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Voting with abstention

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Abstract

Most voting models in the literature neglect abstention, but is such a simplification justified? I investigate this question in a model with outside pressure on voters. For sequential voting (e.g., roll call votes), with and without an abstention option, there is a *unique subgame perfect equilibrium*, which implies that true majorities always succeed. Abstention can be an equilibrium strategy for some voters, in particular under complex decision rules (e.g., weighted voting, double majorities). Simultaneous voting often has a unique pure strategy equilibrium but also a *plethora of mixed and pure/mixed strategy equilibria*. Therefore, only with equilibrium selection, can we evaluate the consequences of neglecting abstention. For equal weight voting, equilibria selected by the procedure of Harsanyi and Selten change completely with an abstention option, even if abstention itself is not or rarely used. With small enough outside pressure, however, the *selected equilibrium honors true majorities in both cases*.

1 | INTRODUCTION

I want to investigate a model which differs in two respects from most other voting models. First, it assumes vote-dependent costs instead of participation costs and, second, it allows abstention.

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On September 3, 2019, the British Conservative Party withdrew the whip from 21 of its MPs because they voted against the party line. This means, they were practically expelled from their own party. Only 10 of the suspended MPs had the whip restored; only four of these could run in the December 2019 election as Tory candidates (retaining their seats); all other “rebels” lost their seats. This is an extreme example of the outside pressure under which not only members of parliament but also members of other voting bodies make their decisions. MPs’ outside pressure stems from party whips, lobbyists, their electorate, and public opinion (media). Therefore, voting according to true preferences is often connected with costs and, I think, an appropriate model of voting on a proposal should consider such costs. *True preferences* are defined as the vote in cases where it is decisive.

I will use this model to investigate the question whether or not a popular simplification of voting models is justified. The bulk of the theoretical literature about voting on a proposal assumes *binary* decisions (*Yes-No* voting) and *binary results* (accepting or rejecting a proposal). Does the introduction of an abstention option have a considerable strategic effect? If not, the concentration on simpler binary decisions is justified. There are voting rules where abstention has the same effect on the voting result as voting *No*, for example if the acceptance of a proposal needs a certain absolute number of *Yes* votes, that is, if an absolute (super-)majority is necessary. For pluralities (= relative majorities) and many other decision rules, however, abstention and *No* votes are not equivalent and the restriction on *Yes-No* voting is a simplifying assumption.

1.1 | Voting beyond equal weights

Under common ownership of firms, the shares of the owners are their weights when they vote on proposals. Decision making in multinational organizations has to take into account considerable asymmetry concerning the population of the countries as well as their economic, political, and military power. This leads to veto rights, weighted voting, and the requirement of multiple (super-)majorities. For examples of votes with complex decision rules, see Posner and Sykes (2014) and Bolle (2018). We will start this investigation with asymmetric voters and general decision rules and then concentrate on almost symmetric voters and relative super-majorities.

1.2 | Vote dependent versus participation costs

In almost all models, voting costs, if introduced at all, are participation costs. In many empirical (e.g., Hodler et al., 2015) as well as theoretical investigations (e.g., Krishna & Morgan, 2012) these are of central interest. Vote dependent costs may be intrinsic (party or personal loyalty, religious requirements, conscience) or extrinsic, caused by the threats of party whips or bribes from lobbyists or their equivalents in international organizations (Dippel, 2009) and board meetings of firms and faculties (colleagues with special knowledge, special resources, formal or informal power to promote people). There is a small number of nonformal articles on vote dependent costs and their influence on voting, for example, Kilgour et al. (2006) and other discussions of the role of *party whips*. Extrinsic voting costs may also stem from the effect of a vote on public opinion and reputation. The benefits of an accepted proposal may as well stem from different intrinsic and extrinsic sources. Costs and benefits have been discussed in detail for voting games without an abstention option in Bolle (2018).

A central problem in models with participation costs is the *paradox of voting*, that is, the question why voters bear participation costs, in particular in general elections, although their influence on the result is negligible. The central problem in models with vote dependent costs is the question whether or not voters follow their true preferences. While, starting with Downs (1957), a large amount of literature with participation costs has emerged, voting with vote dependent costs has been investigated mainly by Groseclose and Milyo (2010, 2013) and Bolle (2018, 2019). In a mainly empirical investigation of roll call votes in the US senate, Spenkuch et al. (2018), find regularities concerning order effects and narrow results. These will be compared with the unique equilibria, which are derived in the section about sequential votes.

1.3 | Results with vote-dependent costs and without abstention

In equal weight voting, pure strategy equilibria are mostly unique, that is, with the exception of two cases where no pure strategy equilibrium exists. In these equilibria, voters decide only according to their costs, however small they are, and not according to the benefits of an accepted proposal. If we rely on these pure strategy equilibria then we accept that “conflicted” voters never follow their true preferences. It is disturbing and hardly believable that voters will select the pure strategy equilibrium even in cases where costs are negligible. Therefore, Bolle (2019) has investigated competing mixed and pure/mixed strategy equilibria. Under the simplification of almost symmetric voters, he found that, according to Harsanyi and Selten (1988), the pure strategy equilibrium applies only for large deviation costs. For small deviation costs, the voters choose a strictly mixed strategy, which converges to a pure strategy if costs converge to zero. The majority side almost certainly follow their true preferences and the minority side join them. Therefore, the result of the vote honors the “true majority.”

1.4 | Why and when abstain?

Abstention and nonparticipation are often used synonymously, in particular with respect to general elections (e.g., Adams et al., 2006). In this paper, however, abstention is an explicit voting option with costs of its own. Strategically, the option of abstention is not necessary for absolute majority voting. There, voting *No* or *Abstain* have the same effect. If we apply the voting game in an environment where a threshold public good (bad) can be produced by voluntary (predetermined) contributions with positive (negative) costs, a “voter” can either contribute or not, but not abstain. Public good models usually assume that the product is either a public good or a public bad (common pool problem) *for all*; the application of the model with vote dependent costs allows investigating cases where the product is a good for some people and a bad for others. In plurality voting and under more complicated decision rules, abstention is strategically different from voting *Yes* or *No*. Plurality is the most used decision rule in voting, and therefore an investigation of the effect of an abstention option seems to be rather important. In addition to the strategic effect, there is a “psychological” effect of abstention. Even if an absolute majority is required, conflicted voters may nonetheless use abstention as a compromise. Government MPs may reject a proposal of the government with a seemingly weaker *Abstain* vote, which may signal their conflict and their regret that they have to vote against the proposal. Members of the opposing party would strategically follow the party line by voting *Abstain*, but they signal their intrinsic disagreement with their vote. These arguments are in line with the literature on *expressive preferences* (Brennan & Lomasky, 1997; Hillman, 2010). In our model, we

will not assume expressive preferences. There are opportunity costs of abstention, which are smaller than the deviation costs for *Yes* or *No*. We will investigate abstention also under absolute majority requirements.

1.5 | Further literature

The literature on simultaneous and sequential voting is vast and differentiated, for a brief description of different strands of models with different research questions see Bolle and Otto (2019). All these models are quite different from the “vote dependent costs model” of this investigation. Possibly closest to the topic of this article is the large literature on *Vote Buying*, mainly in general elections (e.g., Dekel et al., 2009; Finan & Schechter, 2012), but also in parliament (e.g., Dal Bo, 2007) or in shareholder firms (e.g., Dekel & Wolinsky, 2011). With the exception of Dal Bo (2007), in all these investigations, if there is a game, it is between the buyers of votes, while the reaction to the “bribes” is nonstrategic. In this investigation, the “bribes” are given and the behavior of the voters is strategic. The model of Dal Bo (2007) will be characterized below. In addition, many voting models pose special questions. As an example, regard the question whether there are asymmetric equilibria when players are symmetric—in strategic terms but perhaps not concerning the color of their shoes (Bolle, 2019; De Sinopoli & Iannantuoni, 2007).

The number of voting models with abstention is also rather limited. We find such model variants for cooperative Simple Games (e.g., Felsenthal & Machover, 1997; Freixas, 2012; Musegaas et al., 2018) which are mainly concerned with defining and computing power indices. Abstention is also introduced in models with incomplete information. Feddersen and Pesendorfer (1999) investigate voting with abstention in a model without costs and with asymmetric information of the voters. Battaglini (2005) assumes (small) participation costs and information cascades in sequential voting. Morton and Tyran (2015) experimentally investigate the readiness of nonexperts in voting committees to abstain when there is suspicion that the experts are corrupt. In these models, the aggregation of information is the central topic. My model is with complete information and its central question is whether voters bend to outside forces.

1.6 | Outlook

In the next Section, I will set up a general model of voting. Then I will ask whether sequential voting (Section 3) and simultaneous voting (Sections 4, 5, and 6) are crucially influenced by the introduction of an abstention option. The derivation of the unique subgame perfect equilibrium of sequential voting games with general decision rules is surprisingly simple. In Section 4, the often unique pure strategy equilibria of simultaneous voting games are investigated and Section 5 derives attributes of mixed and pure/mixed strategy equilibria. Section 6 applies the equilibrium selection procedure of Harsanyi and Selten (1988) to voting games with almost symmetric voters. Section 7 evaluates the question whether or not we can stick to simplified voting models without abstention.

2 | VOTING GAMES WITH ABSTENTION

Definition 1. In a *Voting Game with Abstention (VGA)*, there are $n \geq 2$ players who simultaneously or sequentially vote on a certain proposal. The preferences of the voters

will be characterized by Assumptions 1 and 2. The acceptance of a proposal is described by the following structure.

$N = \{1, \dots, n\}$ is the player set, Y denotes the set of players voting Yes, A the set of players who choose *Abstain*, and $N - Y - A$ vote *No*. \mathcal{H} designates the set of all combinations (Y, A) which suffice to accept the proposal. It has the following properties:

(*never sufficient*) $(\phi, \phi) \notin \mathcal{H}$ with $\phi =$ empty set

(*always sufficient*) $(N, \phi) \in \mathcal{H}$

(*monotonicity*) If $Y \subset Y'$, $Y \cup A \subset Y' \cup A'$ and $(Y, A) \in \mathcal{H}$, then also $(Y', A') \in \mathcal{H}$.

Definition 1 is an extension of the definition of *Simple Cooperative Games* where the characteristic function takes binary values. Simple cooperative games describe successful coalitions and therefore correspond to binary (Yes-No) voting. The monotonicity condition requires that switching from *No* to *Abstain* and from *Abstain* to *Yes* do not cause a switch from acceptance to rejection. Together with the monotonicity condition, the two other conditions guarantee that the game is not trivial, that is, voting never having any effect. On first glance, the *never sufficient* condition may appear too weak. It might be strengthened by requiring at least one *Yes* vote for the acceptance of the proposal; but as we have announced in the introduction and as we will explain in more detail below, the voting game will be defined as a “residual game.” All voters with dominant strategies have been removed from the game while taking into account their decisions. \mathcal{H} describes the “residual requirements” for the acceptance of the proposal.

