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Insurance demand in the presence of loss-dependent background risk

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Abstract

We analyze insurance demand when insurable losses come with an uninsurable zero-mean background risk that increases in the loss size. If the individual is risk vulnerable, loss-dependent background risk triggers a precautionary insurance motive and increases optimal insurance demand. Prudence alone is sufficient for insurance demand to increase in two cases: the case of fair insurance and the case where the smallest possible loss exceeds a certain threshold value (referred to as the *large loss case*). We derive conditions under which insurance demand increases or decreases in initial wealth. In the large loss case, prudence determines whether changes in the background risk lead to more insurance demand. We generalize this result to arbitrary loss distributions and find conditions based on decreasing third-degree Ross risk aversion, Arrow–Pratt risk aversion, and Arrow–Pratt temperance.

KEYWORDS

background risk, insurance demand, loss-dependent background risk, prudence, risk vulnerability

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1 | MOTIVATION

Insurance is a powerful tool to transfer undesirable risks to third parties. In reality, insurance contracts are often incomplete in the sense that they do not perfectly indemnify each possible loss in any state of the world. This incompleteness has been long recognized in the insurance economics literature since Doherty and Schlesinger (1983b).

In this study, we investigate insurance demand in the presence of an uninsurable loss-dependent background risk. We assume that an individual's loss consists of two components: the insurable component denoted by X and the uninsurable component denoted by η_X . The total loss faced by the individual is given by $X + \eta_X$, but the indemnity is solely based on X as η_X cannot be treated by insurance. We assume that η_X has a zero mean. This rules out any wealth effects due to the uninsurable component. η_X can be interpreted as an observation error between the actual loss size and the insured loss as discussed under the term *approximate insurance* by Gollier (1996). Intuitively, we consider the situation in which only estimated losses are insurable, but not actual losses.

Applications in which such an uninsurable component arises include the following:

- The insurer is not able to perfectly observe the loss. This can be the case when the loss evolves over long periods of time and the indemnity is paid instantly, exposing the decision-maker to price risk.
- The insurer uses a simplified indemnification process, basing the indemnity on estimations rather than actual losses, and insurance policies which by design do not condition on the actual loss. Fixed indemnity insurance like hospital cash benefits serves as an example here.
- The insured loss is subject to an uninsurable exchange rate risk.

In all these examples, the indemnity is solely based on estimated losses X rather than actual losses $X + \eta_X$. In the exchange rate example, η_X describes the random exchange rate component, where $\eta_X = X\eta$, and η is the random exchange rate factor. At the same time, all of these examples have in common that the error η_X typically increases in loss size. Therefore, in this study, we assume that the distribution of η_X undergoes (second-order) increases in risk in the sense of Rothschild and Stiglitz (1970) as the loss X increases. Consequently, the larger the insurable loss X is in absolute terms, the riskier is the uninsurable loss component. For this reason, we coin η_X *loss-dependent background risk*. Due to loss-dependent background risk, the decision-maker is exposed to some risk even when she takes out maximum coverage.

Our study aims to analyze the implications of loss-dependent background risk on insurance demand. Therefore, we derive optimal insurance demand with loss-dependent background risk and benchmark this coverage level to the case in which such a background risk is absent. This approach allows us to study the comparative effect of loss-dependent background risk on optimal insurance demand. This comparison is meaningful in the context of salience: loss-dependent background risk is prone to be neglected, as the decision-maker may not be aware of its existence. If loss-dependent background risk becomes salient, the decision-maker is in a situation where the additional risk suddenly becomes present in her perception of the decision situation. Salience of the loss-dependent background risk may not only change over time for one individual, but may also differ between individuals. We address the question of whether, and under which conditions, salience of a loss-dependent background risk changes insurance demand, either intrapersonal over time or interpersonal for individuals with and without salience of the loss-dependent background risk. Salience of this risk is also relevant from the

perspective of the insurer in the decision problem. If the insurer neglects the fact that the insured face the additional uninsurable risk, expectations on insurance demand are potentially biased.

More generally, imperfections can arise for different reasons: one example of imperfect insurance policies is the case of insurer default, in which the insurer fails to fully pay the agreed-upon indemnity (Doherty & Schlesinger, 1990). The insured is thus exposed to the uninsurable risk of insurer insolvency, irrespective of the selected insurance policy. Another example discussed in the literature is that insurance policies typically exclude losses due to certain events like war or nuclear hazards, leaving the individual fully exposed to these kinds of risks irrespective of the amount of insurance purchased (Doherty & Schlesinger, 1983b). These excluded losses are examples of risks that have been coined background risk in the literature (Gollier & Pratt, 1996). Imperfections due to loss-dependent background risk are different from other sources of imperfections, like, excluded losses or insurer default risk. For instance, losses triggered by uninsurable events are typically independent of losses that can be insured, as the events are separate. Loss-dependent background risk, on the other hand, manifests whenever insurable losses realize and depends on the realization of the insurable loss event. Loss-dependent background risk has an upside, for example, if the actual loss is smaller than the insurable component and is therefore a special case of a basis risk. This is different to insurer default risk, which only has a downside risk.

An important contribution of our paper is that we find that risk vulnerability determines insurance demand even if background risk is not independent¹: We find that a loss-dependent background risk leads to a higher demand for insurance if the individual is risk vulnerable.² Unlike in the case of an independent background risk, a loss-dependent background risk triggers a precautionary insurance motive for risk-vulnerable decision-makers that can potentially lead to more than full coverage being optimal.³

Our paper contributes to the literature on insurance demand in the presence of uninsurable risk. It is most closely related to Eeckhoudt and Kimball (1992) who study insurance demand with a loss-dependent background risk that deteriorates in a third-order stochastic dominance sense as the insurable loss increases. They find that standardness⁴ leads to more insurance demand in the presence of the loss-dependent background risk. The loss-dependent background risk we consider here is a special case of the one studied in Eeckhoudt and Kimball (1992). We show that risk vulnerability is already sufficient for unambiguous results in our setup such that the stronger assumption of decreasing absolute risk aversion together with decreasing absolute prudence is not needed. In the special case of a binary loss distribution, Fei and Schlesinger (2008) show that third-order preferences determine how insurance demand changes if a loss-dependent background risk is present. We show that their result does not necessarily hold for arbitrary loss distributions, but we also discuss assumptions on the loss distribution and the insurance policy under which results from Fei and Schlesinger (2008) continue to hold. To further complement the work

¹In the literature on risk preferences, background risk is usually assumed to be independent of insurable risk. In the case of such an independent background risk, risk vulnerability as introduced by Gollier and Pratt (1996) leads to a more cautious behavior. A risk-vulnerable decision-maker consequently demands more insurance to better cope with an independent background risk.

²The concept of risk vulnerability states that “adding an unfair background risk to wealth makes risk-averse individuals behave in a more risk-averse way with respect to another independent risk” (Gollier & Pratt, 1996, p. 1110).

³In the case of loss-dependent background risk, full insurance refers to a policy that fully covers X , even though the realized loss may be larger or smaller than the actual indemnity.

⁴Kimball (1993) defines standardness as decreasing absolute risk aversion and decreasing absolute prudence.

of Eeckhoudt and Kimball (1992) and Fei and Schlesinger (2008), we conduct comparative statics with respect to wealth levels and changes in risk. Again, we utilize results from the theory of decision-making in the presence of an independent background risk to derive conditions for unambiguous effects of an increase in wealth on optimal insurance demand. We further give conditions that result in higher insurance demand if the insurable loss increases in riskiness. With respect to changes in the background risk, we provide conditions on the decision-maker's preferences such that an increase in the background risk results in nondecreasing insurance demand.

After the introduction, the paper evolves as follows: We first introduce the general model setup and derive the main results with respect to insurance demand in Section 2. Section 3 includes comparative statics with respect to changes in wealth endowment and the distribution of the insurable and the background risk. In Section 4, we discuss related literature. We conclude with a brief outlook.

2 | OPTIMAL INSURANCE DEMAND WITH LOSS-DEPENDENT BACKGROUND RISK

We consider a decision-maker who faces a potential insurable loss given by a random variable $X \in [0, L]$, where L is the maximum possible loss. X is assumed to be distributed according to a cumulative distribution function $F(\cdot)$. The individual's initial wealth is denoted by $W_0 > 0$.⁵ Utility of final wealth is given by a von Neumann–Morgenstern utility function $u(\cdot)$ with $u' > 0$ and $u'' < 0$.⁶ The decision-maker can purchase insurance coverage in the form of a coinsurance policy.⁷

In addition to the insurable loss X , we assume that the decision-maker faces an uninsurable loss component that cannot be treated by insurance. To this end, let $(\eta_x)_{x \geq 0}$ denote a family of random variables all defined on $[a, b]$ for some $a, b \in \mathbb{R}$ such that $E[\eta_x] = 0$ for $x \geq 0$. Let η_x be a random variable such that η_x denotes the random variable conditional on the realization $X = x$ with cumulative distribution function $G(\eta, x)$. For any $X = x$, the distribution of η_x depends on x , which justifies the term *loss-dependent* background risk for the uninsurable component of the loss. In this setup, the individual's total loss is given by $X + \eta_x$.

Following Diamond and Stiglitz (1974) and Gollier (1996), we assume that η_x increases in risk in the sense of Rothschild and Stiglitz (1970) as x increases. Formally, this relationship can be modeled as follows: suppose that for all $x \in [0, L]$ we have

$$\int_a^b G_x(\eta, x) d\eta = 0 \quad \text{and} \quad \int_a^y G_x(\eta, x) d\eta \geq 0 \quad \text{for } a \leq y \leq b, \quad (1)$$

where $G_x(\eta, x)$ captures the change in $G(\eta, x)$ due to an infinitesimal change in x (Mahul, 2000). This characterization thus is the representation of mean-preserving spreads

⁵To avoid negative wealth, we assume W_0 is large enough to cover any potential loss, both from the insurable as well as the background risk.

⁶We assume throughout that u is sufficiently many times differentiable such that the derivatives exist as required.

⁷We address the case of deductible insurance policies in Appendix A. We show that all results derived for coinsurance also continue to hold for the case of deductible insurance.

as in Rothschild and Stiglitz (1970) for infinitesimal increases in insurable risk X and the associated change in the background risk's distribution. We also restrict the analysis to the case in which $\eta_0 \equiv 0$, such that background risk is only present if a positive loss occurs. To avoid the trivial case, we assume that there exist some $x \in [0, L]$ and some $y \in [a, b]$ for which (1) holds with a strict inequality. In the remainder of the study, the term loss-dependent background risk is understood in the sense described in this section.

In a coinsurance policy, the decision-maker chooses the fraction $\alpha \geq 0$ of the loss which is born by the insurance company, in exchange for a premium of size $\alpha P = \alpha \lambda \mathbb{E}[X]$, where $\lambda > 0$ denotes the loading factor. However, we assume that the insurance policy only conditions on X , but not on the additional component η_X . The decision-maker faces η_X irrespective of the insurance coverage decision and the level of coverage is a fraction of the insurable loss X alone. Random final wealth in the presence of a loss-dependent background risk is given by

$$\widetilde{W}(X) = W_0 - X - \alpha P + \alpha X + \eta_X.$$

Expected utility for any given level of insurance demand α is

$$\mathbb{E}[u(\widetilde{W}(X))] = \mathbb{E}[u(W_0 - X - \alpha P + \alpha X + \eta_X)]. \tag{2}$$

Consequently, the optimal level of coinsurance α^{**} solves the following necessary and sufficient first-order condition (FOC):

$$\left. \frac{\partial \mathbb{E}[u(\widetilde{W}(X))]}{\partial \alpha} \right|_{\alpha=\alpha^{**}} = \mathbb{E}[(X - P)u'(W_0 - X - \alpha^{**}P + \alpha^{**}X + \eta_X)] = 0. \tag{3}$$

The second-order condition follows from the concavity of u . Thus, the FOC is necessary and sufficient to characterize the unique maximizer. This modeling approach results in the loss-dependent background risk η_X being uncorrelated to the loss X .⁸ The restriction to background risks with $\eta_0 = 0$ guarantees that background risk only manifests if a loss occurs. Therefore, we do not consider any baseline background risk which the decision-maker is exposed to irrespective whether a loss occurs or not.⁹

In the following, we address the question of how to optimally deal with loss-dependent background risk when making insurance decisions. A natural point of comparison is the optimal coinsurance level when not taking loss-dependent background risk into account. We, therefore, benchmark the optimal level of coverage with loss-dependent background risk to the case in which the insured faces the exact same situation, but with loss-dependent background risk being absent. Given that loss-dependent background risk is zero-mean, taking loss-dependent background risk into account does not change the expected final wealth (holding the level of coverage constant). The comparison of the optimal level of coverage with and without background risk is specifically relevant as

⁸We have $\text{Cov}(X, \eta_X) = \mathbb{E}[X\eta_X] - \mathbb{E}[X]\mathbb{E}[\eta_X] = \mathbb{E}[X\mathbb{E}[\eta_X|X]] - \mathbb{E}[X]\mathbb{E}[\eta_X] = 0$ as $\mathbb{E}[\eta_X] = 0$ for all x .

⁹Nevertheless, some of our results still hold if some amount of background risk is present irrespective whether a loss occurs or not. However, as the background risk is not truly loss-induced and to avoid the need to distinguish cases for which results hold with baseline background risk or not, we do not consider this case any further.

background risk is prone to be neglected in insurance decision-making—by individuals, insurers as well as researchers. If it becomes salient to individuals, insurance demand potentially changes.

