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**Article — Published Version**

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Journal of Time Series Analysis

**Provided in Cooperation with:**

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*Suggested Citation:* Schweikert, Karsten (2021) : Oracle Efficient Estimation of Structural Breaks in Cointegrating Regressions, Journal of Time Series Analysis, ISSN 1467-9892, John Wiley & Sons, Ltd, Oxford, UK, Vol. 43, Iss. 1, pp. 83-104, <https://doi.org/10.1111/jtsa.12593>

This Version is available at:

<https://hdl.handle.net/10419/284831>

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## ORACLE EFFICIENT ESTIMATION OF STRUCTURAL BREAKS IN COINTEGRATING REGRESSIONS

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In this article, we propose an adaptive group lasso procedure to efficiently estimate structural breaks in cointegrating regressions. It is well known that the group lasso estimator is not simultaneously estimation consistent and model selection consistent in structural break settings. Hence, we use a first step group lasso estimation of a diverging number of breakpoint candidates to produce weights for a second adaptive group lasso estimation. We prove that parameter changes are estimated consistently by group lasso and show that the number of estimated breaks is greater than the true number but still sufficiently close to it. Then, we use these results and prove that the adaptive group lasso has oracle properties if weights are obtained from our first step estimation. Simulation results show that the proposed estimator delivers the expected results. An economic application to the long-run US money demand function demonstrates the practical importance of this methodology.

*Received 02 June 2020; Accepted 19 April 2021*

Keywords: Adaptive group lasso; change-points; cointegration; model selection; US money demand

**JEL:** C22; C52

MOS subject classification: 62E20; 62M10; 91B84.

### 1. INTRODUCTION

In this article, we consider modelling cointegration relationships where the long-run equilibrium may differ for subsamples, thereby allowing for (multiple) structural breaks in the cointegrating regression. We assume that cointegration holds over some (fairly long) period of time, but then shifts to a new ‘long-run’ relationship. The number of breaks and their location are unknown to the researcher. Although coefficients of long-run equilibrium equations are relatively persistent by definition, accounting for the possibility of structural breaks is crucial in cointegration analysis, which usually involves long sample periods. On the one hand, long time series are needed to study the long-run behaviour of economic systems, on the other hand, employing long time series increases the likelihood of encountering structural change during the sample period. It is widely known that structural breaks, when present, can mask cointegrating relationships and render cointegration tests uninformative (Campos *et al.*, 1996; Gregory *et al.*, 1996; Qu, 2007). Hence, we propose a two-step approach to detect (multiple) structural breaks in cointegrating regressions using penalized regression techniques.

Since time series used for economic analyses have become very long in some instances, detecting (multiple) structural breaks has emerged as an important problem in the econometrics literature. For comprehensive surveys on structural breaks in time series models (‘change-point’ detection in the statistics literature or ‘pattern recognition’ in the context of signal processing), see, for example, Perron (2006), Aue and Horváth (2013) and Niu *et al.* (2015). While classical structural break models for linear regressions attempt to detect one unknown break via a grid search procedure (Andrews, 1993), it is not feasible to use grid searches for the detection of multiple breaks because the computational cost increases exponentially with the presumed number of breaks (needing least

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squares operations of order  $O(T^m)$  for  $m$  breaks). Addressing this issue, Bai and Perron (1998, 2003) use dynamic programming techniques (henceforth Bai–Perron algorithm), requiring at most least-squares operations of order  $O(T^2)$  for any number of breaks, to add breaks sequentially to the model. Recently, several approaches have been proposed that reframe the task of detecting and estimating structural breaks as a model selection problem employing penalized regressions and related model selection techniques (Davis *et al.*, 2006; Harchaoui and Lévy-Leduc, 2010; Jin *et al.*, 2013; Chan *et al.*, 2014; Ciuperca, 2014; Jin *et al.*, 2016; Qian and Jia, 2016; Qian and Su, 2016; Behrendt and Schweikert, 2021). Instead of grid search procedures which augment linear regression models with parameter changes, model selection procedures take a top-down approach and try to shrink the set of all possible breakpoint candidates to contain only the true breakpoints. These approaches benefit from high computational efficiency and detect structural breaks with high accuracy.

The theory for (multiple) structural breaks in cointegrating regressions is not nearly as developed as the theory for change-points in the statistics and signal processing literature. Most studies are concerned with cointegration testing in the presence of structural instability. One of the most popular cointegration tests with an unknown breakpoint is the one proposed by Gregory and Hansen (1996a, 1996b) in which the location of the break can be estimated via grid search at the minimum of the individual cointegration test statistics. Hatemi-J (2008) extends the test to account for two breaks and Schweikert (2020) allows for the possibility of nonlinear adjustment to the long-run equilibrium. Maki (2012) employs a hybrid procedure, detecting  $m - 1$  breaks by minimizing the sum of squared residuals among all possible sample splits and finally determining the last break by minimizing a cointegration test statistic. Unfortunately, these tests are non-informative about the location of breaks. They optimize the model specification to provide evidence against the null hypothesis of no cointegration, thereby not necessarily finding those breakpoints which optimize the model fit. Maki (2012) determines the first  $m - 1$  breaks based on improving the model fit but does not do so for the last breakpoint. Hence, the set of estimated breakpoints is not completely informative. Other studies with a strong focus on cointegration testing which conduct breakpoint estimation as a by-product are Carrion-i Silvestre and Sanso (2006) and Arai and Kurozumi (2007). They propose a CUSUM-based approach to test the null hypothesis of cointegration with a structural break against the alternative hypothesis of no cointegration. Qu (2007) considers a cointegrated system allowing the cointegrating rank to change during different subsamples so that it is possible to detect cointegrating relationships that exist only in some subsamples. Westerlund and Edgerton (2006) design LM-based test statistics invariant to structural breaks to test the null of no cointegration and Davidson and Monticini (2010) use subsample procedures to account for structural breaks in their cointegration tests.

In contrast, few studies are primarily focussed on modelling structural change in cointegrated systems. Kejriwal and Perron (2008, 2010) propose to estimate the number and location of structural breaks in cointegrating equations by applying the Bai–Perron algorithm. Inference on breakpoints is studied in, among others, Bai *et al.* (1998), Qu and Perron (2007), Kejriwal and Perron (2008, 2010), Li and Perron (2017) and Oka and Perron (2018). Using penalized regression approaches to account for structural breaks in cointegrating regression has not been explored yet in great detail. A similar idea has been proposed by Schmidt and Schweikert (2021) but their procedure is limited to bivariate cointegrating regressions using a modified adaptive lasso estimator. Here, we extend their methodology to cointegrating regressions with multiple regressors and provide a rigorous proof that the adaptive group lasso estimator is oracle efficient in settings with an unknown number of breaks and a diverging number of breakpoint candidates.

The proposed estimation method in this article consists of two main steps: in the first step, we apply the group lasso estimator to a cointegration model with a diverging number of breakpoint candidates. We allow that breaks can occur at any point in time except for some lateral trimming which is mostly needed to identify the baseline coefficients in the first regime. We prove that the group lasso estimator consistently estimates parameter changes. However, it is well known that lasso estimators are not simultaneously parameter estimation consistent and model selection consistent in situations where the restricted eigenvalue condition or related conditions such as the strong irrepresentable condition do not hold (Chan *et al.*, 2014). Under these conditions, we show that the number of selected breaks is greater than the true number of breaks almost surely, but their estimated location is sufficiently close to their true location. In the second step, we use the first step group lasso estimates as weights for the adaptive

group lasso. We provide a rigorous proof that the adaptive group lasso has the oracle properties if the first step algorithm assumes a maximum number of breaks and the distance between breaks depends on the sample size ensuring that the breakpoint candidates for the second step estimation are sufficiently distinct. The number of breaks is then estimated as the number of non-zero groups obtained after adaptive group lasso optimization.

The article is organized as follows. Section 2 describes the proposed adaptive group lasso procedure to estimate structural breaks in cointegrating regressions. Section 3 is devoted to the Monte Carlo simulation study. Section 4 reports the results of an empirical application of our methodology to the US money demand function, and Section 5 concludes. Proofs of all theorems in the article are provided in the Mathematical Appendix in the Supporting Information.

## 2. METHODOLOGY

In the following, we specify a cointegrated system with multiple structural breaks at which it attains new equilibrium states. The cointegrated system does not deviate persistently from each equilibrium until the next break occurs and a new equilibrium is maintained.

### 2.1. Framework

Let  $\{y_t\}_{t=1}^\infty$  denote a scalar process generated by

$$y_t = \sum_{j=1}^{m+1} \left[ \mu + \beta'_j X_t + u_t \right] \mathbb{1}\{t_{j-1} \leq t < t_j\}, \quad t = 1, 2, \dots, \tag{1}$$

where  $t_j, j \in \{0, 1, \dots, m + 1\}$  denote the breakpoints  $1 = t_0 < t_1 < \dots < t_{m+1} = T + 1$ .  $\mu$  is the intercept,  $\beta'_j = (\beta_{j1}, \beta_{j2}, \dots, \beta_{jN})$  are regime-dependent coefficients and  $\{X_t\}_{t=1}^\infty$ , where  $X_t = (X_{1t}, X_{2t}, \dots, X_{Nt})'$ , follows an  $N$ -vector integrated process<sup>1</sup>

$$X_t = X_{t-1} + v_t, \quad t = 1, 2, \dots, \tag{2}$$

where  $X_0 = 0$ .  $\{u_t\}_{t=1}^\infty$  and  $\{v_t\}_{t=1}^\infty$  are mean-zero weakly stationary error processes. For expositional simplicity, we restrict our analysis to cointegrating regressions with a constant intercept across regimes.<sup>2</sup> We make the following assumptions about the vector process  $w_t = (u_t, v_t)'$ :

**Assumption 1.** The vector process  $\{w_t\}_{t=1}^\infty$  satisfies the following conditions:

- (i)  $Ew_t = 0$  for  $t = 1, 2, \dots$
- (ii)  $\{w_t\}_{t=1}^\infty$  is weakly stationary.
- (iii)  $\{w_t\}_{t=1}^\infty$  is strong mixing with mixing coefficients of size  $-p\beta/(p-\beta)$  and  $E|w_t|^p < \infty$  for some  $p > \beta > 5/2$ .

Furthermore, we assume that the long-run covariance matrix  $\Omega_v = \sum_{j=-\infty}^\infty E v_t v'_{t-j}$  is positive definite. In addition, we require that

$$\sup_T E \left| \frac{1}{T} \sum_{i=1}^s X_{ji} u_i \right|^{4+\epsilon} < \infty, \quad \text{for } 1 \leq j \leq N, 1 \leq s \leq T \text{ and some } \epsilon > 0.$$

<sup>1</sup> Note that this specification of the process rules out integrated regressors with a deterministic drift component. Relaxing this assumption would be relatively straightforward.

