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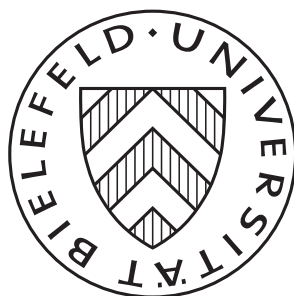
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A Mean-Field Model of Optimal Investment

Alessandro Calvia, Salvatore Federico, Giorgio Ferrari, and Fausto Gozzi



A MEAN-FIELD MODEL OF OPTIMAL INVESTMENT

ALESSANDRO CALVIA^{a,1}, SALVATORE FEDERICO^{b,2}, GIORGIO FERRARI^{c,3}, AND FAUSTO GOZZI^{d,4}

ABSTRACT. We establish the existence and uniqueness of the equilibrium for a stochastic mean-field game of optimal investment. The analysis covers both finite and infinite time horizons, and the mean-field interaction of the representative company with a mass of identical and indistinguishable firms is modeled through the time-dependent price at which the produced good is sold. At equilibrium, this price is given in terms of a nonlinear function of the expected (optimally controlled) production capacity of the representative company at each time. The proof of the existence and uniqueness of the mean-field equilibrium relies on a priori estimates and the study of nonlinear integral equations, but employs different techniques for the finite and infinite horizon cases. Additionally, we investigate the deterministic counterpart of the mean-field game under study.

Keywords: mean-field games; mean-field equilibrium; forward-backward ODEs; optimal investment; price formation.

AMS 2020: 35Q89; 47H10; 49N10; 49N80; 91A07; 91B38; 91B70.

JEL classification: C02; C61; C62; C72; D25; D41.

1. INTRODUCTION

In this paper, we consider a mean-field model of optimal investment with competition à la Cournot, where the price of the good produced by a representative company depends on the aggregate production of the entire economy through a nonlinear inverse demand function. In the absence of interventions, the production capacity of the representative company evolves stochastically over time as a geometric Brownian motion, and its level can be increased through investment, subject to quadratic costs. The representative company discounts profits and costs at a constant rate and aims to maximize total expected profits from production, net of investment costs. Instantaneous profits depend linearly on the company's production capacity (thus, production occurs at full capacity) and on the time-dependent price of the produced good. The mean-field equilibrium investment and average production processes $(\hat{\mathbf{u}}, \hat{q})$ are such that expected total net profits are maximized and, assuming an isoelastic inverse demand function, the price is given in terms of a nonlinear function of the average optimally controlled production at each time (see also Achdou et al. [2] and Remark 2.4 below). We are able to prove the existence and uniqueness of the equilibrium pair $(\hat{\mathbf{u}}, \hat{q})$ when the problem's time horizon is finite or infinite. Furthermore, we demonstrate that the existence and

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uniqueness carry over to the deterministic counterpart of our model, essentially using the same techniques as in the stochastic case.

The investment problem under study falls under the category of mean-field games with scalar interaction. Mean-field games, independently introduced by Lasry and Lions [17], and Huang, Caines, and Malhamé [15], represent limit models of non-cooperative symmetric N -player games with mean-field interaction as the number of players N tends to infinity. An exhaustive review of mean-field models can be found in the two-volume book by Carmona and Delarue [8].

In our context, the consistency condition that the equilibrium price aligns at each time with a decreasing nonlinear function of the expected (optimally controlled) production capacity can be viewed as the limit, as the number N of identical and indistinguishable companies operating in the market diverges, of the requirement that price inversely depends on the aggregate production of the entire economy, scaled by a factor of $1/N$. As discussed in Huang et al. [15], this scaling can be justified by considering situations where "an increasing number of firms join together to serve an increasing number of consumers" (see the discussion in Huang et al. [15], after Equation (2.4) therein).

The equilibrium construction for a given time horizon $T \in (0, +\infty]$ follows a three-step approach. Firstly, given a deterministic path of the average production $q := (q_s)_{s \in [0, T]}$, we solve the representative company's optimal investment problem. Because the production capacity evolves as a geometric Brownian motion and is linearly dependent on the investment process and since the performance criterion has linear dependence on the production capacity and quadratic dependence on the investment costs, the resulting optimal control problem for the representative company is of linear-quadratic type and has an explicit solution. Moreover, the optimal control $\hat{\mathbf{u}}^{(T, q)}$ for fixed average production trajectory q is deterministic. In the second step, we calculate the expectation of the optimally controlled production capacity process, which is easily computable given the explicit representation of the state process. Finally, in the third step, we impose the consistency condition, requiring that this expectation must match q_s for each time $s \in [0, T]$. This condition leads to a nonlinear integral equation for the equilibrium average production trajectory $\hat{q} := (\hat{q}_s)_{s \in [0, T]}$. By construction the control $\hat{\mathbf{u}} := \hat{\mathbf{u}}^{(T, \hat{q})}$ and the function \hat{q} form a mean-field equilibrium.

It is important to note that although the company's optimal control problem for a fixed average production trajectory q is linear-quadratic, due to the nonlinear dependence of the net profit functional on q , the overall mean-field problem is *not* of linear-quadratic type. In this regard, our results differ from those presented in Bensoussan et al. [3], Delarue and Tchuendom [12], Tchuendom [19], among others, which focus on linear-quadratic mean-field games.

While the two-step approach previously outlined works for both the case where $T < +\infty$ and the case where $T = +\infty$, different technical arguments are used to prove the existence and uniqueness of the equilibrium in the finite and infinite horizon cases. As a matter of fact, the integral equation uniquely characterizing the equilibrium average production \hat{q} does not fall into the standard theory of integro-differential equations, as we have an initial-value problem with the integral being backward in time, and thus requires a careful ad hoc analysis. Specifically, the proof of the existence of a solution to it relies on a priori estimates and Schauder's fixed point theorem when $T < +\infty$, whereas it utilizes a priori estimates and the Frechet-Kolmogorov L^p compactness theorem in the case of $T = +\infty$. Uniqueness is then established in both cases through suitable contradiction arguments that exploit the properties of any mean-field equilibrium. Notably, when $T = +\infty$, we can also prove that the time-dependent equilibrium \hat{q} monotonically converges to the stationary average production level. The latter is obtained as the unique constant solution to the integral equation that uniquely determines \hat{q} .

Given the geometric dynamics of the production capacity, the linear dependence of the profit functional on the controlled state variable, the quadratic costs of investment, and the fact that the mean-field interaction is of scalar type and only involves the expected value of the latter, it is not surprising that the mean-field equilibrium is deterministic and does not depend on the volatility coefficient σ appearing in the production capacity's dynamics. Guided by this observation, we also consider the deterministic counterpart of the previously discussed mean-field game and show that a unique equilibrium exists in this setting as well. This equilibrium can indeed be constructed by following exactly the same arguments employed in the stochastic case.

Games of optimal investment in both stochastic and deterministic settings are extensively covered in the Economics literature, and a comprehensive overview of models and results can be found in Vives [20]. Specifically, mean-field problems with Cournot competition have garnered interest in recent literature; see Chan and Sircar [10] for an insightful overview. Graber and Bensoussan [14] study the existence (and, under certain conditions, uniqueness) of a solution to a system of partial differential equations (PDEs) (namely, the Hamilton-Jacobi-Bellman (HJB) and Fokker-Planck equations) associated with a mean-field game involving Bertrand and Cournot competition among a continuum of players. In Graber and Sircar [13], existence and uniqueness of the master equation associated with a mean-field game of controls with absorption are proven, while Chan and Sircar [9] delve into dynamic mean-field games with exhaustible capacities and interactions akin to Cournot and Bertrand competitions. An optimal transport perspective on Cournot-Nash equilibria is explored in Acciaio et al. [1]. Finally, Cao et al. [6] considers stationary discounted and ergodic mean-field games of singular controls motivated by irreversible investment and provide existence and uniqueness results, as well as relations across the two classes of considered problems.

To the best of our knowledge, the existence and uniqueness of the (nonstationary) mean-field equilibrium for a mean-field model of optimal investment with an isoelastic demand function, as discussed in this paper, is presented here for the first time.

The rest of the paper is organized as follows. The stochastic mean-field game is introduced in Section 2. In Section 3 the company's optimal investment problem is solved for both the cases $T < +\infty$ and $T = +\infty$, while existence and uniqueness of the equilibrium is shown in Section 4, again for the finite and infinite time horizon cases. The deterministic version of the mean-field problem is finally considered in Section 5.

1.1. Notation. In this section we collect the main notation used in this work.

- Throughout the paper the set \mathbb{N} denotes the set of natural integers without the zero element, i.e., $\mathbb{N} = \{1, 2, \dots\}$, while \mathbb{R} denotes the set of real numbers. Whenever $T = +\infty$, the notation $[0, T]$ indicates the interval $[0, +\infty)$.
- For any $p \geq 1$, any measure space (E, \mathcal{E}, μ) , and any interval $I \subseteq \mathbb{R}$, we indicate by $L^p((E, \mathcal{E}, \mu); I)$ the set of all functions with values in I that are p -integrable with respect to μ . If $E \subseteq \mathbb{R}$, we take $\mathcal{L}(E)$ as the Lebesgue σ -algebra and as μ the Lebesgue measure, which is denoted by Leb , and we simply write $L^p(E; I)$. When computing integrals with respect to this measure, we simply write dx instead of $\text{Leb}(dx)$.
- For any $\eta > 0$ and $E, I \subseteq \mathbb{R}$, the notation $L_\eta^p(E; I)$ indicates the set of all functions $f: E \rightarrow I$ such that $t \mapsto e^{-\eta t} f(t)$ is p -integrable with respect to the Lebesgue measure.
- Given two intervals $I, J \subseteq \mathbb{R}$, we denote by $C(I; J)$ the set of all continuous functions from I to J , endowed with the usual sup-norm. The notation $C^p(I; J)$, $p \in \mathbb{N} \cup \{+\infty\}$, denotes the set of functions from I to J that are continuously differentiable p times.

2. THE STOCHASTIC MODEL

Let $(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_s)_{s \geq 0}, \mathbb{P})$ be a complete filtered probability space, with \mathbb{F} satisfying the usual assumptions, supporting an \mathbb{F} -adapted standard Brownian motion B and an independent \mathcal{F}_0 -measurable random variable ξ . Throughout the paper, $T \in (0, +\infty) \cup \{+\infty\}$ denotes a finite or infinite time horizon. In what follows, whenever $T = +\infty$ the notation $[0, T]$ indicates the interval $[0, +\infty)$.

We consider a real-valued \mathbb{F} -adapted process $X = (X_s)_{s \in [0, T]}$, satisfying the stochastic differential equation (SDE)

$$\begin{cases} dX_s = -\delta X_s ds + \sigma X_s dB_s + u_s ds, & s \in (0, T], \\ X_0 = \xi, \end{cases} \quad (2.1)$$

where $\delta, \sigma > 0$ are given coefficients, and the control process $\mathbf{u} := (u_s)_{s \in [0, T]}$ is chosen in either of the following two classes of admissible controls: if $T < +\infty$,

$$\mathcal{U}_T := \left\{ \mathbf{u} = (u_s)_{s \in [0, T]} \text{ s.t. } \mathbf{u}: \Omega \times [0, T] \rightarrow [0, +\infty) \text{ is } (\mathcal{F}_s)_{s \in [0, T]} \text{-progressively measurable and} \right.$$

$$\mathbb{E} \left[\int_0^T u_s^2 ds \right] < +\infty \}; \quad (2.2)$$

if, instead, $T = +\infty$,

$$\mathcal{U}_\infty := \left\{ \mathbf{u} = (u_s)_{s \geq 0} \text{ s.t. } \mathbf{u}: \Omega \times [0, \infty) \rightarrow [0, +\infty) \text{ is } \mathbb{F}\text{-progressively measurable,} \right. \\ \left. \mathbb{E} \left[\int_0^t u_s ds \right] < +\infty, \mathbb{P}\text{-a.s.}, \forall t \geq 0, \text{ and } \mathbb{E} \left[\int_0^{+\infty} e^{-\rho s} u_s^2 ds \right] < +\infty \right\}. \quad (2.3)$$

Whenever necessary, to stress the dependence of the solution to (2.1) on the initial condition ξ and on the control \mathbf{u} , we denote it by $X_s^{\xi, \mathbf{u}}$.

We observe that the solution to SDE (2.1) can be written explicitly (cf. [16, Problem 5.6.15]), for each $\mathbf{u} \in \mathcal{U}_T$, as

$$X_s^{\xi, \mathbf{u}} = Y_s \xi + \int_0^s \frac{Y_s}{Y_r} u_r dr, \quad s \in [0, T], \quad (2.4)$$

where

$$Y_s := e^{\sigma B_s - (\delta + \frac{\sigma^2}{2})s}, \quad s \in [0, T]. \quad (2.5)$$

In the mean-field game studied in this paper, the process X describes the private state of a representative player. In particular, it models the evolution of her/his production capacity, which depreciates at a rate δ and can be increased by choosing the investment rate $\mathbf{u} \in \mathcal{U}_T$. Note that the statistical distribution of the state of the other players in the economy does not influence the production capacity level of the representative player.