The benchmark is the optimal coinsurance rate α^* when loss-dependent background risk is absent. Random final wealth is then given by

$$\hat{W}(X) = W_0 - X - \alpha P + \alpha X.$$

The optimal choice of coinsurance rate α^* in this setup is well understood: it is optimal to opt for full coverage in case of actuarially fair prices ($\lambda = 1$), while a proportional loading ($\lambda > 1$) results in partial coverage (Mossin, 1968). The optimal level of coinsurance solves the following necessary and sufficient FOC:

$$\left. \frac{\partial \mathbb{E}[u(\hat{W}(X))]}{\partial \alpha} \right|_{\alpha=\alpha^*} = \mathbb{E}[(X - P)u'(W_0 - X - \alpha^*P + \alpha^*X)] = 0. \quad (4)$$

Our modeling approach is fairly general and allows for different dependence structures between the insurable loss and the loss-dependent background risk. An economically important special case of the loss-dependent background risk we consider here is $\eta_X = X\eta$ for an independent random variable η with $\mathbb{E}[\eta] = 0$. In this case, loss-dependent background risk is proportional to the size of the insurable loss. Such a loss-dependent background risk models a proportional gap between actual loss and indemnity. One example where we expect such a proportional loss-dependent background risk to arise is when the insurer is not able to perfectly observe the loss size. This may occur if the loss evolves over long periods of time, whereas the indemnity is paid instantly. The insurer can only estimate the loss as X , while the actual loss is $X(1 + \eta)$, where η describes the random long-term price evolution. Another example arises when the insurer uses a simplified indemnification process and bases the reimbursement on standardized values (e.g., price assumptions), which do not fully reflect the actual loss values of the individual.

We illustrate an insurance contract with loss-dependent background risk with the following additional example: Consider the case of hospital cash benefit insurance which helps to pay out-of-pocket expenses during a hospital stay. Denote by N the random number of days spent at a hospital and by d the (unbiased) anticipated out-of-pocket expenses for each day at a hospital. At contract inception, the decision-maker contractually agrees on a daily benefit paid for each day spent at a hospital, which does not condition on actual expenses, denoted by b . Writing $b = \alpha d$, we can interpret the daily cash benefit as a coinsurance on anticipated daily costs. Thus, in our modeling setup, the insurable loss is given by $X = Nd$. However, the actual out-of-pocket expenses are not known ex ante (e.g., as prices vary by hospital), such that the insured is exposed to price risk irrespective of the amount of daily cash benefit. The total random cost is given by $X + \eta_X = N(d + \epsilon)$ for some random variable ϵ which denotes the random price factor. As anticipated costs are unbiased, the price risk has a zero mean and increases in the loss size: η_X as above thus models the gap between anticipated expenses and actual expenses and cannot be treated by this type of insurance. The decision-maker is fully exposed to this price risk, irrespective of the amount of coverage purchased. Another application of our modeled risk is technology risk as modeled in Li and Peter (2021) where individuals face the risk of technological uncertainty in the loss case.

2.1 | Optimal coinsurance demand with loss-dependent background risk

In this section, we provide an analysis of the optimal coinsurance demand if a loss-dependent background risk is present. We benchmark this optimal coverage level to the case without loss-dependent background risk. This benchmark case arises when the uninsurable component is either not present or not salient to the individual when deciding on the optimal coverage level. Thus, comparing the two levels of insurance demand indicates whether a salient loss-dependent background risk changes the optimal insurance coverage decision.

With and without loss-dependent background risk being present, the insurable risk is the same in both situations. From a pure insurance perspective, both situations are identical. The only difference is the decision-maker's exposure to η_X , which is not part of the insurance policy. In what follows, we always refer to changes in demand due to the presence of η_X , relative to the benchmark case. We say that the optimal coinsurance demand increases if an individual optimally purchases a higher level of coinsurance in the presence of η_X compared with its absence.

Fei and Schlesinger (2008) consider loss-dependent background risk with binary losses. They show that a prudent decision-maker optimally purchases more insurance with loss-dependent background risk relative to the benchmark. When considering loss-dependent background risk with a general loss distribution X , prudence is not sufficient to sign the effect of loss-dependent background risk. We show this by constructing a counterexample in Appendix B. Hence, the result obtained by Fei and Schlesinger (2008) for binary loss distributions does not generalize to our setup with a general loss distribution and a general loss-dependent background risk. The intuition behind this result is that for a general loss distribution with corresponding loss-dependent background risk, the degree of prudence also matters. Intuitively, if a decision-maker is a lot more prudent for large values of wealth (i.e., insurable losses are low) compared with low values of wealth (insurable losses are high), it may be optimal to *decrease insurance demand* to increase wealth in case losses are low as a response to the introduction of loss-dependent background risk.

In the case of independent background risk, risk vulnerability implies that insurance demand is (weakly) increasing if background risk is present.¹⁰ Since the loss-dependent background risk in (2) is not independent of other sources of risk, it is not clear whether this result also extends to the case of loss-dependent background risk considered here. If we simply mirror the approach of defining the derived utility function, this new utility function $v_x(w) = \mathbb{E}[u(w + \eta_x)]$ is loss-dependent, where $X = x$ denotes the realized loss. Thus, the usual comparative statics approaches (simply substituting utility functions and comparing absolute risk aversion) do not apply. However, the following result shows that for a decision-maker with risk-vulnerable utility function, loss-dependent background risk increases demand for coinsurance relative to the benchmark.

Proposition 1. *Let $\lambda \geq 1$. If the utility function is risk vulnerable, then $\alpha^{**} > \alpha^*$.*

Our result adds an intuitive sufficient condition for insurance demand to be larger with loss-dependent background risk compared with the benchmark, namely risk vulnerability. Risk

¹⁰In the case of actuarially fair premiums, it is optimal to demand full coverage, irrespective of whether an independent background risk is present or not. In case of unfair premiums, the (strictly) risk-vulnerable decision-maker demands more coverage if an independent background risk is present.

vulnerability is itself implied by standardness Gollier and Pratt (1996). In Eeckhoudt and Kimball (1992), Proposition 1 is shown to hold under these two latter conditions. Since in general the degree of prudence at different wealth levels matters, decreasing absolute risk aversion and decreasing absolute prudence (standardness) are indeed sufficient: decreasing absolute prudence guarantees that the degree of prudence cannot be higher for low losses (high wealth) compared with high losses (low wealth). Nevertheless, risk vulnerability is a weaker concept than standardness. In fact, risk vulnerability is also implied by decreasing and convex absolute risk aversion (Gollier & Pratt, 1996). These two sufficient conditions are not equivalent (Gollier, 2001). The contribution of Proposition 1 is twofold: first and more generally, it shows that the concept of risk vulnerability can be a useful characterization of preferences even if background risk is not independent. Second, risk vulnerability as an intuitive concept can replace the more restrictive and less intuitive condition of standardness as shown in Eeckhoudt and Kimball (1992).

Proposition 1 relies on the assumption that the decision-maker is prudent, which is implied by risk vulnerability. In fact, in the special cases considered in this section (Propositions 2 and 3), imprudence results in *less* coinsurance demand rather than more. The assumption that η_x increases in risk is crucial in Proposition 1. Consider the following example: Let $\eta_x = X\eta$ for $0 \leq x \leq P$ for some independent zero-mean random variable η and $\eta_x \equiv 0$ for $x > P$. Loss-dependent background risk increases in risk up to the threshold P , and vanishes beyond this threshold. Thus, background risk is only present for states in which the net insurance payment (i.e., loss—premium) is nonpositive. In this case, it follows with similar reasoning as in Fei and Schlesinger (2008) that a risk-vulnerable (and hence, prudent) decision-maker optimally reduces coinsurance demand to better cope with the additional risk. The reduced coinsurance demand translates into a lower premium payment, thus increasing wealth in the states in which the background risk is present.¹¹

Proposition 1 shows that a loss-dependent background risk triggers a precautionary insurance motive if the individual is risk vulnerable. The optimal level of coinsurance α^{**} is not bounded by unity, such that demanding more than full insurance may be optimal.¹² This is different in the case of an independent background risk: the risk-vulnerable decision-maker's optimal coinsurance demand increases with the addition of background risk, but will never exceed full coverage. Proposition 1 shows that optimal insurance demand changes when taking loss-dependent background risk into account. Intuitively, a salient loss-dependent background risk triggers higher demand for insurance coverage for a risk-vulnerable decision-maker compared with a loss-dependent background risk which is not salient, or no background risk at all.

The special case of a fair premium illustrates the precautionary insurance motive nicely, which arises from the loss dependency of the background risk: without loss-dependent background risk, it is optimal to opt for full coverage and $\alpha^* = 1$. This result continues to hold for an independent background risk. The following proposition now shows that insurance demand is strictly larger with loss-dependent background risk if the decision-maker is prudent (and also that insurance demand is strictly smaller if the decision-maker is imprudent). Contrary to Proposition 1, the restriction on actuarially fair policies in Proposition 2 allows us to relax the assumptions imposed on risk preferences.

¹¹This can formally be seen by modifying the proof of Proposition 3.

¹²The term *full coverage* always refers to full coverage of the insurable loss X .

Proposition 2. *Let $\lambda = 1$. If the decision-maker is prudent (imprudent), more than (less than) full coverage is optimal, that is, $\alpha^{**} > 1$ ($\alpha^{**} < 1$).*

Thus, for an actuarially fair premium, we do not need the more restrictive assumption of risk vulnerability: prudence, independent of the actual degree of prudence at different wealth levels, is enough for coinsurance demand to be unambiguously larger. The reason is that, starting from an optimal full insurance contract with an actuarially fair premium and no background risk, wealth is constant irrespective of the actual loss suffered. If now the loss-dependent background risk is taken into account, the precautionary motive due to prudence increases insurance demand and the actual degree of prudence at different wealth levels is irrelevant. For binary loss distributions, the previous result has already been obtained by Lee (2012) and Fei and Schlesinger (2008).¹³

In general, however, the assumption of prudence is not strong enough for a loss-dependent background risk to increase insurance demand for arbitrary loss distributions when $\lambda \neq 1$. We now consider a class of distributions, for which prudence is sufficient. To this end, we consider a restriction on the distribution of the loss random variable X together with a restriction on the insurance policy. More specifically, we consider the case of insurance against *large* risk in the sense that any nonzero realization of the loss random variable X is larger than the premium P , that is, $\mathbb{P}(0 < X < P) = 0$. If a loss occurs, the net payout of insurance (loss—premium) is positive.

Proposition 3. *Let $\lambda \geq 1$ and $\mathbb{P}(0 < X < P) = 0$. If the decision-maker is prudent (imprudent), more (less) coverage is optimal, that is, $\alpha^{**} > \alpha^*$ ($\alpha^{**} < \alpha^*$).*

The result makes use of the fact that the loss-dependent background risk only realizes in cases in which the decision-maker suffers a strictly positive loss and the restriction to large losses guarantees that the loss-dependent background risk is only relevant for cases in which the net insurance payout (loss—premium) is nonnegative. This means that the decision-maker only has to worry about the uninsurable background risk if the net insurance payout is nonnegative: since loss-dependent background risk is not present for cases with small or no losses, the decision-maker does not have any motive to decrease insurance demand (and thus, increase wealth in these states) to better cope with the background risk. Due to the special structure of the loss distribution, this result is similar to the comparative statics results of Eeckhoudt et al. (1991) who consider local changes in insurable risk.¹⁴

Similarly, if the loss-dependent background risk is such that it is present only in states for which the net payout from insurance is positive, we also get that prudence alone is sufficient to guarantee that demand increases. This is summarized in the following corollary:

Corollary 1. *Let $\lambda \geq 1$. Suppose that $\eta_X = 0$ almost surely if $X < P$. If the decision-maker is prudent (imprudent), more (less) coverage is optimal, that is, $\alpha^{**} > \alpha^*$ ($\alpha^{**} < \alpha^*$).*

All results in this section address the optimal level of coinsurance. These results also imply that if the assumptions of the results hold, the threshold loading above which no insurance is in

¹³A related result is obtained in Adam-Müller and Nolte (2011) in the context of demand for hedges.

¹⁴Proposition 3 also includes the result obtained by Fei and Schlesinger (2008) as a special case: for binary loss distributions, the only possible loss has to be larger than the premium for the decision-maker to buy any coverage at all.

demand is higher if loss-dependent background risk is taken into account. Thus, if λ^* denotes the loading factor such that $\alpha^* = 0$, and λ^{**} the respective factor such that $\alpha^{**} = 0$, then $\lambda^{**} > \lambda^*$.

3 | COMPARATIVE STATICS

In this section, we analyze the effect on optimal coinsurance demand if initial wealth, the insurable risk, or the loss-dependent background risk changes. Throughout this chapter, we assume that the decision-maker is exposed to some loss-dependent background risk that is taken into account when deciding on the optimal level of coverage. We require assumptions on the decision-maker's preferences and on the loss distribution to derive unambiguous results.

3.1 | Changes in initial wealth

One major result in insurance economics is the fact that the slope of absolute risk aversion determines the wealth effect on optimal insurance demand. Without background risk, an increase in initial wealth does not affect demand for fair insurance. For actuarially unfair premiums, insurance is an inferior good under decreasing absolute risk aversion. This is also the case for an independent background risk y , as the derived utility function $v(w) = \mathbb{E}[u(w + y)]$ inherits the property of decreasing absolute risk aversion from u (see, e.g., Kihlstrom et al., 1981; Nachman, 1982; Pratt, 1988). In this section, we look at preferences such that the intuitive inverse relation between wealth and insurance demand continues to hold if the decision-maker is exposed to loss-dependent background risk. Our first result highlights that results in the presence of independent background risk do not necessarily hold with loss-dependent background risk. Proposition 4 shows that we can find conditions such that the decision-maker increases coinsurance demand as a reaction to a wealth increase. In case of independent background risk, no risk-averse decision-maker would change insurance demand as a reaction to increased wealth.¹⁵

Proposition 4. *Let $\lambda = 1$. If the decision-maker is prudent ($u''' > 0$) and intemperate ($u^{(4)} > 0$), optimal coinsurance demand α^{**} increases in initial wealth.*

If initial wealth is increased, two effects determine the decision-maker's reaction. Without loss-dependent background risk, a change in wealth potentially changes the demand for insurance. The direction of this effect is governed by the sign of the coefficient of absolute risk aversion. This effect is also present with loss-dependent background risk. Moreover, the decision-maker's precautionary motive due to the presence of loss-dependent background risk may also change with wealth. Both these effects separately can result in more or less insurance demand as a reaction to an increase in wealth. Thus, if we control for the direction of these two effects, we expect unambiguous results. Unlike in the case of no or loss-independent

¹⁵In a model with a binary loss distribution, it can be shown that decreasing absolute prudence is necessary and sufficient for an increase in wealth to result in a lower coinsurance rate in case of fair premiums, see Eeckhoudt et al. (2003). Decreasing absolute prudence requires temperance, while intemperance implies increasing absolute prudence.

background risk, $\alpha^{**} > 1$ is generally possible, as we have seen in Section 2. As in this case decreasing absolute risk aversion potentially even increases insurance demand, we restrict the following analysis to the case $\alpha^{**} \leq 1$ to derive unambiguous results.