Definition 2. In a VGA with \mathcal{H} designating successful voting combinations we define:

(*Pivot players*) Player i is called a *Yes-Abstain pivot player* with respect to (Y, A) if $(Y \cup \{i\}, A - \{i\}) \in \mathcal{H}$ and $(Y - \{i\}, A \cup \{i\}) \notin \mathcal{H}$. Player i is called an *Abstain-No pivot player* with respect to (Y, A) if $(Y - \{i\}, A \cup \{i\}) \in \mathcal{H}$ and $(Y - \{i\}, A - \{i\}) \notin \mathcal{H}$. Player i is called a *Yes-No pivot player* or simply a *pivot-player* if she is either a *Yes-Abstain* or an *Abstain-No pivot player*, which implies $(Y \cup \{i\}, A - \{i\}) \in \mathcal{H}$ and $(Y - \{i\}, A - \{i\}) \notin \mathcal{H}$.

(*MSC*) We call $(Y, A) \in \mathcal{H}$ a *minimal supporting combination* if all $i \in Y$ are *Yes-Abstain* pivot players and all $i \in A$ *Abstain-No* pivot players.

(*Negligible players*) Player i is called *negligible* if she is not contained in a set of any minimal supporting combination.

(*Absolute majorities*) If $(Y, A) \in \mathcal{H}$ is independent of A , that is, $(Y, A) \in \mathcal{H}$ if and only if $(Y, \phi) \in \mathcal{H}$, then we call the VGA a voting game with absolute majorities.

The most important examples of VGAs are *absolute (relative) k-majority VGAs*, defined by $\mathcal{H} = \{(Y, A) : |Y| \geq k\}$ with $1 \leq k \leq n$ and $|Y| =$ number of members of Y , ($\mathcal{H} = \{(Y, A) : |Y| - |N - Y - A| \geq k\}$ with $-n < k \leq n$). Relative k -majority VGAs are probably the most frequently used decision rule, in parliaments, committees, and faculty meetings. The cases $k = 0$ for absolute majority and $k = -n$ for pluralities are excluded because the proposal would always be accepted.

We now continue the characterization of VGAs as *non-cooperative* games.

Assumption 1 (Preferences of voters in VGAs). If player i votes *No* and if the proposal is not accepted, then her revenue is $R_i = 0$, that is, the status quo is evaluated by 0. Player i bears costs c_i (or d_i) if she votes *Yes* (or *Abstain*) and she enjoys benefits G_i if the proposal is accepted. Players want to maximize their revenues R_i which are *benefits minus costs*, formally $R_i = G_i * 1_{Suc} - c_i * 1_{Yes} - d_i * 1_{Abstain}$ with $1_{Suc} = 1$ if the proposal is accepted and

$1_{Suc} = 0$ otherwise; $1_{Yes} = 1(1_{Abstain} = 1)$ if player i has voted *Yes* (*Abstain*) and $1_{Yes} = 0(1_{Abstain} = 0)$ otherwise.

With three decision options and two results, all utility functions that are additively separable in costs and benefits are equivalent to this simple linear utility function.

Assumption 2 (Parameter restrictions).

- (i) c_i and d_i have the same sign and $0 < |d_i| < |c_i|$.
- (ii) The cases $|d_i| \leq |G_i| \leq |c_i|$ are excluded.

(i) requires abstention to have intermediate costs (between *Yes* and *No*). This should always be the case if outside pressure stems mainly from one group (e.g., party whips) or from several groups with the same goal. The assumption may be violated if outside pressure comes from several groups with different goals or if costs are mainly participation costs. Thus, Assumption 2 (i) separates VGAs from the “participation costs” games in the literature. (ii) excludes two cases to simplify the investigation. A voter with $0 < d_i < G_i < c_i$ would never vote *Yes*, a voter with $c_i < G_i < d_i < 0$ would never vote *No*. Thus, two voter sets with restricted choices are excluded. It is easy to integrate these sets into the analysis, however with more complex characterizations of equilibria.

Six cases remain: If $0 < d_i < c_i < G_i$, player i wants the proposal to be accepted without voting approvingly or abstaining. If $c_i < d_i < 0 < G_i$ or $c_i < d_i < G_i < 0$, i has the dominant strategy to vote *Yes*. If $0 < G_i < d_i < c_i$ or $G_i < 0 < d_i < c_i$, i has the dominant strategy to vote *No*. If $G_i < c_i < d_i < 0$, then i wants to free-ride on the *No* or *Abstention* votes of others.

Lemma 1:

- (i) Voters with dominant strategies vote according to their costs of voting. Voters with negative costs vote *Yes*, voters with positive costs vote *No*.
- (ii) If voters with dominant strategies determine the voting result and if all other voters know this, then all voters vote according to their costs.

(Without proof)

In the following, we neglect voters with dominant strategies after taking their decisions into account. This implies a new definition of (restricted) VGAs.

Definition 3. We define voter sets N^+ and N^- with n^+ and n^- members. For all $i \in N^+$ we have $0 < d_i < c_i < G_i$ and for all $i \in N^-$ we have $G_i < c_i < d_i < 0$. No voter is negligible. All other voters are assumed to follow their dominant strategies. The game with the voter set $N = N^+ \cup N^-$ is called the *reduced simultaneous or sequential voting game with abstention*. A game with $n^+ = 0$ or $n^- = 0$ is denoted as VGA-0. From now on, a VGA denotes all other reduced simultaneous voting games with abstention.

The success of a voting rule may be measured by the probability of producing *true majority results*. Under the naïve assumption that voters always follow their *true preferences*, true majority results are always implied; but this assumption is contrary to almost all equilibria of VGAs. For this statement, however, we need a definition of true preferences and true majorities.

Definition 4. The *true preference* of a voter is defined as her optimal vote in cases where she is decisive with certainty and can choose only Yes or No. Acceptance or rejection has a *true majority* if it results from all voters following true preferences.

The definition implies that voters with dominant strategies always follow their true preferences. The true preference of voters from $N^+(N^-)$ is Yes (No). Acceptance has a true majority if $(N^+, \phi) \in \mathcal{H}$. For equal weight VGAs, this means $n^+ \geq k$ (absolute majority) or $n^+ - n^- \geq k$ (plurality).

A pure strategy equilibrium has a rather simple property: *All voters who follow their true preferences or abstain are pivotal, all others not.* One problem with pure strategy equilibria is finding partitions of players with this property; another is the often-lacking plausibility of these equilibria. In strictly mixed strategy equilibria, all players are pivotal with a certain probability, which has to be equal to a cost/benefit relation. The problem here is the multitude of strictly mixed strategy equilibria and the existence of additional pure/mixed strategy equilibria. Under these circumstances, without equilibrium selection, the contribution of game theory to the evaluation of voting rules would tend to zero.

The central goal of this paper is the comparison of voting games with and without an abstention option. For the latter, we use the same terminology as for VGAs.

Definition 5. A voting game without an abstention option (VG) is a VGA with $A = \phi$ in all definitions. Voters cannot abstain and, therefore, there is only one type of pivotality.

3 | SEQUENTIAL VOTING

The most investigated voting procedure in the US Senate is a roll call vote where the members are required, in alphabetic order, to vote either “yea” or “nay” and where abstention is possible in principle but usually not applied. Roll call votes are sequential although we may argue that, facing a fast sequence of 100 votes, senators have decided on their vote in advance. Therefore, not only abstentions but also dynamics of voting are neglected in most of the econometric work on roll call voting (cf. Clinton et al., 2004). Nonetheless, let us derive the subgame perfect equilibrium of the sequential voting game with abstention. Groseclose and Milyo (2013) have investigated sequential equal weight voting and Bolle (2018) sequential voting under general decision rules, both without the option of abstention. They often find a unique subgame perfect equilibrium.

Let us assume that the order of voters is (player 1, player 2, ..., player n). Again, we disregard all players with dominant strategies after taking their decisions into account so that the sets of remaining necessary votes for the passing of the proposal is again described by \mathcal{H} .

Assumption 3. The voters know whether any other voter is from N^+ or N^- . Voter i knows the votes of voters $j < i$.

The game consists of a sequence of subgames which are essentially described by (Y_i, A_i) , the votes of players 1, 2, ..., $i - 1$.

Proposition 1. *The sequential voting game has a unique subgame perfect equilibrium. All voters decide in accordance to their costs except if they are pivotal under the assumption that all subsequent voters follow their true preferences. The result always honors true majorities. Formally:*

Let (Y^*, A^*) denote the votes cast in the course of the game. We define $Y_i^* = (Y^* \cap \{1, \dots, i-1\}) \cup (\{i+1, \dots, n\} \cap N^+)$ and $A_i^* = A^* \cap \{1, \dots, i-1\}$.

- (i) $i \in N^-$ votes *No* ($i \notin Y^* \cup A^*$) if she is an *Abstain-No* pivot player with respect to (Y_i^*, A_i^*) , she votes *Abstain* ($i \in A^*$) if she is a *Yes-Abstain* pivot player, and otherwise she votes *Yes* ($i \in Y^*$).
- (ii) $i \in N^+$ votes *Abstain* ($i \in A^*$), if she is an *Abstain-No* pivot player with respect to (Y_i^*, A_i^*) , she votes *Yes* ($i \in Y^*$) if she is a *Yes-Abstain* pivot player, and otherwise she votes *No* ($i \notin Y^* \cup A^*$).
- (iii) If $(N^+, \phi) \notin \mathcal{H}$ then the proposal is rejected; all $i \in N^+$ vote *No*.
- (iv) If $(N^+, \phi) \in \mathcal{H}$ then the proposal is accepted; all $i \in N^-$ vote *Yes*.

Proof. (Y^*, A^*) is defined by moving from voter $i = 1$ to voter $i = n$. The proof is by backward induction. Apparently, voter n will stick to the rules (i) and (ii). Then the proposal will be accepted if and only if $(Y_n^* \cup \{n\}, A_n^*) \in \mathcal{H}$. Let us now assume that the proposal will be accepted if and only if $(Y_{i+1}^* \cup \{i+1\}, A_{i+1}^*) \in \mathcal{H}$. Then player $i \in N^+$ will induce $(Y_{i+1}^* \cup \{i+1\}, A_{i+1}^*) \in \mathcal{H}$ if she can and with lowest costs. If she is pivotal with respect to S_i^*, Y_i^* , she votes *Yes* or *Abstain*, otherwise she saves costs and votes *No*. If $i \in N^-$ is pivotal with respect to (S_i^*, A_i) , she votes *Abstain* or *No*, otherwise she incurs negative costs and votes *Yes*. So, (i) and (ii) apply also for player i . (iii) and (iv) follow from (i) and (ii) with $i = 1$.

Spenkuch et al. (2018, p. 1982) find: “In sum, our analysis of sequential roll-call votes in the US Senate documents three empirical regularities. First, the majority party is considerably more likely to barely win a roll-call vote than to just lose it. Second, senators who, by virtue of their last name, get to vote earlier in the order are more likely to deviate from the party line than their colleagues who get to vote late. Third, the vote order affects the choices of senators only on roll calls that end up being close.”

These regularities do not contradict Proposition 1, but they are not implied by it without further assumptions. In Proposition 1, voters with dominant strategies are absent. These always vote according to their costs (following the party line without conflict). Therefore, the vote in the Senate looks always less extreme than it might appear from the viewpoint of Proposition 1. If, after the exclusion of all voters with dominant strategies, the majority party still has a majority, the first regularity is confirmed.

