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The Yule-Frisch-Waugh-Lovell Theorem for Linear Instrumental Variables Estimation

Deepankar Basu*

July 15, 2023

Abstract

In this paper, I discuss three aspects of the Frisch-Waugh-Lovell theorem. First, I show that the theorem holds for linear instrumental variables estimation of a multiple regression model that is either exactly or overidentified. I show that with linear instrumental variables estimation: (a) coefficients on endogenous variables are identical in full and partial (or residualized) regressions; (b) residual vectors are identical for full and partial regressions; and (c) estimated covariance matrices of the coefficient vectors from full and partial regressions are equal (up to a degree of freedom correction) if the estimator of the error vector is a function only of the residual vectors and does not use any information about the covariate matrix other than its dimensions. While estimation of the full model uses the full set of instrumental variables, estimation of the partial model uses the residualized version of the same set of instrumental variables, with residualization carried out, with respect to the set of exogenous variables. Second, I show why the theorem applies in large samples to the limited information maximum likelihood (LIML) estimator but not to linear GMM estimators. Third, I trace the historical and analytical development of the theorem and suggest that it be renamed as the Yule-Frisch-Waugh-Lovell (YFWL) theorem to recognize the pioneering contribution of the statistician G. Udny Yule in its development.

JEL Codes: C26.

Keywords: multiple regression; partial regression; Frisch-Waugh-Lovell theorem; instrumental variables estimation.

1 Introduction

The Frisch-Waugh-Lovell (FWL) theorem is a remarkable result about linear regression models estimated with the method of least squares. The theorem shows that coefficients of variables in a multiple regression are exactly equal to corresponding coefficients in partial regressions that use residualized versions of the dependent and independent variables. While it has been primarily used in econometrics ([Krishnakumar, 2006](#); [Tielens and Van Hove, 2017](#)), it has found wider applications in a variety of disciplines, including statistics ([Arendacka and Puntanen, 2015](#); [Cinelli and Hazlett, 2020](#); [Gross and Moeller, 2023](#)), electrical engineering ([Monsurro and Trifiletti, 2017](#)), computer science ([Ahrens et al., 2021](#)), and genetics & molecular biology ([Reeves et al., 2012](#)), to name just a few. It is now included in many graduate-level textbooks in

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econometrics, including [Davidson and MacKinnon \(1993, p. 19–24\)](#), [Hayashi \(2000, p. 73–74\)](#), ([Davidson and MacKinnon, 2004, p. 62–69](#)), [Angrist and Pischke \(2009, pp. 35–36\)](#), and ([Greene, 2012, p. 33](#)). The theorem is widely used for estimating linear regression models with large number of dummy variables by partialling them out, e.g. in panel data sets with large number of fixed effects ([McCaffrey et al., 2012](#); [Gaure, 2013](#); [Correia, 2016](#)).

The first contribution of this paper is to show that the FWL theorem can be derived for generalized linear instrumental variables estimation. In particular, consider a linear regression model with a mix of endogenous and exogenous variables. Suppose a researcher has a set of valid instrumental variables, where the number of instrumental variables is weakly larger than the number of endogenous variables—thus I allow for both exactly and over-identified models. Now consider two models: a full model and a partial model. In the full model, the researcher estimates all the parameters of the regression function using instrumental variables estimation (with the full set of instrumental variables). In the partial model, the researcher first residualizes the outcome variable, the endogenous regressors and the full set of instrumental variables with the set of exogenous variables, and then estimates a linear regression of the residualized outcome variable on the residualized set of endogenous variables using instrumental variables estimation with the residualized instrumental variables.

In such a context, I show that: (a) the coefficient vector on the endogenous variables are numerically identical for instrumental variables estimation of the full and the partial model; (b) the vector of residuals are numerically identical for instrumental variables estimation of the full and partial model; and (c) estimated covariance matrices of the coefficient vector on the endogenous variables from the full and partial models are identical (up to a degree of freedom adjustment) if the estimator of the vector of regression errors are only functions of the regression residuals (and the only information used from the regressor matrices are their dimensions). Hence, this is the FWL theorem for instrumental variables estimation. This implies, in particular, that the argument about computational efficiency that led to widespread use of the FWL in least squares estimation of linear models with large number of dummy variables ([McCaffrey et al., 2012](#); [Gaure, 2013](#); [Correia, 2016](#)), can also be applied to instrumental variables estimation of such models.

