

Mahayni, Antje; Lubos, Oliver; Offermann, Sascha

**Article — Published Version**

## Minimum return rate guarantees under default risk: optimal design of quantile guarantees

Review of Managerial Science

**Provided in Cooperation with:**

Springer Nature

*Suggested Citation:* Mahayni, Antje; Lubos, Oliver; Offermann, Sascha (2020) : Minimum return rate guarantees under default risk: optimal design of quantile guarantees, Review of Managerial Science, ISSN 1863-6691, Springer, Berlin, Heidelberg, Vol. 15, Iss. 7, pp. 1821-1848, <https://doi.org/10.1007/s11846-020-00410-3>

This Version is available at:

<https://hdl.handle.net/10419/288930>

**Standard-Nutzungsbedingungen:**

Die Dokumente auf EconStor dürfen zu eigenen wissenschaftlichen Zwecken und zum Privatgebrauch gespeichert und kopiert werden.

Sie dürfen die Dokumente nicht für öffentliche oder kommerzielle Zwecke vervielfältigen, öffentlich ausstellen, öffentlich zugänglich machen, vertreiben oder anderweitig nutzen.

Sofern die Verfasser die Dokumente unter Open-Content-Lizenzen (insbesondere CC-Lizenzen) zur Verfügung gestellt haben sollten, gelten abweichend von diesen Nutzungsbedingungen die in der dort genannten Lizenz gewährten Nutzungsrechte.

**Terms of use:**

*Documents in EconStor may be saved and copied for your personal and scholarly purposes.*

*You are not to copy documents for public or commercial purposes, to exhibit the documents publicly, to make them publicly available on the internet, or to distribute or otherwise use the documents in public.*

*If the documents have been made available under an Open Content Licence (especially Creative Commons Licences), you may exercise further usage rights as specified in the indicated licence.*



<https://creativecommons.org/licenses/by/4.0/>



# Minimum return rate guarantees under default risk: optimal design of quantile guarantees

Antje Mahayni<sup>1</sup> · Oliver Lubos<sup>1</sup> · Sascha Offermann<sup>1</sup>

Received: 28 May 2019 / Accepted: 6 August 2020 / Published online: 19 August 2020  
© The Author(s) 2020

## Abstract

The paper analyzes the design of participating life insurance contracts with minimum return rate guarantees. Without default risk, the insured receives the maximum of a guaranteed rate and a participation in the investment returns. With default risk, the payoff is modified by a default put implying a compound option. We represent the yearly returns of the liabilities by a portfolio of plain vanilla options. In a Black and Scholes model, the optimal payoff constrained by a maximal shortfall probability can be stated in closed form. Due to the completeness of the market, it can be implemented for any equity to debt ratio.

**Keywords** Guarantee scheme · Derivatives · Life insurance · Return rate guarantees · Default risk · Regulatory requirements · Utility to the insured

**JEL Classification** G 31 · G 22

---

The authors gratefully acknowledge financial support by the German Insurance Science Association (DVfVW). In addition the authors would like to thank the two anonymous referees for their valuable and helpful suggestions and comments. Furthermore, earlier versions of this paper were presented at the 20th Annual Conference of the Swiss Society for Financial Market Research (Zurich), 24th Annual Meeting of the German Finance Association (Ulm), 1st Frankfurt Insurance Research Workshop (Frankfurt), 10th World Congress of the Bachelier Finance Society (Dublin) and American Risk and Insurance Association Annual Meeting 2018 (Chicago). The authors would like to thank the seminar and conference participants for useful comments and suggestions.

---

✉ Oliver Lubos  
oliver.lubos@uni-due.de

Antje Mahayni  
antje.mahayni@uni-due.de

Sascha Offermann  
sascha.offermann@uni-due.de

<sup>1</sup> Mercator School of Management, University of Duisburg–Essen, Lotharstr. 65, 47057 Duisburg, Germany

## 1 Introduction

The paper analyzes the optimal design of participating life insurance contracts with minimum return rate guarantees (MRRGs) under default risk.<sup>1</sup> The benefits to the insured are linked to an investment strategy which is conducted by the insurer on the financial market as e.g. observed in participating life insurance contracts. Unless there is a default event, the insured receives the maximum of a guaranteed rate and a participation in the investment returns. An optimal contract design implies the highest expected utility to the insured. The focus is on MRRGs which are fairly priced (pricing by no arbitrage condition) and satisfy regulatory requirements posed on the probability that the guarantees are violated (quantile MRRGs).

It is worth mentioning that we merely focus on a savings plan which is motivated by participating life insurance contracts. In reality, these contracts are much more complicated. They also include a term life insurance component and possess several premium payment options to policyholders. It is often criticized, that the underlying of this kind of life insurance product is in reality typically based on book values and not market values like it is suggested in most research papers. However, the main effect is, that the underlying possesses a lower volatility (via “smoothing”) and—*ceteris paribus*—the value of the embedded options is lower. In any case, one can in principle account for this effect via choosing the “appropriate” volatility in the GBM - whenever her model is adjusted to empirical data via time series data. For a detailed description of participating life insurance contracts, we refer e.g., to Grosen and Jørgensen (2000) and Grosen and Jørgensen (2002). Additionally to these facts, the insurance companies even smoothen their asset and liability sides in reality to overcome bad financial years with the surplus of good years.<sup>2</sup> Furthermore, we also define the default event exclusively in terms of the investment returns and do not consider that the insurance company may itself default.

Considering the possibility that the liabilities (guarantees) can not be honored impedes the basic idea of a guarantee. However, in reality there is no guarantee prevailing with probability one. Any guarantee may fail in times of extremely negative market conditions, i.e. guarantees are only valid under sufficiently good scenarios. Thus, one may soften the term guarantee and imagine it as honored with a high probability (quantile guarantee). In the context of participating life insurance contracts the guarantee is secured by regulatory requirements on the maximal shortfall probability. For example, Solvency II contains the condition that the shortfall probability w.r.t. a time horizon of one year is limited to 0.5%. Intuitively, it is clear that the value of a guarantee is decreasing in the shortfall probability. Default risk mitigates the guarantee component (it is less often binding and thus the guarantee

---

<sup>1</sup> In particular, we refer to annual return rate guarantees which are common in German-speaking countries.

<sup>2</sup> A paper on this topic is for example Maurer et al. (2016), where a stylized model with payout smoothing is provided and a literature overview of this topic is given. In addition, Kling et al. (2007) shows an example how smoothing can be modelled when analyzing some question related to participating products.

is cheaper than without default risk). In contrast, control of the shortfall probability makes the guarantee more binding. In summary, the pricing effects due to the impact of default risk are rather obvious. The impacts on the utility to the insured is more ambivalent, unless the insurer implements an optimal investment strategy. Therefore, our main focus is on the optimal contract design in the presence of an upper probability bound on the shortfall probability posed by the regulator, i.e. the optimal design of quantile MRRGs.

We proceed as follows. In the absence of mortality and surrender risk, we discuss the modification of the (return) payoff which arise by introducing default risk referred to a strictly binding guarantee. In a stylized manner, we model the asset side of the insurance company (the contract provider) by means of the value process of an admissible financial market investment strategy, i.e. a self-financing strategy where the initial value is given by the sum of equity and the contributions of the insureds. The liability side, i.e. the benefits to the insured, depends on the guarantee promise as well as on the question how the surpluses, if any, are distributed between the shareholders and the insured. This is modeled by a participation fraction on the investment returns. Considering default risk, the return payoff to the insured also depends on the amount of equity backing up the guarantee. If the intended payoff which is paid in a default free version is not obtained by the investment strategy, the remaining amount is provided by reducing the equity, i.e. unless the equity amount drops to zero.

In summary, the impact of the default risk on the contract pricing is captured by a short position in a default put option. In financial terms, the default put is a compound option (option on an option). The inner option is introduced by the guarantee option of the insured, i.e. arising from the (intended) guarantee. The outer option is implied by the default possibility, i.e. the intended payoff is only honored if the asset/investment performance is sufficiently good. We show that, w.r.t. each annual return payoff, the (return) payoff of the compound option can (for a suitable distinction of the equity to debt ratio compared to a function of the guarantee and participation fraction) be disentangled into a piecewise linear payoff function (of the investment return), i.e. the payoff can be stated in terms of plain vanilla options. The same is true for the liabilities to the insured (Proposition 2). Closed form solutions for pricing the default put and the insurance contract itself are possible in any financial market model setup which provides closed form solutions for plain vanilla options. Closed-form solutions for the return payoff in the context of no default risk but with mortality risk can be found e.g. in Bacinello (2001).

The Cliquet-style contracts can then be solved in closed form in any model/investment setup which implies independent and identically distributed return increments, at least if one assumes a constant or deterministic equity to debt fraction. Some general implications of considering (i) default risk and (ii) regulatory requirements on the shortfall probability can already be derived in a model free manner such that the results are valid in any arbitrage free model setup. We illustrate and quantify the results in a Black and Scholes model setup. This simple model setup in combination with the assumption that the insured is described by a constant relative risk aversion (CRRA) gives further insights on the utility effects from the perspective of the insured.

Due to the completeness of the model setup and the exclusion of mortality and surrender risk, we can even solve the resulting pure portfolio optimization problem and state the expected utility maximizing return payoff under the quantile condition posed by the regulator, i.e. the upper bound on the shortfall probability (Proposition 5).<sup>3</sup> In particular, the derivation of the optimal quantile contract is tractable because of the complete market assumption. W.l.o.g., one can analyze the relevant optimization problem without considering equity, i.e. by means of setting the equity to debt fraction to zero. Once the optimal return distribution is computed without equity, the same return payoff distribution can be implemented in the presence of any equity amount held by the insurance company. We compare the optimal quantile MRRG with the unrestricted solution (no shortfall condition posed by the regulator) as well as with solutions which are based on restrictions on the investment strategy implemented by the insurance company. For example, we consider the case that the insurer is restricted to constant mix strategies. Intuitively it is clear that the upper bound on the shortfall probability (if binding) affords some kind of quantile hedge. The resulting optimal payoff is not attainable without some (synthetic) option positions and can not be contained by a fixed sharing rule between equity and debt. We show that the utility loss to the insured arising if the insurer implements a suboptimal investment strategy can be significant.

The contributions of the paper can be summarized as follows. Based on the distinction between a high and a low equity to debt ratio (compared to the combination of guarantee and participation fraction), we state the return payoff to the insured (Proposition 2) by means of piecewise linear functions of the return of the insurers asset returns. On the one hand, this simplifies the pricing problem under default risk to the pricing of standard call (put) options. On the other hand, this already gives model independent insights, i.e. insights which are true w.r.t. any arbitrage free financial market model setup. For example, a low (high) equity to debt ratio implies a concave (piecewise concave and convex) payoff.<sup>4</sup> Thus, for a low equity to debt ratio, the value of the liabilities is decreasing in the riskiness of the insurer's assets. Consequently, the default risk dominates the guarantee option which contradicts the guarantee concept, i.e. if the admissible asset distributions are not restricted by an upper bound on the shortfall probability (on the guarantee). A further contribution is then given by deriving the optimal return payoff distribution to the insured (Proposition 5). Because of the market completeness, the optimal (return) payoff to the insured can be implemented for any equity to debt ratio. Finally it is important to point out that there are utility losses to the insured (and there is too much equity involved) if the insurer implements a suboptimal investment strategy.

