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Burgard, Jan Pablo; Esteban, María Dolores; Morales, Domingo; Pérez, Agustín

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ORIGINAL PAPER



Small area estimation under a measurement error bivariate Fay-Herriot model

Jan Pablo Burgard¹ : María Dolores Esteban² · Domingo Morales² · Agustín Pérez³

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Abstract

The bivariate Fay–Herriot model is an area-level linear mixed model that can be used for estimating the domain means of two correlated target variables. Under this model, the dependent variables are direct estimators calculated from survey data and the auxiliary variables are true domain means obtained from external data sources. Administrative registers do not always give good auxiliary variables, so that statisticians sometimes take them from alternative surveys and therefore they are measured with error. We introduce a variant of the bivariate Fay–Herriot model that takes into account the measurement error of the auxiliary variables and we give fitting algorithms to estimate the model parameters. Based on the new model, we introduce empirical best predictors of domain means and we propose a parametric bootstrap procedure for estimating the mean squared error. We finally give an application to estimate poverty proportions and gaps in the Spanish Living Condition Survey, with auxiliary information from the Spanish Labour Force Survey.

Keywords Multivariate models \cdot Fay–Herriot model \cdot small area estimation \cdot measurement error \cdot Monte Carlo simulation \cdot poverty proportion \cdot poverty gap

Jan Pablo Burgard burgardj@uni-trier.de

¹ Research Institute for Official and Survey Statistics (RIFOSS), Trier University, Trier, Germany

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² Operations Research Center, University Miguel Hernández de Elche, Elche, Spain

³ Departments of Economic and Financial Studies, University Miguel Hernández de Elche, Elche, Spain

1 Introduction

Survey sample sizes are generally calculated to obtain precise direct estimates of target parameters in planned territories, but they might be not large enough for obtaining reliable estimates in unplanned smaller regions or small areas. Small area estimation (SAE) theory introduces indirect model-based or model-assisted estimators for treating these situations. SAE is an important part of statistical inference in finite populations with applications to social and economic statistics. The monograph of Rao and Molina (2015) contains a general description of SAE.

When auxiliary variables related to the target variable are available at the small area level, the most widely used area-level model in SAE is the Fay–Herriot (FH) model. This model was first proposed by Fay and Herriot (1979) to obtain estimates of mean per capita income in U.S. small areas using survey data. Esteban et al. (2012a, b), Marhuenda et al. (2013) and Morales et al. (2015) apply variants of the basic FH model to the small area estimation of poverty indicators in the Spanish Living Condition Survey (SLCS), with auxiliary information from the Spanish Labour Force Survey (SLFS).

If there is more than one target variables, multivariate area-level mixed models can take into account their correlations. These correlations can be an important additional information for the estimation. Fay (1987) and Datta et al. (1991) showed that small area estimators obtained from multivariate models have, in general, better precision than the ones obtained from univariate models for each response variable. Molina et al. (2007) and López-Vizcaíno et al. (2013), López-Vizcaíno et al. (2015) made use of this idea and applied it to estimate labour force indicators. Several other authors have investigated and applied multivariate Fay–Herriot (MFH) models in the SAE setup, e.g. Datta et al. (1996), González-Manteiga et al. (2008), Porter et al. (2015) or Benavent and Morales (2016).

Under the MFH model, the values of the dependent variable are direct estimates calculated from survey data and the auxiliary variables are "true" domain means obtained from administrative registers. However, it is not always possible to find good auxiliary variables in administrative registers and MFH models are sometimes applied with auxiliary variables measured with errors. Oftentimes, direct estimates obtained from a sample of a different survey are used. The aforementioned applications do not take into account the measurement error of the auxiliary variables. This manuscript addresses this practical issue.

Concerning the contributions to area-level linear mixed models with covariates measured with error, Ybarra and Lohr (2008) introduced a functional measurement error model where the underlying true values of the covariates are fixed but unknown quantities. Their model can be viewed as a generalization of the Fay–Herriot model. They also introduced a new small area estimator that accounts for sampling variability in the auxiliary information, and derive its properties, in particular showing that it is approximately unbiased. They applied their estimator to predict quantities measured in the U.S. National Health and Nutrition Examination Survey, with auxiliary information from the U.S. National Health Interview Survey. Marchetti et al. (2015) presented an application where measures derived from Big

Data are used as covariates in a Fay–Herriot model to estimate poverty indicators, accounting for the presence of measurement error in the covariates. Polettini and Arima (2015) introduced predictors of small area means based on area-level linear mixed models with covariates perturbed for disclosure limitation.

Adopting a Bayesian approach, Arima et al. (2015) rewrote the Ybarra-Lohr measurement error model as a hierarchical model and introduced Bayes predictors. This last work was later extended by Arima et al. (2017) proposing multivariate Fay–Herriot Bayesian predictors of small area means under functional measurement error. On the other hand, Burgard et al. (2019) followed a likelihood-based approach for extending the Ybarra-Lohr model. They proposed residual maximum likelihood (REML) estimators of the model parameters and introduced empirical best predictors and a mean squared error analytical approximation.

Concerning unit-level models, further contributions to the Bayesian SAE literature on measurement error models are Ghosh et al. (2006), Ghosh and Sinha (2007), Torabi et al. (2009), Datta et al. (2010) and Arima et al. (2012). More recently, Torabi (2013) presented an application of data cloning conducting a frequentist analysis of GLMM with covariates subject to the measurement error model.

This paper introduces a three-step bivariate Fay–Herriot model by assuming that the vector of true domain means of auxiliary variables differ from the corresponding vector of direct estimators in a zero-mean multivariate normally distributed random error. The introduced functional measurement error model can be considered as a multivariate adaptation of the Ybarra-Lohr univariate model to a parametric inference setup with multivariate normal measurement errors. The proposed approach can also be considered as the non Bayesian counterpart of the statistical methodology introduced by Arima et al. (2017).

Ybarra and Lohr (2008) did not assume the normality of the measurement errors and they proposed a weighted least squared approach to estimate the model parameters. Arima et al. (2017) assumed the multivariate normality of the measurement errors and they considered a Bayesian approach. They simulated the posterior distributions of the model parameters and calculated the hierarchical Bayes predictors of domain means by applying Markov Chain Monte Carlo algorithms. This paper preserves the likelihood modelling of Arima et al., but applies a non Bayesian approach. The main target is to calculate empirical best predictors of domain means and to estimate the corresponding mean squared errors (MSE).

Assuming that measurement errors have a multivariate normal distribution is a natural choice in practice, as due to the central limit theorem, the distribution of the auxiliary variable estimators has an asymptotic multivariate normal distribution. This adaption, besides giving another motivation of the empirical best predictor provided by Ybarra and Lohr (2008), has two mayor advantages. First, we derive Fisher-scoring algorithms for calculating maximum likelihood (ML) and pseudo-residual maximum likelihood (pseudo-REML) estimators of model parameters. Second, we provide a parametric bootstrap procedure for estimating the mean squared errors.

The rest of the paper is organized as follows. Section 2 introduces the measurement error bivariate Fay-Herriot model. Section 3 derives the best predictors of random effects and target domain parameters. It also calculates the MSEs of the best predictors. Section 4 presents the relative efficiency matrix of the best predictors that takes into account that variables are measured with error compared to the corresponding ones that ignores this information. Section 5 proposes a parametric bootstrap procedure for estimating the mean squared error of the empirical best predictors. Section 6 describes the pseudo-REML method for estimating the model parameters. Section 7 carries out simulation experiments to investigate the behavior of the pseudo-REML fitting algorithm, the empirical best predictors and the bootstrap estimator of the mean squared error of the empirical best predictors. Section 8 gives an application to real data where the target is the small area estimation of poverty proportions and gaps in the SLCS, with auxiliary information from the SLFS. Sections 9 summarizes some conclusions. The paper contains three appendixes that are provided as supplementary file. Appendix A gives some auxiliary results for the calculation of the best predictors and their MSEs. Appendix B shows some tables with results of the application to SLCS data. Appendix C presents the Fisher scoring algorithm for calculating the ML estimators of the model parameters.

2 The measurement error bivariate Fay-Herriot model

Let U be a finite population partitioned into D domains U_1, \ldots, U_D . Let $\mu_d = (\mu_{d1}, \mu_{d2})'$ be a vector of characteristics of interest in the domain d and let $y_d = (y_{d1}, y_{d2})'$ be a vector of direct estimators of μ_d calculated by using the data of the target survey sample. The measurement error bivariate Fay-Herriot (MEBFH) model is defined in three steps. The first step indicates that direct estimators are unbiased and follow the *sampling model*

$$y_d = \mu_d + e_d, \quad d = 1, \dots, D, \tag{1}$$

where the vectors $e_d = (e_{d1}, e_{d2})' \sim N_2(0, V_{ed})$ are independent and the 2 × 2 covariance matrices V_{ed} are known. In most cases, V_{ed} is taken to be the design-based covariance matrix of the direct estimator y_d , d = 1, ..., D.

In the second step the true area characteristic μ_{dk} is assumed to be linearly related to $p_k + q_k$ explanatory variables, k = 1, 2, d = 1, ..., D. Let $\vec{x}_{dk} = (\tilde{x}_{dk1}, ..., \tilde{x}_{dkp_k})$ be a row vector containing the true aggregated (population) values of p_k explanatory variables for μ_{dk} and let $\tilde{X}_d = \text{diag}(\vec{x}_{d1}, \vec{x}_{d2})$ be a $2 \times p$ block-diagonal matrix with $p = p_1 + p_2$. Let $\lambda_k = (\lambda_{k1}, ..., \lambda_{kp_k})'$ be a column vector of size p_k containing regression parameters for μ_{dk} and let $\lambda = (\lambda'_1, \lambda'_2)'_{p \times 1}$. Let $z'_{dk} = (z_{dk1}, ..., z_{dkq_k})$ be a row vector containing the true aggregated (population) values of q_k explanatory variables for μ_{dk} and let $Z_d = \text{diag}(z'_{d1}, z'_{d2})$ be a $2 \times q$ block-diagonal matrix with $q = q_1 + q_2$. Let $\eta_k = (\eta_{k1}, ..., \eta_{kq_k})'$ be a column vector of size q_k containing regression parameters for μ_{dk} and let $\eta = (\eta'_1, \eta'_2)'_{q \times 1}$. The linking model is

$$\mu_d = Z_d \eta + \tilde{X}_d \lambda + u_d, \quad u_d = (u_{d1}, u_{d2})' \sim N_2(0, V_{ud}), \quad d = 1, \dots, D,$$
(2)

where the vectors u_d 's are independent of the vectors e_d 's. The 2 × 2 covariance matrices V_{ud} depend on 3 unknown parameters, $\theta_1 = \sigma_{u1}^2$, $\theta_2 = \sigma_{u2}^2$ and $\theta_3 = \rho$, i.e.

$$V_{ud} = \begin{pmatrix} \sigma_{u1}^2 & \rho \sigma_{u1} \sigma_{u2} \\ \rho \sigma_{u1} \sigma_{u2} & \sigma_{u2}^2 \end{pmatrix}.$$

This manuscript assumes that the \tilde{x}_{dk} 's are unknown random vectors that are predicted from independent data sources. These data sources could be administrative registers or other surveys with larger sample sizes than the target survey. For k = 1, 2, let us define the random measurement error vectors $v'_{dk} = (v_{dk1}, \dots, v_{dkp_k})$. We assume that the vectors $v_d = (v'_{d1}, v'_{d2})'$ are independent with distributions $v_d \sim N_p(0, \Sigma_d)$ and known $p \times p$ covariance matrices

$$\Sigma_d = \begin{pmatrix} \Sigma_{d11} & \Sigma_{d12} \\ \Sigma_{d21} & \Sigma_{d22} \end{pmatrix}, \quad \Sigma_{dk_1k_2} = \text{ cov } (v_{dk_1}, v_{dk_2}), \quad k_1, k_2 = 1, 2, d = 1, \dots, D.$$

The third step considers the functional measurement error model

$$\tilde{x}_{dk} = x_{dk} + v_{dk}, \quad d = 1, \dots, D, \ k = 1, 2,$$
(3)

where x'_{dk} is a row vector containing the unbiased predictors of the components of \vec{x}'_{dk} and the vectors v_d and $x_d = (x'_{d1}, x'_{d2})'$, d = 1, ..., D, are independent. In most cases, x_{dk} is a vector of direct estimators calculated from data of a different survey and $\sum_{dk_1k_2}$ is taken to be the design-based covariance matrix of vectors x_{dk_1} and x_{dk_2} , $k_1, k_2 = 1, 2$.

