sum_{d=0}^{m} P_{\mu,\vartheta}(D=d) f_{\mu,\vartheta}^{\mathbf{S}^{\mathscr{L}_D}|D=d}(\boldsymbol{t})$$
$$= \prod_{j=1}^{m} f_\vartheta(t_j) \sum_{d=0}^{m} \mathbb{1}_{(\nu_{d,d}(\boldsymbol{t}),\nu_{d,d+1}(\boldsymbol{t})]}(T)$$
$$= \prod_{j=1}^{m} f_\vartheta(t_j), \quad 0 < t_1, \ldots, t_m. \tag{3.13}$$

Thus, even though the normalized spacings in (3.9) have random normalizing factors depending on the random number $D = \sum_{j=1}^{m} \mathbb{1}_{(-\infty,T]}(X_{j:m:n}^{\mathscr{R}})$ of failure times before threshold T, they are independent and identically Exp(ϑ)-distributed. Here, Exp(ϑ) = Exp(0, ϑ) denotes the one-parameter exponential distribution with mean ϑ. Summing up, we get the following result which extends the analogous one for standard progressively Type-II censored order statistics (see Balakrishnan and Cramer (2014), Theorem 2.3.2). Note that $\widehat{\vartheta}_1$ depends on μ via the first spacing $S_1^{\mathscr{L}_D} = n(X_{1:m:n} - \mu)$ so that the estimator can not be used in case of an unknown location parameter.

Corollary 3.1. *Given* Exp(μ, ϑ)*-distributed lifetimes with location parameter* μ *and scale parameter* ϑ*, the normalized spacings* $S_1^{\mathscr{L}_D}, \ldots, S_m^{\mathscr{L}_D}$ *as defined in* (3.9) *are independent and identically* Exp(ϑ)*-distributed random variables.*

(i) If the location parameter μ *is supposed to be known, then the MLE* $\widehat{\vartheta}_1$ *of* ϑ *has a scaled* χ^2*-distribution with* $2m$ *degrees of freedom, that is,*

$$2m\widehat{\vartheta}_1/\vartheta \sim \chi^2(2m).$$

(ii) If μ *is unknown, then we have for the MLE* $\widehat{\vartheta}_2$ *that*

$$2m\widehat{\vartheta}_2/\vartheta \sim \chi^2(2m-2).$$

In particular, expectations and variances are given by $\mathrm{E}_\vartheta(\widehat{\vartheta}_1) = \vartheta$, $\mathrm{Var}_\vartheta(\widehat{\vartheta}_1) = \vartheta^2/m$ *and* $\mathrm{E}_\vartheta(\widehat{\vartheta}_2) = \frac{m-1}{m}\vartheta$, $\mathrm{Var}_\vartheta(\widehat{\vartheta}_2) = \frac{m-1}{m^2}\vartheta^2$*, respectively, for* $\vartheta > 0$.

Proof. By analogy with standard progressive Type-II censoring (see Balakrishnan and Cramer, 2014, p. 271), the representation in terms of spacings in (3.8) shows that the MLEs can be written as a sum of the normalized spacings in (3.9). Since these spacings are independent and identically Exp(ϑ)-distributed by (3.13), the scale MLEs have the given χ^2-distributions. ∎

Remark 3.2. Based on the distributional results presented in Corollary 3.1, we discuss properties of the MLEs in the one- and two parameter exponential model (cf. (i) to (iv)). Furthermore, we present inferential results based on these estimators. In particular, this includes exact confidence intervals for the parameter ϑ (cf. (v)) as well as exact confidence regions for (μ, ϑ) in the two-parameter model.

(i) Note that $\widehat{\vartheta}_1$ is an unbiased estimator of ϑ and $\widehat{\vartheta}_2$ is an asymptotically unbiased estimator as m tends to infinity, respectively. Moreover, due to the invariance property of maximum likelihood estimators, the squared MLE $\widehat{\vartheta}_i^2/m$ is the MLE of the variance $\mathrm{Var}_\vartheta(\widehat{\vartheta}_i)$. Ng et al. (2009) pointed out that $\widehat{\vartheta}_i^2/m$ is an estimator of the asymptotic variance.

(ii) The MLEs $\widehat{\mu}$ and $\widehat{\vartheta}_2$ in the two-parameter model are independent and

$$\widehat{\mu}^* = \widehat{\mu} - \frac{m}{(m-1)n}\widehat{\vartheta}_2$$

is an unbiased estimator of μ since $\widehat{\mu} \sim \text{Exp}(\mu, \vartheta/n)$ (see Balakrishnan and Cramer, 2014, p. 271). Moreover, it follows that both $\widehat{\vartheta}_1$ and $(\widehat{\mu}, \widehat{\vartheta}_2)$ are complete sufficient statistics given the respective distributional assumption so that we get directly the uniformly minimum variance unbiased estimators of the parameters in the generalized Ng–Kundu–Chan model, too. Notice that the estimators are not linear estimators in general since the weights $\gamma_{j,D}$ are random variables! Thus, the best linear unbiased estimators can not be obtained from this result.

(iii) In the one-parameter model (assuming $\mu = 0$), a linear unbiased estimator of ϑ is given by $\widehat{\vartheta}_L = nX^{\mathscr{L}_D}_{1:m:n}$, that is, the first normalized spacing which, by definition, has a non-random weight. In order to discuss the problem whether other linear unbiased estimators exist, we consider properties of the spacings in Section 3.1.2.

(iv) Using a moment generating function approach, an explicit expression of the density function of $\widehat{\vartheta}_1$ has been derived in Lin and Huang (2012). The connection to the χ^2-distribution of the maximum likelihood estimator as given in Corollary 3.1, however, is hidden behind the alternating sum structure presented there.

(v) The above estimators in the generalized Ng–Kundu–Chan model are based only on the normalized spacings and, thus, their distribution equals that in case of standard progressive Type-II censoring. Therefore, all the inferential results established under progressive Type-II censoring can directly be applied in the generalized Ng–Kundu–Chan model as long as they are based on these estimators. This particularly applies to confidence intervals and regions, statistical tests, and so forth.

For instance, exact statistical intervals and regions can be directly constructed from the comments given in (ii) (see Chapter 17 in Balakrishnan and Cramer (2014)). For the parameter ϑ, we get confidence intervals based on quantiles of χ^2-distributions. Denoting by $\chi^2_\beta(k)$ the β-quantile of the $\chi^2(k)$-distribution with k degrees of freedom,

$$\left[\frac{2m\widehat{\vartheta}_1}{\chi^2_{1-\alpha/2}(2m)}, \frac{2m\widehat{\vartheta}_1}{\chi^2_{\alpha/2}(2m)}\right],$$
$$\left[\frac{(2m-2)\widehat{\vartheta}_2}{\chi^2_{1-\alpha/2}(2m-2)}, \frac{(2m-2)\widehat{\vartheta}_2}{\chi^2_{\alpha/2}(2m-2)}\right]$$

yield two-sided exact $1-\alpha$ confidence intervals for ϑ. By analogy, one sided confidence intervals are obtained. An exact confidence for μ can be directly taken from Balakrishnan and Cramer (2014, Corollary 17.1.1) Simultaneous confidence regions for (μ, ϑ) can be constructed using results of Wu (2010) (see also Balakrishnan and Cramer (2014, Theorem 17.1.3)). Of course, computational approaches like bootstrap can be used to construct confidence intervals.

All the goodness of fit tests prepared for the exponential distribution under progressive Type-II censoring can be used when they are based on the normalized spacings. A survey is provided in Döring and Cramer (2019). For other topics, we refer to Balakrishnan and Cramer (2014); Balakrishnan and Cramer (2023).

3.1.2 | Non-normalized spacings

For the non-normalized spacings $W^{\mathscr{L}_D}_1, \ldots, W^{\mathscr{L}_D}_m$ as defined in (3.7), the situation becomes different since the normalizing factors of the normalized spacings $S^{\mathscr{L}_D}_1, \ldots, S^{\mathscr{L}_D}_m$ given in (3.9) are random in general. Even though in Section 3.1.1 the normalized spacings were shown to be independent and identically exponentially distributed, we will prove that the non-normalized spacings are generally dependent with marginal distributions given by mixtures of exponential distributions. For this purpose, we will again apply the density transformation theorem and the law of total probability, followed by a rather technical proof of Theorem 3.3, which is moved to the appendix.

First, we get similarly to the case of normalized spacings by the density transformation formula that the conditional joint density function of $\mathbf{W}^{\mathscr{L}_D} = (W^{\mathscr{L}_D}_1, \ldots, W^{\mathscr{L}_D}_m)'$ conditionally on $D = d$ is given by

$$\begin{aligned} f^{\mathbf{W}^{\mathscr{L}_D}|D=d}_{\mu,\vartheta}(\boldsymbol{t}) &= \frac{c(\mathscr{L}_d)}{P_{\mu,\vartheta}(D=d)}\vartheta^{-m} \\ &\quad \exp\left(-\frac{1}{\vartheta}\sum_{j=1}^{m}\gamma_{j,d}t_j\right)\mathbb{1}_{(t_{\bullet_d}, t_{\bullet_{d+1}}]}(T) \\ &= \frac{1}{P_{\mu,\vartheta}(D=d)}\prod_{j=1}^{m}\left(\gamma_{j,d}f_\vartheta(\gamma_{j,d}t_j)\right)\mathbb{1}_{(t_{\bullet_d}, t_{\bullet_{d+1}}]}(T), \\ &\quad 0 < t_1, \ldots, t_m, \end{aligned} \tag{3.14}$$

for $d \in \{0, \dots, m\}$, where by similarity with (3.12)

$$\left(A^{-1}\boldsymbol{t}\right)_j = \sum_{i=1}^{j} t_i = t_{\bullet j}, \quad , j \in \{1, \dots, m\},$$
$$t_{\bullet 0} = 0, t_{\bullet m+1} = \infty, \tag{3.15}$$

so that $0 = t_{\bullet 0} \leq t_{\bullet 1} \leq \cdots \leq t_{\bullet m} < t_{\bullet m+1} = \infty$. In particular,

$$f_{\mu,\vartheta}^{\mathbf{W}^{\mathscr{L}_D}|D=d}(\boldsymbol{t}) = \frac{1}{P_{\mu,\vartheta}(D=d)} \prod_{j=1}^{m} \left(\gamma_{j,d} f_\vartheta(\gamma_{j,d} t_j)\right),$$
$$0 < t_1, \dots, t_m, \quad t_{\bullet d} < T \leq t_{\bullet d+1}. \tag{3.16}$$

Notice that in (3.14) compared to (3.11) the indicator does not depend on the censoring plans anymore but the exponential density function does instead, which fundamentally changes the situation regarding the spacings' unconditional joint and marginal density functions.

Applying again the law of total probability, by (3.14) the (unconditional) joint density function of $\mathbf{W}^{\mathscr{L}_D} = (W_1^{\mathscr{L}_D}, \dots, W_m^{\mathscr{L}_D})'$ is given by

$$\begin{aligned} f_{\mu,\vartheta}^{\mathbf{W}^{\mathscr{L}_D}}(\boldsymbol{t}) &= \sum_{d=0}^{m} P_{\mu,\vartheta}(D=d) f_\vartheta^{\mathbf{W}^{\mathscr{L}_D}|D=d}(\boldsymbol{t}) \\ &= \sum_{d=0}^{m} \mathbb{1}_{(t_{\bullet d}, t_{\bullet d+1}]}(T) \prod_{j=1}^{m} \left(\gamma_{j,d} f_\vartheta(\gamma_{j,d} t_j)\right), \\ &\qquad 0 < t_1, \dots, t_m. \end{aligned} \tag{3.17}$$

This illustrates that the joint density function of the the non-normalized spacings is connected to densities of independent exponential random variables. However, the spacings are obviously not independent except when $\gamma_{j,d} = \gamma_j$ for all $d \in \{0, \dots, m\}$ and $j \in \{1, \dots, m\}$. This holds only when $\mathscr{R} = \mathscr{S}$, that is, the censoring plan is not affected by the threshold T.

The marginal cumulative distribution functions and density functions are given in Theorem 3.3 where $\sum_{d=0}^{-1} \cdots = 0$ is defined. The rather technical proofs are presented in the appendix.

Theorem 3.3. *Let $F_{d:m:n}$ and $f_{d:m:n}$ be the marginal cumulative distribution function and density function of the d-th progressively type-II censored order statistics $Z_{d:m:n}^{\mathscr{R}}$ based on the censoring plan $\mathscr{R}$ and a standard exponential distribution.*

For $k \in \{1, \dots, m\}$, the marginal density function and cumulative distribution function of the k-th spacing are given by

$$\begin{aligned} f_{\mu,\vartheta}^{W_k^{\mathscr{L}_D}}(t) &= \sum_{d=0}^{k-2} (F_{d:m:n}((T-\mu)/\vartheta) - F_{d+1:m:n}((T-\mu)/\vartheta)) \\ &\quad \gamma_{k,d} f_\vartheta(\gamma_{k,d} t) + F_{k-1:m:n}((T-\mu)/\vartheta) \gamma_k f_\vartheta(\gamma_k t), \quad t \geq 0, \\ F_{\mu,\vartheta}^{W_k^{\mathscr{L}_D}}(t) &= \sum_{d=0}^{k-2} (F_{d:m:n}((T-\mu)/\vartheta) - F_{d+1:m:n}((T-\mu)/\vartheta)) \\ &\quad F_\vartheta(\gamma_{k,d} t) + F_{k-1:m:n}((T-\mu)/\vartheta) F_\vartheta(\gamma_k t), \quad t \geq 0, \end{aligned}$$

that is, the distribution of $W_k^{\mathscr{L}_D}$ is a mixture of possibly different exponential distributions. Furthermore, the ℓ-th moment of $W_k^{\mathscr{L}_D}$ is given by

$$\begin{aligned} &E(W_k^{\mathscr{L}_D})^\ell \\ &= \vartheta^\ell \left(\sum_{d=0}^{k-2} \frac{F_{d:m:n}((T-\mu)/\vartheta) - F_{d+1:m:n}((T-\mu)/\vartheta)}{\gamma_{k,d}^\ell} \right. \\ &\quad \left. + \frac{F_{k-1:m:n}((T-\mu)/\vartheta)}{\gamma_k^\ell} \right). \end{aligned}$$

Remark 3.4.