Also the second regularity seems plausible, but again we need additional assumptions. As a counter-example, we require a relative k -majority (as in the US senate) with $n^+ = n^-$ and $k = 0$, that is, the voters from N^+ have a narrow majority. Let us assume that the voters from N^+ vote first, thereafter the voters from N^- . Then all voters vote *Yes*, the first (following their true preferences) because they must vote *Yes* to guarantee the acceptance of the proposal, and the voters from N^- (voting against their true preferences) to incur the negative costs of voting *Yes*. With a reverse order, all voters vote against their true preferences. As an example confirming

the second regularity let us assume $n^+ > k > n^-$. In the unique subgame perfect equilibrium, the voters from N^- vote Yes, but that is not sufficient for the acceptance of the proposal. If the voters from N^- vote first, then only the last $(n^+ + n^- + k)/2$ voters from N^+ follow their true preferences, resulting in a narrow vote. This is also the case if the first positions in the sequence are filled alternately with voters from N^- and N^+ . Also many random distributions of voters will have the same consequences.

The third regularity is not supported by Proposition 1. In our first example, the majority according to true preferences is narrow, but the result may show 100% for the proposal. The second set of examples shows that even a large majority in true preferences can result in a narrow vote. The order of voters is always important—not for the result, but for the decisions of single voters.

Does Proposition 1 imply *Abstain* votes in some situations? If an absolute majority is required, the *Abstain* option is never used. In a plurality k -majority VGA, the relation $n^+ = n^- + k + 1$ with the first player from N^+ is special. In this case, the first player will vote *Abstain*. She need not vote *Yes*, but she must not vote *No*. Also in a subgame with such a relation for the remaining player numbers and the remaining requirement k , the equilibrium choice is *Abstain*. There are no other constellations in games with the equilibrium acceptance of votes where *Abstain* is chosen. Under more complex voting rules, there can be more situations with *Abstain* votes.

The sequential VG with an abstention option has the same nice properties as a sequential VG without. Both have a unique subgame perfect equilibrium with (iii) and (iv) from Proposition 1 (Bolle, 2019). Therefore, in sequential games, the simplification by disregarding the *Abstain* option does not concern the result of the vote.

4 | PURE STRATEGY EQUILIBRIA OF SIMULTANEOUS VOTES

As in the case of sequential votes, pure strategy equilibria are based on ordinal preferences and, therefore, weak information requirements.

Assumption 3'. In the general case, we assume that \mathcal{H} , N^- , and N^+ are common knowledge. For equal weight VGAs, we can reduce this requirement by assuming that k , n^- , and n^+ are common knowledge.

Proposition 2. *Let us assume a VGA with $n^- = 0$, that is, a VGA-0.*

- (i) *Voting according to a minimal supporting combination $(Y, A) \in \mathcal{H}$ is an equilibrium.*
- (ii) *If $(\{i\}, \phi) \notin \mathcal{H}$ for all i , then $(Y, A) = (\phi, \phi)$ describes the only pure strategy equilibrium without the acceptance of the proposal; otherwise no such equilibrium exists.*

Proof. In both cases, no player can gain from changing his decision. In an equilibrium without the acceptance of the proposal, no player would incur costs by voting *Yes* or *Abstain*.

If $\mathcal{H} \neq (N, \phi)$ and every player is nonnegligible, then \mathcal{H} contains more than one minimal supporting combination. In an absolute k -majority VGA with $n^- = 0$, there are $\binom{n}{k}$ minimal combinations with the acceptance of the proposal, all of the form (Y, ϕ) , $|Y| = k$. There are

similar equilibria for relative k-majority VGAs. The equilibria for VGAs, that is, games with $n^- > 0$ and $n^+ > 0$, are completely different. Therefore, the investigation of bribing by Dal Bo (2007) with $n^- = 0$ relies on completely different equilibria than those in a VGA.

Proposition 3 (Necessary and sufficient conditions for pure strategy equilibria of a VGA). *Let (Y^*, A^*) denote the sets of players who vote Yes or Abstain in a pure strategy equilibrium of a VGA. If the proposal is accepted, then*

- (i) $N^- \subset Y^*$
- (ii) No voter $i \in N^-$ is a pivot player, that is, $(Y^* - \{i\}, A^*) \in \mathcal{H}$
- (iii) All voters $i \in (Y^* \cup A^*) \cap N^+$ are pivot players, that is, $(Y^* - \{i\}, A^* \cup \{i\}) \notin \mathcal{H}$ for all $i \in Y^* \cap N^+$ and $(Y^*, A^* - \{i\}) \notin \mathcal{H}$ for all $i \in A^* \cap N^+$

If the proposal is rejected, then

- (iv) $Y^* \cup A^* \subset N^-$
- (v) No voter $i \in N^+$ is a pivot player, that is, $(Y^* \cup \{i\}, A^*) \notin \mathcal{H}$
- (vi) All voters $i \in N^- - Y^*$ are pivot players, that is, $(Y^*, A^* \cup \{i\}) \in \mathcal{H}$ for $i \in N^- - Y^* - A^*$ and $(Y^*, A^* \cup \{i\}) \in \mathcal{H}$ for $i \in A^*$

Proof. The conditions are apparently sufficient: no voter can gain from changing her vote. The necessity of each requirement is shown separately. Ad (i): When the proposal is accepted, every player $i \in N^-$ is, because of her negative costs, better off if she votes Yes. Ad (ii): Because of her negative benefits, every player $i \in N^-$ would withdraw her support if and only if she is a pivot player. Ad (iii): Because of her positive costs, every player $i \in (Y^* \cup A^*) \cap N^+$ would withdraw her support (abstention) if and only if she is not a pivot player. Ad (iv): Because of her positive costs, no player from N^+ would support a rejected proposal. Ad (v): A player from N^+ supports the proposal if and only if she is a pivot player. Ad (vi): All voters from A^* and $N^- - Y^* - A^*$ must not be pivot players because, otherwise, they could prevent negative benefits.

In the following, we characterize cases where $(Y, A) = (N^-, \emptyset)$ describes the unique equilibrium or where no pure strategy equilibrium exists.

Definition 6. Voter j is *replaceable*¹ by i if, for every (Y, A) with $i, j \notin Y \cup A$,

- (i) $(Y \cup \{j\}, A) \in \mathcal{H}$ implies $(Y \cup \{i\}, A) \in \mathcal{H}$
- (ii) $(Y, A \cup \{j\}) \in \mathcal{H}$ implies $(Y, A \cup \{i\}) \in \mathcal{H}$.

i and j are *mutually replaceable* if i is replaceable by j and j is replaceable by i .

All voters in equal weight voting games are mutually replaceable. In general voting games, there may be no replaceability relations between a pair of players or they may be one-sided or

¹For Simple Cooperative Games, Isbell (1958) uses a similar definition but says that “ i is as least as desirable as j .” In the literature on power indices, desirability is used to characterize local monotonicity, that is, if i is as least as desirable as j , then its power index is not lower than j 's index (Freixas & Gambarelli 1997).

mutual. Examples of one-sided replaceability are games where the chairman has tie-breaking power or where shareholders vote with sufficiently unequal shares. The following example without replaceability relations is adopted from Bolle (2018). Double majorities are often required for decisions by condominium owners and in international organizations (Posner & Sykes, 2014).

Example 1 (Double majorities). There are four countries (1, 2, 3, 4) with weights (2, 4, 1, 5) and populations (40, 20, 50, 10). The acceptance of a proposal in an organization of these countries requires aggregate weights of Yes-voters of at least 6 and an aggregate population of at least 60.

In this example, the only minimal supportive combinations of countries are $(\{1, 2\}, \phi)$ and $(\{3, 4\}, \phi)$. Therefore, no player is replaceable by another player.

Proposition 4.

- (i) *If there is an equilibrium (Y^*, A^*) with the acceptance of the proposal and if every $j \in N^+$ is replaceable by a voter $i(j) \in N^-$, then $(Y^*, A^*) = (N^-, \phi)$.*
- (ii) *If there is an equilibrium (Y^*, A^*) with the rejection of the proposal and if every $i \in N^-$ is replaceable by a voter $j(i) \in N^+$, then $(Y^*, A^*) = (N^-, \phi)$.*
- (iii) *If every $i \in N^-$ is replaceable by a voter $j(i) \in N^+$ and if every $j \in N^+$ is replaceable by a voter $i(j) \in N^-$, then there is a unique pure strategy equilibrium $(Y^*, A^*) = (N^-, \phi)$ or no pure strategy equilibrium exists.*

Proof. Ad (i): If a proposal is rejected then, according to Proposition 3 (i), $i \in N^-$ votes Yes. If $j \in N^+$ votes Yes or Abstain then, according to Proposition 3 (iii), j is a pivot player which implies that $i(j)$ is also a pivot player. This contradicts Proposition 3 (ii). Therefore $j \in N^+$ votes No. Ad (ii): similar argument. Ad (iii): (i) and (ii).

Proposition 5 (Sufficient conditions for a unique pure strategy equilibrium (N^-, ϕ)). (Y^*, A^*) denote the sets of players voting Yes or Abstain in a pure strategy equilibrium.

- (i) *If $(N^- - \{i\}, \phi) \in \mathcal{H}$ for all $i \in N^-$, then $(Y^*, A^*) = (N^-, \phi)$ describes the unique pure strategy equilibrium with the acceptance of the proposal. If, in addition, every $j \in N^-$ is replaceable by a player $i(j) \in N^+$, then no pure strategy equilibrium without the acceptance of the proposal exists.*
- (ii) *If $(N^- \cup \{i\}, \phi) \notin \mathcal{H}$ for all $i \in N^+$, then (N^-, ϕ) describes the unique pure strategy equilibrium without the acceptance of the proposal. If, in addition, every $i \in N^+$ is replaceable by a player $i(j) \in N^-$, then no pure strategy equilibrium with the acceptance of the proposal exists.*

Proof. Ad (i): Because of Proposition 3 (i) and $(N^- - \{i\}, \phi) \in \mathcal{H}$, no player from N^- has an incentive to withdraw his support and no player from N^+ has an incentive to vote Yes. Therefore (N^-, ϕ) is the unique equilibrium with the acceptance of the proposal. In an equilibrium (Y^*, A^*) with the rejection of the proposal, $N^- - Y^*$ is not empty because, otherwise, the proposal would be accepted, and, according to Proposition 3 (vi), it contains only pivot players. But as each of these pivot players can be replaced by a player

from N^+ the necessary conditions from Proposition 3 (v) do not apply. Ad (ii): The arguments for (i) can be repeated almost verbally.

Corollary 1.

- (i) In an absolute k -majority VGA with $n^- \neq k - 1$ or k , (N^-, ϕ) is the unique pure strategy equilibrium. There is no pure strategy equilibrium for $n^- = k - 1$ or k .
- (ii) In a relative k -majority VGA with $n^- - n^+ \neq k - 2$ or $k - 1$ or k or $k + 1$, (N^-, ϕ) is the unique pure strategy equilibrium. There is no pure strategy equilibrium for $n^- - n^+ = k - 2$ or $k - 1$ or k or $k + 1$.