Let us also define the $2 \times p$ block diagonal matrices $B_d = \text{diag}(\lambda'_1, \lambda'_2)$ and $X_d = \text{diag}(x'_{d_1}, x'_{d_2})$.

The *measurement error* bivariate Fay-Herriot (MEBFH) model can be expressed as a single model in the form

$$y_d = Z_d \eta + X_d \lambda + B_d v_d + u_d + e_d, \quad d = 1, ..., D,$$
 (4)

or in the matrix form

$$y = Z\eta + X\lambda + Bv + u + e_{z}$$

where $B = \underset{1 \le d \le D}{\text{diag}}(B_d), X = \underset{1 \le d \le D}{\text{col}}(X_d), Z = \underset{1 \le d \le D}{\text{col}}(Z_d)$ and

$$y = \underset{1 \le d \le D}{\text{col}}(y_d), \ u = \underset{1 \le d \le D}{\text{col}}(u_d), \ v = \underset{1 \le d \le D}{\text{col}}(v_d), \ e = \underset{1 \le d \le D}{\text{col}}(e_d).$$

We finally assume that matrices V_{ud} , Σ_d and V_{ed} are invertible and that x_d , v_d , u_d , e_d , d = 1, ..., D, are independent, but we only introduce inference procedures conditionally on *X*. If there are no measurement errors, then the v_d 's are null vectors and the bivariate Fay–Herriot (BFH) model is obtained as a special case of (4). The

MEBFH model can be considered as a multivariate generalization of the Fay– Herriot model with measurement error studied by Ybarra and Lohr (2008) or by Burgard et al. (2019). This approach was also considered by Arima et al. (2017) in a Bayesian context. Note that the MEBFH model is not a linear mixed model as the matrix *B* depends on the vector λ of model parameters. Therefore, the MEBFH model cannot be expressed in the standard form $Y = X\beta + Zu + e$.

It holds that

$$V_{\lambda d} = \operatorname{var} (B_d v_d) = B_d \operatorname{var} (v_d) B'_d = \operatorname{diag} (\lambda'_1, \lambda'_2) \Sigma_d \operatorname{diag} (\lambda_1, \lambda_2)$$
$$= \begin{pmatrix} \lambda'_1 \Sigma_{d11} \lambda_1 & \lambda'_1 \Sigma_{d12} \lambda_2 \\ \lambda'_2 \Sigma_{d21} \lambda_1 & \lambda'_2 \Sigma_{d22} \lambda_2 \end{pmatrix}.$$

Therefore, $V_u = \operatorname{var}(u) = \operatorname{diag}_{1 \le d \le D}(V_{ud}), \quad V_e = \operatorname{var}(e) = \operatorname{diag}_{1 \le d \le D}(V_{ed})$ and var $(Bv) = \operatorname{diag}_{1 \le d \le D}(V_{\lambda d})$. Conditioned to $x_d, d = 1, \dots, D$, the covariance matrix of

y is

$$V = V(\theta, \lambda) = \operatorname{var}(y|X) = \operatorname{diag}_{1 \le d \le D}(V_d) = \operatorname{diag}_{1 \le d \le D}(V_{\lambda d} + V_{ud} + V_{ed}).$$
(5)

Further, it holds that $y_d|_{x_d} \sim N_2(Z_d\eta + X_d\lambda, V_{\lambda d} + V_{ud} + V_{ed})$,

$$y_d|_{x_d,v_d} \sim N_2 (Z_d \eta + X_d \lambda + B_d v_d, V_{ud} + V_{ed}), \text{ and}$$

 $y_d|_{x_d,u_d} \sim N_2 (Z_d \eta + X_d \lambda + u_d, V_{\lambda d} + V_{ed}).$

3 Best predictors under the MEBFH model

This section derives the best predictors (BP) of the random effects v_d and u_d and of the target parameter μ_d . It also calculates variances and expectations of cross products. The proofs of the following propositions are based on the properties of the multivariate normal distribution. We recall that the kernel of the *n*-variate normal distribution is

$$f(y|\mu, \Sigma) = \frac{1}{(2\pi)^{n/2} |\Sigma|^{1/2}} \exp\left\{-\frac{1}{2}(y-\mu)'\Sigma^{-1}(y-\mu)\right\}$$

 $\propto \exp\left\{-\frac{1}{2}y'\Sigma^{-1}y + \mu'\Sigma^{-1}y\right\}.$

The first two propositions deal with the BP of v_d and its basic properties.

Proposition 1 Under model (4), the best predictor of v_d is

$$v_d^{bp} = E[v_d | x_d, y_d] = \Psi_d B'_d (V_{ud} + V_{ed})^{-1} (y_d - Z_d \eta - X_d \lambda), \tag{6}$$

where

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$$\Psi_{d} = \left(B'_{d}(V_{ud} + V_{ed})^{-1}B_{d} + \Sigma_{d}^{-1}\right)^{-1} = \Sigma_{d} - \Sigma_{d}B'_{d}(V_{\lambda d} + V_{ud} + V_{ed})^{-1}B_{d}\Sigma_{d}.$$

Proof The conditional distribution of v_d , given the estimators x_d and y_d , is

$$\begin{split} f_{v_d} &= f(v_d | x_d, y_d) \propto f(y_d | x_d, v_d) f(v_d) = \left(2\pi | V_{ud} + V_{ed} |^{1/2}\right)^{-1} \\ &\quad \exp \left\{ -\frac{1}{2} (y_d - Z_d \eta - X_d \lambda - B_d v_d)' (V_{ud} + V_{ed})^{-1} (y_d - Z_d \eta - X_d \lambda - B_d v_d) \right\} \\ &\quad \cdot \left((2\pi)^{p/2} |\Sigma_d|^{1/2} \right)^{-1} \exp \left\{ -\frac{1}{2} v_d' \Sigma_d^{-1} v_d \right\} \\ &\propto \exp \left\{ -\frac{1}{2} (v_{d1}' \lambda_1, v_{d2}' \lambda_2) (V_{ud} + V_{ed})^{-1} \left(\frac{\lambda_1' v_{d1}}{\lambda_2' v_{d2}} \right) \right. \\ &\quad + \left(v_{d1} \lambda_1', v_{d2}' \lambda_2 \right) (V_{ud} + V_{ed})^{-1} (y_d - Z_d \eta - X_d \lambda) \right\} \exp \left\{ -\frac{1}{2} v_d' \Sigma_d^{-1} v_d \right\}. \end{split}$$

Therefore, we have

$$f_{v_d} = \exp\left\{-\frac{1}{2}(v'_{d1}, v'_{d2})\left(B'_d (V_{ud} + V_{ed})^{-1} B_d + \Sigma_d^{-1}\right) {v_{d1} \choose v_{d2}} + (v'_{d1}, v'_{d2})\Psi_d^{-1} \left[\Psi_d B'_d (V_{ud} + V_{ed})^{-1} (y_d - Z_d \eta - X_d \lambda)\right]\right\}.$$

We have proved that $f(v_d|x_d, y_d)$ is a multivariate normal distribution with parameters

$$\text{var} (v_d | x_d, y_d) = \left(B'_d \left(V_{ud} + V_{ed} \right)^{-1} B_d + \Sigma_d^{-1} \right)^{-1} = \Psi_d, \\ E[v_d | x_d, y_d] = \Psi_d B'_d \left(V_{ud} + V_{ed} \right)^{-1} (y_d - Z_d \eta - X_d \lambda).$$

By applying Lemma A.1 of Appendix A in the supplementary file, the results follows. $\hfill \Box$

Proposition 2 Under model (4), it holds that $E[\hat{v}_d^{bp}|x_d] = 0$ and

$$\operatorname{var} \left(\hat{v}_d^{bp} | x_d \right) = E \left[\hat{v}_d^{bp} \hat{v}_d^{bp\,\prime} | x_d \right] = \Psi_d B'_d (V_{ud} + V_{ed})^{-1} (V_{\lambda d} + V_{ud} + V_{ed}) (V_{ud} + V_{ed})^{-1} B_d \Psi_d.$$

Proof We recall that $y_d - Z_d \eta - X_d \lambda = B_d v_d + u_d + e_d$. Therefore, we have $\operatorname{var}(v_d^{bp}|x_d) = \Psi_d B'_d (V_{ud} + V_{ed})^{-1} E[(B_d v_d + u_d + e_d)(B_d v_d + u_d + e_d)'|x_d]$ $\cdot (V_{ud} + V_{ed})^{-1} B_d \Psi_d$ $= \Psi_d B'_d (V_{ud} + V_{ed})^{-1} (V_{\lambda d} + V_{ud} + V_{ed}) (V_{ud} + V_{ed})^{-1} B_d \Psi_d.$

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The following two propositions derive the BP of u_d , show that it is predictively unbiased and calculate its variance.