Our paper is related to several strands of the literature including the ones on (i) pricing and hedging embedded guarantees/options, (ii) the impact of default risk (emphasizing on participating life insurance contracts), (iii) utility losses caused by

---

<sup>3</sup> Notice, that in general Solvency requirements and Solvency II in particular lead in fact to restrictions when it comes to optimal asset allocation settings

<sup>4</sup> In our setup, a low equity to debt ratio is always implied if there is a return guarantee which gives a return accumulation higher (or equal) one.

guarantees and/or suboptimal investment decisions (conducted by insurance companies or pension funds), (iv) portfolio planning, (v) quantile hedging, and (vi) the analysis of piecewise convex and concave contingent payoffs. Without postulating completeness we only refer to the most related literature and hint at the additional literature given within the mentioned papers. Pricing embedded options by no arbitrage already dates back to Brennan and Schwartz (1976). A more recent paper is Nielsen et al. (2011). Risk management and hedging aspects are discussed in Coleman et al. (2006), Coleman et al. (2007), and Mahayni and Schlögl (2008).

An early paper which already provides tools to determine closed-form solutions for the solvency restriction based on a shortfall concept under certain distribution assumptions (normal and log normal case) is given by Winkler et al. (1972) using partial moments. Non-linear optimization problems under shortfall constraints have already been solved in the past, c.f. McCabe and Witt (1980) who calculated the optimal chance-constrained expected profit of a non-life insurer.

Considering default is, in the context of participating life insurance contracts, firstly analyzed in Briys and de Varenne (1997) and Grosen and Jørgensen (2002). More recent papers are Schmeiser and Wagner (2015) and Hieber et al. (2019). Other papers on participating life insurance contracts excluding default risk are e.g. Bacinello (2001) who discusses amongst other results how a minimum interest rate guarantee (“technical rate”) has to be set, such that the contracts are fairly priced and Gatzert et al. (2012) where the customer value of the policyholder is maximized.

Papers on utility losses caused by (suboptimal) investment strategies include Jensen and Sørensen (2001), Jensen and Nielsen (2016) and Chen et al. (2019).<sup>5</sup> Chen et al. (2019) consider a general utility maximization under fair-pricing and budget constraints in a complete, arbitrage-free Black and Scholes model setup for an CRRA Investor. The payoff function is chosen such that it also includes default risk. They apply their results on equity-like life insurances using a constant mix strategy and examine the effect of taxation.

Literature on portfolio planning with a main focus on insurance contracts with guarantees includes Huang et al. (2008), Milevsky and Kyrychenko (2008), Boyle and Tian (2008) and Mahayni and Schneider (2016). The general idea of maximizing the expected utility of the insured by choosing optimal parameter settings which fulfill fair pricing conditions has been provided in the literature before. The paper of Branger et al. (2010) analyzes different forms of point-to-point guarantees. Cliquet-style options are analyzed in Gatzert et al. (2012) and Schmeiser and Wagner (2015). In contrast to these articles we add the portfolio composition as a decision variable in the optimization problem to determine the overall expected utility maximizing payoff of the insured in quasi-closed form.

Portfolio planning itself dates back to Merton (1971) who, amongst other results, solves the portfolio planning problem for a CRRA investor. The solution for investors who must also manage market-risk exposure using the Value-at-Risk (VaR) is firstly mentioned in Basak and Shapiro (2001). Yiu (2004) solves the problem where

---

<sup>5</sup> In particular, Jensen and Sørensen (2001) analyze wealth losses for pension funds and emphasize that the individual investor can substantially suffer from the investment strategy conducted by the sponsor.

the VaR constraint is posed for the entire investment horizon. More recently, Gao et al. (2016) derive the solution for an investor with a dynamic mean-variance-CVaR and a dynamic mean-variance-safety-first constraint. A joint (terminal) VaR and portfolio insurance constraint is considered in Chen et al. (2018a). Multiple VaR constraints are analyzed in Chen et al. (2018b).

With respect to European and American guarantees, we also refer to El Karoui et al. (2005). Quantile hedging already dates back to Föllmer and Leukert (1999). For an analysis of retail products with investment caps (piecewise convex and concave payoffs) we e.g. refer to Bernard et al. (2009), Bernard and Li (2013), Mahayni and Schneider (2016).

Literature on the insurance demand dates back to Leland (1980) and Benninga and Blume (1985) who show that in a complete financial market setup with risky and risk-free asset investments and a utility function with constant risk aversion the investor will never buy portfolio insurance, instead buys the asset itself directly. Ebert et al. (2012) confirm the result for guarantee contracts, i.e. for CRRA Investors with reasonable risk aversion parameter Cumulative Prospect Theory (CPT) can not explain the demand for complex guarantee contracts. Ruß and Schelling (2018) introduce the concept of Multi Cumulative Prospect Theory (MCPT) which does not only consider the terminal value of the investment but also the annual value change. Under the MCPT the demand for complex guarantee products can be explained.

The rest of the paper is organized as follows. Section 2 describes the contract design. In particular, it is based on a combination of the contract parameters and the equity fraction such that the contract design gives no rise to any arbitrage opportunity. In addition, the contract design must meet some regulatory requirements regarding an upper bound on the shortfall probability. Along the ways, we give some convenient representations of the payoff profiles. We illustrate the contract design and some important properties in a Black and Scholes model setup. In Sect. 3, we derive the optimal contract design (return payoff, respectively) of a quantile minimum return guarantee (MRRG), i.e. a return guarantee which satisfies the fair pricing condition and an upper bound on the shortfall probability, and in view of an insured whose preferences are characterized by a constant relative risk aversion. We illustrate the utility loss to the insured which is caused if the insurer implements a suboptimal investment strategy. Section 4 concludes the paper.

## 2 Contract design, payoffs, and fair pricing

We examine stylized versions of minimum return rate guarantees (MRRGs) which are e.g. observed in participating life insurance contracts. The insured pays a single premium at inception of the contract. The focus is on contracts which grant the insured a participation on positive investment results and include a return guarantee unless there is default risk. Since we abstract from mortality or surrender risk, there is no loss of generality due to a single premium compared to more flexible premium payments. The initial contribution of the insured is denoted by  $P_0$ . The product terminates and pays out to the insured at  $T > 0$ .

### 2.1 Stylized version of MRRG

Throughout the following,  $A_T$  denotes the terminal value of the insurance result (asset result) which is the outcome of an admissible investment strategy with initial investment  $A_0$ . In particular, the initial investment  $A_0$  consists of the existing equity amount  $E_0 \geq 0$  and the contributions of the insureds  $P_0$ , i.e.  $A_0 = E_0 + P_0$ . In particular, we normalize  $P_0 = 1$  and set  $E_0 = \alpha^{(E)}$  where  $\alpha^{(E)} \in [0, 1]$  denotes the equity fraction (equity to debt ratio, respectively).

Along the lines of Schmeiser and Wagner (2015), we assume that the policyholder's account evolves from  $t - 1$  to  $t$  ( $t \in \{1, \dots, T\}$ ) according to

$$P_t = P_{t-1} \left( 1 + \max \left\{ g, \alpha \left( \frac{A_t}{A_{t-1}} - 1 \right) \right\} \right),$$

where  $\alpha$  ( $\alpha \in ]0, 1[$ ) denotes the participation fraction and  $1 + g$  ( $g \geq -1$ ) is the guaranteed accumulation factor granted for one year.<sup>6</sup> The special case  $g = -1$  includes a contract without guarantee. To simplify the expositions, we restrict ourselves to  $T = 1$ , i.e. we refer to the *intended* MRRG payoff  $P_1$  to the insured, i.e. the payoff which is valid without default risk given by

$$P_1 = P_0 \left( 1 + \max \left\{ g, \alpha \left( \frac{A_1}{A_0} - 1 \right) \right\} \right). \tag{1}$$

Using  $1 + \max \left\{ g, \alpha \left( \frac{A_1}{A_0} - 1 \right) \right\} = 1 + g + \alpha \left( \frac{A_1}{A_0} - \left( 1 + \frac{g}{\alpha} \right) \right)^+$  implies the following Lemma.

**Lemma 1** (Intended payoff representation) *For  $P_0 = 1$ , the intended payoff to the insured  $P_1$  can be represented by*

$$P_1 = 1 + g + \alpha \left( \frac{A_1}{A_0} - K \right)^+, \text{ where } K = 1 + \frac{g}{\alpha}. \tag{2}$$

Thus,  $P_1$  can be stated in terms of the payoff of (i) a long position in  $e^{-r}P_0(1 + g)$  zero bonds maturing in one year ( $r$  denotes the c.c. interest rate) and (ii)  $\frac{\alpha P_0}{A_0}$  long calls on the synthetic asset  $A$  with maturity  $T = 1$  and strike  $\tilde{K} = A_0(1 + \frac{g}{\alpha})$ . Without default risk, the MRRG payoff is illustrated in Fig. 1. In particular, by pure dominance arguments, the (arbitrage free) value of a payoff which is always equal or sometimes even above another payoff must be higher than the value of the other payoff. Thus, two equally valuable payoffs  $P_1$  and  $\tilde{P}_1$  with  $\alpha > \tilde{\alpha}$  imply that  $g < \tilde{g}$ .<sup>7</sup>

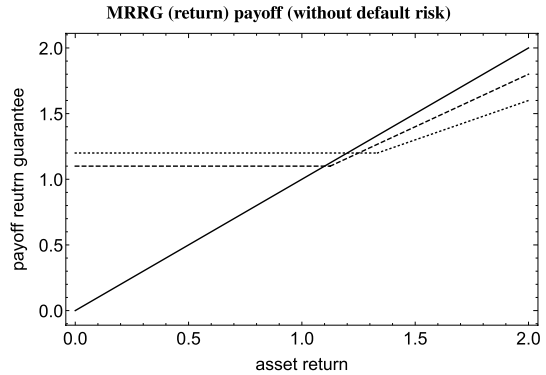
The assumption of a maturity  $T = 1$  implies some simplifications to our model: Because of the one period setting, the insured has no other premium payment option

<sup>6</sup> For different contract specifications within participation life insurance contracts, cf. Nielsen et al. (2011). Further details, in particular w.r.t. participating life insurance contracts with annual return rate guarantees which are common in German-speaking countries are given in Schmeiser and Wagner (2015).

<sup>7</sup> The properties of such contracts are analyzed in detail in Nielsen et al. (2011).