Let us now consider conditions under which insurance demand decreases (does not increase) in initial wealth. The following proposition relies on a result that can be found in a proof in Eeckhoudt et al. (1996) and is stated explicitly in Wang and Li (2014). We denote by $r_A(w) = -\frac{u''(w)}{u'(w)}$ the coefficient of absolute risk aversion and by $t_A(w) = -\frac{u^{(4)}(w)}{u'''(w)}$ the coefficient of absolute temperance. In the following proposition, we refer to decreasing absolute risk aversion in the Arrow–Pratt sense. Let \mathcal{W} denote the set of attainable wealth levels, that is, the set of wealth outcomes the decision-maker may end up in, excluding the background risk.

Proposition 5. *Assume u is nonincreasingly absolute risk-averse and $\alpha^{**} \leq 1$. Suppose there exists a scalar β such that*

$$t_A(w + y) \geq \beta \geq r_A(w + y') \quad \text{for all } w \in \mathcal{W} \text{ and } y, y' \in [a, b]. \tag{5}$$

*Then the optimal coinsurance demand α^{**} is nonincreasing in initial wealth.*

Condition (5) is a global version of one of the local risk vulnerability conditions given in Gollier and Pratt (1996). It states that the coefficient of absolute temperance is bounded away from the coefficient of absolute risk aversion for every possible outcome. This condition is equivalent to *decreasing third-degree Ross risk aversion* and as such a sufficient condition for any increase in independent background risk to raise risk aversion (Wang & Li, 2014). The similar result given in Eeckhoudt et al. (1996) additionally requires decreasing (second-degree) Ross risk aversion as the authors consider second-order stochastic dominance deteriorations rather than mean-preserving spreads. Similar to condition (5), decreasing (second-degree) Ross risk aversion holds if there exists a scalar $\hat{\beta}$

$$p_A(w + y) \geq \hat{\beta} \geq r_A(w + y') \quad \text{for all } w \in \mathcal{W} \text{ and } y, y' \in [a, b], \tag{6}$$

where $p_A = -\frac{u'''(w)}{u''(w)}$ denotes the coefficient of absolute prudence. As pointed out in Eeckhoudt et al. (1996), under decreasing (Arrow–Pratt) absolute prudence, (5) follows from (6), as decreasing absolute prudence implies $t_A(x) > p_A(x)$ for all x .

It is not obvious whether (5) can be satisfied at all. As an illustrative example, consider the case of an isoelastic utility function $u(x) = \frac{x^{1-\gamma}-1}{1-\gamma}$ for some $0 < \gamma \neq 1$. Then (5) holds if $\frac{\gamma+2}{\gamma}(w + a) \geq (w + b)$ for all w (Eeckhoudt et al., 1996). This condition is more likely to be satisfied the larger all values $w \in \mathcal{W}$ are relative to the size of $[a, b]$. Thus, if loss-dependent background risk is small relative to wealth outcomes, (5) holds for isoelastic utility and an increase in wealth weakly reduces optimal demand for coinsurance.

We now consider the case of insurance against large losses. Similar to the case of Proposition 1, it is sufficient to consider that the decision-maker exhibits risk vulnerability rather than the more restrictive assumption imposed in Proposition 5. In particular, this result highlights again that risk vulnerability plays a crucial role in determining comparative behavior in the presence of loss-dependent background risk.

Lemma 1. *Assume that the decision-maker is risk vulnerable and $\alpha^{**} \leq 1$. Then the optimal coinsurance demand α^{**} is nonincreasing in initial wealth W_0 if $\mathbb{P}(0 < X < P) = 0$.*

Lastly, we obtain a similar result if we restrict the background risk to only change once from zero to positive.

Lemma 2. *Assume that the decision-maker is risk vulnerable and $\alpha^{**} \leq 1$. Let $\eta_x \equiv 0$ for $x < (\leq)x_0$ and $\eta_x \equiv \eta$ for $x \geq (>)x_0$, for some $x_0 \in (0, L)$ and a zero-mean risk η . Then the optimal coinsurance demand α^{**} is nonincreasing in initial wealth.*

Lemma 2 again provides a simple sufficient condition for an unambiguous effect of a change of wealth on optimal insurance demand. The result makes specific use of the fact that there is only one change in background risk: it only changes from being nonexistent to being existent for increasing losses.

3.2 | Changes in risks

In this section, we provide comparative statics results for changes in the insurable risk as well as the background risk. To derive unambiguous results, in this section we consider a parametric relationship between the insurable loss X and the loss-dependent background risk η_x . Specifically, we assume that there exists a function $g: \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$ mapping from the nonnegative real numbers onto the same set, such that $g(0) = 0$ and g is nondecreasing. Loss-dependent background risk is now given by $\eta_x = g(X)\eta$, for some zero-mean risk η such that $g(x)\eta \in [a, b]$ for all $x \in [0, L]$. $g(x)\eta$ increases in risk as x increases. Thus, this parametric characterization is a special case of the loss-dependent background risk considered previously. Moreover, we also assume there exists a $c \in [0, g(L)]$ such that $\mathbb{P}(g(X) \leq c) > 0$ and $\mathbb{P}(g(X) > c) > 0$, such that the loss-dependent background risk increases in risk with positive probability. This model approach includes $\eta_x = X\eta$ as a special case.

First, we consider a change in the insurable risk, which also, depending on g , potentially induces a change in the loss-dependent background risk. Even without background risk, a mean-preserving increase in insurable risk could potentially either increase or decrease optimal demand for insurance.¹⁶ If we restrict the analysis to local changes in risk, however, we can obtain unambiguous results. For instance, in the case of a simple coinsurance contract without any background risk and an unfair premium, a prudent decision-maker optimally increases the coinsurance rate if the loss distribution X undergoes a mean-preserving increase in risk only for losses larger than the premium P . This stems from the increase in risk as well as from a precautionary insurance motive: by increasing insurance demand, the prudent decision-maker shifts wealth to the riskier states to better cope with the additional risk.

The next result addresses the question of whether these relationships are still valid if a loss-dependent background risk is present. In general, a change in insurable risk could also affect loss-dependent background risk in a variety of different ways. Focusing on the parametric relationship allows us to derive definitive predictions.

¹⁶While insurance demand increases if relative prudence is bounded by 2 (Menegatti & Peter, 2022), it is unclear how demand reacts if relative prudence is above this threshold.

To obtain clear predictions, we consider changes in the right tail of the loss distribution. For this, we analyze the effect on optimal demand if the distribution of X undergoes a mean-preserving increase in risk in the part of the distribution in which the insurable loss is larger than the premium. As a change in X not only affects the insurable risk but also the uninsurable loss-dependent background risk $g(X)\eta$, we need to restrict the curvature of g to unambiguously determine whether demand for insurance is increasing. If g is concave, an increase in risk X could potentially reduce the demand for insurance. This is because of the decrease in (average) background risk. To illustrate this point, we consider the following example: suppose $g(x) = 0$ for $x < c$ and $g(x) = 1$ for $x \geq c$, for some $c \in \mathbb{R}$. Now assume that X_2 is a mean-preserving spread of X such that $X_2 \stackrel{d}{=} X + \tilde{\epsilon}$, where $\tilde{\epsilon}$ is either $+\epsilon$ or $-\epsilon$ with equal probability if $X = c$, and zero in all other states of X .¹⁷ Hence, the only difference between X and X_2 is that the probability mass in c is shifted to $c + \epsilon$ and $c - \epsilon$. If ϵ is sufficiently small, the effect of substituting X with X_2 in terms of insurance demand against the insurable risk X_2 will be negligible. On the other hand, substituting X with X_2 has a high impact on $g(X)$, as $g(c) = 1$ and $g(c + \epsilon) = 1$ while $g(c - \epsilon) = 0$. Thus, on average, the loss-dependent background risk will be less severe, potentially driving insurance demand down. Hence, to rule out such an effect, we require (weak) convexity of g in the following result (and we also explicitly assume g is at least twice differentiable to facilitate the analysis):

Proposition 6. *Assume $u''' > 0$, $\alpha^{**} \leq 1$ and let $g'' \geq 0$. If the insurable risk becomes riskier with the stochastic change being concentrated on losses larger than the premium, optimal insurance demand increases in the weak sense.*

Coinsurance demand is increasing if $\alpha^{**} < 1$ or $g'(x) > 0$ for some $x > P$.¹⁸ Thus, the qualitative result for the case of coinsurance without background risk still holds if loss-dependent background risk is present. Even without background risk, the mean-preserving increase in the upper tail of insurable risk induces the prudent decision-maker to increase coinsurance to better cope with the risk. This motive is enforced if the loss-dependent background risk becomes more severe as well, which in the previous case follows from the convexity of g .

Lastly, we address the question under which conditions an increase in the background risk η raises insurance demand. We consider that the background risk η undergoes a mean-preserving increase in risk, holding the relationship to the insurable loss, g , constant. Unlike in the binary loss model by Fei and Schlesinger (2008), prudence alone is not strong enough to yield a definitive result in our framework. This follows from Appendix B, as the introduction of loss-dependent background risk is a special case of an increase in risk.

The case of changes in independent background risk is addressed in Wang and Li (2014). We already utilized the condition they derive in our equation (5) for the result that addresses increases in wealth. Given that the loss-dependent background risk could potentially be constant for most loss sizes, a sufficient condition for increasing coinsurance demand after an increase in background risk needs to include the independent case. However, we also need to address the additional effect a change in the background risk η has on demand due to the loss

¹⁷By $\stackrel{d}{=}$ we denote that the random variables are equal in distribution.

¹⁸If both $\alpha^{**} = 1$ and $g'(x) \equiv 0$ for $x > P$, then a change in the loss-distribution does not affect the decision-maker, as the loss is fully insured and background risk does not change either.

size dependency. The following proposition provides sufficient conditions for an increase in η to result in nondecreasing coinsurance demand.

In what follows, we assume that η_2 is riskier than η_1 . We denote the optimal levels of coinsurance in the presence of either background risk by α_2^{**} and α_1^{**} .

Proposition 7. *Let η_2 be a mean-preserving increase in risk of η_1 and let $a, b \in \mathbb{R}$ such that $g(x)\eta_1, g(x)\eta_2 \in [a, b]$ for all $x \in [0, L]$ and assume $\alpha_1^{**} \leq 1$. Suppose that $u''' > 0$, g is differentiable and there exist scalars β_1 and β_2 such that*

$$t_A(w + y) \geq \beta_1 \geq r_A(w + y') \quad \text{for all } w \in \mathcal{W} \text{ and } y, y' \in [a, b], \quad (7)$$

$$yt_A(w + y) \leq \beta_2 \leq y'r_A(w + y') + 2 \quad \text{for all } w \in \mathcal{W} \text{ and } y, y' \in [a, b]. \quad (8)$$

*Then insurance demand is increasing, that is, $\alpha_2^{**} \geq \alpha_1^{**}$.*

Condition (8) can be interpreted as a global version of the precautionary motive condition we implicitly required in Proposition 1 in the form of prudence. In (8), we utilize the coefficient of *partial* relative risk aversion, $xr_A(w + x)$, as well as the coefficient of *partial* relative temperance, $xt_A(w + x)$. Thresholds on these coefficients occur in numerous decision problems, for instance, in optimal portfolio choice, see Chiu et al. (2012). It is not immediately obvious that (7) and (8) can be satisfied at the same time. Consider again the example of an isoelastic utility function. We already know that (7) holds if $\frac{\gamma+2}{\gamma}(w + a) \geq (w + b)$ for all w . Equation (8) becomes $y(\gamma + 2)(w + y)^{-1} \leq y'\gamma(w + y')^{-1} + 2$ for all $y, y' \in [a, b]$. Thus, if $w \in \mathcal{W}$ is large relative to values on the interval $[a, b]$, then both (7) and (8) may hold. Consequently, for an isoelastic utility function, the assumptions of Proposition 7 hold if loss-dependent background risk is small relative to wealth outcomes.

In case of a large loss distribution, we can relax the assumptions imposed on the decision-maker's preferences to derive a sufficient condition: prudence is sufficient such that insurance demand is higher in the presence of η_2 compared with η_1 , as shown in the following result.

Proposition 8. *Assume $\mathbb{P}(0 < X < P) = 0$. For a prudent (imprudent) decision-maker, a mean-preserving increase in risk of η_1 to η_2 increases (decreases) the optimal coinsurance demand, that is, $\alpha_2^{**} > \alpha_1^{**}$.*

Proposition 8 also holds for a general loss distribution X if background risk only is present for sufficiently high losses, that is, if $g(x) = 0$ for $x < P$.

4 | RELATED LITERATURE

Insurance demand with loss-dependent background risk so far has sparked only some interest in the literature. Fei and Schlesinger (2008) consider the case of a loss-dependent background risk for binary loss distributions. As such, their background risk can be thought of as a loss-state-dependent background risk. In this light, our model is a generalization of this loss-state-dependent background risk along two interrelated lines: first, we consider a more general distribution of the insurable loss which potentially captures more than one loss and one no-loss state. Second, given the more general loss distribution, we can also consider more general

dependencies between the (loss-dependent) background risk and the loss and add several additional comparative statics results. Eeckhoudt et al. (2003) introduce a model with fixed reimbursement which is technically equivalent to the loss-state-dependent background risk considered in Fei and Schlesinger (2008). Thus, our study can also be thought of as a generalization of Eeckhoudt et al. (2003), where the fixed reimbursement is based on a fraction of expected losses in a given state of the world.