Proposition 3 Under model (4), the best predictor of u_d is

$$\hat{u}_{d}^{bp} = E[u_{d}|x_{d}, y_{d}] = \Phi_{d} \left(V_{\lambda d} + V_{ed} \right)^{-1} (y_{d} - Z_{d} \eta - X_{d} \lambda),$$

$$\Phi_{d} = \left(\left(V_{\lambda d} + V_{ed} \right)^{-1} + V_{ud}^{-1} \right)^{-1}.$$
(7)

Proof The conditional distribution of u_d , given x_d and y_d , is

$$\begin{split} f_{u_d} &= f(u_d | x_d, y_d) \propto f(y_d | x_d, u_d) f(u_d) = \\ &= \frac{1}{2\pi |V_{\lambda d} + V_{ed}|^{1/2}} \\ &\quad \cdot \exp \Big\{ -\frac{1}{2} (y_d - Z_d \eta - X_d \lambda - u_d)' (V_{\lambda d} + V_{ed})^{-1} (y_d - Z_d \eta - X_d \lambda - u_d) \Big\} \\ &\quad \cdot \frac{1}{2\pi |V_{ud}|^{1/2}} \exp \Big\{ -\frac{1}{2} u'_d V_{ud}^{-1} u_d \Big\} \\ &\quad \propto \exp \Big\{ -\frac{1}{2} u'_d (V_{\lambda d} + V_{ed})^{-1} u_d + u'_d (V_{\lambda d} + V_{ed})^{-1} (y_d - Z_d \eta - X_d \lambda) \Big\} \\ &\quad \cdot \exp \Big\{ -\frac{1}{2} u'_d V_{ud}^{-1} u_d \Big\} \\ &= \exp \Big\{ -\frac{1}{2} u'_d \Big((V_{\lambda d} + V_{ed})^{-1} + V_{ud}^{-1} \Big) u_d \\ &\quad + u'_d \Phi_d^{-1} \Big[\Phi_d (V_{\lambda d} + V_{ed})^{-1} (y_d - Z_d \eta - X_d \lambda) \Big] \Big\}. \end{split}$$

Therefore $f(u_d|x_d, y_d)$ is a multivariate normal distribution with parameters

var
$$(u_d | x_d, y_d) = \left(\left(V_{\lambda d} + V_{ed} \right)^{-1} + V_{ud}^{-1} \right)^{-1} = \Phi_d,$$

 $E[u_d | x_d, y_d] = \Phi_d \left(V_{\lambda d} + V_{ed} \right)^{-1} (y_d - Z_d \eta - X_d \lambda).$

Proposition 4 Under model (4), it holds that $E[\hat{u}_d^{bp}|x_d] = 0$ and

$$\operatorname{var} (\hat{u}_d^{bp} | x_d) = E [\hat{u}_d^{bp} \hat{u}_d^{bp'} | x_d] = \Phi_d (V_{\lambda d} + V_{ed})^{-1} (V_{\lambda d} + V_{ud} + V_{ed}) (V_{\lambda d} + V_{ed})^{-1} \Phi_d.$$

Proof As $y_d - Z_d \eta - X_d \lambda = B_d v_d + u_d + e_d$, we have

$$\text{var} (\hat{u}_{d}^{bp}|x_{d}) = \Phi_{d}(V_{\lambda d} + V_{ed})^{-1} E [(B_{d}v_{d} + u_{d} + e_{d})(B_{d}v_{d} + u_{d} + e_{d})'|x_{d} \\ \cdot (V_{\lambda d} + V_{ed})^{-1} \Phi_{d} \\ = \Phi_{d}(V_{\lambda d} + V_{ed})^{-1} (V_{\lambda d} + V_{ud} + V_{ed})(V_{\lambda d} + V_{ed})^{-1} \Phi_{d}.$$

 \Box The following two propositions give the best predictor of μ_d and its MSE. This section ends by defining the empirical best predictor of μ_d .

Proposition 5 Under model (4), the best predictor (MEBFH-BP) of μ_d is

$$\hat{\mu}_{d}^{bp} = Z_{d}\eta + X_{d}\lambda + V_{\lambda d} (V_{ud} + V_{ed})^{-1} (y_{d} - Z_{d}\eta - X_{d}\lambda) - V_{\lambda d} (V_{\lambda d} + V_{ud} + V_{ed})^{-1} V_{\lambda d} (V_{ud} + V_{ed})^{-1} (y_{d} - Z_{d}\eta - X_{d}\lambda) + \Phi_{d} (V_{\lambda d} + V_{ed})^{-1} (y_{d} - Z_{d}\eta - X_{d}\lambda)$$
(8)

Proof As $\mu_d = Z_d \eta + X_d \lambda + B_d v_d + u_d$, $\Psi_d = \Sigma_d - \Sigma_d B'_d (V_{\lambda d} + V_{ud} + V_{ed})^{-1} B_d \Sigma_d$ and $V_{\lambda d} = B_d \Sigma_d B'_d$, we have

$$\begin{split} \hat{\mu}_{d}^{bp} &= E[\mu_{d}|x_{d}, y_{d}] = Z_{d}\eta + X_{d}\lambda + B_{d}v_{d}^{bp} + \hat{u}_{d}^{bp} \\ &= Z_{d}\eta + X_{d}\lambda + B_{d}\Psi_{d}B_{d}'(V_{ud} + V_{ed})^{-1}(y_{d} - Z_{d}\eta - X_{d}\lambda) \\ &+ \Phi_{d}(V_{\lambda d} + V_{ed})^{-1}(y_{d} - Z_{d}\eta - X_{d}\lambda) \\ &= Z_{d}\eta + X_{d}\lambda + V_{\lambda d}(V_{ud} + V_{ed})^{-1}(y_{d} - Z_{d}\eta - X_{d}\lambda) \\ &- V_{\lambda d}(V_{\lambda d} + V_{ud} + V_{ed})^{-1}V_{\lambda d}(V_{ud} + V_{ed})^{-1}(y_{d} - Z_{d}\eta - X_{d}\lambda) \\ &+ \Phi_{d}(V_{\lambda d} + V_{ed})^{-1}(y_{d} - Z_{d}\eta - X_{d}\lambda). \end{split}$$

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Proposition 6 Under the model (4), the MSE of $\hat{\mu}_d^{bp}$ is

$$\begin{split} MSE(\hat{\mu}_{d}^{bp}|\mathbf{x}_{d}) &= \Big\{ B_{d} \Psi_{d} B_{d}' (V_{ud} + V_{ed})^{-1} (V_{\lambda d} + V_{ud} + V_{ed}) (V_{ud} + V_{ed})^{-1} B_{d} \Psi_{d} B_{d}' \\ &+ \Phi_{d} (V_{\lambda d} + V_{ed})^{-1} (V_{\lambda d} + V_{ud} + V_{ed}) (V_{\lambda d} + V_{ed})^{-1} \Phi_{d} \\ &+ B_{d} \Psi_{d} B_{d}' (V_{ud} + V_{ed})^{-1} (V_{\lambda d} + V_{ud} + V_{ed}) (V_{\lambda d} + V_{ed})^{-1} \Phi_{d} \\ &+ \Phi_{d} (V_{\lambda d} + V_{ed})^{-1} (V_{\lambda d} + V_{ud} + V_{ed}) (V_{ud} + V_{ed})^{-1} B_{d} \Psi_{d} B_{d}' \Big\} \\ &+ \Big\{ V_{\lambda d} + V_{ud} \Big\} \\ &- \Big\{ V_{\lambda d} (V_{ud} + V_{ed})^{-1} B_{d} \Psi_{d} B_{d}' + V_{ud} (V_{\lambda d} + V_{ed})^{-1} \Phi_{d} \\ &+ V_{\lambda d} (V_{\lambda d} + V_{ed})^{-1} \Phi_{d} + V_{ud} (V_{ud} + V_{ed})^{-1} B_{d} \Psi_{d} B_{d}' \Big\} \\ &- \Big\{ B_{d} \Psi_{d} B_{d}' (V_{ud} + V_{ed})^{-1} B_{d} \Sigma_{d} B_{d}' + \Phi_{d} (V_{\lambda d} + V_{ed})^{-1} V_{ud} \\ &+ \Phi_{d} (V_{\lambda d} + V_{ed})^{-1} B_{d} \Sigma_{d} B_{d}' + B_{d} \Psi_{d} B_{d}' (V_{ud} + V_{ed})^{-1} V_{ud} \Big\}. \end{split}$$

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Proof It hold that

$$\begin{split} MSE(\hat{\mu}_{d}^{bp}|x_{d}) &= E\big[(\hat{\mu}_{d}^{bp} - \mu_{d})(\hat{\mu}_{d}^{bp} - \mu_{d})'|x_{d}\big] \\ &= E\big[\hat{\mu}_{d}^{bp}\hat{\mu}_{d}^{bp'}|x_{d}\big] + E\big[\mu_{d}\mu_{d}'|x_{d}\big] - E\big[\mu_{d}\hat{\mu}_{d}^{bp'}|x_{d}\big] - E\big[\hat{\mu}_{d}^{bp}\mu_{d}'|x_{d}\big] \end{split}$$

where $E[\mu_d \mu'_d | x_d] = V_{\lambda d} + V_{ud}$.

The remaining expectations are calculated in Propositions A.2 and A.3 from Appendix A in the supplementary file. By doing the corresponding substitutions, the result follows. \Box

In practice, the BPs are not calculable because the model parameters are not known.

Under model (4), the empirical best predictors (MEBFH-EBP) of v_d , u_d and μ_d are obtained from formulas (6), (7) and (8) by plugging estimators $\hat{\beta}$, $\hat{\sigma}_{u1}^2$, $\hat{\sigma}_{u2}^2$ and $\hat{\rho}$ in the place of β , σ_{u1}^2 , σ_{u2}^2 and ρ respectively. The MEBFH-EBP of μ_d is

$$\hat{\mu}_d^{ebp} = E[\mu_d | x_d, y_d] = Z_d \hat{\eta} + X_d \hat{\lambda} + \text{ diag } (\hat{\lambda}'_1, \hat{\lambda}'_2) \hat{\nu}_d^{ebp} + \hat{u}_d^{ebp}.$$
(9)

Section 6 introduces the maximum likelihood and the pseudo-REML of estimators of model parameters. In the application to real data, the MEBFH-EBP of μ_d is calculated by plugging pseudo-REML estimators.

4 Relative efficiency matrix of best predictors

An important question is, when using estimated auxiliary variables, how much efficiency can be gained by using the MEBFH-BP instead of the BFH-BLUP. We recall that MEBFH-BP denotes the BP of μ_d calculated by assuming that the MEBFH model is the true model. Similarly, BFH-BLUP denotes the best linear unbiased predictor of μ_d calculated by assuming that the BFH model (with no measurement errors) is the true model. The gain in efficiency is measured as the relative reduction of the MSE when using the MEBFH-BP instead of the BFH-BLUP, under the assumption that all model parameters are known and the true model is MEBFH.

We first derive the BFH-BLUP of μ_d and its MSE under the true MEBFH model. If we equate V_{λ} to the null matrix in (8), we obtain the BFH-BLUP, i.e.

$$\hat{\mu}_{d}^{bp_{0}} = Z_{d}\eta + X_{d}\lambda + (V_{ud}^{-1} + V_{ed}^{-1})^{-1}V_{ed}^{-1}\xi_{d} = Z_{d}\eta + X_{d}\lambda + V_{ud}(V_{ud} + V_{ed})^{-1}\xi_{d},$$
(10)

where $\xi_d = y_d - Z_d \eta - X_d \lambda = B_d v_d + u_d + e_d$. The prediction error of the BFH-BLUP is

$$\begin{split} \hat{\mu}_{d}^{bp_{0}} - \mu_{d} &= Z_{d}\eta + X_{d}\lambda + V_{ud}(V_{ud} + V_{ed})^{-1}\xi_{d} - \left(Z_{d}\eta + X_{d}\lambda + B_{d}v_{d} + u_{d}\right) \\ &= V_{ud}(V_{ud} + V_{ed})^{-1}\xi_{d} - B_{d}v_{d} - u_{d}. \end{split}$$

Under the MEBFH model, the BFH-BLUP is predictively unbiased. It holds that

$$E[\xi_d|x_d] = E[B_d v_d + u_d + e_d] = 0, \quad E[\hat{\mu}_d^{bp_0} - \mu_d|x_d] = 0.$$

Under the MEBFH model, we have

$$E[\xi_{d}\xi'_{d}|x_{d}] = B_{d}E[v_{d}v'_{d}|x_{d}]B'_{d} + E[u_{d}u'_{d}|x_{d}] + E[e_{d}e'_{d}|x_{d}]$$