Throughout the paper, we work under the following assumption.

Assumption 2.1. *The initial condition ξ of SDE (2.1) is an \mathcal{F}_0 -measurable, positive, and integrable random variable. More precisely, $\xi > 0$, \mathbb{P} -a.s., and $0 < \mathbb{E}[\xi] < +\infty$.*

The assumption above ensures that, for any $\mathbf{u} \in \mathcal{U}_T$, $X_s^{\xi, \mathbf{u}} \geq 0$, \mathbb{P} -a.s., for all $s \in [0, T]$. Thus, the production capacity level of the representative agent is never negative. Moreover, it grants us the following result, which will be used later on.

Lemma 2.2. *Under Assumption 2.1 and for any $\mathbf{u} \in \mathcal{U}_T$, the unique solution to SDE (2.1) has finite first moment, given by*

$$\mathbb{E}[X_s^{\xi, \mathbf{u}}] = \mathbb{E}[\xi] e^{-\delta s} + \mathbb{E} \left[\int_0^s e^{-\delta(s-r)} u_r dr \right], \quad s \in [0, T]. \quad (2.6)$$

Proof. Fix ξ and \mathbf{u} as above. Using the expression of $X_s^{\xi, \mathbf{u}}$ given in (2.4) we have that

$$\mathbb{E}[X_s^{\xi, \mathbf{u}}] = \mathbb{E}[Y_s \xi] + \mathbb{E} \left[\int_0^s \frac{Y_s}{Y_r} u_r dr \right], \quad s \in [0, T].$$

Since the random variable $Y_s \xi$ is non-negative and Y_s is independent of \mathcal{F}_0 , for all $s \in [0, T]$, we can directly compute the first summand

$$\mathbb{E}[Y_s \xi] = \mathbb{E}[\xi] \mathbb{E}[Y_s] = \mathbb{E}[\xi] e^{-\delta s}.$$

For the second summand, observe that the integrand is non-negative and that, for each fixed $s \in [0, T]$, the random variable $\frac{Y_s}{Y_r}$ is independent of \mathcal{F}_r , for all $r \in [0, s]$. Therefore, applying the Fubini-Tonelli theorem,

$$\mathbb{E} \left[\int_0^s \frac{Y_s}{Y_r} u_r dr \right] = \int_0^s \mathbb{E} \left[\frac{Y_s}{Y_r} u_r \right] dr = \int_0^s \mathbb{E} \left[\frac{Y_s}{Y_r} \right] \mathbb{E}[u_r] dr = \mathbb{E} \left[\int_0^s e^{-\delta(s-r)} u_r dr \right],$$

which is finite thanks to the assumptions on \mathbf{u} . Therefore, we get (2.6). \square

In our mean-field game we assume that every player aims at maximizing the discounted net profit functional

$$J_{T,q}(\xi, \mathbf{u}) := \mathbb{E} \left[\int_0^T e^{-\rho s} \left\{ X_s^{\xi, \mathbf{u}} q_s^{-\beta} - \frac{1}{2} u_s^2 \right\} ds \right], \quad (2.7)$$

where $\rho > 0$ is a discount factor, $\beta > 0$ is a fixed parameter, and $q = (q_s)_{s \in [0, T]}$ is a given deterministic measurable function. At equilibrium, the function q will identify with the average production capacity of the whole population of agents. This is formalized in the next definition.

Definition 2.3. Fix a random variable ξ under Assumption 2.1.

A pair $(\hat{\mathbf{u}}, \hat{q})$, where $\hat{\mathbf{u}} \in \mathcal{U}_T$ and $\hat{q}: [0, T] \rightarrow (0, +\infty)$ is a measurable function, is an equilibrium of the mean-field game if

- (i) $J_{T, \hat{q}}(\xi, \hat{\mathbf{u}}) \geq J_{T, \hat{q}}(\xi, \mathbf{u})$, for all $\mathbf{u} \in \mathcal{U}_T$;
- (ii) $\hat{q}_s = \mathbb{E}[X_s^{\xi, \hat{\mathbf{u}}}]$, for all $s \in [0, T]$.

In the next sections we show that there exists a unique equilibrium by adopting a classic fixed point approach. The first step is to solve an optimization problem for a given measurable function q , representing the evolution of the average production capacity level of the agents in the economy. We show that there exists a unique explicit and deterministic optimal control $\hat{\mathbf{u}} \in \mathcal{U}_T$, depending on q , which provides us with the best response of the representative agent to the distribution of the states of the other agents in the economy, summarized by the average q .

The second step is to determine the optimally controlled dynamics of the production capacity level of the representative agent. Thanks to (2.6), we can compute explicitly $\mathbb{E}[X_s^{\xi, \hat{\mathbf{u}}}]$, $s \in [0, T]$. Since the optimal control $\hat{\mathbf{u}}$ depends on q , the fixed point argument follows from condition (ii) in Definition 2.3. We show that proving that there exists a unique equilibrium reduces to finding the unique solution of an integro-differential equation.

Remark 2.4. The profit function appearing in (2.7) – i.e., the function $(x, p) \mapsto xp^{-\beta}$ – is related to the isoelastic demand obtained from Spence-Dixit-Stiglitz preferences and can be motivated as follows.¹

Assume that each player in our economy is a firm indexed by its productivity (or size) $x > 0$. We can consider the firms' production capacities as a proxy for this index and, thus, this setting is coherent with the model introduced above. Each firm produces a single good and faces the demand function

$$\left(\frac{\pi(x)}{P}\right)^{-\gamma}, \quad \text{with } \gamma > 1,$$

where $\pi(x)$ is the price set by a firm with productivity level x , and $P = [\int_{\mathbb{R}} \pi(x)^{1-\gamma} \mu(dx)]^{1/(1-\gamma)}$ is a price index, which is computed according to the statistical distribution μ of the firms productivity in the economy. Here we are assuming that the price set by two firms having an equal productivity level is the same. In other words, the price policy of each firm is exclusively determined by its productivity.

Each firm (i.e., fix $x > 0$) aims at maximizing the function

$$\pi \mapsto \left\{ \pi \left(\frac{\pi}{P}\right)^{-\gamma} - \frac{1}{x} \left(\frac{\pi}{P}\right)^{-\gamma} \right\}.$$

It is easy to see that the maximum profit is

$$\frac{1}{\gamma-1} \left(\frac{\gamma}{\gamma-1}\right)^{-\gamma} x^{\gamma-1} P^\gamma, \quad (2.8)$$

which is attained setting the price $\pi(x) = \frac{\gamma}{(\gamma-1)x}$.

This entails that $P = \frac{\gamma}{(\gamma-1)} \left(\int_{\mathbb{R}} x^{\gamma-1} \mu(dx)\right)^{1/(1-\gamma)}$, and hence, substituting this expression into (2.8), we get that the profit is

$$\frac{1}{\gamma-1} x^{\gamma-1} p^{-\gamma}, \quad \text{with } p = \left(\int_{\mathbb{R}} x^{\gamma-1} \mu(dx)\right)^{\frac{1}{\gamma-1}}. \quad (2.9)$$

Therefore, setting $\beta = \gamma = 2$, we get the profit function $(x, p) \mapsto xp^{-\beta}$ appearing in our model. Note, however, that in the discounted net profit functional introduced in (2.7) we do not restrict the choice of the parameter β to the one dictated by the economic application outlined above. Instead,

¹See also the discussion in Achdou et al. [2, p. 7, Footnote 5] concerning the the model in Luttmer [18].

we consider any possible value $\beta > 0$, as the mathematical results that we prove can be stated in this more general framework.

The discussion above highlights that it is possible to consider more general forms of interaction between players in the profit function. In this paper we consider the average of the production capacities of the firms in the economy, but in (2.9) the geometric average appears. More generally, one can consider

$$p = F \left(\int_{\mathbb{R}} f(x) \mu(dx) \right),$$

for some $F, f: \mathbb{R} \rightarrow [0, +\infty)$ strictly increasing, see, e.g., [6]. However, as will be clear later on, the analysis carried out in this work heavily relies on the fact that the interaction between players is through the average of the production capacities. Indeed, thanks to this feature, we will find an explicit characterization of the solution to the mean-field game introduced above.

3. THE OPTIMIZATION PROBLEM

In this section we consider the optimization problem associated to the mean-field game described in Section 2. We show that, for each possible choice (in a suitable class of measurable functions) of the function q appearing in the discounted net profit functional (2.7), the corresponding maximization problem has a unique solution and we compute explicitly the associated optimal control.

We divide our analysis into two subsections, the first devoted to the finite time horizon case, the second one to the infinite time horizon case.

3.1. The finite time horizon case. Let us consider the finite time horizon case, i.e., fix $T < +\infty$. For each fixed $q: [0, T] \rightarrow (0, +\infty)$ such that $q^{-\beta} \in L^1((0, T]; (0, +\infty))$, we consider the problem of maximizing the functional

$$J_{T,q}(\xi, \mathbf{u}) := \mathbb{E} \left[\int_0^T e^{-\rho s} \left\{ X_s^{\xi, \mathbf{u}} q_s^{-\beta} - \frac{1}{2} u_s^2 \right\} ds \right], \quad (3.1)$$

where ξ satisfies Assumption 2.1, \mathbf{u} is chosen in the class of admissible controls introduced in (2.2), and $X^{\xi, \mathbf{u}}$ is the unique solution of SDE (2.1). We also introduce the value function corresponding to the optimization problem, namely,

$$V_{T,q}(\xi) := \sup_{\mathbf{u} \in \mathcal{U}_T} J_{T,q}(\xi, \mathbf{u}), \quad \xi \in L^1((\Omega, \mathcal{F}_0, \mathbb{P}); (0, +\infty)). \quad (3.2)$$

In the next proposition we are going to show important properties of the functional $J_{T,q}$.

Proposition 3.1. *Fix a random variable ξ satisfying Assumption 2.1, $\mathbf{u} \in \mathcal{U}_T$, and $q: [0, T] \rightarrow (0, +\infty)$ such that $q^{-\beta} \in L^1((0, T]; (0, +\infty))$. Then, the functional $J_{T,q}$ defined in (3.1) is finite and verifies*

$$J_{T,q}(\xi, \mathbf{u}) = \mathbb{E}[\xi] z_0^{(T,q)} + \mathbb{E} \left[\int_0^T e^{-\rho s} \left\{ z_s^{(T,q)} u_s - \frac{1}{2} u_s^2 \right\} ds \right], \quad (3.3)$$

where $z^{(T,q)}: [0, T] \rightarrow [0, +\infty)$ is the deterministic function given by

$$z_s^{(T,q)} := \int_s^T e^{-(\rho+\delta)(r-s)} q_r^{-\beta} dr, \quad s \in [0, T]. \quad (3.4)$$

Moreover, if ξ, ξ' both verify Assumption 2.1 and are such that $\mathbb{E}[\xi] = \mathbb{E}[\xi']$, then $J_{T,q}(\xi; \mathbf{u}) = J_{T,q}(\xi'; \mathbf{u})$, for any $\mathbf{u} \in \mathcal{U}_T$.

Proof. Fix $\xi \in L^1((\Omega, \mathcal{F}_0, \mathbb{P}); (0, +\infty))$, and $\mathbf{u} \in \mathcal{U}_T$. Since the assumptions of Lemma 2.2 are verified, we can use (2.6) and apply the Fubini-Tonelli theorem (note that the integrand is non-negative) to get

$$\mathbb{E} \left[\int_0^T e^{-\rho s} X_s^{\xi, \mathbf{u}} q_s^{-\beta} ds \right] = \mathbb{E}[\xi] \int_0^T e^{-\rho s} e^{-\delta s} q_s^{-\beta} ds + \mathbb{E} \left[\int_0^T \int_0^s e^{-\rho s} e^{-\delta(s-r)} q_s^{-\beta} u_r dr ds \right].$$

Clearly, the first summand of the last equality is finite, thanks to the assumptions on ξ and q . Moreover,

$$\mathbb{E}[\xi] \int_0^T e^{-\rho s} e^{-\delta s} q_s^{-\beta} ds = \mathbb{E}[\xi] z_0^{(T,q)}.$$

Also the second summand is finite, thanks to the assumptions on q and \mathbf{u} . Exchanging the order of the two time integrals we get

$$\begin{aligned} & \mathbb{E} \left[\int_0^T \int_0^s e^{-\rho s} e^{-\delta(s-r)} q_s^{-\beta} u_r \, dr \, ds \right] = \mathbb{E} \left[\int_0^T e^{\delta r} u_r \int_r^T e^{-(\rho+\delta)s} q_s^{-\beta} \, ds \, dr \right] \\ & = \mathbb{E} \left[\int_0^T e^{-\rho r} u_r \int_r^T e^{-(\rho+\delta)(s-r)} q_s^{-\beta} \, ds \, dr \right] = \mathbb{E} \left[\int_0^T e^{-\rho r} z_r^{(T,q)} u_r \, dr \right]. \end{aligned}$$

It easily follows that $J_{T,q}$ is finite and that Equation (3.3) holds, which also entails the last statement of the Proposition. \square

Remark 3.2. *The last statement in Proposition 3.1 implies that the value function $V_{T,q}$ depends on the initial condition ξ only through its average. More precisely, if ξ, ξ' both verify Assumption 2.1 and are such that $\mathbb{E}[\xi] = \mathbb{E}[\xi']$, then $V_{T,q}(\xi) = V_{T,q}(\xi')$.*

Thanks to the Proposition above we can find the optimal control for our optimization problem and this, as a byproduct, allows us to explicitly compute the value function.

