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Optimal promotions of competing firms in a frictional labour market with organizational hierarchies

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Abstract

We study optimal promotion decisions of hierarchical firms, with one junior and one senior managerial position, which interact in a search and matching labour market. Workers acquire experience over time while being employed in a junior position and the firm has to determine the experience level at which the worker receives a promotion which allows her to fill a senior position. Promoted workers move to the senior position in their current firm, if it is vacant, otherwise they search for senior positions on the market. The promotion cut-offs of the competing firms exhibit strategic complementarity, but we show that generically a unique stable symmetric general equilibrium exists. We find that stronger competition among firms leads to later (earlier) promotions if the initial number of firms is small (large) giving rise to an inverse U-shape relationship. In the presence of two skill groups, stronger competition among firms reduces the importance of skill differences, so the gap in wages and promotion times decreases with the number of firms. The model is compatible with empirical evidence that high-skill workers are promoted faster than the low-skilled and that internal promotions are more frequent than cross-firm moves to a higher hierarchical position.

1 | INTRODUCTION

Empirical evidence suggests that workers progress in their careers by means of internal promotions within firms, job-to-job transitions between firms and experience accumulation. However, existing research analyzes promotions and job-to-job mobility within different strands of literatures. Whereas search and matching studies developed strong techniques for the analytical treatment of on-the-job search and between-firm mobility of workers, research on internal promotions within firms is conducted in the literature on internal labour markets and principal agent models.¹ In this study we develop a unified search and matching framework with hierarchical firms, experience accumulation, job-to-job mobility and internal promotions. A combination of these areas leads to new insights on how the composition of the applicant pool, strategic competition between firms and search frictions influence the optimal timing of promotions.

Modeling the impact of labour market competition on the decisions of firms is especially important in the view of the increasing shortage of (qualified) labour power observed in a number of developed economies. For example, in the

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United States and in Germany aggregate statistical data shows that the number of applicants per vacancy was falling and the average vacancy durations were increasing in the past decade due to the growing disbalance in the supply and demand of labour.² At the same time, a rigorous analytical investigation of the impact of stronger labour market competition among firms on their internal promotion decisions is missing in the literature. Our paper aims to fill this gap by setting up a search and matching model with hierarchical firms and endogenous promotion times chosen by firms competing on the labour market. A novel feature of our model is that it explicitly allows for the possibility of internal within-firm and external between-firm turnover of workers. Thus, we follow the recommendation by Waldman (2013), who writes that “there are a few subjects that deserve more attention at both the theoretical and empirical levels. One subject is the connection between internal-labour-market wage and *promotion dynamics and the turnover decision*” (p. 539).

More specifically, we develop a search and matching model with three hierarchical levels in the career ladder. The first level consists of nonmanagerial jobs available to all workers without frictions. In addition, there are firms in the market consisting of two professional positions: one junior position and one senior position. This structure implies that there are three hierarchical job levels and two submarkets in our model: the primary market for young inexperienced individuals applying for their first junior manager position and a secondary market for experienced workers applying for senior manager positions. Firms with open positions post vacancies in each of the two submarkets respectively. As in Gibbons and Waldman (1999), the productivity of junior managers is growing over time due to experience accumulation and there is complementarity between experience and the hierarchical layer the worker is assigned to.

The main choice variable of the firm is the promotion time. Specifically, firms choose the minimum experience cut-off which is necessary for the junior worker to be internally promoted to the senior level. Workers in junior positions achieve this level of experience by means of on-the-job training provided by the firm. The experience cut-off is set in the beginning of the employment relationship and is a part of the labour contract. Note that the actual promotion can only take place if the junior worker accumulated the minimum experience level set by the firm and there is an open senior position in this firm. This is different from the model of Gibbons and Waldman (1999), where every worker can always be promoted in every firm and promotions do not depend on the availability of open positions at higher hierarchical levels. Hence we extend existing literature by accounting for the slot constraint associated with a promotion decision.

The tradeoff for firms can be characterized in the following way: if the inexperienced worker is promoted too early in his/her career, this worker will have a relatively low productivity after the promotion because this worker's experience is too low for the senior level. In this situation it is a better strategy for the firm to wait and search for a more experienced worker in the secondary submarket for senior managers. This submarket exists because some workers have already reached sufficient experience to be promoted, but there are no open positions in their firms (binding slot constraints). Thus, these workers start searching for senior managerial jobs with alternative employers (on-the-job search). This is different from the classical model of on-the-job search by Burdett and Mortensen (1998), where all employees are searching for better paid jobs, and shows that promotions and on-the-job search are closely linked to each other, moreover, this link is missing in the previous studies.

Based on this model we find that the optimal promotion time of a given firm is increasing in the average promotion time of the market, so there is strategic complementarity between the promotion times of the different firms. This is because the optimal individual promotion time of the firm depends on the distribution of experience of managerial applicants in the secondary submarket, which again is determined by the promotion decisions of the other firms in the market. We account for this competition effect by characterizing Nash equilibrium assuming steady states of the labour flows. We find that there are two symmetric Nash equilibria but only one of them is stable. Focusing on the stable equilibrium, we can show that the model is compatible with the empirical evidence in Lluís (2005), Cassidy et al. (2016), and Frederiksen et al. (2016) that internal promotions are more frequent than across-firm moves to a higher hierarchical level.

When we increase the degree of competition in the market we observe an inverse U-shape relationship between the number of firms and the optimal promotion experience cut-off. If competition is initially weak and gets stronger then firms delay promotions. The reason is that the probability of filling a junior position is very sensitive to the number of firms and falls sharply. Thus, firms respond by delaying promotions and avoiding open vacancies on the junior level. On the contrary, if competition is already strong and gets even stronger then the probability of filling a senior position becomes negligible, thus firms respond by earlier promotions of the internal candidates. In addition, we find that higher search costs lead to earlier promotions but the magnitude of this effect varies a lot with the degree of competition. Specifically, the impact of the search costs is most pronounced and leads to substantially earlier promotions when there is strong competition among firms and filling vacancies is difficult.

We consider two extensions of our benchmark model. First, we consider a setup when additional output is generated if two workers (junior and senior) are working together as a team. We find that such team synergy is associated with earlier promotions. The reason is that search frictions in the senior submarket are more severe, so hiring junior workers is easier for firms than hiring experienced managers. To fill both positions, firms promote their own junior employees earlier compared to the benchmark case and try to hire another junior worker afterwards. This strategy leads to the highest gain from the team synergy for firms.

In the second extension we consider skill heterogeneity of workers, assuming that high skill workers are more productive than low skill workers only in senior managerial jobs. This model extension can explain the empirical evidence that high skill workers are promoted earlier than low skill workers (Baker et al., 1994; Lluís, 2005; McCue, 1996) and that external hires are more educated on average than internally promoted workers (Baker et al., 1994; Bidwell, 2011; DeVaro et al., 2019). Even though there exist other theoretical models of internal labour markets compatible with this evidence (e.g., Bernhardt, 1995; DeVaro & Waldman, 2012; Gibbons & Waldman, 2006) these studies are lacking the equilibrium turnover of workers and firms. Integrating promotions and job-to-job transitions we are able to endogenize the rates at which firms meet workers of a specific type, either for their junior or senior vacancies. We show how firms' promotion strategies are then altered by the general equilibrium effects. This allows us to explore how changes in the distribution of worker types and the degree of firm competition affect promotion timing for all skill groups giving rise to novel testable empirical implications.

In particular, we show that better quality of the labour force associated with a larger fraction of high skill workers induces slower promotions, whereas in a setting with homogeneous labour an increase of the skill level leads to faster promotions. Therefore, an improvement in the average skills of the labour force may have different implications for promotions depending on the exact reason for such an improvement. A larger fraction of high skill workers raises the expected skill of an external hire *relative* to the skill of the incumbent junior worker under consideration for a promotion. In this situation it is optimal for firms to delay promotions gaining more time for an external hire. In the case of homogeneous workers, the skills of all workers are improving—external applicants *and* incumbent candidates—which leads to earlier promotions.

Investigating the impact of competition in a model with heterogeneous workers, we find that differences in the skills lose importance when the number of firms is increasing and filling positions becomes a challenging task for firms. This leads to smaller differences in the promotion times of the two skill groups and has a mitigating impact on the wage gap. Hence, we conclude that stronger competition among firms, leading to a shortage of labour, has a potential of reducing the return to schooling, measured by differences in wages between the high and low skill workers. In how far this mechanism and the shortage of labour in the United States and in Germany can contribute to moderating the observed increase in wage inequality and polarization remains an open question for future empirical research in this field (Antonczyk et al., 2018).

Our study is closely related to the literature on human capital accumulation and job assignments. The seminal contributions here are by Gibbons and Waldman (1999) and Zabojnik and Bernhardt (2001). In Gibbons and Waldman (1999) worker's productivity depends on the individual skill level, accumulated experience and the hierarchical layer the worker is assigned to. As workers accumulate experience and knowledge they are optimally promoted by firms to higher positions due to the assumed complementarity between workers productivity (skills and accumulated experience) and hierarchical layers within the firm. Thus, firms gain higher profits by promoting experienced workers. We keep the same complementarity assumption in our model. Zabojnik and Bernhardt (2001) follow a different approach and assume that junior workers compete for promotions to senior positions via a tournament, so that promotions serve as an incentive device, but *ex post* do not yield higher profits to the employer. Even though both studies are successful in explaining wage dynamics of individuals within firms, whether due to experience accumulation or exerted effort, they are based on restrictive assumptions preventing the turnover of workers across employers, hence, there is no worker mobility between firms in the equilibrium. This highlights the main difference of our approach since firms in our model are embedded in a rich labour market with search frictions and job-to-job transitions of workers. This allows us to study the impact of labour market competition, search frictions and other labour market characteristics on the optimal promotion strategies of firms.

Several other studies combine promotions and across-firm mobility (Ghosh, 2007; Owan, 2004; Waldman, 1990). In Waldman (1990) workers can decide to accumulate high or low human capital in the first period. This decision is observed by the first-period employer who can retain and promote the worker or not in the second period. This employer decision is underlying the involuntary transitions of workers (dismissals). In addition, workers learn their preference for a particular type of employer which gives rise to voluntary transitions (quits). This mechanism of

voluntary quits associated with exogenous preference/disutility realizations at current employer is similar in Ghosh (2007). Further, strategic promotion decisions and worker turnover are also considered in Owan (2004). The model has three time periods and gives rise to two equilibria. In the first one firms promote their high ability workers relatively early, this serves as a positive signal to outside employers and initiates a bidding game for the worker, thus turnover is high in the first equilibrium. In contrast, there are late promotions, no bidding and low labour turnover in the second equilibrium. In a recent study, Cassidy et al. (2016) takes up the idea of “promotions-as-signals” in a two-period model and extends the setup to two gender groups, however, the turnover of workers is modeled as an exogenous shock and does not depend on the number of bidding competitors or their offers.

Even though we also consider strategic promotion decisions by firms in a combination with labour turnover, our model differs from the previous literature in three important dimensions. First, we explicitly allow for a slot constraint on the senior level, meaning that firms cannot promote every worker they consider good enough for a senior position. Second, the probability of a voluntary job change by the worker in our model depends on the tightness of the labour market, thus, workers need time to find alternative employment due to the presence of search frictions. Finally, our model is laid out in continuous time, so firms choose the optimal time for a promotion driven by the accumulated human capital and ability of the worker as well as the probabilities of quitting and filling a senior position externally. Especially, this last effect associated with an endogenous probability of external hiring to a senior position and its impact on the promotion decision is not considered in Waldman (1990) and Owan (2004). We also take into account that the junior position remains vacant after the worker is promoted, and so the time needed to fill this position may influence the promotion decision of the firm. This consideration stemming from the dynamic nature of the model is missing in the previous literature.

Next, our study is conducted in the search and matching framework (Diamond, 1982; Mortensen, 1982; Pissarides, 1985). We model job-to-job transitions following the approach of Burdett and Mortensen (1998). To the best of our knowledge the first study analyzing tenure in a search and matching framework with job-to-job transitions is Pissarides (1994). There are good and bad jobs in his setting, thus unemployed workers accept bad jobs but continue searching for good jobs. An important feature of the model is that workers accumulate job-specific experience and their wage grows over time. In the equilibrium very experienced workers with high wages stop searching at all since the gain from moving to a good job becomes smaller than the cost of searching. The main difference of this study from current work is that we treat experience as transferable across firms while it is completely lost upon the quit in Pissarides (1994). Recent work in this field includes prominent extensions by Burdett and Coles (2003), Burdett et al. (2011), and Bagger et al. (2014). These studies analyse tenure accumulation with on-the-job search, but they do not consider internal promotions.

The rest of the paper is structured as follows. In Section 2 we introduce the economic framework and analyse the dynamics of workers and firms across states. Section 3 presents the value functions of firms and their choice of the optimal promotion time as well as the emerging partial and general equilibrium in the benchmark setting. In Section 4 we extend the model to two skill groups. Section 5 concludes the paper.

2 | THE MODEL

2.1 | The economic framework

Time is continuous with an infinite horizon. There is a continuum of both firms and workers with a total measure of workers normalized to 1. The inflow of new workers into the labour market is denoted by d . In the benchmark model all entering workers are homogeneous with identical skills, however, in the extension we also analyze consequences of skill heterogeneity. Job ladders have three hierarchical levels. All young workers entering the market immediately take simple jobs on the low level. These are subsistence jobs that do not yield any professional experience. All entering firms are identical and every firm is a dyad consisting of two positions: one junior position and one senior (managerial) position. The inflow of new firms is denoted by n . Both positions are empty when the firm enters the market and can be posted simultaneously. Posting an open position (junior or senior) is associated with a flow cost s for the firm. Let e_0 denote the stock of workers in low level jobs, e_1 —are workers employed in junior positions and e_2 denotes managers in senior positions, so that $e_0 + e_1 + e_2 = 1$ due to the normalization.

Once accepted in the junior position young workers start accumulating professional experience x with $\dot{x} = 1$. There are two broad ways of modeling the accumulation of experience/human capital on the job. The first way is experience

accumulation through learning by doing, so that there are no costs for workers and firms beyond those of performing the job by the worker in a standard manner. In this case experience accumulation continues as long as the worker performs the job, so promotions can be modeled as an optimal stopping problem where the firm decides spontaneously about the promotion if the senior position in the firm is open. The other approach understands the process of experience accumulation as costly on-the-job training, involving additional effort on the worker side beyond performing the job in a standard way. This type of experience accumulation has a finite duration even if the worker continues performing the job. In this paper we follow the second approach and assume that every firm i decides about the duration of training \bar{x}_i . Since this type of training is necessary for promotion, it is decided by the firm and communicated to the worker before the beginning of the job.

Intuitively, we think of modern trainee programs offered by large companies, where junior workers have to spend some time in different departments/branches of the firm, for example, production, accounting, marketing departments or regional subsidiaries. Even though this type of program requires some effort, it allows the worker to accumulate job- and industry-specific human capital making the worker eligible for a future promotion to a senior position. Therefore, we assume that experience x is transferable but not observable by other firms in the market until it becomes \bar{x}_i , when the worker receives a certificate of completed training. One immediate implication of this assumption is that workers involved in trainee programs do not search on-the-job, since they can not credibly signal their unfinished training to external employers. Rather they accumulate experience up to the level \bar{x}_i specified in their job contract. Having obtained this experience they are promoted if there is an open senior position in the firm, that is the firm commits to promoting experienced workers with completed training. Otherwise, if the senior position is filled, workers start searching on-the-job and use their certificate as a signal of sufficient managerial experience on the external market.

This model structure leads to the existence of two separate submarkets, one where firms are posting junior positions and anticipate a worker with $x = 0$, and another one where firms are posting their senior positions and anticipate workers searching on-the-job and possessing a proof of sufficient experience. For the purpose of tractability we assume that there are no dismissals. Firms obtain a flow profit $\pi_1 = d_1 + c_1 e^{\gamma x}$ from a filled junior position and a flow profit $\pi_2 = d_2 + c_2 e^{\gamma \bar{x}}$ from a senior position, where $d_1 > d_2$ and $c_1 < c_2$ as in Gibbons and Waldman (1999). Intuitively, this means that the fixed component of output $d_j, j = 1, 2$ is falling with a higher hierarchical level, while experience becomes more important, that is $c_j, j = 1, 2$ is increasing with j . The multiplier parameter γ is a proxy for the return to worker's experience. In a symmetric equilibrium all firms choose an identical promotion/experience cut-off \bar{x} , thus firms correctly anticipate that applicants to senior positions achieved an experience level \bar{x} and the profit from a senior job becomes $d_2 + c_2 e^{\gamma \bar{x}}$. There is no experience accumulation in senior positions in the sense that there is no further training, so output is constant. In addition, we introduce a profit synergy Δ if the firm is employing both workers simultaneously, that is, one junior worker accumulating experience and one senior manager. So the total profit of this firm is given by $d_1 + c_1 e^{\gamma x} + d_2 + c_2 e^{\gamma \bar{x}} + \Delta$. Intuitively, this is a synergy from team work because younger inexperienced workers gain from the advice of senior managers, whereas senior managers may gain from the innovative new ideas of younger workers.

Given that the essence of the trainee program is to let the junior worker learn different functions within the firm, so that no external teachers or coaches are involved, we refrain from adding an additional parameter for the fixed cost of training and focus solely on the indirect cost associated with a foregone profit. We capture the indirect cost by assuming that $d_1 + c_1 < d_2 + c_2$. This means that firms obtain lower profits at $x = 0$ by assigning the young worker to the junior level rather than the senior level. In the following sections of the paper we show that this decision is optimal for firms since there are expected gains associated with having more experienced senior managers in the future. This assumption is different from Gibbons and Waldman (1999) where firms only consider the immediate gain and so the model requires that $d_1 + c_1 > d_2 + c_2$.