= $V_{\lambda_{d}} + V_{u_{d}} + V_{e_{d}} = V_{d},$
 $E[\xi_{d}v'_{d}|x_{d}]B'_{d} = E[(B_{d}v_{d} + u_{d} + e_{d})v'_{d}|x_{d}]B'_{d} = B_{d}E[v_{d}v'_{d}|x_{d}]B'_{d} = V_{\lambda_{d}},$
 $E[\xi_{d}u'_{d}|x_{d}] = E[(B_{d}v_{d} + u_{d} + e_{d})u'_{d}|x_{d}] = E[u_{d}u'_{d}|x_{d}] = V_{u_{d}}.$

Under the MEBFH model, with all model parameters known, the MSE of the BFH-BLUP is

$$\begin{split} MSE(\hat{\mu}_{d}^{bp_{0}}|x_{d}) &= E\left[(\hat{\mu}_{d}^{bp_{0}}-\mu_{d})(\hat{\mu}_{d}^{bp_{0}}-\mu_{d})'|x_{d}\right] \\ &= V_{ud}(V_{ud}+V_{ed})^{-1}E\left[\xi_{d}\xi_{d}'|x_{d}\right](V_{ud}+V_{ed})^{-1}V_{ud} \\ &+ B_{d}E\left[v_{d}v_{d}'|x_{d}\right]B_{d}' + E\left[u_{d}u_{d}'|x_{d}\right] \\ &- V_{ud}(V_{ud}+V_{ed})^{-1}E\left[\xi_{d}v_{d}'|x_{d}\right]B_{d}' \\ &- V_{ud}(V_{ud}+V_{ed})^{-1}E\left[\xi_{d}u_{d}'|x_{d}\right] - B_{d}E\left[v_{d}u_{d}'|x_{d}\right] \\ &- B_{d}E\left[v_{d}\xi_{d}'|x_{d}\right](V_{ud}+V_{ed})^{-1}V_{ud} \\ &- E\left[u_{d}\xi_{d}'|x_{d}\right](V_{ud}+V_{ed})^{-1}V_{ud} - E\left[u_{d}v_{d}'|x_{d}\right]B_{d}' \\ &= V_{ud}(V_{ud}+V_{ed})^{-1}V_{d}(V_{ud}+V_{ed})^{-1}V_{ud} + V_{\lambda_{d}} + V_{u_{d}} \\ &- 2V_{ud}(V_{ud}+V_{ed})^{-1}V_{ud} \\ &- V_{ud}(V_{ud}+V_{ed})^{-1}V_{ud} \end{split}$$

If all MEBFH model parameters are known, the relative efficiency matrix of the MEBFH-BP compared to the BFH-BLUP is

$$RE_{d} = \begin{pmatrix} RE_{d11} & RE_{d12} \\ RE_{d21} & RE_{d22} \end{pmatrix} = \frac{MSE(\hat{\mu}_{d}^{bp}|x_{d})}{MSE(\hat{\mu}_{d}^{bpo}|x_{d})}, \quad d = 1, \dots, D,$$
(11)

where the division of the 2×2 matrices $MSE(\hat{\mu}_d^{bp}|x_d)$ and $MSE(\hat{\mu}_d^{bp_0}|x_d)$ is done component by component. We are mainly interested in the diagonal components RE_{d11} and RE_{d11} ; this is to say, in the relative efficiencies when predicting μ_{d1} and μ_{d2} respectively. We further remark that RE_d does not depend on x_d , η or D.

In what follows, we presents some numerical calculations of the relative efficiencies of the MEBFH model with respect to the BFH model. Let us consider a MEBFH model (4) with $q_1 = q_2 = p_1 = p_2 = 1$, so that the model elements are $Z_d = \text{diag}(z_{d1}, z_{d2}), \quad \eta = (\eta_1, \eta_2)', \quad X_d = \text{diag}(x_{d1}, x_{d2}), \quad \lambda = (\lambda_1, \lambda_2)', B = \text{diag}(\lambda_1, \lambda_2), \quad v_d = (v_{d1}, v_{d2})', \quad u_d = (u_{d1}, u_{d2})', \quad e_d = (e_{d1}, e_{d2})'.$ For $d = 1, \ldots, D$, define $\tau_{d11} = \Sigma_{d11}, \tau_{d21} = \tau_{d12} = \Sigma_{d12}, \tau_{d22} = \Sigma_{d22}$ and

$$\Sigma_d = \begin{pmatrix} \tau_{d11} & \tau_{d12} \\ \tau_{d12} & \tau_{d22} \end{pmatrix}, \quad V_{ud} = \begin{pmatrix} \theta_1 & \theta_3 \sqrt{\theta_1} \sqrt{\theta_2} \\ \theta_3 \sqrt{\theta_1} \sqrt{\theta_2} & \theta_2 \end{pmatrix}, \quad V_{ed} = \begin{pmatrix} 1 & c \\ c & 1 \end{pmatrix},$$

where $\tau_{d12} = \rho_{\tau} \tau_{d11}^{1/2} \tau_{d22}^{1/2}$. Take $\lambda_1 = \lambda_2 = 1$, $\tau = \tau_{d11} = \tau_{d22}$, $\theta_1 = 1$ and $\theta_2 = 3/2$.

Table 1 presents the relative efficiencies RE_{d11} (top) and RE_{d22} (bottom) for the 45 scenarios 1A,...,9E. All the scenarios take $\theta_1 = 1$, $\theta_2 = 3/2$. The row cases A, B, C, D, E take $\tau = 1.81, 1.41, 1.01, 0, 61, 0.21$ respectively. Scenarios 1–9 take the parameters ρ_{τ} , c and θ_3 given by the rows 2, 3 and 4 of Table 1. We observe that the relative efficiencies decrease as τ increases from case E to case A. This is to say, the greater the measurement error variance of the auxiliary variables is, the greater gain of efficiency is obtained by using the best predictor based on the MEBFH model. On the other hand, if the measurement errors are negligible ($\tau \approx 0$), then the gain of efficiency almost null.

The greater values of RE_{d11} and RE_{d22} are in the column case 5. Therefore, the efficiency gain when using the MEBFH model is smaller when the correlations ρ_{τ} , c and θ_3 of the components of the measurement errors v_d , the sampling errors e_d and the random effects u_d respectively, are close to zero. In the limit $\rho_{\tau} = c = \theta_3 = 0$, we get two independent measurement error univariate Fay–Herriot models and it is not possible to transport information from one component to other.

Figure 1 plots the relative efficiencies RE_{d11} and RE_{d22} for $\theta_3 \in \{-0.75, 0.85\}$ and any ρ_{τ} and *c*. The case $\theta_3 = -0.75$ covers the Scenarios 2A and 3A and any other scenario with $\theta_3 = -0.75$, $-0.5 \le \rho_{\tau} \le 0.5$ and $-0.5 \le c \le 0.5$. The case $\theta_3 =$ 0.85 contains the scenarios 8A and 9A as individual points in the right-hand printed surfaces.

Scenario	1	2	3	4	5	6	7	8	9
$ ho_{ au}$	0.85	0.85	0.65	0.45	0.05	- 0.35	- 0.55	- 0.75	- 0.95
с	0.85	0.85	0.65	0.45	0.05	- 0.35	- 0.55	- 0.75	- 0.95
θ_3	- 0.95	- 0.75	-0.75	-0.55	0.05	0.45	0.65	0.65	0.85
А	0.3961	0.4701	0.5069	0.6182	0.7743	0.6687	0.5635	0.5242	0.4170
В	0.4604	0.5399	0.5784	0.6885	0.8290	0.7356	0.6354	0.5960	0.4827
С	0.5494	0.6330	0.6719	0.7735	0.8874	0.8136	0.7260	0.6887	0.5728
D	0.6802	0.6919	0.7959	0.8738	0.9454	0.9007	0.8391	0.8095	0.7029
Е	0.8848	0.9329	0.9486	0.9743	0.9910	0.9812	0.9640	0.9536	0.8991
А	0.4076	0.4831	0.5383	0.6733	0.8633	0.7352	0.6065	0.5473	0.4210
В	0.4733	0.5537	0.6106	0.6885	0.9014	0.7949	0.6774	0.6195	0.4872
С	0.5639	0.6474	0.7032	0.8170	0.9389	0.8610	0.7640	0.7114	0.5778
D	0.6959	0.7060	0.8220	0.9028	0.9726	0.9299	0.8673	0.8281	0.7083
E	0.8971	0.9395	0.9585	0.9817	0.9959	0.9877	0.9726	0.9604	0.9030

Table 1 RE_{d11} (top) and RE_{d22} (bottom) for $\theta_1 = 1$, $\theta_2 = 3/2$ and scenarios 1A,..., 9E.

In summary, Table 1 and Figure 1 show some scenarios where the MSE of the MEBFH-BP is around a half of the MSE of the BFH-EBLUP. They also show some other scenarios where the gain of efficiency is rather small. This information is useful to decide in what situations it is more profitable to use the more complex EBP based on the MEBFH model.

5 Mean squared error estimation

Obtaining an approximation to the MSE of the EBP of μ_d under the MEBFH model requires tedious calculations. Unlike the case of no measurement errors, the obtained approximation will be quite awkward and not very useful to introduce analytic MSE estimators. This is why we propose applying a parametric bootstrap procedure, like the one introduced by González-Manteiga et al. (2008) and later extended by González-Manteiga et al. (2010) to semi-parametric Fay–Herriot models.

Let $\psi = (\eta', \lambda', \theta')'$ be the vector of model parameters, where $\theta = (\sigma_{u1}^2, \sigma_{u2}^2, \rho)'$. The following parametric bootstrap procedure estimates $MSE(\hat{\mu}_d^{ebp})$.



Fig. 1 Relative efficiencies for $\tau = 1.81$, $\theta_1 = 1$, $\theta_2 = 3/2$

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- 1. Calculate the estimates $\hat{\psi}$ and $\hat{V}_{ud} = V_{ud}(\hat{\psi})$ by using the data (y_d, Z_d, X_d) , d = 1, ..., D.
- 2. Repeat *B* times
 - 2.1. For d = 1, ..., D, generate $u_d^{*(b)} \sim N_2(0, \hat{V}_{ud})$, $e_d^{*(b)} \sim N_2(0, V_{ed})$ $v_d^{*(b)} \sim N_2(0, \Sigma_d)$, $y_d^{*(b)} = Z_d \hat{\eta}^{(i)} + X_d \lambda^{(i)} + \text{ diag } (\hat{\lambda}_1^{(i)}, \hat{\lambda}_2^{(i)}) v_d^{*(b)} + u_d^{*(b)}$.
 - 2.2. Calculate the estimator $\hat{\psi}^{*(b)}$ by using the data $(y_d^{*(b)}, Z_d, X_d), d = 1, ..., D$.
 - 2.3 Calculate the EBPs $\hat{\mu}_{dk}^{*(b)}$, d = 1, ..., D, k = 1, 2, under the model of Step 2.1.
- 3 For d = 1, ..., D, k = 1, 2, calculate the MSE estimator of the EBP; i.e.

$$mse_{dk}^{*} = \frac{1}{B} \sum_{b=1}^{B} \left(\hat{\mu}_{dk}^{*(b)} - \mu_{dk}^{*(b)} \right)^{2}.$$

6 Estimation of model parameters

We consider two methods for estimating η , λ , σ_{u1}^2 , σ_{u2}^2 and ρ under model (4): (1) pseudo-residual maximum likelihood, and (2) maximum likelihood. Both method are based on the distribution of *y*|*X*. Appendix C in the supplementary file gives the Fisher-scoring algorithm to calculate the maximum likelihood estimators of the model parameters. However, we we do not implement this last algorithm because it has a greater computational complexity. This section describes the pseudo-residual maximum likelihood method.