Theorem 3.3. *Fix a random variable ξ satisfying Assumption 2.1 and $q: [0, T] \rightarrow (0, +\infty)$ such that $q^{-\beta} \in L^1((0, T]; (0, +\infty))$. Then, $\widehat{\mathbf{u}}^{(T,q)} := z^{(T,q)} \in \mathcal{U}_T$, where $z^{(T,q)}$ is the function defined in (3.4), is an optimal control for problem (3.2), which is deterministic and independent of ξ .*

Moreover, $\widehat{\mathbf{u}}^{(T,q)}$ is essentially unique, i.e., if $\bar{\mathbf{u}}^{(T,q)} \in \mathcal{U}_T$ is an optimal control for problem (3.2) different from $\widehat{\mathbf{u}}^{(T,q)}$, then

$$\bar{u}_s^{(T,q)} = \widehat{u}_s^{(T,q)}, \quad \text{for } \mathbb{P} \otimes \text{Leb-a.e. } (\omega, s) \in \Omega \times [0, T].$$

Finally, the value of the optimization problem admits the explicit expression

$$V_{T,q}(\xi) = \mathbb{E}[\xi] z_0^{(T,q)} + \frac{1}{2} \int_0^T e^{-\rho s} (z_s^{(T,q)})^2 \, ds, \quad (3.5)$$

and the optimally controlled state process $X^{\xi, \widehat{\mathbf{u}}^{(T,q)}}$ is given by

$$X_s^{\xi, \widehat{\mathbf{u}}^{(T,q)}} = Y_s \xi + \int_0^s \frac{Y_s}{Y_r} z_r^{(T,q)} \, dr, \quad s \in [0, T], \quad (3.6)$$

where Y is the process defined in (2.5).

Proof. Fix $\xi \in L^1((\Omega, \mathcal{F}_0, \mathbb{P}); (0, +\infty))$. From (3.3) it immediately follows that

$$V_{T,q}(\xi) = \mathbb{E}[\xi] z_0^{(T,q)} + \sup_{\mathbf{u} \in \mathcal{U}_T} \mathbb{E} \left[\int_0^T e^{-\rho s} \left\{ z_s^{(T,q)} u_s - \frac{1}{2} u_s^2 \right\} \, ds \right]. \quad (3.7)$$

Hence, if we can find $\widehat{\mathbf{u}}^{(T,q)} \in \mathcal{U}_T$ that maximizes the integrand $z_s^{(T,q)} u_s - \frac{1}{2} u_s^2$, for $\mathbb{P} \otimes \text{Leb-a.e. } (\omega, s) \in \Omega \times [0, T]$, then $\widehat{\mathbf{u}}^{(T,q)}$ it must be optimal. Clearly, the integrand is maximized in the sense above if we take $\widehat{\mathbf{u}}^{(T,q)} = z^{(T,q)}$. Its admissibility is a consequence of the fact that $z^{(T,q)}$ is bounded. Therefore, $\widehat{\mathbf{u}}^{(T,q)}$ is optimal, and clearly essentially unique in the sense specified above. Substituting its definition in (3.7) we immediately get (3.5) and (3.6). \square

3.2. The infinite time horizon case. We analyze now the infinite time horizon case, i.e., we set $T = +\infty$. For each fixed $q: [0, +\infty) \rightarrow (0, +\infty)$ such that $q^{-\beta} \in L^1((0, +\infty); (0, +\infty))$, we consider the problem of maximizing the functional

$$J_{\infty,q}(\xi, \mathbf{u}) := \mathbb{E} \left[\int_0^{+\infty} e^{-\rho s} \left\{ X_s^{\xi, \mathbf{u}} q_s^{-\beta} - \frac{1}{2} u_s^2 \right\} \, ds \right], \quad (3.8)$$

where ξ satisfies Assumption 2.1, \mathbf{u} is chosen in the class of admissible controls defined in (2.3), and $X^{\xi, \mathbf{u}}$ is the unique solution of SDE (2.1). We also introduce the value function corresponding to the optimization problem, namely,

$$V_{\infty,q}(\xi) := \sup_{\mathbf{u} \in \mathcal{U}_{\infty}} J_{\infty,q}(\xi, \mathbf{u}), \quad \xi \in L^1((\Omega, \mathcal{F}_0, \mathbb{P}); (0, +\infty)). \quad (3.9)$$

The next result is analogous to Proposition 3.1. The proof proceeds along the same lines and, thus, we omit it.

Proposition 3.4. Fix a random variable ξ satisfying Assumption 2.1 and $\mathbf{u} \in \mathcal{U}_\infty$. Consider a function $q: [0, +\infty) \rightarrow (0, +\infty)$ such that $q^{-\beta} \in L^1_{\rho+\delta}((0, +\infty); (0, +\infty))$ and such that the function $z^{(\infty, q)}: [0, +\infty) \rightarrow (0, +\infty)$ given by

$$z_s^{(\infty, q)} := \int_s^{+\infty} e^{-(\rho+\delta)(r-s)} q_r^{-\beta} dr, \quad s \geq 0, \quad (3.10)$$

is bounded on $[0, +\infty)$.

Then, the functional $J_{\infty, q}$ defined in (3.8) is finite and verifies

$$J_{\infty, q}(\xi, \mathbf{u}) = \mathbb{E}[\xi] z_0^{(\infty, q)} + \mathbb{E} \left[\int_0^{+\infty} e^{-\rho s} \left\{ z_s^{(\infty, q)} u_s - \frac{1}{2} u_s^2 \right\} ds \right]. \quad (3.11)$$

Moreover, if ξ, ξ' both verify Assumption 2.1 and are such that $\mathbb{E}[\xi] = \mathbb{E}[\xi']$, then $J_{\infty, q}(\xi, \mathbf{u}) = J_{\infty, q}(\xi', \mathbf{u})$, for any $\mathbf{u} \in \mathcal{U}_\infty$.

Remark 3.5. Also in this case (cf. Remark 3.2) the last statement in Proposition 3.4 entails that the value function $V_{\infty, q}$ depends on the initial condition ξ only through its average.

Thanks to Proposition 3.4, also in the infinite time horizon case we can find the optimal control for problem (3.9) and the explicit form of the corresponding value function, as stated in the next theorem. Its proof is omitted, being similar to that of Theorem 3.3.

Theorem 3.6. Fix a random variable ξ satisfying Assumption 2.1. Consider a function $q: [0, +\infty) \rightarrow (0, +\infty)$ such that $q^{-\beta} \in L^1_{\rho+\delta}((0, +\infty); (0, +\infty))$ and such that the function $z^{(\infty, q)}$, defined in (3.10) is bounded on $[0, +\infty)$.

Then, $\hat{\mathbf{u}}^{(\infty, q)} := z^{(\infty, q)} \in \mathcal{U}_\infty$ is an optimal control for problem (3.9), which is deterministic and independent of ξ .

Moreover, $\hat{\mathbf{u}}^{(\infty, q)}$ is essentially unique, i.e., if $\bar{\mathbf{u}}^{(\infty, q)} \in \mathcal{U}_\infty$ is an optimal control for problem (3.9) different from $\hat{\mathbf{u}}^{(\infty, q)}$, then

$$\bar{u}_s^{(\infty, q)} = \hat{u}_s^{(\infty, q)}, \quad \text{for } \mathbb{P} \otimes \text{Leb-a.e. } (\omega, s) \in \Omega \times [0, +\infty).$$

Finally, the value function of the optimization problem admits the explicit expression

$$V_{\infty, q}(\xi) = \mathbb{E}[\xi] z_0^{(\infty, q)} + \frac{1}{2} \int_0^{+\infty} e^{-\rho s} (z_s^{(\infty, q)})^2 ds, \quad (3.12)$$

and the optimally controlled state process $X^{\xi, \hat{\mathbf{u}}^{(\infty, q)}}$ is given by

$$X_s^{\xi, \hat{\mathbf{u}}^{(\infty, q)}} = Y_s \xi + \int_0^s \frac{Y_s}{Y_r} z_r^{(\infty, q)} dr, \quad s \geq 0, \quad (3.13)$$

where Y is the process defined in (2.5).

Remark 3.7. All the results of this section can be straightforwardly extended to the case where the optimization problem starts at a time $t > 0$. More precisely, setting the initial condition $X_t = x > 0$ in SDE (2.1), we get that (2.6) becomes

$$\mathbb{E}[X_s^{\xi, \mathbf{u}}] = \mathbb{E}[\xi] e^{-\delta(s-t)} + \mathbb{E} \left[\int_t^s e^{-\delta(s-r)} u_r dr \right], \quad s \in [t, T],$$

while the expressions of the functional $J_{T, q}$ and of the value function $V_{T, q}$ (now both dependent on t) given in (3.3) and (3.5), if $T < +\infty$, and in (3.11) and (3.12), if $T = +\infty$, become

$$J_{T, q}(t, \xi, \mathbf{u}) = \mathbb{E}[\xi] e^{-\rho t} z_t^{(T, q)} + \mathbb{E} \left[\int_t^T e^{-\rho s} \left\{ z_s^{(T, q)} u_s - \frac{1}{2} u_s^2 \right\} ds \right],$$

$$V_{T, q}(t, \xi) = \mathbb{E}[\xi] e^{-\rho t} z_t^{(T, q)} + \frac{1}{2} \int_t^T e^{-\rho s} (z_s^{(T, q)})^2 ds,$$

with optimal control $\hat{\mathbf{u}}^{(T, q)} := z^{(T, q)}$, where

$$z_s^{(T, q)} := \int_s^T e^{-(\rho+\delta)(r-s)} q_r^{-\beta} dr, \quad s \in [t, T].$$

4. EXISTENCE AND UNIQUENESS OF EQUILIBRIA

In this section we discuss the existence and the uniqueness of an equilibrium for the mean-field game introduced in Section 2.

As usual in mean-field games theory, looking for an equilibrium boils down to finding a fixed point of a suitable map. According to Definition 2.3, in our case this map is $(q_s)_{s \in [0, T]} \mapsto \left(\mathbb{E}[X_s^{\xi, \hat{\mathbf{u}}^{(T, q)}}] \right)_{s \in [0, T]}$,

where $\hat{\mathbf{u}}^{(T, q)}$ is the optimal control for the maximization problem studied in Section 3, whose expression is given either in Theorem 3.3, in the case $T < +\infty$, or in Theorem 3.6, in the case $T = +\infty$.

The next result formalizes this fact, showing also that the fixed point map is precisely the solution map of an integral equation.