**Fig. 1** Intended MRRG payoff  $P_1$  depending on the asset return  $\frac{A_1}{A_0}$ . For varying asset return  $\frac{A_1}{A_0}$ , the figures illustrate guarantee return payoffs without default risk. The black solid line refers to  $(\alpha, g) = (1, -1)$  (no guarantee), the black dashed line is given by  $(\alpha, g) = (0.8, 0.1)$ , and the black dotted line is based on  $(\alpha, g) = (0.6, 0.2)$



than an upfront premium. Furthermore the insurer cannot suffer from death or surrender of the policyholder, such that the surrender and mortality risk is excluded from our analysis. Because of this, our optimization problem in the later Section is a purely state dependent portfolio optimization problem without time dependency. In this simplified setting, we find in the next Section model independent insights for any arbitrage free financial market model and in Sect. 3.3, we can derive the utility maximizing return payoff of the insured.<sup>8</sup>

**2.2 MRRG under default risk**

Considering default risk (DR), the insured only receives the payoff  $P_1$  if the asset value  $A_1$  is sufficiently high. The *actual* payoff to the insured under default risk is denoted by  $L_1 = P_1^{With\ DR}$  and is given by

$$L_1 = P_1 - (P_1 - A_1)^+ \text{ where}$$

$$(P_1 - A_1)^+ = \max\{P_1 - A_1, 0\} = \max \left\{ \left( 1 + \max \left\{ g, \alpha \left( \frac{A_1}{A_0} - 1 \right) \right\} \right) - A_1, 0 \right\} \tag{3}$$

can be interpreted as the default put option of the contract provider. Although the default put option is given in terms of a nested version of the max operator (a compound option feature), it is possible to disentangle the payoff in terms of the payoffs of plain vanilla options, only. Notice that the initial value of the asset side is given by  $A_0 = P_0 + E_0$ . Normalizing  $P_0 = 1$  and setting  $E_0 = \alpha^{(E)}$  gives  $A_1 = (1 + \alpha^{(E)}) \frac{A_1}{A_0}$  such that

<sup>8</sup> The assumption of a maturity  $T = 1$  gives us the possibility to state the payoff of the insured in closed-form. For a maturity  $T > 1$  this is not possible anymore. See for example the comment in the paper of Schmeiser and Wagner (2015) on page 669.

$$(P_1 - A_1)^+ = \max \left\{ \left( 1 + \max \left\{ g, \alpha \left( \frac{A_1}{A_0} - 1 \right) \right\} \right) - (1 + \alpha^{(E)}) \frac{A_1}{A_0}, 0 \right\}.$$

In particular, there is only one random variable  $\frac{A_1}{A_0}$  involved. An intuitive interpretation of the payoff  $L_1$  is possible if one considers the payoff of the default put option as a function of the random outcome of the investment decisions of the insurer, i.e. as a function of the asset increment  $\frac{A_1}{A_0}$ , and to distinguish between the strikes  $K_1, K_2$ , and  $K_3$  defined by

$$K_1 := 1 + \frac{g}{\alpha}, \quad K_2 := \frac{1 + g}{1 + \alpha^{(E)}} \text{ and } K_3 := \frac{1 - \alpha}{1 - \alpha + \alpha^{(E)}}. \tag{4}$$

$K_1$  defines the level of  $\frac{A_1}{A_0}$  such that the inner option (the call option of the insured due to the participation in the excess returns) is in the money, i.e. where the intended payoff  $P_1$  pays out  $1 + \alpha \left( \frac{A_1}{A_0} - 1 \right)$  instead of  $1 + g$ . Now, the put option (of the equity holders) can be in the money in both cases, i.e. we can observe (i) the intended payoff  $P_1$  is equal to  $1 + g$ , but the asset side  $A_1$  is lower, i.e.  $A_1 < 1 + g \Leftrightarrow \frac{A_1}{A_0} < \frac{1+g}{1+\alpha^{(E)}} = K_2$ , and (ii) the intended payoff  $P_1$  is equal to  $1 + \alpha \left( \frac{A_1}{A_0} - 1 \right)$ , but the asset side  $A_1$  is lower, i.e.  $A_1 < 1 + \alpha \left( \frac{A_1}{A_0} - 1 \right) \Leftrightarrow \frac{A_1}{A_0} < \frac{1-\alpha}{1-\alpha+\alpha^{(E)}} = K_3$ .

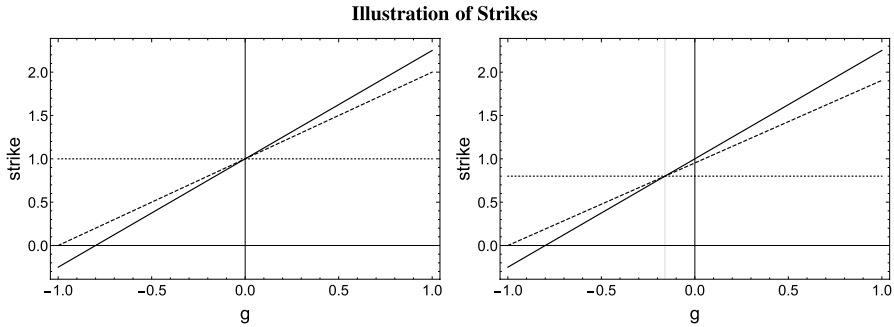
In consequence, we can express the payoff of the default put option by means of piecewise linear functions as follows:

$$\begin{aligned} (P_1 - A_1)^+ &= \left( (1 + g) - (1 + \alpha^{(E)}) \frac{A_1}{A_0} \right) 1_{\left\{ \frac{A_1}{A_0} \leq \min\{K_1, K_2\} \right\}} \\ &\quad + \left( 1 + \alpha \left( \frac{A_1}{A_0} - 1 \right) - (1 + \alpha^{(E)}) \frac{A_1}{A_0} \right) 1_{\left\{ K_1 < \frac{A_1}{A_0} < \max\{K_1, K_3\} \right\}}, \\ \text{i.e. } (P_1 - A_1)^+ &= (1 + \alpha^{(E)}) \left( K_2 - \frac{A_1}{A_0} \right) 1_{\left\{ \frac{A_1}{A_0} \leq \min\{K_1, K_2\} \right\}} \\ &\quad + (1 + \alpha^{(E)} - \alpha) \left( K_3 - \frac{A_1}{A_0} \right) 1_{\left\{ K_1 < \frac{A_1}{A_0} < \max\{K_1, K_3\} \right\}}. \end{aligned}$$

A crucial distinction is given by a different ranking order of the strikes  $K_1, K_2$  and  $K_3$ . However, the relation between the strikes is given by comparing the equity to debt ratio  $\alpha^{(E)}$  to the guarantee  $g$  (and participation rate  $\alpha$ ). A visualization is given in Fig. 2. The result is summarized in the following Lemma.

**Lemma 2 (Strikes)** *Let  $K_1, K_2$ , and  $K_3$  be defined as in Equation (4), then the following relations hold*

- (i)  $K_1 = K_2 = K_3 \Leftrightarrow \alpha^{(E)} = \frac{-g(1-\alpha)}{\alpha+g}$
- (ii)  $K_1 > K_2 > K_3 \Leftrightarrow \alpha^{(E)} > \frac{-g(1-\alpha)}{\alpha+g}$ ,



**Fig. 2** Strikes depending on the guarantee value  $g$ . For varying guarantees  $g$ , the figures illustrate the strikes  $K_1$  (black solid line),  $K_2$  (black dashed line), and  $K_3$  (black dotted line). The left hand figure is based on  $\alpha^{(E)} = 0$  (where the ordering of the strikes changes at  $g = 0$ ) while the right hand figure is based on  $\alpha^{(E)} \geq 0$  (where the ordering of the strikes changes at  $g < 0$ )

(iii)  $K_3 > K_2 > K_1 \Leftrightarrow \alpha^{(E)} < \frac{-g(1-\alpha)}{\alpha+g}$ .

In particular, the relation  $\alpha^{(E)} > \frac{-g(1-\alpha)}{\alpha+g} \Leftrightarrow g > \frac{-\alpha\alpha^{(E)}}{1-\alpha+\alpha^{(E)}} \tag{5}$

and  $g \geq 0$  implies  $\alpha^{(E)} \geq \frac{-g(1-\alpha)}{\alpha+g}$ . In addition, notice that case (ii) ((iii), respectively) in fact means a rather high (low) equity to debt ratio compared to the guarantee  $g$ .

In summary, the payoff (return) of the default put can be represented as follows.

**Proposition 1** (Payoff representation of the defaultable put) *The payoff of the defaultable put can be stated in terms of a piecewise linear function in the asset increment  $\frac{A_1}{A_0}$ , i.e.*

$$(P_1 - A_1)^+ = \begin{cases} (1 + \alpha^{(E)}) \left( K_2 - \frac{A_1}{A_0} \right) 1_{\left\{ \frac{A_1}{A_0} \leq K_1 \right\}} \\ + (1 + \alpha^{(E)} - \alpha) \left( K_3 - \frac{A_1}{A_0} \right) 1_{\left\{ K_1 < \frac{A_1}{A_0} < K_3 \right\}} & \text{for } \alpha^{(E)} \leq \frac{-g(1-\alpha)}{\alpha+g} \\ (1 + \alpha^{(E)}) \left( K_2 - \frac{A_1}{A_0} \right)^+ & \text{for } \alpha^{(E)} > \frac{-g(1-\alpha)}{\alpha+g} \end{cases} \tag{6}$$

An intuitive way to understand the liability side under default risk is analogously given by stating the payoff  $L_1$  depending on the asset increment  $\frac{A_1}{A_0}$ . First notice that, without default risk, the call option of the insured (cf. Lemma 1) is in the money if  $\frac{A_1}{A_0} > K_1 = 1 + \frac{g}{\alpha}$ . Otherwise the intended return is  $1 + g$ . Under default risk, the insured only receives  $1 + g$  if this is possible, i.e. if  $A_1 > P_0(1 + g)$  ( $P_0 = 1, A_0 = 1 + \alpha^{(E)}$ ), or equivalently if  $\frac{A_1}{A_0} > K_2 = \frac{1+g}{1+\alpha^{(E)}}$ . For  $\frac{A_1}{A_0} \leq K_1 = 1 + \frac{g}{\alpha}$ , the insured only receives the minimum of  $1 + g$  and  $A_1 = (1 + \alpha^{(E)}) \frac{A_1}{A_0}$ .

Now, consider the case that  $\frac{A_1}{A_0} > K_1 = 1 + \frac{g}{\alpha}$ , i.e.  $P_1 = 1 + \alpha \left( \frac{A_1}{A_0} - 1 \right)$ . Again, under default risk, the insured nevertheless only receives the lower of  $1 + \alpha \left( \frac{A_1}{A_0} - 1 \right)$

and  $A_1 = (1 + \alpha^{(E)}) \frac{A_1}{A_0}$ , which is defined by the benchmark  $K_3 = \frac{1-\alpha}{1-\alpha+\alpha^{(E)}}$ . In summary, we obtain

$$L_1 = \begin{cases} (1 + \alpha^{(E)}) \frac{A_1}{A_0} & \text{for } \frac{A_1}{A_0} < \min\{K_1, K_2\} \\ 1 + g & \text{for } \min\{K_1, K_2\} \leq \frac{A_1}{A_0} < K_1 \\ (1 + \alpha^{(E)}) \frac{A_1}{A_0} & \text{for } K_1 \leq \frac{A_1}{A_0} < \max\{K_1, K_3\} \\ 1 + \alpha \left( \frac{A_1}{A_0} - 1 \right) & \text{for } \frac{A_1}{A_0} \geq \max\{K_1, K_2, K_3\}. \end{cases}$$

Using Lemma 2 immediately gives the following Proposition.<sup>9</sup>

**Proposition 2** (Payoff representation of liabilities) *Let  $K_1, K_2$  and  $K_3$  be defined as in Equation (4). For  $P_0 = 1$  and  $\alpha^{(E)} = E_0 (A_0 = 1 + \alpha^{(E)})$  it holds*

- (i) *Low equity to debt ratio: For  $\alpha^{(E)} \leq \frac{-g(1-\alpha)}{\alpha+g}$ , the payoff (return) to the insured is given by*

$$L_1 = \begin{cases} (1 + \alpha^{(E)}) \frac{A_1}{A_0} & \text{for } \frac{A_1}{A_0} < K_3 \\ 1 + \alpha \left( \frac{A_1}{A_0} - 1 \right) & \text{for } \frac{A_1}{A_0} \geq K_3, \end{cases} \tag{7}$$

i.e.  $L_1 = (1 + \alpha^{(E)}) \frac{A_1}{A_0} - (1 - \alpha + \alpha^{(E)}) \left( \frac{A_1}{A_0} - K_3 \right)^+$

- (ii) *High equity to debt ratio: For  $\alpha^{(E)} > \frac{-g(1-\alpha)}{\alpha+g}$  it holds*