Workers employed in senior managerial positions retire at an exogenous rate ρ and exit the market. If the manager retires and the junior position is not filled, the firm is empty and closes. It is a simplifying assumption of the model that there is no labour market exit among younger workers employed in low level jobs and junior positions. In our analysis we only consider the steady state, moreover the entry parameter d is chosen to keep the population size constant.

Variable d_{00} denotes the stock of empty new firms in the market, whereas d_{01} is the stock of firms with a senior manager but no junior worker. This gives rise to $d_{00} + d_{01}$ open junior positions. These positions are randomly matched with ze_0 searching inexperienced workers, where z denotes the search effort of workers. More precisely, z is the probability of sending an application. To determine the number of matches in the submarket for junior positions we use an urn-ball matching mechanism. This means that every worker (ball) applies randomly with probability z to one vacancy (urn). This matching function has solid microeconomic foundations and standard macroeconomic

properties (see Albrecht et al., 2003), for example, increasing in both arguments (the number of searching workers and vacancies) and constant returns to scale in the limit when the numbers of vacancies and searching workers are sufficiently large. Suppose some worker sends an application to one randomly chosen firm, then the probability that a given firm does not receive this application is $1 - \frac{1}{d_{00} + d_{01}}$. Since workers send their applications independently without coordination, the probability that this firm does not get any of the ze_0 applications is given by $(1 - \frac{1}{d_{00} + d_{01}})^{ze_0}$. Let q_1 be the job-filling rate resulting from this application process and λ_1 be the job-finding rate for inexperienced workers. They are given by:

$$q_1 = 1 - \left(1 - \frac{1}{d_{00} + d_{01}}\right)^{ze_0} \quad \lambda_1 = z \frac{q_1(d_{00} + d_{01})}{ze_0} = q_1 \frac{(d_{00} + d_{01})}{e_0}. \quad (1)$$

The term $q_1(d_{00} + d_{01})$ is a total number of matches in the junior market, thus $\frac{q_1(d_{00} + d_{01})}{ze_0}$ is a probability of matching for workers conditional on sending an application in a given matching round. Multiplying this conditional matching probability with z we obtain the job-finding rate for junior workers. Further, let d_{10} denote firms with a junior worker but no senior manager. This means that the total number of open managerial positions is given by $d_{00} + d_{10}$. Finally, let d_{11}^N denote the stock of full firms with both employees, where the worker in the junior position is not yet eligible for promotion ($x < \bar{x}$). In a similar way, d_{11}^S —is the stock of full firms, where the junior worker is already eligible for senior positions and searching on-the-job. This means that the stock of applicants in the managerial market is given by zd_{11}^S . So the job-filling rate in the managerial market q_2 and the workers' job-finding rate in this market λ_2 are given by:

$$q_2 = 1 - \left(1 - \frac{1}{d_{00} + d_{10}}\right)^{zd_{11}^S} \quad \lambda_2 = z \frac{q_2(d_{00} + d_{10})}{zd_{11}^S} = q_2 \frac{(d_{00} + d_{10})}{d_{11}^S} \quad (2)$$

Note that we assume the same search intensity parameter z in both markets. This setting can be generalized to different search intensities for experienced and inexperienced workers, however, it is not important for our main results. So we keep the model simple and consider only one search intensity parameter z .

The total number of firms in the market F is given by $d_{00} + d_{01} + d_{10} + d_{11}^N + d_{11}^S$. This notation also allows us to calculate the number of workers, which we normalize to 1:

$$e_0 + d_{10} + d_{01} + 2d_{11}^N + 2d_{11}^S = 1$$

Here $e_1 = d_{10} + d_{11}^N + d_{11}^S$ is the total number of employees in junior positions, and $e_2 = d_{01} + d_{11}^N + d_{11}^S$ is the total number of employees in senior positions.

2.2 | Firm dynamics

Transitions of firms are illustrated in Figure 1. Consider changes in the stock of new empty firms d_{00} . The inflow of new firms into the market is given by n . Since every new firm posts both the junior and the senior position in the respective submarkets it exits the state d_{00} whenever it finds the first employee. So the outflow of firms from d_{00} takes place at rate $q_1 + q_2$. In this paper we restrict our analysis to the steady states and consider a stationary distribution of workers and firms across states. This means that $\dot{d}_{00} = 0$ in the steady state:

$$0 = \dot{d}_{00} = n - (q_1 + q_2)d_{00} \Rightarrow d_{00} = \frac{n}{q_1 + q_2} \quad (3)$$

The entry of firms into the market is given by n , whereas the exit is ρd_{01} . These are the firms that lose their only employee due to retirement, which happens at rate ρ . Thus we get $d_{01} = n/\rho$ to guarantee a constant number of firms in the market.

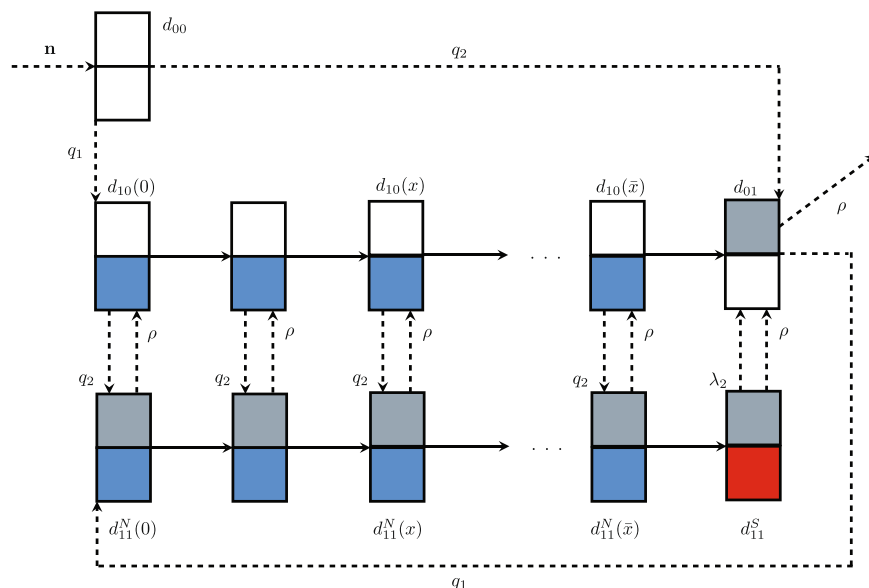


FIGURE 1 Types of firms and their transitions [Color figure can be viewed at wileyonlinelibrary.com]

Further, consider changes in the stocks of firms $d_{10}(x)$ and $d_{11}^N(x)$. Note that workers with experience $0 \leq x \leq \bar{x}$ are not yet searching on-the-job since their experience is not sufficient for managerial positions and they cannot credibly signal unfinished training to external employers. Variable \bar{x} here denotes the equilibrium promotion cut-off and will be determined in Section 3. This means that the inflow of firms into state $d_{10}(x)$ is equal to $\rho d_{11}^N(x)$. These are the firms where the manager retires at rate ρ and they are left with only one junior worker. At the same time $\rho d_{11}^N(x)$ is the outflow of firms from the state $d_{11}^N(x)$. If the manager retires firms post the open position in the second submarket for experienced workers and find a manager at rate q_2 . This means that the outflow of firms from the state $d_{10}(x)$ is equal to $q_2 d_{10}(x)$. This is also the inflow of firms into the state $d_{11}^N(x)$. So we get the following system of two first-order linear differential equations³:

$$\begin{cases} \partial d_{10}(x)/\partial x = -q_2 d_{10}(x) + \rho d_{11}^N(x) \\ \partial d_{11}^N(x)/\partial x = q_2 d_{10}(x) - \rho d_{11}^N(x) \end{cases} \quad (4)$$

Proposition 1 contains solution of this system of differential equations and includes the proof:

Proposition 1. *The stock of firms $d_{10}(x)$ with a junior worker accumulating experience x , and the stock of firms $d_{11}^N(x)$ with a similar junior worker and a senior manager are given by:*

$$\begin{cases} d_{10}(x) = k_1 \rho + k_2 e^{-(\rho+q_2)x} \\ d_{11}^N(x) = k_1 q_2 - k_2 e^{-(\rho+q_2)x} \end{cases} \quad \text{where } k_1 = \frac{q_1 n (\rho + q_1 + q_2)}{\rho (\rho + q_2) (q_1 + q_2)} \quad k_2 = -\frac{(q_1)^2 n}{(\rho + q_2) (q_1 + q_2)}.$$

The stock of firms $d_{10}(x)$ is increasing in x , while $d_{11}^N(x)$ is decreasing in x since $k_2 < 0$.

Proof. The coefficient matrix of the homogeneous system of Equations (4) has eigenvalues 0 and $-(\rho + q_2)$, which gives rise to the general solution summarized above. To find the constant terms k_1 and k_2 we use the following initial conditions: $q_1 d_{00} = d_{10}(0)$ and $q_1 d_{01} = d_{11}^N(0)$. The first condition implies that the stock of firms $d_{10}(0)$ always consists of new firms finding their first junior worker $q_1 d_{00}$. The second condition implies that the stock of firms $d_{11}^N(0)$ consists of firms d_{01} who find a junior worker, that is $q_1 d_{01}$. Using these initial conditions we find the corresponding values of k_1 and k_2 . \square

Proposition 1 shows that the stock of firms $d_{10}(x)$ is increasing with higher experience of the junior worker x , while $d_{11}^N(x)$ is decreasing. To understand this result note that the sum of two variables is a constant, that is $d_{10}(x) + d_{11}^N(x) = k_1(\rho + q_2) \forall x \in [0 \dots \bar{x}]$, so it is always the case that one of them is increasing and the other one is decreasing. The starting ratio of these two stocks is $d_{10}(0)/d_{11}^N(0) = \rho/(q_1 + q_2)$ but the long-run ratio for larger values of x is: $\lim_{x \rightarrow \infty} d_{10}(x)/\lim_{x \rightarrow \infty} d_{11}^N(x) = \rho/q_2$. So the ratio is clearly increasing with higher experience levels. At the same time we know that the sum of these two stocks is fixed and each of them is a monotonous function of x . This confirms again that $d_{10}(x)$ should be increasing. So as workers accumulate more and more experience they are more likely to find themselves in a situation with an open senior position. The reason is that senior managers retire over time, but the probability of substituting them with an external candidate is relatively low.

By integrating variables $d_{10}(x)$ and $d_{11}^N(x)$ over the interval $[0 \dots \bar{x}]$ we find the total stocks of firms d_{10} and d_{11}^N :

$$d_{10} = \int_0^{\bar{x}} d_{10}(x) dx = k_1 \rho \bar{x} + \frac{k_2}{\rho + q_2} (1 - e^{-(\rho + q_2)\bar{x}}), \quad (5)$$

$$d_{11}^N = \int_0^{\bar{x}} d_{11}^N(x) dx = k_1 q_2 \bar{x} - \frac{k_2}{\rho + q_2} (1 - e^{-(\rho + q_2)\bar{x}}) \quad (6)$$

The remaining unknown stock of firms is d_{11}^S . These are the firms with two employees, where the junior one is already searching for jobs with alternative employers. All firms of type $d_{11}^N(\bar{x})$ automatically enter the state d_{11}^S since the junior worker starts searching on-the-job upon attaining experience \bar{x} . This is the inflow of firms into the state d_{11}^S . At rate ρ the senior manager retires and the firm promotes the junior worker to the managerial job. In addition, it can also happen that the junior worker finds a new employer at rate λ_2 . As one can see from Figure 1, in both cases the firm leaves the state d_{11}^S and enters the stock of firms d_{01} . Hence we get:

$$0 = \dot{d}_{11}^S = d_{11}^N(\bar{x}) - (\rho + \lambda_2) d_{11}^S \Rightarrow d_{11}^S = \frac{d_{11}^N(\bar{x})}{\rho + \lambda_2} = \frac{k_1 q_2 - k_2 e^{-(\rho + q_2)\bar{x}}}{\rho + \lambda_2}. \quad (7)$$

Finally, recall that e_0 are the young individuals searching for their first professional job, so that $\dot{e}_0 = d - \lambda_1 e_0$. In the steady state, it should be that the inflow into this state d should be equal to the outflow $\lambda_1 e_0$, where the outflow are young inexperienced workers finding their first employer. So we get $e_0 = d/\lambda_1$. Variable d is the endogenous entry of young individuals, which we can find from normalizing the total population of workers to 1:

$$\frac{d}{\lambda_1} = 1 - (d_{10} + d_{01} + 2d_{11}^N + 2d_{11}^S). \quad (8)$$

Solving jointly the system of Equations (1)–(8), $d_{01} = n/\rho$, $e_0 = d/\lambda_1$ we can find the equilibrium distribution of firms $\{d_{00}, d_{10}, d_{11}^N, d_{11}^S, d_{01}\}$, as well as variables d and e_0 and the equilibrium transition rates λ_j , and q_j , $j = 1, 2$. Note that variable \bar{x} (promotion cut-off) is taken as given at this stage and will be endogenously derived in Section 3.

2.3 | Transition rates

We proceed by illustrating the mechanism of our model with a help of a numerical example which resembles realistic career paths of workers in developed economies. In this section, we focus on the transitions of workers and firms for a given promotion cut-off \bar{x} . One period of time is set to be one quarter. Consider young workers entering the market at the age of 18 years. Variable z is the search intensity parameter which is the driving force behind the job-finding rate λ_1 . We set $z = 0.0146$, this corresponds to $\lambda_1 = 0.0145$ and implies that workers stay in level 0 jobs for approximately $1/\lambda_1 = 69$ quarters or 17.25 years. Intuitively, this means that workers find their first managerial job on level e_1 at the age of 35.25 years on average.⁴ In state e_1 workers start accumulating professional managerial experience x . We consider $\bar{x} = 48$, so that it takes 48 quarters or 12 years for workers to be eligible for the position of a senior manager. Thus, workers reach the prespecified necessary level of experience at the age of 47.25 years on average.

Recall that $d_{10}(\bar{x})$ is a stock of workers who are directly promoted to senior positions within their firm at every point in time. At the same time $d_{11}^N(\bar{x})$ is a stock of workers eligible for promotions, however, they can not be promoted directly within their firm since the senior position is occupied. These workers start searching on-the-job and enter the accumulated pool of workers searching and applying to senior positions d_{11}^S . So the total stock of workers eligible for promotion in a given period of time is $d_{10}(\bar{x}) + d_{11}^N(\bar{x}) + d_{11}^S = k_1(\rho + q_2) + d_{11}^S$. Out of these workers, $d_{10}(\bar{x}) + (\rho + \lambda_2)d_{11}^S$ are actually promoted, where $d_{10}(\bar{x}) + \rho d_{11}^S$ are promoted directly within their firms and $\lambda_2 d_{11}^S$ make a transition to a senior position in another firm. So the average duration of time from the moment of becoming eligible \bar{x} till the actual promotion within or between firms is given by:

$$\frac{k_1(\rho + q_2) + d_{11}^S}{k_1\rho + k_2e^{-(\rho+q_2)\bar{x}} + (\rho + q_2)d_{11}^S}.$$

In our model, this duration is equal to 16.3 quarters or approximately 4 years, so that workers become senior managers at the age of 51.3 years on average. This duration is achieved by setting the number of entering firms n equal to 0.0023. This also implies that the average stock of firms active in the market is equal to 0.55. So there are on average 550 active firms or 1100 positions per 1000 workers. However, not all of these positions are filled due to the search frictions and experience requirements. Further, we set $\rho = 0.015$, so the average time workers spend in senior positions until retirement is $1/\rho = 66.6$ quarters or 16.7 years. So workers retire on average at the age of 68 years. Finally, the total population is normalized to 1. Given that the exit rate of workers is $\rho = 0.015$, constant size of the population can be achieved by setting $d = 0.0051$. This means that 5.1 workers on average enter the market with a population of 1000 workers. Our choice of parameters at this stage is summarized in Table 1. Endogenous values of the quarterly transition rates in the steady-state are summarized on the right side of Table 1.

Table 2 shows the distributions of workers and firms in the steady-state. We can see that 35% of all workers remain on average in simple jobs e_0 . Further, 31% are employed in junior positions e_1 , where 6.7% of workers are searching on-the-job and applying to senior positions (d_{11}^S). A total of 34% of workers occupy senior management positions e_2 . These numbers imply that $p_1 = 0.31/(0.31 + 0.34) = 0.479$, that is 47.9% of workers in professional jobs are employed in junior positions, with the remaining 52.1% being employed in senior positions. Considering transitions of workers, we can see that 1.3% of e_1 workers reach senior positions by changing employers. Another 5.3% of junior workers are internally promoted within their firms per year.

Even though internal mobility of workers is not intensive, these numbers are close to the empirical findings. For example, Lluís (2005) finds that in Germany the annual probability of internal promotions is 5.7% for relatively young workers with less than 10 years of market experience. The same study reports that internal mobility is more intensive

TABLE 1 Values of exogenous parameters and quarterly transition rates

Parameter	Value	Interpretation	Variable	Value	Interpretation
z	0.0146	Search intensity of workers	q_1	0.0212	Job-filling rate, level 1
ρ	0.0150	Exit/retirement rate	q_2	0.0046	Job-filling rate, level 2
n	0.0023	Entry of empty firms	λ_1	0.0145	Job-finding rate, level 1
d	0.0051	Entry of young workers	λ_2	0.0146	Job-finding rate, level 2

TABLE 2 Stationary distributions of workers and firms for parameters from Table 1 and $\bar{x} = 48$

Variable	Value	Variable	Equation	Value
d_{00}	0.0878	Workers in simple jobs e_0	$= 1 - e_1 - e_2$	0.3507
d_{01}	0.1514	Workers in junior jobs e_1	$= d_{10} + d_{11}^N + d_{11}^S$	0.3107
d_{10}	0.1236	Workers in managerial jobs e_2	$= d_{01} + d_{11}^S + d_{11}^N$	0.3385
d_{11}^S	0.0670	Internally promoted (per year)	$= (d_{10}(\bar{x}) + \rho d_{11}^S)/e_1$	0.0527
d_{11}^N	0.1201	Job-to-job movers (per year)	$= \lambda_2 d_{11}^S/e_1$	0.0126

in the United States, with 6.7% for men and 6.2% for women in the same experience category. A more recent study by Cassidy et al. (2016) reports an average probability of internal promotions equal to 4.6% in Finland. Also in the study by Frederiksen et al. (2016) using Danish data internal promotions of men are more intensive (1.27%) compared to the cross-firm moves to a higher hierarchical level (0.29%), though both rates are relatively low. Based on these results, we can conclude that our model successfully captures the empirical evidence that within-firm promotions are more intensive than between-firm promotions.