ML estimators of model parameters have well known asymptotic properties. Under regularity conditions on the auxiliary variables, they are consistent and asymptotically normal. The Fisher-scoring algorithm maximizes the log-likelihood of *y*|*X* by solving the corresponding system of nonlinear equation, i.e. first partial derivatives equated to zero. It is a system with p + q + 3 equations and the Fisher-scoring algorithm inverts matrices of that dimension. If *p* or *q* are big, then the algorithm speed decreases and it might have convergence problems when the number of domains *D* is not big enough.

The REML estimators of the parameters of a linear mixed model are quite attractive. They have similar good asymptotic properties as ML estimators, but their calculation has a lower computational cost because the REML log-likelihood involves only the variance component parameters. Nevertheless, the MEBFH model is not a linear mixed model and the REML method is thus not applicable. This is why, we introduce pseudo-REML approach by treating the components of matrix B_d in (4) as known constants. In that case, the MEBFH model becomes a linear mixed model and the REML method can be applied yielding to the maximization of the derived REML log-likelihood. However, the components of B_d depends on the

unknown vector of parameters λ and, therefore, we are not applying the REML method but a pseudo-REML approach. The small sample properties of the pseudo-REML are empirically investigated in Simulation 1.

Let us define $T = [Z, X] = \underset{1 \le d \le D}{\operatorname{col}}(T_d)$, $T_d = [Z_d, X_d]$ and $\beta = (\eta', \lambda')'$, so that model (4) can be written in the form

$$y_d = T\beta + Bv_d + u_d + e_d, \quad d = 1, ..., D,$$
 (12)

The pseudo-REML log-likelihood of model (12) is

$$l_{reml}(\theta) = -\frac{2D - p - q}{2}\log 2\pi + \frac{1}{2}\log|T'T| - \frac{1}{2}\log|V| - \frac{1}{2}\log|T'V^{-1}T| - \frac{1}{2}y'Py,$$

where $\theta = (\theta_1, \theta_2, \theta_3)$, $\theta_1 = \sigma_{u1}^2$, $\theta_2 = \sigma_{u2}^2$, $\theta_3 = \rho$, PVP = P, PT = 0, $P = V^{-1} - V^{-1}T(T'V^{-1}T)^{-1}T'V^{-1}$, and V is defined in (5). By applying the formulas

$$\frac{\partial \log |V|}{\partial \theta} = \operatorname{tr} \left(V^{-1} \frac{\partial V}{\partial \theta} \right), \quad \frac{\partial V^{-1}}{\partial \theta} = -V^{-1} \frac{\partial V}{\partial \theta} V^{-1},$$

we calculate the first partial derivatives of l_{reml} with respect to θ_{ℓ} , i.e.

$$\begin{split} \frac{\partial l_{reml}(\theta)}{\partial \theta_{\ell}} &= -\frac{1}{2} \operatorname{tr} \left(V^{-1} \frac{\partial V}{\partial \theta_{\ell}} \right) + \frac{1}{2} \operatorname{tr} \left((T'V^{-1}T)^{-1}T'V^{-1} \frac{\partial V}{\partial \theta_{\ell}}V^{-1}T \right) - \frac{1}{2} y' \frac{\partial P}{\partial \theta_{\ell}} y \\ &= -\frac{1}{2} \operatorname{tr} \left(V^{-1} \frac{\partial V}{\partial \theta_{\ell}} \right) + \frac{1}{2} \operatorname{tr} \left(V^{-1}T(T'V^{-1}T)^{-1}T'V^{-1} \frac{\partial V}{\partial \theta_{\ell}} \right) - \frac{1}{2} y' \frac{\partial P}{\partial \theta_{\ell}} y \\ &= -\frac{1}{2} \operatorname{tr} \left(\left[V^{-1} - V^{-1}T(T'V^{-1}T)^{-1}T'V^{-1} \right] \frac{\partial V}{\partial \theta_{\ell}} \right) - \frac{1}{2} y' \frac{\partial P}{\partial \theta_{\ell}} y \\ &= -\frac{1}{2} \operatorname{tr} \left(P \frac{\partial V}{\partial \theta_{\ell}} \right) - \frac{1}{2} y' \frac{\partial P}{\partial \theta_{\ell}} y, \quad \ell = 1, 2, 3. \end{split}$$

Let us define $G = V^{-1}T(T'V^{-1}T)^{-1}$, so that $P = (I - GT')V^{-1} = V^{-1}(I - TG')$. The first partial derivatives of $P = V^{-1} - V^{-1}T(T'V^{-1}T)^{-1}T'V^{-1}$ with respect to θ_{ℓ} are

$$\begin{split} \frac{\partial P}{\partial \theta_{\ell}} &= -V^{-1} \frac{\partial V}{\partial \theta_{\ell}} V^{-1} + V^{-1} \frac{\partial V}{\partial \theta_{\ell}} V^{-1} T (T'V^{-1}T)^{-1} T'V^{-1} \\ &+ V^{-1} T (T'V^{-1}T)^{-1} T'V^{-1} \frac{\partial V}{\partial \theta_{\ell}} V^{-1} \\ &- V^{-1} T (T'V^{-1}T)^{-1} T'V^{-1} \frac{\partial V}{\partial \theta_{\ell}} V^{-1} T (T'V^{-1}T)^{-1} T'V^{-1} \\ &= -V^{-1} \frac{\partial V}{\partial \theta_{\ell}} V^{-1} + V^{-1} \frac{\partial V}{\partial \theta_{\ell}} V^{-1} TG' + GT'V^{-1} \frac{\partial V}{\partial \theta_{\ell}} V^{-1} \\ &- GT'V^{-1} \frac{\partial V}{\partial \theta_{\ell}} V^{-1} TG' \\ &= -(I - GT')V^{-1} \frac{\partial V}{\partial \theta_{\ell}} V^{-1} (I - GT')' = -P \frac{\partial V}{\partial \theta_{\ell}} P, \quad \ell = 1, 2, 3 \end{split}$$

Therefore, the score vector for $\ell = 1, 2, 3$ is

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$$S(\theta, \beta) = (S_1, S_2, S_3)', \ S_{\ell} = S_{\ell}(\theta, \beta) = \frac{\partial I_{reml}}{\partial \theta_{\ell}} = -\frac{1}{2} \operatorname{tr}(PV_{\ell}) + \frac{1}{2} y' PV_{\ell} Py,$$

where $P = P(\theta, \beta) = (P_{d_1d_2})_{d_1, d_2 = 1, \dots, D}, \ V_{\ell} = V_{\ell}(\theta) = \frac{\partial V}{\partial \theta_{\ell}} = \operatorname{diag}_{1 \le d \le D} (V_{ud\ell}) \text{ and}$
 $P_{dd} = V_d^{-1} - V_d^{-1} T_d Q T'_d V_d^{-1}, \quad P_{d_1d_2} = -V_{d_1}^{-1} T_{d_1} Q T'_{d_2} V_{d_2}^{-1}, \quad Q = (T'V^{-1}T)^{-1},$
 $2 V_{\ell} = \left(-\frac{1}{2} - \frac{\rho_{12}\sigma_{u2}}{2} \right) \qquad 2 V_{\ell} = \left(-\frac{1}{2} - \frac{\rho_{12}\sigma_{u1}}{2} \right)$

$$V_{ud1} = \frac{\partial V_{ud}}{\partial \sigma_{u1}^2} = \begin{pmatrix} 1 & \frac{\rho_{12}\sigma_{u2}}{2\sigma_{u1}} \\ \frac{\rho_{12}\sigma_{u2}}{2\sigma_{u1}} & 0 \end{pmatrix}, \quad V_{ud2} = \frac{\partial V_{ud}}{\partial \sigma_{u2}^2} = \begin{pmatrix} 0 & \frac{\rho_{12}\sigma_{u1}}{2\sigma_{u2}} \\ \frac{\rho_{12}\sigma_{u1}}{2\sigma_{u2}} & 1 \end{pmatrix},$$
$$V_{ud3} = \frac{\partial V_{ud}}{\partial \rho_{12}} = \begin{pmatrix} 0 & \sigma_{u1}\sigma_{u2} \\ \sigma_{u1}\sigma_{u2} & 0 \end{pmatrix}.$$

For $\ell = 1, 2, 3$, we have

$$PV_{\ell} = \left[\underset{1 \le d \le D}{\text{diag}} (V_d^{-1}) - \underset{1 \le d \le D}{\text{col}} (V_d^{-1}T_d) \mathcal{Q}_{1 \le d \le D} (T_d'V_d^{-1}) \right] \underset{1 \le d \le D}{\text{diag}} (V_{ud\ell})$$
$$= \underset{1 \le d \le D}{\text{diag}} (V_d^{-1}V_{ud\ell}) - \underset{1 \le d \le D}{\text{col}} (V_d^{-1}T_d) \mathcal{Q}_{1 \le d \le D} (T_d'V_d^{-1}V_{ud\ell}).$$

For a, b = 1, 2, 3, we have

$$\begin{split} PV_{a}PV_{b} &= \bigg[\underset{1 \leq d \leq D}{\text{diag}} (V_{d}^{-1}V_{uda}) - \underset{1 \leq d \leq D}{\text{col}} (V_{d}^{-1}T_{d}) \mathcal{Q}_{1 \leq d \leq D} (T_{d}^{\prime}V_{d}^{-1}V_{uda}) \bigg] \\ & \cdot \bigg[\underset{1 \leq d \leq D}{\text{diag}} (V_{d}^{-1}V_{udb}) - \underset{1 \leq d \leq D}{\text{col}} (V_{d}^{-1}T_{d}) \mathcal{Q}_{1 \leq d \leq D} (T_{d}^{\prime}V_{d}^{-1}V_{udb}) \bigg] \\ &= \underset{1 \leq d \leq D}{\text{diag}} (V_{d}^{-1}V_{uda}V_{d}^{-1}V_{udb}) \\ & - \underset{1 \leq d \leq D}{\text{col}} (V_{d}^{-1}V_{uda}V_{d}^{-1}T_{d}) \mathcal{Q}_{1 \leq d \leq D} (T_{d}^{\prime}V_{d}^{-1}V_{udb}) \\ & - \underset{1 \leq d \leq D}{\text{col}} (V_{d}^{-1}T_{d}) \mathcal{Q}_{1 \leq d \leq D} (T_{d}^{\prime}V_{d}^{-1}V_{udb}) \\ & + \underset{1 \leq d \leq D}{\text{col}} (V_{d}^{-1}T_{d}) \mathcal{Q} \Big(\underset{d = 1}{\overset{D}{\sum}} T_{d}^{\prime}V_{d}^{-1}V_{uda}V_{d}^{-1}T_{d} \Big) \mathcal{Q}_{1 \leq d \leq D} (T_{d}^{\prime}V_{d}^{-1}V_{udb}) \end{split}$$