Our study is most closely related to Eeckhoudt and Kimball (1992) who study insurance demand with a loss-dependent background risk that deteriorates in a third-order stochastic dominance sense as the insurable loss increases. The loss-dependent background risk considered here is a special case of the one studied by Eeckhoudt and Kimball (1992). They derive conditions on the utility function such that insurance demand increases in the presence of a loss-dependent background risk. Their conditions involve the coefficients of absolute risk aversion and absolute prudence, in particular the slope of these coefficients as functions of wealth. In this paper, we relax the conditions that need to be imposed to obtain this result. Moreover, we add new comparative statics results that have not yet been derived in the literature.

In a setup related to ours, Gollier (1996) derives the optimal insurance contract design in the presence of loss-dependent background risk (which he considers as insurance of *approximate losses*). He finds that the optimal indemnity schedule is of the form of a disappearing deductible, that is, above the deductible, the indemnity increases by a factor larger than one in the loss size such that the deductible vanishes as the loss increases. Vanishing deductibles are not common in practice, and coinsurance and deductible policies are still the most frequent types of insurance. This may be due to the fact that if the decision-maker has some influence on the loss size, a disappearing deductible results in a motive to increase losses. Such a policy is therefore prone to moral hazard (Gollier, 1996). For this reason, and in the spirit of related studies like Eeckhoudt and Kimball (1992), we focus on studying coinsurance and deductible policies. In addition, a framework with coinsurance or deductibles allows concise conclusion about the quantity of insurance purchased.

In this paper, we address the question of whether (and under which conditions) taking loss-dependent background risk into account changes the optimal insurance decision. As an indication of the impact of loss-dependent background risk on insurance demand, considering the case of independent background risk seems to be promising. In our notation, this case translates to $\eta_x = \eta$ for all x for some zero-mean risk η . The case of independent background risk is considered in Eeckhoudt and Kimball (1992) and Gollier and Pratt (1996). They show that insurance demand is at least as high in the presence of the independent background risk compared with the absence of the additional risk if the decision-maker is decreasingly absolute risk-averse and decreasingly absolute prudent (Eeckhoudt & Kimball, 1992) or risk vulnerable (Gollier & Pratt, 1996). The latter is a weaker condition than decreasing absolute risk aversion together with decreasing absolute prudence (standardness).

However, the decision about the optimal level of insurance coverage with loss-dependent background risk is different from the situation with independent background risk. For instance, if the decision-maker is risk vulnerable, the presence of independent background risk cannot result in more than full coverage being optimal. As we have seen, more than full coverage can be optimal if the risk-vulnerable decision-maker is exposed to loss-dependent background risk. Thus, loss-dependent background risk may trigger a qualitatively different reaction than independent background risk if the decision-maker is risk vulnerable.

Lastly, we can also interpret our setup as a model addressing insurance demand with random wealth, where the distributions of wealth and the loss are stochastically dependent:

$W_0 + \eta_X$ can be thought of as random initial wealth, clearly being stochastically dependent to X . Insurance demand with random dependent wealth is, for instance, considered in Hong et al. (2011). They model the dependence by either positive or negative expectation dependence.¹⁹ Accordingly, our findings also provide new insights into insurance demand in a random wealth environment, in which the distribution of random wealth depends on the loss outcome.

5 | CONCLUSION

There are many instances in which insurance contracts do not perfectly indemnify the actual loss in each state of the world. Numerous reasons render insurance contracts only imperfect risk management tools. In this study, we analyze the case that insurable losses come with an additional uninsurable background risk that increases in loss size. We coin this risk loss-dependent background risk. Examples for such an additional risk arise, for instance, if the insurance company uses a simplified indemnification process that is based on standardized assumptions of the loss size. As a consequence, insurance indemnities are not based on the actual loss, but only on some estimate of the true loss size.

This study addresses the implications such a loss-dependent background risk has on insurance demand. To this end, we revisit the problem of the impact of loss-dependent background risk on insurance demand originally studied by Eeckhoudt and Kimball (1992) and tie together different studies and concepts addressing background risk: one significant contribution of this paper is that we show that the concept of risk vulnerability is also relevant in risky decisions where the background risk is not independent of the insurable risk. In our insurance application, risk vulnerability guarantees that the addition of a loss-dependent background risk increases insurance demand. However, we observe qualitatively different results: the loss-dependent background risk implies a precautionary insurance motive for a risk-vulnerable decision-maker. This is not necessarily the case for an independent background risk. An interesting direction for further research is to investigate the role of risk vulnerability more generally in the presence of a dependent background risk.

Our results show that a salient loss-dependent background risk increases insurance demand if the individual is risk vulnerable. For binary loss distributions, prudence is shown to be a sufficient assumption on preferences for unambiguous results (Fei & Schlesinger, 2008). While we show that this result does not extend to general loss distributions, we show that it extends to a large class of more general loss distribution-policy combinations: the case of large loss insurance. The only restriction we impose such that prudence is still sufficient is that if a loss occurs, it is larger than the premium. Compared with the case of a general loss distribution, the large loss case requires weaker assumptions on the decision-maker's preferences, as we have shown throughout this study. Many risks faced by individuals are indeed large loss risks, such that the restriction to large loss insurance includes many examples of insurable losses.

¹⁹Following the definition given in Hong et al. (2011), a random variable A is negatively (positively) expectation dependent with respect to another random variable B if $\mathbb{E}[A|B \leq b] \geq (\leq) \mathbb{E}[A]$ for all b . However, their results do not apply to our setup as random wealth $W_0 + \eta_X$ in our model is neither negatively nor positively expectation dependent with the loss X . This can be illustrated by the following example: Suppose X takes on the values 0 with probability $\frac{2}{3}$ and 50 with probability $\frac{1}{3}$ and assume $\eta_X = X\eta$, where η is either 0.3 or -0.3 with the same probabilities. Then $\mathbb{E}[X] = 50/3$.

Since $\mathbb{E}[X|X\eta \leq -1] = 50$ because $X\eta \leq -1$ only if $X = 50$, we have $\mathbb{E}[X|X\eta \leq -1] > \mathbb{E}[X]$, which means that X is not positively expectation dependent with $W_0 + X\eta$. On the other hand, we also have $\mathbb{E}[X|X\eta \leq 0] = 10$ which follows from the fact that $X\eta \leq 0$ only excludes the case $X = 50, \eta = 0.3$, such that X is either 0 (with probability 0.8) or 50 (probability 0.2). Thus, $\mathbb{E}[X|X\eta \leq 0] < \mathbb{E}[X]$ and therefore, X is not negatively expectation dependent with $W_0 + X\eta$.

Moreover, we provide insights into how changes in initial wealth endowment as well as changes in the risk structure affect insurance demand. We derive conditions on the insurable risk as well as higher-order risk preferences, which lead to unambiguously higher or lower insurance demand if initial wealth, the insurable risk or the background risk changes and discuss differences to cases with independent or no background risk.

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APPENDIX A: THE DEDUCTIBLE INSURANCE CASE

In this section, we consider the case of deductible insurance. We show that the results we derived for the case of coinsurance continue to hold for deductible insurance policies. Throughout, we assume the same random loss framework as introduced in Section 2.

In the deductible case, the decision-maker chooses the deductible $D \leq L$ such that she bears all losses below the deductible, while in case of a loss above D , her exposure to losses is capped at D . In this case, the premium for such an insurance policy is assumed to be given by $\phi(D) = \lambda \int_D^L (x - D) dF(x)$ for a loading factor $\lambda \geq 1$. Thus, for a given level of the deductible D , random final wealth is given by

$$\hat{W} = W_0 - \min(X, D) - \phi(D).$$

Similar to the coinsurance case, it is optimal to opt for a zero deductible in case of actuarially fair prices (i.e., $\lambda = 1$), while for a proportional loading ($\lambda > 1$), a strictly positive deductible is optimal (see, e.g., Schlesinger, 2013, Chap. 7). The optimal deductible D^* solves the FOC which follows from the Leibniz integral rule

$$\begin{aligned} \frac{\partial \mathbb{E}[u(\hat{W}(X))]}{\partial D} \Big|_{D=D^*} &= \int_0^{D^*} -\phi'(D^*) u'(W_0 - X - \phi(D^*)) dF(x) \\ &+ \int_{D^*}^L -(\phi'(D^*) + 1) u'(W_0 - D^* - \phi(D^*)) dF(x) = 0. \end{aligned} \quad (\text{A1})$$

Similar to Eeckhoudt et al. (1991) and Doherty and Schlesinger (1983a), throughout this study, we assume the second-order conditions for the deductible cases to hold.²⁰

In the presence of loss-dependent background risk, random final wealth is given by

$$\widetilde{W}(X) = W_0 - \min(X, D) - \phi(D) + \eta_X.$$

Consequently, the optimal deductible D^{**} solves the following FOC

$$\begin{aligned} \left. \frac{\partial E[u(\widetilde{W}(X))]}{\partial D} \right|_{D=D^{**}} &= \int_0^{D^{**}} -\phi'(D^{**})E[u'(W_0 - X - \phi(D^{**}) + \eta_X)]dF(x) \\ &+ \int_{D^{**}}^L -(\phi'(D^{**}) + 1)E[u'(W_0 - D^{**} - \phi(D^{**}) + \eta_X)]dF(x) = 0. \end{aligned} \tag{A2}$$

Optimal choice of deductible

In this section, we show that the results we obtained for the case of a coinsurance policy also qualitatively hold for deductible insurance policies.

Again, we start with the most general case of an arbitrary loss distribution X and a general loss-dependent background risk η_X with the properties outlined in Section 2. For the case of optimal coinsurance demand, we have seen that risk vulnerability is sufficient to determine that demand is increasing if loss-dependent background risk is taken into account. The following result shows that a risk-vulnerable decision-maker decreases the optimal level of the deductible if a loss-dependent background risk is taken into account, at least as long as the deductible can be decreased any further. If the optimal deductible without any background risk is zero (which is the case for an actuarially fair premium), the optimal deductible is zero after the introduction of the loss-dependent background risk.

Proposition 9. *Let $\lambda \geq 1$. If the utility function is risk vulnerable, then $D^{**} \leq D^*$ (with $D^{**} < D^*$ if $D^* > 0$).*

In the case of optimal coinsurance demand, we have seen that for an actuarially fair premium, a prudent decision-maker has a precautionary motive which induces him to buy more than full coverage. Similarly, for the case of deductible insurance, we can show that the FOC with loss-dependent background risk at 0 is negative, that is, the decision-maker also has a precautionary motive and could improve expected utility by transferring wealth from the no-loss state to the loss states. If we restrict our analysis to potential deductibles $D \in [0, L]$, however, we have the following result:

Proposition 10. *Let $\lambda = 1$. $D^{**} = 0$ ($D^{**} > 0$) is optimal if the decision-maker is prudent (imprudent).*

If we restrict our analysis to loss distributions such that any potential positive loss is at least as large as the premium paid, Proposition 3 shows that prudence is sufficient for optimal

²⁰See Schlesinger (1981) for a discussion of the second-order condition for the deductible case.

coinsurance demand to be increasing when loss-dependent background risk is taken into account. In the case of deductible insurance, we can prove a similar result. Unlike the coinsurance case, in which the threshold for large loss insurance is exogenously given by the premium P , in the case of a deductible insurance policy, the effect of insurance on wealth in the loss states is positive once the loss reaches the deductible. Thus, as the optimal level of the deductible is endogenously chosen by the decision-maker, the threshold depends on the decision-maker's preferences. We restrict the next result to only actuarially unfair premiums as the case of actuarially fair premiums is already addressed in Proposition 10.

Proposition 11. *Let $\lambda > 1$ and $\mathbb{P}(0 < X < D^*) = 0$. $D^{**} < D^*$ ($D^{**} > D^*$) is optimal if the decision-maker is prudent (imprudent).*

Thus, similar to the coinsurance case, Proposition 11 also includes the result obtained by Fei and Schlesinger (2008) as a special case since for binary loss distributions, the only possible loss has to be larger than the deductible for the decision-maker to demand any insurance coverage at all.

Again, Proposition 11 makes use of the fact that the loss-dependent background risk only manifests in cases in which the decision-maker suffers a loss above the deductible. Similarly, if we restrict the loss-dependent background risk to be present only in states for which the loss is sufficiently large such that the net payout from insurance is positive, that is, the loss is above the deductible, prudence alone is sufficient to guarantee that the deductible decreases if loss-dependent background risk is taken into account. This finding is summarized in the following:

Corollary 2. *Let $\lambda > 1$. Suppose that $\eta_x = 0$ almost surely if $X \leq D^*$. $D^{**} < D^*$ ($D^{**} > D^*$) is optimal if the decision-maker is prudent (imprudent).*

Comparative statics

The comparative statics results that we derived for the coinsurance case also qualitatively hold for the deductible insurance case. We briefly state the results for the deductible case in the following without proof.²¹

Proposition 12. *Assume that the decision-maker is risk vulnerable and $D^{**} \geq 0$. Then the optimal level of deductible D^{**} is nondecreasing in initial wealth W_0 if $\mathbb{P}(0 < X < D^{**}) = 0$.*

Proposition 13. *Assume that the decision-maker is risk vulnerable and $D^{**} \geq 0$. Let $\eta_x \equiv 0$ for $x < (\leq) x_0$ and $\eta_x \equiv \eta$ for $x \geq (>) x_0$, for some $x_0 \in (0, L)$ and a zero-mean risk η . Then the optimal level of deductible D^{**} is nondecreasing in initial wealth.*

Proposition 14. *Assume u is nonincreasingly absolute risk-averse and $D^{**} \geq 0$. Suppose there exists a scalar β such that*

$$t_A(w + y) \geq \beta \geq r_A(w + y') \quad \text{for all } w \in \mathcal{W} \text{ and } y, y' \in [a, b]. \quad (\text{A3})$$

*Then the optimal level of deductible D^{**} is nondecreasing in initial wealth.*

²¹The proofs are very similar to the coinsurance case. Only slight modifications of the proofs of Section 3 yield similar FOCs, and the arguments established in the proofs of Section 3 carry over.