Theorem 4.1. *Let us fix a random variable ξ satisfying Assumption 2.1.*

Consider the mean-field game introduced in Section 2 in the finite time horizon case, i.e., $T < +\infty$. Then,

(i) *If there exists an equilibrium $(\hat{\mathbf{u}}, \hat{q})$ in the sense of Definition 2.3, such that*

$$\hat{q}^{-\beta} \in L^1((0, T]; (0, +\infty)), \quad (4.1)$$

then \hat{q} is a solution to the integral equation

$$y_s = e^{-\delta s} \mathbb{E}[\xi] + \int_0^s e^{-\delta(s-r)} \int_r^T e^{-(\rho+\delta)(u-r)} y_u^{-\beta} du dr, \quad s \in [0, T]. \quad (4.2)$$

(ii) *Vice versa, if there exist a unique solution \hat{q} to (4.2) satisfying (4.1), then there exists a unique equilibrium $(\hat{\mathbf{u}}, \hat{q}) = (z^{(T, \hat{q})}, \hat{q})$ of the mean-field game among all equilibria $(\tilde{\mathbf{u}}, \tilde{q})$ such that \tilde{q} verifies (4.1), where $z^{(T, q)}$ is the function defined in (3.4).*

Consider, instead, the same mean-field game in the infinite time horizon case, i.e., $T = +\infty$. Then,

(iii) *If there exists an equilibrium $(\hat{\mathbf{u}}, \hat{q})$ in the sense of Definition 2.3, such that*

$$\hat{q}^{-\beta} \in L_{\rho+\delta}^1((0, +\infty); (0, +\infty)) \quad \text{and} \quad z^{(\infty, \hat{q})} \text{ is bounded on } [0, +\infty), \quad (4.3)$$

where $z^{(\infty, q)}$ is the function defined in (3.10), then \hat{q} is a solution to the integral equation

$$y_s = e^{-\delta s} \mathbb{E}[\xi] + \int_0^s e^{-\delta(s-r)} \int_r^{+\infty} e^{-(\rho+\delta)(u-r)} y_u^{-\beta} du dr, \quad s \geq 0. \quad (4.4)$$

(iv) *Vice versa, if there exist a unique solution \hat{q} to (4.4) satisfying (4.3), then there exists a unique equilibrium $(\hat{\mathbf{u}}, \hat{q}) = (z^{(\infty, \hat{q})}, \hat{q})$ of the mean-field game among all equilibria $(\tilde{\mathbf{u}}, \tilde{q})$ such that \tilde{q} verifies (4.3).*

Proof. We prove only the first two statements in the finite time horizon case, as the case $T = +\infty$ is analogous.

(i). Let $(\hat{\mathbf{u}}, \hat{q})$ be an equilibrium of the mean-field game, with $\hat{\mathbf{u}} \in \mathcal{U}_T$ and $\hat{q}: [0, T] \rightarrow (0, +\infty)$, such that $\hat{q}^{-\beta} \in L^1((0, T]; (0, +\infty))$. Since point (i) of Definition 2.3 holds, from Theorem 3.3 we have that $\hat{\mathbf{u}} = z^{(T, \hat{q})}$, for $\mathbb{P} \otimes \text{Leb}$ -a.e. $(\omega, s) \in \Omega \times [0, T]$. Combining the expression of $z^{(T, q)}$, given in (3.4), and (2.6), we get

$$\begin{aligned} \mathbb{E}[X_s^{\xi, \hat{\mathbf{u}}}] &= \mathbb{E}[\xi] e^{-\delta s} + \mathbb{E} \left[\int_0^s e^{-\delta(s-r)} \hat{u}_r dr \right] = \mathbb{E}[\xi] e^{-\delta s} + \mathbb{E} \left[\int_0^s e^{-\delta(s-r)} z_r^{(T, \hat{q})} dr \right] \\ &= e^{-\delta s} \mathbb{E}[\xi] + \int_0^s e^{-\delta(s-r)} \int_r^T e^{-(\rho+\delta)(u-r)} \hat{q}_u^{-\beta} du dr, \quad s \in [0, T]. \end{aligned} \quad (4.5)$$

Therefore, by point (ii) of Definition 2.3, we get that \hat{q} is a solution to (4.2).

(ii). Let \hat{q} be the unique solution to (4.2) such that $\hat{q}^{-\beta} \in L^1((0, T]; (0, +\infty))$. Define $\hat{\mathbf{u}} := z^{(T, \hat{q})}$ according to (3.4). Then, by Theorem 3.3, we know that $\hat{\mathbf{u}}$ is optimal for the problem of maximizing $J_{T, \hat{q}}$, i.e., that point (i) of Definition 2.3 holds. We also have that equation (4.5) above is verified, and hence, since we assumed that \hat{q} solves (4.2), we get that $\hat{q}_s = \mathbb{E}[X_s^{\xi, \hat{\mathbf{u}}}]$, for all $s \in [0, T]$, i.e.,

that point (ii) of Definition 2.3 also holds. This means that $(\widehat{\mathbf{u}}, \widehat{q})$ is an equilibrium of the mean-field game.

If $(\widetilde{\mathbf{u}}, \widetilde{q})$ is any other equilibrium of the mean-field game, such that $\widetilde{q}^{-\beta} \in L^1((0, T]; (0, +\infty))$, then necessarily $\widetilde{\mathbf{u}} = z^{(T, \widetilde{q})}$, for $\mathbb{P} \otimes \text{Leb}$ -a.e. $(\omega, s) \in \Omega \times [0, T]$, by Theorem 3.3. Therefore, we can repeat the same argument of the proof of point (i) to get that \widetilde{q} is a solution to (4.2), and hence that $(\widetilde{\mathbf{u}}, \widetilde{q}) = (\widehat{\mathbf{u}}, \widehat{q})$, since we assumed that (4.2) has a unique solution. \square

Thanks to Theorem 4.1 we can reduce our search for a unique equilibrium of our mean-field game to proving existence and uniqueness of a solution to the integral equation (4.2), in the finite time horizon case, or (4.4), in the infinite time horizon case. This, in turn, is equivalent to showing existence and uniqueness of a solution $y: [0, T] \rightarrow (0, +\infty)$ to the integro-differential equation

$$\begin{cases} \frac{d}{ds} y_s = -\delta y_s + \int_s^T e^{-(\rho+\delta)(u-s)} y_u^{-\beta} du, & s \in [0, T], \\ y_0 = x. \end{cases} \quad (4.6)$$

for any initial datum $x > 0$.

Indeed, this equivalence follows from the next two definitions and from Proposition 4.4 below, whose proof is omitted, being a standard result.

Definition 4.2. *A function $y \in C^1([0, T]; (0, +\infty))$ that satisfies (4.6) for all $s \in [0, T]$ is called a classical solution to (4.6).*

Definition 4.3. *A function $y \in C([0, T]; (0, +\infty))$ that satisfies the integral equation*

$$y_s = e^{-\delta s} x + \int_0^s e^{-\delta(s-r)} \int_r^T e^{-(\rho+\delta)(u-r)} y_u^{-\beta} du dr, \quad s \in [0, T],$$

is called a mild solution to (4.6).

Proposition 4.4. *Any classical solution $y \in C^1([0, T]; (0, +\infty))$ to (4.6) is also a mild solution. Vice versa, any mild solution $y \in C([0, T]; (0, +\infty))$ to (4.6) is continuously differentiable in $[0, T]$ and also a classical solution.*

Remark 4.5. *From now on, we will just speak of a solution (in either sense) to (4.6), unless otherwise specified.*

It is convenient to introduce a further integro-differential equation, namely

$$\begin{cases} \frac{d}{ds} z_s = e^{(\rho+2\delta)s} \int_s^T e^{-(\rho+\delta-\delta\beta)u} z_u^{-\beta} du, & s \in [0, T], \\ z_0 = x, \end{cases} \quad (4.7)$$

and the corresponding integral equation

$$z_s = x + \int_0^s e^{(\rho+2\delta)r} \int_r^T e^{-(\rho+\delta-\delta\beta)u} z_u^{-\beta} du dr, \quad s \in [0, T], \quad (4.8)$$

which we consider for any initial datum $x > 0$.

The notions of classical and mild solution to (4.7) are analogous to those given in Definitions 4.2 and 4.3. Also in this case, a result similar to Proposition 4.4 shows that these two solution concepts are equivalent.

The following result, whose proof is omitted being based on routine computations, shows that solutions to (4.6) can be expressed in terms of those of (4.7) and vice versa. Thus, we can study either of these two equations, depending on convenience.

Proposition 4.6. *If $y: [0, T] \rightarrow (0, +\infty)$ is a solution to (4.6), then $z_s := e^{\delta s} y_s$, $s \in [0, T]$ is a solution to (4.7). Vice versa, if $z: [0, T] \rightarrow (0, +\infty)$ is a solution to (4.7), then $y_s := e^{-\delta s} z_s$, $s \in [0, T]$ is a solution to (4.6).*

Remark 4.7. *Note that equations (4.6) and (4.7) do not fall into the standard theory of integro-differential equations, as we have an initial-value problem with the integral being backward in time.*

Now, we proceed to study existence and uniqueness of a solution to (4.6) or, equivalently, (4.7). As in the previous section, we are going to divide our analysis into two subsections, the first devoted to the finite time horizon case, the second one to the infinite time horizon case.

4.1. The finite time horizon case. Let us fix $T < +\infty$ and $x > 0$. We observe, first, that solutions of (4.7) satisfy suitable a priori estimates.

Proposition 4.8. *Let $z \in C([0, T]; (0, +\infty))$ be a solution to (4.7). Then, for all $s \in [0, T]$,*

$$z_s \geq \underline{\alpha}_s^x, \quad \text{and} \quad z_s \leq \bar{\alpha}_s^x, \quad (4.9)$$

where

$$\underline{\alpha}_s^x := x, \quad s \in [0, T], \quad (4.10)$$

and, for any $s \in [0, T]$,

$$\bar{\alpha}_s^x := \begin{cases} x - \frac{x^{-\beta}}{\rho + \delta - \delta\beta} \left\{ \frac{1 - e^{\delta(1+\beta)s}}{\delta(1+\beta)} - e^{-(\rho+\delta-\delta\beta)T} \frac{1 - e^{(\rho+2\delta)s}}{\rho + 2\delta} \right\}, & \text{if } \beta \neq 1 + \frac{\rho}{\delta}, \\ x - \frac{x^{-\beta}}{(\rho + 2\delta)^2} \left\{ [1 + (\rho + 2\delta)(T - s)]e^{(\rho+2\delta)s} - [1 + (\rho + 2\delta)T] \right\}, & \text{if } \beta = 1 + \frac{\rho}{\delta}. \end{cases} \quad (4.11)$$

Proof. Since $z_s > 0$, for all $s \in [0, T]$, the right hand side of (4.8) is positive, and hence, again for all $s \in [0, T]$,

$$z_s \geq x, \quad \text{and} \quad z_s^{-\beta} \leq x^{-\beta}, \quad (4.12)$$

which provides us with the lower bound. To get the upper bound, it is enough to substitute the second inequality in (4.12) into the right hand side of (4.8) and compute the integrals. \square

Given the result above our goal will be to show that (4.7) admits a unique solution in the (nonempty) set

$$\mathcal{C}_x := \{f \in C([0, T]; (0, +\infty)) : \underline{\alpha}_s^x \leq f_s \leq \bar{\alpha}_s^x, \forall s \in [0, T]\}. \quad (4.13)$$

Theorem 4.9. *For each fixed $x > 0$, there exists a unique solution to the integro-differential equation (4.7) in the set \mathcal{C}_x defined in (4.13), which is also twice continuously differentiable. Consequently, the integro-differential equation (4.6) admits a unique solution y . Such solution belongs to $C^2([0, T]; (0, +\infty))$ and satisfies*

$$e^{-\delta s} x \leq y_s \leq e^{-\delta s} \bar{\alpha}_s^x, \quad s \in [0, T]. \quad (4.14)$$

Proof. Fix $x > 0$. Showing that (4.7) admits a unique solution is equivalent to proving that the operator Γ , defined on $C([0, T]; (0, +\infty))$ and given by

$$\Gamma(f)_s := x + \int_0^s e^{(\rho+2\delta)r} \int_r^T e^{-(\rho+\delta-\delta\beta)u} f_u^{-\beta} du dr, \quad s \in [0, T], \quad (4.15)$$

admits a unique fixed point in \mathcal{C}_x .

It is easy to verify that \mathcal{C}_x is a closed, bounded, and convex subset of $C([0, T]; (0, +\infty))$ (which is endowed with the usual sup-norm). Moreover, Γ maps \mathcal{C}_x into itself. Indeed, from (4.15) we readily verify that, for all $f \in \mathcal{C}_x$ and all $s \in [0, T]$, $\Gamma(f)_s \geq \underline{\alpha}_s^x$, since f is a positive function; moreover, noting that for all $f, g \in \mathcal{C}_x$

$$f_s \leq g_s \implies \Gamma(f)_s \geq \Gamma(g)_s, \quad s \in [0, T],$$

we immediately check that $\Gamma(f)_s \leq \Gamma(\bar{\alpha}^x)_s = \bar{\alpha}_s^x$. Finally, Γ is compact in \mathcal{C}_x as a consequence of the Ascoli-Arzelà theorem. Indeed, any sequence in \mathcal{C}_x is equi-bounded, thanks to the definition of \mathcal{C}_x , and equi-continuous since, for all $f \in \mathcal{C}_x$, $\Gamma(f)$ is differentiable and

$$\sup_{f \in \mathcal{C}} \sup_{s \in [0, T]} |\Gamma(f)'_s| = \sup_{f \in \mathcal{C}} \sup_{s \in [0, T]} \left\{ e^{(\rho+2\delta)s} \int_s^T e^{-(\rho+\delta-\delta\beta)u} f_u^{-\beta} du \right\} \leq \sup_{s \in [0, T]} \frac{d}{ds} \bar{\alpha}_s^x < +\infty.$$

Therefore, by Schauder's fixed point theorem the map Γ has a fixed point $z \in \mathcal{C}_x$, i.e., z verifies (4.8). Hence, z is a solution of equation (4.7). The fact that $z \in C^2([0, T]; (0, +\infty))$ simply follows by differentiating twice (4.8).