If we change the perspective and consider firms, then the share of senior vacancies filled externally can be calculated as: $\lambda_2 d_{11}^S / (d_{10}(\bar{x}) + \rho d_{11}^S + \lambda_2 d_{11}^S)$. This share is equal to 19.3% in the benchmark calibration, meaning that 80.7% of vacancies are filled internally on the senior level. At the same time, junior positions act as a port of entry in the model with 100% of all positions filled externally. These values are close to the empirical findings in Baker et al. (1994) where the fraction of external hires on the bottom of the firm hierarchy is 98.6% and it falls down to around 25% in the following levels. Also Kauhanen and Napari (2012) and DeVaro et al. (2019) report that internal hiring increases in importance at higher hierarchical levels, yet outside hiring remains substantial even at the top level.

The left panel of Figure 2 shows changes in the mobility of workers between levels 1 and 2 with respect to the promotion cut-off \bar{x} . Later promotions reduce the intensity of transitions from junior to senior positions. Both internal promotions and job-to-job transitions are less frequent with a higher promotion cut-off. This means that workers have to complete longer training in the junior position. The same figure (right axis) also illustrates the relative fraction of internally promoted workers, we obtain it by dividing the number of promoted workers $d_{10}(\bar{x}) + \rho d_{11}^S$ with a total number of workers making it to the senior position $\lambda_2 d_{11}^S + d_{10}(\bar{x}) + \rho d_{11}^S$. We can see that this relative fraction is increasing from 76% when $\bar{x} = 28$ to 85% when $\bar{x} = 68$. This reveals an unusual general equilibrium effect in our model. If some firm i decides to delay internal promotions and wants to hire more senior managers on the external market it sets a higher cut-off value \bar{x}_i . However, if all firms follow the same strategy and set a higher cut-off \bar{x} then the relative fraction of senior managers reaching senior positions via internal promotions is increasing. Thus internal promotions become a more important source of upward mobility for workers even though the individual intention of every firm is different. The reason is that with a higher experience requirement \bar{x} , there are less applicants in the external market, so the job-to-job mobility rate declines stronger than the internal promotion rate.

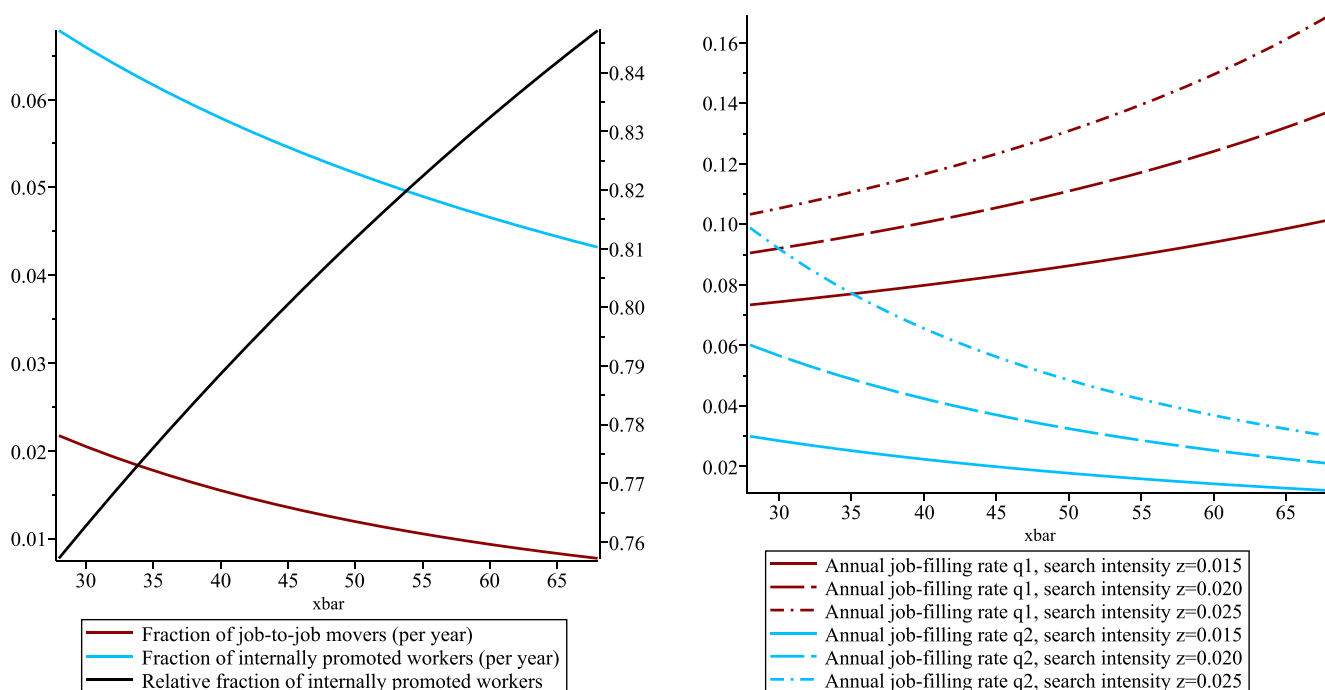


FIGURE 2 Selected variables for different values of the promotion cut-off \bar{x} and search intensity z . Left panel: Fractions of internally promoted workers $(d_{10}(\bar{x}) + \rho d_{11}^S)/e_1$ and job-to-job movers $\lambda_2 d_{11}^S/e_1$ per year. Right panel: Annual job-filling rates q_1 and q_2 . [Color figure can be viewed at wileyonlinelibrary.com]



The right panel of Figure 2 shows changes in the job-filling rates q_1 and q_2 . More intensive job search by workers makes it easier for firms to fill their open positions, so q_1 and q_2 are both increasing in z . But there are adverse effects of the promotion cut-off \bar{x} . Later promotions reduce the pool of competing vacancies on level 1. Reduced competition of firms in this submarket improves their hiring chances, so the job-filling rate q_1 is increasing with \bar{x} . There is an opposite effect in the second submarket for experienced workers. Delayed promotions reduce the pool of applicants for senior positions which leads to the lower job-filling rate q_2 .

The preceding analysis was conducted for a given promotion cut-off \bar{x} , which allowed us to illustrate adjustments in the transition rates and the equilibrium stocks of workers and firms induced by the exogenous changes in \bar{x} . In the next section, we endogenize the promotion strategies and investigate consequences of strategic competition in promotions among firms.

3 | OPTIMAL PROMOTION BY FIRMS

In this section, we analyze the optimal promotion strategy of firms in several steps. First, in Section 3.1 we determine the best response of an individual firm to a given promotion threshold used by all other firms. Second, in Section 3.2 we do a partial equilibrium analysis and show that for our calibration of the model there is a unique stable fixed point of the best response map for fixed values of the job-filling and job-finding rate. We then show that the transition rates generated under the (partial) equilibrium value of the promotion threshold actually coincide with the values underlying our partial equilibrium analysis. Hence, the obtained promotion threshold also constitutes a symmetric general equilibrium of our model. In Sections 3.3 and 3.4 we then explore the implications of changes in key parameters on the optimal firm promotion threshold disentangling partial and general equilibrium effects.

3.1 | Firm's best response

As a first step, we characterize in this section the optimal promotion time chosen by an individual firm for a given promotion threshold of all competitors and for given job filling rates for junior and senior positions. Denoting by $J_{00}(\bar{x}_i, \bar{x})$ the present value of a firm starting to search for a worker, that is, a firm with neither a junior nor a senior level worker, which uses a promotion threshold \bar{x}_i , whereas all other firms in the market promote at \bar{x} . When a new firm opens it has to choose its promotion strategy and the optimal choice is given by

$$\bar{x}_i^*(\bar{x}) = \arg \max_{\bar{x}_i \geq 0} J_{00}(\bar{x}_i, \bar{x}). \quad (9)$$

To analyze this optimization problem the value function J_{00} has to be determined. When entering the market the firm has two open positions—one junior and one senior—so the firm is searching for workers in both markets simultaneously and has a double cost $2s$. Therefore,

$$rJ_{00}(\bar{x}_i, \bar{x}) = -2s + q_1(J_{10}(0|\bar{x}_i, \bar{x}) - J_{00}) + q_2(J_{01}(\bar{x}|\bar{x}_i, \bar{x}) - J_{00}),$$

where $J_{10}(x|\bar{x}_i, \bar{x})$ is the present value for a firm with only one junior worker, whose experience is x , and no senior level worker and $J_{01}(y|\bar{x}_i, \bar{x})$ is the present value for a firm with only one senior level worker, whose experience is y , and no junior worker. If the firm first finds an inexperienced worker, which happens at rate q_1 it moves to the state $J_{10}(0)$, since we know that $x = 0$. In contrast, if the firm first finds a senior manager which happens at rate q_2 it moves to the state $J_{01}(\bar{x})$ since we know that all managers in the senior market have experience \bar{x} .⁵

To determine $J_{10}(x|\bar{x}_i, \bar{x})$ let $J_{11}^N(x, y|\bar{x}_i, \bar{x})$ be the present value of profits for a firm with a worker, whose experience is x , and a manager with experience y . Note that both value functions indirectly depend on the promotion cut-off \bar{x}_i chosen by firm i and in the market experience level \bar{x} chosen by competing firms. Recall that $\pi_1(x) = (d_1 + c_1 e^{\gamma x})$ and $\pi_2(y) = (d_2 + c_2 e^{\gamma y})$ denote the flow profits obtained by the firm from a filled junior and senior position respectively. The present value $J_{10}(x|\bar{x}_i, \bar{x})$ is given by the following equation:

$$rJ_{10}(x|\bar{x}_i, \bar{x}) = \pi_1(x) - s + q_2 \left(J_{11}^N(x, \bar{x}|\bar{x}_i, \bar{x}) - J_{10}(x|\bar{x}_i, \bar{x}) \right) + \frac{\partial J_{10}(x|\bar{x}_i, \bar{x})}{\partial x}. \quad (10)$$

The firm receives a flow profit $\pi_1(x)$ by employing a junior worker accumulating experience x . In addition, the firm pays a flow cost s for posting a vacancy in the market for experienced workers. At rate q_2 the firm is successful in this market and moves to the state $J_{11}^N(x, \bar{x}|\bar{x}_i, \bar{x})$, where \bar{x} is the market level of experience set by other firms and guaranteeing workers' eligibility for senior positions. For the ease of exposition in the following, we use $J_{10}(x)$ for $J_{10}(x|\bar{x}_i, \bar{x})$ and $J_{11}^N(x, y)$ for $J_{11}^N(x, y|\bar{x}_i, \bar{x})$ and omit the indirect dependence on $\{\bar{x}_i, \bar{x}\}$ in other value functions. We come back to the explicit notation when we determine the optimal promotion time \bar{x}_i^* of firm i and the equilibrium value of \bar{x} in the end of this section.

Next consider the present value $J_{11}^N(x, y)$, where x is the current experience of the worker in the junior position and y is the constant experience level of the manager. Note that $y = \bar{x}$ if the manager was hired in the market but it can be different from \bar{x} if the manager was promoted within the firm:

$$rJ_{11}^N(x, y) = \pi_1(x) + \Delta + \pi_2(y) - \rho \left(J_{11}^N(x, y) - J_{10}(x) \right) + \frac{\partial J_{11}^N(x, y)}{\partial x}. \quad (11)$$

Here the firm receives additional profit Δ from teamwork, but may lose the manager due to retirement which happens at rate ρ .

In the next step, we write down the equation for $J_{01}(y)$, which is the present value of profits for a firm with only one senior manager, whose experience level is y :

$$rJ_{01}(y) = \pi_2(y) - \rho J_{01}(y) - s + q_1 \left(J_{11}^N(0, y) - J_{01}(y) \right).$$

The firm receives the flow profit $\pi_2(y) = (d_2 + c_2 e^{\gamma y})$ generated by the manager and is continuously posting a vacancy in the market for junior workers, which is associated with a flow cost s . At rate q_1 the firm is successful in this market and moves to the state $J_{11}^N(0, y)$. This is because applicants to junior positions are young and inexperienced with $x = 0$. Finally, at rate ρ the firm may lose the senior manager and exit the market.

The last state for the firm is when the junior worker has already accumulated experience necessary for promotion. Recall that \bar{x}_i denotes promotion cut-off of an arbitrary firm i . This means that the junior worker obtains a certificate of completed training and becomes eligible for senior positions having accumulated experience \bar{x}_i . This promotion cut-off is chosen by the firm upon signing the employment contract. If the senior position is open in firm i , the worker with $x = \bar{x}_i$ is promoted immediately. However, it is also possible that the senior position is occupied, so the worker starts searching for alternative employment. Let $J_{11}^S(\bar{x}_i, y)$ be the present value of profits for a firm with a searching worker whose experience is (\bar{x}_i) and a manager (y) :

$$rJ_{11}^S(\bar{x}_i, y) = \pi_1(\bar{x}_i) + \Delta + \pi_2(y) - \rho \left(J_{11}^S(\bar{x}_i, y) - J_{01}(\bar{x}_i) \right) - \lambda_2 \left(J_{11}^S(\bar{x}_i, y) - J_{01}(y) \right).$$

This equation shows the following. The firm obtains the flow profit generated by both workers $\pi_1(\bar{x}_i) + \pi_2(y)$ and additional profit Δ from teamwork. At rate ρ the manager may retire, so the searching worker is promoted to the senior position and the firm moves to the state $J_{01}(\bar{x}_i)$. Alternatively, it may happen that the worker finds alternative employment and quits at rate λ_2 . In this case, the firm is left with only one manager and the present value of profits is $J_{01}(y)$.

Summarizing, there are five different states for every firm with the corresponding values $J_{00}(\bar{x}_i, \bar{x})$ for a new firm, $J_{10}(x, y)$ and $J_{11}^N(x, y)$ for firms with learning young workers as well as $J_{11}^S(\bar{x}_i, y)$ and $J_{01}(y)$ for firms with experienced workers. The two present values $J_{10}(x, y)$ and $J_{11}^N(x, y)$, where young workers are still accumulating experience x necessary for a promotion are given by the system of first-order linear differential Equations (10) and (11). In Proposition 2 we characterize the general solution of this system for the case when the senior manager was hired from the market and has experience $y = \bar{x}$:

Proposition 2. *The present value of profits $J_{10}(x)$ for a firm with a junior worker, and the present value of profits $J_{11}^N(x, \bar{x})$ for a firm with a junior worker and a senior manager hired from the market are given by:*

$$J_{10}(x) = \frac{d_1 - s}{r} + q_2 \frac{(\pi_2(\bar{x}) + \Delta + s)}{r(r + \rho + q_2)} + Ae^{rx} + \frac{c_1 e^{\gamma x}}{r - \gamma} - \frac{q_2 K e^{(r + \rho + q_2)x}}{\rho + q_2},$$

$$J_{11}^N(x, \bar{x}) = \frac{(\pi_2(\bar{x}) + \Delta + s)(r + q_2)}{r(r + \rho + q_2)} + \frac{\rho K e^{(r + \rho + q_2)x}}{\rho + q_2} + \frac{d_1 - s}{r} + Ae^{rx} + \frac{c_1 e^{\gamma x}}{r - \gamma},$$

where variables A and K are the integration constants and can be found from the two boundary conditions: $J_{10}(\bar{x}_i) = J_{01}(\bar{x}_i)$ and $J_{11}^N(\bar{x}_i, \bar{x}) = J_{11}^S(\bar{x}_i, \bar{x})$. Detailed expressions for A and K are presented in the appendix.

Proof. Appendix. □

Proposition 2 illustrates general solutions for the present values of profits $J_{10}(x)$ and $J_{11}^N(x, \bar{x})$. In addition, it shows that the two integration constants A and K , and therefore, the two specific solutions can be derived from the two boundary conditions. The first condition says that firms are committed to promote the worker upon experience \bar{x}_i if the senior position is open, so the present value of the firm changes from $J_{10}(\bar{x}_i)$ to $J_{01}(\bar{x}_i)$. The second condition says that workers with experience \bar{x}_i stop accumulating experience and start searching for alternative jobs at \bar{x}_i if the senior position is filled, so the present value of the firm is changing from $J_{11}^N(\bar{x}_i, \bar{x})$ to $J_{11}^S(\bar{x}_i, \bar{x})$. Solving these two boundary conditions for A and K we show in the appendix that both variables depend on the individual decision of firm i and on the behavior of other firms \bar{x} , that is $A(\bar{x}_i, \bar{x})$ and $K(\bar{x}_i, \bar{x})$.

Based on the above analysis we can now write the firm's optimization problem (9) as

$$\bar{x}_i^*(\bar{x}) = \arg \max_{\bar{x}_i \geq 0} [q_1 J_{10}(0|\{\bar{x}, A(\bar{x}_i, \bar{x}), K(\bar{x}_i, \bar{x})\}) + q_2 J_{01}(\bar{x}|\{\bar{x}, A(\bar{x}_i, \bar{x}), K(\bar{x}_i, \bar{x})\})],$$

where we show explicitly the arguments of functions $J_{10}(0)$ and $J_{01}(\bar{x})$. The solution of this maximization problem gives the optimal response function $\bar{x}_i(\bar{x})$ of firm i . Since firms are homogeneous with respect to their profit functions, they all have identical optimal response functions. In light of this in what follows we restrict our attention to symmetric Nash equilibria and impose the equilibrium condition $\bar{x}_i^*(\bar{x}) = \bar{x}$ to find the equilibrium promotion time \bar{x} .