Lastly, we consider changes in the risks. For this, we again restrict the analysis to parametric relationships between the insurable loss X and the loss-dependent background risk: $\eta_X = g(X)\eta$ with g and η as in Section 3.2. In the case of a deductible insurance policy without any background risk, a mean-preserving increase in insurable risk above the optimal deductible does not change the optimal level of the deductible (Eeckhoudt et al., 1991). The rationale for this result is that the mean-preserving transformation does not change the premium, and the increase in risk does not affect the decision-maker, as it only occurs above the deductible. A loss-independent background risk y can be incorporated by simply replacing the utility function u with $v(w) = \mathbb{E}[u(w + y)]$. Thus, the results also extend to the case in which loss-independent background risk is present. This is not true any longer for a loss-dependent background risk.

Proposition 15. *Assume $u''' > 0$, $D^{**} > 0$ and let $g'' \geq 0$. If X undergoes a mean-preserving increase in risk to Y with cumulative distribution function G then the optimal level of deductible D^{**} is decreasing if $F(x) = G(x)$ for all $x \leq D^{**}$.*

Thus, for the deductible case, the convexity of g even creates the motive to decrease the deductible, since, as pointed out in Eeckhoudt et al. (1991), this motive is not present without background risk.

We also obtain the following result which is qualitatively identical to Proposition 7. We denote by D_1^{**} and D_2^{**} the optimal levels of deductible for background risks $g(x)\eta_1$ and $g(x)\eta_2$.

Proposition 16. *Let η_2 be a mean-preserving increase in the risk of η_1 and let $a, b \in \mathbb{R}$ such that $g(x)\eta_1, g(x)\eta_2 \in [a, b]$ for all $x \in [0, L]$ and assume $D_1^{**} > 0$. Suppose that $u''' > 0$, g is differentiable and there exist scalars β_1 and β_2 such that*

$$\begin{aligned} t_A(w + y) &\geq \beta_1 \geq r_A(w + y') \quad \text{for all } w \in \mathcal{W} \text{ and } y, y' \in [a, b], \\ y t_A(w + y) &\leq \beta_2 \leq y' r_A(w + y') + 2 \quad \text{for all } w \in \mathcal{W} \text{ and } y, y' \in [a, b]. \end{aligned}$$

*Then $D_2^{**} \leq D_1^{**}$.*

For the large loss case, we accordingly obtain the following result:

Proposition 17. *For a prudent (imprudent) decision-maker, a mean-preserving increase in risk for η decreases (increases) the optimal level of deductible D^{**} if $\mathbb{P}(0 < X < D^{**}) = 0$.*

APPENDIX B: COUNTEREXAMPLE

Fei and Schlesinger (2008) show that for binary loss distributions, a prudent decision-maker optimally demands more coinsurance coverage if a background risk is added to the loss state. This section aims to illustrate an example which shows that this result does not need to hold any longer if more than one loss state (and thus, more than one state with background risk) is possible.

Specifically, we look at an example in which a prudent decision-maker optimally buys less coinsurance if he faces an additional loss-dependent background risk compared with a situation without any background risk. As for fair insurance premiums, prudence alone is sufficient to guarantee that coinsurance demand increases if loss-dependent background risk is introduced, we need to consider the case of unfair insurance where the price of insurance involves a positive loading (see Proposition 2). Moreover, Proposition 1 shows that a risk-vulnerable decision-maker

optimally increases coinsurance demand if loss-dependent background risk is introduced. Thus, we need to consider a utility function that exhibits prudence but is not risk vulnerable.

To come up with a suitable utility function, we consider a solution to the following third-order differential equation: $u(x) - u'''(x) = 0$. Given that the differential equation specifies that the function itself equals its third derivative, the fourth derivative equals its first derivative, and so forth.²² The set of solutions for this differential equation has the following form

$$u(x) = c_1 \exp(x) + c_2 \exp\left(-\frac{x}{2}\right) \cos\left(\frac{\sqrt{3}}{2}x\right) + c_3 \exp\left(-\frac{x}{2}\right) \sin\left(\frac{\sqrt{3}}{2}x\right). \quad (\text{B1})$$

Using $c_1 = \frac{1}{2}$, $c_2 = 2$, and $c_3 = 40$ and rescaling x to $0.01x$, we get a utility function on $[0, 120]$ which satisfies all required properties. Figure B1 shows plots of the function $u(x)$ and its first three derivatives for the interval $[0, 120]$. The third derivative is qualitatively identical to the function $u(x)$, and the same also holds by the construction of u for the first and fourth derivatives, and so forth. We have $u(x) > 0$, $u'(x) > 0$, $u''(x) < 0$, $u'''(x) > 0$, and also $u^{(4)} > 0$ for $x \in [0, 120]$, respectively. Hence, a decision-maker with utility function $u(x)$ is risk-averse, prudent, and intemperate (at least on the interval $[0, 120]$). However, the decision-maker is not risk vulnerable, as this necessarily requires temperance (Gollier & Pratt, 1996).

Now consider a decision-maker with utility function $u(x)$ and initial wealth of $W_0 = 110$, who faces an insurable loss X such that $\mathbb{P}(X = 99) = 0.89$, $\mathbb{P}(X = 60) = 0.1$, and $\mathbb{P}(X = 0) = 0.01$. Thus, the expected loss is $\mathbb{E}[X] = 94.11$.

The decision-maker may buy coinsurance coverage at the (actuarially unfair) price of $P = 96.2$. Without background risk, the optimal level of coinsurance is given by $\alpha^* = 0.1571$. If we now include a loss-dependent background risk to the random loss such that $\eta_X = X\eta$ for an η that is either 0.1 or -0.1 with equal probability, the optimal coinsurance demand drops slightly to $\alpha^{**} = 0.1568$. Thus, although the decision-maker is prudent, the optimal coinsurance demand decreases if loss-dependent background risk is taken into account.

The reason for the decrease in optimal demand in this case is that the function $V(x) = \frac{\mathbb{E}[u'(W_0 - x - \alpha^*P + \alpha^*x + x\eta)]}{u'(W_0 - x - \alpha^*P + \alpha^*x)}$, which is derived in the proof of Proposition 1, does not increase monotonically in loss size x . Figure B2 plots the function $V(x)$ against losses for the same parameters as above. As can be seen from the figure, the function increases, then reaches a maximum, and decreases thereafter. Moreover, the figure also shows the values of $V(x)$ for the potential loss sizes, namely 60 and 99. Since $V(60)$ is larger than $V(99)$, more weight is put on the case where the net payment from (full) insurance is negative compared with the case where the net payment from (full) insurance is positive. The reason for the shape of the weight function is that there are two opposing effects: as losses increase, the loss-dependent background risk increases, which results in an increase of V . On the other hand, since the utility function is intemperate, and thus, increasingly absolute prudent, the effect of the background risk decreases in loss size (as wealth decreases), which results in a decrease of V . The overall effect is shown graphically in Figure B2. Consequently, we have an overall negative effect of loss-dependent background risk on optimal demand for coinsurance and therefore, the decision-maker optimally decreases the coinsurance rate.

²²For our numerical example, we rescale the function by the factor 0.01, such that the third derivative is actually not identical to the function, but is scaled downwards by the factor 10^{-6} . This results in a quantitatively different function, but qualitatively, the functions are identical. Moreover, the signs of the derivatives are not affected by rescaling, which means that our analyses are not affected either.

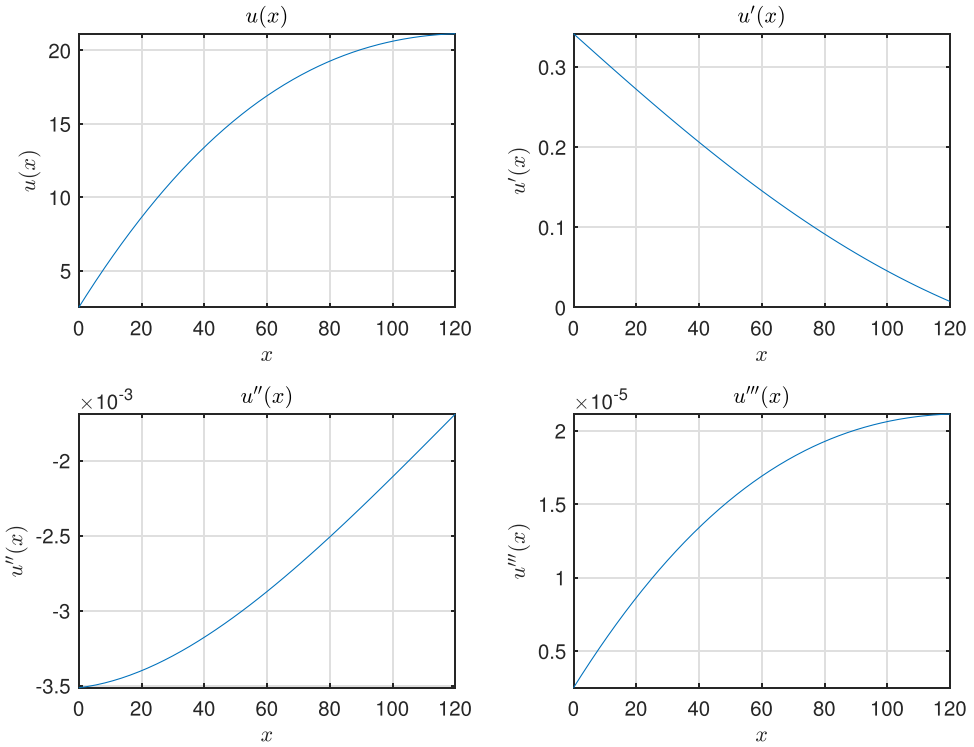


FIGURE B1 Utility function $u(x)$ and its first three derivatives. This figure shows the utility function u and the first three derivatives of u on the interval $[0, 120]$.

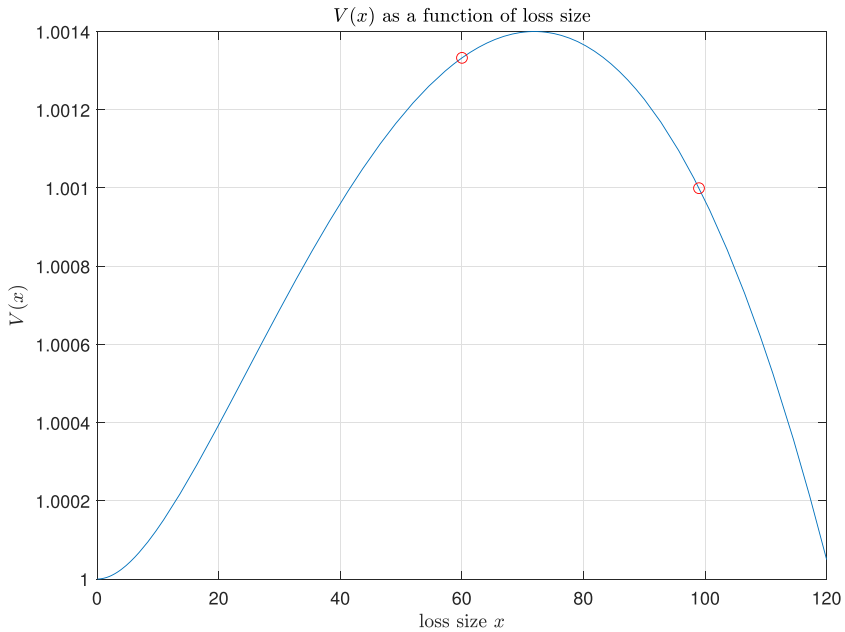


FIGURE B2 $V(x)$ as a function of loss size. This figure shows the function $V(x)$ as a function of the loss size $X = x$.

APPENDIX C: PROOFS

The following elementary lemma, which formalizes an approach frequently used in insurance economics (see, e.g., Schlesinger, 2013, Chap. 7) will prove useful in some results:

Lemma 3. *Let $h : \mathbb{R}^+ \rightarrow \mathbb{R}$ be a function such that $\int_0^L h(x)dF(x) = 0$ and $h(x) \leq 0$ for $x \leq x_0$ and $h(x) > 0$ for $x > x_0$. If $g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is nondecreasing (increasing), then*

$$\int_0^L h(x)g(x)dF(x) \geq (>)0.$$

A strict inequality holds if g is not constant on $[0, L]$.

Proof. Since g is nondecreasing (increasing), we have $g(x) \geq (>)g(x_0)$ for $x > x_0$ as well as $g(x_0) \geq (>)g(x)$ for $x_0 > x$. Thus, we have

$$\begin{aligned} \int_{x_0}^L h(x)g(x)dF(x) &\geq (>) \int_{x_0}^L h(x)g(x_0)dF(x) \\ &= g(x_0) \int_{x_0}^L h(x)dF(x) \\ &= -g(x_0) \int_0^{x_0} h(x)dF(x) \geq (>) - \int_0^{x_0} h(x)g(x)dF(x), \end{aligned}$$

that is, $\int_0^L h(x)g(x)dF(x) = \int_{x_0}^L h(x)g(x)dF(x) + \int_0^{x_0} h(x)g(x)dF(x) \geq (>)0$. □

Proofs of Section 2

Proof of Proposition 1. α^* solves the FOC without background risk (4). We now plug α^* into the FOC with loss-dependent background risk and check whether the sign of this FOC is positive in α^* . As expected utility in (2) is concave in α , a positive sign indicates the optimal insurance demand α^{**} to be higher than α^* .