<sup>2</sup> Our main results generally hold if the coefficients of included deterministic components, for example, linear and quadratic trend terms, do not change over the sampling period. A discussion of breaks in the intercept is included in the Supporting Information of this article.

While the first three conditions of Assumption 1 are standard in cointegration analysis, assuming that  $\Omega_v$  is positive definite implies that  $X_t$  is non-cointegrated. We denote the number of structural breaks by  $m$ . While the number of true structural breaks  $m_0$  is unknown, we assume that the maximum number of structural breaks  $m^*$  is known to the researcher. The estimated number of breakpoints is denoted by  $\hat{m}$ . The locations of breakpoints relative to sample size, so-called break fractions, are denoted by  $\tau_j = t_j/T, j \in \{0, 1, \dots, m + 1\}$ .

Throughout this article, we use the following notation to present our main results: let  $y_T = (y_1, y_2, \dots, y_T)'$  denote the vector containing  $T$  observations of our response variable and  $u_T = (u_1, u_2, \dots, u_T)'$  denotes the error term vector. The vector of  $T$  observations for the  $N$ -dimensional variable  $X_t$  is denoted by  $X = (X_1, \dots, X_T)'$ . Our design matrix  $Z_T$  is an  $T \times TN$  matrix defined by

$$Z_T = \begin{pmatrix} X'_1 & 0 & 0 & \dots & 0 \\ X'_2 & X'_2 & 0 & \dots & 0 \\ X'_3 & X'_3 & X'_3 & \dots & 0 \\ \vdots & & & & \\ X'_T & X'_T & X'_T & \dots & X'_T \end{pmatrix}, \tag{3}$$

and we define the Gram matrix  $\Sigma = Z'_T Z_T / T^2$ . Adjacent columns of  $Z_T$  differ only by one entry which means that the columns are almost identical for  $T \rightarrow \infty$ . Consequently,  $\Sigma$  does not converge to a positive definite asymptotic counterpart. It follows that the restricted eigenvalue condition (Bickel *et al.*, 2009) does not hold and we cannot establish our consistency proofs based on this assumption. See Chan *et al.* (2014) for a thorough discussion of this issue.

We set  $\theta_1 = \beta_1$  and

$$\theta_i = \begin{cases} \beta_{j+1} - \beta_j, & \text{when } i = t_j, \\ \mathbf{0}, & \text{otherwise,} \end{cases} \tag{4}$$

for  $i = 2, \dots, T$ . For the remainder of this article,  $\theta_i = \mathbf{0}$  means that  $\theta_i$  has all entries equalling zero and  $\theta_i \neq \mathbf{0}$  means that  $\theta_i$  has at least one non-zero entry. The coefficient vector  $\theta(T) = (\theta_1, \theta_2, \dots, \theta_T)'$  is of length  $TN$  and contains all time-specific parameter changes. Because we treat structural breaks as rare events and assume that parameter changes persist for some time, the number of non-zero elements in  $\theta(T)$  is assumed to be small, that is smaller than  $m^* + 1$  groups of size  $N$ .

We denote the true value of a parameter with a 0 superscript.  $\{\tau_j^0, j = 1, \dots, m_0\}$  denotes the set of true break fractions and  $\beta_j^0, j = 1, \dots, m_0 + 1$  defines the true coefficient of the  $j$ th regime. For technical reasons, we additionally set  $\beta_0^0 = 0$ . We define the index sets  $\bar{\mathcal{A}} = \{1 \leq i \leq T : \theta_i^0 \neq \mathbf{0}\}$  denoting the indices of truly non-zero coefficients (including the baseline coefficient) and  $\mathcal{A} = \{i \geq 2 : \theta_i^0 \neq \mathbf{0}\}$  denoting the non-zero parameter changes. The index set obtained from our first step estimation belonging to all estimated non-zero parameter changes is denoted by  $\mathcal{A}_T = \{i \geq 2 : \hat{\theta}_i \neq \mathbf{0}\}$ . We note that the first regime's coefficient (before the first breakpoint) is not allowed to be zero.<sup>3</sup> Since we indicate breakpoints with non-zero coefficients in our penalized regression approach, the set  $\mathcal{A} = \{t_1^0, t_2^0, \dots, t_{m_0}^0\}$  is also used to denote true breakpoints. Similarly, the set  $\mathcal{A}_T = \{\hat{t}_1, \hat{t}_2, \dots, \hat{t}_{\hat{m}}\}$  denotes estimated breakpoints, that is, indices of those coefficients which are estimated to be non-zero.  $|\mathcal{A}|$  denotes the cardinality of the set  $\mathcal{A}$  and  $\mathcal{A}^c$  denotes the complementary set. We use those sets to index rows and columns of vectors and matrices. For example, let  $Z_{T,\mathcal{A}}, Z_{T,\mathcal{A}^c}$  contain the columns of  $Z_T$  and  $\theta_{\mathcal{A}}(T), \theta_{\mathcal{A}^c}(T)$  contain the rows of  $\theta(T)$  associated with active and inactive breakpoints respectively.

<sup>3</sup> While the cointegrating vector  $(1, \mathbf{0})'$  in principle ensures that  $y_t = \mu + u_t$  is stationary under our assumptions, we exclude this case to simplify our exposition. In the following, we need a clear distinction between zero and non-zero coefficients to decide whether their indices belong into the sets  $\bar{\mathcal{A}}$  or  $\mathcal{A}^c$ . Allowing zero baseline coefficients would require several case-by-case considerations.

For notational convenience, we use ‘ $\Rightarrow$ ’ to signify weak convergence of the associated probability measures and  $\xrightarrow{p}$  to denote convergence in probability. Continuous stochastic processes such as a Brownian motion  $B(s)$  on  $[0,1]$  are simply written as  $B$  if no confusion is caused. We also write integrals with respect to the Lebesgue measure such as  $\int_0^1 B(s)ds$  simply as  $\int_0^1 B$ . Throughout the article, several (distinct) large constants are all denoted with  $C$ , while small constants are denoted by  $\epsilon$ .

Using these definitions, our cointegration model described in (1) can be expressed as a high-dimensional regression model in matrix form

$$y_T = \mathbf{Z}_T \boldsymbol{\theta}(T) + u_T. \quad (5)$$

Since only  $m_0+1$  groups within  $\boldsymbol{\theta}(T)$  are truly non-zero, we need to obtain a sparse solution to the high-dimensional regression problem in (5). This means we frame the detection of structural breaks as a model selection problem and use available methods from this strand of the literature. To reduce the dimensionality of the estimation problem, we assume that breaks occur for all coefficients simultaneously. This allows us to treat all regressors at each point in time as one group. We can therefore apply the group lasso estimator proposed by Yuan and Lin (2006) to achieve a sparse solution. As our first step, we minimize the objective function,

$$Q^*(\boldsymbol{\theta}(T)) = \frac{1}{T} \|y_T - \mathbf{Z}_T \boldsymbol{\theta}(T)\|^2 + \lambda_T \sum_{i=1}^T \|\boldsymbol{\theta}_i\|, \quad (6)$$

to obtain the group lasso estimator for  $\boldsymbol{\theta}(T)$  which is henceforth denoted by  $\tilde{\boldsymbol{\theta}}(T) = \arg \min_{\boldsymbol{\theta}(T)} Q^*$ .  $\lambda_T$  is the tuning parameter and  $\|\cdot\|$  denotes the  $L_2$ -norm. Unfortunately, the group lasso estimator inherits the same problems, namely estimation inefficiency and model selection inconsistency, as the plain lasso estimator. Similar to the idea first presented in Zou (2006), we reestimate the objective function with individual coefficient weights to alleviate this problem and to try to reduce the number of falsely detected breaks. The statistical properties of adaptive group lasso estimators for a fixed number of groups are investigated in Wang and Leng (2008). Since we have a diverging set of breakpoint candidates, least squares estimation of the full model is not feasible. However, we show that group lasso is a consistent estimator for non-zero parameter changes giving us appropriate weights for a second step adaptive group lasso estimation. This approach is similar to the ideas put forth in Wei and Huang (2010), Horowitz and Huang (2013), Schmidt and Schweikert (2021) and Behrendt and Schweikert (2021).

As will be demonstrated later, the group lasso estimator only slightly overselects breaks under the right tuning. The algorithm employed to estimate  $\tilde{\boldsymbol{\theta}}(T)$  allows to prespecify the maximum number of breakpoint candidates  $M$ , that is the maximum number of non-zero groups in  $\tilde{\boldsymbol{\theta}}(T)$ , and the minimum distance between breaks. Since the group lasso overselects breaks in the first step,  $M$  should be set large enough to encompass all true breakpoints and some additional falsely selected non-zero groups. This condition guarantees that  $\tilde{\boldsymbol{\theta}}(T)$  always contains  $MN$  elements. In turn,  $TN - MN$  columns of  $\mathbf{Z}_T$  corresponding to zero coefficients are eliminated during the first step to result in the  $T \times MN$  design matrix  $\mathbf{Z}_S$ . Hence, for given  $M \ll T$ , the column size of the new design matrix is substantially smaller than the original size  $TN$  and does not longer depend on the sample size. This allows us to further assume that all eigenvalues of  $\boldsymbol{\Sigma}_S = \mathbf{Z}_S' \mathbf{Z}_S / T^2$  are contained in the interval  $[c_*, c^*]$ , where  $c_*$  and  $c^*$  are two positive constants. This means that we can relate to a restricted eigenvalue condition similar to Bickel *et al.* (2009) for the second step estimation. While the restricted eigenvalue condition in general does not hold for change-point settings, the dimension reduction of the first step allows us to postulate this assumption for our reduced design matrix. It should be noted that our assumption for the second step estimation is not restrictive for empirical applications because the notion of a long-run equilibrium relationship implies a maximum number of breaks and a minimum regime length. A minimum regime length is further justified by the minimum subsample size needed to precisely estimate parameter changes. Consequently,  $M$  should be chosen so that the average regime length in case of equidistantly spaced breaks still guarantees enough observations per regime to estimate all coefficient changes.