The second contribution of the paper is to investigate if the FWL theorem applies to two closely related estimators: the limited information maximum likelihood (LIML) estimator and the linear generalized method of moments (GMM), especially the optimal or two step generalized method of moments (2SGMM) estimator. I show that the FWL theorem applies in large samples to all consistent K-class estimators (the LIML estimator being an important member of this class) but does *not* apply to linear GMM estimators (for

overidentified models). The reason for this has to do with whether the estimator in question can be written in terms of the projection matrix that carries out the relevant partialling out. I show that while it is possible to express all consistent K-class estimators in terms of a projection matrix that is approximately equal to the projection matrix that carries out the partialling out, it is not possible to do so in case of linear GMM estimators. That is why the FWL theorem applies, approximately, to all consistent K-class estimators, including the LIML estimator, but does not apply to linear GMM estimators (for overidentified models). I give an empirical example to illustrate this point for both LIML and the optimal 2SGMM estimators. Of course, for an exactly identified model, the linear GMM estimator is the IV estimator, and in this case, the FWL theorem applies.

The third contribution of the paper is to offer a historical narrative about the analytical development of the FWL theorem. Through this analytical narrative, I wish to highlight two points. The first point relates to the contribution of the statistician G. Udny Yule in the development of the FWL theorem. In the econometrics literature, the theorem is understood to have originated in a 1933 paper by Ragnar Frisch and Frederick V. Waugh in the first volume of *Econometrica* (Frisch and Waugh, 1933), which was later generalized by Michael C. Lovell (Lovell, 1963). In fact, the result was proved more than two and a half decade ago by Yule in a 1907 paper (Yule, 1907). This seems to be well known in statistics (Agresti, 2015, p. 60) and should be recognized in econometrics as well. To recognize Yule's pioneering contribution to the development of this important result, I suggest that we refer to it as the Yule-Frisch-Waugh-Lovell (YFWL) theorem, rather than the currently used Frisch-Waugh-Lovell (FWL) theorem.

The second point is to trace out the analytical *development* of the theorem through the decades highlighting the contribution at each stage of its development. In Yule (1907), who proved the result using basic algebra, the partial regressions are always bivariate regressions of residual vectors. In Frisch and Waugh (1933), where the proof relied on some basic properties of determinants and Cramer's rule for finding solutions of linear systems of equations, the partial regressions are themselves multiple regressions but the residuals are computed with bivariate regressions. Thus, Yule (1907) allows multiple variables in the conditioning set, but conceives of the partial regressions as bivariate regressions. On the other hand, Frisch and Waugh (1933) allows only one variable (the linear time trend) in the conditioning set, but allows the partial regressions to be multiple regressions.

Thirty years later, Lovell (1963) generalizes the theorem significantly. He allows multiple variables in the conditioning set and allows the partial regressions to themselves be multiple regressions.¹ In terms of

¹Lovell (1963, p. 1000) points out that Tintner (1957, p. 304) had extended the Frisch-Waugh result to the case of polynomial time trends.

methodology, [Lovell \(1963\)](#) introduces the use of projection matrices from linear algebra, making the proof compact and elegant. Current presentations of the theorem closely follow Lovell's exposition ([Davidson and MacKinnon, 2004](#); [Greene, 2012](#)). Finally, [Lovell \(1963\)](#) shows that the vector of residuals from the full and partial regressions are identical, a result that neither [Yule \(1907\)](#) nor [Frisch and Waugh \(1933\)](#) had investigated. This finding led Lovell to comment briefly on degrees of freedom adjustment necessary for statistical inference.

Till the 1960s, the YFWL theorem was used in the context of linear regression models estimated with least squares. While [Yule \(1907\)](#) and [Frisch and Waugh \(1933\)](#) had considered ordinary least squares (OLS) estimation, [Lovell \(1963\)](#) had pointed out that the theorem could be extended easily to generalized least squares estimation. Could the same set of results be derived for instrumental variables estimation? [Giles \(1984\)](#) provided an answer in the affirmative for an exactly identified model (where the number of instrumental variables are exactly equal to the number of endogenous variables in the model). [Giles \(1984\)](#) showed that both the coefficient vector and the residuals would be identical from instrumental variables estimations of the full and partial models.

The issue of statistical inference had not yet been investigated thoroughly, i.e. would researchers be able to conduct inference about the subset of coefficients in a multiple regression using the standard errors (or the covariance matrix) estimated from the partial regression? It is true that [Lovell \(1963\)](#) had addressed this issue briefly by pointing out that *estimated* standard errors from the multiple and partial regressions are equal up to a degree of freedom adjustment when the true error is homoskedastic. But he left it at that. It is only in [Ding \(2021\)](#) that we find a systematic treatment of the issue. [Ding \(2021\)](#) has demonstrated that, in the context of least squares estimation, estimates of the covariance matrices from the multiple and partial regressions are equal up to a degree of freedom adjustment in the case of homoskedastic errors and, quite surprisingly, are exactly equal for some variants of heteroskedasticity consistent covariance matrices (HCCM), heteroskedasticity and autocorrelation consistent (HAC) covariance matrices, and for some variants of clustered robust covariance matrices.