To show uniqueness, assume that two different solutions $z, \tilde{z} \in \mathcal{C}_x$ of (4.7) exist. It is convenient to write the integro-differential equation satisfied by $w_s := z_{T-s}$ and $\tilde{w}_s := \tilde{z}_{T-s}$, $s \in [0, T]$,

$$\begin{cases} \frac{d}{ds} w_s = -e^{\delta(1+\beta)(T-s)} \int_0^s e^{-(\rho+\delta-\delta\beta)(s-r)} w_r^{-\beta} dr, & s \in [0, T], \\ w_T = x. \end{cases} \quad (4.16)$$

From this equation we deduce that

$$w'_s - \tilde{w}'_s = -e^{\delta(1+\beta)(T-s)} \int_0^s e^{-(\rho+\delta-\delta\beta)(s-r)} \left[w_r^{-\beta} - \tilde{w}_r^{-\beta} \right] dr, \quad s \in [0, T]. \quad (4.17)$$

The following two cases may occur:

Case 1. Consider $w_0 \neq \tilde{w}_0$ and assume, without loss of generality, that $w_0 > \tilde{w}_0$. Let us define

$$s^* := \inf\{u > 0 : w_u = \tilde{w}_u\}.$$

For each fixed $s \in [0, s^*)$, we have that $w_s > \tilde{w}_s$ and, hence, $w_r^{-\beta} - \tilde{w}_r^{-\beta} < 0$, for all $r \in [0, s]$; thus, we deduce from (4.17) that $w'_s - \tilde{w}'_s > 0$, i.e., that the distance between the two solutions increases in time. Therefore, $s^* = +\infty$, but this contradicts the fact that $w_T = \tilde{w}_T$.

Case 2. Consider $w_0 = \tilde{w}_0$ and note that both w and \tilde{w} are bounded below by the constant x , since $z, \tilde{z} \in \mathcal{C}_x$. Observe, also, that the function $w \mapsto w^{-\beta}$ is Lipschitz in $[x, +\infty)$, with Lipschitz constant $L := x^{-\beta-1}$. Therefore, integrating (4.17) on $[0, s]$ we get

$$w_s - \tilde{w}_s = - \int_0^s e^{\delta(1+\beta)(T-u)} \int_0^u e^{-(\rho+\delta-\delta\beta)(u-r)} \left[w_r^{-\beta} - \tilde{w}_r^{-\beta} \right] dr du, \quad s \in [0, T],$$

whence

$$|w_s - \tilde{w}_s| \leq L \int_0^s \left\{ e^{\delta(1+\beta)(T-u)} \int_0^u e^{-(\rho+\delta-\delta\beta)(u-r)} dr \right\} \sup_{r \in [0, u]} |w_r - \tilde{w}_r| du, \quad s \in [0, T].$$

Since $s \mapsto \sup_{r \in [0, s]} |w_r - \tilde{w}_r|$ is bounded on $[0, T]$, applying Gronwall's lemma we get that $w_s = \tilde{w}_s$ for all $s \in [0, T]$, which contradicts the assumption that w and \tilde{w} are different.

Therefore, equation (4.7) admits a unique solution in \mathcal{C}_x . Finally, from Proposition 4.6, we immediately deduce that also (4.6) has a unique solution in $C^2([0, T]; (0 + \infty))$ satisfying the bounds in (4.14). \square

Thanks to the result above, we can establish the following.

Proposition 4.10. *For each fixed random variable ξ satisfying Assumption 2.1, there exists a unique equilibrium (\hat{u}, \hat{q}) for the mean-field game with finite time horizon, among all equilibria verifying the integrability condition (4.1).*

More precisely, $\hat{u} = z^{(T, \hat{q})}$, where $z^{(T, \hat{q})}$ is the function defined in (3.4), and \hat{q} is the unique solution to (4.6), with initial condition $x = \mathbb{E}[\xi]$.

Proof. Let \hat{q} be the unique solution to (4.6), with $x = \mathbb{E}[\xi]$. Since \hat{q} satisfies the bounds (4.14) and since the functions $s \mapsto e^{-\delta s} x$ and $s \mapsto e^{-\delta s} \bar{\alpha}_s^x$ are continuous on $[0, T]$, hence therein bounded, we deduce that also \hat{q} is bounded on $[0, T]$. More precisely,

$$0 < m \leq \hat{q}_s \leq M, \quad \forall s \in [0, T], \quad \text{with } m := e^{-\delta T} x \text{ and } M := \sup_{s \in [0, T]} e^{-\delta s} \bar{\alpha}_s^x. \quad (4.18)$$

Therefore, \hat{q} satisfies the integrability condition (4.1) and applying Theorem 4.1-(ii) we get the result. \square

4.2. The infinite time horizon case. We discuss now the infinite time horizon case, i.e., we fix $T = +\infty$. Here we work under the assumption $\beta < 1 + \frac{\rho}{\delta}$, which will be used to guarantee convergence of the integral appearing in (4.7).

A reasoning similar to the one used to prove Proposition 4.8 shows that the following holds.

Proposition 4.11. *Let $z \in C([0, +\infty); (0, +\infty))$ be a solution to (4.7), with $T = +\infty$, and assume that $\beta < 1 + \frac{\rho}{\delta}$. Then, for all $s \geq 0$,*

$$z_s \geq \underline{\alpha}_s^{x, \infty}, \quad \text{and} \quad z_s \leq \bar{\alpha}_s^{x, \infty}. \quad (4.19)$$

where

$$\underline{\alpha}_s^{x, \infty} := x, \quad s \geq 0, \quad (4.20)$$

and

$$\bar{\alpha}_s^{x, \infty} := x - \frac{x^{-\beta}}{\rho + \delta - \delta\beta} \left\{ \frac{1 - e^{\delta(1+\beta)s}}{\delta(1+\beta)} \right\}, \quad s \geq 0. \quad (4.21)$$

We aim at showing that (4.7) admits a unique solution in the (nonempty) set

$$\mathcal{C}_{x, \infty} := \{f \in C([0, +\infty); (0, +\infty)) : \underline{\alpha}_s^{x, \infty} \leq f_s \leq \bar{\alpha}_s^{x, \infty}, \forall s \geq 0\}. \quad (4.22)$$

Note, in particular, that for any $f \in \mathcal{C}_{x, \infty}$ we have that

$$\int_s^{+\infty} e^{-(\rho+\delta-\delta\beta)u} f_u^{-\beta} du \leq \int_s^{+\infty} e^{-(\rho+\delta-\delta\beta)u} x^{-\beta} du = \frac{x^{-\beta}}{\rho + \delta - \delta\beta} e^{-(\rho+\delta-\delta\beta)s} < +\infty, \quad \forall s \geq 0,$$

as f is bounded below by the constant x and we assumed $\beta < 1 + \frac{\rho}{\delta}$.

Theorem 4.12. *Let $T = +\infty$ and assume that $\beta < 1 + \frac{\rho}{\delta}$. Then, for each fixed $x > 0$, there exists a unique solution to the integro-differential equation (4.7) in the set $\mathcal{C}_{x, \infty}$ defined in (4.22), which is also twice continuously differentiable. Moreover, the integro-differential equation (4.6) admits a unique solution y . Such solution belongs to $C^2([0, +\infty); (0, +\infty))$ and satisfies*

$$e^{-\delta s} x \leq y_s \leq e^{-\delta s} \left[x - \frac{x^{-\beta}}{\delta(1+\beta)(\rho + \delta - \delta\beta)} \right] + \frac{e^{\delta\beta s} x^{-\beta}}{\delta(1+\beta)(\rho + \delta - \delta\beta)}, \quad s \geq 0. \quad (4.23)$$

Finally, if $x = y_\infty := (\delta(\rho + \delta))^{-\frac{1}{\beta+1}}$, then the function $y_s = y_\infty$, for all $s \geq 0$, is the unique constant solution to (4.6). If, instead, $x \neq y_\infty$, then the corresponding solution y to (4.6) is strictly monotone and such that

$$\lim_{s \rightarrow +\infty} y_s = y_\infty.$$

Proof. We divide the proof as follows. First, we show existence and uniqueness of a solution to (4.7) in the set $\mathcal{C}_{x, \infty}$, for each fixed $x > 0$. As in the finite time horizon case, the fact that such solution is twice continuously differentiable follows by differentiation of (4.8). Then, it immediately follows from Proposition 4.6 that (4.6) admits a unique solution $y \in C^2([0, +\infty); (0, +\infty))$ satisfying the bounds given in (4.23). Finally, we will show that the constant function equal to y_∞ is the unique constant solution to (4.6), and that the solutions starting at $x \neq y_\infty$ are monotone in time and converge to y_∞ .

Existence. Fix $x > 0$, and consider the family of solutions $\mathcal{Z} := \{z^{(n)}\}_{n \geq 1}$, where $z^{(n)}$ is the solution to the finite horizon problems with terminal time $T = n$, i.e.,

$$\begin{cases} \frac{d}{ds} z_s^{(n)} = e^{(\rho+2\delta)s} \int_s^n e^{-(\rho+\delta-\delta\beta)u} (z_u^{(n)})^{-\beta} du, & s \in [0, n], \\ z_0 = x. \end{cases} \quad (4.24)$$

We extend each of them by continuity taking them to be constant over $(n, +\infty)$, so that $z^{(n)} \in \mathcal{C}_{x, \infty}$, as $\bar{\alpha}_s^x \leq \bar{\alpha}_s^{x, \infty}$, for all $s \in [0, n]$. Differently from the finite horizon case, here we cannot directly apply the Ascoli-Arzelà theorem, as the infinite horizon does not allow to get the same equicontinuity estimates as in the proof of Theorem 4.9. Hence, use a weaker compactness criterion based on the Fréchet-Kolmogorov theorem.

Let us define the set

$$\mathbb{L}_{\rho+2\delta}^1([0, \infty); \mathbb{R}) := \left\{ f : [0, \infty) \rightarrow \mathbb{R} : \int_0^\infty e^{-(\rho+2\delta)s} |f_s| ds < +\infty \right\},$$

endowed with the usual weighted norm $\|\cdot\|_{1, \rho+2\delta}$. Clearly $\mathcal{Z} \subset \mathbb{L}_{\rho+2\delta}^1([0, \infty); \mathbb{R})$.

We observe that, given the bounds in (4.22), the family \mathcal{Z} is equibounded. We want to show that it is also equi-integrable in $L^1_{\rho+2\delta}$. For any $s \geq 0$ and $h > 0$ we have that

$$z_{s+h}^{(n)} - z_s^{(n)} = \mathbf{1}_{[0, n-h]}(s) \int_s^{s+h} (z_r^{(n)})' dr + \mathbf{1}_{[n-h, n]}(s) \int_s^n (z_r^{(n)})' dr.$$

Let $K := \frac{x^{-\beta}}{\rho+\delta-\delta\beta}$. Using the fact that $0 < (z_s^{(n)})' \leq Ke^{\delta(1+\beta)s}$, for all $s \geq 0$, and all $n \geq 1$, we get that, for any $h > 0$ small enough,

$$\begin{aligned} \|z_{\cdot+h}^{(n)} - z_{\cdot}^{(n)}\|_{1, \rho+2\delta} &= \int_0^{+\infty} e^{-(\rho+2\delta)s} |z_{s+h}^{(n)} - z_s^{(n)}| ds \\ &\leq \int_0^{n-h} e^{-(\rho+2\delta)s} \int_s^{s+h} (z_r^{(n)})' dr ds + \int_{n-h}^n e^{-(\rho+2\delta)s} \int_s^n (z_r^{(n)})' dr \\ &\leq \frac{K(e^{\delta(1+\beta)h} - 1)}{\delta(1+\beta)} \int_0^{n-h} e^{-(\rho+\delta-\delta\beta)s} ds \\ &\quad + \frac{Ke^{\delta(1+\beta)n}}{\delta(1+\beta)} \int_{n-h}^n e^{-(\rho+2\delta)s} ds - \underbrace{\frac{K}{\delta(1+\beta)} \int_{n-h}^n e^{-(\rho+\delta-\delta\beta)s} ds}_{\leq 0} \\ &\leq \frac{K(e^{\delta(1+\beta)h} - 1)}{\delta(1+\beta)(\rho+\delta-\delta\beta)} \underbrace{[1 - e^{-(\rho+\delta-\delta\beta)(n-h)}]}_{\leq 0} + \frac{Ke^{-(\rho+\delta-\delta\beta)n}}{\delta(1+\beta)(\rho+2\delta)} [e^{(\rho+2\delta)h} - 1] \\ &\leq \frac{K(e^{\delta(1+\beta)h} - 1)}{\delta(1+\beta)(\rho+\delta-\delta\beta)} + \frac{K}{\delta(1+\beta)(\rho+2\delta)} [e^{(\rho+2\delta)h} - 1], \quad \forall n \geq 1. \end{aligned}$$

Therefore, we obtain that

$$\lim_{h \rightarrow 0^+} \sup_{n \geq 1} \|z_{\cdot+h}^{(n)} - z_{\cdot}^{(n)}\|_{1, \rho+2\delta} = 0,$$

i.e., that the family \mathcal{Z} is equi-integrable in $L^1_{\rho+2\delta}$.