We follow Wang and Leng (2008) and define the adaptive group lasso objective function

$$Q(\theta_S) = \frac{1}{T} \|y_T - Z_S \theta_S\|^2 + \lambda_S \sum_{i=1}^M w_i \|\theta_{S,i}\|, \tag{7}$$

where  $\gamma > 0$  and  $w_i$  are the group-specific weights assigned as follows

$$w_i = \begin{cases} \|\tilde{\theta}_{S,i}\|^{-\gamma}, & \text{if } \tilde{\theta}_{S,i} \neq 0, \\ \infty, & \text{if } \tilde{\theta}_{S,i} = 0, \end{cases}$$

and set  $0 \times \infty = 0$ .  $\tilde{\theta}_{S,i}$ ,  $i = 1, \dots, |\mathcal{A}_T| + 1$  denotes the non-zero group lasso coefficient estimates obtained from optimizing the objective function in (6). The remaining  $M - |\mathcal{A}_T| - 1$  group elements of  $\tilde{\theta}_S$  can be filled with zero groups as long as their selected indices lead to  $\Sigma_S$  being a positive definite matrix for all  $T$ .

We denote the estimator minimizing  $Q(\theta_S)$  with  $\hat{\theta}_S = \arg \min_{\theta_S} Q$ . The weight of the first coefficient is usually set to zero to ensure that the system is cointegrated with a cointegrating vector different from  $(1, \mathbf{0})'$  if no structural break occurs. Eliminating columns from the initial design matrix requires a mapping of our second step indices to recover the original indices. For notational convenience, we use the mapping  $g : \mathbb{N} \rightarrow \mathbb{N}$ ,  $i \mapsto g(i) = t_i$ , where  $t_i$  is the breakpoint corresponding to the index  $i$ , for this purpose and define the index set  $\bar{\mathcal{A}}^*$  ( $\mathcal{A}^*$ ) to pick out the elements that correspond to truly non-zero coefficients (parameter changes).

We note that the major computing cost comes from the first step group lasso estimation considering a large number of observations as potential breakpoints. The second step represents a marginal addition to the total computing time if the first step estimation was sufficiently successful in eliminating inactive breakpoint candidates. The interested reader may consult Chan *et al.* (2014) for a detailed discussion of computational complexity in this context.<sup>4</sup>

## 2.2. Asymptotic Properties

In the following, we study the asymptotic properties of our adaptive group lasso estimator. To discuss asymptotic properties, we need to impose some further assumptions about the location and magnitude of active breakpoints.

- Assumption 2.** (i)  $I_{\min} = \min_{1 \leq j \leq m_0+1} |t_j^0 - t_{j-1}^0| > \zeta T$  for some  $\zeta > 0$ , where  $I_{\min}$  is the minimum break interval.  
 (ii) The break magnitudes are bounded to satisfy  $m_\beta = \min_{1 \leq j \leq m_0+1} \|\beta_j^0 - \beta_{j-1}^0\| > \nu$  for some  $\nu > 0$  and  $M_\beta = \max_{1 \leq j \leq m_0+1} \|\beta_j^0 - \beta_{j-1}^0\| < \infty$ .  
 (iii) There exists a constant  $C > 0$  such that

$$m' \Sigma m \geq C \sum_{j \in \bar{\mathcal{A}}} \|m_j\|^2,$$

for all  $TN \times 1$  vectors  $m = (m_1, m_2, \dots, m_T)'$  whenever  $\sum_{j \in \bar{\mathcal{A}}^c} \|m_j\| \leq 2 \sum_{j \in \bar{\mathcal{A}}} \|m_j\|$ .

Assumption 2(i) requires that the length of the regimes between breaks increases with the sample size and in the same proportions to each other. This allows us to consistently detect and estimate the true break fractions as it

<sup>4</sup> While the Bai–Perron algorithm needs at most  $O(T^2)$  operations, the group LARS algorithm used to solve (6) has computational burden of order  $O(M^3 + MT)$ . Hence, the group LARS algorithm has a stronger dependence on the maximum number of breaks, whereas the Bai–Perron algorithm only depends on the number of observations. This implies that the Bai–Perron algorithm is better suited for small to moderate samples with a potentially large number of breaks, often found in linear regressions modelling short-run relationships. Instead, the group LARS algorithm is well-suited for large sample sizes and a small to moderate number of structural breaks which is often found for long-run relationships in the presence of structural change.

makes the break dates asymptotically distinct (Perron, 2006). The first inequality of Assumption 2(ii) is a necessary condition to ensure that a structural break occurs at  $t_j^0$ . We do not consider small breaks with local-to-zero behaviour in this setting (see Bai *et al.* (1998) for assumptions used in this context). This assumption is not believed to be restrictive for the intended empirical applications where applied researchers aim to estimate the long-run equilibrium to obtain the error correction term, that is, the cointegration residuals, for their follow-up analysis. Essentially, they need optimal in-sample forecasts in terms of mean squared error of the cointegrating regression under structural instability to consistently estimate these residuals. Boot and Pick (2020) show that in-sample forecasts are largely unaffected by local-to-zero breaks. The second part excludes the possibility of infinitely large parameter changes. Assumption 2(iii) implies that the number of active breaks is less than the number of observations and the smallest eigenvalue of  $\Sigma_{\mathcal{A}}$  is greater than or equal to  $C$  by letting  $\mathbf{m}_j = 0$  for  $j \in \mathcal{A}^c$ . Consequently, Assumption 2(iii) ensures that  $\Sigma_{\mathcal{A}}$  is positive definite for all  $T$ , which is only then the case if  $\mathbf{Z}_{T,\mathcal{A}}$  contains columns which are sufficiently distinct. This in turn means that the intervals between breaks need to be sufficiently large for all  $T$ . It is important to note that Assumption 2(i) can be deduced as an implication of Assumption 2(iii) and we need Assumption 2(iii) exclusively for the first step estimation. Our second step estimation requires only Assumption 2(i) and (ii) as long as consistent weights are available.

First, we need to show that the initial estimator provides consistent weights for the second step adaptive lasso procedure (Huang *et al.*, 2008). Theorem 1 provides a consistency result for the group lasso estimator in cointegrating regressions with (possibly) multiple structural breaks.

**Theorem 1.** Under Assumptions 1 and 2, if  $\lambda_T = 2Nc_0T^\delta$  for some  $c_0 > 0$  and  $3/4 < \delta < 1$ , then there exists some  $C > 0$  such that with probability greater than  $1 - \frac{C}{c_0^2T^{2\delta-1}}$ ,

$$\|\tilde{\theta}(T) - \theta^0(T)\| \leq \frac{2}{T^{(1-\delta)/2}} \sqrt{\frac{Nc_0(m_0 + 1)M_\beta}{C}}.$$

**Remark 1.** The specification of  $\lambda_T$  implies that  $\lambda_T \rightarrow \infty$  for  $T \rightarrow \infty$ . This means we have to apply a stricter penalty for increasing sample sizes to discard a larger set of inactive candidate breaks searching for a fixed number of  $m_0$  active breaks. On the other hand,  $\lambda_T$  fulfils the condition  $\lambda_T/T \rightarrow 0$  so that the tuning parameter cannot grow too fast avoiding to ignore active breaks. Since the convergence rate of the group lasso coefficients depends inversely on  $\delta$ , it is useful to employ a selection rule for  $\lambda_T$  where  $\delta$  is small.

**Remark 2.** Given that  $\lambda_T$  is set optimally such that  $\delta$  is only slightly above  $3/4$ , the convergence rate of our first step group lasso estimator is slightly slower than  $T^{1/8}$ . This means that we lose a substantial portion of the convergence rate which is  $T$  for fixed breaks under complete information on their location. The reduced convergence rate can be considered the cost for an estimator which is robust against (multiple) structural breaks with unknown location. For comparison, the convergence rate of in-sample predictions for white noise processes with mean shifts reported in Harchaoui and Lévy-Leduc (2010) is  $(T/\log T)^{1/4}$ . Instead, Chan *et al.* (2014) find that in-sample predictions for piecewise stationary autoregressive processes have a faster convergence rate which amounts to  $\sqrt{T/\log T}$ , but this result is based on white-noise assumptions for the error term process.

Theorem 1 shows that it is crucial to let the tuning parameter  $\lambda_T$  grow at the right rate. However, this rate provides only limited practical guidance towards the choice of  $\lambda_T$ . We follow Kock (2016), Qian and Su (2016) and Schmidt and Schweikert (2021) and propose to select  $\lambda_T$  by minimizing an information criterion in the form of

$$IC^*(\lambda_T) = \log\left(\frac{SSR}{T}\right) + \rho_T|\mathcal{A}_T|, \quad (8)$$



where  $SSR$  is the sum of squared residuals resulting from the group lasso estimation of (6) and  $|\mathcal{A}_T|$  gives the number of non-zero breakpoint candidates. The penalty function  $\rho_T$  allows for different choices. While Kock (2016) suggests to use the BIC for potentially non-stationary autoregressive models which corresponds to  $\rho_T = \log(T)/T$ , Qian and Su (2016) propose to use  $\rho_T = 1/\sqrt{T}$  for the estimation of structural breaks in stationary time series regressions. In this article, we follow Schmidt and Schweikert (2021) and employ a modified BIC according to Wang *et al.* (2009) which incorporates the additional factor  $\log \log d_T^*$  where  $d_T^*$  denotes the total amount of coefficients in the full model. This modification of the BIC accounts for the fact that the true model must be found in situations where the number of coefficients diverges.

For the next theorem, we temporarily assume that the exact number of breaks is known. This assumption will help us to provide an important consistency result for the estimated location of breakpoints. We note that this temporary assumption will be relaxed for our main results.

**Theorem 2.** Under Assumptions 1 and 2, if  $m_0$  is fixed and  $|\mathcal{A}_T| = m_0$ , then for all  $\epsilon > 0$

$$P \left( \max_{1 \leq j \leq m_0} |\hat{t}_j - t_j^0| \leq T\epsilon \right) \rightarrow 1, \quad \text{as } T \rightarrow \infty.$$

**Remark 3.** Dividing by  $T$  on both sides of the inequality in Theorem 2 shows that each break fraction can be detected within an  $\epsilon$ -neighbourhood of its true location. Hence, the convergence rate is similar to the one found in Davis *et al.* (2006) who use identical assumptions on the minimum break interval. Harchaoui and Lévy-Leduc (2010), allowing for a maximum number of location shifts in white noise processes, report a slightly faster convergence rate. Similarly, Chan *et al.* (2014) apply group lasso to piecewise stationary autoregressive processes with a potentially diverging number of true breakpoints and report the nearly optimal convergence rate  $\log T/T$  if errors are Gaussian.

The previous result is an important building block for our main results. Next, we prove that the group lasso estimator yields a set of estimated breakpoints for which the number of selected breaks is greater than the true number of breaks almost surely when the exact number of breakpoints is unknown. Furthermore, we evaluate the consistency of estimated breakpoints using the Hausdorff distance between the set of estimated breakpoints and the set of true breakpoints. We follow Boysen *et al.* (2009) and define  $d_H(A, B) = \max_{b \in B} \min_{a \in A} |b - a|$  with  $d_H(A, \emptyset) = d_H(\emptyset, B) = 1$ , where  $\emptyset$  is the empty set. Theorem 3 shows that the set of estimated breakpoints converges to the set of true breakpoints under the Hausdorff distance.