Proof. Ad (i): This statement has been proved by Groseclose and Milyo (2010). In our notation, it is easy to check. According to Proposition 4, (N^-, ϕ) is the only possible equilibrium. According to Proposition 5, (N^-, ϕ) is an equilibrium except, possibly, in cases $n^- = k$ or $k - 1$. If $n^- = k$, every $i \in N^-$ has an incentive to switch to Abstain, if $n^- = k - 1$, then every $i \in N^+$ has an incentive to switch to Yes. Therefore, in both cases, (N^-, ϕ) is not an equilibrium. Ad (ii): Arguments as in (i).

Therefore, if we rely on pure strategy equilibria, for simultaneous voting with equal weights we can disregard an abstention option; but, as a general advice, this result is delusive. First, it is not true for more complex voting rules. Second, in many cases these equilibria are not the most plausible candidates for equilibrium selection. In addition to the general *preference falsification* as predicted by Corollary 1, these equilibria are particularly doubtful if they do not imply true majority results although costs are negligible. In such cases, there are competing “almost” pure strategy equilibria with highly probable true majority results. For VGs, Bolle (2019) has shown the existence of such equilibria and their selection by the Harsanyi and Selten (1988) equilibrium selection theory. But before we turn to mixed strategy equilibria, let us briefly consider an example of more complicated voting rules.

Example 2. *The UN security council* has 15 members (countries), five permanent members (set P) and 10 nonpermanent members. A proposal is accepted if at least nine members vote Yes and none of the permanent members votes No. Assume that all permanent members are in N^+ and 5 nonpermanent members (set nP^-) are in N^- and the other 5 (set nP^+) are in N^+ . Therefore, acceptance has a true majority.

With and without abstention, there is an equilibrium $(Y^*, A^*) = (nP^-, \emptyset)$ without the acceptance of the proposal (Proposition 5 (ii)). $(Y^*, A^*) = (nP^- \cup P, \emptyset)$ with the acceptance of the proposal is an equilibrium only without abstention because, only then, the permanent members are pivot players (required by Proposition 3 (iii)). Therefore, with the introduction of an abstention option, the set of pure strategy equilibria can change without abstention being used.

5 | MIXED STRATEGY EQUILIBRIA

In the following two sections, arguments and goals completely change. While pure strategy equilibria are typically unique or nonexistent, there is a plethora of mixed and pure/mixed strategy equilibria. Therefore, if we do not assume pure strategy equilibria to be the only

relevant equilibria, the most important problem is equilibrium selection. There are several methods of equilibrium selection with a (generically) unique result, but their application is connected with a large number of cases and subcases. Therefore, in Section 6, we will restrict our attention to “almost symmetric” relative k -majority VGAs with either “sufficiently” large or small cost/benefit ratios. In the first case, pure strategy equilibria are selected. In the second case, there is still a large number of subcases, but equilibria are selected where either voters from N^+ or N^- play a pure strategy and the others use a mixture between only two strategies. Because of the small cost/benefit ratios, the latter is close to a pure strategy. Nonetheless, these equilibria are completely different from pure strategy equilibria.

It turns out that the description of cases and subcases of equilibrium selection is a bit easier if we take Abstain as the residual option. This slight change of perspective compared with the previous sections will not lead to any confusion because only results are carried over. The players' probabilities of voting *Yes* and *No* are γ_i and ν_i with $\gamma_i + \nu_i \leq 1$. The probability of abstention is $1 - \gamma_i - \nu_i$. We define $p = ((\gamma_i, \nu_i))_{i=1, \dots, n}$ and $p_{-i} = ((\gamma_1, \nu_1), \dots, (\gamma_{i-1}, \nu_{i-1}), (\gamma_{i+1}, \nu_{i+1}), \dots, (\gamma_n, \nu_n))$.

Definition 7. Player i plays a *strictly mixed strategy* if $0 < \gamma_i < 1$, $0 < \nu_i < 1$, and $0 < \gamma_i + \nu_i < 1$. A *partially mixed strategy* has $\gamma_i = 0$ or $\nu_i = 0$ or $\gamma_i + \nu_i = 1$. In a *strictly mixed strategy equilibrium*, all players play strictly mixed strategies. If some players play mixed strategies but not all players play *strictly* mixed strategies we call the equilibrium a *pure/mixed strategy equilibrium*. The probability w_i that voter i is an *Abstain-No* pivot player is called i 's *Abstain-No* decisiveness. The probability x_i that i is a *Yes-Abstain* pivot player is called i 's *Yes-Abstain* decisiveness.

Assumption 3''. In addition to Assumption 3', voters know all cost/benefit ratios.

$Q = Q(p)$ denotes the probability of success, that is, that voting with p results in a combination $(Y, A) \in \mathcal{H}$. Q_{-i} , Q_{0i} , and Q_{+i} denote the probability of success if i votes *No*, *Abstain*, or *Yes* with certainty. The monotonicity condition in Definition 1 implies $Q_{-i} \leq Q_{0i} \leq Q_{+i}$. We have

$$Q = \gamma_i Q_{+i} + \nu_i Q_{-i} + (1 - \gamma_i - \nu_i) Q_{0i} = Q_{0i} + \gamma_i (Q_{+i} - Q_{0i}) - \nu_i (Q_{0i} - Q_{-i})$$

$x_i = Q_{+i} - Q_{0i} \geq 0$ and $w_i = Q_{0i} - Q_{-i} \geq 0$ are the *Yes-Abstain* and the *Abstain-No* decisiveness of player i . Player i 's expected revenue is

$$\begin{aligned} R_i(p) &= G_i Q(p) - \gamma_i c_i - (1 - \nu_i - \gamma_i) d_i \\ &= G_i Q_{0i} + \gamma_i [G_i x_i - c_i + d_i] - \nu_i [G_i w_i - d_i]. \end{aligned} \quad (1)$$

Because Q_{+i} , Q_{0i} , Q_{-i} , x_i , and w_i depend only on p_{-i} , we get

$$\partial R_i / \partial \gamma_i = G_i x_i - e_i \text{ with } e_i = c_i - d_i \quad (2)$$

$$\partial R_i / \partial \nu_i = -(G_i w_i - d_i). \quad (3)$$

A mixed strategy equilibrium requires that R_i is independent of γ_i and ν_i , that is, $\partial R_i / \partial \gamma_i = 0$ and $\partial R_i / \partial \nu_i = 0$. If, in equilibrium, $\partial R_i / \partial \gamma_i > 0$ then *Yes* is preferred to *Abstain*: *Yes* $>$ *Abstain*. $\partial R_i / \partial \gamma_i < 0$ implies *Yes* $<$ *Abstain*. $\partial R_i / \partial \nu_i > (<) 0$ implies *Abstain* $> (<)$ *No*.

In three of the four combinations of the signs of the two derivatives, one of the possible votes is preferred to the other two. If $\partial R_i / \partial \gamma_i < 0$ and $\partial R_i / \partial \nu_i < 0$, then *Abstain* $>$ *Yes*, *Abstain* $>$

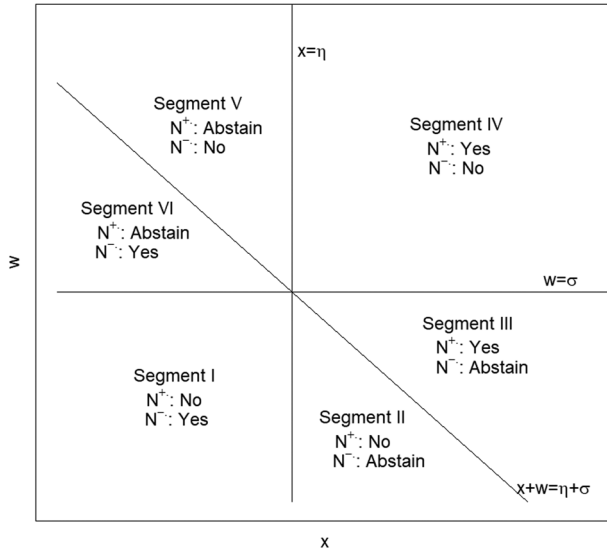


FIGURE 1 The best reply map (Proposition 6). $x_i = x, w_i = w$ and $\sigma_i = \sigma, \eta_i = \eta$.

No and, therefore $\gamma_i = v_i = 0$. If $\partial R_i / \partial \gamma_i > 0$ and $\partial R_i / \partial v_i < 0$, then Yes $>$ Abstain $>$ No and, therefore, $\gamma_i = 1$. If $\partial R_i / \partial \gamma_i < 0$ and $\partial R_i / \partial v_i > 0$, then No $>$ Abstain $>$ Yes and, therefore, $v_i = 1$. In the remaining case $\partial R_i / \partial \gamma_i > 0$ and $\partial R_i / \partial v_i > 0$, Abstain will never be chosen. Yes is optimal if $\partial R_i / \partial \gamma_i > \partial R_i / \partial v_i$, that is, $G_i x_i - e_i > -(G_i w_i - d_i)$ or

$$G_i(x_i + w_i) - (e_i + d_i) > 0 \tag{4}$$

Proposition 6 (Best replies to (x_i, w_i)).

- (i) Strictly mixed strategies require $G_i x_i - e_i = 0$ and $G_i w_i - d_i = 0$
- (ii) If $G_i x_i - e_i < 0$ and $G_i w_i - d_i > 0$ then Abstain is played with certainty.
- (iii) If $G_i x_i - e_i > 0$ and $G_i w_i - d_i > 0$ then Yes is played with certainty.
- (iv) If $G_i x_i - e_i < 0$ and $G_i w_i - d_i < 0$ then No is played with certainty
- (v) If $G_i x_i - e_i > 0$ and $G_i w_i - d_i < 0$, then Abstain will never be chosen. Yes (No) is played with certainty if $G_i(x_i + w_i) - (e_i + d_i) > (<)0$.

Proof. (2) and (3).

Let us define $\eta_i = e_i / G_i$ and $\sigma_i = d_i / G_i$. Then Proposition 6 (i) is $x_i = \eta_i$ and $w_i = \sigma_i$. For the inequalities in Proposition 6, not only the cost/benefit ratios are important but also the signs. Therefore, using cost/benefit ratios, the other conditions of Proposition 6 apply differently to players from N^+ and N^- . The best reply map (Figure 1) serves to visualize conditions for pure, mixed, and partially mixed equilibrium strategies. (x_i, w_i) are implications of the equilibrium strategies and the equilibrium strategies must be best replies to these (x_i, w_i) . The unique pure strategy equilibria of Corollary 1 have $(x_i, w_i) = (0, 0)$, strictly mixed strategy equilibria must imply $(x_i, w_i) = (\eta_i, \sigma_i)$. Partially mixed equilibrium strategies need decisiveness combinations on the lines in Figure 1, which are characterized by equalities. Between these lines the best replies are pure strategies; that is, equilibrium strategies with such decisiveness combinations are pure strategies. As an example, let us take Segment IV with $x_i > \eta_i$ and $w_i > \sigma_i$. For players from N^+ , Proposition 6 (iii) applies, that is, Yes is a best reply. For players from N^- , Proposition 6 (iv) applies, that is, No is a best reply.