For $\ell = 1, 2, 3$, we have

$$\begin{aligned} \operatorname{tr} \ (PV_{\ell}) &= \sum_{d=1}^{D} \operatorname{tr} \ (V_{d}^{-1}V_{ud\ell}) - \sum_{d=1}^{D} \operatorname{tr} \ (V_{d}^{-1}T_{d}\mathcal{Q}T_{d}'V_{d}^{-1}V_{ud\ell}) = \sum_{d=1}^{D} \operatorname{tr} \ (P_{dd}V_{ud\ell}), \\ y'PV_{\ell}Py &= \sum_{d=1}^{D} y_{d}'V_{d}^{-1}V_{ud\ell}V_{d}^{-1}y_{d} - 2\sum_{d_{1}=1}^{D} \sum_{d_{2}=1}^{D} y_{d_{1}}'V_{d_{1}}^{-1}V_{ud_{1}\ell}V_{d_{1}}^{-1}T_{d_{1}}\mathcal{Q}T_{d_{2}}'V_{d_{2}}^{-1}y_{d_{2}} \\ &+ \sum_{d_{1}=1}^{D} \sum_{d_{2}=1}^{D} y_{d_{1}}'V_{d_{1}}^{-1}T_{d_{1}}\mathcal{Q}\Big(\sum_{d_{1}=1}^{D} T_{d}'V_{d}^{-1}V_{ud\ell}V_{d}^{-1}T_{d}\Big)\mathcal{Q}T_{d_{2}}'V_{d_{2}}^{-1}y_{d_{2}}. \end{aligned}$$

For a, b = 1, 2, 3, the second partial derivatives of the REML log-likelihood function are

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$$\begin{aligned} \frac{\partial l_{reml}^2(\theta)}{\partial \theta_a \partial \theta_b} &= \frac{1}{2} \quad \text{tr} \, \left(PV_b PV_a \right) - \frac{1}{2} \, y'(PV_b PV_a P + PV_a PV_b P) y \\ &= \frac{1}{2} \quad \text{tr} \, \left(PV_a PV_b \right) - y' PV_a PV_b Py, \end{aligned}$$

where the last equality follows from the fact that V_{ℓ} is symmetric, $\ell = 1, 2, 3$. By changing the sign, taking expectations and applying PT = 0, $PV = I - V^{-1}TQT'$ and the formula

$$E[y'Ay] = \operatorname{tr} (A \operatorname{var} (y)) + E[y]'AE[y],$$

we get the components of the Fisher information matrix, i.e.

$$F_{ab} = F_{ab}(\theta) = -\frac{1}{2} \operatorname{tr} (PV_a PV_b) + \operatorname{tr} (PV_a PV_b PV) + \beta' T' V_a PV_b PT\beta$$

$$= -\frac{1}{2} \operatorname{tr} (PV_a PV_b) + \operatorname{tr} (PV_a PV_b [I - V^{-1}TQT'])$$

$$= \frac{1}{2} \operatorname{tr} (PV_a PV_b) + \operatorname{tr} (PV_a PV_b V^{-1}TQT') = \frac{1}{2} \operatorname{tr} (PV_a PV_b), \quad a, b = 1, 2, 3.$$

Therefore, the Fisher information matrix is

$$F(\theta, \beta) = (F_{a,b})_{a,b=1,2,3}, \quad F_{ab} = F_{ab}(\theta, \beta) = \frac{1}{2} \text{ tr } (PV_a PV_b), \quad a, b = 1, 2, 3.$$

The trace of PV_aPV_b can be calculated as

$$\operatorname{tr} (PV_{a}PV_{b}) = \sum_{d=1}^{D} \operatorname{tr} \left(V_{d}^{-1}V_{uda}V_{d}^{-1}V_{udb} \right) - \sum_{d=1}^{D} \operatorname{tr} \left(V_{udb}V_{d}^{-1}V_{uda}V_{d}^{-1}T_{d}QT_{d}'V_{d}^{-1} \right) - \sum_{d=1}^{D} \operatorname{tr} \left(V_{uda}V_{d}^{-1}V_{udb}V_{d}^{-1}T_{d}QT_{d}'V_{d}^{-1} \right) + \operatorname{tr} \left(\left\{ \sum_{d=1}^{D} T_{d}'V_{d}^{-1}V_{udb}V_{d}^{-1}T_{d} \right\} Q \left\{ \sum_{d=1}^{D} T_{d}'V_{d}^{-1}V_{uda}V_{d}^{-1}T_{d} \right\} Q \right).$$

The pseudo-REML Fisher-scoring algorithm is

- 1. Set the initial values $\beta^{(0)}$, $\theta^{(0)}$, and $\varepsilon_i > 0$, $j = 1, \dots, p + q + 3$.
- 2. Repeat the following steps until the tolerance or the boundary conditions are fulfilled.
 - (a) Updating equation for θ : Do $\theta^{(i+1)} = \theta^{(i)} + F^{-1}(\theta^{(i)}, \beta^{(i)})S(\theta^{(i)}, \beta^{(i)})$.
 - (b) Boundary condition: If $\theta_1^{(i+1)} > 0$, $\theta_2^{(i+1)} > 0$ and $|\theta_3^{(i+1)}| < 1$, continue. Otherwise, do $\hat{\theta} = \theta^{(i)}$ and stop.
 - (c) Updating equation for β :Do $\beta^{(i+1)} = (T'V^{-1}(\theta^{(i+1)}, \beta^{(i)})T)^{-1}T'V^{-1}(\theta^{(i+1)}, \beta^{(i)})y$.

- (d) Tolerance condition: If $|\theta_{\ell}^{(i+1)} \theta_{\ell}^{(i)}| < \varepsilon_{p+q+\ell}$, $|\beta_{j}^{(i+1)} \beta_{j}^{(i)}| < \varepsilon_{j}$, $j = 1, ..., p + q, \ \ell = 1, 2, 3$, do $\hat{\theta}_{\ell} = \theta_{\ell}^{(i+1)}$, $\hat{\beta} = \beta^{(i+1)}$ and stop. Otherwise, continue.
- 3. Output: $\hat{\theta}$, $\hat{\beta}$, $F^{-1}(\hat{\theta}, \hat{\beta})$.

The asymptotic distributions of the REML estimators $\hat{\theta}$ and $\hat{\beta}$,

$$\hat{\theta} \sim N_2(\theta, F^{-1}(\theta, \beta)), \quad \hat{\beta} \sim N_{p+q}(\beta, (T'V^{-1}(\theta, \beta)T)^{-1}),$$

can be used to construct $(1 - \alpha)$ -level confidence intervals for the components θ_{ℓ} of θ and β_i of β , i.e.

$$\hat{\theta}_{\ell} \pm z_{\alpha/2} \, v_{\ell\ell}^{1/2}, \ \ell = 1, 2, 3, \quad \hat{\beta}_j \pm z_{\alpha/2} \, g_{jj}^{1/2}, \ j = 1, \dots, p+q, \tag{13}$$

where $F^{-1}(\hat{\theta}, \hat{\beta}) = (v_{ab})_{a,b=1,2,3}$, $(T'V^{-1}(\hat{\theta}, \hat{\beta})T)^{-1} = (g_{ab})_{a,b=1,\dots,p+q}$ and z_{α} is the α -quantile of the N(0, 1) distribution. For $\hat{\beta}_j = \beta_0$, the *p*-value for testing the hypothesis H_0 : $\beta_j = 0$ is

p-value =
$$2P_{H_0}(\hat{\beta}_j > |\beta_0|) = 2P(N(0,1) > |\beta_0|/\sqrt{g_{jj}}).$$
 (14)

We remark that we have changed the notation in (13) and (14), where β_j denotes the *j*-th component of the vector β and not the vector of regression parameters of the *j*-th category.

7 Simulations

This section presents simulation experiments for assessing the performance of the fitting method, the EBP estimator, and the MSE estimator. The objective is to show how the new methodology works in realistic (not extreme) scenarios. In practice, some explanatory variables can be taken from an auxiliary survey that is different from the target survey and that has bigger sample sizes. This is the case of the SLCS (target survey) and the SLFS (auxiliary survey). The second one has bigger sample sizes than the first. Therefore, it is natural to choose scenarios where the variances of the measurement errors are lower than the variances of the sampling errors and lower than the variances of the random effects.

In the calculation of Sect. 4, the variances of the sampling errors are $\sigma_{e1}^2 = \sigma_{e2}^2 = 1$ and the variances of the random effects are $\sigma_{u1}^2 = \theta_1 = 1$ and $\sigma_{u2}^2 = \theta_2 = 3/2 = 0.66$. Therefore, it is quite natural to choose scenario E, where the measurement error variances are $\tau_{d11} = \tau_{d22} = 0.21$ Concerning the random effect correlations, we are mainly interested in the case $\theta_3 = \text{corr } (u_{d1}, u_{d2}) = 0.45$ because θ_3 is positive in the application to real data. This is why, we consider that the scenario 6E is the most close to the application to real data presented in Sect. 8.

For the sake of completeness, we also run simulations under the scenarios 5E and 4E with correlations $\theta_3 = 0.05$ and $\theta_3 = -0.55$.

The data for the simulation experiments is generated as follows. We take $q_1 = q_2 = p_1 = p_2 = 1$, so that the elements of the MEBFH model (4) are $Z_d = \text{diag}(z_{d1}, z_{d2}), \quad \eta = (\eta_1, \eta_2)', \quad X_d = \text{diag}(x_{d1}, x_{d2}), \quad \lambda = (\lambda_1, \lambda_2)', B = \text{diag}(\lambda_1, \lambda_2), \quad v_d = (v_{d1}, v_{d2})', \quad u_d = (u_{d1}, u_{d2})', \quad e_d = (e_{d1}, e_{d2})'.$ Take $z_{d1} = z_{d2} = 1, \quad \eta_1 = \eta_2 = \lambda_1 = \lambda_2 = 1$ and

$$x_{d1} = \mu_{x1} + \sigma_{x1}^{1/2} S_{d1}, \quad x_{d2} = \mu_{x2} + \sigma_{x2}^{1/2} S_{d2}, \quad S_{dk} \stackrel{\text{ind}}{\sim} U(0,1), \quad k = 1, 2, \ d = 1, \dots, D,$$

with $\mu_{x1} = \mu_{x2} = 1$, $\sigma_{x1} = 1.2$ and $\sigma_{x2} = 2.4$. Note that

$$\operatorname{var}_{U(0,1)}(x_{d12}) = \frac{\sigma_{x1}}{12} = 0.1, \quad \operatorname{var}_{U(0,1)}(x_{d22}) = \frac{\sigma_{x2}}{12} = 0.2.$$

For d = 1, ..., D, generate $v_d \sim N_2(0, \Sigma_d)$, $u_d \sim N_2(0, V_{ud})$ and $e_d \sim N_2(0, V_{ed})$, where

$$\Sigma_{d} = \begin{pmatrix} \tau_{d11} & \tau_{d12} \\ \tau_{d21} & \tau_{d22} \end{pmatrix}, \quad V_{ud} = \begin{pmatrix} \theta_{1} & \theta_{3}\sqrt{\theta_{1}}\sqrt{\theta_{2}} \\ \theta_{3}\sqrt{\theta_{1}}\sqrt{\theta_{2}} & \theta_{2} \end{pmatrix}, \quad V_{ed} = \begin{pmatrix} 1 & c \\ c & 1 \end{pmatrix}$$

 $\tau_{d12} = \rho_{\tau} \tau_{d11}^{1/2} \tau_{d22}^{1/2}$, $\tau_{d11} = \tau_{d22} = \tau = 0.21$, $\rho_{\tau} = c = -0.35$, $\theta_1 = 1$ and $\theta_2 = 1.5$. Concerning the random effect correlation, we consider $\theta_3 = 0.45$, $\theta_3 = 0.05$ and $\theta_3 = -0.55$. This is to say, we take the same model parameters as in Scenarios 6E, 5E and 4E of Sect. 4.