In fact, the analysis in [Ding \(2021\)](#) highlights a general principle: if the estimate of error covariance matrix is a function only of the residuals and does not depend on the matrix of covariates beyond its dimensions, then estimated covariance matrices from multiple and partial regressions are equal, either exactly or with a degree of freedom correction. This principle can be leveraged to offer a computational solution for cases where [Ding \(2021\)](#)'s results do not hold, i.e. estimates and other information from partial regressions are sufficient to compute correct covariance matrices of coefficient vectors in the multiple regression. If only

a subset of coefficients is of interest then estimation and proper statistical inference can be conducted by working with the partial regression alone.

The current paper is most closely related to [Giles \(1984\)](#) and [Ding \(2021\)](#). I extend [Giles \(1984\)](#) in two ways. First, while [Giles \(1984\)](#) had demonstrated the YFWL theorem for instrumental variables estimation of an exactly identified model, I extend the result to over-identified models. I offer a general proof that nests both cases of instrumental variables estimation, and of course least squares estimation (because, in this case, the regressors become their own instruments). Second, [Giles \(1984\)](#) had not investigated properties of the *estimated* covariance matrices from full and partial models. I draw on [Ding \(2021\)](#) to do so, and in so doing I also extend the analysis in [Ding \(2021\)](#) from ordinary least squares estimation to generalized linear instrumental variables estimation.

I would like to end this introductory section by noting a limitation of the YFWL for generalized instrumental variables estimation. The key results of the theorem, i.e. equality of the coefficient vectors and residual vectors from full and partial models can only be derived when then conditioning set (the set of covariates used for residualizing) does not contain any endogenous variables. The reason has to do with the necessity of projection matrices in deriving key results of the YFWL theorem. If endogenous variables appear in the conditioning set, the relevant projection matrices are no longer available for the analysis. But this negative result is less restrictive than appears at first sight, as I discuss in [section 3.7](#).

The rest of the paper is organized as follows: in [section 2](#), I introduce the set-up and pose the main question; in [section 3](#), I present the YFWL for generalized instrumental variables estimation; in [section 4](#), I discuss why the YFWL theorem applies, in large samples, to all consistent K-class estimators (including the LIML estimator) but does not apply to linear GMM estimators (including the two step GMM estimator); in [section 5](#), I present a narrative account of the analytical development of the theorem through the decades; I conclude in [section 6](#) by highlighting the usefulness of, and highlighting a puzzle about the intellectual history of, the YFWL theorem. Proofs are collected together in the appendix.

2 The set-up and the question

Consider a linear regression of an outcome variable, Y , on a set of k covariates,

$$Y = W\beta + \varepsilon, \tag{1}$$

where Y is a $N \times 1$ vector of the outcome variable, W is a $N \times k$ matrix of covariates, β is a $k \times 1$ of parameters, and ε is a $N \times 1$ vector of the stochastic error term.

Suppose on the basis of a specific question under investigation, it is possible for the researcher to partition the set of regressors into two groups, W_1 and W_2 , where the former is a $N \times k_1$ matrix and the latter is a $N \times k_2$ matrix, with $k = k_1 + k_2$. The model in (1) can then be written as

$$Y = W_1\beta_1 + W_2\beta_2 + \varepsilon, \quad (2)$$

where β_1 and β_2 are $k_1 \times 1$ and $k_2 \times 1$ vectors of parameters. I would like to ensure that the model satisfies the identification condition (Greene, 2012, p. 19, 223) for both sets of parameters, β_1 and β_2 , using the following assumption.

Assumption 1. *The matrices W_1 and W_2 are both of full column rank.*

Suppose the researcher is only interested in the second group of regressors, W_2 , i.e. she is only interested in estimating and conducting inference about β_2 . But, of course, she does not want to completely throw away the information in W_1 because the variables in this subset do impact Y and are likely to also be correlated to the variables in W_2 . Hence, while she wants to condition her analysis on W_1 , she is not interested in their partial effects on the outcome (e.g. fixed effects in a panel data regression model).

Since the researcher is only interested in the parameters on W_2 , she could consider the following *partial* regression model,

$$\tilde{Y} = \tilde{W}_2\tilde{\beta}_2 + \tilde{\varepsilon}, \quad (3)$$

where \tilde{Y} and \tilde{W}_2 are residualized (or partialled out) versions of the outcome variable and the variables of interest in (2), respectively, where residualization has been carried out with respect to W_1 (the exogenous covariates in the model), i.e.

$$\tilde{Y} = M_{W_1}Y, \text{ and } \tilde{W}_2 = M_{W_1}W_2, \quad (4)$$

where

$$M_{W_1} = I - P_{W_1} \text{ and } P_{W_1} = W_1(W_1'W_1)^{-1}W_1', \quad (5)$$

are the hat and residual maker matrices, respectively, for the set of exogenous regressors, W_1 .