**Theorem 3.** If Assumptions 1 and 2 hold, then as  $T \rightarrow \infty$

$$P(|\mathcal{A}_T| \geq m_0) \rightarrow 1,$$

and for all  $\epsilon > 0$

$$P(d_H(\mathcal{A}_T, \mathcal{A}) \leq T\epsilon) \rightarrow 1.$$

**Remark 4.** The first part of Theorem 3 yields the familiar result that the group lasso estimator is not model selection consistent in settings where the restricted eigenvalue condition (Bickel *et al.*, 2009) or the irrerepresentable condition (Meinshausen and Bühlmann, 2006; Zhao and Yu, 2006) do not hold for the full design matrix. The estimator tends to overselect breakpoints. We note that Assumption 2(iii) is slightly different from the restricted eigenvalue condition used in Bickel *et al.* (2009) and restricts only the design submatrix generated from columns containing active breakpoints. This result shows that we do not systematically select too few breaks which is crucial for the intended second step estimation using weights obtained by group lasso estimation. Ignored breaks would directly result in infinite weights for the second step which would mean that these breaks could not be recovered.

**Remark 5.** The second part of Theorem 3 implies that the Hausdorff distance from the set of estimated breakpoints to the true breakpoints diverges slower than the sample size. Consequently, the Hausdorff distance as a percentage of the sample size is bounded by a constant. This provides us with a consistency result for the estimated break fractions and gives us justification to consider multiple structural breaks at once, since the Hausdorff distance evaluates the joint location of all breakpoints.

Finally, we consider the asymptotic properties of the adaptive group lasso estimator with weights obtained from our first step estimation. We note that Theorem 3 allows us to bound the number of breakpoint candidates by a constant. Hence, the dimensionality of the model selection problem no longer depends on the sample size.

**Theorem 4.** If Assumptions 1 and 2 hold,  $\lambda_S \rightarrow 0$ ,  $\lambda_S^2 T^{(1-\delta)\gamma} \rightarrow \infty$  for  $3/4 < \delta < 1$  and  $\gamma > 0$ , then

- (a) Consistency:  $\|\hat{\theta}_S - \theta_S^0\| = O_p(T^{-1})$
- (b) Model selection:  $P(g(\{j \geq 2 : \|\hat{\theta}_{S_j}\| \neq 0\}) = \mathcal{A}) \rightarrow 1$
- (c) Distribution:

$$T(\hat{\theta}_{S, \bar{\mathcal{A}}^*} - \theta_{S, \bar{\mathcal{A}}^*}^0) \Rightarrow \left[ \int_0^1 \mathbf{B}'_{\tau, \bar{\mathcal{A}}^*} \mathbf{B}_{\tau, \bar{\mathcal{A}}^*} \right]^{-1} \left[ \int_0^1 \mathbf{B}_{\tau, \bar{\mathcal{A}}^*} dU + \Lambda_{\bar{\mathcal{A}}^*}^* \right], \tag{9}$$

$$\Lambda_{\bar{\mathcal{A}}^*}^* = [\Lambda, (1 - \tau_1)\Lambda, \dots, (1 - \tau_{m_0})\Lambda]', \quad \Lambda = \sum_{t=0}^{\infty} E(v_0 u_t),$$

where  $\mathbf{B}_{\tau, \bar{\mathcal{A}}^*}$  and  $U$  are defined in the proof.

**Remark 6.** Although the second step tuning parameter  $\lambda_S$  can be chosen by a selection rule independent of the first step tuning parameter  $\lambda_T$ , its value depends on  $\delta$ , that is how effective additional coefficients are penalized in the first step and consequently how many truly inactive breakpoint candidates remain in our second step design matrix. Since the number of parameters in the full model can now be limited by a prespecified maximum number of breaks, we suggest to use an information criterion like the BIC, which has performed quite well in our simulation experiments.

**Remark 7.** Combining parts (a)–(c) of Theorem 4 shows that the adaptive group lasso estimator has oracle properties. This means that the adaptive group lasso performs correct model selection and has the same asymptotic distribution as the least squares estimator if the breaks’ location would have been known beforehand. Since our regression involves non-stationary components, the asymptotic distribution of the least squares estimator is naturally given as a functional of Brownian motions. Schmidt and Schweikert (2021) use the term ‘non-standard oracle property’ to distinguish it from the term used in Fan and Li (2001). The asymptotic bias term  $\Lambda$  originating from the dependency between increments of the regressors and the error term of the cointegrating regression can be eliminated using dynamic augmentation according to Saikkonen (1991) and Stock and Watson (1993).

**Remark 8.** It is notable that our estimator has non-standard oracle properties although the convergence rate of the group lasso estimator is slower than  $T^{1/8}$ . Zou (2006) argues that the convergence rate of the initial estimator is allowed to be substantially slower than the desired convergence rate of the adaptive lasso estimator if the tuning parameter is specified accordingly.

### 3. SUMMARY OF MONTE CARLO EXPERIMENTS

We conduct simulation experiments to assess the adequacy of our technical results in Section 2. We investigate the finite sample performance of our adaptive group lasso procedure with respect to the accuracy in finding the exact

number of breaks, their location and the magnitude of parameter changes. We consider model specifications with one, two and four breakpoints respectively. The following DGP is employed to model a multi-variate cointegrated system with multiple structural breaks,

$$\begin{aligned} y_t &= \mu + \beta_t X_t + \vartheta_t & \vartheta_t &\sim N(0, \sigma_\vartheta^2), \\ X_t &= X_{t-1} + \omega_t & \omega_t &\sim N(0, \Sigma), \end{aligned} \quad (10)$$

where  $X_t = (X_{1t}, X_{2t}, \dots, X_{Nt})'$  and  $\Sigma = \text{diag}(\sigma_\omega^2)$ , that is the innovations of our generated random walk processes have identical normal distributions.  $\mu$  is a non-zero intercept and  $\beta_t = (\beta_{1t}, \beta_{2t}, \dots, \beta_{Nt})$  is a time-varying slope coefficient vector with non-zero baseline value and a finite number of breaks. We note that  $\text{cov}(\vartheta_t, \omega_t) = 0$ , that is our regressors are strictly exogenous and the asymptotic bias reported in Theorem 4 is non-existent.

Naturally, the ability of all structural break estimators to detect breaks depends on the overall signal strength. Niu *et al.* (2015) define signal strength in change-point models by  $S = m_\beta^2 I_{\min}$ , where  $I_{\min} = \min_{1 \leq j \leq m_0+1} |t_j - t_{j-1}|$  is the minimum distance between breaks and  $m_\beta = \min_{1 \leq j \leq m_0+1} \|\beta_j - \beta_{j-1}\|$  is the minimum jump size. For our main simulations concerned with consistency of the adaptive group lasso estimator, we use equal jump sizes for multiple breaks and locate the breaks with equidistant spacing between them. Hence, overall signal strength is a linear function of the sample size in our simulations. We use a baseline value of two and a jump size of two which is equal to the standard deviation of the regression error term. Simulations with a better signal-to-noise ratio yield more precise estimates for all sample sizes.

In Table I, we report our results for  $N = 2$  regressors. We specify our model for one break located at  $\tau = 0.5$ , two breaks at  $\tau = (0.33, 0.67)$  and four breaks at  $\tau = (0.2, 0.4, 0.6, 0.8)$  to have an equidistant spacing on the unit interval. We first compute the percentages of correct estimation (pce) of the number of breaks  $m$  and measure the accuracy of the break date estimation conditional on the correct estimation of  $m$ . For this matter, we compute the average Hausdorff distance and divide it by  $T$ , henceforth denoted  $\text{hd}/T$ , to compare the values across different sample sizes. The corresponding figures in our tables are reported in percentages. As  $T$  grows larger, the number of breaks is detected with increasing precision and the distance between estimated breakpoints and true breakpoints declines to nearly zero. Parameter estimates are already very accurate at small sample sizes. As expected, the parameter changes of models with fewer breakpoints can be estimated more precisely than those of models with a larger number of breakpoints, as indicated by larger standard deviations obtained for the latter at all sample sizes.<sup>5</sup> Comparing these results with those obtained for the Bai–Perron algorithm,<sup>6</sup> where the number of breaks is determined via the BIC, we find that both approaches perform similarly well. The results are reported in Table II. While the Bai–Perron algorithm estimates the true break fractions slightly more accurately, parameter changes on average have larger standard deviations at all samples sizes. The number of structural breaks is estimated with identical accuracy.<sup>7</sup>

Next, we investigate if dynamic augmentation according to Saikkonen (1991) and Stock and Watson (1993) yields consistent coefficient estimates if the strict exogeneity condition of our main results is violated. To do so, we follow Kejriwal and Perron (2008) and draw the vector  $(\vartheta_t, \omega_{1t}, \omega_{2t})'$  jointly from a multi-variate normal

<sup>5</sup> Results for  $N = 1$  reported in Schmidt and Schweikert (2021) show a very similar pattern. We find that it is slightly more difficult to detect the correct number of breaks in regressions with multiple regressors although the jump size measured as the Euclidean distance is equal for both settings.

<sup>6</sup> Kejriwal and Perron (2008, 2010) obtain estimates of the parameters using the dynamic programming algorithm of Bai and Perron (2003) with no modification since the algorithm itself is valid irrespective of the nature of the regressors and errors given that it detects break dates that minimize the global sum of squared residuals in a regression.

<sup>7</sup> We can confirm the theoretical claims made about the computational complexity of the Bai–Perron algorithm in comparisons to the group LARS algorithm in Section 2.1. We obtain the following computational times (in seconds) for both algorithms. First, using the simulation set-up for Table I and a sample size of  $T = 1000$ , we have  $M = 1$ : (gLARS: 2.61, BP: 11.03),  $M = 2$ : (gLARS: 9.24, BP: 11.41),  $M = 4$ : (gLARS: 21.27, BP: 15.36). Here, we find that the Bai–Perron algorithm is more robust to a larger number of breaks in terms of computational time. Second, we increase the sample size to  $T = 10000$  and record the following times,  $M = 1$ : (gLARS: 4.04, BP: 1013.15),  $M = 2$ : (gLARS: 27.35, BP: 1317.45),  $M = 4$ : (gLARS: 35.23, BP: 1563.55). In this case, we can confirm that the group LARS algorithm is much more computationally efficient for large sample sizes. All simulations are computed on a computer with an Intel i5-6500 CPU at 3.20 GHz and 16 GB RAM.