The best reply map has been derived for one player. If, however, replaceable voters play the same strategy, then the same point on the best reply represents their decisiveness. If voters within N^+ and N^- play the same strategies but between N^+ and N^- different strategies, then their decisiveness is represented by two (mostly) different points on the best reply map. If all players play different strategies, then every player has her individual (x_i, w_i) . In the following, we are particularly interested in cases where voters are almost symmetric and N^+ and N^- play different strategies. As an example, let voters from N^+ play the pure strategy *No* and let voters from N^- mix between *Yes* and *Abstain*. These strategies are equilibria only if the decisiveness (x_i, w_i) of players from N^- lies on the line $x = \eta$ (in Figure 1, indices are suppressed), and the decisiveness of players from N^+ is anywhere in segments I or II.

5.1 | VGAs with almost symmetric players

VGAs often have an extreme multiplicity of equilibria. In these cases, the question which of these equilibria applies is the crucial question for applications. Unique pure strategy equilibria are salient candidates; but for small cost/benefit ratios the implied deviations from true majority voting are particularly disturbing. Equilibrium selection by “plausibility” or with formal procedures is difficult to apply in a general VDA. Therefore, in the following, we concentrate on almost symmetric players.

Definition 8 *Two replaceable players are called symmetric if they have the same costs and benefits. They are called almost symmetric if they have the same cost/benefit ratios $\eta_i = \eta$ and $\sigma_i = \sigma$. A game is called almost symmetric if all player pairs are almost symmetric. An almost symmetric relative k -majority VGA, where all voters from N^+ and N^- are restricted to play the same strategies $s^+ = (\gamma^+, v^+)$ and $s^- = (\gamma^-, v^-)$, is denoted as $\Gamma(\eta, \sigma, k, n^+, n^-)$.*

Because symmetric players cannot be distinguished, it is practically impossible for them to coordinate on an equilibrium where they play different strategies. Therefore, the restriction to $\Gamma(\eta, \sigma, k, n^+, n^-)$ is plausible for symmetric players. Harsanyi and Selten (1988) regard players as symmetric if their utilities are equal after a linear positive transformation, that is, all players with the same sign who are almost symmetric according to Definition 8 are symmetric according to Harsanyi and Selten (1988). They require symmetric players to play the same strategy; that is, following their requirement, we should investigate games $\Gamma(\eta, \sigma, k, n^+, n^-)$.

It is easy to derive from Proposition 6 (i) that $\Gamma(\eta, \sigma, 0, 1, 1)$ has a unique *strictly mixed strategies equilibrium* and that $\Gamma(\eta, \sigma, 0, 2, 1)$ with $\eta > 1/2$ has no strictly mixed strategies equilibrium. With numerical methods we find at least three strictly mixed strategy equilibria for $\Gamma(0.3, 0.1, 1, 2, 1)$, namely $(\gamma^*, v^*) = (0.11, 0.44)$, $(\gamma^{**}, v^{**}) = (0.23, 0.55)$, and $(\gamma^{***}, v^{***}) = (0.73, 0.20)$ played by all players. Generically, however, strictly mixed strategy equilibria do not play any role in the following equilibrium selection procedure. Nonetheless, these examples illustrate that the set of pure, strictly mixed, and pure/mixed equilibria may be large and diverse. Lemmas 2, 3, and 4 in the appendix report restrictions to equilibria with partially mixed strategies.

6 | EQUILIBRIUM SELECTION FOR GAMES $\Gamma(\eta, \sigma, k, n^+, n^-)$

The discussion of social interactions by applying game theory usually needs equilibrium selection criteria, in particular if parameter (policy) changes are to be discussed. Assuming a unique pure strategy equilibrium (if existent) to be played is one possibility. Another is the application of a

formal equilibrium selection theory. From the multitude of the existing concepts, I have chosen Harsanyi and Selten's (1988) selection of an equilibrium by the *Linear Tracing Procedure*. In my view, there are mainly two prominent competitors of this concept. One is selection via *Global Games* (Carlsson & Van Damme, 1993) where common knowledge games are transformed into games of incomplete information with private and correlated signals. While incomplete information (noise) vanishes, equilibrium play converges, under certain conditions, to one of the Nash equilibria of the original game. The Global Games selection has turned out to be easily applicable only for games where strategies are strategic complements. Frankel et al. (2003) show that, otherwise, the result may depend on the distribution of "noise." In voting games, however, equilibria coexist where strategies are strategic complements or substitutes (see Bolle, 2019). A further prominent method of equilibrium selection was suggested by McKelvey and Palfrey (1995) as a limit of their Quantal Response Equilibria (QRE) which assume all strategies (from a finite set) to be played with probabilities that are ordered according to the utilities these strategies gain against the strategies of the other players. When random deviations from the best response vanish, QRE (generically) converges to a Nash equilibrium.² For voting games, the QRE selection suffers from the problem that QREs depend not only on cost/benefit ratios but also on the magnitude of costs and benefits. This also applies to limits of QRE. Therefore, our choice of the QRE selection depends on the question whether we assume that players with $(c_i, e_i, G_i) = (2, 1, 4)$ and $(c_i, e_i, G_i) = (4, 2, 8)$ behave equally or differently. If we assume that linear positive transformations of utility functions do not matter, we should not apply the QRE selection to games $\Gamma(\eta, \sigma, k, n^+, n^-)$. Investigating equilibrium selection in the class of almost symmetric games via QRE seems to be an awkward choice. Thus, I argue that, while all three prominent equilibrium selection theories have general advantages and disadvantages, for $\Gamma(\eta, \sigma, k, n^+, n^-)$ only the Harsanyi and Selten theory, which requires the same strategies within the player sets N^+ and N^- , is a sensible choice.

6.1 | The tracing procedure

Harsanyi and Selten (1988) select equilibria mainly by the application of the *Linear Tracing Procedure* which describes a process of expectation formation of the players. The procedure starts from the *centroid* of a game Γ which consists of the strategy profile where every pure strategy is played with the same probability. In our games with three pure strategies, all strategies are played with probability $1/3$. In the *tracing procedure*, for every $0 \leq t \leq 1$, equilibria are determined in a game $\Gamma(t)$. Best replies at t are derived under the assumption that other players play random with probability $1 - t$ and follow a certain strategy with probability t . In an equilibrium of $\Gamma(t)$ all players play such best replies. If there is a unique continuous path of equilibria from $t = 0$ to $t = 1$, then the equilibrium at $t = 1$ is selected. Harsanyi and Selten (1988) and Herings (2000) prove the general feasibility of the linear tracing procedure.

In the following, all variables which are indicated as dependent on t are related to $\Gamma(t)$. Without such an indication, a variable is related to $\Gamma = \Gamma(1)$. The tracing procedure can be

²In addition to these concepts, there is the proposal by Güth and Kalkofen (1989), there are approaches with learning to play equilibria (e.g., Berninghaus & Ehrhart, 1998) or other dynamic approaches (Binmore & Samuelson, 1999). Kim (1996) investigates further proposals for equilibrium selection in symmetric normal form games with Pareto-ranked equilibria. Our equilibria are usually not Pareto-ranked.

TABLE 1 Thresholds k without $(x_i^+, w_i^+) = (x_i^-, w_i^-) = (0, 0)$ if voters decide according to (w, x) in the best reply map

Segment	$x^+ = 1$	$w^+ = 1$	$x^- = 1$	$w^- = 1$
I	$n^- - n^+ + 2$	$n^- - n^+ + 1$	$n^- - n^+$	$n^- - n^+ - 1$
II	$-n^+ + 2$	$-n^+ + 1$	$-n^+ + 1$	$-n^+$
III	n^+	$n^+ - 1$	$n^+ + 1$	n^+
IV	$n^+ - n^-$	$n^+ - n^- - 1$	$n^+ - n^- + 2$	$n^+ - n^- + 1$
V	$-n^- + 1$	$-n^-$	$-n^- + 2$	$-n^- + 1$
VI	$n^- + 1$	n^-	n^-	$n^- - 1$

illustrated by paths $q(t) = (X(t), W(t))$ of decisiveness scores on the best reply map (Figure 1). The tracing procedure starts with decisiveness values implied by the centroid.

$$q(0) = q_i(0) = (X_i(0), W_i(0)) \quad (5)$$

where $X_i(0)$ is the probability that, for $t = 0$, the number of Yes votes minus the number of No votes is $k - 1$, and $W_i(0)$ is the probability that this difference is k .

Let us assume that $q(0)$ is an interior point in one of the segments in Figure 1. Then the best replies to $q(0)$ in game Γ are the unique pure strategies indicated in the best reply map. In Segment I, for example, the best reply to $q(0)$ is *No* for $i \in N^+$, that is, $(y_i^+, v_i^+) = (0, 1)$ and *Yes* for $i \in N^-$, that is, $(y_i^-, v_i^-) = (1, 0)$. This is the only equilibrium of $\Gamma(0)$. For $t > 0$, we have to take into account the decisiveness scores (x_i, w_i) in game Γ , caused by these strategies. The decisiveness scores in game $\Gamma(t)$ are

$$q_i(t) = (X_i(t), W_i(t)) = (1 - t)q(0) + t(x_i, w_i). \quad (6)$$

Let us assume $q(0)$ to be in segment I (Case 0), that is, voters from N^- vote Yes and voters from N^+ vote No. Therefore, players from N^- (N^+) expect $n^- - n^+ - 1$ ($n^- - n^+ + 1$) Yes minus No votes from the other voters. Excluding the cases from Table 1, we have $(x_i, w_i) = (0, 0)$ and $q_i(t)$ moves on a line from $q(0)$ to the origin.

The introductory example of the 21 Tory rebels seems to be a typical example with $q(0)$ in Segment I, provided the rebels had rational expectations. However, after declining so many previous proposals, mainly from the May government, the rebels may have developed an illusion of impunity. Therefore, this is an example with (surprisingly) high costs, but not necessarily for Case 0 and an incorrect theoretical prediction.

Because pure strategy equilibria are particularly implausible for small cost/benefit ratios, we are mainly interested in cases where $q(0)$ lies in Segment IV. When it is necessary for the exclusion of further subcases, we will always assume that the cost/benefit ratios are “sufficiently” small. Our concentration on this scenario helps a lot to keep this investigation tractable, but nonetheless there are still many subcases with different relations between k , n^+ , and n^- .