(i) Plots of the densities of the spacings $W_k^{\mathscr{L}_D}$, $k = 1, \dots, 6$, in the generalized Ng–Kundu–Chan model are presented in Figure 4 for the censoring plans $\mathscr{R} = (1, 2, 3, 5, 1, 4)$, $\mathscr{S} = (6, 3, 0, 1, 4, 2)$ as considered in Example 2.4. The mean ϑ of the exponential distribution and the time threshold T are chosen as $\vartheta = 1$ and $T = .1$, respectively. The densities of the first spacing, which by definition is the density of the first failure time itself, are obviously identical. However, for the other spacings, the difference between the (strictly decreasing) density function of the spacing $W_k^{\mathscr{L}_D}$, $2 \leq k \leq 6$, and the density of the exponential distribution with parameter $1/\gamma_k(\mathscr{R})$ resp. $1/\gamma_k(\mathscr{S})$ becomes visible, illustrating that the marginal distribution of the respective spacing is indeed not an exponential density. Differences of the densities with respect to the exponential densities are shown in Figure 5 for easier comparison.

Note that the densities of the spacings are strictly decreasing on $(0, \infty)$. Furthermore, it follows that they are log-convex (see An, 1998, Proposition 3) and completely monotone as a mixture of exponentials (see Bernstein, 1929, Feller 1971, p. 439). Since the density function of a spacing is decreasing, we conclude from Bagnoli and Bergstrom (2005, Corollary 1) that the cumulative distribution function of a spacing is log-concave.

(ii) Furthermore, we get stochastic ordering of the spacings if $\mathscr{R}$ and $\mathscr{S}$ are regularly paired as introduced in Remark 2.2. According to (2.6), $\gamma_{j,d} = \gamma_j$, $1 \leq j \leq d+1$, and

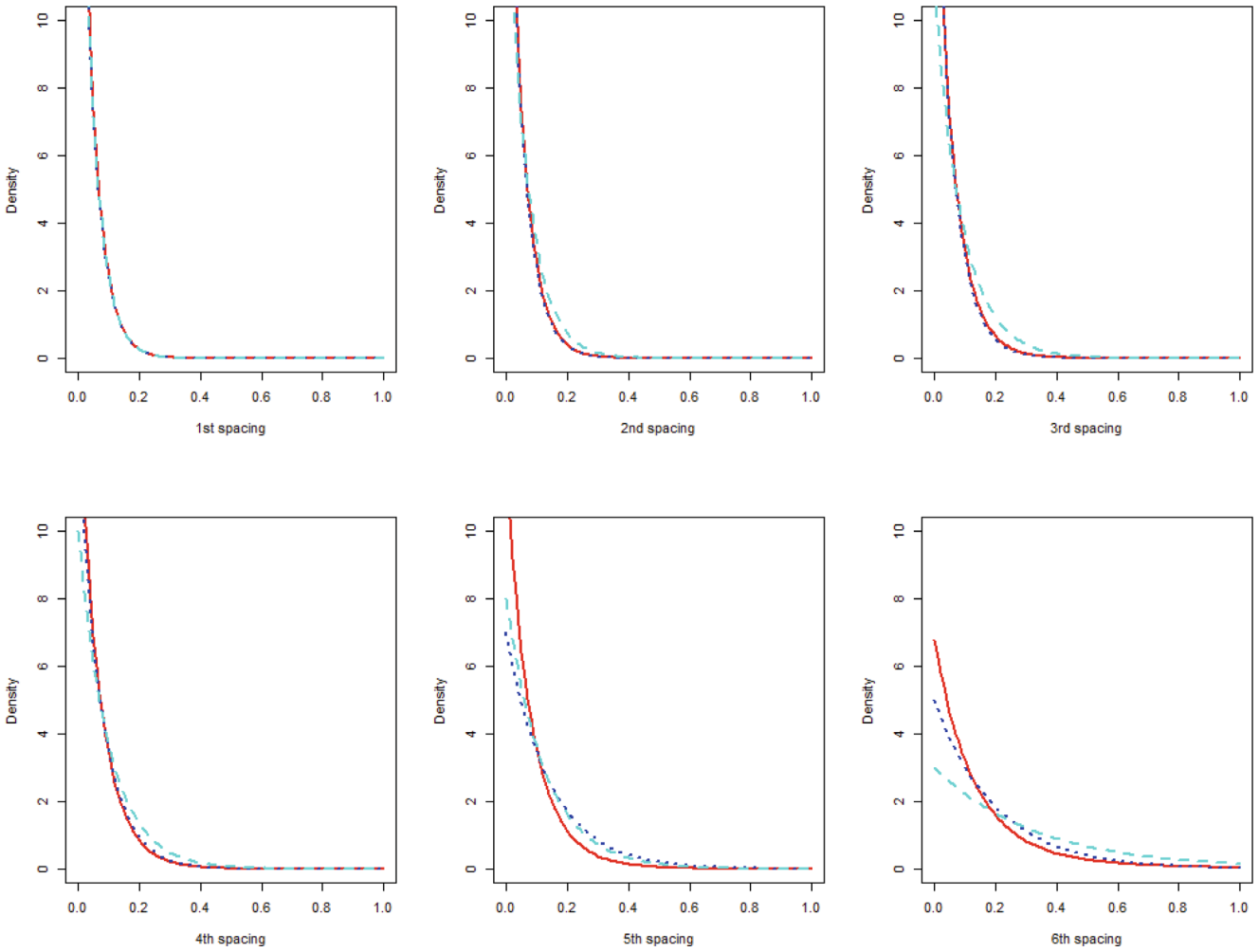

FIGURE 4 Plots of the densities of the (non-normalized) spacings $W_1^{\mathscr{D}_D}, \ldots, W_6^{\mathscr{D}_D}$, in the generalized Ng–Kundu–Chan model for the censoring plans $\mathscr{R} = (1, 2, 3, 5, 1, 4)$, $\mathscr{S} = (6, 3, 0, 1, 4, 2)$, threshold $T = .1$ and distribution parameter $\vartheta = 1$ (solid lines). For comparison the exponential densities with parameters $1/\gamma_k(\mathscr{R})$ (dotted lines) and $1/\gamma_k(\mathscr{S})$ (dashed lines), respectively, are also depicted.

$$\gamma_{j,d} = n - j + 1 - R_{\bullet d} - \sum_{i=d+1}^{j-1} S_i = \gamma_j + \sum_{i=d+1}^{j-1} (R_i - S_i), \quad d+2 \le j \le m.$$

Therefore, for all $d \in \{0, \ldots, m-2\}$ and $j \in \{d+2, \ldots, m\}$, we get

$$\gamma_{j,d} \ge \gamma_j \iff \sum_{i=d+1}^{j-1} (R_i - S_i) \ge 0.$$

This condition is equivalent to $S_i \le R_i$, $1 \le i \le m-1$. From Theorem 3.3, we get for $t \ge 0$ in this case

$$\begin{aligned} F_{\mu,\vartheta}^{W_k^{\mathscr{D}_D}}(t) &\ge \sum_{d=0}^{k-2} (F_{d:m:n}((T-\mu)/\vartheta) \\ &\quad - F_{d+1:m:n}((T-\mu)/\vartheta) F_\vartheta(\gamma_k t)) \\ &\quad + F_{k-1:m:n}((T-\mu)/\vartheta) F_\vartheta(\gamma_k t) \\ &= F_\vartheta(\gamma_k t). \end{aligned}$$

Hence, the k-th spacing $W_k^{\mathscr{D}_D}$ is stochastically larger than an $\mathrm{Exp}(\vartheta/\gamma_k)$-distribution, $k \in \{1, \ldots, m\}$. Of course, for $k = 1$, it equals an $\mathrm{Exp}(\vartheta/n)$-distribution.

(iii) Some plots of the means (i.e., for $\ell = 1$) of the spacings (divided by ϑ) as a function of ϑ are presented in Figure 6, illustrating that the means are not linear in the parameter ϑ.

Remark 3.5. In order find a computable expression for the density function, we use the following well-known representations which hold in case of the standard exponential distribution (see Balakrishnan and Cramer (2014), eqs. 2.19 and 2.20; $1 \le r \le m$):

$$\begin{aligned} F_{r:m:n}(t) &= 1 - \left(\prod_{i=1}^{r} \gamma_i\right) \sum_{j=1}^{r} a_{j,r} \frac{e^{-\gamma_j t}}{\gamma_j}, \\ f_{r:m:n}(t) &= \left(\prod_{i=1}^{r} \gamma_i\right) \sum_{j=1}^{r} a_{j,r} e^{-\gamma_j t}, \quad t > 0, \end{aligned} \tag{3.18}$$

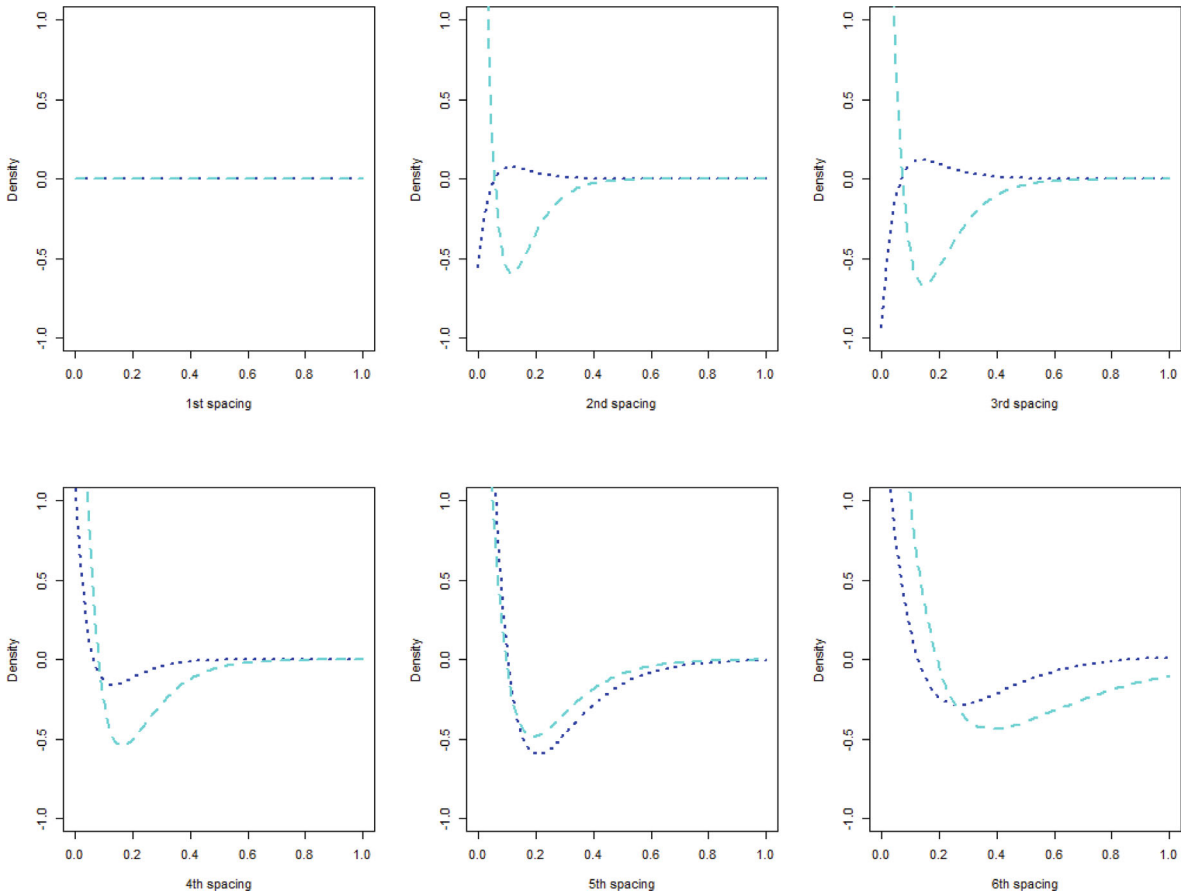

FIGURE 5 Plots of the differences of the densities of the spacings $W_k^{\mathscr{L}_D}$, $k = 1, \ldots, 6$, in the generalized Ng–Kundu–Chan model for the censoring plans $\mathscr{R} = (1, 2, 3, 5, 1, 4)$, $\mathscr{S} = (6, 3, 0, 1, 4, 2)$, threshold $T = .1$ and distribution parameter $\vartheta = 1$ with respect to the exponential densities with parameters $1/\gamma_k(\mathscr{R})$ (dotted lines) and $1/\gamma_k(\mathscr{S})$ (dashed lines), respectively.

with $a_{j,r} = \prod_{i=1, i\neq j}^{r} (\gamma_i - \gamma_j)^{-1}$, $j = 1, \ldots, r$. Moreover, as pointed out in Balakrishnan and Cramer (2014), for $d \in \{1, \ldots, m-1\}$, the following identity holds in case of standard exponentially distributed lifetimes

$$F_{d:m:n}(x) - F_{d+1:m:n}(x) = \frac{1}{\gamma_{d+1}} f_{d+1:m:n}(x), \quad x \geq 0.$$

3.1.3 | Application to linear inference

In the next theorem, we present results on unbiased estimators of the parameter ϑ in the generalized Ng–Kundu–Chan model assuming Exp(ϑ)-distributed lifetimes. Notice that $nW_1^{\mathscr{L}_D} = nX_{1:m:n}^{\mathscr{R}}$ is always a linear unbiased estimator of ϑ so that the existence of a linear unbiased estimator is guaranteed.