Table I. Estimation of (multiple) structural breaks in the slope coefficients (adaptive group lasso)

SB1: $\mu = 2, \theta_k = 2, k = \{1, 2\}, (\tau = 0.5)$											
$T$	$pce$	$hd/T$	$\tau$	$\theta_{1,1}$	$\theta_{1,2}$	$\theta_{2,1}$	$\theta_{2,2}$	$\theta_{1,3}$	$\theta_{2,1}$	$\theta_{2,2}$	$\theta_{2,3}$
100	100	0.48	0.501 (0.022)	2.01 (0.144)	1.99 (0.168)	2.01 (0.155)	1.98 (0.168)		2.02 (0.217)	2.01 (0.269)	1.97 (0.321)
200	100	0.17	0.500 (0.009)	2.01 (0.073)	2.00 (0.087)	2.00 (0.073)	1.99 (0.086)		2.01 (0.093)	2.00 (0.116)	2.00 (0.105)
400	100	0.08	0.500 (0.004)	2.00 (0.042)	2.00 (0.057)	2.00 (0.040)	2.00 (0.053)		2.00 (0.046)	2.00 (0.058)	2.00 (0.055)
SB2: $\mu = 2, \theta_k = 2, k = \{1, 2, 3\}, (\tau_1 = 0.33, \tau_2 = 0.67)$											
$T$	$pce$	$hd/T$	$\tau_1$	$\tau_2$	$\theta_{1,1}$	$\theta_{1,2}$	$\theta_{1,3}$	$\theta_{1,3}$	$\theta_{2,1}$	$\theta_{2,2}$	$\theta_{2,3}$
100	99.8	0.69	0.327 (0.024)	0.672 (0.024)	2.01 (0.221)	2.00 (0.254)	1.98 (0.306)		2.02 (0.217)	2.01 (0.269)	1.97 (0.321)
200	100	0.25	0.329 (0.010)	0.670 (0.006)	2.00 (0.092)	2.00 (0.117)	2.00 (0.114)		2.01 (0.093)	2.00 (0.116)	2.00 (0.105)
400	100	0.06	0.330 (0.004)	0.670 (0.002)	2.00 (0.047)	2.00 (0.059)	2.00 (0.053)		2.00 (0.046)	2.00 (0.058)	2.00 (0.055)
SB4: $\mu = 2, \theta_k = 2, k = \{1, \dots, 5\}, (\tau_1 = 0.2, \tau_2 = 0.4, \tau_3 = 0.6, \tau_4 = 0.8)$											
$T$	$pce$	$hd/T$	$\tau_1$	$\tau_2$	$\tau_3$	$\tau_4$	$\tau_4$	$\tau_4$	$\tau_4$	$\tau_4$	$\tau_4$
100	96.5	1.66	0.201 (0.024)	0.402 (0.032)	0.601 (0.026)	0.801 (0.016)					
200	100	0.57	0.199 (0.008)	0.400 (0.006)	0.600 (0.006)	0.800 (0.009)					
400	100	0.16	0.200 (0.006)	0.400 (0.003)	0.600 (0.002)	0.800 (0.003)					
$T$	$\theta_{1,1}$	$\theta_{1,2}$	$\theta_{1,3}$	$\theta_{1,4}$	$\theta_{1,5}$	$\theta_{1,5}$	$\theta_{1,5}$	$\theta_{1,5}$	$\theta_{1,5}$	$\theta_{1,5}$	$\theta_{1,5}$
100	2.04 (0.380)	2.01 (0.497)	1.98 (0.544)	1.96 (0.556)	1.99 (0.504)						
200	2.02 (0.191)	1.99 (0.204)	2.00 (0.213)	2.00 (0.203)	2.00 (0.203)						
400	2.00 (0.074)	2.00 (0.096)	2.00 (0.092)	2.00 (0.087)	2.00 (0.091)						
$T$	$\theta_{2,1}$	$\theta_{2,2}$	$\theta_{2,3}$	$\theta_{2,4}$	$\theta_{2,5}$	$\theta_{2,5}$	$\theta_{2,5}$	$\theta_{2,5}$	$\theta_{2,5}$	$\theta_{2,5}$	$\theta_{2,5}$
100	2.04 (0.358)	2.01 (0.426)	1.96 (0.495)	1.97 (0.532)	1.99 (0.447)						
200	2.02 (0.211)	1.99 (0.222)	2.00 (0.208)	1.99 (0.203)	2.00 (0.203)						
400	2.00 (0.074)	2.00 (0.097)	2.00 (0.090)	2.00 (0.087)	2.00 (0.087)						

Note: We use 1000 replications of the data-generating process given in (10). The variance of the error terms is  $\sigma_\epsilon^2 = 1$  and  $\sigma_\theta^2 = 4$  respectively. The first panel reports the results for one active breakpoint at  $\tau = 0.5$ , the second panel considers two active breakpoints at  $\tau_1 = 0.33$  and  $\tau_2 = 0.67$  and the third panel has four active breakpoints at  $\tau_1 = 0.2, \tau_2 = 0.4, \tau_3 = 0.6$  and  $\tau_4 = 0.8$ . The baseline coefficients and parameter changes at all breakpoints take the value 2. Standard deviations are given in parentheses.

Table II. Estimation of (multiple) structural breaks in the slope coefficients (Bai–Perron)

SBI: $\mu = 2, \theta_k = 2, k = \{1, 2\}, (\tau = 0.5)$											
$T$	$pce$	$hd/T$	$\tau$	$\theta_{1,1}$	$\theta_{1,2}$	$\theta_{2,1}$	$\theta_{2,2}$	$\theta_{1,3}$	$\theta_{2,1}$	$\theta_{2,2}$	$\theta_{2,3}$
100	100	0.23	0.500 (0.009)	2.00 (0.155)	2.00 (0.214)	2.00 (0.156)	1.99 (0.207)				
200	100	0.07	0.500 (0.004)	2.00 (0.077)	2.00 (0.109)	2.00 (0.076)	2.00 (0.106)				
400	100	0.03	0.500 (0.002)	2.00 (0.039)	2.00 (0.054)	2.00 (0.040)	2.00 (0.054)				
SB2: $\mu = 2, \theta_k = 2, k = \{1, 2, 3\}, (\tau_1 = 0.33, \tau_2 = 0.67)$											
$T$	$pce$	$hd/T$	$\tau_1$	$\tau_2$	$\theta_{1,1}$	$\theta_{1,2}$	$\theta_{1,3}$	$\theta_{2,1}$	$\theta_{2,2}$	$\theta_{2,3}$	
100	100	0.40	0.330 (0.008)	0.670 (0.008)	2.00 (0.238)	2.01 (0.322)	1.99 (0.338)	2.01 (0.239)	1.99 (0.338)	1.99 (0.327)	
200	100	0.15	0.330 (0.004)	0.670 (0.003)	2.00 (0.120)	1.99 (0.165)	2.00 (0.164)	2.00 (0.119)	2.01 (0.165)	2.00 (0.159)	
400	100	0.06	0.330 (0.002)	0.670 (0.002)	2.00 (0.059)	2.00 (0.080)	2.00 (0.082)	2.00 (0.058)	2.00 (0.078)	2.00 (0.082)	
SB4: $\mu = 2, \theta_k = 2, k = \{1, \dots, 5\}, (\tau_1 = 0.2, \tau_2 = 0.4, \tau_3 = 0.6, \tau_4 = 0.8)$											
$T$	$pce$	$hd/T$	$\tau_1$	$\tau_2$	$\tau_3$	$\tau_4$					
100	96.7	0.86	0.200 (0.011)	0.401 (0.009)	0.600 (0.011)	0.799 (0.011)					
200	100	0.32	0.200 (0.005)	0.400 (0.004)	0.600 (0.004)	0.800 (0.004)					
400	100	0.13	0.200 (0.002)	0.400 (0.002)	0.600 (0.001)	0.800 (0.002)					
$T$	$\theta_{1,1}$	$\theta_{1,2}$	$\theta_{1,3}$	$\theta_{1,4}$	$\theta_{1,5}$						
100	1.97 (0.400)	2.05 (0.558)	2.00 (0.550)	2.00 (0.587)	1.98 (0.575)						
200	2.00 (0.194)	1.99 (0.266)	2.00 (0.266)	2.00 (0.279)	1.98 (0.271)						
400	2.00 (0.096)	2.00 (0.134)	2.00 (0.133)	2.00 (0.135)	2.00 (0.132)						
$T$	$\theta_{2,1}$	$\theta_{2,2}$	$\theta_{2,3}$	$\theta_{2,4}$	$\theta_{2,5}$						
100	2.02 (0.434)	2.00 (0.539)	1.98 (0.559)	1.99 (0.556)	2.01 (0.559)						
200	2.00 (0.202)	2.00 (0.274)	2.01 (0.251)	2.00 (0.272)	2.02 (0.270)						
400	2.00 (0.095)	2.00 (0.136)	2.00 (0.135)	2.00 (0.133)	2.01 (0.136)						

Note: We use 1000 replications of the data-generating process given in (10). The variance of the error terms is  $\sigma^2 = 1$  and  $\sigma_0^2 = 4$  respectively. The first panel reports the results for one active breakpoint at  $\tau = 0.5$ , the second panel considers two active breakpoints at  $\tau_1 = 0.33$  and  $\tau_2 = 0.67$  and the third panel has four active breakpoints at  $\tau_1 = 0.2, \tau_2 = 0.4, \tau_3 = 0.6$  and  $\tau_4 = 0.8$ . The baseline coefficients and parameter changes at all breakpoints take the value 2. Standard deviations are given in parentheses.

distribution with zero mean and covariance matrix

$$V = \begin{pmatrix} \sigma_{\beta}^2 & 0.5 & 0.5 \\ 0.5 & \sigma_{\omega}^2 & 0 \\ 0.5 & 0 & \sigma_{\omega}^2 \end{pmatrix}. \quad (11)$$

Using this configuration, the strict exogeneity condition is violated for both regressors but the regressors are still generated by independent processes. If we attempt to detect and estimate structural breaks without dynamic augmentation, we still detect breakpoints precisely but obtain strongly biased coefficient estimates. In Table III, we find the corresponding results after the inclusion of  $l = 1$  and  $l = 2$  leads and lags. Now, we can recover the number, location and magnitude of all breakpoints with similar accuracy compared to our simulations under strict exogeneity.