6.2 | The tracing procedure with sufficiently small cost/benefit ratios

The tracing procedure is investigated in more detail in the appendix; here, only the essential ideas are presented. In the following, let us suppress the indices i of players. We distinguish only voters

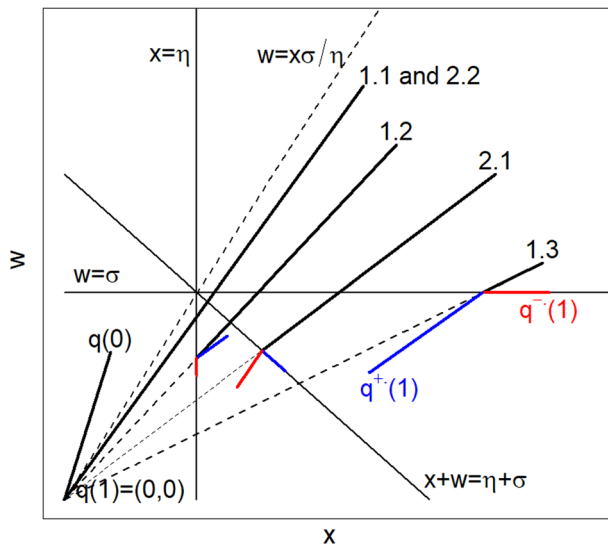


FIGURE 2 $q(t)$ in the cases of Proposition 8, blue lines $q^+(t)$ for voters from N^+ , red lines $q^-(t)$ for voters from N^- , and black lines for $q^+(t) = q^-(t)$. The endpoints $q(1)$ or $q^+(1)$ and $q^-(1)$ describe the decisiveness scores in the selected equilibrium

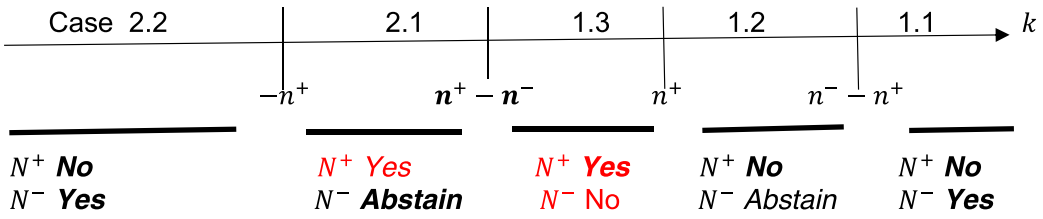
from N^+ and N^- . Let us assume that $q(0)$ lies in Segment IV and let us exclude the special cases indicated in Table 1 for Segments I, II, III, and IV. Then, $(x^+, w^+) = (x^-, w^-) = (0, 0)$ applies and $q(t) = q^+(t) = q^-(t)$ moves on a line to the origin until it intersects the line $w = \sigma$. While voters from N^+ keep their *Yes* vote in Section III, voters from N^- switch from *No* to *Abstain*.

$q(t)$ can move further on the line to the origin only if, at the intersection and for constant t , the players from N^- continuously increase their probability of voting *Abstain* from 0 to 1. This is possible if $k \geq n^+ + 2$ or $k \leq n^+ - n^- - 2$. In the former case, the proposal is always rejected, independent of the probability of the voters from N^- voting *Abstain* and independent of whether one of the players completely changes her vote. Therefore $(x, w) = (x^+, w^+) = (x^-, w^-) = (0, 0)$ is maintained. As players from N^- are indifferent in game $\Gamma(t)$, the transition constitutes a continuous path of equilibria. In the case $k \leq n^+ - n^- - 2$, in game Γ , the proposal is always accepted. The development of $q(t)$ is described in Figure 2. In Cases 1.1 and 2.2, $(x, w) = (x^+, w^+) = (x^-, w^-) = (0, 0)$ applies on the whole path $q(t)$ to the origin. Therefore, in these cases, again the pure strategy equilibrium is selected. The difference to the case with $q(0)$ in Section I is that the pure strategy equilibrium is assumed only if the true majority result is implied, in Cases 1.1 rejection and in Case 2.2 acceptance of the proposal.

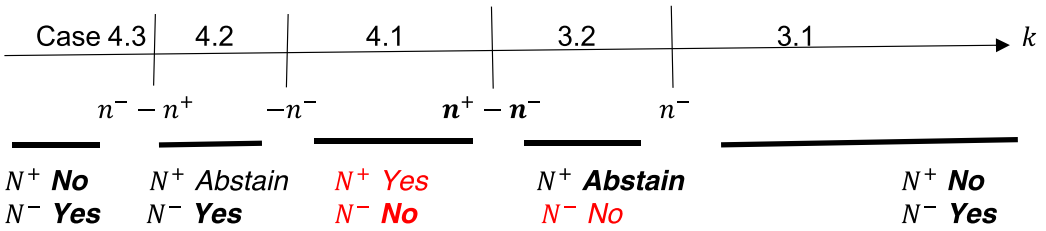
In the remaining cases, $q(t)$ cannot cross all lines to the origin. In Case 1.3, already $w = \sigma$ cannot be crossed. When $q(t)$ intersects this line, the voters from N^- have to start increasing the probability of voting *Abstain* from zero to positive values; but this cannot happen with a constant t . Because of $w^- > 0$, $q^-(t)$ would leave the line $w = \sigma$ and the voters from N^- would no longer be indifferent between *No* and *Abstain*. To continuously increase the probability of voting *Abstain*, $q^-(t)$ must stay on the line $w = \sigma$. Equation (6) shows that we need increasing t to balance the increasing w^- . Then $q^-(t)$ and $q^+(t)$ separate with $q^-(t)$ moving along the line $w = \sigma$. In the appendix, it is shown that, for sufficiently small σ and η , $q^+(t)$ moves downwards but does not reach the line $w + x = \sigma + \eta$. Proposition 8 in the appendix discusses all cases in Figure 2 in detail. Proposition 9 in the appendix covers cases with $q(0)$ in Segment IV but above

With abstention option:

$q(0)$ below $w = x\sigma/\eta$



$q(0)$ above $w = x\sigma/\eta$:



Without Abstention option:

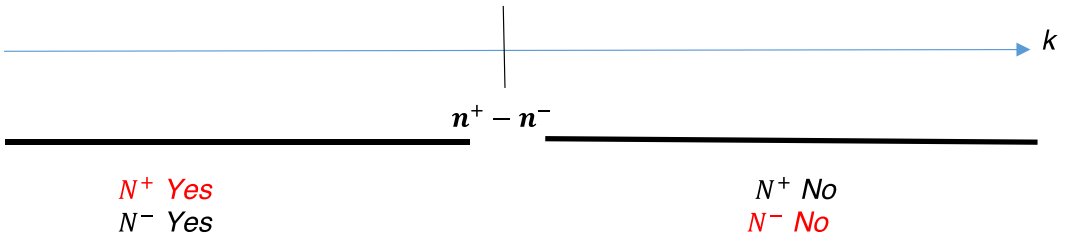


FIGURE 3 Voting with sufficiently small η and σ . Red (black): voting according to (against) true preferences. Bold type: Certainty instead of high probability. *Explanatory remarks:* Figure 3 is a visual representation of Propositions 7, 8, and 9 (the latter two in the appendix). Around the indicated borders of cases, some k values are excluded. Either Case 1.2 or 2.1 is empty; if $n^- < 5$, Case 3.1 is empty. Either Case 3.2 or 4.2 is empty; if $n^+ < 5$, Case 4.1 is empty. In all cases covered by Propositions 7, 8, and 9, the voting result reflects, with high probability, true majorities

the line $w = x\sigma/\eta$. Proposition 7 will describe the results of voting without abstention. Figure 3 illustrates Propositions 8, 9, and 7, thus allowing an easy comparison of the selected equilibria with and without an Abstain option.

Considering so many cases and subcases is tedious but seems to be inevitable because different relations between k , n^+ , and n^- cause different voting behavior. The common property of all these cases (with “sufficiently” small σ and η) is that true majorities are honored with certainty or high probability.

6.3 | Voting without abstention

In the best reply map, we have to consider only the line $x + w = \eta + \sigma$. Above this line, voters from N^+ vote Yes and voters from N^- vote No; vice versa below the line. With $q(0)$ above this line, $q(t)$ moves on a line to the origin until it intersects $x + w = \eta + \sigma$ at $t = t_1$. Until this point, the majority voters (for $k > n^+ - n^-$, the voters from N^- , otherwise those from N^+) determine the result with certainty. Therefore and because they are indifferent, the minority voters can continuously change their strategy to that of the majority voters. Then, for $t > t_1$ both play the symmetric mixed strategy equilibrium with $\pi(t) =$ probability of the majority decision, starting with $\pi(t_1) = 1$. $q(t)$ remains on the line $x + w = \eta + \sigma$ until $t = 1$. For $\eta + \sigma$ sufficiently small, $\pi(1)$ is arbitrarily close to 1. This implies the voting behavior described in Figure 3. The tedious part of the proof is the exclusion of other paths $q(t)$.

Proposition 7. *Without an abstention option and for sufficiently small $\eta + \sigma$, the selected equilibrium for $n^+ - n^- > k$ is symmetric, all players voting Yes with high probability; for $n^+ - n^- < k - 1$, all players vote No with high probability.*

Proof. Bolle (2019).

There are two differences between the tracing procedures with and without abstention. First, on the lines $x = \eta$ and $w = \sigma$, transitions are “smooth,” namely from *Yes* to *Abstain* or from *Abstain* to *No* compared with “abrupt” transitions from *Yes* to *No* on the line $x + w = \eta + \sigma$ which is the only critical line without abstention. Second, with abstention, only one of the player groups is indifferent on a line. Therefore, at every intersection of $q(t)$ with one of the lines, only one side changes their strategy.

6.4 | The excluded cases

The investigation of cases where $q(0)$ lies in Segments II, III, V, and VI, follows the same lines as in the other cases, but medium η and σ require considering more subcases. The omission of the special values k in Table 1 cause “gaps” between the main cases (see Figure 3). Their discussion can but need not be difficult. If, for example, $q(0)$ lies in Segment I and if $k = n^- - n^+ + 2$, then $x^+ = 1$ and $w^+ = x^- = w^- = 0$ (see Table 1). $q^-(t)$ moves along the line to the origin and $q^+(t)$ downward to the right until it possibly intersects the line $x = \eta$.

7 | CONCLUSION: CAN WE NEGLECT ABSTENTION?

For sequential voting with general decision rules, we get unique subgame perfect equilibria with and without abstention, both honoring true majorities. Therefore, we can neglect abstention if we are interested only in the result of a vote. In equal weight voting, abstention is rarely used.

For simultaneous voting, uniqueness of equilibria and independence from an abstention option applies only for pure strategy equilibria of equal weight majority voting. In these pure strategy equilibria, all voters vote against true preferences with the consequence that results

often do not reflect true majorities. Such results are implausible for small voting costs. At least then, the plethora of mixed and pure/mixed strategy equilibria of simultaneous voting games should be taken into account. Applying the selection (tracing) procedure of Harsanyi and Selten (1988) to equal weight VGAs with almost symmetric voters, we get more plausible results. Pure strategy equilibria (possibly with results against true majorities) are selected if the costs of voting according to true preferences are large enough; for small enough costs, pure strategy equilibria are selected only if they imply true majority results. In the other cases with small enough costs, an equilibrium is selected where either the voters from N^+ or N^- play a pure strategy and the others a mixture between only two of the three options. In VGs with low enough costs, equilibria are selected with all voters playing the same strategy; also these equilibria honor true majorities with high probability.

By applying the tracing procedure of Harsanyi and Selten (1988) to simultaneous equal weight voting with small voting costs, we partially reproduce the results for sequential voting: true majorities are honored with high probability. Therefore, evaluating our extant voting rules by equilibria selected with the tracing procedure instead of pure strategy equilibria is connected with much more trust in these democratic institutions.