Similar to (4), we have

$$\begin{aligned} \left. \frac{\partial \mathbb{E}[u(\widetilde{W}(X))]}{\partial \alpha} \right|_{\alpha=\alpha^*} &= \mathbb{E}[(X - P)u'(W_0 - X - \alpha^*P + \alpha^*X + \eta_X)] \\ &= \int_0^L [(x - P)u'(W_0 - x - \alpha^*P + \alpha^*x) \\ &\quad \times \frac{\mathbb{E}[u'(W_0 - x - \alpha^*P + \alpha^*x + \eta_x)]}{u'(W_0 - x - \alpha^*P + \alpha^*x)}] dF(x). \end{aligned} \tag{C1}$$

The first part of (C1) without the fraction is just the FOC for the nonbackground risk case (4), which equals zero at α^* . We now further investigate the fraction in (C1).

$$\begin{aligned} & \frac{\partial}{\partial x} \frac{\mathbb{E}[u'(W_0 - x - \alpha^*P + \alpha^*x + \eta_x)]}{u'(W_0 - x - \alpha^*P + \alpha^*x)} \\ &= \frac{\partial}{\partial x} \frac{\int_a^b u'(W_0 - x - \alpha^*P + \alpha^*x + \eta) dG(\eta, x)}{u'(W_0 - x - \alpha^*P + \alpha^*x)} = \frac{1}{(u'(W_0 - x - \alpha^*P + \alpha^*x))^2} \\ & \times \left[\left(\int_a^b (-1 + \alpha^*) u''(W_0 - x - \alpha^*P + \alpha^*x + \eta) dG(\eta, x) \right. \right. \\ & \left. \left. + \int_a^b u'(W_0 - x - \alpha^*P + \alpha^*x + \eta) dG_x(\eta, x) \right) \times u'(W_0 - x - \alpha^*P + \alpha^*x) \right. \\ & \left. - \int_a^b u'(W_0 - x - \alpha^*P + \alpha^*x + \eta) dG(\eta, x) (-1 + \alpha^*) u'' \right. \\ & \left. (W_0 - x - \alpha^*P + \alpha^*x) \right]. \end{aligned}$$

First, observe that $\int_a^b u'(W_0 - x - \alpha^*P + \alpha^*x + \eta_x) dG_x(\eta, x) \geq 0$ as $u''' > 0$ and η_x undergoes an increase in risk as x increases (with a strict inequality for strict changes). This can be seen by partially integrating the term twice. Thus, for the fraction to be nondecreasing in x , it is sufficient if

$$\begin{aligned} & \int_a^b (-1 + \alpha^*) u''(W_0 - x - \alpha^*P + \alpha^*x + \eta) dG(\eta, x) u'(W_0 - x - \alpha^*P + \alpha^*x) \\ & \geq \int_a^b u'(W_0 - x - \alpha^*P + \alpha^*x + \eta) dG(\eta, x) (-1 + \alpha^*) u'' \\ & (W_0 - x - \alpha^*P + \alpha^*x), \end{aligned}$$

which is equivalent to

$$\frac{(-1 + \alpha^*) \int_a^b u''(W_0 - x - \alpha^*P + \alpha^*x + \eta) dG(\eta, x)}{\int_a^b u'(W_0 - x - \alpha^*P + \alpha^*x + \eta) dG(\eta, x)} \geq \frac{(-1 + \alpha^*) u''(W_0 - x - \alpha^*P + \alpha^*x)}{u'(W_0 - x - \alpha^*P + \alpha^*x)},$$

which holds due to risk vulnerability as $\mathbb{E}[\eta_x] = 0$ for all x and $\alpha^* \leq 1$.

As a result, $\frac{\mathbb{E}[u'(W_0 - x - \alpha^*P + \alpha^*x + \eta_x)]}{u'(W_0 - x - \alpha^*P + \alpha^*x)}$ is nondecreasing in x and increasing at least for some x by assumption (as we assume an increase in risk as x increases at least at some levels of x). By Lemma 3, it follows that (C1) is positive and we have $\alpha^{**} > \alpha^*$. \square

Proof of Proposition 2. First consider $u'''(\cdot) > 0$. As before, we evaluate the FOC with loss-dependent background risk at $\alpha^* = 1$ and check the sign. Since we are now in the case of actuarially fair premiums, (C1) simplifies to

$$\left. \frac{\partial \mathbb{E}[u(\widetilde{W}(X))]}{\partial \alpha} \right|_{\alpha=1} = \int_0^L (x - P) u'(W_0 - P) \frac{\mathbb{E}[u'(W_0 - P + \eta_x)]}{u'(W_0 - P)} dF(x). \tag{C2}$$

We have $\frac{\mathbb{E}[u'(W_0 - P + \eta_x)]}{u'(W_0 - P)} \geq 1$ due to prudence and Jensen's inequality. Using the same arguments as in the proof of Proposition 1 and taking into account that $u'(W_0 - P)$ is constant in x , a sufficient condition for (C2) to be positive is that $\mathbb{E}[u'(W_0 - P + \eta_x)]$ is nondecreasing in x (with at least some x for which it is increasing). Taking the first derivative, we have

$$\begin{aligned} \frac{\partial}{\partial x} \mathbb{E}[u'(W_0 - P + \eta_x)] &= \frac{\partial}{\partial x} \int_a^b u'(W_0 - P + \eta) dG(\eta, x) \\ &= \int_a^b u'(W_0 - P + \eta) dG_x(\eta, x) \geq 0, \end{aligned}$$

which follows from $u''' > 0$, with a strict inequality at least for some x by assumption.

Similarly, we can show the result for $u''' < 0$. □

Proof of Proposition 3. Assume $u'''(\cdot) > 0$. As before, we consider the FOC in (C1). Since by assumption $(X - P) \leq 0$ implies $X = 0$ almost surely and thus also $\eta_x = 0$ almost surely, we have in (C1) that for all nonpositive values of $(x - P)$

$$\frac{\mathbb{E}[u'(W_0 - x - \alpha^*P + \alpha^*x + \eta_x)]}{u'(W_0 - x - \alpha^*P + \alpha^*x)} = \frac{u'(W_0 - \alpha^*P)}{u'(W_0 - \alpha^*P)} = 1,$$

while for $(x - P) > 0$, we have

$$\frac{\mathbb{E}[u'(W_0 - x - \alpha^*P + \alpha^*x + \eta_x)]}{u'(W_0 - x - \alpha^*P + \alpha^*x)} \geq 1,$$

due to $u'''(\cdot) > 0$ and Jensen's inequality, with a strict inequality at least for some sufficiently large x by assumption.

Thus, all nonpositive values in (C1) remain unchanged, while (some) positive values are multiplied with a factor strictly larger than one. As a result, given the FOC in (4), the positive part dominates, such that (C1) turns positive, which in turn yields $\alpha^{**} > \alpha^*$.

Similarly, for $u'''(\cdot) < 0$, the above reasoning also yields $\alpha^{**} < \alpha^*$. □

Proofs of Section 3

Proof of Proposition 4. We want to determine the sign of

$$\frac{\partial \alpha}{\partial W_0} = - \frac{\frac{\partial^2 \mathbb{E}[u(\widetilde{W}(X))]}{\partial \alpha \partial W_0}}{\frac{\partial^2 \mathbb{E}[u(\widetilde{W}(X))]}{\partial \alpha^2}}. \tag{C3}$$

Since $P = \mathbb{E}[X]$, we know that for the optimal coinsurance rate, we have $\alpha^{**} > 1$ if the individual is prudent. As we know that $\frac{\partial^2 \mathbb{E}[u(\widetilde{W}(X))]}{\partial \alpha^2} < 0$, we need to sign

$$\begin{aligned} \left. \frac{\partial^2 \mathbb{E}[u(\widetilde{W}(X))]}{\partial \alpha \partial W_0} \right|_{\alpha=\alpha^{**}} &= \mathbb{E}[(X - P)u''(W_0 - X - \alpha^{**}P + \alpha^{**}X + \eta_x)] \\ &= \int_0^L (x - P)\mathbb{E}[u''(W_0 - x - \alpha^{**}P + \alpha^{**}x + \eta_x)]dF(x). \end{aligned} \tag{C4}$$

We have $P = \mathbb{E}[X]$, that is, $\mathbb{E}[X - P] = 0$. By (a modified version of) Lemma 3, (C4) is positive if $\mathbb{E}[u''(W_0 - x - \alpha^{**}P + \alpha^{**}x + \eta_x)]$ is increasing in x . Taking the first derivative with respect to x , we obtain

$$\begin{aligned} &\frac{\partial}{\partial x}\mathbb{E}[u''(W_0 - x - \alpha^{**}P + \alpha^{**}x + \eta_x)] \\ &= \frac{\partial}{\partial x} \int_a^b u''(W_0 - x - \alpha^{**}P + \alpha^{**}x + \eta)dG(\eta, x) \\ &= (-1 + \alpha^{**}) \int_a^b u'''(W_0 - x - \alpha^{**}P + \alpha^{**}x + \eta)dG(\eta, x) \\ &\quad + \int_a^b u''(W_0 - x - \alpha^{**}P + \alpha^{**}x + \eta)dG_x(\eta, x) \geq 0, \end{aligned}$$

which follows from $\alpha^{**} > 1$, $u''' > 0$, and $u^{(4)} > 0$. By assumption, we have a strict inequality for at least some x , such that (C4) is positive. \square

Proof of Proposition 5. Similar to (C4), we consider

$$\begin{aligned} \left. \frac{\partial^2 \mathbb{E}[u(\widetilde{W}(X))]}{\partial \alpha \partial W_0} \right|_{\alpha=\alpha^{**}} &= \int_0^L (x - P)\mathbb{E}[u''(W_0 - x - \alpha^{**}P + \alpha^{**}x + \eta_x)]dF(x) \\ &= \int_0^L (x - P)\mathbb{E}[u'(W_0 - x - \alpha^{**}P + \alpha^{**}x + \eta_x)] \\ &\quad \times \frac{\mathbb{E}[u''(W_0 - x - \alpha^{**}P + \alpha^{**}x + \eta_x)]}{\mathbb{E}[u'(W_0 - x - \alpha^{**}P + \alpha^{**}x + \eta_x)]}dF(x). \end{aligned} \tag{C5}$$

We need (C5) to be nonpositive. For this it is sufficient if

$$\frac{\mathbb{E}[u''(W_0 - x - \alpha^{**}P + \alpha^{**}x + \eta_x)]}{\mathbb{E}[u'(W_0 - x - \alpha^{**}P + \alpha^{**}x + \eta_x)]} < 0 \tag{C6}$$

is nonincreasing in x .

Consider arbitrary $x_1, x_2 \in [0, L]$ such that $x_1 < x_2$. Then we have

$$\frac{\mathbb{E}[u''(W_0 - x_1 - \alpha^{**}P + \alpha^{**}x_1 + \eta_{x_1})]}{\mathbb{E}[u'(W_0 - x_1 - \alpha^{**}P + \alpha^{**}x_1 + \eta_{x_1})]} \geq \frac{\mathbb{E}[u''(W_0 - x_2 - \alpha^{**}P + \alpha^{**}x_2 + \eta_{x_1})]}{\mathbb{E}[u'(W_0 - x_2 - \alpha^{**}P + \alpha^{**}x_2 + \eta_{x_1})]},$$

because for a fixed η_{x_1} , $\mathbb{E}[u(w + \eta_{x_1})]$ is nonincreasingly absolute risk-averse as u is by assumption nonincreasingly absolute risk-averse (Pratt, 1988). We require $\alpha^{**} \leq 1$.

Now the change from η_{x_1} to η_{x_2} is a second-degree increase in risk. By assumption (5), we have that u exhibits decreasing third-degree Ross risk aversion (see, e.g., Proposition 2.5 in Wang & Li, 2014). Nonincreasing absolute risk aversion implies prudence, that is, $u''' > 0$, which means that u is strictly third-degree risk-averse as required in the proposition of Wang and Li (2014). Now it follows from Proposition 3.2 in Wang and Li (2014) that

$$\frac{\mathbb{E}[u''(W_0 - x_2 - \alpha^{**}P + \alpha^{**}x_2 + \eta_{x_1})]}{\mathbb{E}[u'(W_0 - x_2 - \alpha^{**}P + \alpha^{**}x_2 + \eta_{x_1})]} \geq \frac{\mathbb{E}[u''(W_0 - x_2 - \alpha^{**}P + \alpha^{**}x_2 + \eta_{x_2})]}{\mathbb{E}[u'(W_0 - x_2 - \alpha^{**}P + \alpha^{**}x_2 + \eta_{x_2})]}.$$

Thus, taken together, it follows that

$$\frac{\mathbb{E}[u''(W_0 - x_1 - \alpha^{**}P + \alpha^{**}x_1 + \eta_{x_1})]}{\mathbb{E}[u'(W_0 - x_1 - \alpha^{**}P + \alpha^{**}x_1 + \eta_{x_1})]} \geq \frac{\mathbb{E}[u''(W_0 - x_2 - \alpha^{**}P + \alpha^{**}x_2 + \eta_{x_2})]}{\mathbb{E}[u'(W_0 - x_2 - \alpha^{**}P + \alpha^{**}x_2 + \eta_{x_2})]},$$

and (C6) is nonincreasing in loss size x , which in turn means that (C5) is nonpositive. \square

Proof of Lemma 1. Again, we need (C5) to be nonpositive. Equation (C5) without the fraction is just the FOC and thus zero. By assumption, $(X - P) \leq 0$ implies that $X = 0$ almost surely and consequently $\eta_x = \eta_0 \equiv 0$. In this case, we have

$$\frac{\mathbb{E}[u''(W_0 - x - \alpha^{**}P + \alpha^{**}x + \eta_x)]}{\mathbb{E}[u'(W_0 - x - \alpha^{**}P + \alpha^{**}x + \eta_x)]} = \frac{u''(W_0 - \alpha^{**}P)}{u'(W_0 - \alpha^{**}P)}.$$

On the other hand, for $x > P$, we have

$$\frac{\mathbb{E}[u''(W_0 - x - \alpha^{**}P + \alpha^{**}x + \eta_x)]}{\mathbb{E}[u'(W_0 - x - \alpha^{**}P + \alpha^{**}x + \eta_x)]} \leq \frac{u''(W_0 - \alpha^{**}P)}{u'(W_0 - \alpha^{**}P)},$$

which follows from risk vulnerability for the (nonpositive-mean) risk $(-1 + \alpha^{**})x + \eta_x$. Thus, $\frac{\mathbb{E}[u''(W_0 - x - \alpha^{**}P + \alpha^{**}x + \eta_x)]}{\mathbb{E}[u'(W_0 - x - \alpha^{**}P + \alpha^{**}x + \eta_x)]}$ is more negative if $(x - P) > 0$ compared with when $(x - P) \leq 0$. Consequently, (C5) is nonpositive, that is, the optimal level of coinsurance is nonincreasing in W_0 , and we have a strict inequality if risk vulnerability is strict. \square

Proof of Lemma 2. Again, we need (C5) to be nonpositive. We apply Lemma 3 and need to show that the fraction in (C5) is nonincreasing in x . For $x \leq x_0$, we have $\eta_x \equiv 0$ and thus

$$\frac{\mathbb{E}[u''(W_0 - x - \alpha^{**}P + \alpha^{**}x + \eta_x)]}{\mathbb{E}[u'(W_0 - x - \alpha^{**}P + \alpha^{**}x + \eta_x)]} = \frac{u''(W_0 - x - \alpha^{**}P + \alpha^{**}x)}{u'(W_0 - x - \alpha^{**}P + \alpha^{**}x)}.$$

From nonincreasing absolute risk aversion (implied by risk vulnerability), it follows due to $\alpha^{**} \leq 1$ that for $x \leq x_0$

$$\frac{u''(W_0 - x - \alpha^{**}P + \alpha^{**}x)}{u'(W_0 - x - \alpha^{**}P + \alpha^{**}x)}$$

is nonincreasing in x .