Theorem 3.6. *Given $k \in \{1, \ldots, m\}$ and censoring plans $\mathscr{R}$ and $\mathscr{S}$, the expectation of $W_k^{\mathscr{L}_D}$ is a linear function in the parameter $\vartheta > 0$ if an only if $\gamma_j(\mathscr{S}) = \gamma_j(\mathscr{R})$, $j \in \{1, \ldots, k\}$. In this case, $EW_k^{\mathscr{L}_D} = \frac{\vartheta}{\gamma_k}$, $\vartheta > 0$.*

In general, the expectation of a linear estimator $\widehat{L} = \boldsymbol{a}'\mathbf{W}^{\mathscr{L}_D}$ is linear in the parameter $\vartheta > 0$ only if the equation

$$\Lambda \boldsymbol{a}_* = 0 \tag{3.19}$$

holds where $\boldsymbol{a}_* = (a_2, \ldots, a_m)'$ and

$$\Lambda = \begin{pmatrix} \frac{1}{\gamma_2} - \frac{1}{\gamma_{2,0}} & \frac{1}{\gamma_3} - \frac{1}{\gamma_{3,0}} & & \frac{1}{\gamma_k} - \frac{1}{\gamma_{k,0}} & & \cdots & \frac{1}{\gamma_m} - \frac{1}{\gamma_{m,0}} \\ 0 & \frac{1}{\gamma_3} - \frac{1}{\gamma_{3,1}} & & \frac{1}{\gamma_k} - \frac{1}{\gamma_{k,1}} & & \cdots & \frac{1}{\gamma_m} - \frac{1}{\gamma_{m,1}} \\ \vdots & & & & & & \vdots \\ 0 & 0 & \ldots\ 0 & \frac{1}{\gamma_k} - \frac{1}{\gamma_{k,k-2}} & & & \frac{1}{\gamma_m} - \frac{1}{\gamma_{m,k-2}} \\ \vdots & & & & & \ddots & \vdots \\ \vdots & & & & 0 & \frac{1}{\gamma_{m-1}} - \frac{1}{\gamma_{m-1,m-3}} & \frac{1}{\gamma_m} - \frac{1}{\gamma_{m,m-3}} \\ 0 & 0 & \ldots\ \ldots\ \ldots\ \ldots & & & 0 & \frac{1}{\gamma_m} - \frac{1}{\gamma_{m,m-2}} \end{pmatrix}. \tag{3.20}$$

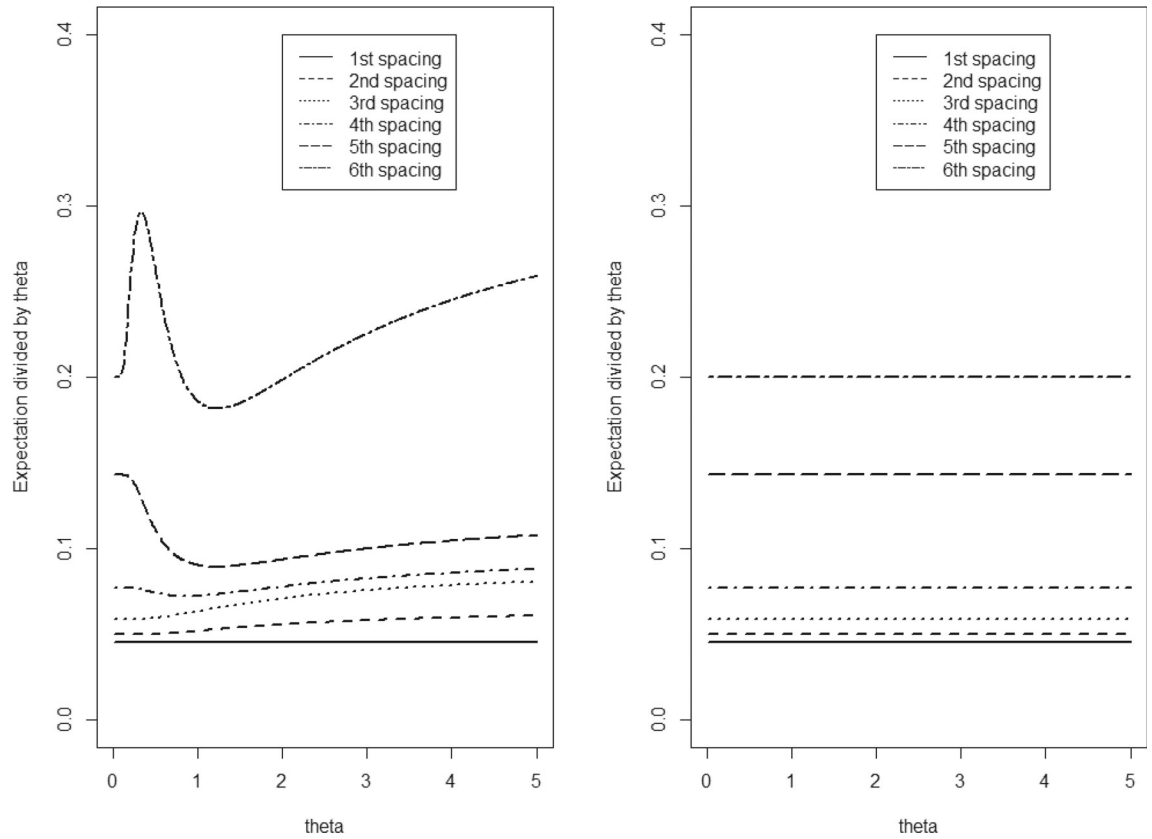

FIGURE 6 Plots of the means of the (non-normalized) spacings $W_1^{\mathscr{L}_D}, \ldots, W_6^{\mathscr{L}_D}$, divided by ϑ, in the generalized Ng–Kundu–Chan model with threshold $T = .1$ as a function of ϑ for (A) on the left the censoring plans $\mathscr{R} = (1, 2, 3, 5, 1, 4)$, $\mathscr{S} = (6, 3, 0, 1, 4, 2)$ and (B) on the right for equal censoring plans $\mathscr{R} = \mathscr{S} = (1, 2, 3, 5, 1, 4)$.

If $\boldsymbol{a} \in \mathbb{R}^m$ *satisfies* (3.19), *then the linear estimator* $\widehat{L}$ *is unbiased for* ϑ *if*

$$a_1 = \gamma_1 - \gamma_1 \sum_{j=2}^{m} \frac{a_j}{\gamma_{j,0}}, \tag{3.21}$$

where $\gamma_1 = \gamma_1(\mathscr{R}) = \gamma_1(\mathscr{S}) = n$ *and* $\gamma_{j,0} = \gamma_j(\mathscr{S})$, $j = 2, \ldots, m$.

Proof. From Theorem 3.3, we find that the expectation of a linear estimator of progressively Type-II censored order statistics in the generalized Ng–Kundu–Chan model, that is, of

$$\widehat{L} = \sum_{j=1}^{m} b_j X_{j:m:n}^{\mathscr{L}_D} = \boldsymbol{b}' \mathbf{X}^{\mathscr{L}_D} = \boldsymbol{b}' A^{-1} \mathbf{W}^{\mathscr{L}_D} = \boldsymbol{a}' \mathbf{W}^{\mathscr{L}_D} \tag{3.22}$$

with A as in (3.10) and $\boldsymbol{a}' = \boldsymbol{b}' A^{-1}$, is given by $E\widehat{L} = \boldsymbol{a}' E\mathbf{W}^{\mathscr{L}_D}$. Therefore, it is sufficient to study the linear estimators in terms of $\mathbf{W}^{\mathscr{L}_D}$. Since $W_1^{\mathscr{L}_D} = X_{1:m:n}^{\mathscr{R}}$, the case $k = 1$ can be excluded from the discussion so that we can assume $k \geq 2$ subsequently. First, we consider the expectation of a single spacing $W_k^{\mathscr{L}_D}$, $k \in \{2, \ldots, m\}$.

Clearly, we get from Theorem 3.3

$$EW_k^{\mathscr{L}_D} = \vartheta \left(\sum_{d=0}^{k-2} \frac{F_{d:m:n}(T/\vartheta) - F_{d+1:m:n}(T/\vartheta)}{\gamma_{k,d}} + \frac{F_{k-1:m:n}(T/\vartheta)}{\gamma_k} \right) = \vartheta \, c_k(\vartheta), \quad \text{say.}$$

Therefore, $EW_k^{\mathscr{L}_D}$ is a linear function of ϑ if and only if the term in the brackets denoted by $c_k(\vartheta)$ is constant w.r.t. ϑ, that is,

$$\begin{aligned} c_k(\vartheta) \equiv c_k &= \sum_{d=0}^{k-2} \frac{F_{d:m:n}(T/\vartheta) - F_{d+1:m:n}(T/\vartheta)}{\gamma_{k,d}} + \frac{F_{k-1:m:n}(T/\vartheta)}{\gamma_k} \qquad (3.23) \\ &= \frac{1}{\gamma_{k,0}} + \sum_{d=1}^{k-2} F_{d:m:n}(T/\vartheta) \left(\frac{1}{\gamma_{k,d}} - \frac{1}{\gamma_{k,d-1}} \right) \\ &\quad + F_{k-1:m:n}(T/\vartheta) \left(\frac{1}{\gamma_k} - \frac{1}{\gamma_{k,k-2}} \right), \qquad (3.23) \end{aligned}$$

where $\sum_{d=1}^{k-2} \cdots = 0$ when $k = 2$. Notice that $F_{1:m:n}, \ldots, F_{k-1:m:n}$ are linearly independent functions since $\gamma_1 > \cdots > \gamma_m$ and

$$\operatorname{span}\{1, g_{\gamma_1}, \ldots, g_{\gamma_{k-1}}\} = \operatorname{span}\{1, F_{1:m:n}, \ldots, F_{k-1:m:n}\}$$

with $g_\ell(t) = e^{-\ell t}$, $t \geq 0$; see, e.g., (3.18). Thus, $c_k(\vartheta)$ is constant w.r.t. ϑ if and only if the coefficients in (3.23) are zero for $d \geq 1$, that is,

$$\begin{aligned} &\frac{1}{\gamma_{k,d}} - \frac{1}{\gamma_{k,d-1}} = 0, \quad d = 1, \dots, k-2, (\text{when } k \geq 3) \\ &\frac{1}{\gamma_k} - \frac{1}{\gamma_{k,k-2}} = 0. \end{aligned} \tag{3.25}$$

This obviously holds iff

$$\gamma_{k,0} = \cdots = \gamma_{k,k-2} = \gamma_k.$$

Furthermore, by construction of the censoring plans, we know that $\gamma_k(\mathcal{S}) = \gamma_{k,0}$ and $\gamma_k = \gamma_k(\mathcal{R})$ and we conclude from Lemma 2.3 that $\gamma_j(\mathcal{S}) = \gamma_j(\mathcal{R}), j \in \{1, \dots, k\}$. Therefore, we obtain that

$$EW_k^{\mathscr{L}_D} = c_k\vartheta \quad \forall\, \vartheta > 0$$

$$\Leftrightarrow \gamma_j(\mathcal{S}) = \gamma_j(\mathcal{R}), j \in \{1, \dots, k\}. \tag{3.26}$$

In this case, $c_k = 1/\gamma_k$ so that $\gamma_k W_k^{\mathscr{L}_D}$ is an unbiased estimator of ϑ. In other words, due to $n = \gamma_1(\mathcal{S}) = \gamma_1(\mathcal{R})$, condition (3.26) is generally satisfied for $W_1^{\mathscr{L}_D} = X_{1:m:n}^{\mathscr{L}_D}$. For $W_2^{\mathscr{L}_D}$, we need that $R_1 = S_1$. For $W_3^{\mathscr{L}_D}$, the condition can be written as $R_1 + R_2 = S_1 + S_2$, and so forth.