7.1 Simulation 1

The target of Simulation 1 is to check the behavior of the pseudo-REML Fisherscoring algorithm for fitting the MEBFH model. The steps of Simulation 1 are

- 1. Generate z_{dk} , x_{dk} , d = 1, ..., D, k = 1, 2.
- 2. Repeat I = 1000 times (i = 1, ..., 1000)

2.1. Generate
$$v_d^{(i)} \sim N_2(0, \Sigma_d), u_d^{(i)} \sim N_2(0, V_{ud}), e_d^{(i)} \sim N_2(0, V_{ed})$$
 and
 $y_d^{(i)} = Z_d \eta + X_d \lambda + B v_d^{(i)} + u_d^{(i)} + e_d^{(i)}, \quad d = 1, \dots, D.$ (15)

- 2.2. For every model parameter $\gamma \in \{\eta_1, \lambda_1, \eta_2, \lambda_2, \theta_1, \theta_2, \theta_3\}$, calculate the corresponding REML estimator $\hat{\gamma}^{(i)} \in \{\hat{\eta}_1^{(i)}, \hat{\lambda}_1^{(i)}, \hat{\eta}_2^{(i)}, \hat{\lambda}_2^{(i)}, \hat{\theta}_1^{(i)}, \hat{\theta}_2^{(i)}, \hat{\theta}_3^{(i)}\}$.
- 3. Output (empirical biases and root-MSEs):

$$BIAS(\hat{\gamma}) = \frac{1}{I} \sum_{i=1}^{I} (\hat{\gamma}^{(i)} - \gamma), \quad RMSE(\hat{\gamma}) = \left(\frac{1}{I} \sum_{i=1}^{I} (\hat{\gamma}^{(i)} - \gamma)^2\right)^{1/2}.$$

For the sake of brevity, Table 2 presents only the results of Simulation 1 under Scenario 6E. The column labeled by γ contains the values of the true model parameters. Simulation 1 shows that the REML Fisher scoring algorithm works properly because BIAS and RMSE decrease as *D* increases. Similar results are obtained under Scenarios 4E and 5E.

7.2 Simulation 2

Simulation 2 investigates the performance of the EBPs of the mean parameters μ_{dk} . The steps of Simulation 2 are

- 1. Generate z_{dk} , x_{dk} , d = 1, ..., D, k = 1, 2. Take $D \in \mathcal{D} = \{50, 100, 200\}$.
- 2. Repeat $I = 10^4$ times (i = 1, ..., I)
 - 2.1. Generate $\{(e_d^{(i)}, u_d^{(i)}, v_d^{(i)}, y_d^{(i)}) : d = 1, ..., D\}$ from the MEBFH model (15).
 - 2.2. Calculate the true means $\mu_d^{(i)} = Z_d \eta + X_d \lambda + B v_d^{(i)} + u_d^{(i)}, d = 1, \dots, D.$
 - 2.3. Fit the MEBFH model to the simulated data $(y_d^{(i)}, Z_d, X_d), d = 1, ..., D$. Calculate the EBP $\hat{\mu}_d^{(i)}$ under the MEBFH model.
- 3. For d = 1, ..., D, k = 1, 2, calculate the relative performance measures

3.1
$$E_{dk} = \left(\frac{1}{I}\sum_{i=1}^{I}(\hat{\mu}_{dk}^{(i)} - \mu_{dk}^{(i)})^{2}\right)^{1/2}, B_{dk} = \frac{1}{I}\sum_{i=1}^{I}(\hat{\mu}_{dk}^{(i)} - \mu_{dk}^{(i)}), \\ \mu_{dk} = \frac{1}{I}\sum_{i=1}^{I}\mu_{dk}^{(i)}.$$

3.2.
$$RE_{dk} = 100\frac{E_{dk}}{\mu_{dk}}, RB_{dk} = 100\frac{B_{dk}}{\mu_{dk}}, RE_{k} = \frac{1}{D}\sum_{d=1}^{D}RE_{dk}, \\ RB_{k} = \frac{1}{D}\sum_{d=1}^{D}|RB_{dk}|.$$

Table 3 presents the relative performance measures RB_k (left) and RE_k (right) of EBPs under Scenarios 4E, 5E and 6E. As expected, the EBPs have almost no bias in all cases. We also note that root-MSEs decrease slowly as the number of domains increases. This is somewhat reasonable, since increasing *D* also increases the number of quantities μ_d to be predicted.

7.3 Simulation 3

Simulation 3 investigates the performance of the parametric bootstrap estimator of the MSE of the EBPs. For D = 100, the steps of Simulation 3 are

		$BIAS(\hat{\gamma})$			$RMSE(\hat{\gamma})$		
	γ	D = 50	D = 100	D = 200	D = 50	D = 100	D = 200
η_1	1	0.0474	0.0243	0.0163	0.9989	0.6843	0.5105
λ_1	1	-0.0278	- 0.0137	-0.0072	0.6228	0.4365	0.3208
η_2	1	0.0082	-0.0248	0.0320	1.0128	0.6510	0.4635
λ_2	1	-0.0012	0.0154	- 0.0189	0.5294	0.3567	0.2546
θ_1	1	- 0.0253	- 0.0296	- 0.0130	0.4658	0.3465	0.2497
θ_2	1.5	- 0.0315	- 0.0229	0.0058	0.5732	0.4213	0.2833
θ_3	0.45	0.0319	0.0339	0.0177	0.2973	0.2322	0.1691

Table 2 Empirical biases (left) and root-MSEs (right) for scenario 6E

- 1. Take $mse_{dk} = E_{dk}^2$, k = 0, 1, d = 1, ..., D, from the output of Simulation 2.
- 2. Generate z_{dk} , x_{dk} , d = 1, ..., D, k = 1, 2.
- 3. Repeat $I = 10^2$ times (i = 1, ..., I)
 - 3.1. Generate $\{(e_d^{(i)}, u_d^{(i)}, v_d^{(i)}, y_d^{(i)}) : d = 1, ..., D\}$ from the MEBFH model (15).
 - 3.2. Calculate the REML estimators $\hat{\gamma}^{(i)} \in \{\hat{\eta}_1^{(i)}, \hat{\lambda}_1^{(i)}, \hat{\eta}_2^{(i)}, \hat{\lambda}_2^{(i)}, \hat{\theta}_1^{(i)}, \hat{\theta}_2^{(i)}, \hat{\theta}_3^{(i)}\}$ by using the sample data $(y_d^{(i)}, Z_d, X_d), d = 1, ..., D$.
 - 3.3. Repeat B times
 - (a) Generate $u_d^{*(ib)} \sim N_2(0, \hat{V}_{ud}^{(i)}), e_d^{*(ib)} \sim N_2(0, V_{ed}), v_d^{*(ib)} \sim N_2(0, \Sigma_d)$ and

$$y_d^{*(ib)} = \mu_d^{*(ib)} + e_d^{*(ib)},$$

$$\mu_d^{*(ib)} = Z_d \hat{\eta}^{(i)} + X_d \lambda^{(i)} + \text{ diag } (\hat{\lambda}_1^{(i)}, \hat{\lambda}_2^{(i)}) v_d^{*(ib)} + u_d^{*(ib)}, \ d = 1, \dots, D.$$
(16)

- (b) Calculate the REML estimators $\hat{\eta}_1^{*(ib)}, \hat{\lambda}_1^{*(ib)}, \hat{\eta}_2^{*(ib)}, \hat{\lambda}_2^{*(ib)}, \hat{\theta}_1^{*(ib)}, \hat{\theta}_2^{*(ib)}, \hat{\theta}_3^{*(ib)}$ by using the sample data $(y_d^{*(ib)}, Z_d, X_d), d = 1, \dots, D.$
- (c) Calculate the EBP $\hat{\mu}_{dk}^{*(ib)}$ under the MEBFH model (16).
- 3.4. For d = 1, ..., D, k = 1, 2, calculate the bootstrap MSE estimator of the EBP, i.e.

$$mse_{dk}^{*(i)} = \frac{1}{B} \sum_{b=1}^{B} \left(\hat{\mu}_{dk}^{*(ib)} - \mu_{dk}^{*(ib)} \right)^2.$$

Scenario	k	D = 50	D = 100	D = 200	D = 50	D = 100	D = 200
4E	1	0 2220	0 1936	0.2123	28,1546	27.5489	27 0206
	2	0.2295	0.2290	0.2192	27.1848	27.1914	27.1846
5E	1	0.2293	0.2013	0.2369	30.5311	30.0734	29.6244
	2	0.2566	0.2435	0.2308	29.5998	29.7698	29.8576
6E	1	0.2113	0.1908	0.2271	29.0125	28.4583	27.9936
	2	0.2439	0.2175	0.2100	28.0417	28.1359	28.1733

Table 3 Relative measures RB_k (left) and RE_k (right) of EBPs (in %)

4. For d = 1, ..., D, k = 1, 2, calculate the relative performance measures $RE_{dk}^* = 100E_{dk}^*/mse_{dk}$ and $RB_{dk}^* = 100B_{dk}^*/mse_{dk}$, where

$$E_{dk}^* = \left(\frac{1}{I}\sum_{i=1}^{I}(mse_{dk}^{*(i)} - mse_{dk})^2\right)^{1/2}, \ B_{dk}^* = \frac{1}{I}\sum_{i=1}^{I}(mse_{dk}^{*(i)} - mse_{dk})^2$$

5. For k = 1, 2, calculate the averages across domains of the relative performance measures, i.e. $RB_k^* = \frac{1}{D} \sum_{d=1}^{D} |RB_{dk}^*|$, $RE_k^* = \frac{1}{D} \sum_{d=1}^{D} RE_{dk}^*$.

Table 4 presents the relative performance measures RB_k^* and EB_k^* (in %) under Scenarios 4E, 5E and 6E, for D = 100. Figures 2 and 3 presents the boxplots of the absolute performance measures B_{dk}^* 's and E_{dk}^* 's respectively under scenario 6E. Similar boxplots are constructed for Scenarios 4E and 5E, but they are not presented here. They show that the parametric bootstrap estimators of the MSEs of the EBPs have a negative bias in the simulated scenarios. Nevertheless, the observed biases are rather small in comparison with the calculated empirical root-MSEs. Simulation 3 suggests that running a parametric bootstrap algorithm with a number of replicates *B* between 200 and 400 give a considerably good approximation to estimate the MSE of the EBP.

8 Estimation of poverty proportions and gaps in Spanish provinces

This manuscript presents an application of the MEBFH model to the estimation of poverty proportions and gaps in Spanish provinces by sex. Spain is divided in 52 provinces (including the autonomous cities of Ceuta and Melilla) leading to D = 104 target domains (provinces crossed by sex) of known sizes N_d , d = 1, ..., D.