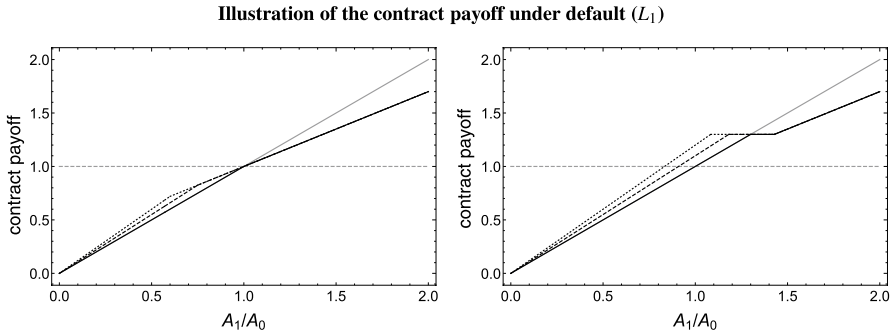
$$L_1 = \begin{cases} (1 + \alpha^{(E)}) \frac{A_1}{A_0} & \text{for } \frac{A_1}{A_0} < K_2 \\ 1 + g & \text{for } K_2 \leq \frac{A_1}{A_0} < K_1 \\ 1 + \alpha \left( \frac{A_1}{A_0} - 1 \right) & \text{for } \frac{A_1}{A_0} \geq K_1, \end{cases} \tag{8}$$

i.e.  $L_1 = (1 + \alpha^{(E)}) \frac{A_1}{A_0} - (1 + \alpha^{(E)}) \left( \frac{A_1}{A_0} - K_2 \right)^+ + \alpha \left( \frac{A_1}{A_0} - K_1 \right)^+$

<sup>9</sup>

$$\min\{K_1, K_2\} = \begin{cases} K_1 & \text{for } \alpha^{(E)} \leq \frac{-g(1-\alpha)}{\alpha+g} \\ K_2 & \text{for } \alpha^{(E)} > \frac{-g(1-\alpha)}{\alpha+g} \end{cases}, \quad \max\{K_1, K_3\} = \begin{cases} K_3 & \text{for } \alpha^{(E)} \leq \frac{-g(1-\alpha)}{\alpha+g} \\ K_1 & \text{for } \alpha^{(E)} > \frac{-g(1-\alpha)}{\alpha+g} \end{cases},$$

$$\max\{K_1, K_2, K_3\} = \begin{cases} K_3 & \text{for } \alpha^{(E)} \leq \frac{-g(1-\alpha)}{\alpha+g} \\ K_1 & \text{for } \alpha^{(E)} > \frac{-g(1-\alpha)}{\alpha+g} \end{cases}.$$



**Fig. 3** Contract payoffs  $L_1$  depending on the asset return  $\frac{A_1}{A_0}$ . For varying the asset return  $\frac{A_1}{A_0}$ , the figures illustrate the payoff  $L_1 = P_1 - (P_1 - A_1)^+$ . It holds  $0 = \alpha_1^{(E)} < \alpha_2^{(E)} < \alpha_3^{(E)}$ . The black solid line refer to  $\alpha^{(E)} = 0$ , the black dashed line to  $\alpha^{(E)} = \alpha_2^{(E)}$ , and the dotted line to  $\alpha^{(E)} = \alpha_3^{(E)}$ . The left hand figure is based on  $\alpha^{(E)} \leq \frac{-g(1-\alpha)}{\alpha+g}$  (low equity fraction) while the right hand figure is based on  $\alpha^{(E)} > \frac{-g(1-\alpha)}{\alpha+g}$  (high equity fraction) while. In particular, the payoffs on the left hand side are piecewise concave and convex while the payoffs on the right hand side are concave

For a low equity to debt ratio (Case (i)), the above Proposition states that the liabilities of the insured are given by the payoff of

- (i)  $\frac{1+\alpha^{(E)}}{A_0}$  long positions in the insurer’s assets  $A$  and
- (ii)  $\frac{1-\alpha+\alpha^{(E)}}{A_0}$  short calls with strike  $K = K_3 A_0 = \frac{1-\alpha}{1-\alpha+\alpha^{(E)}} A_0$ .

For a high equity to debt ratio (Case (ii)), the above Proposition states that the liabilities of the insured are given by the payoff of

- (i)  $\frac{1+\alpha^{(E)}}{A_0}$  long positions in the insurer’s assets  $A$ ,
- (ii)  $\frac{1+\alpha^{(E)}}{A_0}$  short positions in a call on  $A$  with strike  $K = K_2 A_0 = \frac{1+g}{1+\alpha^{(E)}} A_0$  and
- (iii)  $\frac{\alpha}{A_0}$  long calls with strike  $K = K_1 A_0 = \left(1 + \frac{g}{\alpha}\right) A_0$ .

In addition, the above Proposition immediately implies the following important properties of the liability payoffs.

**Corollary 1** (Properties of the liability payoff) *Let  $L_1$  be the liability payoff stated in Proposition 2, then it holds*

- (i)  $L_1$  is increasing in  $g$  and  $\alpha^{(E)}$ . For  $g > 0$ ,  $L_1$  is increasing in  $\alpha$ .
- (ii) For  $\alpha^{(E)} \leq \frac{-g(1-\alpha)}{\alpha+g}$ ,  $L_1$  is concave in  $\frac{A_1}{A_0}$ .
- (iii) For  $\alpha^{(E)} > \frac{-g(1-\alpha)}{\alpha+g}$ ,  $L_1$  is piecewise concave and piecewise convex in  $\frac{A_1}{A_0}$ .

An illustration of  $L_1$  is given in Fig. 3. The left hand figure is based on  $\alpha^{(E)} \leq \frac{-g(1-\alpha)}{\alpha+g}$  (low equity fraction) while the right hand figure is based on the case

$\alpha^{(E)} > \frac{-g(1-\alpha)}{\alpha+g}$  (*high equity fraction*). In particular, the payoffs on the left hand side are concave while the payoffs on the right hand side are piecewise concave and convex while. Intuitively, it is clear that a higher amount of equity means that the real degree of guarantee is, *ceteris paribus*, higher than for a lower amount of equity. This is resembled in the payoff profiles, i.e. a higher amount of equity gives more convexity to the payoff profile (implying a more valuable guarantee).

### 2.3 Fair pricing and regulatory requirements

Throughout the following analysis, we make some assumptions on the contract design (and the model setup for the financial market). We assume that the financial market model is arbitrage free. Furthermore, we assume that, because of competition, the contracts are fairly priced such that no arbitrage is introduced (among the insurers and between the insurance products and the financial market products):

**Assumption 1** (*No arbitrage*) We assume that the financial market model is arbitrage free. Thus, the fundamental theorem of asset pricing implies the existence of an equivalent pricing measure  $\mathbb{P}^*$  such that the price of any traded asset  $X$  with payoff  $X_T$  at  $T > 0$  is given by the expected discounted payoff under  $\mathbb{P}^*$ , i.e.

$$X_0 = \mathbb{E}_{\mathbb{P}^*} \left[ e^{-\int_0^T \tilde{r}_u du} X_T \right], \quad (9)$$

where  $\tilde{r}_u$  denotes the forward rate, such that  $\int_0^T \tilde{r}_u du$  is the continuously compounded interest rate prevailing at time  $T$ .

**Assumption 2** (*Fair pricing*) We assume competition between the insurance companies (and with the opportunity to invest in the financial market). In particular, we thus assume that the insurance contracts are fairly priced, i.e. depending on the investment decisions which are carried out by the insurer on the financial market, the contract prices are given by the arbitrage free (financial market) prices.<sup>10</sup>

**Assumption 3** (*Stakeholders*) The policyholders are not able to participate at the arbitrage free financial market, such that they cannot replicate future cash-flows. They just have the possibility to invest in the asset side of the insurance company. The insurer itself, resp. its shareholders, of course have this access to the market.<sup>11</sup>

In addition, we assume later that an admissible contract design must honor regulatory requirements as e.g. posed by an upper bound on the shortfall probability.

<sup>10</sup> It should be mentioned that in practice it would not be possible to e.g. make sure that all these contracts are initially fair: Rather, in practice, cross-subsidizing effects are unavoidable (cf. e.g. Hieber et al. (2015)).

<sup>11</sup> This assumption is reasonable and has often been used in other literature dealing with this topic, e.g. Schmeiser and Wagner (2015) or Briys and de Varenne (1997).

First, we consider the assumption on the contract pricing and the implications of postulating an arbitrage free financial model setup. Subsequently, we introduce the regulatory requirement and represent the shortfall probability in terms of the strikes introduced above.

Along the lines of Proposition 2, the arbitrage free value of the liabilities (and the default put, respectively) is given by the (arbitrage free) value of the corresponding portfolio of plain vanilla options. To simplify the exposition, we refer to a one year horizon, i.e. the call (or put) options have a maturity of  $T = 1$ . The (arbitrage free) value of a call (put) option (with maturity  $T = 1$ ) and strike  $K$  is denoted by  $Call(K)$  ( $Put(K)$ ). W.l.o.g., we refer to the options written on the increments  $\frac{A_1}{A_0}$  (which is implied by the investment strategy of the insurer), i.e. we use the relation

$$\left(\frac{A_1}{A_0} - K\right)^+ = \frac{1}{A_0} (A_1 - KA_0)^+.$$

To be more precise,  $Call(K)$  ( $Put(K)$ ) denotes the  $t = 0$  value of the  $T = 1$  payoff  $\left(\frac{A_1}{A_0} - K\right)^+$  ( $\left(K - \frac{A_1}{A_0}\right)^+$ , respectively).

**Proposition 3** (Fair pricing conditions) *Assume that the asset  $A$  can be synthesized by a financial market strategy, i.e. the  $t = 0$  price of the payoff  $A_1$  is  $A_0$  ( $A$  is an asset paying no dividends). In addition, assume that the financial market is arbitrage free. Then, the fair pricing condition (posed by the normalization  $P_0 = 1$ ) is given by the condition that the market consistent price of the payoff  $L_1$  is equal to  $P_0 = 1$ . In particular, depending on the equity fraction  $\alpha^{(E)}$ , the guarantee  $g$ , and the participation rate  $\alpha$ , the following pricing conditions hold:*

(i) **Low equity to debt ratio:** For  $\alpha^{(E)} \leq \frac{-g(1-\alpha)}{\alpha+g}$ , it holds

$$1 = 1 + \alpha^{(E)} - (1 - \alpha + \alpha^{(E)})Call(K_3). \tag{10}$$

(ii) **High equity to debt ratio:** For  $\alpha^{(E)} > \frac{-g(1-\alpha)}{\alpha+g}$ , it holds

$$1 = 1 + \alpha^{(E)} - (1 + \alpha^{(E)})Call(K_2) + \alpha Call(K_1). \tag{11}$$

where the strikes  $K_1, K_2$  and  $K_3$  are defined as in Equation (4).

**Corollary 2** (Properties of fair contracts under default risk) *The fair pricing conditions imply the following properties*

(i) For  $\alpha^{(E)} = 0$ , a fair contract implies  $\alpha^{fair} = 1$ .

(ii) In the special case that  $g = -1$  (no guarantee) it also holds  $\alpha^{fair} = 1$ .

The proof is straightforward and the results are intuitive: Part (i) states that without equity, the insured face the whole risk of the asset investments, i.e. the (fair)

liabilities are given by  $L_1 = \frac{A_1}{A_0}$ . In particular, without further restrictions on the distribution of  $\frac{A_1}{A_0}$ , i.e. restrictions on the riskiness of the investment strategy, there is no guarantee without equity. The interpretation of part (ii) is analogous. Since there is no guarantee if  $g = -1$ , a fair contract must imply  $L_1 = \frac{A_1}{A_0}$ .