While the voting outcome is not affected, voting behavior in simultaneous equal weight games depends a lot on the existence of an abstention option. According to Figure 3, abstention is used with high probability only in some subcases; but this does not imply that voting behavior is otherwise not affected. In all cases, either the voters from N^+ or N^- play a different strategy when the abstention option is available. An additional effect of an abstention option is that, for small cost/benefit ratios, the selected equilibria are not only close to pure strategy equilibria, but that many mixture probabilities are exactly 0 or 1 (bold types in Figure 3). Without abstention, all players play the same strictly mixed strategy; with abstention, either voters from N^+ or N^- play a pure strategy and the others use a mixture between only two strategies.

Therefore, if we are interested only in the voting outcome, we can simplify our models by neglecting abstention; if we are interested also in voting behavior, we cannot, even if abstention itself is not used. The difference is much more pronounced in simultaneous than in sequential voting.

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DATA AVAILABILITY STATEMENT

All data will be made publicly available.

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APPENDIX A

For the proof of Propositions 8 and 9, we need Lemmas 2, 3 and 4.

$$B(\pi, m; j) = \binom{m}{j} \pi^j (1 - \pi)^{m-j}. \quad (\text{A1})$$

is the binomial distribution with $\pi =$ success probability and $j =$ number of successes $0 \leq j \leq m$. B is a unimodal function of π with the maximum $\max(m, j)$ at $\pi = j/m$. For every $0 \leq \delta < \max(m, j)$ and $0 < j < m$, $B(\pi, m; j) = \delta$ has two solutions; for $\delta = 0$ they are $\pi = 0$ and $\pi = 1$, for increasing δ they continuously increase (decrease). For $0 \leq \pi \leq 1$, $B(\pi, m; 0)$ decreases from 1 to 0 and $B(\pi, m; m)$ increases from 0 to 1.

Lemma 2. *Let us assume $n^+ - n^- + 3 \leq k \leq n^+ - 2$ (Case 1.3 of Proposition 8). A partially mixed strategy equilibrium where voters from N^+ play a mixture of Yes (prob = γ) and No (prob = $1 - \gamma$) and where players from N^- play a mixture of Abstain (prob = $1 - \nu$) and No (prob = ν) does not exist with γ and ν arbitrarily close to 1.*

Proof. The equilibrium requires

$$w^-(\nu, \gamma) = \sigma \quad (\text{A2})$$

$$w^+(\nu, \gamma) + x^+(\nu, \gamma) = \eta + \sigma \quad (\text{A3})$$

with

$$w^-(\nu, \gamma) = \sum_{(h,m) \in K} B(\nu, n^- - 1; m) * B(\gamma, n^+; h) \quad (\text{A4})$$

$$x^+(\nu, \gamma) = \sum_{(h,m) \in L} B(\nu, n^-; m) * B(\gamma, n^+ - 1; h) \quad (\text{A5})$$

$$w^+(\nu, \gamma) = \sum_{(h,m) \in M} B(\nu, n^-; m) * B(\gamma, n^+ - 1; h). \quad (\text{A6})$$

The respective decisiveness is 0 if K or L or M is empty. K determines the number of Yes votes h from N^+ and the number m of No votes from $N^- - \{i\}$ which lead to a number of Yes minus No votes k so that i is an Abstain-No pivot player, that is, $h - (n^+ - h) - m = k$.

$$K = \{(h, m) \in \mathbb{N}^2 : 0 \leq h \leq n^+, 0 \leq m \leq n^- - 1, 2h - m = k + n^+\} \tag{A7}$$

With similar arguments, we get

$$L = \{(h, m) \in \mathbb{N}^2 : 0 \leq h \leq n^+ - 1, 0 \leq m \leq n^-, 2h - m = k + n^+ - 2\} \tag{A8}$$

$$(M = \{(h, m) \in \mathbb{N}^2 : 0 \leq h \leq n^+ - 1, 0 \leq m \leq n^-, 2h - m = k + n^+ - 1\}) \tag{A9}$$

K and M are not empty because, because $h = n^+$ is feasible in K and $h = n^+ - 1$ in L and M because of the restrictions of Case 1.3. These maximal h -values imply maximal m values m^K, m^L, m^M in the three sets with $m^K = m^L = n^+ - k$ and $m^M = n^+ - k - 1$. For small $1 - \nu$ and small $1 - \gamma$, (A4), (A5), and (A6) can be approximated by

$$w^-(\nu, \gamma) = \binom{n^- - 1}{m^K} \nu^{m^K} (1 - \nu)^{n^- - 1 - m^K} \gamma^{n^+} (1 - \gamma)^0 + \varepsilon_1 \tag{A10}$$

where ε_1 contains summands with $(1 - \pi)^s (1 - \gamma)^r, s \geq n^- - 1 - m^K, r \geq 0$, and $r + s > n^- - 1 - m^K$. For small $1 - \nu$ and $1 - \gamma$, ε_1 can be neglected. The same is true for

$$w^+(\nu, \gamma) = \binom{n^-}{m^L} \nu^{m^L} (1 - \nu)^{n^- - m^L} \gamma^{n^+ - 1} (1 - \gamma)^0 + \varepsilon_2 \tag{A11}$$

$$x^+(\nu, \gamma) = \binom{n^-}{m^M} \nu^{m^M} (1 - \nu)^{n^- - m^M} \gamma^{n^+ - 1} (1 - \gamma n^- - m^L)^0 + \varepsilon_3. \tag{A12}$$

Therefore, (A2) and (A3) imply

$$\frac{\sigma}{\eta + \sigma} \approx \frac{c_1(1 - \nu)^j}{c_2(1 - \nu)^{j+1} + c_3(1 - \nu)^{j+1}} \tag{A13}$$

with $j = n^- - 1 - n^+ + k \geq 2$ because of the restrictions of this case and constants c_i which are independent of ν .

While the left side of (A13) is bounded by 1, the right-hand side takes arbitrarily large values for ν approaching 1. This contradiction proves the Lemma.

Lemma 3. *Let us assume $-n^+ + 2 \leq k \leq n^+ - n^- - 2$ (Case 2.1 of Proposition 8). A partially mixed strategy equilibrium where voters from N^+ play a mixture of Yes (prob = γ) and No (prob = $1 - \gamma$) and where players from N^- play a mixture of Yes (prob = β) and Abstain (prob = $1 - \beta$) does not exist for γ arbitrarily close to 1 and β arbitrarily close to zero.*

Proof. We can follow the lines of the proof of Lemma 2. In the equivalents of (A4), (A5), and (A6), we substitute ν by β and we compute x^- instead of w^- . After determining new sets K, L, M , which again have maximal h values of n^+ or $n^+ - 1$, we estimate $x^- \approx const * \beta * (1 - \gamma)^0, x^+ \approx const * \beta^2 * (1 - \gamma)^1$, and $w^+ \approx const * \beta^3 * (1 - \gamma)^1$. Then, the equivalent of (A13) with $\frac{\eta}{\eta + \sigma}$ is contradictory for arbitrarily small β and $1 - \gamma$.

Lemma 4. *Let us assume $n^+ + 2 \leq k \leq n^- - n^+ - 2$ (Case 1.2 of Proposition 9). A partially mixed strategy equilibrium where voters from N^+ play a mixture of Yes (prob = γ) and No (prob = $1 - \gamma$) and where players from N^- play a mixture of Yes (prob = β) and Abstain (prob = $1 - \beta$) does not exist for sufficiently small γ if β .*

Proof. As for Lemmas 2 and 3.

Proposition 8. For sufficiently large η and σ , $q(0)$ lies in Segment I of the best reply map.

Case 0. $q(0)$ lies in Segment I. With the exception of the four k values in Table 1, the pure strategy equilibrium is selected; voters from N^+ vote Yes and those from N^- vote No. All voters vote against their true preferences. The true majority result may or may not be implied.

Cases 1 and 2. For sufficiently small η and σ , $q(0)$ lies in Segment IV. The linear tracing procedure results in the following equilibrium selection. $q(0)$ lies in Segment IV below the line $w = x\sigma/\eta$. For Subcases 1.2, 1.3, and 2.1, sufficiently small η and σ are assumed. With high probability or certainty, the selected equilibrium implies the true majority result, namely rejection in cases 1 ($k > n^+ - n^-$) and acceptance in cases 2 ($k \leq n^+ - n^-$). Depending on n^+ and n^- , some of the following Subcases may be empty, that is, without feasible thresholds k . The special values of Table 1 are excluded. Therefore, subcases need not be adjacent to one another.

Case 1.1: $k \geq n^+ + 2$ and $k \geq n^- - n^+ + 3$. The pure strategy equilibrium is selected.

Case 1.2: $n^+ + 2 \leq k \leq n^- - n^+ - 2$. Voters from N^+ vote Yes and those from N^- vote Abstain with high probability.

Case 1.3: $n^+ - n^- + 3 \leq k \leq n^+ - 2$. For sufficiently small σ , voters from N^+ vote Yes with certainty and voters from N^- vote No with high probability.

Case 2.1: $-n^+ + 2 \leq k \leq n^+ - n^- - 2$. Voters from N^+ vote Yes and voters from N^- vote Abstain with high probability.

Case 2.2: $k \leq -n^+ - 1$ and $k \leq n^+ - n^- - 2$. The pure strategy equilibrium is selected.

Proof. The general feasibility of the tracing procedure has been proved by Harsanyi and Selten (1988) and Herings (2000). Therefore, we need not prove the existence of solutions of the following systems of equations but we will only investigate their properties. First, the simple Cases 0, 1.1, and 2.2 are examined. Then, we investigate Case 1.3., where a mixed strategy is played at the first intersection of $q(t)$ with one of the critical lines, that is, with $w = \sigma$. Thereafter, we investigate the Cases 2.1 and 1.2 where playing mixed strategies start at the lines $w + x = \eta + \sigma$ and $x = \eta$ (see Figure 2). We suppress the indices i of players.

Case 0. For $q(0)$ in Segment I and excluding the four special values of k (Table 1), we have $(x, w) = (x^+, w^+) = (x^-, w^-) = (0, 0)$ and, therefore, $q(t)$ moves on the line to the origin which is reached at $t = 1$, that is, the pure strategy equilibrium is selected.

Case 1.1. $k \geq n^+ + 2$, $k \geq n^- - n^+ + 3$.

Case 2.2: $k \leq -n^+ - 1$, $k \leq n^+ - n^- - 2$. For these k values, $(x_i, w_i) = (0, 0)$ applies on the whole path to the origin. On the lines between two segments and without changing t , either voters from N^+ or voters from N^- continuously change their strategy to that of the new segment. In Segments IV and III, voters from N^+ vote Yes. On the line $w = \sigma$, Yes is never a best reply for voters from N^- . When they start to mix between No and Abstain, for a player from N^- the minimum number of Yes minus No votes from other players is $n^+ - n^- + 1$. The maximum number is n^+ . The restrictions of the two cases imply that

players from N^- are never decisive: either they cannot prevent acceptance or they cannot prevent rejection of the proposal. For a player from N^+ , these minimum and maximum numbers are $n^+ - n^- - 1$ and $n^+ - 1$. Therefore, $q(t) = q^+(t) = q^-(t)$ proceed along the line to the origin. The same procedure applies when $q(t)$ intersects the lines $w + x = \eta + \sigma$ and $x = \eta$. Note that $q^+(t)$ or $q^-(t)$ cannot stay on the lines because, with $(x_i, w_i) = (0, 0)$, the decreasing $(1 - t) * q(0)$ cannot be balanced. Therefore, in these cases, again the pure strategy equilibrium is selected. The difference to Case 0 is that the pure strategy equilibrium is assumed only if the true majority result is implied, in Case 1.1 rejection and in 2.2 acceptance of the proposal.