In $x = x_0$, we have by risk vulnerability

$$\frac{u''(W_0 - x_0 - \alpha^{**}P + \alpha^{**}x_0)}{u''(W_0 - x_0 - \alpha^{**}P + \alpha^{**}x_0)} \geq \frac{\mathbb{E}[u''(W_0 - x_0 - \alpha^{**}P + \alpha^{**}x_0 + \eta)]}{\mathbb{E}[u'(W_0 - x_0 - \alpha^{**}P + \alpha^{**}x_0 + \eta)]}.$$

Moreover, as shown by Pratt (1988), the derived utility function $v(w) = \mathbb{E}[u(w + \eta)]$ is nonincreasingly absolute risk-averse as u has this property. Thus, we also have for $x > x_0$ that

$$\frac{\mathbb{E}[u''(W_0 - x - \alpha^{**}P + \alpha^{**}x + \eta)]}{\mathbb{E}[u'(W_0 - x - \alpha^{**}P + \alpha^{**}x + \eta)]}$$

is nonincreasing. Collecting all our arguments yields that for this specific choice of η_x

$$\frac{\mathbb{E}[u''(W_0 - x - \alpha^{**}P + \alpha^{**}x + \eta_x)]}{\mathbb{E}[u'(W_0 - x - \alpha^{**}P + \alpha^{**}x + \eta_x)]}$$

is indeed nonincreasing in x on $[0, L]$. Slight modifications of the proof also reveal that the result also holds for $\eta_x \equiv 0$ for $x < x_0$ and $\eta_x \equiv \eta$ for $x \geq x_0$. □

Proof of Proposition 6. Let Y with cumulative distribution function G be a mean-preserving increase in the risk of X with cumulative density function F . Then it follows from the FOC for the optimal coinsurance rate α^{**} for X that

$$\begin{aligned} & \int_0^L (x - P)\mathbb{E}[u'(W_0 - x - \alpha^{**}P - \alpha^{**}x + g(x)\eta)]dG(x) \\ &= \int_0^L (x - P)\mathbb{E}[u'(W_0 - x - \alpha^{**}P - \alpha^{**}x + g(x)\eta)]d[G(x) - F(x)]. \end{aligned} \tag{C7}$$

By assumption $F(x) = G(x)$ for $x \leq P$ and it is enough to consider the integral in (C7) only for the interval $(P, L]$. Equation (C7) is nonnegative if the integrated function $h(x)$ is convex, where

$$h(x) = (x - P)\mathbb{E}[u'(W_0 - x - \alpha^{**}P + \alpha^{**}x + g(x)\eta)].$$

Then it follows for the second derivative of h

$$\begin{aligned}
 h''(x) = & 2\mathbb{E}[(-1 + \alpha^{**} + g'(x)\eta)u''(W_0 - x - \alpha^{**}P + \alpha^{**}x + g(x)\eta)] \\
 & + (x - P)\mathbb{E}[g''(x)\eta u''(W_0 - x - \alpha^{**}P + \alpha^{**}x + g(x)\eta)] \\
 & + (-1 + \alpha^{**} + g'(x)\eta)^2 u'''(W_0 - x - \alpha^{**}P + \alpha^{**}x + g(x)\eta)].
 \end{aligned}
 \tag{C8}$$

Due to $u''' > 0$, we have

$$\mathbb{E}[(-1 + \alpha^{**} + g'(x)\eta)u''(W_0 - x - \alpha^{**}P + \alpha^{**}x + g(x)\eta)] \geq 0,$$

with a strict inequality if $g' > 0$ for some $x > P$, as well as (since $g'' \geq 0$)

$$\mathbb{E}[g''(x)\eta u''(W_0 - x - \alpha^{**}P + \alpha^{**}x + g(x)\eta)] \geq 0.$$

Thus, (C8) is nonnegative, that is, h is convex and thus (C7) is nonnegative. □

Proof of Proposition 7. Let H_1 and H_2 denote the cumulative distribution functions of η_1 and η_2 , respectively. For ease of exposition, we denote expectations taken with respect to either H_1 and H_2 by \mathbb{E}_{H_1} and \mathbb{E}_{H_2} and omit to distinguish between the random variables η_1 and η_2 in the argument of the expectation operators.

As α_1^{**} is optimal in the presence of η_1 , we have the following FOC:

$$\int_0^L (x - P)\mathbb{E}_{H_1}[u'(W_0 - x - \alpha_1^{**}P + \alpha_1^{**}x + g(x)\eta)]dF(x).$$

We now consider the FOC in the presence of η_2 evaluated at α_1^{**} :

$$\begin{aligned}
 & \int_0^L (x - P)\mathbb{E}_{H_2}\left[u'(W_0 - x - \alpha_1^{**}P + \alpha_1^{**}x + g(x)\eta)\right]dF(x) \\
 & = \int_0^L (x - P)\mathbb{E}_{H_1}\left[u'(W_0 - x - \alpha_1^{**}P + \alpha_1^{**}x + g(x)\eta)\right] \\
 & \quad \times \frac{\mathbb{E}_{H_2}\left[u'(W_0 - x - \alpha_1^{**}P + \alpha_1^{**}x + g(x)\eta)\right]}{\mathbb{E}_{H_1}\left[u'(W_0 - x - \alpha_1^{**}P + \alpha_1^{**}x + g(x)\eta)\right]}dF(x).
 \end{aligned}
 \tag{C9}$$

Equation (C9) is nonnegative (positive) if $\frac{\mathbb{E}_{H_2}[u'(W_0 - x - \alpha_1^{**}P + \alpha_1^{**}x + g(x)\eta)]}{\mathbb{E}_{H_1}[u'(W_0 - x - \alpha_1^{**}P + \alpha_1^{**}x + g(x)\eta)]}$ is nondecreasing (increasing) in x (Lemma 3). Taking the first derivative yields

$$\begin{aligned}
 & \frac{\partial}{\partial x} \frac{\mathbb{E}_{H_2}[u'(W_0 - x - \alpha_1^{**}P + \alpha_1^{**}x + g(x)\eta)]}{\mathbb{E}_{H_1}[u'(W_0 - x - \alpha_1^{**}P + \alpha_1^{**}x + g(x)\eta)]} \\
 & = \frac{1}{(\mathbb{E}_{H_1}[u'(W_0 - x - \alpha_1^{**}P + \alpha_1^{**}x + g(x)\eta)]^2} \\
 & \quad \times \left(\mathbb{E}_{H_2}[(-1 + \alpha_1^{**} + g'(x)\eta)u''(W_0 - x - \alpha_1^{**}P + \alpha_1^{**}x + g(x)\eta)] \right. \\
 & \quad \times \mathbb{E}_{H_1}[u'(W_0 - x - \alpha_1^{**}P + \alpha_1^{**}x + g(x)\eta)] \\
 & \quad - \mathbb{E}_{H_1}[(-1 + \alpha_1^{**} + g'(x)\eta)u''(W_0 - x - \alpha_1^{**}P + \alpha_1^{**}x + g(x)\eta)] \\
 & \quad \left. \times \mathbb{E}_{H_2}[u'(W_0 - x - \alpha_1^{**}P + \alpha_1^{**}x + g(x)\eta)] \right).
 \end{aligned}$$

Thus, we have

$$\begin{aligned}
 & \frac{\partial}{\partial x} \frac{\mathbb{E}_{H_2} \left[u' \left(W_0 - x - \alpha_1^{**} P + \alpha_1^{**} x + g(x) \eta \right) \right]}{\mathbb{E}_{H_1} \left[u' \left(W_0 - x - \alpha_1^{**} P + \alpha_1^{**} x + g(x) \eta \right) \right]} \geq 0 \\
 & \Leftrightarrow \frac{\mathbb{E}_{H_2} \left[\left(-1 + \alpha_1^{**} + g'(x) \eta \right) u'' \left(W_0 - x - \alpha_1^{**} P + \alpha_1^{**} x + g(x) \eta \right) \right]}{\mathbb{E}_{H_2} \left[u' \left(W_0 - x - \alpha_1^{**} P + \alpha_1^{**} x + g(x) \eta \right) \right]} \quad (C10) \\
 & \geq \frac{\mathbb{E}_{H_1} \left[\left(-1 + \alpha_1^{**} + g'(x) \eta \right) u'' \left(W_0 - x - \alpha_1^{**} P + \alpha_1^{**} x + g(x) \eta \right) \right]}{\mathbb{E}_{H_1} \left[u' \left(W_0 - x - \alpha_1^{**} P + \alpha_1^{**} x + g(x) \eta \right) \right]}
 \end{aligned}$$

which in turn is equivalent to

$$\begin{aligned}
 & \left(-1 + \alpha_1^{**} \right) \frac{\mathbb{E}_{H_2} \left[u'' \left(W_0 - x - \alpha_1^{**} P + \alpha_1^{**} x + g(x) \eta \right) \right]}{\mathbb{E}_{H_2} \left[u' \left(W_0 - x - \alpha_1^{**} P + \alpha_1^{**} x + g(x) \eta \right) \right]} \\
 & + g'(x) \frac{\mathbb{E}_{H_2} \left[\eta u'' \left(W_0 - x - \alpha_1^{**} P + \alpha_1^{**} x + g(x) \eta \right) \right]}{\mathbb{E}_{H_2} \left[u' \left(W_0 - x - \alpha_1^{**} P + \alpha_1^{**} x + g(x) \eta \right) \right]} \quad (C11) \\
 & \geq \left(-1 + \alpha_1^{**} \right) \frac{\mathbb{E}_{H_1} \left[u'' \left(W_0 - x - \alpha_1^{**} P + \alpha_1^{**} x + g(x) \eta \right) \right]}{\mathbb{E}_{H_1} \left[u' \left(W_0 - x - \alpha_1^{**} P + \alpha_1^{**} x + g(x) \eta \right) \right]} \\
 & + g'(x) \frac{\mathbb{E}_{H_1} \left[\eta u'' \left(W_0 - x - \alpha_1^{**} P + \alpha_1^{**} x + g(x) \eta \right) \right]}{\mathbb{E}_{H_1} \left[u' \left(W_0 - x - \alpha_1^{**} P + \alpha_1^{**} x + g(x) \eta \right) \right]}.
 \end{aligned}$$

As in the proof of Proposition 5, we get from Wang and Li (2014) due to prudence, $\alpha_1^{**} \leq 1$ and condition (7) that

$$\begin{aligned}
 & \left(-1 + \alpha_1^{**} \right) \frac{\mathbb{E}_{H_2} \left[u'' \left(W_0 - x - \alpha_1^{**} P + \alpha_1^{**} x + g(x) \eta \right) \right]}{\mathbb{E}_{H_2} \left[u' \left(W_0 - x - \alpha_1^{**} P + \alpha_1^{**} x + g(x) \eta \right) \right]} \\
 & \geq \left(-1 + \alpha_1^{**} \right) \frac{\mathbb{E}_{H_1} \left[u'' \left(W_0 - x - \alpha_1^{**} P + \alpha_1^{**} x + g(x) \eta \right) \right]}{\mathbb{E}_{H_1} \left[u' \left(W_0 - x - \alpha_1^{**} P + \alpha_1^{**} x + g(x) \eta \right) \right]}.
 \end{aligned}$$

If $g'(x) = 0$, then the second term on either side of (C11) vanishes and the above condition implies (C10). Thus, in the following we assume $g'(x) > 0$ as well as $g(x) > 0$. Using an approach similar to Eeckhoudt et al. (1996) and Wang and Li (2014), we now show that prudence and condition (8) yield

$$\frac{\mathbb{E}_{H_2} \left[\eta u'' \left(W_0 - x - \alpha_1^{**} P + \alpha_1^{**} x + g(x) \eta \right) \right]}{\mathbb{E}_{H_2} \left[u' \left(W_0 - x - \alpha_1^{**} P + \alpha_1^{**} x + g(x) \eta \right) \right]} \geq \frac{\mathbb{E}_{H_1} \left[\eta u'' \left(W_0 - x - \alpha_1^{**} P + \alpha_1^{**} x + g(x) \eta \right) \right]}{\mathbb{E}_{H_1} \left[u' \left(W_0 - x - \alpha_1^{**} P + \alpha_1^{**} x + g(x) \eta \right) \right]},$$

such that (C10) holds. For ease of exposition, let $w := W_0 - x - \alpha_1^{**}P + \alpha_1^{**}x$ and $c := g(x)$. Moreover, assume $\eta_1, \eta_2 \in [\hat{a}, \hat{b}]$. Then we above condition reads

$$\begin{aligned} \frac{\mathbb{E}_{H_2}[\eta u''(w + c\eta)]}{\mathbb{E}_{H_2}[u'(w + c\eta)]} &\geq \frac{\mathbb{E}_{H_1}[\eta u''(w + c\eta)]}{\mathbb{E}_{H_1}[u'(w + c\eta)]} \\ &\Leftrightarrow \int_{\hat{a}}^{\hat{b}} \eta u''(w + c\eta) dH_2(\eta) - \int_{\hat{a}}^{\hat{b}} \eta u''(w + c\eta) dH_1(\eta) \\ &\geq \frac{\mathbb{E}_{H_1}[\eta u''(w + c\eta)]}{\mathbb{E}_{H_1}[u'(w + c\eta)]} \left(\int_{\hat{a}}^{\hat{b}} u'(w + c\eta) dH_2(\eta) \right. \\ &\quad \left. - \int_{\hat{a}}^{\hat{b}} u'(w + c\eta) dH_1(\eta) \right). \end{aligned} \tag{C12}$$

Let $H_i^2(x) := \int_{\hat{a}}^x H_i(\eta) d\eta$. Performing integration by parts twice yields

$$\begin{aligned} &\int_{\hat{a}}^{\hat{b}} \eta u''(w + c\eta) dH_2(\eta) - \int_{\hat{a}}^{\hat{b}} \eta u''(w + c\eta) dH_1(\eta) \\ &= \int_{\hat{a}}^{\hat{b}} [2cu'''(w + c\eta) + c^2\eta u^{(4)}(w + c\eta)] [H_2^2(\eta) - H_1^2(\eta)] d\eta \end{aligned}$$

and

$$\begin{aligned} &\int_{\hat{a}}^{\hat{b}} u'(w + c\eta) dH_2(\eta) - \int_{\hat{a}}^{\hat{b}} u'(w + c\eta) dH_1(\eta) \\ &= \int_{\hat{a}}^{\hat{b}} c^2 u'''(w + c\eta) [H_2^2(\eta) - H_1^2(\eta)] d\eta > 0, \end{aligned}$$

where the inequality follows from prudence.