Now consider the condition that there is a regulatory requirement on the shortfall probability. Assume that the regulator requires an upper bound  $\epsilon$  for the probability that the intended *guaranteed* accumulation  $P_1$  is not honored because the asset value  $A_1$  is lower, i.e.

$$\mathbb{P}(A_1 < P_1) \leq \epsilon. \tag{12}$$

Again, normalizing  $P_0 = 1$  and using  $A_1 = (1 + \alpha^{(E)})\frac{A_1}{A_0}$  implies that the event  $\{A_1 < P_1\}$  can be represented in terms of the strikes  $K_1 = 1 + \frac{g}{\alpha}$ ,  $K_2 = \frac{1+g}{1+\alpha^{(E)}}$  and  $K_3 = \frac{1-\alpha}{1-\alpha+\alpha^{(E)}}$ :

$K_1$  defines the level of  $\frac{A_1}{A_0}$  such that the inner option is in the money, i.e. where the intended payoff  $P_1$  pays out  $1 + \alpha\left(\frac{A_1}{A_0} - 1\right)$  instead of  $1 + g$ . The strike  $K_2$  defines the level of  $\frac{A_1}{A_0}$  such that the put option is in the money, i.e. the intended Payoff  $P_1$  is equal to  $1 + g$ , but the asset side  $A_1$  is lower.  $K_3 = \frac{1-\alpha}{1-\alpha+\alpha^{(E)}}$  defines the level of  $\frac{A_1}{A_0}$  where the liabilities can not be satisfied if the inner option is in the money, i.e.

$$\{A_1 < P_1\} = \left\{ \frac{A_1}{A_0} \leq K_1; \frac{A_1}{A_0} < K_2 \right\} \cup \left\{ \frac{A_1}{A_0} > K_1; \frac{A_1}{A_0} < K_3 \right\}. \tag{13}$$

With Lemma 2 and the representation of the shortfall event in Equation (13), we immediately obtain the following Proposition.

**Proposition 4** (Shortfall probability) *The shortfall probability  $SFP := \mathbb{P}(A_1 < P_1)$  is given by*

$$\begin{aligned} SFP &= \mathbb{P}\left(\frac{A_1}{A_0} < \min\{K_1, K_2\}\right) + \mathbb{P}\left(K_1 \leq \frac{A_1}{A_0} \leq \max\{K_1, K_3\}\right) \\ &= \mathbb{P}\left(\frac{A_1}{A_0} < K_3\right) 1_{\{\alpha^{(E)} \leq \frac{-g(1-\alpha)}{\alpha+g}\}} + \mathbb{P}\left(\frac{A_1}{A_0} \leq K_2\right) 1_{\{\alpha^{(E)} > \frac{-g(1-\alpha)}{\alpha+g}\}}. \end{aligned} \tag{14}$$

It is worth to emphasize that, e.g. in the context of Solvency II, the upper bound on the shortfall probability determines the amount of equity which is needed to assure the solvency to a high degree, i.e. to honor the liabilities to the insured. Recall that  $K_2 = \frac{1+g}{1+\alpha^{(E)}}$  and  $K_3 = \frac{1-\alpha}{1-\alpha+\alpha^{(E)}}$ . Obviously, the lower the strike is, the lower is the probability that the value of a given investment strategy drops below the strike. Since the above strikes are decreasing in the equity fraction  $\alpha^{(E)}$ , a higher equity fraction is able to reduce the shortfall probability.<sup>12</sup>

<sup>12</sup> However, if one assumes a complete financial market model, any reduction in the shortfall probability can also be implemented by a change in the asset distribution by means of a suitable investment strategy.



### 2.4 Black and Scholes model setup and illustration

Along the lines of the previous subsections, the contracts can be fairly priced in closed form in any arbitrage free model setup which allows closed form solutions of plain vanilla options. For the sake of simplicity, we place ourselves in a Black and Scholes model setup to give some illustrations. The financial market model over the filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$  is given by the Black and Scholes model, i.e. there are two investment possibilities, a risky asset  $S$  and a risk-free asset  $B$  which accumulates according to a constant interest rate  $r$ . The filtration  $(\mathcal{F}_t)_{t \in [0, T]}$  is generated by the standard Brownian motion  $(W_t)_{t \in [0, T]}$ . Because of the completeness of the Black and Scholes model, there exists a uniquely determined equivalent martingale measure  $\mathbb{P}^*$  under which the process  $(W_t^*)_{t \in [0, T]}$  defines a standard Brownian motion. In particular, the risky asset  $(S_t)_{t \in [0, T]}$  and risk free bond dynamics  $(B_t)_{t \in [0, T]}$  are given by

$$\begin{aligned} dS_t &= S_t(\mu dt + \sigma dW_t) = S_t(r dt + \sigma dW_t^*), \quad S_0 = s \\ dB_t &= B_t r dt, \quad B_0 = b. \end{aligned}$$

Under the real world probability measure  $\mathbb{P}$ , the asset price follows a geometric Brownian motion with constant drift  $\mu$  ( $\mu > r$ ) and constant volatility  $\sigma$  ( $\sigma > 0$ ). Under the uniquely defined equivalent martingale measure (pricing measure)  $\mathbb{P}^*$ , the asset price follows a geometric Brownian motion with constant drift  $r$  and constant volatility  $\sigma$  ( $\sigma > 0$ ). The risk free bond  $B$  grows at a constant interest rate  $r$ .

#### 2.4.1 Constant mix strategies

Assuming that the insurer decides to implement an investment strategy which is described by a constant fraction of wealth  $m^{(A)}$  invested in the risky asset (and the remaining fraction  $1 - m^{(A)}$  is invested in the risk free bond) implies that the asset process is also given by a lognormal process, i.e.

$$dA_t = A_t \left( m^{(A)} \frac{dS_t}{S_t} + (1 - m^{(A)})r dt \right).$$

Thus, w.r.t. an investment horizon of  $T = 1$ , it holds

$$\begin{aligned} A_1 &= A_0 e^{\mu_A^{(RW)} - \frac{1}{2}\sigma_A^2 + \sigma_A W_1} = A_0 e^{r - \frac{1}{2}\sigma_A^2 + \sigma_A W_1^*} \\ \text{where } \mu_A^{(RW)} &= m^{(A)}\mu + (1 - m^{(A)})r \text{ and } \sigma_A = m^{(A)}\sigma. \end{aligned}$$

$\mu^{(RW)}$  denotes the drift of the asset dynamics under the real word measure  $\mathbb{P}$ . Under the pricing measure  $\mathbb{P}^*$ , the drift is equal to  $r$ . In particular, let  $N(\mu, \sigma^2)$  denote the normal distribution with mean  $\mu$  and variance  $\sigma^2$  and  $\Phi(\cdot)$  the cumulative distribution function of the standard normal distribution. Then it holds

$$\ln \frac{A_1}{A_0} \sim N\left(\mu_A - \frac{1}{2}\sigma_A^2, \sigma_A^2\right) \text{ under } \mathbb{P}, \quad \ln \frac{A_1}{A_0} \sim N\left(r - \frac{1}{2}\sigma_A^2, \sigma_A^2\right) \text{ under } \mathbb{P}^*.$$

In consequence, the arbitrage free (competitive) price of the liabilities  $L_1$  (the default put, respectively) can be derived by means of Proposition 3 where the call price formula  $Call(K) = Call^{(BS)}(K, \sigma_A)$  is given by the Black and Scholes pricing formula (w.r.t. the returns), i.e.

$$Call^{(BS)}(K, \sigma_A) = \Phi(d_1(K, \sigma_A)) - e^{-r}K\Phi(d_2(K, \sigma_A)),$$

$$\text{where } d_1(K, \sigma_A) = \frac{-\ln K + r + \frac{1}{2}\sigma_A^2}{\sigma_A} \text{ and } d_2(K, \sigma_A) = d_1(K, \sigma_A) - \sigma_A. \tag{15}$$

Figure 4 gives an illustration of fair contract designs. The left figure illustrates fair tuples of the contract parameter  $(\alpha, g)$ . Along the lines of the model free results, the (return) payoff of the MRRG under default risk is increasing in  $\alpha$  and  $g$ . Thus, in order to stay on a fair contract design, an increasing guarantee  $g$  must be compensated by decreasing the participation rate  $\alpha$ . In addition, the fair  $(\alpha, g)$  combinations are lower for higher equity fractions, i.e. the black line refers to  $\alpha^{(E)} = \alpha_1^{(E)} = 0.01$ , the black dashed line to  $\alpha^{(E)} = \alpha_2^{(E)} = 0.02$ , and the dotted line to  $\alpha^{(E)} = \alpha_3^{(E)} = 0.05$ . This result is straightforward and can, for example, be found in Grosen and Jørgensen (2002). An interesting effect arises in view of the piecewise concave and piecewise convex payoff structures (implied by  $g > 0$  and  $\alpha^{(E)} > 0$ , cf. Corollary 1). Although the contract value is increasing in the equity fraction  $\alpha^{(E)}$ , this is not necessarily true with respect to the riskiness of the investments, i.e. w.r.t.  $m^{(A)}$  (the volatility  $\sigma_A = m^{(A)}\sigma$ , respectively). Thus, for a fixed equity fraction  $\alpha^{(E)}$ , there may be two investment fractions  $m^{(A,1)}$  and  $m^{(A,2)}$  such that the contract is fairly priced. This is illustrated in the right hand plot of Fig. 4 which depicts fair contracts for the benchmark case in terms of fair combinations of the equity fraction  $\alpha^{(E)}$  and the investment fraction  $m^{(A)}$  (defining the volatility of the assets, i.e.  $\sigma_A = m^{(A)}\sigma$ ). The solid line refers to  $\alpha = 0.9$ , the dashed line refers to a lower participation fraction  $\alpha = 0.85$  and the dotted line refers to  $\alpha = 0.8$ . For the shortfall probability given in Proposition 4, the Black and Scholes model setup immediately implies

$$SFP = \Phi(\bar{d}_0(K_3))1_{\{\alpha^{(E)} \leq \frac{-g(1-\alpha)}{\alpha+g}\}} + \Phi(\bar{d}_0(K_2))1_{\{\alpha^{(E)} > \frac{-g(1-\alpha)}{\alpha+g}\}},$$

$$\text{where } \bar{d}_0(K) := \frac{\ln K - (\mu_A - \frac{1}{2}\sigma_A^2)}{\sigma_A}. \tag{16}$$

Again, notice that, e.g. in the context of Solvency II, the upper bound on the shortfall probability is posed to determine the amount of equity which is needed to assure the solvency to a high degree, i.e. to honor the liabilities to the insured. Recall that  $K_2 = \frac{1+g}{1+\alpha^{(E)}}$  and  $K_3 = \frac{1-\alpha}{1-\alpha+\alpha^{(E)}}$ . Obviously, the lower the strike is, the lower is the

probability of a constant mix strategy that its terminal value drops below the strike. Since the above strikes are decreasing in the equity fraction  $\alpha^{(E)}$ , a higher equity fraction is able to reduce the shortfall probability, cf. Figure 5 for an illustration. It is worth noticing that any reduction of the shortfall probability can also be obtained by suitably adjusting the investment strategy, i.e. the distribution of  $\frac{A_1}{A_0}$ .