In Table IV, we consider partial breaks in the cointegrating vector. We use a model specification according to the DGP in (10) with  $N = 2$  regressors and induce partial structural breaks through  $\beta_{1t}$  only. Our estimator is applied estimating a full structural change model without prior knowledge that  $\beta_{2t}$  is constant over the sampling period. Again, we observe that the number of breaks, their timing and their magnitude is consistently estimated.<sup>8</sup> The distance between the set of estimated breakpoints and true breakpoints is larger than in the full break setting in Table I. This result is not surprising considering that the break magnitudes for partial breaks are smaller making it more difficult for the adaptive group lasso procedure to detect the true location of the breaks. Consequently, these results also help us to assess how the break magnitude influences the detection rates. Reducing the Euclidean distance from 2 to  $\sqrt{2}$ , roughly doubles the average Hausdorff distance. The convergence rates for zero parameter changes in  $\beta_{2t}$  are almost identical to the convergence rate observed for the non-zero parameter changes in  $\beta_{1t}$ . This is naturally driven by the joint evaluation of all regressors in each group. Unlike bi-level estimators proposed in Huang *et al.* (2009) and Breheny and Huang (2009), the adaptive lasso procedure is not able to shrink coefficients within active groups to zero. Hence, the usual convergence rate for non-zero coefficients applies. In these cases, the convergence rate for  $\beta_{2t}$  could in principle be increased if our procedure was extended to feature bi-level shrinkage. However, this is beyond the scope of this article and is not investigated further at this point.

Finally, we investigate how sensitive our procedure is to break fractions located near the boundary of the unit interval. While the properties of tests for structural changes in the literature depend strongly on the trimming parameter (Bai and Perron, 2006), our method to recover breaks should be more robust in this regard. We only need some lateral trimming to ensure that the first and last regimes identified by our adaptive lasso procedure comprise a sufficiently large number of observations to estimate regime-dependent coefficients.<sup>9</sup> The results for breaks near the boundary are summarized in Table V. The first and second panels consider one break located at  $\tau = 0.1$  and  $\tau = 0.9$  respectively. The pce and average Hausdorff distances over all sample sizes clearly show that a break located close to the beginning of the sample is more difficult to detect than a break located at the end of the sample. Gregory and Hansen (1996a) and Schweikert (2020) report similar findings for their grid search algorithms. To investigate this further, we consider two breaks located at  $\tau = (0.1, 0.9)$  in panel three of Table V. Here, we find that the pce is quite low compared to our main results with equidistant spacing of breakpoints. The first break is estimated less accurately than the second break which can be explained by the fact that parameter changes are measured from one regime to the next and that only a relatively small number of observations is available to estimate the break at  $\tau = 0.1$ .

The results of our first series of boundary experiments imply that it might be possible to relax our trimming restrictions and assume an asymmetric lateral trimming where the first regime must contain sufficiently many observation, say 5% of the sample, while the end of the sample does not necessarily have to be excluded. We

<sup>8</sup> Comparing the results with those obtained for the Bai–Perron algorithm (not reported), we again find that both approaches detect the number of structural breaks with identical accuracy. The Bai–Perron algorithm estimates the true break fractions slightly more accurately, but parameter changes have larger standard deviations at for samples sizes  $T = 200$  and  $T = 400$ .

<sup>9</sup> For our main results, reported in Tables I and II, we follow Kejriwal and Perron (2008) and use a 15% lateral trimming to compare both methods.

Table III. Endogeneity correction via dynamic augmentation

SBI1: $\mu = 2, \theta_k = 2, k = \{1, 2\}, (\tau = 0.1)$										
$T$	$pce$	$hd/T$	$\tau$	$\theta_{1,1}$	$\theta_{1,2}$	$\theta_{2,1}$	$\theta_{2,2}$	$\theta_{2,1}$	$\theta_{2,2}$	$\theta_{2,3}$
100	99.9	0.54	$l = 1$ 0.500 (0.025)	2.00 (0.153)	1.99 (0.170)	2.01 (0.147)	1.99 (0.170)			
200	100	0.14	0.500 (0.008)	2.00 (0.075)	2.00 (0.088)	2.00 (0.069)	2.00 (0.084)			
100	99.9	0.48	$l = 2$ 0.500 (0.022)	2.00 (0.169)	1.99 (0.201)	2.01 (0.170)	1.99 (0.212)			
200	100	0.16	0.500 (0.009)	2.00 (0.075)	2.00 (0.087)	2.00 (0.071)	2.00 (0.085)			
SB2: $\mu = 2, \theta_k = 2, k = \{1, 2, 3\}, (\tau_1 = 0.33, \tau_2 = 0.67)$										
$T$	$pce$	$hd/T$	$\tau_1$	$\tau_2$	$\theta_{1,1}$	$\theta_{1,2}$	$\theta_{1,3}$	$\theta_{2,1}$	$\theta_{2,2}$	$\theta_{2,3}$
100	99.2	0.93	$l = 1$ 0.329 (0.025)	0.672 (0.019)	2.03 (0.292)	1.97 (0.329)	1.99 (0.274)	2.03 (0.285)	1.98 (0.328)	1.98 (0.284)
200	100	0.35	0.330 (0.008)	0.670 (0.010)	2.00 (0.099)	2.00 (0.128)	2.00 (0.141)	2.01 (0.134)	2.00 (0.128)	2.00 (0.158)
100	99.4	1.06	$l = 2$ 0.329 (0.029)	0.670 (0.026)	2.03 (0.290)	1.97 (0.348)	1.99 (0.283)	2.03 (0.299)	1.97 (0.364)	1.99 (0.310)
200	100	0.37	0.329 (0.008)	0.671 (0.011)	2.00 (0.101)	2.00 (0.135)	2.00 (0.123)	2.00 (0.109)	2.00 (0.138)	2.00 (0.133)
SB4: $\mu = 2, \theta_1 = 2, \theta_{2,j} = 2, \theta_{2,j} = 0, j = \{2, \dots, 5\}, (\tau_1 = 0.2, \tau_2 = 0.4, \tau_3 = 0.6, \tau_4 = 0.8)$										
$T$	$pce$	$hd/T$	$\tau_1$	$\tau_2$	$\tau_3$	$\tau_4$	$\tau_3$	$\tau_4$	$\tau_3$	$\tau_4$
100	98.3	1.42	$l = 1$ 0.203 (0.024)	0.402 (0.023)	0.600 (0.022)	0.800 (0.017)				
200	99.4	0.59	0.199 (0.010)	0.400 (0.006)	0.600 (0.009)	0.800 (0.009)				
100	97.9	1.52	$l = 2$ 0.202 (0.022)	0.401 (0.023)	0.600 (0.023)	0.799 (0.019)				
200	99.7	0.64	0.199 (0.011)	0.400 (0.008)	0.600 (0.011)	0.800 (0.011)				
$T$	$\theta_{1,1}$	$\theta_{1,2}$	$\theta_{1,3}$	$\theta_{1,4}$	$\theta_{1,5}$	$\theta_{2,1}$	$\theta_{2,2}$	$\theta_{2,3}$	$\theta_{2,4}$	$\theta_{2,5}$
100	2.04 (0.386)	2.00 (0.422)	$l = 1$ 1.95 (0.487)	1.98 (0.457)	2.01 (0.417)	2.05 (0.403)	1.99 (0.476)	1.99 (0.503)	1.95 (0.474)	1.99 (0.417)
200	2.01 (0.160)	2.00 (0.190)	1.99 (0.185)	2.01 (0.258)	1.99 (0.258)	2.01 (0.157)	2.01 (0.193)	1.99 (0.197)	2.01 (0.261)	1.99 (0.257)
100	2.04 (0.397)	2.00 (0.415)	$l = 2$ 2.01 (0.460)	1.97 (0.452)	2.02 (0.373)	2.05 (0.409)	1.98 (0.437)	2.01 (0.480)	1.94 (0.485)	2.01 (0.380)
200	2.01 (0.191)	2.00 (0.195)	1.98 (0.242)	2.01 (0.203)	1.99 (0.181)	2.01 (0.174)	2.01 (0.199)	1.98 (0.261)	2.00 (0.215)	1.99 (0.181)

Note: We use 1000 replications of the data-generating process given in (10) with an endogenous error term specification. The covariance matrix of the error terms is specified according to (11) with  $\sigma_\omega^2 = 1$  and  $\sigma_\theta^2 = 4$  respectively. We denote the number of leads and lags with  $l$ . The first panel reports the results for one active breakpoint at  $\tau = 0.5$ , the second panel considers two active breakpoints at  $\tau_1 = 0.33$  and  $\tau_2 = 0.67$  and the third panel has four active breakpoints at  $\tau_1 = 0.2, \tau_2 = 0.4, \tau_3 = 0.6$  and  $\tau_4 = 0.8$ . The baseline coefficients and parameter changes at all breakpoints take the value 2. Standard deviations are given in parentheses.

Table IV. Estimation of partial structural breaks

SB1: $\mu = 2, \theta_1 = 2, \theta_{2,1} = 2, \theta_{2,2} = 0$ ( $\tau = 0.5$ )										
$T$	$pce$	$hd/T$	$\tau$	$\theta_{1,1}$	$\theta_{1,2}$	$\theta_{2,1}$	$\theta_{2,2}$	$\theta_{2,1}$	$\theta_{2,2}$	$\theta_{2,3}$
100	99.6	0.80	0.499 (0.033)	2.01 (0.145)	1.98 (0.186)	2.00 (0.137)	-0.01 (0.158)			
200	100	0.26	0.501 (0.013)	2.00 (0.078)	1.99 (0.104)	2.00 (0.074)	0.00 (0.082)			
400	100	0.09	0.500 (0.005)	2.00 (0.040)	2.00 (0.053)	2.00 (0.034)	0.00 (0.042)			
SB2: $\mu = 2, \theta_1 = 2, \theta_{2,j} = 2, \theta_{2,j} = 0, j = \{2, 3\}, (\tau_1 = 0.33, \tau_2 = 0.67)$										
$T$	$pce$	$hd/T$	$\tau_1$	$\tau_2$	$\theta_{1,1}$	$\theta_{1,2}$	$\theta_{1,3}$	$\theta_{2,1}$	$\theta_{2,2}$	$\theta_{2,3}$
100	98.2	1.58	0.328 (0.034)	0.672 (0.032)	2.03 (0.221)	1.99 (0.270)	1.97 (0.293)	2.00 (0.190)	-0.01 (0.230)	0.00 (0.226)
200	100	0.57	0.328 (0.012)	0.671 (0.015)	2.01 (0.105)	1.99 (0.134)	2.00 (0.130)	2.00 (0.092)	0.00 (0.117)	0.00 (0.105)
400	100	0.22	0.330 (0.005)	0.670 (0.005)	2.00 (0.049)	2.00 (0.091)	2.00 (0.078)	2.00 (0.047)	0.00 (0.058)	0.00 (0.055)
SB4: $\mu = 2, \theta_1 = 2, \theta_{2,j} = 0, j = \{2, \dots, 5\}, (\tau_1 = 0.2, \tau_2 = 0.4, \tau_3 = 0.6, \tau_4 = 0.8)$										
$T$	$pce$	$hd/T$	$\tau_1$	$\tau_2$	$\tau_3$	$\tau_4$				
100	94.9	2.86	0.205 (0.035)	0.401 (0.037)	0.600 (0.040)	0.797 (0.033)				
200	99.4	1.10	0.200 (0.015)	0.400 (0.013)	0.600 (0.015)	0.800 (0.015)				
400	100	0.37	0.200 (0.005)	0.400 (0.004)	0.600 (0.004)	0.800 (0.005)				
$T$	$\theta_{1,1}$	$\theta_{1,2}$	$\theta_{1,3}$	$\theta_{1,4}$	$\theta_{1,5}$					
100	2.07 (0.408)	2.00 (1.436)	1.93 (1.527)	1.96 (0.691)	1.98 (0.539)					
200	2.03 (0.217)	1.99 (0.245)	1.98 (0.276)	2.00 (0.236)	2.00 (0.211)					
400	2.01 (0.085)	1.99 (0.104)	2.00 (0.101)	2.00 (0.097)	2.00 (0.096)					
$T$	$\theta_{2,1}$	$\theta_{2,2}$	$\theta_{2,3}$	$\theta_{2,4}$	$\theta_{2,5}$					
100	2.01 (0.313)	-0.02 (0.533)	-0.01 (0.527)	0.00 (0.502)	0.01 (0.448)					
200	2.00 (0.154)	0.00 (0.188)	0.00 (0.183)	0.00 (0.187)	0.01 (0.177)					
400	2.00 (0.074)	0.00 (0.094)	0.00 (0.089)	0.00 (0.087)	0.01 (0.087)					