For a general linear estimator, the situation is somewhat more involved compared to the situation assumed in (3.25), since eliminations of non-linear terms may be possible. For a linear estimator $\widehat{L}$ as in (3.22), we get the expectation

$$E\widehat{L} = \boldsymbol{a}' E\mathbf{W}^{\mathscr{L}_D} = \vartheta \boldsymbol{a}' \Delta \eta(\vartheta)$$

with the lower triangular matrix $\Delta \in \mathbb{R}^{m\times m}$, that is,

$$\Delta = \begin{pmatrix} \frac{1}{\gamma_1} & 0 & 0 & \dots & & & \dots & 0 \\ \frac{1}{\gamma_{2,0}} & \frac{1}{\gamma_2} - \frac{1}{\gamma_{2,0}} & 0 & \dots & & & \dots & 0 \\ \frac{1}{\gamma_{3,0}} & \frac{1}{\gamma_{3,1}} - \frac{1}{\gamma_{3,0}} & \frac{1}{\gamma_3} - \frac{1}{\gamma_{3,1}} & 0 & \dots & & \dots & 0 \\ \vdots & & & & & & & \vdots \\ \frac{1}{\gamma_{k,0}} & \frac{1}{\gamma_{k,1}} - \frac{1}{\gamma_{k,0}} & \frac{1}{\gamma_{k,2}} - \frac{1}{\gamma_{k,1}} & \dots & \frac{1}{\gamma_{k,k-2}} - \frac{1}{\gamma_{k,k-3}} & \frac{1}{\gamma_k} - \frac{1}{\gamma_{k,k-2}} & 0 & 0 \\ \vdots & & & & & & \ddots & \vdots \\ \frac{1}{\gamma_{m,0}} & \frac{1}{\gamma_{m,1}} - \frac{1}{\gamma_{m,0}} & \frac{1}{\gamma_{m,2}} - \frac{1}{\gamma_{m,1}} & \dots & \dots & \dots & \dots & \frac{1}{\gamma_{m,m-2}} - \frac{1}{\gamma_{m,m-3}} \quad \frac{1}{\gamma_m} - \frac{1}{\gamma_{m,m-2}} \end{pmatrix}, \tag{3.27}$$

and $\eta(\vartheta)' = (1, F_{1:m:n}(T/\vartheta), \dots, F_{m-1:m:n}(T/\vartheta))$. From the linear independence of $F_{1:m:n}, \dots, F_{m-1:m:n}$, we get the linear equations

$$\frac{a_1}{\gamma_1} + \sum_{j=2}^{m} \frac{a_j}{\gamma_{j,0}} = 1, \quad \boldsymbol{a}_*' \Delta_* = 0 \tag{3.28}$$

which have to be satisfied in order to get a linear unbiased estimator. Here, Δ_* is constructed from Δ by deleting the first column and first row; and it is $\boldsymbol{a}_*' = (a_2, \dots, a_m)$. Therefore, a linear unbiased estimator of ϑ must satisfy the condition in (3.28). Clearly, this implies (3.21). The rank of Δ_* depends on the censoring plans $\mathcal{R}$ and $\mathcal{S}$. For instance, for $j \in \{1, \dots, m-1\}$, the j-th column of Δ_* is zero iff $R_j = S_j$. To be more precisely, we have

$$\text{rank } \Delta_* = m - 1 - \sum_{j=2}^{m} \mathbb{1}_{\{\gamma_j\}}(\gamma_{j,j-2}). \tag{3.29}$$

In order to proceed, we transpose Δ_*. Then, $\boldsymbol{a}_*$ must be an element of the kernel of Δ_*', which has dimension $\sum_{j=2}^{m} \mathbb{1}_{\{\gamma_j\}}(\gamma_{j,j-2})$. That is, we get the equation

$$\Delta_*' \boldsymbol{a}_* = \begin{pmatrix} \frac{1}{\gamma_2} - \frac{1}{\gamma_{2,0}} & \frac{1}{\gamma_{3,1}} - \frac{1}{\gamma_{3,0}} & & \frac{1}{\gamma_{k,1}} - \frac{1}{\gamma_{k,0}} & \dots & \frac{1}{\gamma_{m,1}} - \frac{1}{\gamma_{m,0}} \\ 0 & \frac{1}{\gamma_3} - \frac{1}{\gamma_{3,1}} & & \frac{1}{\gamma_{k,2}} - \frac{1}{\gamma_{k,1}} & \dots & \frac{1}{\gamma_{m,2}} - \frac{1}{\gamma_{m,1}} \\ \vdots & & & & & \vdots \\ 0 & 0 & \dots\ 0 & \frac{1}{\gamma_k} - \frac{1}{\gamma_{k,k-2}} & & \frac{1}{\gamma_{m,k-1}} - \frac{1}{\gamma_{m,k-2}} \\ \vdots & & & & \ddots & \vdots \\ \vdots & & & 0 & \frac{1}{\gamma_{m-1}} - \frac{1}{\gamma_{m-1,m-3}} & \frac{1}{\gamma_{m,m-2}} - \frac{1}{\gamma_{m,m-3}} \\ 0 & 0 & \dots\ \dots\ \dots\ \dots & 0 & & \frac{1}{\gamma_m} - \frac{1}{\gamma_{m,m-2}} \end{pmatrix} \begin{pmatrix} a_2 \\ \vdots \\ a_m \end{pmatrix} = 0.$$

Now, multiplying the equation $\Delta_*' \boldsymbol{a}_* = 0$ from the left by the regular matrix $B = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 0 & 1 & \dots & 1 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & 1 \end{pmatrix}$, we get $\Lambda = B\Delta_*'$ with Λ as in (3.20) and the equivalent equation given in (3.19). ∎

Remark 3.7.

(i) Theorem 3.6 shows that, except for $nW_1^{\mathscr{L}_D}$, the single spacings usually do not lead to an unbiased estimator of ϑ. In fact, this only holds for the k-th spacing $W_k^{\mathscr{L}_D}$ when the censoring plans $\mathscr{R}$ and $\mathscr{S}$ are identical up to censoring number $k-1$.

For a general linear estimator, this condition may not hold as there may be cancellation effects. Therefore, the condition for a linear unbiased estimator becomes somewhat more complicated, and the respective single spacings may be involved even though they may not be unbiased themselves. An example is provided in Example 3.8.

(ii) Note that the condition in Equation (3.21) can be written as

$$\sum_{j=1}^{m} \frac{a_j}{\gamma_j(\mathscr{S})} = 1.$$

Moreover, due to condition (3.21), we conclude that an unbiased estimator of ϑ exists provided that a linear estimator with mean linear in ϑ exists.

(iii) Clearly, we have in (3.19) either $a_m = 0$ or $\gamma_m = \gamma_{m,m-2}$. If $a_m = 0$, then we can delete the last column of the matrix and get a reduced equation involving only $a_2, \dots, a_{m-1}$. If $a_m \neq 0$, then $\gamma_m = \gamma_{m,m-2}$ must hold which, by Lemma 2.3, means that $R_{m-1} = S_{m-1}$.

(iv) Suppose that the censoring plans are regular paired according to Remark 2.2. Then, for $d \in \{0, \dots, k-2\}$ and $k \in \{2, \dots, m\}$, we find the identity

$$\begin{aligned} \frac{1}{\gamma_k} - \frac{1}{\gamma_{k,d}} &= \frac{1}{\gamma_k \gamma_{k,d}} (\gamma_{k,d} - \gamma_k) \\ &= \frac{1}{\gamma_k \gamma_{k,d}} \sum_{i=d+1}^{k-1} (R_i - S_i) \\ &= \frac{\delta_{k,d}}{\gamma_k \gamma_{k,d}}, \quad \text{say,} \end{aligned}$$

with $\delta_{k,d} = \sum_{i=d+1}^{k-1}(R_i - S_i)$ measuring the differences between the censoring plans $\mathscr{R}$ and $\mathscr{S}$ once the threshold has been exceeded. Therefore, we have the representation

$$\Lambda \boldsymbol{a}_* = \begin{pmatrix} \frac{\delta_{2,0}}{\gamma_2\gamma_{2,0}} & \frac{\delta_{3,0}}{\gamma_3\gamma_{3,0}} & & \frac{\delta_{k,0}}{\gamma_k\gamma_{k,0}} & \dots & \frac{\delta_{m,0}}{\gamma_m\gamma_{m,0}} \\ 0 & \frac{\delta_{3,1}}{\gamma_3\gamma_{3,1}} & & \frac{\delta_{k,1}}{\gamma_k\gamma_{k,1}} & \dots & \frac{\delta_{m,1}}{\gamma_m\gamma_{m,1}} \\ \vdots & & & & & \vdots \\ 0 & 0 & \dots\ 0 & \frac{\delta_{k,k-2}}{\gamma_k\gamma_{k,k-2}} & & \frac{\delta_{m,k-2}}{\gamma_m\gamma_{m,k-2}} \\ \vdots & & & & \ddots & \vdots \\ \vdots & & & 0 & \frac{\delta_{m-1,m-3}}{\gamma_{m-1}\gamma_{m-1,m-3}} & \frac{\delta_{m,m-3}}{\gamma_m\gamma_{m,m-3}} \\ 0 & 0 & \dots\ \dots\ \dots & \dots & 0 & \frac{\delta_{m,m-2}}{\gamma_m\gamma_{m,m-2}} \end{pmatrix} \boldsymbol{a}_* = 0,$$

where the matrix components are functions of the differences $\sum_{i=d+1}^{k-1}(R_i - S_i)$ and the diagonal elements $\Lambda_{i,i}$ are equal to a positive factor times $R_i - S_i$, $i \in \{1, \dots, m-1\}$. Hence, these differences determine the rank of Λ, that is,

$$\text{rank } \Lambda = m - 1 - \sum_{j=1}^{m-1} \mathbb{1}_{\{S_j\}}(R_j). \quad (3.30)$$

This leads to a simplified version of (3.29) and the dimension of the kernel is given by $\sum_{j=1}^{m-1} \mathbb{1}_{\{S_j\}}(R_j)$.

Furthermore, the above representation can be used to calculate coefficients $\boldsymbol{a}$ leading to an unbiased estimator. For illustration, we sketch the first steps. If $a_m \neq 0$ holds for the linear estimator then, as mentioned above, we must have $R_{m-1} = S_{m-1}$ or $\delta_{m,m-2} = 0$ in order to get unbiasedness. Then, row $m-2$ of Λ yields the equation

$$\begin{aligned} &\frac{\delta_{m-1,m-3}}{\gamma_{m-1}\gamma_{m-1,m-3}} a_{m-1} + \frac{\delta_{m,m-3}}{\gamma_m\gamma_{m,m-3}} a_m = 0 \\ \iff &\frac{R_{m-2} - S_{m-2}}{\gamma_{m-1}\gamma_{m-1,m-3}} a_{m-1} \\ &+ \frac{R_{m-2} - S_{m-2}}{\gamma_m\gamma_{m,m-3}} a_m = 0 \end{aligned}$$

Thus, we have two cases. First, $R_{m-2} = S_{m-2}$, that is, there is no restriction on a_{m-1}, a_m, or, secondly, for $R_{m-2} \neq S_{m-2}$,

$$a_{m-1} = -\frac{\gamma_{m-1}\gamma_{m-1,m-3}}{\gamma_m \gamma_{m,m-3}} a_m.$$

Thus, we can freely choose the variables a_j for those j with $R_j = S_j$. The others are then determined by the respective equations.

Example 3.8.

(i) For $m = 2$, $\Lambda = \frac{1}{\gamma_2} - \frac{1}{\gamma_{2,0}} \in \mathbb{R}$. Thus, we find that either $a_2 = 0$ or $\gamma_2 = \gamma_{2,0}$ must hold. If $a_2 = 0$ then $\widehat{L} = a_1 W_1^{\mathscr{L}_D}$ must hold. If $a_2 \neq 0$ then $R_1 = S_1$, which means that the censoring plans $\mathscr{R}$ and $\mathscr{S}$ are identical. Therefore, for $R_1 \neq S_1$, $\widehat{L} = nW_1^{\mathscr{L}_D}$ is the unique linear unbiased estimator whereas, for $R_1 = S_1$, any linear estimator satisfying Equation (3.21) is a linear unbiased estimator. However, in this case, we are in the model of standard progressive Type-II censoring.

(ii) Let $m = 3$. Now, for regularly paired censoring plans, we have

$$\Lambda = \begin{pmatrix} \frac{\delta_{2,0}}{\gamma_2\gamma_{2,0}} & \frac{\delta_{3,0}}{\gamma_3\gamma_{3,0}} \\ 0 & \frac{\delta_{3,1}}{\gamma_3\gamma_{3,1}} \end{pmatrix} = \begin{pmatrix} \frac{R_1 - S_1}{\gamma_2\gamma_{2,0}} & \frac{R_1+R_2-S_1-S_2}{\gamma_3\gamma_{3,0}} \\ 0 & \frac{R_2-S_2}{\gamma_3\gamma_{3,1}} \end{pmatrix} \in \mathbb{R}^{2\times 2}.$$

Now, with $\mathscr{R} = (1, 2, 1)$ and $\mathscr{S} = (0, 2, 2)$, we have $\boldsymbol{\gamma}(\mathscr{R}) = (7, 5, 2)$, $\boldsymbol{\gamma}(\mathscr{S}) = (7, 6, 3)$ and

$$\Lambda = \begin{pmatrix} \frac{1}{5\cdot 6} & \frac{1}{2\cdot 3} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{30} & \frac{1}{6} \\ 0 & 0 \end{pmatrix}$$

so that a solution of $\Lambda \boldsymbol{a}_* = 0$ is given by $\boldsymbol{a}_*' = (a_2, a_3) = (-30, 6)$. Then, it is easy to see that the weights of

$$\widehat{L} = 28W_1^{\mathscr{L}_D} - 30W_2^{\mathscr{L}_D} + 6W_3^{\mathscr{L}_D}$$

satisfy (3.19) and (3.21) so that $\widehat{L}$ is a linear unbiased estimator of ϑ although $EW_2^{\mathscr{L}_D}$ and $EW_3^{\mathscr{L}_D}$ are not linear in ϑ (see Figure 7). This illustrates that the non-linear parts of these expectations cancel out.

Using the result in Theorem 3.6, we can find the best linear unbiased estimator in the generalized Ng–Kundu–Chan model under certain restrictions on the censoring plans.

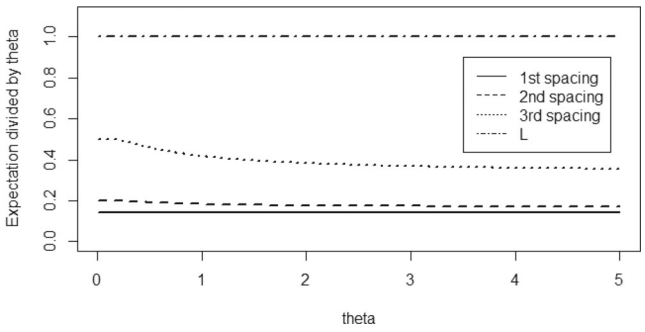

FIGURE 7 Plots of the means of the spacings $W_k^{\mathscr{L}_D}$, $k = 1, 2, 3$, and $\widehat{L}$ divided by ϑ in Example 3.8 with threshold $T = .1$ as a function of ϑ for censoring plans $\mathscr{R} = (1, 2, 1)$, $\mathscr{S} = (0, 2, 2)$.