### 3 Optimal design of quantile guarantees

The following section discusses, from the perspective of the insured, the optimal design of a MRRG under default risk and an upper bound on the shortfall probability. A fair contract design which provides a higher (expected) utility to the insured is also beneficial to the insurance company. The contract provider competes with other insurers and the financial market. Choosing among different contracts, the insured selects the contract which provides herself the highest (expected) utility. Throughout the following, we assume that the preferences of the insured are described by a utility function  $u = u^{(CRRA)}$  implying a constant relative risk aversion (CRRA) denoted by  $\gamma$ , i.e.  $u^{(CRRA)}(x) = \frac{x^{1-\gamma}}{1-\gamma}$  ( $\gamma > 1$ ) and  $u^{(CRRA)}(x) = \ln x$  ( $\gamma = 1$ ). Assuming CRRA preferences has its merits. There are empirical investigations which justify CRRA preference, cf. e.g. Chiappori and Paiella (2011). In addition, CRRA utility allows that the analysis is based on returns.<sup>13</sup> The relevant optimization problem is posed by maximizing the expected utility of the insured under constraints posed by a competitive market (fair pricing) and the restrictions posed by the regulator.<sup>14</sup> In the first instance, we formulate the optimization problem without stating the optimization arguments, i.e.

$$\max \mathbb{E}_{\mathbb{P}} [u(L_1)] \text{ s.t. } \mathbb{P}(A_1 < P_1) \leq \epsilon, \mathbb{E}_{\mathbb{P}^*} [e^{-r}A_1] = 1 + \alpha^{(E)} \text{ and } \mathbb{E}_{\mathbb{P}^*} [e^{-r}L_1] = 1. \quad (17)$$

The first condition states the regulatory requirement on the upper bound on the shortfall of the intended payoff (guarantee)  $P_1$ . The second condition ensures that the asset value  $A_1$  is obtainable by a self-financing investment strategy with initial investment  $A_0 = 1 + \alpha^{(E)}$ , and the third part captures the fair pricing of the liabilities. To shed further light on the (overall) optimal design of quantile guarantees, we

<sup>13</sup> It is worth mentioning that CRRA preferences can not explain the existence of (quantile) guarantees using, cf. Leland (1980). However one can understand that policy makers provide tax advantages for products with downside protection for old-age provision to reduce the risk of poverty among the elderly and possible implications for tax payers - even if downside protection reduces utility on the individual level for CRRA-type policyholders. For the effect of taxation on equity-linked life insurance we refer to Chen et al. (2019)

<sup>14</sup> The optimization procedure with a value at risk restriction can be referred to as a chance-constrained approach. It is transferable in a non-linear (deterministic) optimization program of normal or log normal returns are assumed (cf. McCabe and Witt (1980)). Basically, we also consider log normal payoffs for  $t = 1, 2, \dots$  under a Geometric Brownian Motion (GBM) assumption. However we have added the assumption that the insured is described by a constant relative risk aversion (CRRA) which gives further insights on the utility effects from the perspective of the insured.

discuss and compare (in the Black and Scholes model setup) different approaches concerning the arguments which are optimally chosen in the maximization problem (17) in order to maximize the utility which is provided to the insured. As a benchmark, we consider the optimal unconstrained strategy (no upper bound on the shortfall probability). For  $\alpha^{(E)} = 0$ , this is the classic Merton problem (cf. Merton (1971)). The solution implies the highest possible utility and thus provides an upper bound of the expected utility of all contract designs.

We also comment on an approach suggested in Schmeiser and Wagner (2015) who assume that the insurer implements a constant mix strategy, but can decide on the fraction of asset wealth which is invested riskily. The insurer simultaneously determines the equity fraction  $\alpha^{(E)}$  and the investment fraction  $m^{(A)}$  such that the pricing and shortfall constraints are satisfied for a given guarantee  $g$ . The utility to the insured is then maximized by selecting the guarantee  $g$  which gives the highest expected utility.

Finally, we consider the optimal solution under the pricing and shortfall constraints (without restricting the insurer’s investment strategy to constant mix strategies).

### 3.1 The Merton solution as a benchmark

Assume that the insured is not committed to select among MRRG contracts, only. Instead, assume that she can, without transaction costs, dynamically trade on the financial market. In terms of the MRRG contracts, this is the special case that  $\alpha^{(E)} = 0$  (the insured owns the asset side herself) and a vanishing shortfall probability bound  $\epsilon = 1$  (she is not restricted by the regulator). The optimization problem (17) then boils down to

$$\max_{A_1} \mathbb{E}_{\mathbb{P}} \left[ u \left( \frac{A_1}{A_0} \right) \right] \text{ s.t. } \mathbb{E}_{\mathbb{P}^*} \left[ e^{-r} \frac{A_1}{A_0} \right] = 1,$$

i.e. the investor chooses the optimal payoff  $L_1 = A_1$  (return, respectively,  $A_0 = P_0 = 1$ ).<sup>15</sup> Assuming a Black and Scholes model setup to describe the financial market model, gives the classic Merton problem. The solution is firstly stated in Merton (1971). Under the real world measure  $\mathbb{P}$ , the optimal payoff  $L_1^* = \frac{A_1^*}{A_0}$  is given by

$$\frac{A_1^*}{A_0} = e^{\mu_A^{(RW)} - \frac{1}{2}\sigma_A^2 + \sigma_A W_1},$$

where  $\mu_A^{(RW)} = m^{(A)}\mu + (1 - m^{(A)})r$ ,  $\sigma_A = m^{(A)}\sigma$  and  $m^{(A)} = \frac{\mu - r}{\gamma\sigma^2} =: m^{(Mer)}$ .

(18)

In the optimum, the investor uses a constant mix strategy where the fraction  $m^{(A)}$  of portfolio wealth which is invested riskily is given by the quotient of the (local) excess

<sup>15</sup> Recall that  $\alpha^{(E)} = 0$  implies  $\alpha = 1$ , cf. Corollary 2. With  $A_0 = 1$  it follows  $L_1 = A_1$ .

return  $(\mu - r)$  and the squared asset volatility scaled by the parameter of relative risk aversion  $\gamma\sigma^2$ . The certainty equivalent wealth/return  $CE$  which makes the investor indifferent to the Merton payoff is defined by the condition  $u(CE) = \mathbb{E}_{\mathbb{P}}[u(A_1)]$ , i.e.  $CE = u^{-1}(\mathbb{E}_{\mathbb{P}}[u(A_1)])$ . Straightforward calculations imply

$$CE^* = e^{r + \frac{(\mu-r)^2}{2\gamma\sigma^2}} =: CE^{(Mer)} \text{ and } y^{CE^*} = \ln CE^* = r + \frac{(\mu - r)^2}{2\gamma\sigma^2}, \tag{19}$$

where  $y^{CE^*}$  denotes the (optimal Merton) savings rate. Notice that the above  $CE^*$  defines an upper bound to all certainty equivalents which are implied by (admissible) MRRG contracts and refer to the upper bound by  $CE^{(Mer)}$ . Analogously, we refer to the optimal Merton payoff (fraction) by  $A_1^{(Mer)}$  ( $m^{(Mer)}$ ).

### 3.2 Upper bound on SFP and restriction to constant mix strategies

Schmeiser and Wagner (2015) consider the optimization problem under a SFP condition but assume that the insurer implements a constant mix strategy. In consequence, the insurer does not consider a quantile hedge to honor the guarantee. To ensure the SFP condition for a given guarantee, the insurer is restricted to suitable combinations of investment fractions and equity capital. Amongst other results, Schmeiser and Wagner (2015) consider the optimization problem

$$\max_{g \in \mathcal{G}} \mathbb{E}_{\mathbb{P}}[u(L_1)],$$

where  $\mathcal{G}$  denotes the set of admissible guarantee rates and where the equity fraction  $\alpha^{(E)}$  and the investment fraction of the asset side  $m^{(A)}$  are determined simultaneously by the conditions<sup>16</sup>

$$\mathbb{P}(A_1 < P_1) \leq \epsilon \text{ and } \mathbb{E}_{\mathbb{P}^*}[e^{-r}L_1] = 1.$$

Notice that  $\mathbb{P}(A_1 < P_1) = SFP$  is analytically given by Equation (16). The liability value  $\mathbb{E}_{\mathbb{P}^*}[e^{-r}L_1]$  is stated in Proposition 3 in combination with Equation (15).<sup>17</sup> A few comments are worth mentioning here: Schmeiser and Wagner (2015) consider the exact fulfillment of the shortfall probability corresponding to the minimum safety requirement where the ruin probability  $SFP$  is equal to the upper bound  $\epsilon$ . Intuitively, this is meaningful if the shortfall constraint is binding in the case without equity capital, i.e. if the upper bound on the shortfall probability  $\epsilon$  is sufficiently low compared to the lowest guarantee contained in the set  $\mathcal{G}$ . In addition, the authors consider an exogenously given participation fraction  $\alpha$  (e.g.  $\alpha = 0.9$  as implied by German legislation). However,  $\alpha$  ( $1 - \alpha$ , respectively) implicitly defines a guarantee

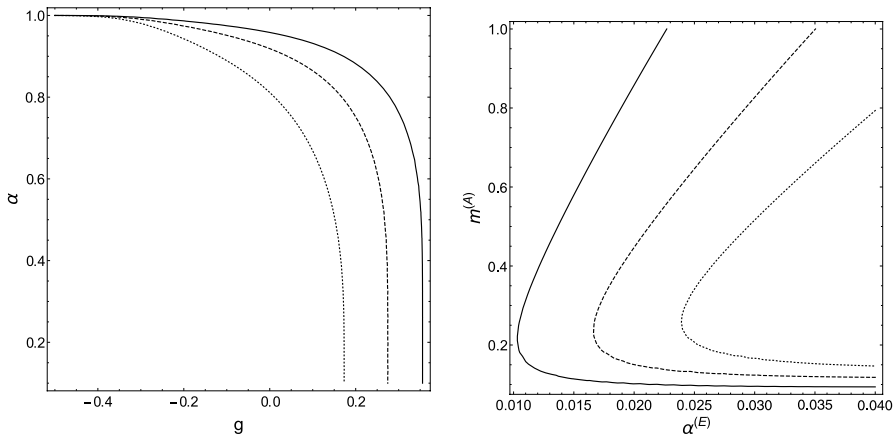
<sup>16</sup> Notice that the condition  $\mathbb{E}_{\mathbb{P}^*}[e^{-r}A_1] = 1 + \alpha^{(E)}$  is ensured since the insurer implements a constant mix strategy with initial investment  $1 + \alpha^{(E)}$ .

<sup>17</sup> Once the equity fraction  $\alpha^{(E)}$  and the investment fraction of the asset side  $m^{(A)}$  are determined, the expected utility (and CE) can be stated in quasi closed form. Schmeiser and Wagner (2015) determine the solution by Monte Carlo simulations.