Thus, (C12) can equivalently be written as

$$\begin{aligned} &\frac{\int_{\hat{a}}^{\hat{b}} [2cu'''(w + c\eta) + c^2\eta u^{(4)}(w + c\eta)] [H_2^2(\eta) - H_1^2(\eta)] d\eta}{\int_{\hat{a}}^{\hat{b}} c^2 u'''(w + c\eta) [H_2^2(\eta) - H_1^2(\eta)] d\eta} \\ &\geq \frac{\mathbb{E}_{H_1}[\eta u''(w + c\eta)]}{\mathbb{E}_{H_1}[u'(w + c\eta)]} = \frac{\int_{\hat{a}}^{\hat{b}} \eta u''(w + c\eta) dH_1(\eta)}{\int_{\hat{a}}^{\hat{b}} u''(w + c\eta) dH_1(\eta)} \\ &\Leftrightarrow \int_{\hat{a}}^{\hat{b}} \frac{2cu'''(w + c\eta) + c^2\eta u^{(4)}(w + c\eta)}{c^2 u'''(w + c\eta)} \frac{c^2 u'''(w + c\eta) [H_2^2(\eta) - H_1^2(\eta)]}{\int_{\hat{a}}^{\hat{b}} c^2 u'''(w + c\eta) [H_2^2(\eta) - H_1^2(\eta)] d\eta} d\eta \\ &\geq \int_{\hat{a}}^{\hat{b}} \frac{\eta u''(w + c\eta)}{u'(w + c\eta)} \frac{u'(w + c\eta)}{\int_{\hat{a}}^{\hat{b}} u'(w + c\eta) dH_1(\eta)} dH_1(\eta). \end{aligned} \tag{C13}$$

Both $\frac{c^2 u'''(w + c\eta)[H_2^2(\eta) - H_1^2(\eta)]}{\int_a^b c^2 u'''(w + c\eta)[H_2^2(\eta) - H_1^2(\eta)] d\eta}$ and $\frac{u'(w + c\eta)}{\int_a^b u'(w + c\eta) dH_1(\eta)}$ are probability density functions.

Thus, the left-hand side and the right-hand side of (C13) can be interpreted as expected values with respect to the induced probability measures. Since an expected value is bounded from below by the infimum possible realization as well as from above by the supremum possible realization, it follows that a sufficient condition for (C13) is

$$\frac{2cu'''(w + cz) + c^2zu^{(4)}(w + cz)}{c^2u'''(w + cz)} \geq \frac{z'u''(w + cz')}{u'(w + cz')} \quad \text{for all } w \text{ and for all } z, z' \in [\hat{a}, \hat{b}].$$

Simple manipulations yield that this holds if there exists a scalar $\hat{\beta}_2$ such that

$$-cz \frac{u^{(4)}(w + cz)}{u'''(w + cz)} \leq \hat{\beta}_2 \leq -cz' \frac{u''(w + cz')}{u'(w + cz')} + 2 \quad \text{for all } w \text{ and for all } z, z' \in [\hat{a}, \hat{b}].$$

As $c = g(x)$ is not fix but changes in x , this relationship needs to hold for all possible values of $g(x)$. As by assumption $g(x)\eta_1, g(x)\eta_2 \in [a, b]$, it is sufficient if there exists a scalar β_2

$$-y \frac{u^{(4)}(w + y)}{u'''(w + y)} \leq \beta_2 \leq -y' \frac{u''(w + y')}{u'(w + y')} + 2 \quad \text{for all } w \text{ and for all } y, y' \in [a, b]. \quad \square$$

Proof of Proposition 8. Let η_2 denote the random variable resulting from a mean-preserving increase in risk for η_1 and α_1^{**} is the optimal level of coinsurance in the presence of η_1 . Let H_1 and H_2 denote the cumulative distribution functions of η_1 and η_2 , respectively.

First assume $u''' > 0$. Now consider the FOC in the presence of η_2 evaluated at α_1^{**} :

$$\begin{aligned} & \int_0^L (x - P) \mathbb{E} \left[u' \left(W_0 - x - \alpha_1^{**}P + \alpha_1^{**}x + g(x)\eta_2 \right) \right] dF(x) \\ &= \int_0^L \int_a^b (x - P) u' \left(W_0 - x - \alpha_1^{**}P + \alpha_1^{**}x + g(x)\eta \right) d[H_2(\eta) - H_1(\eta)] dF(x). \end{aligned} \tag{C14}$$

Now we have for any $x \geq P$ and from prudence:

$$\begin{aligned} & \frac{\partial^2}{\partial \eta^2} (x - P) u' \left(W_0 - x - \alpha_1^{**}P + \alpha_1^{**}x + g(x)\eta \right) \\ &= (x - P) g(x)^2 u''' \left(W_0 - x - \alpha_1^{**}P + \alpha_1^{**}x + g(x)\eta \right) \geq 0, \end{aligned}$$

that is, $(x - P)u'(W_0 - x - \alpha_1^{**}P + \alpha_1^{**}x + g(x)\eta)$ is convex for $x \in [P, L]$, which results in

$$\int_a^b (x - P)u'(W_0 - x - \alpha_1^{**}P + \alpha_1^{**}x + g(x)\eta)d[H_2(\eta) - H_1(\eta)] \geq 0,$$

with a strict inequality if $g(x) > 0$ which by assumption holds for all $x > c$ for some $c \in [P, L]$.

Since $g(0) = 0$, it follows for $x = 0$ that

$$\int_a^b -Pu'(W_0 - \alpha^{**}P + 0\eta)d[H_2(\eta) - H_1(\eta)] = 0,$$

Consequently, (C14) is positive and the optimal coinsurance demand increases for a prudent decision-maker. Similarly, for an imprudent decision-maker, optimal coinsurance demand decreases if the loss-dependent background risk undergoes a mean-preserving increase in risk. □

Proofs of Appendix A

Proof of Proposition 9. Similar to the coinsurance case, D^* solves the FOC without background risk (A1). We now plug D^* into the FOC with loss-dependent background risk (A2) and check whether the sign of this FOC is negative in D^* .

From Leibniz' integral rule it follows that

$$\phi'(D) = \lambda(F(D) - 1).$$

Thus, and by using the law of total expectation, the FOC in (A2) with background risk evaluated at D^* can also be written as

$$\begin{aligned} \left. \frac{\partial \mathbb{E}[u(\tilde{W}(X))]}{\partial D} \right|_{D=D^*} &= \lambda(1 - F(D^*)) \int_0^{D^*} \mathbb{E}[u'(W_0 - x - \phi(D^*) + \eta_x)]dF(x) \\ &\quad + (\lambda(1 - F(D^*)) - 1) \int_{D^*}^L \mathbb{E}[u'(W_0 - D^* - \phi(D^*) + \eta_x)] \\ &\quad dF(x) \\ &= \lambda(1 - F(D^*)) \int_0^{D^*} u'(W_0 - x - \phi(D^*)) \\ &\quad \frac{\mathbb{E}[u'(W_0 - x - \phi(D^*) + \eta_x)]}{u'(W_0 - x - \phi(D^*))}dF(x) \\ &\quad + (\lambda(1 - F(D^*)) - 1) \int_{D^*}^L u'(W_0 - D - \phi(D^*)) \\ &\quad \frac{\mathbb{E}[u'(W_0 - D^* - \phi(D^*) + \eta_x)]}{u'(W_0 - D - \phi(D^*))}dF(x). \end{aligned} \tag{C15}$$

Without the two fractions, (C15) is the FOC without background risk. Thus, as D^* is optimal, the first term in (C15) is positive, while the second is negative. If we show that

$$\frac{\mathbb{E}[u'(W_0 - x - \phi(D^*) + \eta_x)]}{u'(W_0 - x - \phi(D^*))} \text{ for any } x \in [0, D^*] \tag{C16}$$

is not larger than

$$\frac{\mathbb{E}[u'(W_0 - D^* - \phi(D^*) + \eta_x)]}{u'(W_0 - D^* - \phi(D^*))} \text{ for any } x \in [D^*, L], \tag{C17}$$

then the negative component of (C15) is multiplied with a factor that is uniformly at least as large as the factor of the positive component, with a strict ranking at least for some x , such that expected utility in (C15) is negative, that is, it is optimal to decrease the level of deductible.

It follows easily from the arguments made in the proof of Proposition 1 that (C16) is nondecreasing in x . Thus, the fraction is largest for $x = D^*$.

Second, the two fractions in (C15) are identical at $x = D^*$. Thus, it is sufficient if (C17) is nondecreasing in x on $[D^*, L]$, because this implies that the fraction in (C17) is uniformly at least as large as the fraction in (C16). Equation (C17) is nondecreasing due to $u''' > 0$ and the fact that η_x undergoes increases in risk as x increases. Thus, the fractions are nondecreasing. The assumption on η_x implies that at least for some $x \in [0, L]$, at least one of the fractions needs to strictly increase, such that for sufficiently large values of x , the fraction in (C17) is strictly larger than the fraction (C16) for sufficiently small values. Consequently, the negative term in (C15) dominates the positive term, that is, (C15) is negative. Thus, the optimal level of deductible is smaller in the presence of loss-dependent background risk. If $D^* = 0$, the FOC with loss-dependent background risk evaluated at 0 is negative, that is, $D^{**} = 0$ is the optimal attainable level of deductible. □

Proof of Proposition 10. Without background risk, for the case of actuarially fair premiums, $D^* = 0$ is optimal. Thus, using $\eta_0 = 0$, the FOC in (C15) becomes

$$\begin{aligned} \left. \frac{\partial \mathbb{E}[u(\widetilde{W}(X))]}{\partial D} \right|_{D=0} &= (1 - F(0))F(0)u'(W_0 - \phi(0)) - F(0) \int_0^L \mathbb{E} \\ &\quad [u'(W_0 - \phi(0) + \eta_x)] dF(x) \\ &= (1 - F(0))F(0)u'(W_0 - \phi(0)) \\ &\quad - F(0) \int_0^L u'(W_0 - \phi(0)) \frac{\mathbb{E}[u'(W_0 - \phi(0) + \eta_x)]}{u'(W_0 - \phi(0))} dF(x) \end{aligned} \tag{C18}$$

Without the fraction, (C18) is the FOC without background risk and thus zero. Hence, introducing loss-dependent background risk only changes the second, negative part of the FOC without background risk.

If the decision-maker is prudent, it follows from Jensen's inequality that

$$\frac{\mathbb{E}[u'(W_0 - \phi(0) + \eta_x)]}{u'(W_0 - \phi(0))} \geq 1$$

with a strict inequality for at least some $x \in [0, L]$ by assumption. As a result, (C18) is negative. As we restrict the analysis to deductibles on $[0, L]$, 0 is still the optimal level of deductible in the presence of loss-dependent background risk.

Similarly, if the decision-maker is imprudent, we have that

$$\frac{\mathbb{E}[u'(W_0 - \phi(0) + \eta_x)]}{u'(W_0 - \phi(0))} \leq 1$$

with a strict inequality for at least some $x \in [0, L]$ by assumption. As a result, (C18) is positive and the optimal deductible is strictly larger than 0. \square

Proof of Proposition 11. The assumption that $X \leq D^*$ implies $X = 0$ almost surely together with $\eta_0 = 0$ means that the FOC in (C15) changes accordingly to

$$\begin{aligned} \frac{\partial \mathbb{E}[u(\widetilde{W}(X))]}{\partial D} \Big|_{D=D^*} &= \lambda(1 - F(D^*))F(0)u'(W_0 - \phi(D^*)) \\ &\quad + (\lambda(1 - F(D^*)) - 1) \int_{D^*}^L u'(W_0 - D^* - \phi(D^*)) \\ &\quad \times \frac{\mathbb{E}[u'(W_0 - D^* - \phi(D^*) + \eta_x)]}{u'(W_0 - D^* - \phi(D^*))} dF(x), \end{aligned} \tag{C19}$$

that is, the loss-dependent background risk only changes the negative part of the FOC. Since prudence implies

$$\frac{\mathbb{E}[u'(W_0 - D^* - \phi(D^*) + \eta_x)]}{u'(W_0 - D^* - \phi(D^*))} \geq 1$$

and a strict inequality holds at least for some $x \in [D^*, L]$ by the assumption of η_x , (C19) is negative and the optimal level of deductible is lower in the presence of loss-dependent background risk compared with the case of no background risk. \square