Note: We use 1000 replications of the data-generating process given in (10). The variance of the error terms is  $\sigma_{\omega}^2 = 1$  and  $\sigma_{\beta}^2 = 4$  respectively. The first panel reports the results for one active breakpoint at  $\tau = 0.5$ , the second panel considers two active breakpoints at  $\tau_1 = 0.33$  and  $\tau_2 = 0.67$  and the third panel has four active breakpoints at  $\tau_1 = 0.2, \tau_2 = 0.4, \tau_3 = 0.6$  and  $\tau_4 = 0.8$ . The baseline coefficients and take the value 2. We induce partial structural breaks through a change in  $\theta_1$  only. Standard deviations are given in parentheses.



Table V. Break fractions located near the boundaries of the unit interval

SB1: $\mu = 2, \theta_k = 2, k = \{1, 2\}, (\tau = 0.1)$										
$T$	$pce$	$hd/T$	$\tau$	$\theta_{1,1}$	$\theta_{1,2}$	$\theta_{2,1}$	$\theta_{2,2}$			
100	89.0	0.19	0.100 (0.013)	2.01 (0.622)	2.00 (0.621)	2.04 (0.581)	1.96 (0.584)			
200	96.6	0.18	0.098 (0.007)	2.00 (0.300)	2.00 (0.302)	2.01 (0.276)	1.99 (0.276)			
400	99.7	0.15	0.099 (0.007)	2.01 (0.150)	2.00 (0.151)	2.00 (0.142)	2.00 (0.142)			
SB1: $\mu = 2, \theta_k = 2, k = \{1, 2\}, (\tau = 0.9)$										
$T$	$pce$	$hd/T$	$\tau$	$\theta_{1,1}$	$\theta_{1,2}$	$\theta_{2,1}$	$\theta_{2,2}$			
100	95.2	0.21	0.901 (0.008)	2.01 (0.090)	1.98 (0.477)	2.00 (0.081)	1.94 (0.493)			
200	98.0	0.18	0.901 (0.008)	2.00 (0.042)	1.99 (0.239)	2.00 (0.044)	2.00 (0.242)			
400	99.5	0.08	0.901 (0.005)	2.00 (0.022)	1.99 (0.132)	2.00 (0.021)	2.00 (0.124)			
SB2: $\mu = 2, \theta_k = 2, k = \{1, 2, 3\}, (\tau_1 = 0.1, \tau_2 = 0.9)$										
$T$	$pce$	$hd/T$	$\tau_1$	$\tau_2$	$\theta_{1,1}$	$\theta_{1,2}$	$\theta_{1,3}$	$\theta_{2,1}$	$\theta_{2,2}$	$\theta_{2,3}$
100	86.4	0.50	0.100 (0.016)	0.901 (0.013)	2.01 (0.623)	2.00 (0.624)	1.98 (0.490)	2.04 (0.587)	1.95 (0.589)	1.95 (0.480)
200	94.8	0.47	0.099 (0.017)	0.901 (0.010)	2.00 (0.278)	2.00 (0.280)	1.98 (0.248)	2.02 (0.287)	1.98 (0.288)	1.99 (0.244)
400	99.3	0.28	0.099 (0.008)	0.901 (0.005)	2.01 (0.158)	1.99 (0.159)	2.00 (0.132)	2.01 (0.147)	1.99 (0.146)	2.00 (0.123)
SB2: $\mu = 2, \theta_k = 2, k = \{1, 2, 3\}, (\tau_1 = 0.1, \tau_2 = 0.95)$										
$T$	$pce$	$hd/T$	$\tau_1$	$\tau_2$	$\theta_{1,1}$	$\theta_{1,2}$	$\theta_{1,3}$	$\theta_{2,1}$	$\theta_{2,2}$	$\theta_{2,3}$
100	83.9	0.49	0.102 (0.023)	0.950 (0.012)	2.02 (0.640)	1.99 (0.638)	1.95 (0.876)	2.05 (0.597)	1.95 (0.599)	1.98 (0.855)
200	92.8	0.40	0.098 (0.008)	0.950 (0.018)	2.00 (0.270)	2.00 (0.270)	1.97 (0.513)	2.02 (0.275)	1.98 (0.277)	1.97 (0.516)
400	97.7	0.26	0.099 (0.007)	0.951 (0.005)	2.01 (0.154)	1.99 (0.154)	1.99 (0.251)	2.01 (0.143)	2.00 (0.143)	2.00 (0.244)

Note: We use 1000 replications of the data-generating process given in (10). The variance of the error terms is  $\sigma_\epsilon^2 = 1$  and  $\sigma_\theta^2 = 4$  respectively. The first panel reports the results for one active breakpoint near the left boundary at  $\tau = 0.1$ , the second panel considers one active breakpoint near the right boundary at  $\tau = 0.9$ . The third panel has two active breakpoints at  $\tau_1 = 0.1$  and  $\tau_2 = 0.9$  and the fourth panel features breakpoints at  $\tau_1 = 0.1$  and  $\tau_2 = 0.95$ . The baseline coefficients and parameter changes at all breakpoints take the value 2. Standard deviations are given in parentheses.

apply a 0.05/0 trimming and estimate breaks located at  $\tau = (0.1, 0.95)$ . The results for this trimming strategy are presented in panel four of Table V. The break at  $\tau = 0.95$  can still be accurately detected, however the standard errors of the parameter changes increase due to the smaller number of observations in the last regime. We conclude that trimming is not necessary to detect breaks located at the end of the sample. Still, we suggest to set a minimum number of observations per regime to ensure that parameter changes are estimated precisely.

#### 4. APPLICATION: US MONEY DEMAND

We apply our proposed methodology to the US money demand function. Particularly, we estimate a long-run money demand specification and investigate the presence of long-run instabilities in a cointegrating framework. Juselius (2006) considers the condition  $M/P = L(Y, R)$  for equilibrium in the money market, which relates  $M/P$ , the ratio of nominal money balances to price levels, to real income  $Y$  and the short term nominal interest rate  $R$ . Two competing empirical specifications are considered in the literature, namely, the semi-log and the log–log specification. The latter is given by  $L(Y, R) = \alpha Y^{\beta_1} R^{\beta_2}$ , where  $\alpha$  is a constant,  $\beta_1$  is the income-elasticity assumed to be unity and  $\beta_2 < 0$  is the interest-elasticity.<sup>10</sup> For our empirical application, we choose a log–log specification which has been found to fit quite well to US data (Lucas, 2000; Bae and Jong, 2007; Ireland, 2009; Mogliani and Urga, 2018). We extend the dataset used by Maki (2012) to span the period from January 1959 to December 2018. Monthly data are obtained from the Federal Reserve Bank of St. Louis. We consider the empirical US money demand function,

$$m_t^* = \mu + \beta_1 y_t + \beta_2 r_t + u_t, \quad (12)$$

where  $m_t^*$  and  $y_t$  denote the natural logarithm of the ratio of nominal money balances to price levels, and the natural logarithm of real income respectively. According to the log–log specification, we employ the natural logarithm of the short term nominal interest rate, denoted by  $r_t$ .  $u_t$  denotes the equilibrium error of the money demand function if the system is cointegrated. We use M2 as nominal money, the consumer price index as prices, and the index of industrial production as real income. For the interest rate, we use the 6-month Treasury bill rate. All time series are tested for a unit root using the Dickey–Fuller test. The results, which are not reported, support the assumption that all variables are integrated of order one which means that we can continue our cointegration analysis.

First, we assume constancy of the parameters and ignore potential structural breaks. Estimation of the long-run equilibrium equation yields coefficients  $\hat{\mu} = -0.05$ ,  $\hat{\beta}_1 = 0.80$  and  $\hat{\beta}_2 = -0.08$ . Dynamic augmentation of the cointegrating regression with two leads and lags each, does not change the coefficient values. The Engle–Granger test based on an ADF regression yields the  $t$ -ratio  $-0.063$  which does not lead to a rejection of the null hypothesis at the 10% level. Similar results can be obtained for the Phillips–Ouliaris test and the Johansen test. Although it is implausible from a theoretical standpoint that the system is not cointegrated, at least our estimated coefficients have the expected sign and magnitude for post-war data. The estimated income-elasticity measured by  $\beta_1$  is slightly below the theoretically expected value. The interest-elasticity of money demand, measured by  $\beta_2$  is expected to be negative. Lucas (2000) considers  $-0.3$ ,  $-0.5$  and  $-0.7$  as values of  $\beta_2$  and finds that  $\beta_2 = -0.5$  gives the best fit for US data. Meltzer (1963), Lucas (1988), Hoffman and Rasche (1991) and Stock and Watson (1993) find empirical evidence consistent with the theoretical expectation that income-elasticity of money demand is unity and interest-elasticity is relatively high. Ball (2001) studies subperiods from 1903 to 1994 and argues against a stable long-run money demand. Further empirical studies have pointed out the presence of structural instability in US money demand for sample periods including data from the 1990s and 2000s (Teles and Zhou, 2005; Wang, 2011; Lucas and Nicolini, 2015). Potential nonlinearities in the functional form are investigated, for example, by Chen and Wu (2005) and Jawadi and Sousa (2013). However, we take the perspective that the linear cointegrating regression in (12) approximates the data well if we simultaneously account for (multiple) parameter changes during the sample period.