Case 1.3. $n^+ - n^- + 3 \leq k \leq n^+ - 2$. This case exists only if $n^- \geq 5$.

We define t_1 by the intersection of the line (6) with $(x, w) = (0,0)$ with the line $w = \sigma$. For $t > t_1$, voters from N^- start to play a mixture of Abstain ($prob = 1 - \nu(t)$) and No ($prob = \nu(t)$). They choose $\nu(t)$ so that

$$W^-(t) = (1 - t) * W(0) + t * w^- = \sigma \tag{A14}$$

remains constant. The *Abstain-Yes* decisiveness w^- of voters from N^- is the probability that the number of Yes minus the number of No votes is k , that is,

$$w^-(\nu) = B(\nu, n^- - 1; n^+ - k) \tag{A15}$$

Because of the restrictions of this case, we have $1 \leq n^+ - k \leq n^- - 4$. Therefore, if this case is not empty, $B(1, n^- - 1; n^+ - k) = 0$. From (A14) and (A15) follows:

$$B(\nu, n^- - 1; n^+ - k) = \delta(t) = \frac{\sigma - (1 - t)W(0)}{t} \tag{A16}$$

Because of the definition of t_1 , $\delta(t_1) = 0$, and $\nu(t_1) = 1$ is a solution of equation. For larger t , $\delta(t)$ increases and $\nu(t)$ decreases continuously until $\nu(1)$, provided $\sigma \leq \max(n^- - 1; n^+ - k)$. For small enough σ , $\nu(1)$ defined by $B(\nu(1), n^- - 1; n^+ - k) = \sigma$, is arbitrarily close to 1.

The *Yes-Abstain* decisiveness

$$x^-(\nu(t)) = B(\nu(t), n^- - 1; n^+ - k + 1) \tag{A17}$$

has a smaller exponent of $1 - \nu$. Therefore, close to $\nu = 1$, x^- is much larger than w^- and, in the best reply map, $q^-(t)$ moves along the line $w = \sigma$ to the right. Because of the same argument, close to $\nu = 1$,

$$w^+(\nu(t)) = B(\nu(t), n^-; n^+ - k - 1) \tag{A18}$$

is much smaller than $w^-(\alpha)$ and

$$x^+(\nu(t)) = B(\nu(t), n^-; n^+ - k) \tag{A19}$$

is much smaller than $x^-(\alpha)$. As a consequence, $q^+(t)$ moves close to the line to the origin until either $t = 1$ or until $t = t_2 < 1$, when it intersects the line $w + x = \eta + \sigma$. The latter case will be excluded. For sufficiently small σ , $\nu(1)$ and, therefore, $\nu(t_2)$ is arbitrarily close to 1.

On the line $w + x = \eta + \sigma$, voters from N^+ can mix *Yes* and *No*. Starting such a mixed strategy cannot take place with constant $t \geq t_2$ because of $w^+ + x^+ > 0$. Therefore, for $t > t_2$, players from N^- play a mixture of *Abstain* ($prob = 1 - \nu(t)$) and *No* ($prob = \nu(t)$) and voters from N^+ play a mixture of *Yes* ($prob = \gamma(t)$) and *No* ($prob = 1 - \gamma(t)$). As the voters from N^+

vote Yes in Segment III, they start, at $t = t_2$, with $\gamma = 1$. According to Lemma 1, however, a partially mixed equilibrium of Yes and No for players from N^+ and No and Abstain for players from N^- does not exist for ν and γ close to 1. Therefore $q^+(t)$ will never intersect the line $w + x = \eta + \sigma$.

We conclude that, if σ is sufficiently small, in the selected equilibrium, voters from N^- vote No with high probability and voters from N^+ vote Yes with certainty.

Case 2.1. $-n^+ + 2 \leq k \leq n^+ - n^- - 2$. Therefore, $2n^+ \geq n^- + 4$ is required.

$q(t)$ crosses the line $w = \sigma$ because of $k \leq n^+ - n^-$ (see Cases 1.1 and 2.2). At $t = t_1$, it intersects the line $w + x = \eta + \sigma$. Voters from N^- vote Abstain in Segments II and III. On the line, voters from N^+ can mix Yes ($prob = \gamma$) and No ($prob = 1 - \gamma$) to maintain

$$W^+(t) + X^+(t) = (1 - t) * (W(0) + X(0)) + t * (w^+(\gamma) + x^+(\gamma)) = \eta + \sigma. \quad (\text{A20})$$

For a Yes-Abstain pivot player i , Abstain would mean rejection, that is, $h - (n^+ - 1 - h) < k$ when h players from $N^+ - \{i\}$ vote Yes, and i voting Yes would mean acceptance of the proposal, that is, $h - (n^+ - 1 - h) + 1 \geq k$. $k + n^+ - 1$. If $k + n^+ - 1$ is even, then these two inequalities cannot be fulfilled by an integer h . If $k + n^+ - 1$ is odd, we find a critical $h = (k + n^+ - 2)/2$. Vive versa, for w^+ .

$$w^+(\gamma) = B(\gamma, n^+ - 1; (k + n^+ - 1)/2) \text{ and } x^+(\gamma) = 0 \text{ for } k + n^+ - 1 \text{ even} \quad (\text{A21})$$

$$w^+(\gamma) = 0 \text{ and } x^+(\gamma) = B(\gamma, n^+ - 1; (k + n^+ - 2)/2) \text{ for } k + n^+ - 1 \text{ odd.} \quad (\text{A22})$$

As either $x^+ = 0$ or $w^+ = 0$, $q(t)$ moves down the line $W^+(t) + X^+(t) = \eta + \sigma$. At t_1 the mixed strategy of voters from N^+ starts with $\gamma(t_1) = 1$ and, for small $\eta + \sigma$, even $\gamma(1)$ has to be close to 1.

Because of the restrictions of this case, $w^+(\gamma) + x^+(\gamma) \approx const * (1 - \gamma)^j$ with $j = n^+ - 1 - (k - n^+ - 1)/2 > 0$ or $j = n^+ - 1 - (k - n^+ - 2)/2 > 0$.

With respective arguments, the decisiveness scores of voters from N^- are

$$w^-(\gamma) = 0 \text{ and } x^-(\gamma) = B(\gamma, n^+; (k - n^+ - 1)/2) \text{ for even } k + n^+ - 1 \quad (\text{A23})$$

$$w^-(\gamma) = B(\gamma, n^+; (k - n^+)/2) \text{ and } x^-(\gamma) = 0 \text{ for odd } k + n^+ - 1, \quad (\text{A24})$$

None-zero x^- and w^- are about $const * (1 - \gamma)^{j'}$ with $j' > j$. Therefore, for γ close to 1, $(w^-(\gamma), x^-(\gamma))$ are small compared with $(w^+(\gamma), x^+(\gamma))$ and $q^-(t)$ moves downwards below the line $W^+(t) + X^+(t) = \eta + \sigma$ until $t = 1$ or it intersects the line $x = \eta$ at $t = t_2$. There voters from N^- have to switch from Abstain to Yes, starting with $\beta = prob(Yes) = 0$. This is excluded by Lemma 3.

We conclude that, if η and σ are sufficiently small, in the selected equilibrium, voters from N^- vote Abstain with certainty and voters from N^+ vote Yes with high probability.

Case 1.2. $n^+ + 2 \leq k \leq n^- - n^+ - 2$. This case requires $2n^+ + 4 \leq n^-$.

In this case, because of $n^+ + 2 \leq k$, $q(t)$ crosses the lines $w = \sigma$ and $w + x = \eta + \sigma$ as described above, but cannot cross $x = \eta$ because, there, $x_i^- > 0$ when voters from N^- change their strategy. Let us assume that $q(t)$ intersects $x = \eta$ at $t = t_1$. In Segments I and II, voters from N^+ vote No. For $t > t_1$, voters from N^- never vote No but mix between Yes and Abstain.

They continuously change the probability $\beta(t_1) = 0$ for *Yes* and $1 - \beta(t_1) = 1$ for *Abstain* to higher $\beta(t)$. $q^-(t)$ staying on the line $x = \eta$ requires

$$X^-(t) = (1 - t) * X(0) + t * x^- = \eta. \tag{A25}$$

For a voter from N^- we get

$$x^-(\beta) = B(\beta, n^- - 1; k + n^+ - 1) \tag{A26}$$

$$w^-(\beta) = B(\beta, n^- - 1; k + n^+), \tag{A27}$$

$w^-(\beta)$ is proportional to β^{k+n^+} and x^- to β^{k+n^+-1} . Therefore, for small β , w^- is much smaller than x^- and $q^-(t)$ moves along the line $x = \eta$ downwards (See Figure 2).

$$w^+(\beta) = B(\beta, n^-; k + n^+ - 1), \tag{A28}$$

$$x^+(\beta) = B(\beta, n^-; k + n^+ - 2) \tag{A29}$$

Imply, for small β , $w^+(\beta) > w^-(\beta)$ and $x^+(\beta) > x^-(\beta)$. Therefore, $q^+(t)$ moves, for increasing t , upwards to the right (Figure 2) until $t = 1$ or it intersects the line $x + w = \sigma + \eta$ where voters from N^+ start mixing strategies *Yes* (initially with probability $\gamma = 0$) and *No*. Using Lemma 4, the second case can be excluded.

In the selected equilibrium of this case, voters from N^+ vote *No* with certainty and N^- vote *Abstain* with high probability.

Proposition 9. *For sufficiently small η and σ , $q(0)$ lies in Segemnt IV of the best reply map. If $q(0)$ lies above the line $w = x\sigma/\eta$, the linear tracing procedure results in the following equilibrium selection.*

Cases 3 and 4. *For Subcases 3.2, 4.1, and 4.2, sufficiently small η and σ are assumed. With high probability or certainty, the selected equilibrium implies the true majority result, namely rejection in Cases 3 ($k > n^- - n^+$) and acceptance in Cases 4 ($k \leq n^- - n^+$). Depending on n^+ and n^- , some of the following subcases may be empty, that is, without feasible thresholds k . Subcases need not be adjacent to one another.*

Case 3.1: $k \geq n^- + 2$ and $k \geq n^+ - n^- + 3$. The pure strategy equilibrium is selected.

Case 3.2: $n^+ - n^- + 2 \leq k \leq n^- - 2$. Voters from N^+ vote *Abstain* with certainty and those from N^- vote *No* with high probability.

Case 4.1: $-n^- + 3 \leq k \leq n^+ - n^- - 2$. Voters from N^+ vote *Yes* with high probability and voters from N^- vote *No* with certainty.

Case 4.2: $n^- - n^+ + 2 \leq k \leq -n^- - 2$. Voters from N^+ vote *Abstain* with high probability and those from N^- vote *Yes* with certainty.

Case 4.3: $k \leq -n^- - 1$ and $k \leq n^- - n^+ - 2$. The pure strategy equilibrium is selected.

Proof. As for Proposition 8.