3.2 | Partial and general equilibrium

The complexity of the expressions derived for J_{10} and J_{01} makes an explicit analytical characterization of the best response function and the resulting equilibrium infeasible, even if we consider a partial equilibrium with fixed transition rates. Therefore, we illustrate the main properties of the best response function and the equilibrium by extending the calibration of our model developed in Section 2.3 and carrying out a numerical analysis. First, we consider a partial equilibrium framework with fixed transition rates $\{q_1, q_2, \lambda_1, \lambda_2\}$, with the corresponding values from Table 1. We choose the annual discount rate equal to 4%, so that $r = 0.01$. Next we normalize the slope parameter $c_2 = 1$. Parameters $d_2 < d_1$ and $c_2 > c_1$ are calibrated so that $\bar{x} = 48$, corresponding to a promotion time of 12 years, is an equilibrium outcome of the overall model. Even though it is an endogenous variable in the complete model, we keep it fixed in this section and analyze the optimal response of a single firm i . The flow cost of an open vacancy is set low ($s = 0.1$). Later we consider a variation in this parameter in the range $[0.06...0.1]$. We start with a benchmark value $\Delta = 0$ and postpone the analysis of production complementarities to the next section.

The multiplier parameter γ can be seen as a proxy for the return to worker's tenure. We set $\gamma = 0.003$ which corresponds to the rate of return to tenure at 1.2% per year. This estimate is in the middle range of the estimates observed in developed economies (Farber, 1999; Bingley & Westergaard-Nielsen, 2003; Iftikhar & Zaharieva, 2019). The second set of parameters relevant for the present value equations is summarized in Table 3.

Figure 3 shows the objective function of firm i — $J_{00}(\bar{x}_i)$ —for a fixed market promotion time $\bar{x} = 48$ and for fixed transition rates $\{q_1, q_2, \lambda_1, \lambda_2\}$ (left panel). We can see that promoting junior workers too early is not optimal for the firm. This is despite the fact that $d_1 + c_1 < d_2 + c_2$, which means that the flow profit of the firm is higher in the senior

TABLE 3 Values of exogenous parameters

	Value	Interpretation		Value	Interpretation
γ	0.003	Quarterly return to tenure	s	0.100	Flow cost of an open vacancy
c_1	0.250	Slope parameter, level 1	c_2	1.000	Slope parameter, level 2
d_1	0.100	Intercept parameter, level 1	d_2	0.050	Intercept parameter, level 2

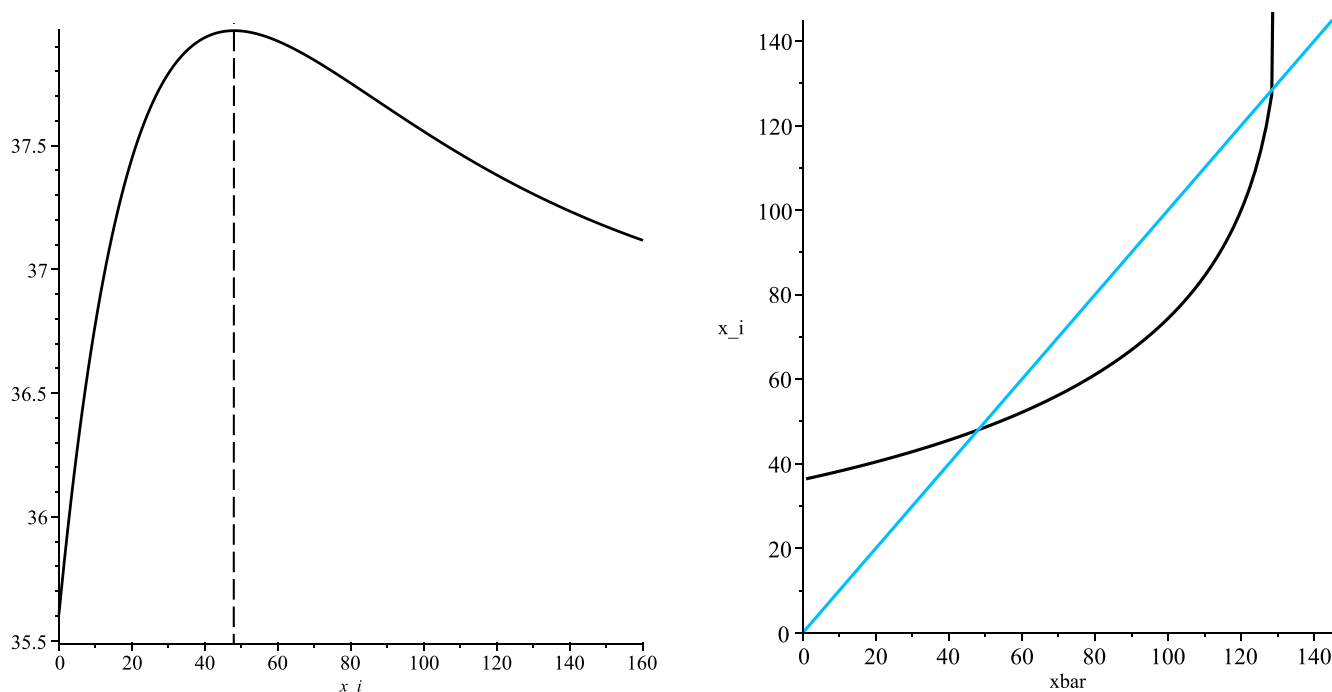


FIGURE 3 Left panel: Objective function of firm i and the optimal choice $\bar{x}_i^*(\bar{x})$ for a fixed market promotion cut-off $\bar{x} = 48$ and fixed transition rates. Right panel: Optimal response function $\bar{x}_i^*(\bar{x})$ for different values of \bar{x} . [Color figure can be viewed at wileyonlinelibrary.com]

position even if the worker does not possess any managerial experience and $x = 0$. So there is an indirect cost for the firm of assigning the worker to the training program in a junior position. The reason is that firms are forward-looking and anticipate a larger gain from promotion once the worker accumulated sufficient managerial experience. At the same time waiting too long is also suboptimal for the firm because the foregone profit associated with a senior position is increasing over time. This is the indirect cost of delayed promotions. In addition, there is the direct flow cost of an open vacancy in the senior position s . As can be clearly seen for our considered parameter values the optimal promotion time is $\bar{x}_i^*(48) = 48$.

The right panel of Figure 3 shows the optimal response function $\bar{x}_i^*(\bar{x})$ for different values of the market promotion time \bar{x} and fixed transition rates (black solid curve). We can see that firm i has strong incentives to delay promotions if other firms in the market promote their junior workers later. Higher \bar{x} implies that managers applying externally are more experienced, so the quality of the candidate pool in the managerial market is better. In this situation it is optimal for firm i to wait longer because the marginal gain from waiting is increasing with \bar{x} due to the better quality of external candidates. Hence, we find that there is strategic complementarity between the promotion times of the different firms in the market.

The same figure shows that $\bar{x}_i^* = \bar{x}^{pe} = 48$ for all firms i is a symmetric partial equilibrium of the model for the given transition rates (values from Table 1). But it is not unique, since there exists a second partial equilibrium with $\bar{x}_h^{pe} = 128.6$ for these transition rates. In light of the strategic complementarity between the optimal promotion times of the firms it is not surprising that multiple equilibria exist in our model, however, only the low equilibrium is strategically stable. Any best response dynamics initialized with a market promotion level $\bar{x} \in [0, \bar{x}_h^{pe})$ converges to the lower equilibrium $\bar{x}_l^{pe} = 48$.

In Section 2.3 we have shown that if all firms use a promotion threshold of $\bar{x} = 48$, then the transition rates under the stationary distribution are given by $\{q_1 = 0.0212, q_2 = 0.0046, \lambda_1 = 0.0145, \lambda_2 = 0.0146\}$ (see Table 1). Since these are exactly the transition rates under which we have carried out the partial equilibrium analysis above, it follows directly that $\bar{x}_i^* = \bar{x}_i^{pe} = 48, i \in [0, 1]$ is also a general equilibrium of the model. Similarly to the partial equilibrium setting, also with endogenous transition rates a second equilibrium with a very high promotion threshold exists, which however is unstable. Hence in what follows we focus on the lower equilibrium and in the following section examine how the equilibrium promotion threshold changes in response to a variation of key parameters in the model.

3.3 | Comparative statics: Partial and general equilibrium effects

Based on the numerical solution of a unique stable symmetric general equilibrium characterized in the previous section we now address two key questions of our study: (1) how promotion chances of junior workers are affected if there exist production complementarities and synergies from the team work and (2) what is the link between the optimal promotion time and the skill level of the worker? The third key question concerning the impact of competition on promotion strategies of firms is conducted in the next section.

To address the first question, we gradually increase the synergy parameter Δ , which was fixed at 0 in the benchmark case. This is illustrated on the left panel of Figure 4. If the synergy parameter is increasing from 0 to 0.6 the promotion cut-off \bar{x}^{ge} in the general equilibrium is decreasing from 48 down to 46.5. Stronger complementarities in the production process create stronger incentives for firms to employ a full team of two employees rather than having open vacancies. In our setting the job-filling rate in the junior market $q_1 = 0.0212$ is substantially higher than the job-filling rate in the senior market $q_2 = 0.0046$ which means that hiring junior workers is easier than senior managers. In this situation, firms prefer earlier promotions of junior employees in the hope that the junior position will be filled faster than the senior position and the firm can gain additional profits from the team production process. Note that this gain comes at the expense of accepting less experienced senior managers.

Further, we decompose this effect into three parts. We write the individually optimal promotion threshold $\bar{x}_i^*(\bar{x}, \zeta, \Delta)$ as a function of the market promotion level \bar{x} as well as the vector of transition rates ζ and the synergy parameter Δ . Furthermore, $\bar{x}^{pe}(\zeta, \Delta)$ denotes the (partial) equilibrium market cutoff under transitions rates ζ and ζ^Δ

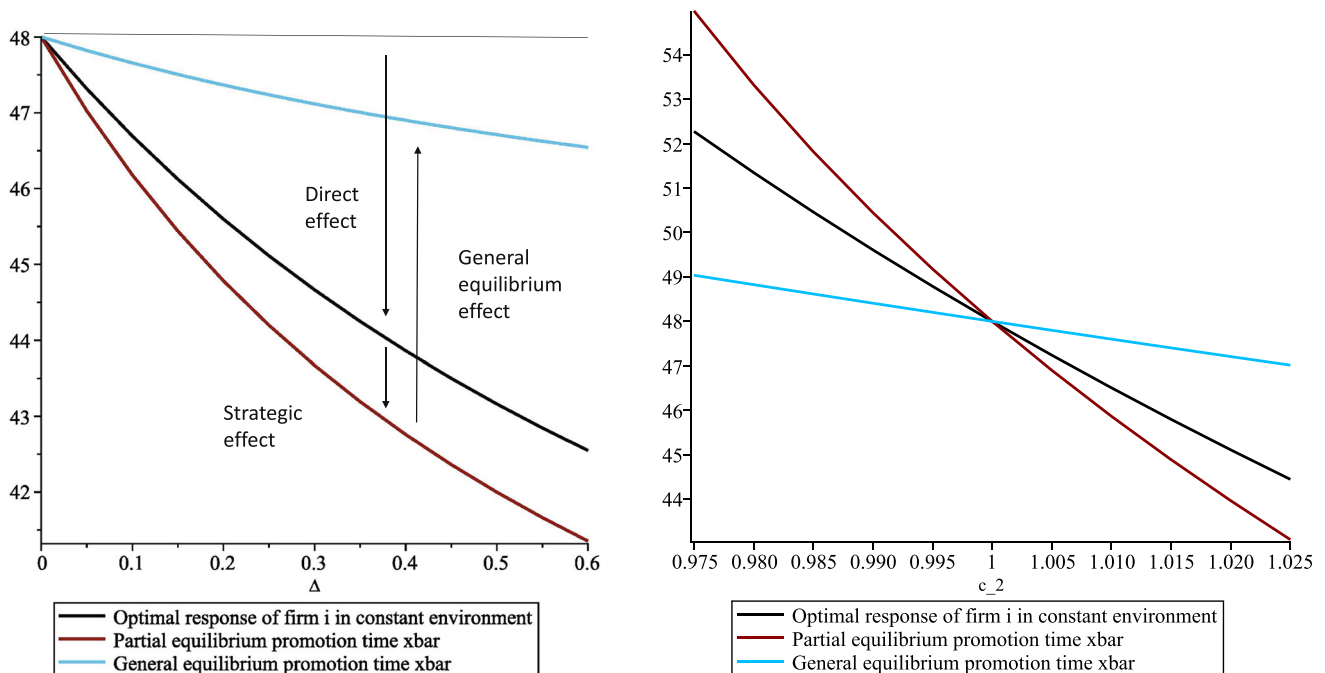


FIGURE 4 Left panel: Equilibrium promotion time $\bar{x}_i(\cdot)$ as a function of the synergy parameter Δ . Right panel: Equilibrium promotion time $\bar{x}_i(\cdot)$ as a function of the skill parameter c_2 . [Color figure can be viewed at wileyonlinelibrary.com]

the general equilibrium transition rates for the synergy parameter Δ . The general equilibrium cutoff under synergy Δ is then denoted as $\bar{x}^{ge}(\Delta) := \bar{x}^{pe}(\zeta^\Delta, \Delta)$. Hence $\bar{x}_i^*(\bar{x}^{pe}(\zeta^0, 0), \zeta^0, 0) = \bar{x}^{pe}(\zeta^0, 0) = \bar{x}^{ge}(0) = 48$. Using this notation we obtain the following decomposition of the effect of a change in Δ :

$$\begin{aligned} \bar{x}^{ge}(0) - \bar{x}^{ge}(\Delta) &= \bar{x}^{pe}(\zeta^0, 0) - \bar{x}^{pe}(\zeta^\Delta, \Delta) = \bar{x}_i^*(\bar{x}^{pe}(\zeta^0, 0), \zeta, 0) - \bar{x}_i^*(\bar{x}^{pe}(\zeta^\Delta, \Delta), \zeta^\Delta, \Delta) = \\ &= \underbrace{\left[\bar{x}_i^*(\bar{x}^{pe}(\zeta^0, 0), \zeta^0, 0) - \bar{x}_i^*(\bar{x}^{pe}(\zeta^0, 0), \zeta^0, \Delta) \right]}_{\text{Direct effect}} \\ &\quad + \underbrace{\left[\bar{x}_i^*(\bar{x}^{pe}(\zeta^0, 0), \zeta^0, \Delta) - \bar{x}_i^*(\bar{x}^{pe}(\zeta^0, \Delta), \zeta^0, \Delta) \right]}_{\text{Strategic effect}} \\ &\quad + \underbrace{\left[\bar{x}_i^*(\bar{x}^{pe}(\zeta^0, \Delta), \zeta^0, \Delta) - \bar{x}_i^*(\bar{x}^{pe}(\zeta^\Delta, \Delta), \zeta^\Delta, \Delta) \right]}_{\text{General equilibrium effect}} \end{aligned}$$

First, Figure 4 (left panel) shows the *direct effect*, this is a change in the optimal promotion time of firm i as a function of Δ in a setting with constant environment. As we can see from the figure, the firm has very strong incentives to promote earlier. If the synergy parameter is increasing from 0 to 0.6 the optimal promotion cut-off of firm i is decreasing from 48 down to 42.5 (black curve). So the direct effect for $\Delta = 0.6$ is equal to $5.5 = 48 - 42.5$. Second, we allow for changes in the behavior of competing firms $\bar{x}^{pe}(\zeta^0, \Delta)$ but keep the set of transition rates ζ^0 fixed. This is the *strategic effect* reflecting strategic competition between firms. We already know from Figure 3 that earlier promotions by the competitors lead to earlier promotions of firm i . This is illustrated by the red curve on Figure 4. If the synergy parameter is increasing from 0 to 0.6 and the firm takes earlier promotions of competitors into account, then the optimal promotion cut-off is decreasing even stronger from 42.5 down to 41.4, so the strategic effect is equal to $1.1 = 42.5 - 41.4$. It makes promotions more sensitive to the production complementarity Δ . The sum of these two effects would be observed in a partial equilibrium setting, in which the transition rates are kept constant. Third, we analyze the *general equilibrium effect* and allow for the endogenous changes in the transition rates. From Figure 2 we already know that if all firms set earlier promotion times then q_1 is decreasing and q_2 is increasing. Intuitively, this means that earlier promotions make it easier for firms to hire senior managers but hiring junior workers becomes more difficult. This general equilibrium effect mitigates the incentives of firm i to promote earlier and makes promotions less sensitive to the production complementarity Δ . The general equilibrium effect is illustrated by the blue curve and is equal to $-5.1 = 41.4 - 46.5$. Based on this decomposition we can conclude that both the direct effect (5.5) and the general equilibrium effect (-5.1) are larger in absolute terms than the strategic effect (1.1) in our setting.

Next, we turn to the effect of worker skills/education. We proxy this effect by changes in the parameter c_2 . The intuition behind this proxy is that more educated workers with higher skills will be more productive in senior positions than low-skill workers even if they have similar practical experience. This is due to the methodological competence, broader knowledge, and problem-solving skills associated with higher education. Following this logic we assume that higher c_2 corresponds to the labour market with more educated workers but there are no productivity differences in junior jobs (c_1). The right panel of Figure 4 shows changes in the promotion times where $c_2 = 1$ is the benchmark case in the middle of the figure. We can see that higher education generally leads to earlier promotions. The effects are reversed when the labour force is less qualified: if c_2 is decreasing from 1 to 0.975, firm i responds by setting the promotion time equal to 52.3 in a constant environment. If all competitors follow the same strategy and set longer promotion times the partial equilibrium is achieved at $\bar{x}_i^*(\bar{x}^{pe}) = \bar{x}^{pe} = 55.0$. The decomposition reveals again that the general equilibrium effect dampens the direct effect of the parameter change on the optimal promotion time and makes it less sensitive to the education parameter. We obtain for $c_2 = 0.975$ a general equilibrium cut-off of $\bar{x}^{ge} = 49.0$. Even though this result is suggestive of the positive link between education and the speed of promotions, it is only a comparative statics result and it is not clear if it will be confirmed in a setting where two skill types are mixed in the same labour market. We provide this analysis in Section 4.