**Table 1** Benchmark parameter setting

Model parameter			Contract parameter				Upper bound on SFP
$r$	$\mu$	$\sigma$	$P_0$	$A_0$	$\alpha$	$g$	$\epsilon$
0.03	0.07	0.2	1	$1+\alpha^{(E)}$	0.9	0.0175	0.005

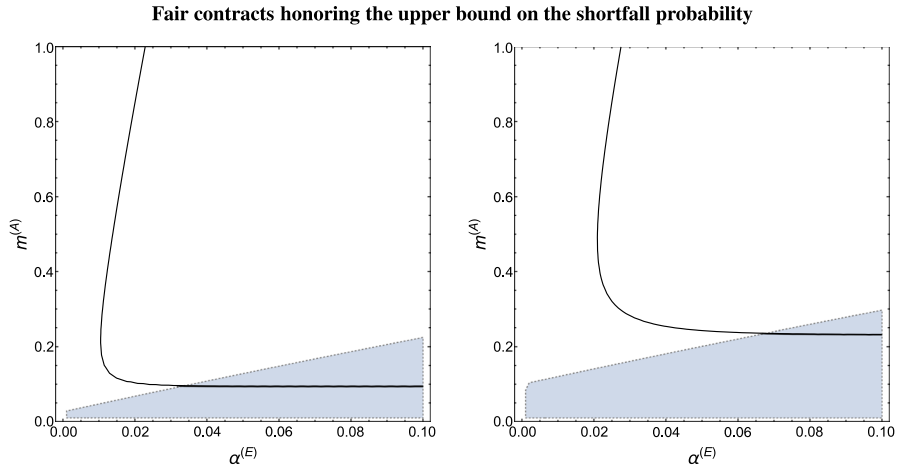
**Illustration of fair contracts (constant mix strategies)**



**Fig. 4** The contract and model parameters are given as in Table 1. The left figures illustrate fair tuples of the contract parameter  $(\alpha, g)$ . The black line refers to  $\alpha^{(E)} = \alpha_1^{(E)} = 0.01$ , the black dashed line to  $\alpha^{(E)} = \alpha_2^{(E)} = 0.02$ , and the dotted line to  $\alpha^{(E)} = \alpha_3^{(E)} = 0.05$ . The figure on the right hand side (the black line, respectively) depicts fair contracts for the benchmark case in terms of fair combinations of the equity fraction  $\alpha^{(E)}$  and the investment fraction  $m^{(A)}$  (defining the volatility of the assets, i.e.  $\sigma_A = m^{(A)}\sigma$ ). The solid line refers to  $\alpha = 0.9$ , the dashed line refers to a lower participation fraction  $\alpha = 0.85$  and the dotted line refers to  $\alpha = 0.8$

fee, i.e. the insured gives up some upside participation for downside protection. In particular, if  $\alpha$  is already sufficiently low (compared to  $g$ ), there does not exist an equity fraction  $\alpha^{(E)} \geq 0$  such that the (fair) pricing condition can be satisfied, cf. Fig. 4 and the results in Schmeiser and Wagner (2015).

As a numerical example, we refer to the benchmark parameter setting summarized in Table 1 and consider the above optimization problem for the guarantees  $g$ , taking the values  $g \in \mathcal{G} = \{-0.1, -0.095, \dots, 0.02, 0.025\}$  and a shortfall probability bound given by  $\epsilon = 0.005$ . For each  $g \in \mathcal{G}$ , Table 2 summarizes the combination of equity fraction  $\alpha^{(E)}$  and investment fraction  $m^{(A)}$  (implying that the SFP is exactly met and the contract is fairly priced) as well as the certainty equivalent contract wealths  $CEs$  of insureds which are described by three different levels of relative risk aversion ( $\gamma = 2, 3.56$ , and  $5.94$ ). In addition, the Merton solution is summarized in the upper line. For each level of relative risk aversion, the highest certainty equivalent (CE) is marked which implies the optimal guarantee rate. Observe that the  $CEs$  obtained by the (optimal) contracts are close to (but below) the Merton solution. In addition, the corresponding investment fractions  $m^{(A)}$  are close to (but above) the Merton fractions. Intuitively, this is explained by



**Fig. 5** If not otherwise mentioned, the contract and model parameters are given as in Table 1. The black lines depict the fair contracts in terms of fair combinations of the equity fraction  $\alpha^{(E)}$  and the investment fraction  $m^{(A)}$  (defining the volatility of the assets, i.e.  $\sigma_A = m^{(A)}\sigma$ ). The shaded region is the region where the upper bound on the shortfall probability ( $\epsilon = 0.005$ ) is honored. While the figure on the left hand side refers to the benchmark guarantee  $g = 0.0175$ , the right hand side is implied by  $g = -0.0175$  (color figure online)

the participation fraction  $\alpha$  which is (along the lines of the benchmark parametrization) equal to  $\alpha = 0.9$ , i.e. the investor gives up 10% of the upside returns.

### 3.3 Optimal quantile payoff

As mentioned above, the Black and Scholes model is complete such that any state dependent payoff is attainable, i.e. it can be synthesized by a self-financing strategy in the asset  $S$  and the risk free investment opportunity  $B$ . In addition with the assumption that the contracts are fairly priced, we can obtain the utility maximizing quantile guarantee payoff  $L_1$  with an initial investment of  $P_0 = 1$ , i.e. the optimal payoff is independent of the equity fraction  $\alpha^{(E)}$ . Thus, w.l.o.g. we can set  $\alpha^{(E)} = 0$ . Recall from Corollary 2 that for  $\alpha^{(E)} = 0$ , a fair contract implies  $\alpha = 1$ , i.e.  $L_1 = A_1 = \frac{A_1}{A_0}$  (since  $P_0 = 1$  and  $A_0 = 1 + \alpha^{(E)} = 1$ ), such that the optimization problem (17) simplifies to

$$\max_{A_1} \mathbb{E}_{\mathbb{P}} [u(A_1)] \text{ s.t. } \mathbb{P}(A_1 < 1 + g) \leq \epsilon \text{ and } \mathbb{E}_{\mathbb{P}^*} [e^{-r}A_1] = 1. \tag{20}$$

The solution to this problem can already fully be traced back to Basak and Shapiro (2001) who state the optimal payoff (in dependence of the state prices) under a terminal VaR constraint.<sup>18</sup>

<sup>18</sup> Basak and Shapiro (2001) state the optimal solution in dependence of the state prices for a general class of utility functions in a dynamic complete market setup where the investor can choose between one risk-less bond and several risky stocks.

**Table 2** The table states, for the benchmark parameter setting summarized in Table 1, the results of the optimization problem constrained to constant mix strategies for the set of guarantees  $g \in \mathcal{G} = \{-0.1, -0.095, \dots, 0.02, 0.025\}$  and a shortfall probability bound given by  $\epsilon = 0.005$

$g$	$\alpha^{(E)}$	$m^{(A)}$	$L_0$	SFP	$CE^{\gamma=2}$ $m^{(Mer)} = 0.5$ $CE^{(Mer)} = 1.0408$	$CE^{\gamma=3.56}$ $m^{(Mer)} = 0.28$ $CE^{(Mer)} = 1.0363$	$CE^{\gamma=5.94}$ $m^{(Mer)} = \mathbf{0.169}$ $CE^{(Mer)} = 1.0339$
<i>Certainty equivalents of quantile MRRGs under the additional restriction to constant mix strategies (<math>\epsilon = 0.005</math>)</i>							
-0.100	0.1285	0.5277	1	0.005	<b>1.0405</b> <i>(1.0406)</i>	1.0341	1.0247
-0.095	0.1250	0.5101	1	0.005	1.0405	1.0345	1.0257
-0.090	0.1211	0.4921	1	0.005	1.0404	1.0348	1.0266
-0.085	0.1175	0.4745	1	0.005	1.0403	1.0351	1.0275
-0.080	0.1140	0.4571	1	0.005	1.0401	1.0353	1.0283
-0.075	0.1105	0.4397	1	0.005	1.0400	1.0355	1.0290
-0.070	0.1073	0.4229	1	0.005	1.0398	1.0357	1.0297
-0.065	0.1034	0.4047	1	0.005	1.0396	1.0359	1.0304
-0.060	0.1000	0.3876	1	0.005	1.0394	1.0360	1.0310
-0.055	0.0970	0.3710	1	0.005	1.0392	1.0361	1.0315
-0.050	0.0925	0.3521	1	0.005	1.0389	1.0361	1.0320
-0.045	0.0890	0.3347	1	0.005	1.0386	<b>1.0361</b> <i>(1.0362)</i>	1.0324
-0.040	0.0850	0.3165	1	0.005	1.0383	1.0361	1.0328
-0.035	0.0812	0.2987	1	0.005	1.0380	1.0360	1.0331
-0.030	0.0775	<b>0.2811</b>	1	0.005	1.0377	1.0359	1.0334
-0.025	0.0738	0.2634	1	0.005	1.0373	1.0358	1.0336
-0.020	0.0694	0.2443	1	0.005	1.0369	1.0356	1.0337
-0.015	0.0653	0.2259	1	0.005	1.0365	1.0354	1.0338
-0.010	0.0611	0.2074	1	0.005	1.0360	1.0351	<b>1.0338</b> <i>(1.0338)</i>
-0.005	0.0569	0.1887	1	0.005	1.0356	1.0349	1.0338
0.000	0.0519	<b>0.1684</b>	1	0.005	1.0350	1.0345	1.0336
0.005	0.0471	0.1485	1	0.005	1.0345	1.0341	1.0334
0.010	0.0419	0.1278	1	0.005	1.0339	1.0336	1.0331
0.015	0.0362	0.1063	1	0.005	1.0333	1.0331	1.0328
0.020	0.0299	0.0833	1	0.005	1.0326	1.0325	1.0323
0.025	0.0219	0.0567	1	0.005	1.0318	1.0317	1.0317

In particular, for each  $g \in \mathcal{G}$ , the combination of equity fraction  $\alpha^{(E)}$  and investment fraction  $m^{(A)}$  (implying that the SFP is exactly met and the contract is fairly priced) are given in columns two and three. The last three columns summarize the associated certainty equivalent contract wealths  $CEs$  of insureds described by three different levels of relative risk aversion ( $\gamma = 2, 3.56, \text{ and } 5.94$ ). In addition, the Merton solution is given in the upper line. For each level of relative risk aversion, the highest certainty equivalent (CE) which can be obtained by optimally choosing the guarantee is marked. For these cases, the CE which can be obtained without a restriction to constant mix strategies is included in italics



**Proposition 5** (Optimal quantile return payoff) *If the shortfall probability is not binding, i.e. if  $\mathbb{P}\left(\frac{A_1^{(Mer)}}{A_0} \leq 1 + g\right) \leq \epsilon$ , the optimal solution coincides with the Merton solution. If the shortfall probability is binding, i.e. if  $\mathbb{P}\left(\frac{A_1^{(Mer)}}{A_0} \leq 1 + g\right) > \epsilon$ , the optimal return payoff w.r.t. the optimization problem (20) is given as follows*

$$\frac{A_1^*}{A_0} = \beta \frac{A_1^{(Mer)}}{A_0} + \left(1 + g - \beta \frac{A_1^{(Mer)}}{A_0}\right) 1_{\left\{\underline{K} < \beta \frac{A_1^{(Mer)}}{A_0} \leq \bar{K}\right\}},$$

where  $0 \leq \underline{K} \leq \bar{K} := 1 + g$ .  $\underline{K}$  is determined by the SFP bound  $\epsilon$  and  $\beta$  by the pricing condition, i.e.

$$\mathbb{P}\left(\frac{A_1^{(Mer)}}{A_0} \leq \frac{\underline{K}}{\beta}\right) = \epsilon \text{ and } 1 - \beta = e^{-r} \mathbb{E}_{\mathbb{P}^*} \left[ \left(1 + g - \beta \frac{A_1^{(Mer)}}{A_0}\right) 1_{\left\{\underline{K} < \beta \frac{A_1^{(Mer)}}{A_0} \leq \bar{K}\right\}} \right].$$

In the limiting cases  $\epsilon \rightarrow 1$  (no constraint on the shortfall probability) and  $\epsilon \rightarrow 0$  (full guarantee) it holds