Theorem 3.9. *Given regular paired censoring plans $\mathscr{R}$ and $\mathscr{S}$ with*

$$R_j = S_j, j \in \{1, \ldots, k_0\}, \quad R_j \neq S_j, j \in \{k_0 + 1, \ldots, m - 1\},$$

for some $k_0 \in \{0, \ldots, m-1\}$, where $\{1, \ldots, k_0\} = \emptyset$ if $k_0 = 0$, the best linear unbiased estimator of the parameter $\vartheta > 0$ in the generalized Ng–Kundu–Chan model is given by

$$\widehat{\vartheta}_L = \frac{1}{k_0+1} \sum_{j=1}^{k_0+1} \gamma_j W_j^{\mathscr{L}_D} = \frac{1}{k_0+1} \left(\sum_{j=1}^{k_0} (R_j + 1) X_j^{\mathscr{L}_D} + \gamma_{k_0+1} X_{k_0+1}^{\mathscr{L}_D} \right).$$

Furthermore,

$$\mathrm{Var}(\widehat{\vartheta}_L) = \frac{\vartheta^2}{k_0+1}, \quad \frac{k_0+1}{\vartheta} \widehat{\vartheta}_L \sim \chi^2(2(k_0+1)).$$

Proof. According to Theorem 3.6 and Remark 3.7, a linear unbiased estimator must only be based on the spacings $W_1^{\mathscr{L}_D}, \ldots, W_{k_0+1}^{\mathscr{L}_D}$. However, due to the assumption, these spacings coincide with progressively Type-II censored order statistics based on the parameters $\gamma_1, \ldots, \gamma_{k_0+1}$. Hence, the corresponding best linear unbiased estimator can be directly taken from this model (see, e.g., Balakrishnan and Cramer, 2014, p. 253). ∎

Example 3.10.

(i) Given the censoring plans in Example 2.4, we get that $k_0 = 0$. Hence, the best linear unbiased estimator of ϑ is given by $\widehat{\vartheta}_L = \gamma_1 X_1^{\mathscr{L}_D}$. In fact, $\widehat{\vartheta}_L$ is the only linear unbiased estimator of ϑ given these censoring plans.

(ii) For the censoring plans given in Table 3, we have $k_0 = 2$ so that the best linear

TABLE 3 Adaptive censoring plans with initial plan $\mathscr{R}$ and $\mathscr{S}$ as desired plan after the change time T.

Change number	Adapted censoring plan	Vector of γ_d Minimum requirement: (6, 5, 4, 3, 2, 1)					
d	$\mathscr{L}_d$	$\gamma_{1,d}$	$\gamma_{2,d}$	$\gamma_{3,d}$	$\gamma_{4,d}$	$\gamma_{5,d}$	$\gamma_{6,d}$
0	$(1, 2, 0, 0, 0, 13) = \mathscr{S}$	22	20	17	16	15	14
1	$(1, 2, 0, 0, 0, 13) = \mathscr{S}$	22	20	17	16	15	14
2	$(1, 2, 0, 0, 0, 13) = \mathscr{S}$	22	20	17	16	15	14
3	$(1, 2, 3, 0, 0, 10)$	22	20	17	13	12	11
4	$(1, 2, 3, 5, 0, 5)$	22	20	17	13	7	6
5	$(1, 2, 3, 5, 1, 4) = \mathscr{R}$	22	20	17	13	7	5
6	$(1, 2, 3, 5, 1, 4) = \mathscr{R}$	22	20	17	13	7	5

Note: The parts of the adapted censoring plans are marked in the respective color of the original censoring plans.

unbiased estimator is given by

$$\begin{aligned}\widehat{\vartheta}_L &= \frac{1}{3}\sum_{j=1}^{3}\gamma_j W_j^{\mathscr{L}_D} \\ &= \frac{1}{3}\Big((R_1+1)X_1^{\mathscr{L}_D} + (R_2+1)X_2^{\mathscr{L}_D} + \gamma_3 X_3^{\mathscr{L}_D}\Big).\end{aligned}$$

3.1.4 | Bayesian inference

For exponentially distributed life times, the prior

$$\pi_{\lambda,\beta}(\vartheta) = \frac{\lambda^\beta}{\Gamma(\beta)}\vartheta^{-(\beta-1)}\mathrm{e}^{-\lambda/\vartheta}, \quad \vartheta > 0,$$

with hyper-parameters $\beta > 0$ and $\lambda > 0$ can be used to obtain explicit Bayesian estimates. Then, from the likelihood function (3.4) and (3.3), the posterior density function $\pi^*_{\lambda,\beta}(\cdot|\text{data})$ of ϑ is obtained as

$$\pi^*_{\lambda,\beta}(\vartheta|\text{data}) = \frac{(\mathrm{TTT}+\lambda)^{m+\beta}}{\Gamma(m+\beta)}\vartheta^{-(m+\beta-1)}\mathrm{e}^{-(\mathrm{TTT}+\lambda)/\vartheta}, \vartheta > 0, \tag{3.31}$$

where the total time on test TTT is defined in (3.6). The Bayesian estimator of ϑ under the squared-error loss function is simply obtained as the posterior mean (which is immediate from (3.31))

$$\widehat{\vartheta}_{\mathrm{B}} = \frac{\mathrm{TTT}+\lambda}{m+\beta-1}. \tag{3.32}$$

Note that it exists for any $\beta, \lambda > 0$.

3.2 | Application to Ng–Kundu–Chan model

Given a censoring plan $\mathscr{R} \in \mathfrak{C}_{m,n}$, the Ng-Kundu-Chan model is included in the presented model by choosing $\mathscr{S} = \mathscr{O}_m = (0, \ldots, 0, n-m)$, that is, right censoring is applied as second plan. In fact, this reflects the strategy that the experiment should be terminated as fast as possible after passing the threshold T. From (2.4) and (2.6), we get

$$\gamma_{j,d} = \begin{cases} n-j+1-R_{\bullet j-1}, & 1 \le j \le d+1 \\ n-j+1-R_{\bullet d}, & d+2 \le j \le m \end{cases}.$$

The above mentioned estimators can thus be applied in this model with obvious adaptions of the censoring plan.

For instance, the likelihood function simplifies to

$$\begin{aligned}\mathcal{L}(\boldsymbol{\theta}|x_1, \ldots, x_m) = c(\mathscr{L}_d)\cdot\left(\prod_{j=1}^{m} f_{\boldsymbol{\theta}}(x_j)\right) \\ \cdot\left(\prod_{j=1}^{d}\big(1-F_{\boldsymbol{\theta}}(x_j)\big)^{R_j}\right)\cdot(1-F_{\boldsymbol{\theta}}(x_m))^{n-m-R_{\bullet d}},\end{aligned}$$

with normalizing constant $c(\mathscr{L}_d) = \prod_{j=1}^{m}\gamma_{j,d} = \prod_{j=1}^{m}(n-j+1-R_{\bullet\min(d,j-1)})$. It is worth to note that no conditional inference by conditioning on the number of observed failures $D = d$ is needed for the above result, as was indicated in Ng et al. (2009). Note that, for example, both the maximum likelihood estimator in (3.6) and the Bayesian estimator in (3.32) have the same form but with appropriately adapted total time on test statistic TTT (which depends on the censoring plans). In the Ng–Kundu–Chan model, it is given by

$$\begin{aligned}\mathrm{TTT} = \sum_{j=1}^{D}(R_j+1)\,X_{j:m:n} + \sum_{j=D+1}^{m-1} X_{j:m:n}^{\mathscr{L}_D} \\ + (n-m+1-R_{\bullet D})X_{m:m:n}^{\mathscr{L}_D}.\end{aligned}$$

Regarding linear estimation, note first that censoring plans $\mathscr{R}$ and $\mathscr{O}_m$ are always regularly paired in the sense of Remark 2.2. The rank condition in (3.30) on Λ simplifies to

$$\operatorname{rank}\ \Lambda = m-1-\sum_{j=1}^{m-1}\mathbb{1}_{\{0\}}(R_j) = \sum_{j=1}^{m-1}\mathbb{1}_{[1,\infty)}(R_j), \tag{3.33}$$

that is, the rank is given by the number of positive censoring numbers $R_j, j \in \{1, \ldots, m-1\}$.

In particular, assuming $R_j > 0$ for every $j \in \{1, \ldots, m\}$, we get rank $\Lambda = m-1$ and the kernel $\{0\}$. Therefore, $\widehat{\vartheta}_L = nW_1^{\mathscr{L}_D}$ is the only unbiased linear estimator of ϑ and, thus, trivially the best one. Of course, the condition can be slightly extended as stated in Theorem 3.9 meaning that the initial part of the censoring plan $\mathscr{R}$ consists of zeros until component k_0.

The density functions of the spacings do not exhibit a particularly simpler structure so that the formulas remain the same except for inserting particular γ's. Note that, according to Remark 2.5, we have $\gamma_{j,d} \ge \gamma_j$ in the Ng–Kundu–Chan model. Hence, the spacings are always stochastically larger

than an exponential distribution. Some plots of the cumulative distribution functions are given in Figure 8. The ordering of the cumulative distribution function is illustrated by these plots. The difference between the distribution function of the spacing and the exponential one becomes larger for an increasing number of the spacing.

4 | ILLUSTRATIVE EXAMPLES AND SIMULATIONS

In this section, we first provide an algorithm to generate random numbers in the generalized Ng–Kundu–Chan model and, second, we present selected simulation results.

4.1 | Simulation algorithm

The generation of non-adaptive progressively Type-II censored order statistics was introduced in Balakrishnan and Sandhu (1995), who proposed a popular generation algorithm in the case of underlying independent and identically distributed random variables. Different approaches to simulate progressively Type-II censored order statistics are summarized and discussed in Balakrishnan and Cramer (2014, Chapter 8). We follow one of these procedures in the following that is based on independent standard exponential random variables $Z_1, \dots Z_m$. That is, we propose a procedure to generate a sample from $X^{\mathscr{L}_D}_{1:m:n}, \dots, X^{\mathscr{L}_D}_{m:m:n}$ as follows, with F being the underlying distribution function having quantile function $F^{\leftarrow}$.

Procedure 4.1. Generation of adapted progressively Type-II censored data (from cdf F)

Input: Censoring plans $\mathscr{R}, \mathscr{S} \in \mathfrak{C}_{m,n}$; threshold $T \in \mathbb{R}$.

Output: Adaptive progressively Type-II censored order statistics $X^{\mathscr{L}_d}_{r:m:n}$, $r = 1, \dots, m$, and applied censoring plan $\mathscr{L}_d$.

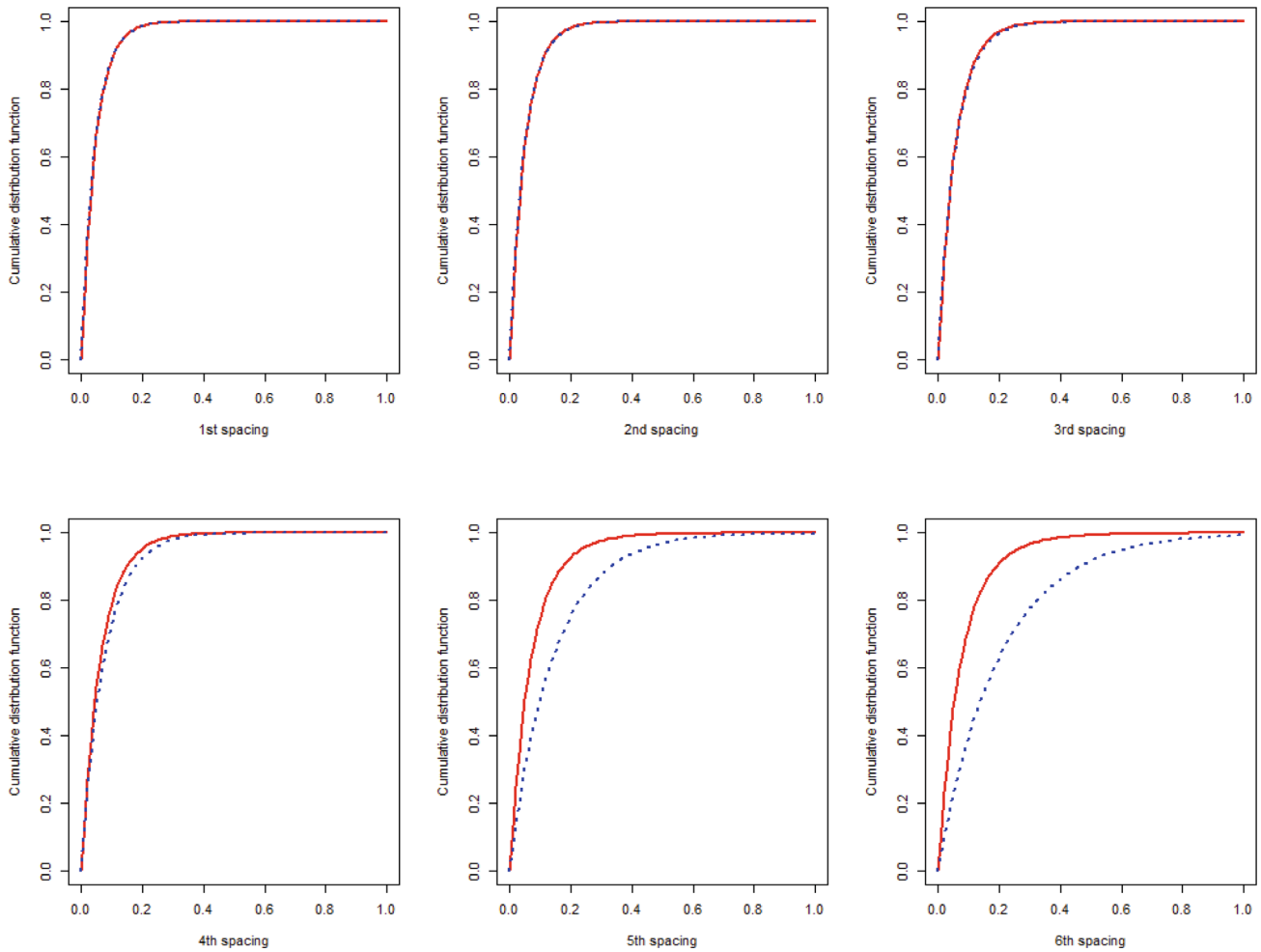

FIGURE 8 Plots of the cumulative distribution functions of the spacings $W_k^{\mathscr{L}_D}$, $k = 1, \dots, 6$, in the (generalized) Ng–Kundu–Chan model for the censoring schemes $\mathscr{R} = (1, 2, 3, 5, 1, 4)$, $\mathscr{S} = (0, 0, 0, 0, 0, 16)$, threshold $T = .1$ and distribution parameter $\vartheta = 1$ (solid lines), compared to the cumulative distribution function of the Exp$(1/\gamma_k(\mathscr{R}))$-distribution (dotted lines).