3.4 | Number of firms and competition

In this section, we analyze the impact of stronger competition in the labour market on the equilibrium promotion time \bar{x} . We capture the effect of stronger competition by increasing the entry of firms n . Given that the exit of firms is driven

by the exogenous retirement of their employees ρ , a more intensive firm entry leads to a larger number of firms in the steady state F . In Figure 5 we illustrate the relationship between the equilibrium promotion cutoff $\bar{x} = \bar{x}^{ge}$ and the number of firms in the market (F) by varying F in the range $[0.35...0.57]$.⁶ We can see that the effect is non-monotone: the optimal promotion time is increasing (decreasing) if the degree of competition among firms is low (high) with a maximum point $\bar{x} = 52.7$ at $F = 0.44$. Recall that the benchmark case is $F = 0.55$, that is there are 550 two-position firms per 1000 workers.

The right panel of Figure 5 reveals the underlying mechanism. It shows a response of the job-filling rates to a larger number of firms F . We can see that both rates are declining. The reason is that a larger number of firms leads to a larger stock of positions and vacancies in both submarkets, worsening the hiring chances of every single firm. This theoretical experiment resembles the situation in Germany and in the United States in the last decade, which we mentioned in Section 1, namely, that filling vacancies becomes increasingly more difficult for firms. From Figure 2 (right panel) we already know that lower q_1 leads to delayed promotions of workers, and at the same time a lower rate q_2 leads to earlier promotions. When the number of firms in the market is small and increasing the job-filling rate q_1 drops sharply, whereas changes in q_2 are rather moderate, so the impact of lower q_1 is dominating when the degree of competition is low leading to delayed promotions. On the contrary, when the degree of competition is high the job-filling rate q_1 exhibits moderate declines, whereas q_2 becomes extremely small. This means that the expected search cost s/q_2 becomes extremely high. In this situation making promotions earlier reduces the pressure on firms associated with hiring in the senior submarket. This explains the non-monotone relationship between the optimal promotion time and the number of firms.

In addition to the effect of competition, the left part of Figure 5 shows comparative statics with respect to the flow cost of applicant search s . We vary s between 0.06 and the benchmark value 0.1. We can see that a higher search cost s always leads to earlier promotions. Even though a higher search cost s has a negative impact on the profits of firms on both submarkets, the expected cost s/q_j , $j = 1, 2$, is increasing stronger in the senior submarket making unfilled senior positions more costly relatively to the open junior positions and leading to earlier promotions. At the same time, the impact of the search cost s is negligibly small when competition is weak, but it becomes more substantial when firms compete stronger for applicants. This is intuitive, since the search cost is less relevant when the probability of filling an open position is high but it becomes more important when firms are struggling to fill their vacancies.

Our results from this section suggest that firms promote their employees later and let them accumulate more experience when the pressure of competition and the search costs are low. On the contrary, firms use earlier promotions when the degree of competition and the search costs are high. Earlier promotions in this case also mean

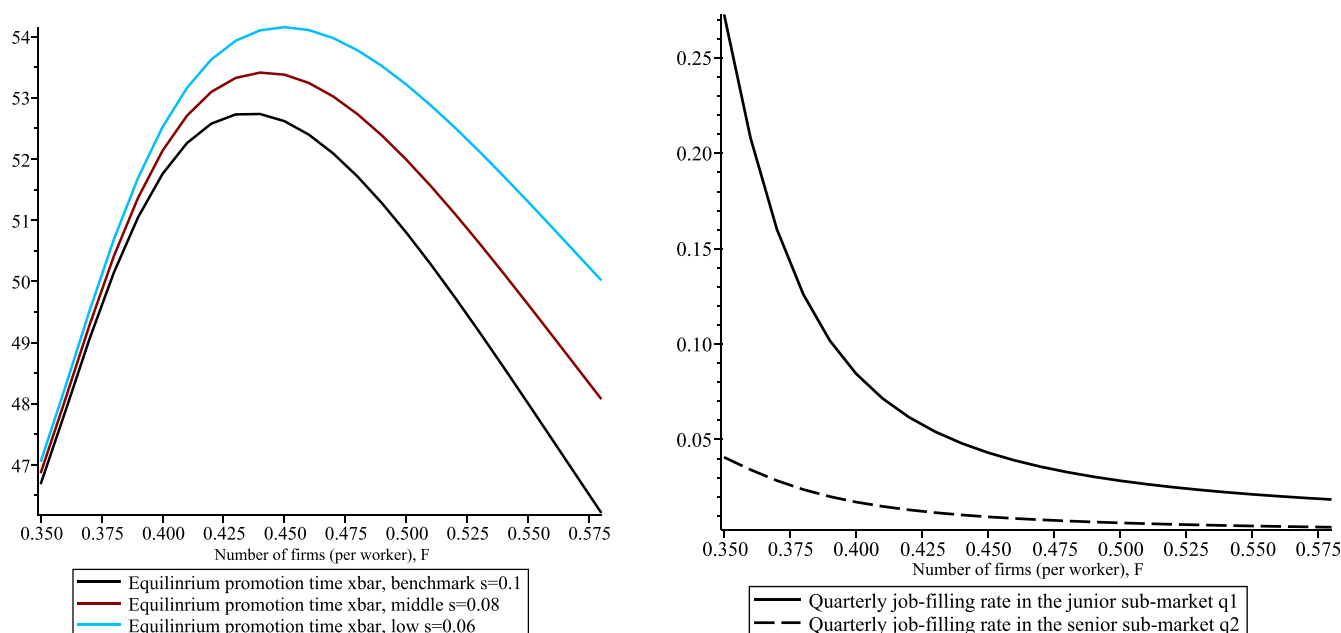


FIGURE 5 Left panel: The equilibrium promotion time \bar{x} as a function of the number of firms F and the search cost s . Right panel: Quarterly job-filling rates q_1 and q_2 . [Color figure can be viewed at wileyonlinelibrary.com]

that workers obtain less training on-the-job, thus their experience in the senior level is lower. Therefore, one important finding stemming from our model is that strong competition among firms in the labour market may force them to promote workers with relatively low experience to senior managerial positions. In the next section, we extend the model to a setup with two different worker groups and investigate whether our results continue to hold in the presence of worker heterogeneity. Moreover, this extension allows us to study the impact of competition and promotions on wage inequality, which is not possible in a model with homogeneous workers.

4 | TWO SKILL LEVELS

4.1 | Optimal promotion with two skill levels

In this section, we extend the model to a setting with two skill groups and analyze the spillover effects that the presence of one skill group imposes on the other group. To keep the model tractable, we refrain from the synergy effect and set $\Delta = 0$ throughout this extension. Let c_2^L be the education parameter of low-skill workers. Once employed in the senior job, they generate the flow profit $\pi_2^L(x) = (d_2 + c_2^L e^{x^s})$ for the firm. Further, $c_2^H > c_2^L$ denotes the education parameter of high-skill workers, so they generate the flow profit $\pi_2^H(x) = (d_2 + c_2^H e^{x^s})$. We assume that the difference between c_2^L and c_2^H is sufficiently small so that firms do not reject low-skill applicants. Moreover, c_1 remains the same for both worker groups, indicating that high and low-skill workers are equally productive when performing junior-level jobs. It is the difference in managerial abilities that we want to capture in this extension. Let a denote the fraction of low-skill workers in the population. Variables \bar{x}_i^L and \bar{x}_i^H denote the promotion times set by firm i for each skill group respectively. As before, this decision is made upon entry and there is full commitment on the side of the firm.

Further, let α_1 denote the fraction of low-skill applicants in the junior market and α_2 be the fraction of low-skill applicants in the senior market. Both variables are closely related to the share of low-skill workers in the population a but can be larger or smaller than a . The entering firm solves the optimization problem

$$\begin{aligned} \{\bar{x}_i^{L*}, \bar{x}_i^{H*}\} = \arg \max_{\bar{x}_i^L, \bar{x}_i^H} & \left[\alpha_1 J_{L0} \left(0 | \{\bar{x}_i^L, \bar{x}_i^H, \bar{x}_L, \bar{x}_H\} \right) + (1 - \alpha_1) J_{H0} \left(0 | \{\bar{x}_i^L, \bar{x}_i^H, \bar{x}_L, \bar{x}_H\} \right) \right] \\ & + q_2 \left[\alpha_2 J_{0L} \left(\bar{x}_L | \{\bar{x}_i^L, \bar{x}_i^H, \bar{x}_L, \bar{x}_H\} \right) + (1 - \alpha_2) J_{0H} \left(\bar{x}_H | \{\bar{x}_i^L, \bar{x}_i^H, \bar{x}_L, \bar{x}_H\} \right) \right], \end{aligned} \quad (12)$$

where $\{\bar{x}_i^{L*}, \bar{x}_i^{H*}\}$ denote the optimal choices, \bar{x}_j is the market experience level of applicants in the managerial market with a skill level $j = L, H$, J_{j0} is a firm with an inexperienced worker of skill $j = L, H$ and an open senior vacancy while J_{0f} is a firm with a senior worker of skill $f = L, H$ and a junior vacancy. The corresponding Bellman equations and the solution procedure for the two skill level case are shown in the appendix. As before, we consider symmetric equilibria, so that $\bar{x}_i^{L*}(\bar{x}_L, \bar{x}_H) = \bar{x}_L$ and $\bar{x}_i^{H*}(\bar{x}_L, \bar{x}_H) = \bar{x}_H$, which guarantees that firms do not have incentives to deviate.⁷

4.2 | Partial equilibrium

To illustrate the implications of skill heterogeneity for our results, we first consider again a partial equilibrium framework with fixed transition rates from Table 1. We set $c_2^L = 0.975$ and $c_2^H = 1.025$, so that high-skill workers are more productive than low-skill workers in senior jobs. We start with a situation where $\alpha_1 = \alpha_2 = 0.8$, which implies that 80% (20%) of workers in both the junior and senior market are low (high) skilled. This roughly corresponds to the share of university/college master graduates in developed economies, for example, this share was 13.04% of the population aged 25 and above in the United States in 2018.⁸ Figure 6 shows the objective function of the firm for the default transition rates. We find that the partial equilibrium is achieved for $\bar{x}_L^{pe} = 65.4$ and $\bar{x}_H^{pe} = 24.4$, which implies that high-skill workers are promoted much earlier than low-skill workers. Intuitively, a firm with a low-skill worker in a junior position has a strong incentive to delay the promotion of this worker because this delay increases the chance for the firm to hire a high-skill worker from the market for the senior position. Quite on the contrary, if the junior

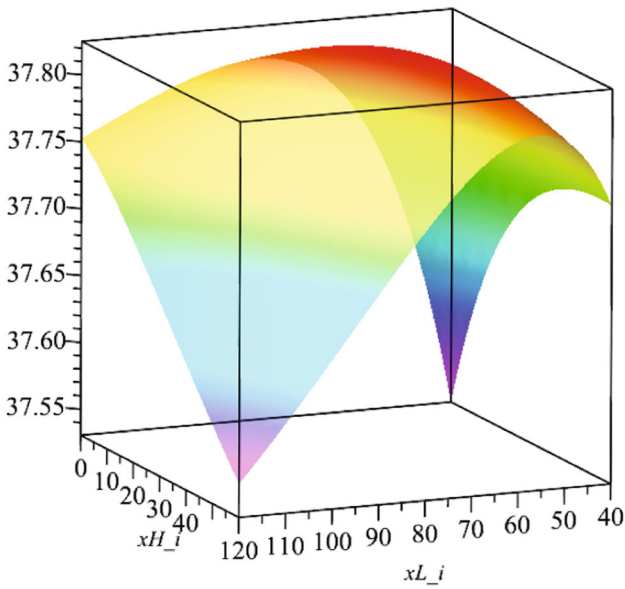


FIGURE 6 Two-dimensional objective function of the firm in the space $\{\bar{x}_i^L, \bar{x}_i^H\}$ for $\alpha_1 = \alpha_2 = 0.8$ and market promotion cut-offs of $\bar{x}_L = 65.4, \bar{x}_H = 24.4$. [Color figure can be viewed at wileyonlinelibrary.com]

worker has high skills, then it is profitable for the firm to exploit these skills in the senior position rather than hiring from the market which comes at the risk of putting a low-skill worker into the senior position. The fact that high skill workers are promoted faster is supported by the empirical evidence (Baker et al., 1994; Lluís, 2005; McCue, 1996) and indicates a proper setup of our model.

If we increase $\alpha_1 = \alpha_2$ to 0.9 we find the equilibrium promotion cut-offs $\bar{x}_L^{pe} = 59.8, \bar{x}_H^{pe} = 22.5$, thus both types of workers are promoted earlier. In the limiting case when $\alpha_1 = \alpha_2 = 1$ we arrive at the economy with only low-skill workers with productivity level $c_2^L = 0.975$ and the corresponding equilibrium threshold is $\bar{x}_L^{pe} = 49$ (see Section 3.3). Hence, a lower average skill level in the labour force (due to a larger share of low-skill workers) is associated with earlier promotions of both worker groups in a partial equilibrium setting. In the next section, we check if this result will persist after the general equilibrium adjustment in the transition rates.

4.3 | General equilibrium

To find the general equilibrium in the model with heterogeneous skills, we need to characterize the steady-state distributions of firms and workers. Let d_{ij}^f be a firm with a junior worker with skill level $i = L, H$ and a senior worker with skill $j = L, H$. In case one of the positions is vacant, the corresponding subscript becomes 0 as in the benchmark model. Also, $f = N, S$ indicates whether the junior worker is already searching for a job on the senior market or not and this superscript is dropped if the junior position is vacant. Further, let e_0^L denote the stock of low-skilled inexperienced workers searching for their first professional job. Similarly, let e_0^H denote the stock high skill workers who are not yet employed in a hierarchical firm. Re-writing the equations for the transition rates to take into account worker skill heterogeneity then yields the following expressions for the vacancy filling and job finding rates:

$$q_1 = 1 - \left(1 - \frac{1}{d_{00} + d_{0L} + d_{0H}}\right)^{z(e_0^L + e_0^H)} \quad \lambda_1 = q_1 \frac{(d_{00} + d_{0L} + d_{0H})}{e_0^L + e_0^H},$$

$$q_2 = 1 - \left(1 - \frac{1}{d_{00} + d_{L0} + d_{H0}}\right)^{z(d_{LL}^S + d_{LH}^S + d_{HH}^S + d_{HL}^S)} \quad \lambda_2 = q_2 \frac{(d_{00} + d_{L0} + d_{H0})}{d_{LL}^S + d_{LH}^S + d_{HH}^S + d_{HL}^S}.$$

As mentioned above, α_2 is the fraction of low skill applicants on the senior market and α_1 is the fraction of low skill applicants on the junior market. These rates can be obtained as:

$$\alpha_1 = \frac{e_0^L}{e_0^L + e_0^H} \quad \alpha_2 = \frac{d_{LL}^S + d_{LH}^S}{d_{LL}^S + d_{LH}^S + d_{HH}^S + d_{HL}^S}.$$

Recall that a is the number of low-skill workers in the population. This means that $a = e_0^L + d_{L0} + 2d_{LL}^N + d_{LH}^N + d_{HL}^N + 2d_{LL}^S + d_{LH}^S + d_{HL}^S + d_{0L}$. As in our benchmark model total population is normalized to 1 such that $e_0^H + d_{H0} + 2d_{HH}^N + d_{HL}^N + d_{LH}^N + 2d_{HH}^S + d_{HL}^S + d_{LH}^S + d_{0H} = 1 - a$, that is, the number of high skill workers equals $1 - a$. The steady-state distribution of firms and workers is found by setting $\dot{d}_{ij}^f = 0$, $i = 0, L, H, j = 0, L, H, f = N, S$. The total number of firms F is given by:

$$d_{00} + d_{L0} + d_{H0} + d_{LL}^N + d_{LH}^N + d_{HL}^N + d_{HH}^N + d_{LL}^S + d_{LH}^S + d_{HL}^S + d_{HH}^S + d_{0L} + d_{0H} = F.$$

The differential equations for all types of firms and the steady-state solution are described in detail in the appendix.