Let z_{dj} be the normalized net annual income of the household where the individual *j* of domain *d* lives. Let z_0 be the poverty line, so that individuals with $z_{dj} < z_0$ are considered as poor. The main goal of this section is to jointly estimate poverty proportions and gaps

$$\bar{Y}_{d1} = \frac{1}{N_d} \sum_{j=1}^{N_d} y_{d1j}, \quad \bar{Y}_{d2} = \frac{1}{N_d} \sum_{j=1}^{N_d} y_{d2j}, \quad d = 1, \dots, D,$$
 (17)

where $y_{d1j} = I(z_{dj} < z_0)$, $I(z_{dj} < z_0) = 1$ if $z_{dj} < z_0$, $I(z_{dj} < z_0) = 0$ otherwise and $y_{d2j} = z_0^{-1}(z_0 - z_{dj})I(z_{dj} < z_0).$

The Spanish Statistical Office calculates z_{dj} by summing up the net annual incomes of the household members and by dividing by its normalized size. Later, the same value of the normalized net annual income of the household is assigned to all the individuals j of the household. Therefore z_{di} is constant within the household. The aim of normalizing the household income is to adjust for the varying size and composition of households. The definition of the total number of normalized household members is the modified OECD scale used by EUROSTAT. This scale gives a weight of 1.0 to the first adult, 0.5 to the second and each subsequent person aged 14 and over and 0.3 to each child aged under 14 in the household h. The normalized size of a household is the sum of the weights assigned to each person. So the total number of normalized household members is

$$H_{dh} = 1 + 0.5(H_{dh \ge 14} - 1) + 0.3H_{dh < 14},$$

where $H_{dh \ge 14}$ is the number of people aged 14 and over and $H_{dh < 14}$ is the number of children aged under 14. Following the standards of the Spanish Statistical Office, the poverty threshold is fixed as the 60% of the median of the normalized net annual incomes in Spanish households. The Spanish poverty threshold (in euros) in 2008 is $z_{2008} = 7488.65$. We deal with data from the SLCS of 2008 with sample size 35967. This is an annual survey where the planned domains are the regions (autonomous communities), so that sample sizes are selected to obtain precise direct estimates at the region level. As Spain is hierarchically partitioned in regions, provinces, counties (comarcas) and municipalities, estimating province-sex poverty proportions is a small area estimation problem.

able 4 Relative measures of SE estimators for $D = 100$ (in)							
	Scenario	Measure	k	B = 50	B = 100	B = 200	B = 400
	4E	RB_k^*	1	4.7738	3.3389	2.0169	2.8475
			2	3.6455	2.8645	1.4327	2.0248
		RE_k^*	1	17.7969	16.5755	14.3046	14.0318
			2	18.0551	15.7436	13.4592	10.4429
	5E	RB_k^*	1	1.8112	0.9627	1.3113	2.9361
			2	0.9266	1.7964	1.7394	1.0371
		RE_k^*	1	17.2544	13.8403	11.5956	10.9446
			2	17.4716	13.3175	10.5300	8.1644
	6E	RB_k^*	1	2.7367	4.2101	2.8832	0.0457
			2	1.6416	3.4513	3.0222	0.0168
		RE_k^*	1	17.4342	15.1222	12.5930	13.1017
			2	16.9305	14.9749	12.1429	9.8613

Ta Μ %



Fig. 2 Boxplots of biases $\{B^*_{dk} : d = 1, ..., D\}, k = 1, 2$



Fig. 3 Boxplots of root-MSEs $\{E_{dk}^*: d = 1, ..., D\}, k = 1, 2$

The direct estimators of the size N_d , the total $Y_{dk} = \sum_{j=1}^{N_d} y_{dkj}$ and the mean $\overline{Y}_{dk} = Y_{dk}/N_d$ are

$$\hat{N}_{d}^{dir} = \sum_{j \in s_{d}} w_{dj}, \quad \hat{Y}_{dk}^{dir} = \sum_{j \in s_{d}} w_{dj} y_{dkj}, \quad \hat{Y}_{dk}^{dir} = \hat{Y}_{dk}^{dir} / \hat{N}_{d}^{dir}, \quad k = 1, 2,$$

where s_d is the domain sample of size n_d and the w_{dj} 's are the official calibrated sampling weights which take non response into account. The direct estimates of the domain means are used as responses in the area-level Fay–Herriot model. The design-based covariance between $\hat{Y}_{dk_1}^{dir}$ and $\hat{Y}_{dk_2}^{dir}$, $k_1, k_2 = 1, 2$, can be estimated by

$$\hat{\operatorname{cov}}_{\pi}(\hat{Y}_{dk_{1}}^{dir},\hat{Y}_{dk_{2}}^{dir}) = \frac{1}{(\hat{N}_{d}^{dir})^{2}} \sum_{j \in s_{d}} w_{dj}(w_{dj}-1)(y_{dk_{1}j}-\hat{Y}_{dk_{1}}^{dir})(y_{dk_{2}j}-\hat{Y}_{dk_{2}}^{dir}), \quad (18)$$

where the case $k_1 = k_2 = k$ denotes estimated variance, i.e. $\hat{Y}_{dk}^{dir} = \hat{Y}_{dk}^{dir}$ ($\hat{Y}_{dk}^{dir}, \hat{Y}_{dk}^{dir}$). The last formulas are obtained from Särndal et al. (1992), pp. 43, 185 and 391, with the simplifications $w_{dj} = 1/\pi_{dj}$, $\pi_{dj,dj} = \pi_{dj}$ and $\pi_{di,dj} = \pi_{di}\pi_{dj}$, $i \neq j$ in the second order inclusion probabilities.

We take data from the SLFS of 2008 to construct the data file of aggregated auxiliary variables. The SLFS is a quarterly survey with provinces as planned domains. Within each quarter, the SLFS sample sizes at the province level are fixed a priori. They are selected high enough to have precise direct estimates. By putting together the data files of the 4 quarters, the SLFS direct estimators of means at domain (province crossed by sex) level are even more precise for 2008 than using only the data of one quarter. Additionally, by doing this aggregation all the calculated SLFS direct estimators of domain means have non zero estimated variances.

The file of auxiliary variables is constructed from the SLFS data of 2008 (SLFS2008). It contains the direct estimates of the domain means by the categories of the considered auxiliary variables. It also contains the variance and covariance estimates of the calculated direct estimators. The auxiliary variables are:

Nationality, with categories Spanish and Foreigner.

Education, with categories LowEdu (less than secondary level completed), HighEdu (secondary or superior level completed)

Age, with categories age1 (\leq 15), age2 (16-24), age3 (25-49), age4 (50-64), age5 (>64).

Labour situation, with categories ≤ 15 , Employed, Unemployed, Inactive.

Table 5 presents the pseudo-REML estimates of the regression parameters of the selected MEBFH model. It also contains the corresponding *p*-values. We note that domains with higher proportions of Spanish or unemployed people tends to have higher poverty proportions. On the other hand, the level of education is negatively related with the poverty proportion. We note that foreigners tend to live and work in rich provinces. Therefore, the obtained results are in agreement with the Spanish socioeconomic situation of 2008.

Table 6 contains the estimates of the variance component parameters and the corresponding asymptotic 95% confidence intervals.

Figure 4 (left) plots the MEBFH model residuals of poverty proportions versus the corresponding EBPs. Figure 4 (right) plots the MEBFH model residuals of poverty gaps versus their EBPs.

We observe that the residuals present greater variability on the right hand side of the OX axis. This is a natural phenomenon, as the EBPs have to smooth the behavior of the direct estimators. The plots show that EBPs tend to be smaller than direct estimates when direct estimates are large. This is due to two main reasons. First, it is part of the smoothing work that EBPs have to do. Second, the variance of the directly estimated proportion is approximately $(1 - n_d/N_d)/(n_d - 1)\hat{Y}_{dk}^{dir}(1 - \hat{Y}_{dk}^{dir})$,

Table 5	Regression parameters	Parameter	Estimate	Std.error	z-value	<i>p</i> -value
		Intercept.prop	- 0.0100	0.0984	0.1021	0.9187
		Spanish.prop	0.3576	0.0837	4.2747	0.0000
		HighEdu.prop	-0.6540	0.0933	7.0080	0.0000
		Unemployed.prop	p 1.7440	0.2939	5.9348 0.2738 2.2605 3.9690	0.0000
		Intercept.gap	- 0.0115	0.0421 0.0363 0.0380		0.7842
		Spanish.gap	0.0822			0.0238
		HighEdu.gap	- 0.1509			0.0001
		Unemployed.gap	0.8373	0.1181	7.0874	0.0000
Table 6 Variance component						
paramet	ers	θ		Lower.lim		Upper.lim
		σ_{u1}^2 0	.00171	0.00102		0.00241
		$\sigma_{u2}^2 = 0$.00028	0.00017		0.00040
		ρ 0	.79750	0.68389		0.91110

which takes its maximum at $\hat{Y}_{dk}^{dir} = 0.5$, for n_d, N_d fixed. Allowing in this context area specific V_{ud} is an interesting future research task, that possibly could further reduce the MSE of the EBP.

Figure 5 plots the EBPs and direct estimates of poverty proportions (left) and gaps (right) for men and women. We observe that both estimators tend to coincide as the sample size increases.

Figure 6 plots the estimated model-based root-MSEs of the EBPs and the estimated design-based root-MSEs of the direct estimators of poverty proportions (left) and gaps (right) for men and women. The first root-MSE is estimated by



Fig. 4 Model residuals versus EBPs of poverty proportions (left) and gaps (right)



Fig. 5 EBPs and direct estimates of men (left) and women (right) poverty proportions



Fig. 6 Root-MSEs of EBPs and direct estimators for men (left) and women (right)

parametric bootstrap under the fitted MEBFH model. The second one is calculated by applying the estimator (18) for design-based variances. The EBPs have lower root-MSEs than the direct estimators. When the sample size increases the root-MSEs of all the estimators become almost equal.

Figures 7 and 8 plot the Spanish provinces in 4 colors depending on the values of the EBPs of the poverty proportions and poverty gaps in %. We observe that the Spanish provinces where the proportion of the population under the poverty line is



Fig. 7 Estimated poverty proportions for men (left) and women (right) in 2008



Fig. 8 Estimated poverty gaps for men (left) and women (right) in 2008

smallest are those situated in the north and east. On the other hand, the Spanish provinces with higher poverty proportions are those situated in the center and south.

Appendix B in the supplementary file contains 2 tables with basic numerical results. To give a brief summary of the obtained numerical results, we order the Spanish provinces (including the cities of Ceuta and Melilla) by sample size and we select one in five. Table B.1 presents the EBPs and direct (dir) estimates of poverty proportions for men and women respectively. Table B.2 presents the EBPs and direct (dir) estimates of poverty gaps for men and women respectively. These tables also give the estimated root-MSEs (rmse.eblup and rmse.dir), for both types of estimators. The column N_d presents the estimates of the domain sizes calculated by using the data of the 4 quarters of the SLFS in 2008. The columns *province* and n_d contain the province name and the sample size respectively.

9 Conclusions

In many applications the auxiliary information used in bivariate Fay–Herriot models is not measured exactly. Under this setting, the EBLUP based on the bivariate Fay–Herriot model is not the EBP anymore, as this model assumes the exact measurement of the auxiliary variables.

By calculating relative efficiencies, we showed that not taking into account the measurement errors may lead to predictions of target parameters with greater mean squared errors. Therefore, we extended the bivariate Fay–Herriot model by allowing for multivariate normal distributed random error on the auxiliary variables. This reflects the typical situation of estimated auxiliary information. Both, the mean squared error and the bias of the new EBP are reduced with respect to the classical EBLUP.

For fitting the new model, the pseudo-REML estimation procedure showed to be stable. Further a second fitting algorithm was introduced but not implemented. In the case of exactly measured auxiliary variables, the measurement error bivariate Fay–Herriot model reduces to the classical bivariate Fay–Herriot model and, therefore, they basically give the same predictions. Finally, we recommend to use the proposed measurement error Fay–Herriot model when the auxiliary variables are estimated.

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