① Generate m independent Exp(1)-distributed random variables $Z_1, \ldots Z_m$;
② Let $Y_0^{\mathscr{R}} := 0$. For $k = 1, \ldots, m$:
  (a) calculate $Y_k^{\mathscr{R}} = Z_k/\gamma_k$, where $\gamma_k = \gamma_k(\mathscr{R})$ is obtained from initial censoring plan $\mathscr{R}$;
  (b) calculate $Z_{k:m:n}^{\mathscr{R}} = Y_k^{\mathscr{R}} + Z_{k-1:m:n}^{\mathscr{R}}$;
③ Determine $d = \sum_{r=1}^{m} \mathbb{1}_{(-\infty, -\ln(1-F(T))}(Z_{r:m:n}^{\mathscr{R}})$;
④ If $d \in \{m-1, m\}$, then
  (a) save $Z_{1:m:n}^{\mathscr{R}}, \ldots, Z_{m:m:n}^{\mathscr{R}}$ as $Z_{1:m:n}^{\mathscr{L}_d}, \ldots, Z_{m:m:n}^{\mathscr{L}_d}$ and continue with step ⑥;
  (b) else, that is, $d \in \{0, \ldots, m-2\}$, save $Z_{1:m:n}^{\mathscr{R}}, \ldots, Z_{d+1:m:n}^{\mathscr{R}}$ as $Z_{1:m:n}^{\mathscr{L}_d}, \ldots, Z_{d+1:m:n}^{\mathscr{L}_d}$ and continue with step ⑤;
⑤ Let $Y_{d+1}^{\mathscr{L}_d} := Z_{d+1:m:n}^{\mathscr{R}}$. For $k = d+2, \ldots, m$:
  (a) determine $\gamma_k(\mathscr{L}_d) = \gamma_{k,d}$ according to Procedure 4.2;

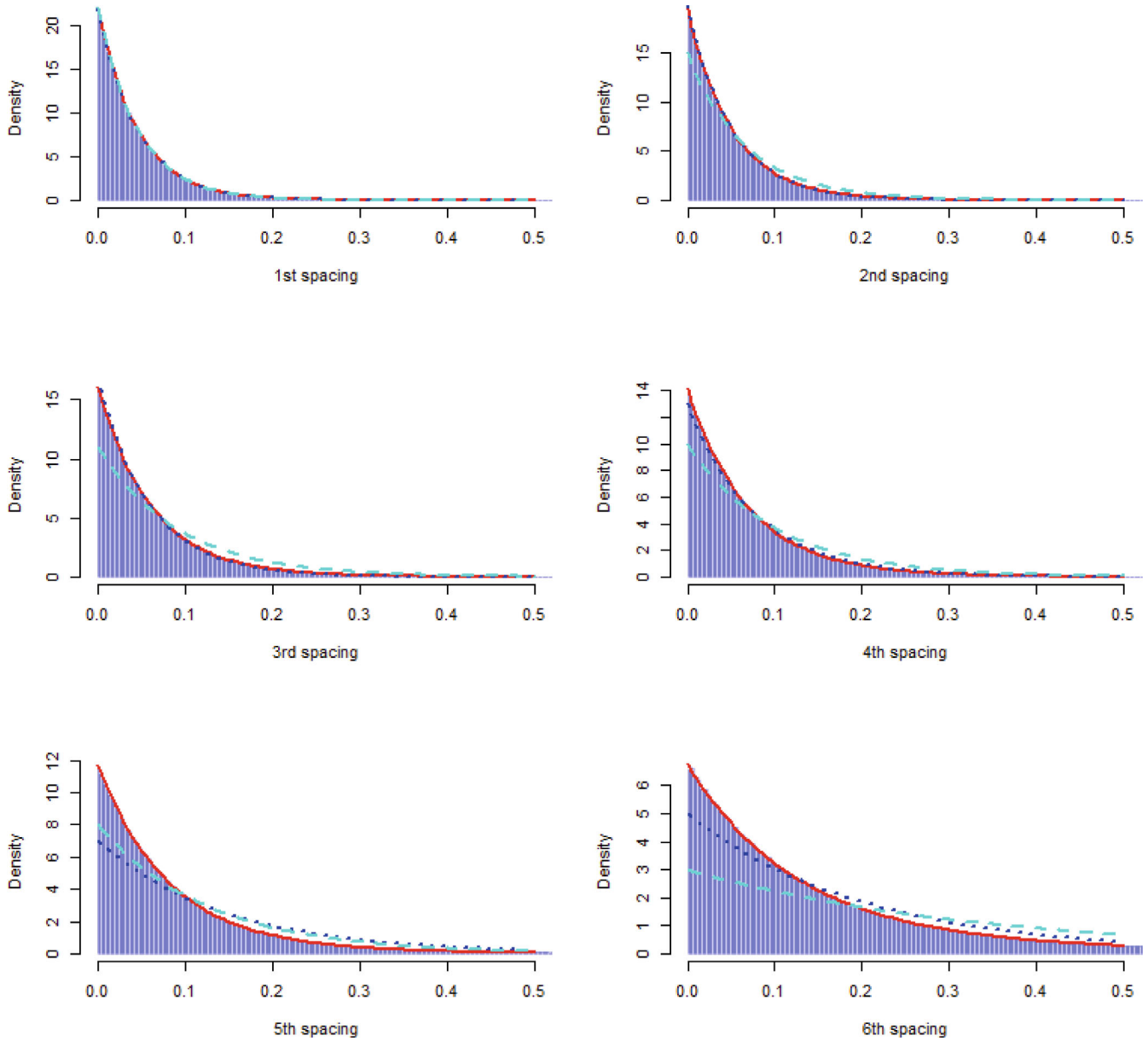

FIGURE 9 Plots of the densities of the spacings $W_k^{\mathscr{L}_D}$, $k = 1, \ldots, 6$, in the generalized Ng–Kundu–Chan model for the censoring plans $\mathscr{R} = (1, 2, 3, 5, 1, 4)$, $\mathscr{S} = (6, 3, 0, 1, 4, 2)$, threshold $T = .1$ and distribution parameter $\vartheta = 1$ (solid lines) compared to the results of 1,000,000 simulations as well as to the exponential densities with parameters $1/\gamma_k(\mathscr{R})$ (dotted line) and $1/\gamma_k(\mathscr{S})$ (dashed line), respectively. Note the different scales of the vertical axis.

(b) calculate $Y_k^{\mathscr{L}_d} = Z_k/\gamma_{k,d}$;
(c) calculate $Z_{k:m:n}^{\mathscr{L}_d} = Y_k^{\mathscr{L}_d} + Z_{k-1:m:n}^{\mathscr{L}_d}$;

⑥ Let $X_{r:m:n}^{\mathscr{L}_d} = F^{\leftarrow}(1 - \exp(-Z_{r:m:n}^{\mathscr{L}_d}))$, $r = 1, \ldots, m$.

Note that step ⑥ can be omitted if the underlying distribution is standard exponential. The same holds in the case of an Exp(ϑ)-baseline distribution if in step ①, instead of independent Exp(1)-, independent Exp(ϑ)-distributed random variables are generated.

Procedure 4.2 (Sub-procedure to determine $\gamma_k(\mathscr{L}_d) = \gamma_{k,d}$, $k = d+2, \ldots, m$, in step ⑤ of Procedure 4.1).

Input: Censoring plans $\mathscr{R}, \mathscr{S} \in \mathfrak{C}_{m,n}$; number $d \in \{0, \ldots, m-2\}$
Output: $\gamma_k(\mathscr{L}_d) = \gamma_{k,d}$, $k = d+2, \ldots, m$.

① If $d = 0$, then let $\gamma_k(\mathscr{L}_d) = \gamma_k(\mathscr{S})$, $k = 2, \ldots, m$, where $\gamma_k(\mathscr{S})$ is obtained from censoring plan $\mathscr{S}$ and stop the procedure; else, that is, $d \in \{1, \ldots, m-2\}$, continue with step ②;

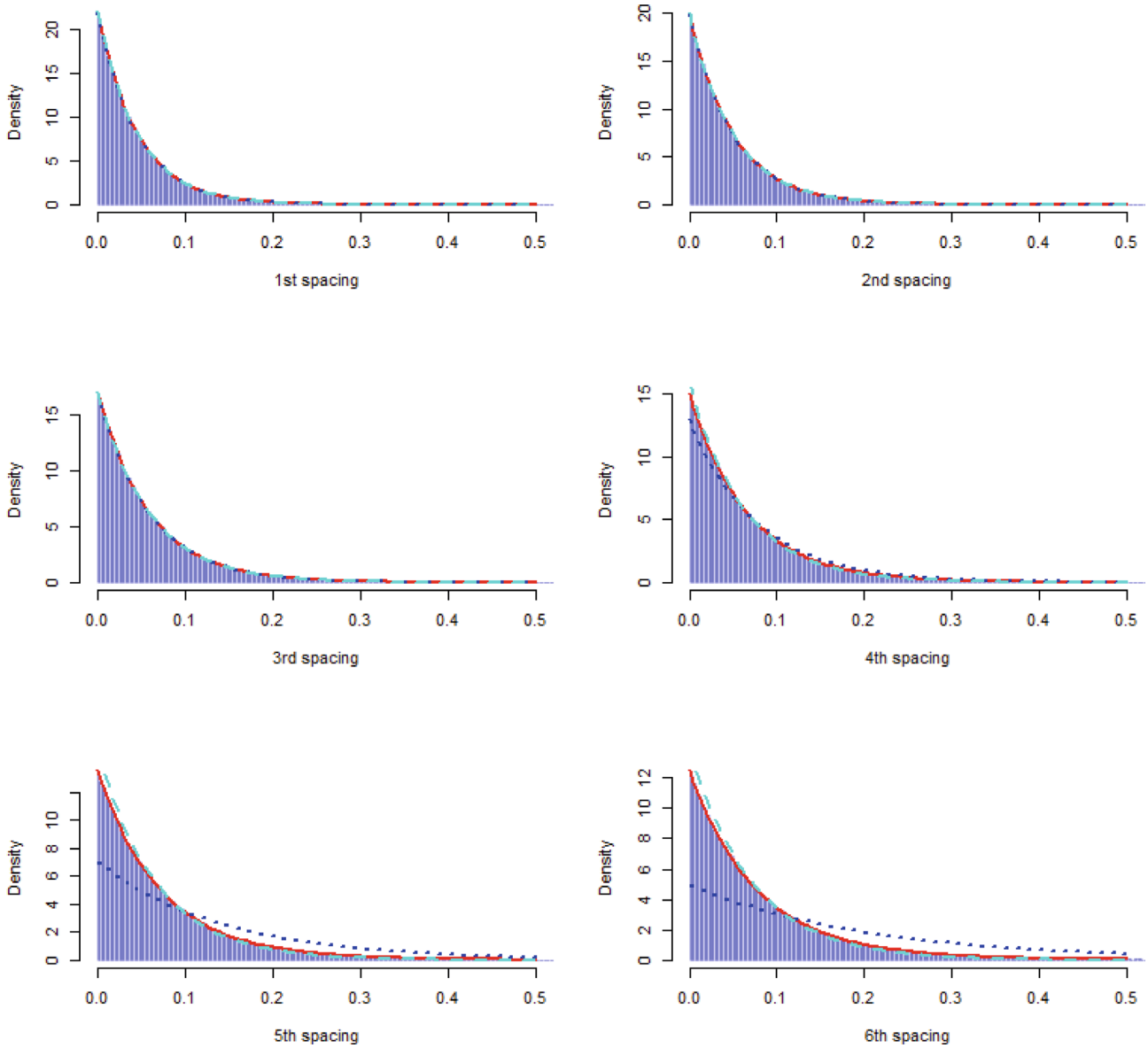

FIGURE 10 Plots of the densities of the spacings $W_k^{\mathscr{L}_D}$, $k = 1, \ldots, 6$, in the generalized Ng–Kundu–Chan model for the censoring plans $\mathscr{R} = (1, 2, 3, 5, 1, 4)$, $\mathscr{S} = (1, 2, 0, 0, 0, 13)$, threshold $T = .1$ and distribution parameter $\vartheta = 1$ (solid lines) compared to the results of 1,000,000 simulations as well as to the exponential densities with parameters $1/\gamma_k(\mathscr{R})$ (dotted line) and $1/\gamma_k(\mathscr{S})$ (dashed line), respectively. Note the different scales of the vertical axis.