- (i) For  $\epsilon \rightarrow 1$  (and/or  $\mathbb{P}\left(\frac{A_1^{(Mer)}}{A_0} \leq 1 + g\right) \leq \epsilon$ ), it holds  $\beta = 1$ , and  $\underline{K} = \bar{K}$ , i.e. the optimal (return) payoff is given by the Merton solution  $\left(\frac{A_1^*}{A_0} = \frac{A_1^{(Mer)}}{A_0}\right)$ .
- (ii) For  $\epsilon \rightarrow 0$ , it holds  $\underline{K} = 0$  (and  $\bar{K} = 1 + g$ ) such that

$$\frac{A_1^*}{A_0} = (1 + g) + \left(\beta \frac{A_1^{(Mer)}}{A_0} - (1 + g)\right)^+,$$

where  $\beta$  solves

$$1 = e^{-r}(1 + g) + \beta \text{Call}^{(BS)}\left(\frac{1 + g}{\beta}, \sigma_A^{(Mer)}\right)$$

and  $\text{Call}^{(BS)}$  is given by Equation (15).<sup>19</sup>

Instead of explicitly stating the adoption to our setup, it is worth to comment on the intuition behind the result. Obviously, if the quantile constraint is not binding, the optimal solution is given by the Merton solution. W.r.t. the other limiting case where the return payoff is constrained by a shortfall probability of zero ( $\epsilon \rightarrow 0$ ), we also refer to El Karoui et al. (2005). The optimal unconstrained payoff is a

<sup>19</sup> Notice that the pricing condition is, by means of the put call parity, now given in terms of the call price.

modification of the Merton solution (unconstrained solution).<sup>20</sup> Intuitively, it is clear that a full hedge of the guarantee features a put option. Notice that

$$(1 + g) + \left( \beta \frac{A_1^{(Mer)}}{A_0} - (1 + g) \right)^+ = \beta \frac{A_1^{(Mer)}}{A_0} + \left( (1 + g) - \beta \frac{A_1^{(Mer)}}{A_0} \right)^+,$$

i.e. the return of the Merton solution is backed up by a put option with strike  $K = 1 + g$ . The put payoff gives the tightest (and thus cheapest) possibility to obtain a full hedge of the guarantee. Thus, it enables the investor to obtain the tightest modification of the unconstrained optimal payoff. To honor the pricing condition, i.e. the value of the payoff must be equal to one, the investor can no longer obtain the full Merton return but only a fraction  $\beta$  of it. In particular, while the value of  $\frac{A_1^{(Mer)}}{A_0}$  is equal to one, the investor now receives only a fraction of the return, i.e. in the presence of a (non vanishing) guarantee, her investment amount which is not needed to finance the put is only a fraction  $\beta$  ( $0 < \beta < 1$ ). In summary, the fraction  $\beta$  is determined by a fix point problem which is due to the condition that the value of the put on the return  $\beta \frac{A_1^{(Mer)}}{A_0}$  must be equal to the reduction of the initial investment  $1 - \beta$  (i.e. both sides depend on  $\beta$ ). Intuitively it is now clear that any deviation from a perfect guarantee ( $\epsilon \rightarrow 0$ ), an admissible shortfall probability which is higher than zero gives rise to lower hedging costs than the solution characterized above. While in the case of a zero shortfall probability the optimal payoff is given by

$$\beta \frac{A_1^{(Mer)}}{A_0} + \left( (1 + g) - \beta \frac{A_1^{(Mer)}}{A_0} \right) 1_{\left\{ \underline{K} < \beta \frac{A_1^{(Mer)}}{A_0} \leq \bar{K} \right\}},$$

where  $\underline{K} = 0$  and  $\bar{K} = 1 + g$ , the investor is now allowed to implement a smaller *guarantee interval*  $[\underline{K}, \bar{K}]$  where  $0 \leq \underline{K} < \bar{K} \leq 1 + g$ . Notice that the upper bound on the shortfall probability implies that fixing either  $\underline{K}$  or  $\bar{K}$  implies the other strike such that  $\beta$  is determined by the resulting fix point problem. However, the cheapest way to do so is by setting  $\bar{K} = 1 + g$ , i.e. starting with the high asset prices (Merton returns, respectively) which are linked to the cheapest states (to be hedged). In summary, the optimal quantile hedge is a scaled version of the Merton solution overlaid by the (cheapest) quantile hedge which honors the SFP bound.<sup>21</sup> In order to illustrate the improvement obtained by the optimal quantile hedge, we add in Table 2 the CE's associated with the optimal quantile guarantees, cf. italic numbers in brackets below the bold faced numbers referring to the optimal values under the restriction to constant mix strategies (and choosing the guarantee). Again, it is worth to emphasize

<sup>20</sup> In fact, the result does not depend on the Black and Scholes model which implies the Merton solution.

<sup>21</sup> W.r.t. quantile hedges, the interested reader is referred to Föllmer and Leukert (1999) who show how to obtain the highest success probability when hedging a claim with a lower initial investment than the one needed for a full hedge (or the other way round).

that the optimal quantile payoff can be implemented for any equity fraction  $\alpha^{(E)}$  of the insurer.

## 4 Conclusion

The paper analyzes the optimal design of participating life insurance contracts with minimum return rate guarantees under default risk. The benefits to the insured depend on the performance of an investment strategy which is conducted by the insurer. This strategy is initialized by an amount given by the sum of equity and the contributions of the insured. Unless there is a default event, the insured receives the maximum of a guaranteed rate and a participation in the returns. Considering default risk modifies the payoff of the insured by means of a default put implying a compound option feature (nested maximum). Based on yearly returns, we show that, in spite of the compound option feature, the (yearly return) payoff of the default put (and the liabilities to the insured) can be represented by piecewise linear functions of the investment return, i.e. the payoff of a portfolio of plain vanilla options. Thus, the liabilities are easily priced in any model setup which gives closed form solutions for standard options. In a complete market setup, we then derive the optimal (expected utility maximizing) quantile guarantee payoff of an investor/insured with constant relative risk aversion. Because of the completeness assumption, the return payoff can be implemented by the insurance company for any equity to debt ratio. We illustrate the utility loss which arise if the insurer implements a suboptimal investment strategy.

**Funding** Open Access funding provided by Projekt DEAL.

**Open Access** This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

## References

- Bacinello AR (2001) Fair pricing of life insurance participating policies with a minimum interest rate guarantee. *Astin Bull* 31(2):275–297
- Basak S, Shapiro A (2001) Value-at-risk-based risk management: optimal policies and asset prices. *Rev Financ stud* 14(2):371–405
- Benninga S, Blume M (1985) On the optimality of portfolio insurance. *J Finance* 40(5):1341–1352
- Bernard C, Li WV (2013) Pricing and hedging of cliquet options and locally capped contracts. *SIAM J Financ Math* 4(1):353–371
- Bernard C, Boyle PP, Gornall W (2009) Locally-capped investment products and the retail investor. *J Deriv* 18(4):72–88

- Boyle P, Tian W (2008) The design of equity-indexed annuities. *Insur Math Econ* 43(3):303–315
- Branger N, Mahayni A, Schneider JC (2010) On the optimal design of insurance contracts with guarantees. *Insur Math Econ* 46(3):485–492
- Brennan M, Schwartz E (1976) The pricing of equity-linked life insurance policies with an asset value guarantee. *J Financ Econ* 3:195–213
- Briys E, de Varenne F (1997) On the risk of insurance liabilities: debunking some common pitfalls. *J Risk Insur* 64(4):673–694
- Chen A, Nguyen T, Stadjé M (2018a) Optimal investment under VaR-Regulation and minimum insurance. *Insur Math Econ* 79:194–209
- Chen A, Nguyen T, Stadjé M (2018b) Risk management with multiple VaR constraints. *Math Methods Oper Res* 88(2):297–337
- Chen A, Hieber P, Nguyen T (2019) Constrained non-concave utility maximization: an application to life insurance contracts with guarantees. *Eur J Oper Res* 273(3):1119–1135
- Chiappori P-A, Paiella M (2011) Relative risk aversion is constant: evidence from panel data. *J Eur Econ Assoc* 9(6):1021–1052
- Coleman T, Kim Y, Li Y, Patron M (2007) Robustly hedging variable annuities with guarantees under jump and volatility risks. *J Risk Insur* 74(2):347–376
- Coleman T, Li Y, Patron M (2006) Hedging guarantees in variable annuities using risk minimization under both the equity and interest rate risk. *Insur Math Econ* 38(2):215–228
- Ebert S, Koos B, Schneider JC (2012) On the optimal type and level of guarantees for prospect theory investors. In: Paris December 2012 Finance Meeting EUROFIDAI-AFFI Paper
- El Karoui N, Jeanblanc M, Lacoste V (2005) Optimal portfolio management with American capital guarantee. *J Econ Dyn Control* 29:449–468
- Föllmer H, Leukert P (1999) Quantile hedging. *Finance Stoch* 3(3):251–273
- Gao J, Xiong Y, Li D (2016) Dynamic mean-risk portfolio selection with multiple risk measures in continuous-time. *Eur J Oper Res* 249(2):647–656
- Gatzert N, Holzmüller I, Schmeiser H (2012) Creating customer value in participating life insurance. *J Risk Insur* 79(3):645–670
- Grosen A, Jørgensen PL (2000) Fair valuation of life insurance liabilities: the impact of interest rate guarantees, surrender options and bonus policies. *Insur Math Econ* 26(1):37–57
- Grosen A, Jørgensen PL (2002) Life insurance liabilities at market value: an analysis of insolvency risk, bonus policy, and regulatory intervention rules in a barrier option framework. *J Risk Insur* 69(1):63–91
- Hieber P, Korn R, Scherer M (2015) Analyzing the effect of low interest rates on the surplus participation of life insurance policies with different annual interest rate guarantees. *Eur Actuar J* 5(1):11–28
- Hieber P, Natolski J, Werner R (2019) (2019), Fair valuation of cliquet-style return guarantees in (homogeneous and) heterogeneous life insurance portfolios. *Scand Actuar J* 6:478–507
- Huang H, Milevsky M, Wang J (2008) Portfolio choice and life insurance: the CRRA case. *J Risk Insur* 74(4):847–872
- Jensen BA, Nielsen JA (2016) How suboptimal are linear sharing rules? *Ann Finance* 12(2):221–243
- Jensen BA, Sørensen C (2001) Paying for minimum interest rate guarantees: who should compensate who?. *Eur Financ Manag* 7(2):183–211
- Kling A, Richter A, Ruß J (2007) The interaction of guarantees, surplus distribution, and asset allocation in with-profit life insurance policies. *J Risk Insur* 74(3):571–589
- Leland H (1980) Who should buy portfolio insurance?. *J Finance* 35(2):581–594
- Mahayni A, Schlögl E (2008) The risk management of minimum return guarantees. *BuR Bus Res* 1(1):55–76
- Mahayni A, Schneider JC (2016) Minimum return guarantees, investment caps, and investment flexibility. *Rev Deriv Res* 19(2):85–111
- Maurer R, Mitchell O, Rogalla R, Sieglin I (2016) Accounting and actuarial smoothing of retirement payouts in participating life annuities. *Insur Math Econ* 71:268–283
- McCabe M, Witt R (1980) Insurance pricing and regulation under uncertainty: a chance-constrained approach. *J Risk Insur* 47(4):607–635
- Merton R (1971) Optimal consumption and portfolio rules in a continuous time model. *J Econ Theory* 3:373–413
- Milevsky M, Kyrzhenko V (2008) Portfolio choice with puts: evidence from variable annuities. *Financ Anal J* 64(3):80–95

- Nielsen JA, Sandmann K, Schlögl E (2011) Equity-linked pension schemes with guarantees. *Insur Math Econ* 49:547–564
- Ruß J, Schelling S (2018) Multi cumulative prospect theory and the demand for cliquet-style guarantees. *J Risk Insur* 85(4):1103–1125
- Schmeiser H, Wagner J (2015) A proposal on how the regulator should set minimum interest rate guarantees in participating life insurance contracts. *J Risk Insur* 82(3):659–686
- Winkler R, Roodman G, Britney R (1972) The determination of partial moments. *Manag Sci* 19(3):290–296
- Yiu K-FC (2004) Optimal portfolios under a value-at-risk constraint. *J Econ Dyn Control* 28(7):1317–1334

**Publisher's Note** Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.