By the Frechet-Kolmogorov theorem, it follows that \mathcal{Z} is relatively compact, so we can extract a subsequence $\{z^{(n)}\}_{n \geq 1} \subset \mathcal{Z}$, still labeled (with an abuse of notation) by $\{z^{(n)}\}_{n \geq 1}$, such that $z^{(n)} \rightarrow z^{(\infty)}$ with respect to $\|\cdot\|_{1, \rho+2\delta}$, as $n \rightarrow \infty$. Then, we can extract a sub-subsequence, still labeled (with an abuse of notation) by $\{z^{(n)}\}_{n \geq 1}$, such that $z_s^{(n)} \rightarrow z_s^{(\infty)}$, for almost every $s \geq 0$, as $n \rightarrow \infty$. Clearly, the limit $z^{(\infty)}$ belongs to $L^1_{\rho+2\delta}$ and, since $z^{(n)} \in \mathcal{C}_{x, \infty}$, for any $n \geq 1$, we have that $\underline{\alpha}_s^{x, \infty} \leq z_s^{(\infty)} \leq \bar{\alpha}_s^{x, \infty}$, for almost every $s \geq 0$, where $\underline{\alpha}^{x, \infty}$ and $\bar{\alpha}^{x, \infty}$ are the functions defined in (4.20) and (4.21). Now we use (4.7) to get that also the derivative $\{(z^{(n)})'\}$ of the last subsequence $\{z^{(n)}\}$ converges a.e. to the function

$$w_s^{(\infty)} := e^{(\rho+2\delta)s} \int_s^{+\infty} e^{-(\rho+\delta-\delta\beta)u} (z_u^{(\infty)})^{-\beta} du, \quad s \geq 0,$$

which belongs to $L^1_{\rho+2\delta}$. By standard computations, it follows that $\{(z^{(n)})'\} \subset L^1_{\rho+2\delta}$ and that

$$0 < (z_s^{(n)})' \leq \frac{x^{-\beta}}{\rho+\delta-\delta\beta} e^{\delta(1+\beta)s} =: g_s, \quad \forall s \geq 0, \forall n \geq 1,$$

with $g \in L^1_{\rho+2\delta}$. Therefore, the sequence of derivatives $\{(z^{(n)})'\}$ converges in $L^1_{\rho+2\delta}$ to $w^{(\infty)}$. We deduce that the subsequence $\{z^{(n)}\}$ converges in the weighted Sobolev space $W_{\rho+2\delta}^{1,1}$ (cf. [4, Chapter 8, Remark 4]). By completeness of this space, we get that $w_s^{(\infty)} = (z_s^{(\infty)})'$, for a.e. $s \geq 0$. Moreover, using the same argument as in the proof of [4, Theorem 8.2], we get that there exists a function $\bar{z}^{(\infty)} \in C([0, +\infty); (0, +\infty))$ such that $\bar{z}_s^{(\infty)} = z_s^{(\infty)}$, for almost every $s \geq 0$, and

$$\bar{z}_s^{(\infty)} = x + \int_0^s (z_u^{(\infty)})' du = x + \int_0^s w_u^{(\infty)} du, \quad \forall s \geq 0.$$

Substituting the expression of $w^{(\infty)}$ in the previous equality we get that

$$\bar{z}_s^{(\infty)} = x + \int_0^s e^{(\rho+2\delta)r} \int_r^{+\infty} e^{-(\rho+\delta-\delta\beta)u} \left(\bar{z}_u^{(\infty)} \right)^{-\beta} du dr \quad s \geq 0,$$

that is, $\bar{z}^{(\infty)} \in \mathcal{C}_{x,\infty}$ and solves (4.7).

Uniqueness. As we observed at the beginning of the proof, any solution z to (4.7) is (at least) twice differentiable. Differentiating (4.7) with respect to the time variable and denoting by $'$ and $''$ first- and second-order derivatives with respect to this variable, we have that z verifies

$$\begin{cases} z_s'' = (\rho + 2\delta)z_s' - e^{\delta(1+\beta)s} z_s^{-\beta}, & s \geq 0, \\ z_0 = x. \end{cases} \quad (4.25)$$

The equation above is a second-order ODE with locally Lipschitz coefficients over $(s, z) \in [0, +\infty) \times (0, +\infty)$, to which we can associate the following initial value problem

$$\begin{cases} z_s'' = (\rho + 2\delta)z_s' - e^{\delta(1+\beta)s} z_s^{-\beta}, & s \geq 0, \\ z_0 = x, \\ z_0' = \zeta > 0. \end{cases} \quad (4.26)$$

Note that we only need to consider $\zeta > 0$, since solutions to (4.7) are such that their first derivative is strictly positive for all $s \geq 0$. For each fixed $x > 0$ and $\zeta > 0$, (4.26) has a unique solution $z^{x,\zeta}$ on the maximal interval $[0, \tau^*(x, \zeta))$, with $\tau^*(x, \zeta) \leq +\infty$.

Since any solution z to (4.7) also verifies (4.25), it also satisfies (4.26) for some $\bar{\zeta} > 0$. Therefore, if we show that for each given $x > 0$, there exists a unique $\zeta > 0$ such that (4.26) has a unique global solution $z^{x,\zeta} \in \mathcal{C}_{x,\infty}$, then necessarily $\zeta = \bar{\zeta}$ and $z^{x,\zeta}$ must also be the unique solution to (4.7). The idea is, thus, to study, for each fixed $x > 0$, the dependence of the solution $z^{x,\zeta}$ to (4.26) on $\zeta > 0$.

Let us fix $x > 0$. Standard results (see, e.g., [11, Theorem 7.5, Chapter 1]) ensure that the solution $z^{x,\zeta}$ to (4.26) and its first- and second-order derivatives with respect to the time variable are differentiable with respect to ζ and that the order of differentiation can be exchanged.

Therefore, defining $\hat{z}_s(\zeta) := \frac{\partial}{\partial \zeta} z_s^{x,\zeta}$, for all $s \geq 0$, we get from (4.26) that $\hat{z}(\zeta)$ solves the initial value problem

$$\begin{cases} \hat{z}_s''(\zeta) = (\rho + 2\delta)\hat{z}_s'(\zeta) + \beta e^{\delta(1+\beta)s} z_s^{x,\zeta}{}^{-\beta-1} \hat{z}_s(\zeta), & s \geq 0, \\ \hat{z}_0(\zeta) = 0, \\ \hat{z}_0'(\zeta) = 1, \end{cases} \quad (4.27)$$

where $z_s(\zeta) := z_s^{x,\zeta}$. Let

$$s^*(\zeta) := \inf\{s \geq 0: \hat{z}_s'(\zeta) \leq 0\} > 0.$$

By definition $\hat{z}_s'(\zeta) > 0$, for all $s \in [0, s^*(\zeta))$, which implies

$$\hat{z}_s(\zeta) > 0, \quad \forall s \in [0, s^*(\zeta)).$$

Therefore, using (4.27) we get that $\hat{z}_s''(\zeta) > 0$, for all $s \in [0, s^*(\zeta))$. It follows that

$$\hat{z}_s'(\zeta) \geq 1, \quad \forall s \in [0, s^*(\zeta)),$$

which implies that $s^*(\zeta) = +\infty$ and, in turn, that $\hat{z}_s(\zeta) \geq s$, for all $s \geq 0$.

Using again (4.27), we then get

$$\begin{cases} \hat{z}_s''(\zeta) \geq (\rho + 2\delta)\hat{z}_s'(\zeta), & s \geq 0, \\ \hat{z}_0'(\zeta) = 1, \end{cases} \quad (4.28)$$

which, implies, by comparison,

$$\hat{z}_s'(\zeta) \geq e^{(\rho+2\delta)s}, \quad \forall s \geq 0,$$

and, consequently,

$$\hat{z}_s(\zeta) \geq \frac{1}{\rho + 2\delta} [e^{(\rho+2\delta)s} - 1], \quad \forall s \geq 0.$$

Thus, for every $a < b$, we have

$$z_s(b) - z_s(a) \geq \int_a^b \hat{z}_s(\zeta) \, d\zeta \geq (b-a) \frac{1}{\rho+2\delta} [e^{(\rho+2\delta)s} - 1], \quad \forall s \geq 0.$$

We know that, for some $\bar{\zeta} > 0$, there exists a global solution $z(\bar{\zeta}) \in \mathcal{C}_{x,\infty}$ to (4.26), since any solution z to (4.7) also verifies this equation. Using the estimate above and (4.22), we get, for all $b > a := \bar{\zeta}$,

$$z_s(b) \geq z_s(\bar{\zeta}) + (b - \bar{\zeta}) [e^{(\rho+2\delta)s} - 1] \geq x + (b - \bar{\zeta}) \frac{1}{\rho+2\delta} [e^{(\rho+2\delta)s} - 1], \quad \forall s \geq 0.$$

This implies that $z(b)$ would grow exponentially at a rate at least $\rho + 2\delta > \delta(1 + \beta)$, which contradicts the fact that solutions in (4.22) need to grow at a rate at most $\delta(1 + \beta)$.

Similarly, we get, for all $a < \bar{\zeta} =: b$,

$$z_s(a) \leq z_s(\bar{\zeta}) - (\bar{\zeta} - a) [e^{(\rho+2\delta)s} - 1] \leq \bar{\alpha}_s^{x,\infty} - (\bar{\zeta} - a) \frac{1}{\rho+2\delta} [e^{(\rho+2\delta)s} - 1], \quad \forall s \geq 0.$$

This implies that $z(a)$ becomes negative in finite time, which contradicts the fact that solutions in (4.22) are positive.

By arbitrariness of $b > \bar{\zeta}$ and $a < \bar{\zeta}$, we conclude.

Convergence. As recalled at the beginning of the proof, uniqueness of the solution to (4.7) implies uniqueness of the solution to (4.6), which is also twice continuously differentiable. Differentiating (4.6), we get the second-order ODE

$$y_s'' = \rho y_s' + (\rho + \delta) \delta y_s - y_s^{-\beta}, \quad s \geq 0. \quad (4.29)$$

Hence, any solution to (4.6) is also a solution to (4.29). We observe that y_∞ is the unique solution to the algebraic equation

$$(\rho + \delta) \delta y - y^{-\beta} = 0. \quad (4.30)$$

Hence, the function $y_s = y_\infty$, for all $s \geq 0$, is the unique stationary solution to (4.6).

Now, we show monotonicity and convergence of the solution to (4.6) when $x \neq y_\infty$. We prove the case $x > y_\infty$, as the other one can be established by similar arguments.

Let y be the unique solution to (4.6) and fix $x > y_\infty$. We prove, first, that $y_s > y_\infty$, for all $s \geq 0$. Let us define

$$s_0 := \inf\{s \geq 0 : y_s = y_\infty\},$$

and assume, by contradiction, that $s_0 < +\infty$. We have two cases:

- (i) $y_{s_0}' < 0$;
- (ii) $y_{s_0}' = 0$.

Consider the first case. By the flow property, the function $\tilde{y}_s := y_{s-s_0}$, $s \geq s_0$, solves (4.6) with initial condition equal to y_∞ and it satisfies the a priori bounds given in (4.23) (with $x = y_\infty$). However, this solution is different from the constant one, which is the unique solution to (4.6) with initial condition y_∞ and satisfying the aforementioned bounds. Hence we have a contradiction.

The second case leads to a contradiction as well. Indeed, we would have two different solutions in the interval $[0, s_0]$ to the Cauchy problem (4.29), with terminal conditions $y_{s_0} = y_\infty$ and $y_{s_0}' = 0$, which clearly admits unique solution, that is, $y_s = y_\infty$, for all $s \geq 0$.