TABLE 4 Adaptive censoring plans with initial plan $\mathscr{R}$ and $\mathscr{S}$ as desired plan after the change time T.

Change number	Adapted censoring plan	Vector of γ_d Minimum requirement: (6, 5, 4, 3, 2, 1)					
d	$\mathscr{L}_d$	$\gamma_{1,d}$	$\gamma_{2,d}$	$\gamma_{3,d}$	$\gamma_{4,d}$	$\gamma_{5,d}$	$\gamma_{6,d}$
0	$(0,0,0,0,0,16) = \mathscr{S}$	22	21	20	19	18	17
1	$(16,0,0,0,0,0) = \mathscr{R}$	22	5	4	3	2	1
2	$(16,0,0,0,0,0) = \mathscr{R}$	22	5	4	3	2	1
3	$(16,0,0,0,0,0) = \mathscr{R}$	22	5	4	3	2	1
4	$(16,0,0,0,0,0) = \mathscr{R}$	22	5	4	3	2	1
5	$(16,0,0,0,0,0) = \mathscr{R}$	22	5	4	3	2	1
6	$(16,0,0,0,0,0) = \mathscr{R}$	22	5	4	3	2	1

Note: The parts of the adapted censoring plans are marked in the respective color of the original censoring plans.

② Determine $\gamma_k(\mathscr{L}_d) = \eta_{k,d} \vee (m-k+1)$, $k = d+2, \ldots, m$, with $\eta_{k,d} = n-k+1-R_{\bullet d} - \sum_{i=d+1}^{k-1} S_i$ according to Equations (2.6) and (2.4).

4.2 | Simulation results and illustrative examples

In that what follows, we present simulation results for selected illustrative cases.

Example 4.3. We first take up Example 2.4 and consider the censoring plans $\mathscr{R} = (1,2,3,5,1,4)$ and $\mathscr{S} = (6,3,0,1,4,2)$ with $n = 22$ and $m = 6$. The corresponding censoring plans $\mathscr{L}_d$ and gamma schemes $\boldsymbol{\gamma}_d$ for any $d \in \{0, \ldots, 6\}$ are given in Table 1. While $\gamma_{1,d}$, that is, the number of items in the experiment immediately before the first failure occurs, does not depend on d, the number $\gamma_{k,d}$ of items in the experiment immediately before the k-th failure does for any $k \in \{2, \ldots, 6\}$. This can be easily seen from Table 1.

In Figure 9, we plot the densities of the spacings $W_k^{\mathscr{L}_D}$, $k = 1, \ldots, 6$, for the censoring plans $\mathscr{R}$ and $\mathscr{S}$, threshold $T = .1$, and distribution parameter $\vartheta = 1$ (see also Figure 4) against the results of 1,000,000 simulations according to Procedure 4.1. It can be seen that the simulations results for each spacing $W_k^{\mathscr{L}_D}$ correspond well to the marginal density function $f_{\mu,\vartheta}^{W_k^{\mathscr{L}_D}}$ given in Theorem 3.3, the latter being plotted as solid line.

For each spacing $W_k^{\mathscr{L}_D}$ also the exponential density curves with parameters $1/\gamma_k(\mathscr{R})$ (dotted line) resp. $1/\gamma_k(\mathscr{S})$ (dashed line) are shown for comparison. As the first spacing $W_1^{\mathscr{L}_D}$ is exponentially distributed with parameter $1/n$ (cf. proof of Theorem 3.3), the density curves do not differ for the first spacing since $n = \gamma_1(\mathscr{R}) = \gamma_1(\mathscr{S})$. For other spacings, the marginal density functions (solid lines) are initially quite similar to the densities of the exponential distribution with parameter $1/\gamma_k(\mathscr{R})$. However, the plots show that the marginal densities and the exponential densities can differ significantly for both exponential distributions with parameters $1/\gamma_k(\mathscr{S})$ (dashed lines) and $1/\gamma_k(\mathscr{R})$ (dotted lines); see also Figure 5.

Example 4.4. Compared to Example 4.3, we keep the first censoring plan $\mathscr{R} = (1,2,3,5,1,4)$ and change the second one to $\mathscr{S} = (1,2,0,0,0,13)$. Thus, as can been seen from Table 3, the applied adapted censoring plan $\mathscr{L}_d$ remains stable for $d \in \{0,1,2\}$ and differs for change numbers d afterwards. Correspondingly, the number of items $\gamma_{k,d}$ in the experiment immediately before the k-th failure does not depend on d for $k \in \{1,2,3\}$. In contrast, the numbers $\gamma_{k,d}$ differ depending on d for any $k \in \{4,5,6\}$ (see table columns referring to $\gamma_{4,d}$ to $\gamma_{6,d}$).

In Figure 10, we plot again the densities of the spacings $W_k^{\mathscr{L}_D}$, $k = 1, \ldots, 6$, for the censoring plan $\mathscr{R}$ and the modified plan $\mathscr{S}$, threshold $T = .1$ and distribution parameter $\vartheta = 1$ against the results of 1,000,000 simulations as well as the exponential density curves with parameter $1/\gamma_k(\mathscr{R})$ (dotted lines) and $1/\gamma_k(\mathscr{S})$ (dashed lines). Then, the density curves do not differ for the first, second and third spacing. However, for higher spacings the density curves become quite different. In this example, the density function of the 5th resp. 6th spacing differ significantly from the exponential density with parameters $1/\gamma_5(\mathscr{R})$ resp. $1/\gamma_6(\mathscr{R})$ (dotted lines). By contrast, the deviation from the density function of the exponential function with parameters $1/\gamma_5(\mathscr{S})$ resp. $1/\gamma_6(\mathscr{S})$ (dashed lines) is not so pronounced. The latter is due to the fact that censoring according

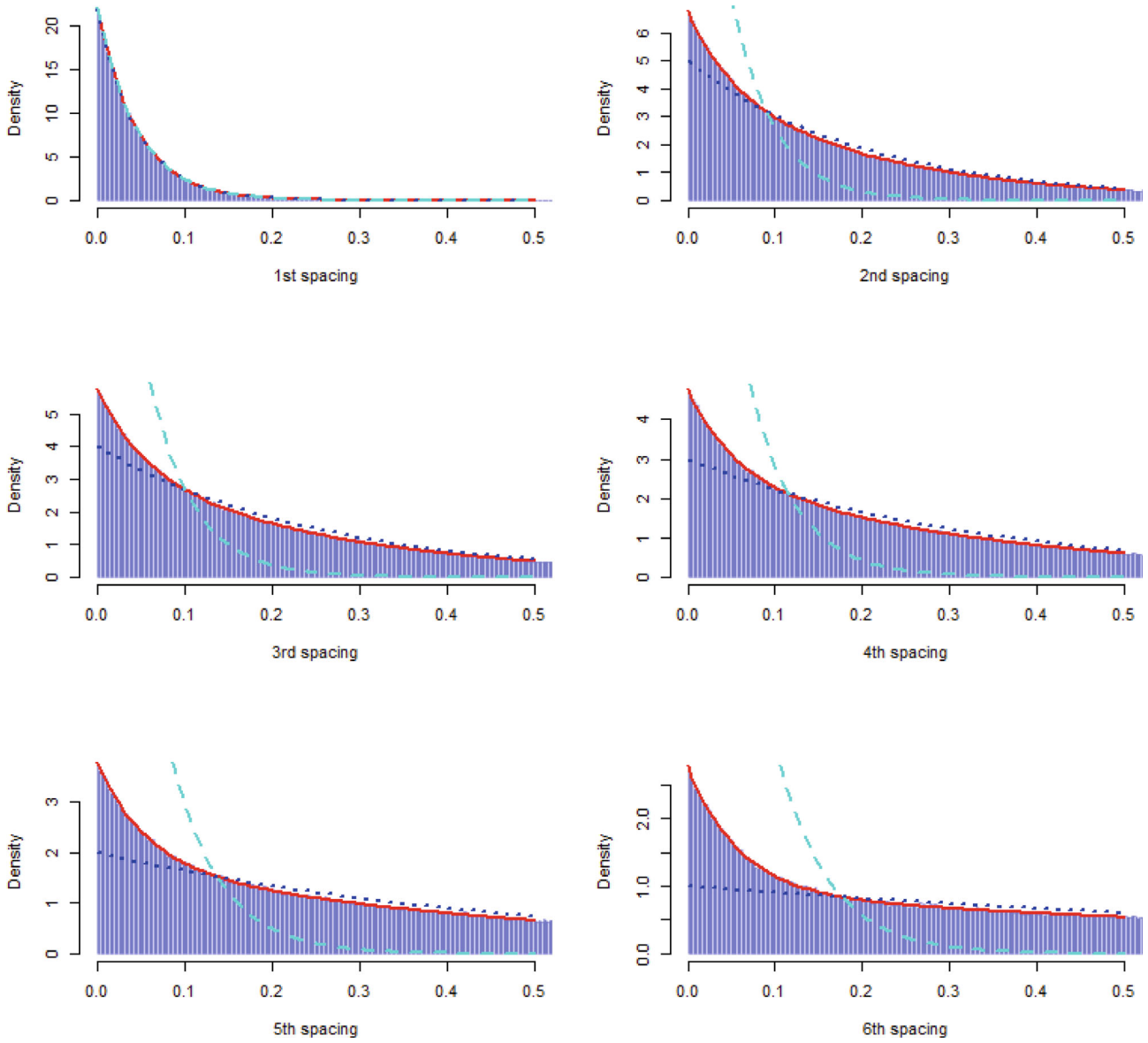

FIGURE 11 Plots of the densities of the spacings $W_k^{\mathscr{D}_D}$, $k = 1, \ldots, 6$, in the generalized Ng–Kundu–Chan model for the censoring plans $\mathscr{R} = (16, 0, 0, 0, 0, 0)$, $\mathscr{S} = (0, 0, 0, 0, 0, 16)$, threshold $T = .1$ and distribution parameter $\vartheta = 1$ (solid lines) compared to the results of 1,000,000 simulations as well as to the exponential densities with parameters $1/\gamma_k(\mathscr{R})$ (dotted line) and $1/\gamma_k(\mathscr{S})$ (dashed line), respectively. Note the different scales of the vertical axis.

to plan $\mathscr{S}$ takes place not only for $d = 0$ but also for $d = 1, 2$.

Example 4.5. In this last example, we consider two opposite one-step censoring plans $\mathscr{R} = (16, 0, 0, 0, 0, 0)$ and $\mathscr{S} = (0, 0, 0, 0, 0, 16)$, that is first-step censoring (see Balakrishnan et al. (2008); Park and Ng (2012)) and right censoring. Obviously, see Table 4, the first applied censoring plan $\mathscr{R} = (16, 0, 0, 0, 0, 0)$ is maintained regardless of d except for the case $d = 0$, that is, except for the case that no failure has been observed before the predefined threshold T. Thus, the number of units in the experiment immediately before the k-th failure $\gamma_{k,d}$ already differs significantly from $\gamma_{1,d}$ for $k = 2$ and decreases steadily by one afterwards for $k = 3, \ldots, 6$.

In Figure 11 it is thus seen that for any spacing $W_k^{\mathscr{D}_D}$, $k \in \{2, \ldots, 6\}$, the marginal density curves (solid lines) differ significantly from both exponential density curves with parameters $1/\gamma_k(\mathscr{R})$ (dotted line) and $1/\gamma_k(\mathscr{S})$ (dashed line), respectively.

5 | CONCLUSION

In the present paper, we have extended the model of adaptive Type-II progressive censoring introduced by Ng et al. (2009)

by allowing to switch from a given initial censoring plan to any new plan of the same length once a failure time exceeds a given threshold. We have described this so-called generalized Ng–Kundu–Chan model in detail, paying particular attention to the construction of the adaptive censoring plan, which depends on the observed number of observations that do not exceed the prefixed threshold, in order to ensure that the desired number of measurements is made. Taking into account that the censoring plan is generally random, it has been shown that the joint distribution of the adaptive progressively Type-II censored order statistics is defined stepwise based on the joint density functions of standard progressively Type-II censored order statistics with non-random censoring plans. Since the number of observations not exceeding the threshold is obtained directly from the sample, we have discussed that likelihood based inference such as the computation of maximum likelihood (and Bayesian) estimates is along the same lines as for non-adaptive progressive Type-II censoring.

Assuming exponentially distributed failure times, we have further discussed the distributional properties of maximum likelihood estimators. It was found that the *normalized* spacings are independent and identically exponentially distributed, as for standard progressive Type-II censored order statistics. This result holds even though the normalized spacings in the generalized Ng-Kundu-Chan model have normalizing factors that depend on the random number D of failure times before the threshold and are thus themselves random. However, the situation is different for non-normalized spacings. It turned out that spacings themselves are generally not independent and have marginal distributions that are mixtures of exponentials. This result has been used to discuss linear inference in the generalized Ng-Kundu-Chan-Model. While the first normalized spacing is always a linear estimator for the exponential rate parameter, it was proved that other single spacings do not generally lead to an unbiased estimator. However, we have derived conditions which ensure the existence of an unbiased estimator when higher spacings are involved, even though they themselves are biased. This is due to cancellation effects of non-linear parts in the expectation of a single spacing. We have also discussed restrictions on the censoring plans that enable us to find a best linear unbiased estimator in the generalized Ng-Kundu-Chan model. Applying our findings to the standard Ng-Kundu-Chan model, we find, for example, that assuming that censoring after the first observed failure time is foreseen according to the initial censoring plan, the first normalized spacing is the only and hence the best linear unbiased estimator of the exponential rate parameter.