In Table 4 the general equilibrium promotion strategies, the corresponding transition rates and the senior level employment are displayed for different fractions of low-skill workers (a) in the population and for different numbers of firms (F). We find that the equilibrium is compatible with the following competition range $F \in [0.47...0.57]$. If the

TABLE 4 Optimal promotion thresholds $\{\bar{x}_L, \bar{x}_H\}$ varying the number of low skill workers in the population a and the number of firms F

		$a = 0.9$	$a = 0.8$
$F = 0.475$	$\{\bar{x}_L, \bar{x}_H\}$	{58.4, 13.5}	{65.2, 16.1}
	ζ^{ge}	$\alpha_1 = 0.8802$; $\alpha_2 = 0.8369$	$\alpha_1 = 0.7624$; $\alpha_2 = 0.6898$
		$q_1 = 0.03498$; $q_2 = 0.00747$	$q_1 = 0.03564$; $q_2 = 0.00751$
	Senior level	$e^{2L} = 0.318$; $e^{2H} = 0.390$	$e^{2L} = 0.309$; $e^{2H} = 0.385$
	Wages	$\bar{w}_L = 1.550$; $\bar{w}_H = 1.713$	$\bar{w}_L = 1.541$; $\bar{w}_H = 1.705$
	Wage gap	10.5%	10.6%
$F = 0.5$	$\{\bar{x}_L, \bar{x}_H\}$	{56, 20.3}	{60.99, 22.4}
	ζ^{ge}	$\alpha_1 = 0.8844$; $\alpha_2 = 0.8510$	$\alpha_1 = 0.7706$; $\alpha_2 = 0.7132$
		$q_1 = 0.02877$; $q_2 = 0.00617$	$q_1 = 0.02895$; $q_2 = 0.00619$
	Senior level	$e^{2L} = 0.324$; $e^{2H} = 0.381$	$e^{2L} = 0.316$; $e^{2H} = 0.377$
	Wages	$\bar{w}_L = 1.560$; $\bar{w}_H = 1.704$	$\bar{w}_L = 1.552$; $\bar{w}_H = 1.697$
	Wage gap	9.2%	9.3%
$F = 0.525$	$\{\bar{x}_L, \bar{x}_H\}$	{54, 23.3}	{58.1, 25.3}
	ζ^{ge}	$\alpha_1 = 0.8865$; $\alpha_2 = 0.8575$	$\alpha_1 = 0.7749$; $\alpha_2 = 0.7247$
		$q_1 = 0.02453$; $q_2 = 0.00524$	$q_1 = 0.0246$; $q_2 = 0.00526$
	Senior level	$e^{2L} = 0.328$; $e^{2H} = 0.378$	$e^{2L} = 0.322$; $e^{2H} = 0.374$
	Wages	$\bar{w}_L = 1.568$; $\bar{w}_H = 1.703$	$\bar{w}_L = 1.561$; $\bar{w}_H = 1.697$
	Wage gap	8.6%	8.7%
$F = 0.55$	$\{\bar{x}_L, \bar{x}_H\}$	{52.3, 24.6}	{55.8, 26.4}
	ζ^{ge}	$\alpha_1 = 0.8878$; $\alpha_2 = 0.8612$	$\alpha_1 = 0.7775$; $\alpha_2 = 0.7312$
		$q_1 = 0.02143$; $q_2 = 0.00453$	$q_1 = 0.02147$; $q_2 = 0.00455$
	Senior level	$e^{2L} = 0.332$; $e^{2H} = 0.378$	$e^{2L} = 0.327$; $e^{2H} = 0.374$
	Wages	$\bar{w}_L = 1.576$; $\bar{w}_H = 1.707$	$\bar{w}_L = 1.570$; $\bar{w}_H = 1.701$
	Wage gap	8.3%	8.4%

Note: ζ^{ge} is the general equilibrium transition rates. e^{2L} and e^{2H} are the fractions of low and high skill workers employed in senior jobs, respectively: $e^{2L} = (d_{LL}^N + d_{HL}^N + d_{LL}^S + d_{HL}^S + d_{0L})/a$ and $e^{2H} = (d_{HH}^N + d_{LH}^N + d_{HH}^S + d_{LH}^S + d_{0H})/(1 - a)$. \bar{w}_L and \bar{w}_H are the average skill-specific wages.

number of firms is below $F = 0.47$, then the model exhibits a corner solution, so that high-skill workers are promoted immediately. On the contrary, if $F > 0.57$, then the competition is too strong and the gains associated with a search for external candidates are too low, so firms stop searching for external candidates at some point in time. Thus, only if $F < 0.57$ the equilibrium involves external and internal worker mobility.

Focusing on the benchmark case $F = 0.55$ we can compare the general equilibrium thresholds with the partial equilibrium values discussed in Section 4.2. We observe that the promotion thresholds for high-skill workers are hardly affected by general equilibrium effects, whereas the promotion thresholds for low-skill worker are significantly lower in general equilibrium compared to the partial equilibrium. For the case of $a = 0.8$, $F = 0.55$ we obtain $\bar{x}_L(\zeta^{ge}) = 55.8$ in general equilibrium compared to a threshold of $\bar{x}_L^{pe}(\zeta^0) = 65.4$ found for the partial equilibrium. Intuitively, the reason for this difference is that under the partial equilibrium values $(\bar{x}_L^{pe}, \bar{x}_H^{pe}) = (65.4/24.4)$ the firm's actual job-filling rate for senior positions in the market q_2 is lower and that for junior positions q_1 is higher compared to the value assumed in the partial equilibrium. As we know from Figure 3, this induces the firm to promote the majority group of low-skill workers earlier and as a result \bar{x}_L is lower. Thus, we can conclude that the general equilibrium reinforces the partial equilibrium effect and leads to even earlier promotions of low-skill workers.

Considering empirical implications of the model with heterogeneous workers we compare the share of high-skill applicants to senior positions hired externally ($1 - \alpha_2 = 0.27$ in the benchmark case) with their share among internally promoted workers, which can be calculated as

$$\frac{d_{H0}(\bar{x}_H) + \rho(d_{HL}^S + d_{HH}^S)}{d_{L0}(\bar{x}_L) + \rho(d_{LL}^S + d_{LH}^S) + d_{H0}(\bar{x}_H) + \rho(d_{HL}^S + d_{HH}^S)} = 0.21.$$

This calculation shows that external hires are more educated on average (27% high skilled) than the internal candidates (21% high skilled). This prediction of the model is strongly supported by the empirical evidence provided in Baker et al. (1994), Bidwell (2011), and DeVaro et al. (2019). For example, in Baker et al. (1994) workers hired externally to senior positions have 18 years of schooling, whereas the average is only 17 years among the internally promoted candidates. The intuition for this model prediction is as follows. Suppose that two junior workers (high and low skilled) start a job in a firm with a senior manager. Senior managers retire at a constant rate ρ . High-skill workers reach their experience requirement much faster than the low-skilled since $\bar{x}_H < \bar{x}_L$, so the probability that the senior position is already vacant when the high-skilled junior worker becomes sufficiently experienced is relatively low (53%) compared to the probability that the senior position is already vacant when the low skilled junior worker becomes sufficiently experienced (63%). Hence, a larger fraction of high-skill workers have to search on the external market and enter vacancies this way. So the result is driven by the selection of applicants with heterogeneous skills.

Analyzing the impact of a , we can see that qualitatively, the result that higher share of low-skill workers is associated with earlier promotions remains unchanged after endogenizing the transition rates. Recall that in Section 3.3 we have shown that lower quality of the homogeneous labour force is associated with later promotions. How can these two findings be reconciled? The key difference between these settings is that under worker heterogeneity an increase of the fraction of low-skill workers reduces the expected skill of a worker hired from the market *relative* to the skill of the junior worker under consideration for internal promotion, regardless of the actual type of the junior worker. So the internal candidate becomes better in relative terms compared to the average external candidate. This induces earlier internal promotions. With homogeneous workers by definition the skill of an outside hire is always identical to that of an internal candidate. So when the skill level is falling, firms want to compensate for the lower qualification of their internal candidates and let them accumulate more experience by delaying internal promotions. Thus, changes in the quality of the labour force can have principally different implications for promotions in the two settings with homogeneous and heterogeneous workers. Overall, this discussion highlights that explicitly considering potential heterogeneities in the workforce is essential for understanding the relationship between the (average) skill level in the worker population and the firms' optimal promotion thresholds.

Next, we study the implications of stronger competition F . We can see that a larger number of competitors in the market leads to earlier promotions of low-skill workers and later promotions of high-skilled workers. So the gap in the promotion times between the two groups of workers becomes smaller with a higher F . Intuitively this happens because profit/productivity differences between the two groups become relatively less important when the competition gets stronger and filling positions becomes increasingly more difficult.

Finally, we use the model to analyze the implications of stronger competition for wage inequality. For this consider a stylized case when firms and workers share the output equally,⁹ so the wage is equal to the profit, which means $w_1(x) = \pi_1(x) = d_1 + c_1 e^{\gamma x}$ and $w_2^i = \pi_2^i(\bar{x}_i) = d_2 + c_2^i e^{\gamma \bar{x}_i}$, $i = L, H$. Note further that the number of workers accumulating experience $d_{i0}(x) + d_{iH}^N(x) + d_{iL}^N(x)$ is independent of x , so the distribution of workers with a skill level $i = L, H$ accumulating experience in junior positions is simply $1/\bar{x}_i$. This means that average skill-specific wages can be calculated as:

$$\underbrace{\left(\int_0^{\bar{x}_i} w_1(x) \frac{1}{\bar{x}_i} dx \right)}_{\text{in junior position, not searching}} \cdot \frac{d_{i0} + d_{iH}^N + d_{iL}^N}{e^{1i} + e^{2i}} + \underbrace{w_1(\bar{x}_i) \frac{d_{iL}^S + d_{iH}^S}{e^{1i} + e^{2i}}}_{\text{in junior position, searching}} + \underbrace{w_2^i \frac{e^{2i}}{e^{1i} + e^{2i}}}_{\text{in senior position}}$$

where the average wage of junior workers accumulating experience contained in the round bracket is equal to $(d_1 + \frac{c_1}{\gamma \bar{x}_i}(e^{\gamma \bar{x}_i} - 1))$ and $e^{1i} + e^{2i}$ is a total number of workers of type i employed in professional jobs. For the ease of exposition the wage of workers employed in simple jobs is normalized to zero.

Our final results concerning wages are also contained in Table 4. If high-skill workers are promoted later, then their average share in senior jobs e^{2H} is decreasing with F leading to a lower average wage. There is the opposite effect for low-skill workers pushing the wages of low-skill workers up. Combining these two effects we find that stronger competition F has a mitigating impact of the inequality of wages. This happens because the role of skill differences between the two groups of workers is diminishing under the pressure of competition when hiring workers of any type becomes the main concern of employers. In other words, firms pay less attention to the qualification of applicants when filling senior positions becomes a challenging task, this leads to the lower average quality of senior managers in the market and a smaller wage gap.

5 | CONCLUSIONS AND DISCUSSION

In this paper, we develop and analyze a model which incorporates the choice of optimal promotion times by hierarchical firms in a search and matching labour market with on-the-job search and captures the option of a firm to fill senior positions through outside recruiting rather than internal promotion. Embedding firms and their individual decisions into a rich labour market with search frictions and heterogeneous worker groups allows us to study the impact of competition and labour force composition on the timing of promotions. Our findings about the effect of the level and heterogeneity of worker skills and of the strategic competition between firms on optimal promotion times deliver innovative insights into the determinants of firm behavior on the labour market and into the resulting implications for labour flows. They provide theory-based explanations for empirical observations, for example, that across-firm job mobility is more rare than within-firm promotions, that high-skill workers are promoted earlier than low-skill workers, and that external hires are more educated on average than internally promoted workers. Our insight that the effects of parameter changes on promotion cut-offs are typically much smaller in a general equilibrium framework than under the assumption of fixed job-filling/job-finding rates at the different hierarchical levels, highlights the importance of endogenizing the supply side of the labour market when analyzing the design of promotion strategies. Furthermore, our results also give rise to several innovative testable implications about the impact of different factors on promotion strategies, which can be used as the theoretical basis for future empirical work in this area. These include:

- Promotion times of competing firms exhibit strategic complementarity meaning that a firm has incentives to delay promotions and provide longer training to junior workers if competitors follow this strategy;
- Improvements in the quality of the labour force can have adverse effects on the timing of promotions. If the corresponding improvement is caused by better qualifications of homogeneous workers, this leads to earlier promotions. On the contrary, if the quality improvement is caused by the composition effect, that is, there is a larger fraction of high-skill workers in the market, this leads to delayed promotions;
- Stronger competition among firms leads to delayed (earlier) promotions if the number of firms in the market is small (large) giving rise to the inverse U-shape pattern.

- Higher search costs and stronger team synergies lead to earlier promotions. The impact of the search cost is more pronounced when the number of competing firms in the labour market is large and filling positions is difficult.
- In the presence of heterogeneous workers, stronger competition reduces differences in the promotions times and wage inequality.

The analysis presented in this paper can be extended in several promising directions. Apart from empirical work building on our results, endogenizing wage bargaining and considering a simultaneous setting of wages and promotion cut-offs may provide further economic insights. Moreover, extending the firm architecture to pyramidal structures, will make it possible to study the role of the promotion channel for transforming skill heterogeneities into wage inequalities under different hierarchical compositions of firms. Some initial analysis of promotion decisions in pyramidal firms is contained in a working paper version of our study (Dawid et al., 2019). Finally, also the role of professional networks for job transitions and emerging wage inequality might be considered. These networks might evolve endogenously through employment at the same company and influence the potential of workers for finding senior positions outside the own firm.

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DATA AVAILABILITY STATEMENT

The data that support the findings of this study, in particular the code underlying the numerical calculations, are available from the corresponding author upon reasonable request.

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ENDNOTES

¹ Excellent surveys on both research directions are Rogerson et al. (2005) and Waldman (2013), respectively

² In Germany there was a dramatic change in the average duration of vacancies from 57 days in November 2010 to 126 days in November 2020. In the same period the number of unemployed persons per vacancy fell from 7.9 to 4.7. (Bundesagentur für Arbeit. Saisonbereinigte Zeitreihen). Also in the United States the number of unemployed persons per job opening fell substantially from 4.8 in November 2010 to 2 in September 2020 (BLS JOLTS).

³ In general the stock variable $d_{10}(x, t)$ may depend on time t , so the total derivative is given by:

$$\frac{\partial d_{10}(x, t)}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial d_{10}(x, t)}{\partial t} = -q_2 d_{10}(x) + \rho d_{11}^N(x).$$

Since the distribution of firms $d_{10}(x, t)$ is stationary in the steady state we set the time derivative $\dot{d}_{10} = \frac{\partial d_{10}(x, t)}{\partial t}$ equal to zero. Moreover, experience x is accumulating one to one with the time because $\dot{x} = \partial x / \partial t = 1$.

⁴ If the search intensity z is low meaning that the number of applying workers ze_0 is low compared to the number of vacancies, then search frictions are low from the perspective of workers and almost all of their applications are successful meaning that $ze_0 \approx q_1(d_{00} + d_{01})$. This implies that $\lambda_1 \approx z$ and $q_1 \approx \frac{ze_0}{d_{00} + d_{01}}$. This setup allows us to focus on the impact of search frictions on firms rather than workers and to calibrate a realistic entry age into junior management jobs.

⁵ It should be noted that the probability that the firm finds only a junior worker respectively only a senior worker in a small time interval $[t, t + \epsilon]$ is of order ϵ , whereas the probability to find both a junior and senior worker in that time interval is of order ϵ^2 . Hence, in our continuous time formulation the term describing the continuation value after both positions have been filled simultaneously vanishes when calculating the right-hand side of the Hamilton–Jacobi–Bellman equation determining J_{00} .

⁶ For $F > 0.575$ firms have incentives to stop searching for an external manager because the gain from searching is too low (q_2 is too low). We are thankful to the anonymous referee for attracting our attention to this point.

⁷ Here we want to note that the equilibrium in a model with two skill groups is based on the assumption that firms never demote their senior employees to a junior position irrespective of their skills. Yet, potentially there could be a gain from demotions if both positions are filled with experienced workers and their formal skills are different. The gain from switching the employees is $|(c_2^H e^{\gamma \bar{x}_H} - c_2^L e^{\gamma \bar{x}_L})|$. In this situation, demotions could be precluded if the cost associated with a demotion is larger than the gain. Intuitively, demotions could be costly since demoted employees lose their motivation and reduce their working effort. We are thankful to the anonymous referee for attracting our attention to this point.

⁸ “Educational Attainment in the United States: 2018”. US Census Bureau.

⁹ More generally, if β denotes the share of output accruing to workers, then the wage expressions would be $w_1(x) = \frac{\beta}{1-\beta} \pi_1(x)$ and $w_2^i = \frac{\beta}{1-\beta} \pi_2^i(\bar{x}_i)$. We consider the standard case $\beta = 0.5$.