Hence, we have proved that $y_s > y_\infty$, for all $s \geq 0$. This implies, by strict monotonicity of the map $y \mapsto (\rho + \delta) \delta y - y^{-\beta}$ and (4.30), that

$$(\rho + \delta) \delta y_s - y_s^{-\beta} > (\rho + \delta) \delta y_\infty - y_\infty^{-\beta} = 0, \quad \forall s \geq 0, \quad (4.31)$$

and hence, from (4.29), that

$$y_s'' > \rho y_s', \quad \forall s \geq 0. \quad (4.32)$$

Now we show that $y_s' < 0$, for all $s \geq 0$. Let

$$s_1 := \inf\{s \geq 0 : y_s' \geq 0\}.$$

Assume, by contradiction, that $s_1 < +\infty$. Then, from (4.32) we get that $y''_{s_1} > 0$. It follows that there exist $\eta, \varepsilon > 0$ such that $y'_{s_1+\varepsilon} = \eta > 0$. Using (4.32) again, we get

$$\begin{cases} y''_s > \rho y'_s, & \forall s \geq s_1 + \varepsilon, \\ y'_{s_1+\varepsilon} = \eta > 0. \end{cases} \quad (4.33)$$

Therefore, solving the corresponding Cauchy problem and using the comparison, we get

$$y'_s \geq \eta e^{\rho[s-(s_1+\varepsilon)]} > 0, \quad s \geq s_1 + \varepsilon. \quad (4.34)$$

Next, differentiating (4.29), we get that $y \in C^3([0, +\infty); (0, +\infty))$ and that it satisfies

$$y'''_s = \rho y''_s + (\rho + \delta)\delta y'_s + \beta y_s^{-\beta-1} y'_s, \quad s \geq 0. \quad (4.35)$$

Since, by (4.34), we know that $y'_s > 0$ for all $s \geq s_1 + \varepsilon$, we deduce that

$$\begin{cases} y'''_s > \rho y''_s + (\rho + \delta)\delta y'_s, & \forall s \geq s_1 + \varepsilon, \\ y'_{s_1+\varepsilon} = \eta, \\ y''_{s_1+\varepsilon} = \kappa, \end{cases}$$

with $\eta > 0$ and $\kappa > \eta\rho > 0$, by (4.33). Solving the corresponding Cauchy problem and using the comparison, we get

$$y'_s > \frac{\eta\delta + \kappa}{\rho + 2\delta} e^{(\rho+\delta)[s-(s_1+\varepsilon)]} + \frac{\eta(\rho + \delta) - \kappa}{\rho + 2\delta} e^{-\delta[s-(s_1+\varepsilon)]}, \quad \forall s \geq s_1 + \varepsilon.$$

This implies that y grows, in the long run, at least with rate $\rho + \delta$ which is strictly bigger than $\delta\beta$, as prescribed by the admissibility condition in (4.23). The contradiction follows, and hence we proved that $y'_s < 0$, for all $s \geq 0$.

We are now going to show that $y''_s > 0$, for all $s \geq 0$. Since we proved that $y'_s < 0$, for all $s \geq 0$, we deduce from (4.35) that

$$y'''_s < \rho y''_s, \quad \forall s \geq 0. \quad (4.36)$$

Assume by contradiction that $y''_{s_2} \leq 0$ for some $s_2 \geq 0$. Then, from (4.36) we get that $y'''_{s_2} < 0$. It follows that there exist $\vartheta < 0$, $\tau > 0$ such that $y''_{s_2+\tau} = \vartheta < 0$. Using again (4.36), we get

$$\begin{cases} y'''_s < \rho y''_s, & \forall s \geq s_2 + \tau, \\ y''_{s_2+\tau} = \vartheta. \end{cases}$$

Therefore, $y''_s \leq \vartheta e^{\rho s} < 0$, for all $s \geq s_2 + \tau$. Considering that $y'_s < 0$, for all $s \geq 0$, this implies that the graph of $s \mapsto y_s$ lies below a straight line with strictly negative slope, contradicting the fact that $y_s > y_\infty$, for all $s \geq 0$, as previously established.

So, we have proved that, for all $s \geq 0$,

$$y_s > y_\infty, \quad y'_s < 0, \quad y''_s > 0.$$

This implies that there exists $\bar{y} := \lim_{s \rightarrow \infty} y_s \geq y_\infty$ and that $\lim_{s \rightarrow \infty} y'_s = 0$. However, it cannot be that $\bar{y} > y_\infty$, as it would imply, using (4.29), that $\lim_{s \rightarrow \infty} y''_s > 0$, contradicting the fact that $y'_s \rightarrow 0$, as $s \rightarrow +\infty$. Therefore, $\bar{y} = y_\infty$. \square

We are now ready to establish existence and uniqueness of an equilibrium for the mean-field game.

Proposition 4.13. *For each fixed random variable ξ satisfying Assumption 2.1, there exists a unique equilibrium $(\hat{\mathbf{u}}, \hat{q})$ for the mean-field game with infinite time horizon, among all equilibria verifying condition (4.3).*

More precisely, $\hat{\mathbf{u}} = z^{(\infty, \hat{q})}$, where $z^{(\infty, q)}$ is the function defined in (3.10), and \hat{q} is the unique solution to (4.6), with initial condition $x = \mathbb{E}[\xi]$.

Proof. Let \hat{q} be the unique solution to (4.6), with $x = \mathbb{E}[\xi]$. We need to check that condition (4.3) is satisfied. We have the following three cases.

Case $x = y_\infty$. From Theorem 4.12, we know that $\hat{q}_s = y_\infty$, for all $s \geq 0$. Therefore,

$$z_s^{(\infty, \hat{q})} = K := \frac{y_\infty^{-\beta}}{\rho + \delta}, \quad \forall s \geq 0, \quad (4.37)$$

and hence \hat{q} verifies condition (4.3).

Case $x > y_\infty$. From Theorem 4.12, we know that \hat{q} is monotone decreasing and converges to y_∞ . This implies that, $0 < y_\infty < \hat{q}_s \leq x$, for all $s \geq 0$, and that $z^{(\infty, \hat{q})}$ is positive and bounded above by the constant K defined in (4.37). Therefore, \hat{q} satisfies condition (4.3).

Case $x < y_\infty$. From Theorem 4.12, we know that \hat{q} is monotone increasing and converges to y_∞ . This implies that, $0 < x \leq \hat{q}_s < y_\infty$, for all $s \geq 0$, and that $z^{(\infty, \hat{q})}$ is positive and bounded above by $\frac{x^{-\beta}}{\rho + \delta}$. Therefore, \hat{q} satisfies condition (4.3).

Applying Theorem 4.1-(iv) we get the result. \square

Remark 4.14. *Also in this case the results of this section can be easily extended to the case where we consider an initial time $t > 0$.*

Clearly, we need to adapt (in an obvious way) the definition of equilibrium given in Definition 2.3 to include the initial time t . The statement of Theorem 4.1 is also adapted accordingly.

The integro-differential equation (4.6) becomes

$$\begin{cases} \frac{d}{ds} y_s = -\delta y_s + \int_s^T e^{-(\rho+\delta)(u-s)} y_u^{-\beta} du, & s \in [t, T], \\ y_t = x, \end{cases} \quad (4.38)$$

while equation (4.7) becomes

$$\begin{cases} \frac{d}{ds} z_s = e^{(\rho+2\delta)s} \int_s^T e^{-(\rho+\delta-\delta\beta)u} z_u^{-\beta} du, & s \in [t, T], \\ z_t = e^{\delta t} x. \end{cases} \quad (4.39)$$

The sets in which we look for solutions to (4.39) (now dependent on t) are

$$\mathcal{C}_{t,x} := \{f \in C([t, T]; (0, +\infty)) : e^{\delta t} \underline{\alpha}_s^x \leq f_s \leq e^{\delta t} \bar{\alpha}_s^x, \forall s \in [t, T]\}, \quad (4.40)$$

in the finite time horizon case, and

$$\mathcal{C}_{t,x,\infty} := \{f \in C([t, +\infty); (0, +\infty)) : e^{\delta t} \underline{\alpha}_s^{x,\infty} \leq f_s \leq e^{\delta t} \bar{\alpha}_s^{x,\infty}, \forall s \geq t\}, \quad (4.41)$$

in the infinite time horizon case.

The statements of Theorems 4.9 and 4.12, and of Propositions 4.10 and 4.13, remain the same (except for minor modifications). The bounds given in (4.14) become

$$e^{-\delta(s-t)} x \leq y_s \leq e^{-\delta(s-t)} \bar{\alpha}_s^x, \quad s \in [t, T], \quad (4.42)$$

while those given in (4.23) become

$$e^{-\delta(s-t)} x \leq y_s \leq e^{-\delta(s-t)} \left[x - \frac{x^{-\beta}}{\delta(1+\beta)(\rho+\delta-\delta\beta)} \right] + \frac{e^{\delta\beta(s-t)} x^{-\beta}}{\delta(1+\beta)(\rho+\delta-\delta\beta)}, \quad s \geq t. \quad (4.43)$$

5. THE DETERMINISTIC MODEL

In this section we discuss the deterministic version of the mean-field game problem introduced in Section 2. The structure of the problem will allow us to exploit the results given in Sections 3 and 4, to get the unique equilibrium of the mean-field game in an explicit form.

Let us consider the ordinary differential equation (ODE)

$$\begin{cases} \frac{d}{ds} X_s = -\delta X_s + u_s, & s \in (0, T], \\ X_0 = x, \end{cases} \quad (5.1)$$

where $\delta > 0$ is a given coefficient and the control $\mathbf{u} := (u_s)_{s \in [0, T]}$ is chosen in either of the following two classes of admissible controls: if $T < +\infty$,

$$\mathcal{U}_T := \left\{ \mathbf{u}: [0, T] \rightarrow [0, +\infty) \text{ measurable and s.t. } \int_0^T u_s^2 ds < +\infty \right\}; \quad (5.2)$$

if, instead, $T = +\infty$,

$$\mathcal{U}_\infty := \left\{ \mathbf{u}: [0, \infty) \rightarrow [0, +\infty) \text{ measurable and s.t. } \int_0^s u_r dr < +\infty, \forall s \geq 0, \right. \\ \left. \text{and } \int_0^{+\infty} e^{-\rho s} u_s^2 ds < +\infty \right\}. \quad (5.3)$$

Note that, for any $x \in \mathbb{R}$ and any $\mathbf{u} \in \mathcal{U}_T$, Equation (5.1) has a unique solution $X^{x, \mathbf{u}}$, given by

$$X_s^{x, \mathbf{u}} = e^{-\delta s} x + \int_0^s e^{-\delta(s-r)} u_r dr, \quad s \in [0, T]. \quad (5.4)$$

As in the stochastic case, X describes the evolution of the production capacity of a representative agent, which depreciates at a rate δ and can be increased by choosing the investment rate $\mathbf{u} \in \mathcal{U}_T$.

Next, we consider the discounted net profit functional

$$J_{T, q}(x, \mathbf{u}) := \int_0^T e^{-\rho s} \left\{ X_s^{x, \mathbf{u}} q_s^{-\beta} - \frac{1}{2} u_s^2 \right\} ds, \quad (5.5)$$

where $\rho > 0$ is a discount factor, $\beta > 0$ is a fixed parameter, and $q = (q_s)_{s \in [0, T]}$ is a given deterministic measurable function.

We introduce the following assumption.

Assumption 5.1. *The initial distribution of the agents in the economy ν_0 has a density m_0 with respect to the Lebesgue measure on \mathbb{R} , is supported on $(0, +\infty)$, and has finite first moment. More precisely, there exists $m_0 \in L^1((0, +\infty); (0, +\infty))$ such that*

$$\nu_0(A) = \int_A m_0(x) dx, \quad \forall A \in \mathcal{L}(\mathbb{R}), \quad \int_0^{+\infty} m_0(x) dx = 1, \quad \int_0^{+\infty} x m_0(x) dx < +\infty.$$

Since the representative agent is chosen randomly by picking her/his initial production capacity level x according to the initial distribution ν_0 , the assumption above ensures that, for any $\mathbf{u} \in \mathcal{U}_T$, $X_s^{x, \mathbf{u}} \geq 0$, for all $s \in [0, T]$. Thus, the production capacity level of the representative agent is never negative.

To introduce the definition of equilibrium for the deterministic mean-field game, we need to consider the so-called *continuity equation* (also known as Liouville equation, cf. [5]) associated to ODE (5.1)

$$\begin{cases} \frac{\partial}{\partial s} p(s, x) + \frac{\partial}{\partial x} (p(s, x) [-\delta x + u_s]) = 0, & (s, x) \in (0, T] \times (0, +\infty), \\ p(0, x) = p_0(x), & x \in (0, +\infty), \end{cases} \quad (5.6)$$

where $\mathbf{u} = (u_s)_{s \in [0, T]} \in \mathcal{U}_T$ and p_0 is a given initial probability density function. This equation describes the evolution of p_0 under the flow determined by ODE (5.1).

Solutions to (5.6) are understood in the weak sense, according to the following definition.