Our findings have been illustrated by several examples throughout the paper. Finally, we have also presented an algorithm to generate random failure times in the generalized Ng-Kundu-Chan model along with selected illustrative simulation results.

Adaptive versions of progressive censoring has been considered in the literature before. To our best knowledge, modifications of adaptive progressive Type-II censored models that would lead to non-deterministic sample size have not been considered so far. In the generalized Ng-Kundu-Chan-model, for example, it would be of interest in experimental design to consider the initial and second censoring plan as possibly being of different lengths. Similarly, the intended sampling according to the switch from the first to the second censoring plan once the threshold has been exceeded, without adapting the censoring plan as described in detail in the present paper, would lead to a random sample size that is at most the length of the the initial censoring plan. Such extensions of the generalized Ng-Kundu-Chan model are subject of our future research.

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DATA AVAILABILITY STATEMENT

This paper does not concern real data sets. Instead, various examples are presented to demonstrate the findings. An algorithm to generate random failure times in the generalized Ng–Kundu-Chan model is also provided, alongside selected simulation results for illustration.

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APPENDIX

Proof of Theorem 3.3. In order to derive the marginal density functions, we use similar arguments to those applied in Cramer and Balakrishnan (2013). First, from the density property of $f_\vartheta^{\mathbf{W}^{\mathscr{L}_D}|D=d}$ we get from (3.16) and Lemma 2.3 for arbitrary $T > 0$ and $0 \le d \le m-1$

$$\begin{aligned}
&\int_{\mathscr{W}_d^{(m)}(T)} \prod_{j=1}^{m} \left(\gamma_{j,d} f_\vartheta(\gamma_{j,d} t_j)\right) \, d\boldsymbol{t}_m \\
&= \int_{\mathscr{W}_d^{(d+1)}(T)} \prod_{j=1}^{d+1} \left(\gamma_j f_\vartheta(\gamma_j t_j)\right) \, d\boldsymbol{t}_{d+1} \\
&= P(D=d) = P_T(D=d), \qquad \text{(A1)}
\end{aligned}$$

where $\mathscr{W}_d^{(m)}(T) = \{\boldsymbol{x} = (x_1, \dots, x_m) \in \mathbb{R}^m_{>0} | x_{\bullet d} < T \le x_{\bullet d+1}\}$, $\mathscr{W}_d^{(d+1)}(T) = \{\boldsymbol{x} = (x_1, \dots, x_{d+1}) \in \mathbb{R}^{d+1}_{>0} | x_{\bullet d} < T \le x_{\bullet d+1}\}$ and $P_T(D=d)$ as in (2.2). Notice that the variables $t_{d+2}, \dots, t_m$ are not involved in the condition so that the respective part of the integral leads to the factor one. For $d = m$, we get

$$\begin{aligned}
&\int_{\mathscr{W}_m^{(m)}(T)} \prod_{j=1}^{m} \left(\gamma_{j,m} f_\vartheta(\gamma_{j,m} t_j)\right) \, d\boldsymbol{t}_m \\
&= \int \mathbb{1}_{(0,T]}(t_{\bullet m}) \prod_{j=1}^{m} \left(\gamma_j f_\vartheta(\gamma_j t_j)\right) \, d\boldsymbol{t}_m \\
&= P(D=m) = P_T(D=m), \quad \text{say.} \qquad \text{(A2)}
\end{aligned}$$

First, notice that $W_1^{\mathscr{L}_D} = X^{\mathscr{R}}_{1:m:n}$, which means that $W_1^{\mathscr{L}_D} \sim \text{Exp}(\vartheta/n)$. For $k \in \{2, \dots, m\}$ fixed, we have to consider various subcases. Depending on $d \in \{0, \dots, m\}$, we have to evaluate the integrals

$$f_{\vartheta}^{\mathbf{W}_k^{\mathscr{F}_D}|D=d}(t_k) = \frac{1}{P(D=d)} \int_{\mathscr{W}_{d,k}^{(m-1)}(T,t_k)} \prod_{j=1}^{m} \left(\gamma_{j,d} f_{\vartheta}(\gamma_{j,d} t_j)\right) dt_m \ \dots \ dt_{k+1} dt_{k-1} \ \dots \ dt_1, \quad t_k > 0, \tag{A3}$$

for $k \in \{1, \dots, m\}$ and $d \in \{0, \dots, m-1\}$ with

$$\mathscr{W}_{d,k}^{(m-1)}(T,t) = \begin{cases} \left\{\widetilde{\boldsymbol{x}}_k \in \mathbb{R}_{>0}^{m-1} | x_{\bullet d} < T \le x_{\bullet d+1}\right\} & ① \ d \le k-2 \\ \left\{\widetilde{\boldsymbol{x}}_k \in \mathbb{R}_{>0}^{m-1} | x_{\bullet d} < T \le x_{\bullet d} + t\right\} & ② \ d = k-1 \\ \left\{\widetilde{\boldsymbol{x}}_k \in \mathbb{R}_{>0}^{m-1} | \sum_{j=1, j\neq k}^{d} x_j + t < T \le \sum_{j=1, j\neq k}^{d+1} x_j + t\right\} & ③ \ d \ge k \end{cases}. \tag{A4}$$

and, for $d = m$,

$$\mathscr{W}_{m,k}^{(m-1)}(T,t) = \begin{cases} \left\{\widetilde{\boldsymbol{x}}_k \in \mathbb{R}_{>0}^{m-1} | \sum_{j=1, j\neq k}^{d} x_j + t < T\right\} & ④ \ k \le m-1 \\ \left\{\widetilde{\boldsymbol{x}} \in \mathbb{R}_{>0}^{m-1} | x_{\bullet m-1} + t < T\right\} & ⑤ \ k = m \end{cases}. \tag{A5}$$

Hereby the notation $\widetilde{\boldsymbol{x}}_k \in \mathbb{R}^{m-1}$ means that the k-th component x_k of the vector $\boldsymbol{x} \in \mathbb{R}^m$ is omitted, that is, $\widetilde{\boldsymbol{x}}_k = (x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_m) \in \mathbb{R}^{m-1}$.

Taking into account the cases ① – ⑤, the integration in (A3) can be written as

$$f_{\vartheta}^{\mathbf{W}_k^{\mathscr{F}_D}|D=d}(t) = \begin{cases} \frac{\gamma_{k,d} f_{\vartheta}(\gamma_{k,d} t)}{P_T(D=d)} \int_{\mathscr{W}_d^{(d+1)}(T)} \prod_{j=1}^{d+1} \left(\gamma_{j,d} f_{\vartheta}(\gamma_{j,d} t_j)\right) dt_{d+1} \ \dots \ dt_1, & d \le k-2 \\ \frac{\gamma_{k,d} f_{\vartheta}(\gamma_{k,d} t)}{P_T(D=d)} \int \mathbb{1}_{(T-t,T)}(t_{\bullet k-1}) \prod_{j=1}^{k-1} \left(\gamma_{j,d} f_{\vartheta}(\gamma_{j,d} t_j)\right) dt_{k-1} \ \dots \ dt_1, & d = k-1 \\ \frac{\gamma_{k,d} f_{\vartheta}(\gamma_{k,d} t)}{P_T(D=d)} \int_{\mathscr{W}_{d-1}^{(d)}(T-t)} \prod_{j=1, j\neq k}^{d+1} \left(\gamma_{j,d} f_{\vartheta}(\gamma_{j,d} t_j)\right) dt_{d+1} \ \dots \ dt_{k+1} dt_{k-1} \ \dots \ dt_1, & k \le d \le m-1 \\ \frac{\gamma_{k,m} f_{\vartheta}(\gamma_{k,m} t)}{P_T(D=m)} \int \mathbb{1}_{(0,T-t]}(t_{\bullet m-1}^{(k)}) \prod_{j=1, j\neq k}^{m} \left(\gamma_{j,m} f_{\vartheta}(\gamma_{j,m} t_j)\right) dt_m \ \dots \ dt_{k+1} dt_{k-1} \ \dots \ dt_1, & k < d = m \\ \frac{\gamma_{m,m} f_{\vartheta}(\gamma_{m,m} t)}{P_T(D=m)} \int \mathbb{1}_{(0,T-t]}(t_{\bullet m-1}) \prod_{j=1}^{m-1} \left(\gamma_{j,d} f_{\vartheta}(\gamma_{j,m} t_j)\right) dt_{m-1} \ \dots \ dt_1, & k = d = m \end{cases}$$

where $t_{\bullet m-1}^{(k)} = \sum_{j=1, j\neq k}^{m} t_j$. In case ①, we get from (A1) that the integral equals $P_T(D=d)$ so that

$$f_{\vartheta}^{\mathbf{W}_k^{\mathscr{F}_D}|D=d}(t) = \gamma_{k,d} f_{\vartheta}(\gamma_{k,d} t), \quad t > 0.$$

In case ③, the integral equals the probability $P_{T-t}(D_{k,d} = d-1) = F_{d-1:d:n}^{(k)}(T-t) - F_{d:d:n}^{(k)}(T-t)$ where the parameters are given by (cf. Lemma 2.3)

$$(\gamma_{1,d}, \dots, \gamma_{k-1,d}, \gamma_{k+1,d}, \dots, \gamma_{d+1,d}) = (\gamma_1, \dots, \gamma_{k-1}, \gamma_{k+1}, \dots, \gamma_{d+1}).$$

Obviously, it is zero for $T < t$. The same holds for case ④. Here we get $P_{T-t}(D_{k,m} = m-1) = F_{m-1:m-1:n}^{(k)}(T-t)$. In case ⑤, the probability is given by $P_{T-t}(D_m = m-1) = F_{m-1:m-1:n}(T-t)$. In case ②, we have $d = k-1$ so that we can write

$$\int \mathbb{1}_{(T-t,T)}(t_{\bullet k-1}) \prod_{j=1}^{k-1} \left(\gamma_{j,k-1} f_{\vartheta}(\gamma_{j,k-1} t_j)\right) dt_{k-1} \ \dots \ dt_1$$

using properties of the indicator function and Lemma 2.3 for $j \le k-1$

$$\begin{aligned} &= \int \mathbb{1}_{(0,T]}(t_{\bullet k-1}) \prod_{j=1}^{k-1} \left(\gamma_j f_{\vartheta}(\gamma_j t_j)\right) dt_{k-1} \ \dots \ dt_1 \\ &\quad - \int \mathbb{1}_{(0,T-t]}(t_{\bullet k-1}) \prod_{j=1}^{k-1} \left(\gamma_j f_{\vartheta}(\gamma_j t_j)\right) dt_{k-1} \ \dots \ dt_1 \\ &= P(X_{k-1:m:n} \le T) - P(X_{k-1:m:n} \le T-t) = F_{k-1:m:n}(T) - F_{k-1:m:n}(T-t) \end{aligned}$$

Summing up, we get for $0 \le t \le T$ and $1 \le k \le m-1$ with the definition $F_{0:m:n}(T) \equiv 1$ and $F_{k-1:m:n}(T-t) = F_{k-1:k:n}^{(k)}(T-t)$ that

$$\begin{aligned} f_{\vartheta}^{W_k^{\mathscr{F}_D}}(t) &= \sum_{d=0}^{k-2} (F_{d:m:n}(T) - F_{d+1:m:n}(T)) \gamma_{k,d} f_{\vartheta}(\gamma_{k,d} t) \\ &\quad + (F_{k-1:m:n}(T) - F_{k-1:m:n}(T-t)) \gamma_k f_{\vartheta}(\gamma_k t) \end{aligned}$$

$$+\sum_{d=k}^{m-1}(F^{(k)}_{d-1:d:n}(T-t)-F^{(k)}_{d:d:n}(T-t))\gamma_{k,d}f_\vartheta(\gamma_{k,d}t)+F^{(k)}_{m-1:m-1:n}(T-t)\gamma_{k,m}f_\vartheta(\gamma_{k,m}t)$$
$$=\sum_{d=0}^{k-2}(F_{d:m:n}(T)-F_{d+1:m:n}(T))\gamma_{k,d}f_\vartheta(\gamma_{k,d}t)+F_{k-1:m-1:n}(T)\gamma_k f_\vartheta(\gamma_k t),$$

using that $\gamma_{k,d}=\gamma_k$, $d=k,\ \dots\ ,m$ (see Lemma 2.1). For $k=m$, we find

$$f_\vartheta^{W_m^{\mathscr{F}_D}}(t)=\sum_{d=0}^{m-2}(F_{d:m:n}(T)-F_{d+1:m:n}(T))\gamma_{m,d}f_\vartheta(\gamma_{m,d}t)+F_{m-1:m-1:n}(T)\gamma_m f_\vartheta(\gamma_m t)),$$

For $T<t$, we get by analogy

$$f_\vartheta^{W_k^{\mathscr{F}_D}}(t)=\sum_{d=0}^{k-2}(F_{d:m:n}(T)-F_{d+1:m:n}(T))\gamma_{k,d}f_\vartheta(\gamma_{k,d}t)+F_{k-1:m:n}(T)\gamma_k f_\vartheta(\gamma_k t).$$

This yields the mixture representation of the density function. The expressions for cumulative distribution function and moments follows directly from properties of the mixture.

■