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APPENDIX

A.1 | Benchmark case

Proof of proposition 2. Let $\Delta J(x, \bar{x}) = J_{11}^N(x, \bar{x}) - J_{10}(x)$ be the capital gain of the firm from filling a senior position in the market which guarantees experience $y = \bar{x}$, so that

$$(r + \rho + q_2)\Delta J(x, \bar{x}) = \pi_2(\bar{x}) + \Delta + s + \frac{\partial \Delta J(x, \bar{x})}{\partial x}.$$

The general solution of this first-order linear differential equation is given by:

$$\Delta J(x, \bar{x}) = \frac{\pi_2(\bar{x}) + \Delta + s}{r + \rho + q_2} + Ke^{(r+\rho+q_2)x},$$

where K is the integration constant. Next insert $\Delta J(x, \bar{x})$ into Equation (10), this yields:

$$rJ_{10}(x) = \pi_1(x) - s + q_2 \frac{(\pi_2(\bar{x}) + \Delta + s)}{r + \rho + q_2} + q_2 Ke^{(r+\rho+q_2)x} + \frac{\partial J_{10}(x)}{\partial x}. \quad (A1)$$

This allows us to find the general solution for the present value of profits $J_{10}(x)$ (with A denoting the integration constant) and $J_{11}^N(x, \bar{x})$. Recall that $J_{11}^N(x, \bar{x}) = \Delta J(x, \bar{x}) + J_{10}(x)$, so we get solution presented in proposition 2. To find the two integration constants A and K we use the following two boundary conditions: $J_{10}(\bar{x}_i) = J_{01}(\bar{x}_i)$ and $J_{11}^N(\bar{x}_i, \bar{x}) = J_{11}^S(\bar{x}_i, \bar{x})$. The first boundary condition $J_{10}(\bar{x}_i) = J_{01}(\bar{x}_i)$ can be written as:

$$\begin{aligned} J_{10}(\bar{x}_i) &= \frac{d_1 - s}{r} + q_2 \frac{\pi_2(\bar{x}) + \Delta + s}{r(r + \rho + q_2)} + Ae^{r\bar{x}_i} + \frac{c_1 e^{\gamma \bar{x}_i}}{r - \gamma} - \frac{q_2 Ke^{(r+\rho+q_2)\bar{x}_i}}{\rho + q_2} \\ &= \frac{\pi_2(\bar{x}_i) - s + q_1 J_{11}^N(0, \bar{x}_i)}{r + \rho + q_1} = J_{01}(\bar{x}_i). \end{aligned}$$

The second boundary condition $J_{11}^N(\bar{x}_i, \bar{x}) = J_{11}^S(\bar{x}_i, \bar{x})$ becomes:

$$\begin{aligned} J_{11}^N(\bar{x}_i, \bar{x}) &= \frac{(\pi_2(\bar{x}) + \Delta + s)(r + q_2)}{r(r + \rho + q_2)} + \frac{\rho Ke^{(r+\rho+q_2)\bar{x}_i}}{\rho + q_2} + \frac{d_1 - s}{r} + Ae^{r\bar{x}_i} + \frac{c_1 e^{\gamma \bar{x}_i}}{r - \gamma} \\ &= \frac{\pi_1(\bar{x}_i) + \Delta + \pi_2(\bar{x})}{r + \rho + \lambda_2} + \frac{\rho(\pi_2(\bar{x}_i) - s + q_1 J_{11}^N(0, \bar{x}_i))}{(r + \rho + \lambda_2)(r + \rho + q_1)} + \frac{\lambda_2(\pi_2(\bar{x}) - s + q_1 J_{11}^N(0, \bar{x}))}{(r + \rho + \lambda_2)(r + \rho + q_1)} = J_{11}^S(\bar{x}_i, \bar{x}) \end{aligned}$$

Note that one term which is still unknown in both boundary conditions is $J_{11}^N(0, \bar{x}_i)$. To find it, we solve Equation (10), which is a first-order linear differential equation. This equation has the form $J'_{10}(x) = rJ_{10}(x) + g(x)$, where $g(x)$ is given by:

$$-g(x) = (d_1 + c_1 e^{\gamma x}) - s + q_2 \frac{(\pi_2(\bar{x}) + \Delta + s)}{r + \rho + q_2} + q_2 K e^{(r+\rho+q_2)x}.$$

With A denoting the integration constant the general solution of this equation is given by $J_{10}(x) = A e^{rx} + e^{rx} \int g(x) e^{-rx}$. The second part of this expression is given by:

$$\begin{aligned} e^{rx} \int g(x) e^{-rx} &= -e^{rx} \left[\int \left(d_1 - s + q_2 \frac{(\pi_2(\bar{x}) + \Delta + s)}{r + \rho + q_2} \right) e^{-rx} dx \right. \\ &\quad \left. + \int c_1 e^{(\gamma-r)x} dx + \int q_2 K e^{(\rho+q_2)x} dx \right] \\ &= \frac{d_1 - s}{r} + q_2 \frac{(\pi_2(\bar{x}) + \Delta + s)}{r(r + \rho + q_2)} - \frac{e^{rx} c_1 e^{(\gamma-r)x}}{\gamma - r} - \frac{e^{rx} q_2 K e^{(\rho+q_2)x}}{\rho + q_2} \\ &= \frac{d_1 - s}{r} + q_2 \frac{(\pi_2(\bar{x}) + \Delta + s)}{r(r + \rho + q_2)} + \frac{c_1 e^{\gamma x}}{r - \gamma} - \frac{q_2 K e^{(r+\rho+q_2)x}}{\rho + q_2}. \end{aligned}$$

Further, we determine the function $J_{11}^N(0, x_i)$. To do so recall that $J_{11}^N(x, y)$ is given by:

$$rJ_{11}^N(x, y) = \pi_1(x) + \Delta + \pi_2(y) - \rho \left(J_{11}^N(x, y) - J_{10}(x) \right) + \frac{\partial J_{11}^N(x, y)}{\partial x}.$$

Inserting $J_{10}(x)$ into this equation we get:

$$\begin{aligned} (r + \rho)J_{11}^N(x, y) &= \pi_1(x) + \Delta + \pi_2(y) + \frac{\partial J_{11}^N(x, y)}{\partial x} \\ &\quad + \rho \left[\frac{d_1 - s}{r} + q_2 \frac{\pi_2(\bar{x}) + \Delta + s}{r(r + \rho + q_2)} + A e^{rx} + \frac{c_1 e^{\gamma x}}{r - \gamma} - \frac{q_2 K e^{(r+\rho+q_2)x}}{\rho + q_2} \right]. \end{aligned}$$

The general solution of this linear first-order differential equation is given by:

$$\begin{aligned} J_{11}^N(x, y) &= \frac{s(r + q_2)}{r(r + \rho + q_2)} + \frac{\rho K e^{(r+\rho+q_2)x}}{\rho + q_2} + \frac{d_1 - s}{r} + A e^{rx} + \frac{c_1 e^{\gamma x}}{r - \gamma} \\ &\quad + \frac{\pi_2(y) + \Delta}{r + \rho} + \frac{\rho q_2 (\pi_2(\bar{x}) + \Delta)}{r(r + \rho + q_2)(r + \rho)} + D e^{(r+\rho)x}. \end{aligned}$$

with D being the integration constant. Evaluating this equation at $y = \bar{x}$, we should get $J_{11}^N(x, \bar{x})$, which implies that $D = 0$, because:

$$\frac{(\pi_2(\bar{x}) + \Delta)}{r + \rho} + \frac{\rho q_2 (\pi_2(\bar{x}) + \Delta)}{r(r + \rho + q_2)(r + \rho)} = \frac{(\pi_2(\bar{x}) + \Delta)(r + q_2)}{r(r + \rho + q_2)}.$$

Inserting $x = 0$ and $y = x_i$, we get the function $J_{11}^N(0, x_i)$:

$$J_{11}^N(0, x_i) = \frac{s(r + q_2)}{r(r + \rho + q_2)} + \frac{\rho K}{\rho + q_2} + \frac{d_1 - s}{r} + A + \frac{c_1}{r - \gamma} + \frac{(\pi_2(x_i) + \Delta)}{r + \rho} + \frac{\rho q_2 (\pi_2(\bar{x}) + \Delta)}{r(r + \rho + q_2)(r + \rho)}. \quad \square$$

A.2 | Two skill levels: Present value equations

Consider some firm with an inexperienced worker of skill $j = L, H$ employed in the junior position and an open vacancy on the senior level. The present value of discounted future profits of this firm is denoted by J_{j0} and given by:

$$rJ_{j0}(x) = \pi_1(x) - s + q_2 \left[\alpha_2 J_{jL}^N(x, \bar{x}_L) + (1 - \alpha_2) J_{jH}^N(x, \bar{x}_H) - J_{j0}(x) \right] + \frac{\partial J_{j0}(x)}{\partial x}. \quad (A2)$$

With probability α_2 the firm will hire another low skill worker for the senior position, which generates the present value of profits $J_{jL}^N(x, \bar{x}_L)$, while with the opposite probability $1 - \alpha_2$ the firm will hire a high skill worker which generates the present value of profits $J_{jH}^N(x, \bar{x}_H)$. Recall that \bar{x}_j denotes the market experience level of applicants in the managerial market with a skill level $j = L, H$. Variables $J_{jf}^N(x, y), j, f = L, H$ can be found as:

$$rJ_{jf}^N(x, y) = \pi_1(x) + \pi_2^f(y) - \rho \left(J_{jf}^N(x, y) - J_{j0}(x) \right) + \frac{\partial J_{jf}^N(x, y)}{\partial x}. \quad (A3)$$

Here $\pi_2^f(y)$ is the flow profit generated by the senior manager who may retire and exit the market at rate ρ . In this case, the firm is left with the inexperienced junior worker and the corresponding present value $J_{j0}(x)$. Further, we define an auxiliary variable $\bar{J}_j(x) \equiv \alpha_2 J_{jL}^N(x, \bar{x}_L) + (1 - \alpha_2) J_{jH}^N(x, \bar{x}_H)$ which is a weighted average between the two present values and is given by:

$$r\bar{J}_j(x) = \pi_1(x) + \alpha_2 \pi_2^L(\bar{x}_L) + (1 - \alpha_2) \pi_2^H(\bar{x}_H) - \rho (\bar{J}_j(x) - J_{j0}(x)) + \frac{\partial \bar{J}_j(x)}{\partial x}. \quad (A4)$$

Note that formally, $\bar{J}_j(x, \bar{x}_L, \bar{x}_H)$ depends on \bar{x}_L and \bar{x}_H but this dependence is suppressed for the ease of exposition. Equation (A2) can then be written as:

$$rJ_{j0}(x) = \pi_1(x) - s + q_2 [\bar{J}_j(x) - J_{j0}(x)] + \frac{\partial J_{j0}(x)}{\partial x}. \quad (A5)$$

In addition, define another auxiliary variable $\Delta J_j(x) \equiv \bar{J}_j(x) - J_{j0}(x)$, this is the average present value gain of finding a manager. Taking difference between Equations (A4) and (A5) it becomes:

$$(r + \rho + q_2) \Delta J_j(x) = \alpha_2 \pi_2^L(\bar{x}_L) + (1 - \alpha_2) \pi_2^H(\bar{x}_H) + s + \frac{\partial \Delta J_j(x)}{\partial x}.$$

The general solution of this first-order linear differential equation is:

$$\Delta J_j(x) = \frac{\alpha_2 \pi_2^L(\bar{x}_L) + (1 - \alpha_2) \pi_2^H(\bar{x}_H) + s}{r + \rho + q_2} + K_j e^{(r+\rho+q_2)x}, \quad (A6)$$

where K_j is the integration constant. Let $\bar{\pi}_2(\bar{x}_L, \bar{x}_H) = \alpha_2 \pi_2^L(\bar{x}_L) + (1 - \alpha_2) \pi_2^H(\bar{x}_H)$ denote the average flow profit of the firm associated with hiring a manager in the market. With this notation we can rewrite Equation (A5) for $J_{j0}(x)$ by inserting $\Delta J_j(x)$ in the following way:

$$rJ_{j0}(x) = \pi_1(x) - s + q_2 \left[\frac{\bar{\pi}_2(\bar{x}_L, \bar{x}_H) + s}{r + \rho + q_2} + K_j e^{(r+\rho+q_2)x} \right] + \frac{\partial J_{j0}(x)}{\partial x}. \quad (A7)$$

With A_j denoting the integration constant, the general solution of this differential equation can be written as:

$$J_{j0}(x) = \frac{d_1 - s}{r} + q_2 \frac{\bar{\pi}_2(\bar{x}_L, \bar{x}_H) + s}{r(r + \rho + q_2)} + A_j e^{rx} + \frac{c_1 e^{rx}}{r - \gamma} - \frac{q_2 K_j e^{(r+\rho+q_2)x}}{\rho + q_2}. \quad (A8)$$

Finally, inserting $J_{j0}(x)$ into Equation (A3) we get the last differential equation for $J_{jf}^N(x, y)$ which allows us to solve the main part of the model. The differential equation for $J_{jf}^N(x, y)$ is given by:

$$(r + \rho)J_{jf}^N(x, y) = \pi_1(x) + \pi_2^f(y) + \frac{\rho(d_1 - s)}{r} + \rho q_2 \frac{\bar{\pi}_2(\bar{x}_L, \bar{x}_H) + s}{r(r + \rho + q_2)} \quad (A9)$$

$$+ \rho A_j e^{rx} + \frac{\rho c_1 e^{\gamma x}}{r - \gamma} - \frac{\rho q_2 K_j e^{(r+\rho+q_2)x}}{\rho + q_2} + \frac{\partial J_{jf}^N(x, y)}{\partial x}. \quad (A10)$$

It can be rewritten as:

$$(r + \rho)J_{jf}^N(x, y) = d_1 + \pi_2^f(y) + \frac{\rho(d_1 - s)}{r} + \rho q_2 \frac{\bar{\pi}_2(\bar{x}_L, \bar{x}_H) + s}{r(r + \rho + q_2)} + \rho A_j e^{rx} + \frac{(\rho + r - \gamma)c_1 e^{\gamma x}}{r - \gamma} - \frac{\rho q_2 K_j e^{(r+\rho+q_2)x}}{\rho + q_2} + \frac{\partial J_{jf}^N(x, y)}{\partial x}.$$

Let D_{jf} denote the integration constant, so the general solution of the above equation becomes:

$$\begin{aligned} J_{jf}^N(x, y) &= \frac{d_1 + \pi_2^f(y)}{r + \rho} + \frac{\rho(d_1 - s)}{r(r + \rho)} + \rho q_2 \frac{\bar{\pi}_2(\bar{x}_L, \bar{x}_H) + s}{r(r + \rho + q_2)(r + \rho)} \\ &\quad + \frac{\rho A_j e^{rx}}{r + \rho - r} + \frac{(\rho + r - \gamma)c_1 e^{\gamma x}}{(r - \gamma)(r + \rho - \gamma)} - \frac{\rho q_2 K_j e^{(r+\rho+q_2)x}}{(\rho + q_2)(r + \rho - (r + \rho + q_2))} + D_{jf} e^{(r+\rho)x} \\ &= \frac{d_1}{r} + \frac{\pi_2^f(y)}{r + \rho} + \frac{\rho q_2 \bar{\pi}_2(\bar{x}_L, \bar{x}_H)}{r(r + \rho + q_2)(r + \rho)} + \left[\frac{-s}{r} + \frac{rs}{r(r + \rho)} + \frac{\rho q_2 s}{r(r + \rho + q_2)(r + \rho)} \right] \\ &\quad + A_j e^{rx} + \frac{c_1 e^{\gamma x}}{(r - \gamma)} + \frac{\rho K_j e^{(r+\rho+q_2)x}}{(\rho + q_2)} + D_{jf} e^{(r+\rho)x} \\ &= \frac{d_1 - s}{r} + \frac{\pi_2^f(y)}{r + \rho} + \frac{\rho q_2 \bar{\pi}_2(\bar{x}_L, \bar{x}_H)}{r(r + \rho + q_2)(r + \rho)} + \frac{s(r + q_2)}{r(r + \rho + q_2)} \\ &\quad + A_j e^{rx} + \frac{c_1 e^{\gamma x}}{(r - \gamma)} + \frac{\rho K_j e^{(r+\rho+q_2)x}}{(\rho + q_2)} + D_{jf} e^{(r+\rho)x}, \end{aligned}$$

where D_{jf} is the corresponding integration constant. Next we combine Equations (A6) and (A8) to find solution for the auxilliary variable $\bar{J}_j(x) = J_{j0}(x) + \Delta J_j(x)$:

$$\bar{J}_j(x) = \frac{d_1 - s}{r} + (r + q_2) \frac{\bar{\pi}_2(\bar{x}_L, \bar{x}_H) + s}{r(r + \rho + q_2)} + A_j e^{rx} + \frac{c_1 e^{\gamma x}}{r - \gamma} + \frac{\rho K_j e^{(r+\rho+q_2)x}}{\rho + q_2}. \quad (A11)$$

Evaluating $J_{jL}^N(x, y)$ at $y = \bar{x}_L$ with the corresponding term D_{jL} , $J_{jH}^N(x, y)$ at $y = \bar{x}_H$ with the corresponding term D_{jH} and taking a weighted average between the two we get $(1 - \alpha_2)J_{jH}^N(x, \bar{x}_H) = \bar{J}_j(x) - \alpha_2 J_{jL}^N(x, \bar{x}_L)$. The right-hand side of this equation is given by:

$$\begin{aligned} \bar{J}_j(x) - \alpha_2 J_{jL}^N(x, \bar{x}_L) &= (r + q_2) \frac{\bar{\pi}_2(\bar{x}_L, \bar{x}_H)}{r(r + \rho + q_2)} - \alpha_2 \frac{\pi_2^L(\bar{x}_L)}{r + \rho} - \alpha_2 \frac{\rho q_2 \bar{\pi}_2(\bar{x}_L, \bar{x}_H)}{r(r + \rho + q_2)(r + \rho)} - \alpha_2 D_{jL} e^{(r+\rho)x} \\ &\quad + (1 - \alpha_2) \left[\frac{d_1 - s}{r} + \frac{(r + q_2)s}{r(r + \rho + q_2)} + A_j e^{rx} + \frac{c_1 e^{\gamma x}}{r - \gamma} + \frac{\rho K_j e^{(r+\rho+q_2)x}}{\rho + q_2} \right]. \end{aligned}$$

Consider the first four terms of this equation:

$$\begin{aligned} & \frac{\bar{\pi}_2(\bar{x}_L, \bar{x}_H)}{r + \rho} + \frac{\rho q_2 \bar{\pi}_2(\bar{x}_L, \bar{x}_H)}{r(r + \rho + q_2)(r + \rho)} - \alpha_2 \frac{\pi_2^L(\bar{x}_L)}{r + \rho} - \alpha_2 \frac{\rho q_2 \bar{\pi}_2(\bar{x}_L, \bar{x}_H)}{r(r + \rho + q_2)(r + \rho)} - \alpha_2 D_{jL} e^{(r+\rho)x} \\ & = (1 - \alpha_2) \frac{\rho q_2 \bar{\pi}_2(\bar{x}_L, \bar{x}_H)}{r(r + \rho + q_2)(r + \rho)} + (1 - \alpha_2) \frac{\pi_2^H(\bar{x}_H)}{r + \rho} - \alpha_2 D_{jL} e^{(r+\rho)x} \end{aligned}$$

because $\bar{\pi}_2(\bar{x}_L, \bar{x}_H) - \alpha_2 \pi_2^L(\bar{x}_L) = (1 - \alpha_2) \pi_2^H(\bar{x}_H)$. Comparing $\bar{J}_j(x) - \alpha_2 J_{jL}^N(x, \bar{x}_L)$ with $(1 - \alpha_2) J_{jH}^N(x, \bar{x}_H)$ we can see that $(1 - \alpha_2) D_{jH} e^{(r+\rho)x} = -\alpha_2 D_{jL} e^{(r+\rho)x}$, so that $(1 - \alpha_2) D_{jH} + \alpha_2 D_{jL} = 0$.