Definition 5.2. *A function $p \in L^1([0, T] \times (0, +\infty); (0, +\infty))$ is a weak solution to (5.6) if, for any function $\varphi \in C^\infty([0, T] \times (0, +\infty))$ with compact support, and any $t > 0$, we have*

$$\begin{aligned} & \int_0^{+\infty} \varphi(t, x) p(t, x) dx \\ &= \int_0^{+\infty} \varphi(0, x) p_0(x) dx + \int_0^t \int_0^{+\infty} \left\{ \frac{\partial}{\partial s} \varphi(s, x) + [-\delta x + u_s] \frac{\partial}{\partial x} \varphi(s, x) \right\} p(s, x) dx ds. \end{aligned} \quad (5.7)$$

Lemma 5.3. *For any $\mathbf{u} \in \mathcal{U}_T$ and any initial probability density p_0 , supported on $(0, +\infty)$ and with finite first moment, the continuity equation (5.6) has a unique solution, given by*

$$p^{p_0, \mathbf{u}}(s, x) = e^{\delta s} p_0 \left(e^{\delta s} x - \int_0^s e^{\delta r} u_r dr \right), \quad (s, x) \in [0, T] \times (0, +\infty). \quad (5.8)$$

Moreover, for each $s \in [0, T]$, the probability density $p^{p_0, \mathbf{u}}(s, \cdot)$ has finite first moment, given by

$$\int_0^{+\infty} x p^{p_0, \mathbf{u}}(s, x) dx = e^{-\delta s} \int_0^{+\infty} x p_0(x) dx + \int_0^s e^{-\delta(s-r)} u_r dr \quad s \in [0, T]. \quad (5.9)$$

Proof. Fix $\mathbf{u} \in \mathcal{U}_T$ and p_0 as above. The fact that $p^{p_0, \mathbf{u}}$ is a solution to (5.6) can be verified *via* standard computations (see, e.g., [5, Example 2.5.2] or [7, Lemma 4.15]). Uniqueness follows from an argument similar to the one used in the proof of [7, Lemma 4.16].

Finally, using the change of variables $y = e^{\delta s} x - \int_0^s e^{\delta r} u_r dr$, we get that, for each $s \in [0, T]$,

$$\begin{aligned} \int_0^{+\infty} x p^{p_0, \mathbf{u}}(s, x) dx &= \int_0^{+\infty} x e^{\delta s} p_0 \left(e^{\delta s} x - \int_0^s e^{\delta r} u_r dr \right) dx \\ &= \int_0^{+\infty} e^{-\delta s} \left(y + \int_0^s e^{\delta r} u_r dr \right) p_0(y) dy = e^{-\delta s} \int_0^{+\infty} y p_0(y) dy + \int_0^s e^{-\delta(s-r)} u_r dr, \end{aligned}$$

which is finite thanks to the assumptions on p_0 and \mathbf{u} . Thus, $p^{p_0, \mathbf{u}}(s, \cdot)$ has finite first moment, for any $s \in [0, T]$, which satisfies (5.9). \square

We are now ready to state the definition of equilibrium for our deterministic mean-field game.

Definition 5.4. *Fix an initial distribution ν_0 under Assumption 5.1 and let m_0 be its density with respect to the Lebesgue measure.*

A pair $(\hat{\mathbf{u}}, \hat{q})$, where $\hat{\mathbf{u}} \in \mathcal{U}_T$ and $\hat{q}: [0, T] \rightarrow (0, +\infty)$ is a measurable function, is an equilibrium of the deterministic mean-field game if

- (i) $J_{T, \hat{q}}(x, \hat{\mathbf{u}}) \geq J_{T, \hat{q}}(x, \mathbf{u})$, for all $\mathbf{u} \in \mathcal{U}_T$ and ν_0 -a.e. $x > 0$;
- (ii) $\hat{q}_s = \int_0^{+\infty} x m^{m_0, \hat{\mathbf{u}}}(s, x) dx$, for all $s \in [0, T]$, where $m^{m_0, \hat{\mathbf{u}}}$ is the unique solution of the continuity equation (5.6), given by (5.8), with initial condition m_0 and control $\hat{\mathbf{u}}$.

Also in this case, we can adopt a fixed point argument to show that there exists a unique equilibrium, by solving, first, the problem of maximizing (5.5) for a given measurable function q , then determining the optimally controlled dynamics of the production capacity level of the representative agent and, finally, setting the fixed point argument from condition (ii) in Definition 5.4.

5.1. The optimization problem. Let us consider the optimization problem

$$V_{T, q}(x) := \sup_{\mathbf{u} \in \mathcal{U}_T} J_{T, q}(x, \mathbf{u}), \quad x > 0, \quad (5.10)$$

where $J_{T, q}$ is the discounted net profit functional defined in (5.5), for any given and fixed q measurable and deterministic function.

In the finite time horizon case, i.e. $T < +\infty$, the following result holds, which is analogous to Theorem 3.3.

Theorem 5.5. *Fix $x > 0$ and $q: [0, T] \rightarrow (0, +\infty)$ such that $q^{-\beta} \in L^1((0, T]; (0, +\infty))$. Then, $\hat{\mathbf{u}}^{(T, q)} := z^{(T, q)} \in \mathcal{U}_T$, where $z^{(T, q)}$ is the function defined in (3.4), is an optimal control for problem (5.10).*

Moreover, $\hat{\mathbf{u}}^{(T, q)}$ is essentially unique, i.e., if $\bar{\mathbf{u}}^{(T, q)} \in \mathcal{U}_T$ is an optimal control for problem (5.10) different from $\hat{\mathbf{u}}^{(T, q)}$, then

$$\bar{u}_s^{(T, q)} = \hat{u}_s^{(T, q)}, \quad \text{for Leb-a.e. } s \in [0, T].$$

Finally, the value function of the optimization problem admits the explicit expression

$$V_{T, q}(x) = x z_0^{(T, q)} + \frac{1}{2} \int_0^T e^{-\rho s} (z_s^{(T, q)})^2 ds, \quad (5.11)$$

and the optimally controlled state $X^{x, \hat{\mathbf{u}}^{(T,q)}}$ is given by

$$X_s^{x, \hat{\mathbf{u}}^{(T,q)}} = e^{-\delta s} x + \int_0^s e^{-\delta(s-r)} z_r^{(T,q)} dr, \quad s \in [0, T]. \quad (5.12)$$

Proof. Replicating the argument of the proof of Proposition 3.1, we get that

$$J_{T,q}(x, \mathbf{u}) = x z_0^{(T,q)} + \int_0^T e^{-\rho s} \left\{ z_s^{(T,q)} u_s - \frac{1}{2} u_s^2 \right\} ds.$$

Then, the result follows using the same reasoning as in the proof of Theorem 3.3. \square

In the infinite time horizon case, i.e. $T = +\infty$, we have the following statement, which is the deterministic counterpart of Theorem 3.6. We state it without proof.

Theorem 5.6. Fix $x > 0$ and consider a function $q: [0, +\infty) \rightarrow (0, +\infty)$ such that $q^{-\beta} \in L^1_{\rho+\delta}((0, +\infty); (0, +\infty))$ and such that the function $z^{(\infty,q)}$, defined in (3.10) is bounded on $[0, +\infty)$.

Then, $\hat{\mathbf{u}}^{(\infty,q)} := z^{(\infty,q)} \in \mathcal{U}_\infty$ is an optimal control for problem (5.10). Moreover, $\hat{\mathbf{u}}^{(\infty,q)}$ is essentially unique, i.e., if $\bar{\mathbf{u}}^{(\infty,q)} \in \mathcal{U}_\infty$ is an optimal control for problem (5.10) different from $\hat{\mathbf{u}}^{(\infty,q)}$, then

$$\bar{u}_s^{(\infty,q)} = \hat{u}_s^{(\infty,q)}, \quad \text{for Leb-a.e. } s \geq 0.$$

Finally, the value function of the optimization problem admits the explicit expression

$$V_{\infty,q}(x) = x z_0^{(\infty,q)} + \frac{1}{2} \int_0^{+\infty} e^{-\rho s} (z_s^{(\infty,q)})^2 ds, \quad (5.13)$$

and the optimally controlled state $X^{x, \hat{\mathbf{u}}^{(\infty,q)}}$ is given by

$$X_s^{x, \hat{\mathbf{u}}^{(\infty,q)}} = e^{-\delta s} x + \int_0^s e^{-\delta(s-r)} z_r^{(\infty,q)} dr, \quad s \geq 0. \quad (5.14)$$

5.2. Existence and uniqueness of equilibria. Also in the deterministic mean-field game previously introduced, the search for an equilibrium boils down to finding a fixed point of a suitable map. According to Definition 5.4, this map is $(q_s)_{s \in [0, T]} \mapsto \left(\int_0^{+\infty} x m^{m_0, \hat{\mathbf{u}}^{(T,q)}}(s, x) dx \right)_{s \in [0, T]}$, where $\hat{\mathbf{u}}^{(T,q)}$ is the optimal control for the optimization problem (5.10) – whose expression is given either in Theorem 5.5, in the case $T < +\infty$, or in Theorem 5.6, in the case $T = +\infty$ – and $m^{m_0, \hat{\mathbf{u}}^{(T,q)}}$ is the unique solution to (5.6), with initial probability density m_0 and control $\hat{\mathbf{u}}^{(T,q)}$.

The next result, which is analogous to Theorem 4.1, shows that also in the deterministic case the fixed point map is the solution map of the same integral equation studied in Section 4. We omit its proof, which is a straightforward adaptation of the proof of Theorem 4.1.

Theorem 5.7. Let us fix an initial distribution ν_0 satisfying Assumption 5.1, and let m_0 be its density with respect to the Lebesgue measure.

Consider the deterministic mean-field game in the finite time horizon case, i.e., $T < +\infty$. Then,

(i) If there exists an equilibrium $(\hat{\mathbf{u}}, \hat{q})$ in the sense of Definition 5.4, such that

$$\hat{q}^{-\beta} \in L^1((0, T]; (0, +\infty)), \quad (5.15)$$

then \hat{q} is a solution to the integral equation

$$y_s = e^{-\delta s} \int_0^{+\infty} y m_0(y) dy + \int_0^s e^{-\delta(s-r)} \int_r^T e^{-(\rho+\delta)(u-r)} y_u^{-\beta} du dr, \quad s \in [0, T]. \quad (5.16)$$

(ii) Vice versa, if there exist a unique solution \hat{q} to (5.16) satisfying (5.15), then there exists a unique equilibrium $(\hat{\mathbf{u}}, \hat{q}) = (z^{(T, \hat{q})}, \hat{q})$ of the mean-field game among all equilibria $(\bar{\mathbf{u}}, \bar{q})$ such that \bar{q} verifies (5.15), where $z^{(T, \hat{q})}$ is the function defined in (3.4).

Consider, instead, the deterministic mean-field game in the infinite time horizon case, i.e., $T = +\infty$. Then,

(iii) If there exists an equilibrium $(\hat{\mathbf{u}}, \hat{q})$ in the sense of Definition 5.4, such that

$$\hat{q}^{-\beta} \in L^1_{\rho+\delta}((0, +\infty); (0, +\infty)) \quad \text{and} \quad z^{(\infty, \hat{q})} \text{ is bounded on } [0, +\infty), \quad (5.17)$$

where $z^{(\infty, \hat{q})}$ is the function defined in (3.10), then \hat{q} is a solution to the integral equation

$$y_s = e^{-\delta s} \int_0^{+\infty} y m_0(y) dy + \int_0^s e^{-\delta(s-r)} \int_r^{+\infty} e^{-(\rho+\delta)(u-r)} y_u^{-\beta} du dr, \quad s \geq 0. \quad (5.18)$$

(iv) Vice versa, if there exist a unique solution \hat{q} to (5.18) satisfying (5.17), then there exists a unique equilibrium $(\hat{\mathbf{u}}, \hat{q}) = (z^{(\infty, \hat{q})}, \hat{q})$ of the mean-field game among all equilibria $(\tilde{\mathbf{u}}, \tilde{q})$ such that \tilde{q} verifies (5.17).

Finally, we get the following result, which establishes existence and uniqueness of an equilibrium also for the deterministic mean-field game. We omit its proof, since it follows the same lines of the proofs of Propositions 4.10 and 4.13, except for minor adaptations.

Proposition 5.8. *For each fixed ν_0 satisfying Assumption 5.1, there exists a unique equilibrium $(\hat{\mathbf{u}}, \hat{q})$ for the deterministic mean-field game with finite time horizon (resp., infinite time horizon), among all equilibria verifying the integrability condition (5.15) (resp., (5.17)).*

More precisely, $\hat{\mathbf{u}} = z^{(T, \hat{q})}$, where $z^{(T, \hat{q})}$ is the function defined in (3.4) (resp. (3.10)), and \hat{q} is the unique solution to (4.6), with initial condition $x = \int_0^{+\infty} y m_0(y) dy$, where m_0 is the density of ν_0 with respect to the Lebesgue measure.

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