<sup>10</sup> Recall that in general, coefficients of log-transformed variables in cointegrating regressions should not be interpreted as elasticities (Johansen, 2005). Only in the special case when those variables are strongly exogeneous, it is allowed to use a ceteris paribus interpretation for the corresponding coefficients.

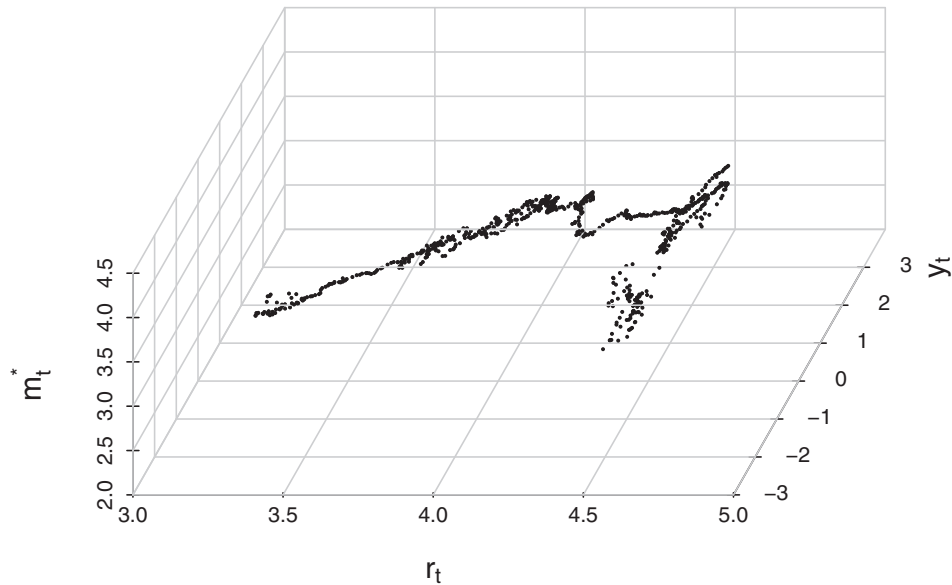


Figure 1. Three-dimensional scatterplot of  $r_t$  (x-axis),  $y_t$  (y-axis) and  $m_t^*$  (z-axis)

A three-dimensional scatterplot of the data in Figure 1 reveals that the relationship between  $r_t$ ,  $y_t$  and  $m_t^*$  has changed during the sampling period. We observe at least three two-dimensional surfaces which correspond to distinct long-run levels from which  $m_t^*$  does not persistently deviate. However, if we consider linear cointegration without the possibility of structural breaks, we infer from Figure 2 that the residual series exhibits a clear trend during the latter half of the sample. We note that the presence of structural breaks might mask the cointegrating relationship. Next, we compare several previously mentioned structural break models with our model selection approach. The Gregory and Hansen (1996a) test indicates a breakpoint at 2008 m06 but does not reject the null hypothesis at the 10% level. Because the GH-test does not model structural breaks under the null hypothesis, this means that the timing of the indicated breakpoint is not informative. The Hatemi-J (2008) test indicates two breakpoints at 1992 m01 and 2008 m06. The null hypothesis of no cointegration can be rejected at the 5% level if these breakpoints are taken into account. The maximum number of breaks chosen for the Maki (2012) test is five. It selects the breakpoints at 1986 m05, 1992 m04, 2004 m05, 2008 m11, 2014 m03 and rejects the null hypothesis of no cointegration at the 1% level. We initially also start with a maximum of five breakpoints for our adaptive group lasso procedure. However, imposing a minimum regime length of 1 year to precisely estimate the parameter changes and dynamically augmenting the cointegrating regression results in a model specification with three breakpoints. The final estimates yield break dates 1992 m07, 2005 m12, and 2015 m11.<sup>11</sup>

The income-elasticity from 1959 m01 to 1992 m07 is estimated to be 0.95 and the interest-elasticity amounts to  $-0.10$  for the same period. These estimates correspond to the theoretical predictions formulated in Juselius (2006) and to the results reported in empirical papers considering this sample period (Lucas, 1988, 2000; Stock and Watson, 1993). The first breakpoint leads to an income-elasticity reduction from 0.95 to 0.89 while the interest-elasticity remains largely unchanged. A partial decoupling of money demand from income might be explained by the begin of the costly Gulf War and a sharp increase in US debt. In turn, the second breakpoint at 2005 m12 has a negligible effect on the income-elasticity (0.89 to 0.90) but results in a larger reduction of the interest-elasticity from  $-0.10$  to  $-0.07$ . This breakpoint can be related to the beginning Global Financial Crisis of 2007–2008. It must be emphasized at this point that estimated break dates might be affected by the usual lead and

<sup>11</sup> The corresponding breakpoint estimates using the Bai–Perron algorithm are almost identically located at 1991 m10, 2004 m07 and 2014 m06.

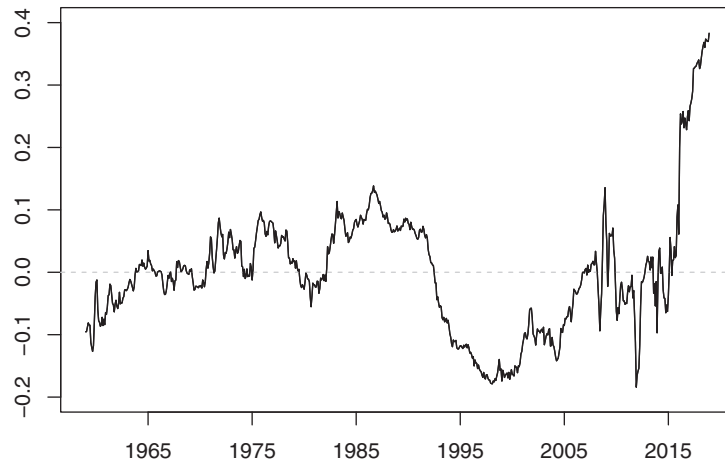


Figure 2. Residual series obtained from least squares estimation

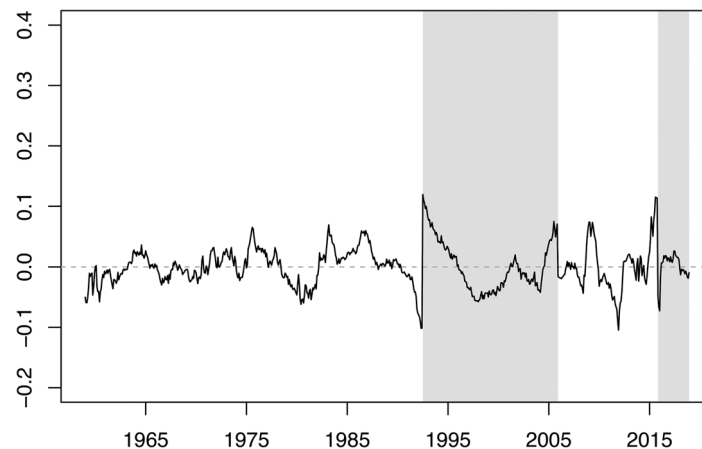


Figure 3. Post-lasso residual series. Estimated regimes are marked by grey and white areas

lag effects, since parameter changes are representative for the following regime. In the aftermath of the Global Financial Crisis, the Federal Reserve implemented a zero interest rate policy. Consequently, the variation in the interest rate for this period approached zero which naturally reduced the interest-elasticity of money demand. After 2015 m11, the expected interest-elasticity does no longer achieve a good fit to the data and increases to 0.01. In contrast, the income-elasticity is very close to unity (0.97).

Accounting for structural breaks, as indicated by the adaptive group lasso procedure, yields a residual series which much more resembles being generated by a stationary process than the original OLS residual. Figure 3 illustrates that the residual series does not exhibit a visible trend. The speed of adjustment after equilibrium errors is now  $-0.097$  which means that roughly 10% of long-run deviations are corrected each period.

## 5. CONCLUSION

In this article, we propose a penalized regression approach to the problem of detecting an unknown number of structural breaks and their location in cointegrating regressions. Our estimator eliminates irrelevant breakpoints from a set of candidate breakpoints and, hence, follows a top-down approach regarding the estimation of structural

breaks. Practitioners should apply this new methodology in complement to the Bai–Perron algorithm which follows a bottom-up approach, that is sequentially increasing the number of breaks. Due to the importance of finding the right model specification with respect to the number and location of structural breaks, either approach can serve as a valuable robustness check of the model specification chosen by the other approach. Ideally both approaches should indicate the same breakpoints which would mean that the chosen model specification is sufficiently sparse (bottom-up) and does not ignore important breaks (top-down).

We can show the important theoretical result that the adaptive group lasso estimator has non-standard oracle properties in settings with a diverging number of breakpoint candidates. This means that the estimator determines the true number of non-zero parameter changes with probability tending to one and consistently estimates their location. The corresponding parameter changes are estimated with the same convergence rate that least squares estimators would have under full information of the number and location of breaks.

This article does not consider cointegration testing. It is unclear how optimal cointegration test can be constructed from the proposed penalized regression approach. An attempt to design such cointegration tests has been made by Schmidt and Schweikert (2021) for a single regressor. Our results depend critically on the stationarity assumption about the error term. Hence, it is required to establish the existence of a cointegration relationship before the penalized regression is estimated. Practitioners should employ cointegration tests which are robust to the presumed number of breaks during the sample period.

Further extensions include the use of bi-level selection via the group fused lasso (Huang *et al.*, 2009; Breheny and Huang, 2009) to estimate partial breaks more efficiently, and the possibility to detect structural breaks in system-based approaches with multiple equilibria (Bai *et al.*, 1998; Qu, 2007).

#### ACKNOWLEDGEMENTS

The author thanks Florian Stark, Alexander Schmidt, Markus Mößler, Timo Dimitriadis, Robert Jung, the editor, Robert Taylor, two anonymous referees as well as the participants of the Doctoral Seminar in Econometrics in Tübingen, German Statistical Week in Trier, ZU Methodenkolloquium in Friedrichshafen, THE Christmas Workshop in Stuttgart, Seminar at Maastricht University, and the 2nd CSL Symposium in Stuttgart for valuable comments and suggestions. Furthermore, the author thanks Maike Becker and Manuel Huth for excellent research assistance. Open Access funding enabled and organized by Projekt DEAL.

#### DATA AVAILABILITY STATEMENT

Data available on request from the authors.

#### SUPPORTING INFORMATION

Additional supporting information may be found online in the supporting information tab for this article.

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