In the next step, we consider the last Bellman equations for firms with experienced junior workers and senior managers. Let $J_{0f}(y)$ denote the present value of future profits for a firm with only one senior manager whose experience is y :

$$rJ_{0f}(y) = \pi_2^f(y) - s - \rho J_{0f}(y) + q_1 \left[\alpha_1 J_{Lf}^N(0, y) + (1 - \alpha_1) J_{Hf}^N(0, y) - J_{0f}(y) \right].$$

With probability α_1 the firm fills its junior position with a low skill worker, while with probability $(1 - \alpha_1)$ the open position is filled with a high skill worker. The last state that we have to take into account is $J_{jf}^S(x, y)$, where the junior worker accumulated sufficient experience and is already searching for senior positions in competing firms. It is given by:

$$rJ_{jf}^S(x, y) = \pi_1(x) + \pi_2^f(y) - \rho \left(J_{jf}^S(x, y) - J_{0j}(x) \right) - \lambda_2 \left(J_{jf}^S(x, y) - J_{0f}(y) \right).$$

If the senior manager retires, the remaining worker is promoted to the senior position, so the firm ends up with a present value of profits $J_{0j}(x)$. In contrast, if the junior worker quits the firm ends up with a present value of profits $J_{0f}(y)$.

As before we impose several boundary conditions:

$$J_{j0}(\bar{x}_i^j) = J_{0j}(\bar{x}_i^j), \quad J_{jf}^N(\bar{x}_i^j, \bar{x}_f) = J_{jf}^S(\bar{x}_i^j, \bar{x}_f) \quad j, f = L, H$$

These conditions imply that firms commit to promoting workers whenever they reach a prespecified skill-specific experience level \bar{x}_i^j depending on their skills $j = L, H$. However, if the senior position is filled the worker starts searching on-the-job. Combining this set of six equations with two equations $\alpha_2 D_{jL} + (1 - \alpha_2) D_{jH} = 0$ we can find a vector of eight integration constants $\{A_j, K_j, D_{jf}\}$ for the optimal skill-specific promotion times \bar{x}_i^j of firm i and market experience cut-offs \bar{x}_j .

In the final step, we consider the objective function of firm i . Given that the firm has to determine its strategy upon the entry, it aims at maximizing the present value of expected future profits J_{00} given by:

$$rJ_{00} = -2s + q_1 [\alpha_1 J_{L0}(0) + (1 - \alpha_1) J_{H0}(0) - J_{00}] + q_2 [\alpha_2 J_{0L}(\bar{x}_L) + (1 - \alpha_2) J_{0H}(\bar{x}_H) - J_{00}].$$

This equation shows that there are four sources of uncertainty for the firm at this stage: which position will be filled first—junior or senior—and which type of worker will be hired—high or low skilled. The choice variables of the firm are \bar{x}_i^L and \bar{x}_i^H which are the promotion cut-offs for each of the two skill groups.

A.3 | Two skill levels: Differential equations

Similarly to the case with homogeneous workers, in the steady state we have $d_{00} = \frac{n}{q_1 + q_2}$. Let $d_{H0}(x)$ be the number of firms which have a high-skill junior worker with experience x and a senior vacancy. Next, $d_{HL}^N(x)$ ($d_{HH}^N(x)$) are the firms with a high skill junior workers with experience x who is not searching and low (high) skill senior worker. Consider the changes in the stock of those firms. The inflow into state $d_{H0}(x)$ is equal to $\rho (d_{HH}^N(x) + d_{HL}^N(x))$ which shows that whenever the senior worker in firm $d_{HL}^N(x)$ or $d_{HH}^N(x)$ retires the firm transitions into $d_{H0}(x)$ state. At the same time $\rho d_{HL}^N(x)$ and $\rho d_{HH}^N(x)$ are the rates of outflow of $d_{HL}^N(x)$ and $d_{HH}^N(x)$ firms, respectively. On the other hand, $d_{H0}(x)$ firms find a senior worker from the external market at rate q_2 which implies that the outflow of this state is $q_2 d_{H0}(x)$. Then the inflow into the state $d_{HL}^N(x)$ is equal to $q_2 \alpha_2 d_{H0}(x)$, that is, the firm that is in state $d_{H0}(x)$ finds a low-skill senior

worker from the external market and becomes of type $d_{HL}^N(x)$. Similarly, $(1 - \alpha_2)$ is the fraction of high-skill applicants to senior positions which implies that the inflow into state $d_{HH}^N(x)$ is equal to $q_2(1 - \alpha_2)d_{H0}(x)$. We then get the following system of three first-order linear differential equations:

$$\begin{cases} \partial d_{H0}(x)/\partial x = -q_2 d_{H0}(x) + \rho(d_{HH}^N(x) + d_{HL}^N(x)) \\ \partial d_{HH}^N(x)/\partial x = q_2(1 - \alpha_2)d_{H0}(x) - \rho d_{HH}^N(x) \\ \partial d_{HL}^N(x)/\partial x = q_2\alpha_2 d_{H0}(x) - \rho d_{HL}^N(x) \end{cases}$$

The coefficient matrix is given by:

$$\begin{pmatrix} -q_2 & \rho & \rho \\ q_2(1 - \alpha_2) & -\rho & 0 \\ q_2\alpha_2 & 0 & -\rho \end{pmatrix}.$$

The eigenvalues are: $r_1 = 0$, $r_2 = -(q_2 + \rho)$, $r_3 = -\rho$ with corresponding eigenvectors:

$$v_1 = \begin{pmatrix} \frac{\rho}{q_2\alpha_2} \\ \frac{1 - \alpha_2}{\alpha_2} \\ 1 \end{pmatrix}; v_2 = \begin{pmatrix} -\frac{1}{\alpha_2} \\ \frac{1 - \alpha_2}{\alpha_2} \\ 1 \end{pmatrix}; v_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}.$$

Thus the general solution is:

$$\begin{cases} d_{H0}(x) = k_1^H \frac{\rho}{\alpha_2 q_2} - k_2^H \frac{1}{\alpha_2} e^{-(\rho+q_2)x} \\ d_{HH}^N(x) = k_1^H \frac{(1 - \alpha_2)}{\alpha_2} + k_2^H \frac{(1 - \alpha_2)}{\alpha_2} e^{-(\rho+q_2)x} - k_3^H e^{-\rho x} \\ d_{HL}^N(x) = k_1^H + k_2^H e^{-(\rho+q_2)x} + k_3^H e^{-\rho x} \end{cases}$$

We find k_1^H, k_2^H and k_3^H using the initial conditions: $d_{H0}(0) = q_1(1 - \alpha_1)d_{00}$, $d_{HH}^N(0) = q_1(1 - \alpha_1)d_{0H}$ and $d_{HL}^N(0) = q_1(1 - \alpha_1)d_{0L}$. The first initial condition implies that the stock of $d_{H0}(0)$ consists of new firms which just hired a junior worker and this worker happens to be high skilled. The second and third initial conditions show that the stock of firms $d_{HH}^N(0)$ ($d_{HL}^N(0)$) equals the firms with a junior vacancy and a high (low) skill senior worker who filled their junior position with a high skill worker. This gives the following results:

$$\begin{aligned} k_1^H &= \frac{q_1 q_2 \alpha_2 (1 - \alpha_1)}{q_2 + \rho} (d_{00} + d_{0H} + d_{0L}), \\ k_2^H &= \frac{q_1 \alpha_2 (1 - \alpha_1)}{q_2 + \rho} (\rho(d_{0H} + d_{0L}) - q_2 d_{00}), \\ k_3^H &= q_1(1 - \alpha_1)((1 - \alpha_2)d_{0L} - \alpha_2 d_{0H}). \end{aligned}$$

Let $d_{L0}(x)$ be the number of firms with a low skill junior worker with experience x and a senior vacancy. Further, $d_{LL}^N(x)$ and $d_{LH}^N(x)$ denote the firms with a junior worker with experience x and senior worker with low and high skill, respectively. The dynamic equations for these firms are defined analogously to the ones for $d_{H0}(x)$, $d_{HH}^N(x)$, and $d_{HL}^N(x)$ and lead to the following system of first-order differential equations:



$$\begin{cases} \partial d_{L0}(x)/\partial x = -q_2 d_{L0}(x) + \rho(d_{LL}^N(x) + d_{LH}^N(x)) \\ \partial d_{LL}^N(x)/\partial x = q_2 \alpha_2 d_{L0}(x) - \rho d_{LL}^N(x) \\ \partial d_{LH}^N(x)/\partial x = q_2(1 - \alpha_2) d_{L0}(x) - \rho d_{LH}^N(x) \end{cases}$$

The coefficient matrix is given by:

$$\begin{pmatrix} -q_2 & \rho & \rho \\ q_2 \alpha_2 & -\rho & 0 \\ q_2(1 - \alpha_2) & 0 & -\rho \end{pmatrix}.$$

The eigenvalues are the same as for the previous system: $r_1 = 0$, $r_2 = -(q_2 + \rho)$, $r_3 = -\rho$ and the corresponding eigenvectors are given by:

$$v_1 = \begin{pmatrix} \frac{\rho}{q_2(1 - \alpha_2)} \\ \frac{\alpha_2}{1 - \alpha_2} \\ 1 \end{pmatrix}; v_2 = \begin{pmatrix} -\frac{1}{1 - \alpha_2} \\ \frac{\alpha_2}{1 - \alpha_2} \\ 1 \end{pmatrix}; v_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}.$$

The general solution is thus:

$$\begin{cases} d_{L0}(x) = k_1^L \frac{\rho}{q_2(1 - \alpha_2)} - k_2^L \frac{1}{1 - \alpha_2} e^{-(\rho+q_2)x} \\ d_{LL}^N(x) = k_1^L \frac{\alpha_2}{(1 - \alpha_2)} + k_2^L \frac{\alpha_2}{(1 - \alpha_2)} e^{-(\rho+q_2)x} - k_3^L e^{-\rho x} \\ d_{LH}^N(x) = k_1^L + k_2^L e^{-(\rho+q_2)x} + k_3^L e^{-\rho x} \end{cases}$$

We then find k_1^L , k_2^L and k_3^L using the following initial conditions: $d_{L0}(0) = q_1 \alpha_1 d_{00}$, $d_{LL}^N(0) = q_1 \alpha_1 d_{0L}$ and $d_{LH}^N(0) = q_1 \alpha_1 d_{0H}$. The initial conditions are interpreted analogously to the ones for k_1^H , k_2^H and k_3^H . The results are given by:

$$\begin{aligned} k_1^L &= \frac{q_1 q_2 \alpha_1 (1 - \alpha_2)}{q_2 + \rho} (d_{00} + d_{0H} + d_{0L}) \\ k_2^L &= \frac{q_1 \alpha_1 (1 - \alpha_2)}{q_2 + \rho} (\rho (d_{0H} + d_{0L}) - q_2 d_{00}) \\ k_3^L &= q_1 \alpha_1 (\alpha_2 d_{0H} - (1 - \alpha_2) d_{0L}) \end{aligned}$$

Further, the steady state values for the stocks of d_{H0} , d_{HH}^N , and d_{HL}^N firms are found by integrating $d_{H0}(x)$, $d_{HH}^N(x)$, and $d_{HL}^N(x)$ over the interval $[0, \bar{x}_H]$:

$$\begin{aligned} d_{H0} &= \int_0^{\bar{x}_H} d_{H0}(x) dx = \frac{k_1^H \rho}{q_2 \alpha_2} \bar{x}_H - \frac{k_2^H}{\alpha_2 (\rho + q_2)} (1 - e^{-(\rho+q_2)\bar{x}_H}) \\ d_{HH}^N &= \int_0^{\bar{x}_H} d_{HH}^N(x) dx = \frac{k_1^H (1 - \alpha_2)}{\alpha_2} \bar{x}_H + \frac{k_2^H (1 - \alpha_2)}{\alpha_2 (\rho + q_2)} (1 - e^{-(\rho+q_2)\bar{x}_H}) - \frac{k_3^H}{\rho} (1 - e^{-\rho \bar{x}_H}) \\ d_{HL}^N &= \int_0^{\bar{x}_H} d_{HL}^N(x) dx = k_1^H \bar{x}_H + \frac{k_2^H}{\rho + q_2} (1 - e^{-(\rho+q_2)\bar{x}_H}) + \frac{k_3^H}{\rho} (1 - e^{-\rho \bar{x}_H}) \end{aligned}$$

Similarly we derive the steady state equations for the stock of d_{L0} , d_{LL}^N , and d_{LH}^N firms by integrating $d_{L0}(x)$, $d_{LL}^N(x)$, and $d_{LH}^N(x)$ over the interval $[0, \bar{x}_L]$:

$$\begin{aligned}
d_{L0} &= \int_0^{\bar{x}_L} d_{L0}(x) dx = \frac{k_1^L \rho}{q_2(1 - \alpha_2)} \bar{x}_L - \frac{k_2^L}{(1 - \alpha_2)(\rho + q_2)} (1 - e^{-(\rho + q_2)\bar{x}_L}) \\
d_{LL}^N &= \int_0^{\bar{x}_L} d_{LL}^N(x) dx = \frac{k_1^L \alpha_2}{(1 - \alpha_2)} \bar{x}_L + \frac{k_2^L \alpha_2}{(1 - \alpha_2)(\rho + q_2)} (1 - e^{-(\rho + q_2)\bar{x}_L}) - \frac{k_3^L}{\rho} (1 - e^{-\rho \bar{x}_L}) \\
d_{LH}^N &= \int_0^{\bar{x}_L} d_{LH}^N(x) dx = k_1^L \bar{x}_L + \frac{k_2^L}{\rho + q_2} (1 - e^{-(\rho + q_2)\bar{x}_L}) + \frac{k_3^L}{\rho} (1 - e^{-\rho \bar{x}_L})
\end{aligned}$$

Next, let us consider the firms which have a junior worker who is already searching in the external market: d_{LL}^S , d_{HL}^S , d_{HH}^S , and d_{LH}^S . The inflow into these states happens whenever the junior worker of a d_{ij}^N ($i = L, H, j = L, H$) firm reaches the promotion cut-off \bar{x}_L or \bar{x}_H , depending on the worker's type. The outflow of these states occurs either when the senior worker retires which happens at a rate ρ or when the searching junior worker finds a senior job at a different firm which occurs at a rate λ_2 . Hence, we get:

$$\begin{aligned}
d_{LL}^S &= d_{LL}^N(\bar{x}_L) - (\rho + \lambda_2) d_{LL}^S & d_{HL}^S &= d_{HL}^N(\bar{x}_H) - (\rho + \lambda_2) d_{HL}^S \\
d_{HH}^S &= d_{HH}^N(\bar{x}_H) - (\rho + \lambda_2) d_{HH}^S & d_{LH}^S &= d_{LH}^N(\bar{x}_L) - (\rho + \lambda_2) d_{LH}^S
\end{aligned}$$

Inserting $d_{LL}^N(x)$ evaluated at \bar{x}_L and solving for $d_{LL}^S = 0$ we find the steady state equation for the stock of d_{LL}^S firms. Analogously, we also find the steady state equations for d_{HL}^S , d_{HH}^S , and d_{LH}^S firms:

$$\begin{aligned}
d_{LL}^S &= \frac{k_1^L \alpha_2 + k_2^L \alpha_2 e^{-(\rho + q_2)\bar{x}_L} - k_3^L (1 - \alpha_2) e^{-\rho \bar{x}_L}}{(1 - \alpha_2)(\rho + \lambda_2)} & d_{HL}^S &= \frac{k_1^H + k_2^H e^{-(\rho + q_2)\bar{x}_H} + k_3^H e^{-\rho \bar{x}_H}}{\rho + \lambda_2} \\
d_{HH}^S &= \frac{k_1^H (1 - \alpha_2) + k_2^H (1 - \alpha_2) e^{-(\rho + q_2)\bar{x}_H} - k_3^H \alpha_2 e^{-\rho \bar{x}_H}}{\alpha_2(\rho + \lambda_2)} & d_{LH}^S &= \frac{k_1^L + k_2^L e^{-(\rho + q_2)\bar{x}_L} + k_3^L e^{-\rho \bar{x}_L}}{(\rho + \lambda_2)}
\end{aligned}$$

Finally, let us consider the firms which have only a senior worker. Firms in state $d_{H0}(\bar{x}_H)$ enter state d_{0H} because the junior worker is promoted. Also, firms in state d_{00} enter state d_{0H} when they find a high-skill senior worker from the external market which happens at a rate $q_2(1 - \alpha_2)$. Further, at a rate ρ the senior worker in firms d_{HL}^S or d_{HH}^S retires and the junior worker is automatically promoted. Then these firms also enter state d_{0H} . Next, at a rate λ_2 the searching worker in firms d_{LH}^S or d_{HH}^S finds a job in a new firm and leaves so that the firms transition into state d_{0H} . On the other hand, the outflow from state d_{0H} happens either when the senior worker retires at a rate ρ or when the firm finds a junior worker from the market which occurs at a rate q_1 . This leads to the following dynamic equation:

$$\dot{d}_{0H} = d_{H0}(\bar{x}_H) + q_2(1 - \alpha_2)d_{00} + \rho d_{HL}^S + \lambda_2 d_{LH}^S + (\rho + \lambda_2) d_{HH}^S - (\rho + q_1) d_{0H}.$$

Similarly we can write the dynamic equation for d_{0L} firms:

$$\dot{d}_{0L} = d_{L0}(\bar{x}_L) + q_2 \alpha_2 d_{00} + \rho d_{LH}^S + \lambda_2 d_{HL}^S + (\rho + \lambda_2) d_{LL}^S - (\rho + q_1) d_{0L}.$$

Substituting in all expressions we have already derived and solving for $\dot{d}_{0H} = \dot{d}_{0L} = 0$ then gives us the steady state values for the stock of d_{0H} and d_{0L} firms.