The matrices P_{W_1} and M_{W_1} play important roles in the whole analysis. They are both symmetric and idempotent. They are referred to, respectively, as the ‘hat matrix’ and the ‘residual maker matrix’ because

P_{W_1} projects any vector in \mathbb{R}^N onto the column space of W_1 , and M_{W_1} projects the same vector onto the orthogonal complement of the column space of W_1 . Thus, P_{W_1} generates the vector of predicted values and M_{W_2} generates the vector of least squares residuals for a regression of any vector in \mathbb{R}^N on W_1 .

The researcher faces an additional challenge: while the conditioning variables in the model, W_1 , are exogenous, i.e. $E(\varepsilon|W_1) = 0$, the variables of interest, W_2 , are endogenous, i.e. $E(\varepsilon|W_2) \neq 0$. Since the model has endogenous regressors, the parameters cannot be estimated consistently by the method of least squares. Instead, the researcher turns to instrumental variables estimation using a set of k_3 instrumental variables, represented by the $n \times k_3$ matrix, Z_2 , with $k_3 \geq k_2$.

Assumption 2. *The set of instruments, Z_2 , satisfy the following conditions (Greene, 2012, p. 223–224):*

1. *Exogeneity: The instrumental variables are uncorrelated with the error term in the full model, i.e.*

$$\text{Cov}(\varepsilon, Z_2) = 0;$$

2. *Relevance: The instrumental variables are correlated with the regressors. In particular, $(Z_2'W_1)$ has rank k_1 , and $(Z_2'W_2)$ has rank k_2 .²*

Given her interest and the set-up, can the researcher avoid estimating the full model in (2)? Can she work with the partial model (3) and yet consistently estimate and conduct proper statistical inference on β_2 ? The YFWL theorem provides an answer in the affirmative.

3 The YFWL theorem for linear instrumental variables estimation

I will now derive a YFWL theorem for instrumental variables estimation, i.e. I will show: (a) the instrumental variables estimator of β_2 from the full regression model is equal to the the instrumental variables estimator of β_2 from the partial model; and (b) I will be show that the residuals from (2) are equal to the residuals from (3). Since instrumental variables estimators are method of moment estimators (Greene, 2012, p. 469), I will work with the moment conditions defining the estimators to develop the proof.

3.1 The full regression

Let b_1 and b_2 be the instrumental variables estimators of β_1 and β_2 , respectively, in the full regression model (2). By construction, the set of exogenous variables, W_1 , and the set of instrumental variables, Z_2 , are both

²Note that both the exogeneity and relevance conditions can also be stated in asymptotic terms using probability limits.

orthogonal to the regression residual vector $u = Y - W_1 b_1 - W_2 b_2$. Hence, we have the following $k_1 + k_3$ moment condition equations,

$$Z'(Y - W_1 b_1 - W_2 b_2) = 0, \quad (6)$$

where

$$Z_{(n \times (k_1 + k_3))} = \begin{bmatrix} W_1 & : & Z_2 \\ (n \times k_1) & & (n \times k_3) \end{bmatrix} \quad (7)$$

is the matrix of instrumental variables. Writing out the equations for the two components of Z separately, we have,

$$W_1'(Y - W_1 b_1 - W_2 b_2) = 0. \quad (8)$$

and

$$Z_2'(Y - W_1 b_1 - W_2 b_2) = 0 \quad (9)$$

From the first equation, we get

$$b_1 = (W_1'W_1)^{-1} W_1'Y - (W_1'W_1)^{-1} W_1'W_2 b_2,$$

where $(W_1'W_1)^{-1}$ exists because W_1 is of full column rank (assumption 1). This expression can be plugged into the second equation and rearranged to give us a linear system of k_3 equations for determining the $k_2 \times 1$ vector b_2 as

$$Z_2' M_{W_1} W_2 b_2 = Z_2' M_{W_1} Y. \quad (10)$$

Thus, b_2 , the instrumental variables estimators of β_2 in the full regression model (2) is the solution of the linear equation system in (10). Note that $Z_2' M_{W_1} W_2$ is a $k_3 \times k_2$ matrix, where $k_3 \geq k_2$. Hence, in general (10) cannot be inverted.

3.2 The partial regression

Let \tilde{b}_2 denote the instrumental variables estimator of $\tilde{\beta}_2$ in the the partial regression model (3). Since Z_2 is a set of valid instrumental variables for W_2 , the residualized part of Z_2 , i.e. $\tilde{Z}_2 = M_{W_1} Z_2$, is a valid instrumental variable for \tilde{W}_2 , the residualized part of W_2 , appearing as the regressors in the partial regression (3). Thus, \tilde{Z}_2 is orthogonal to the residual vector of the partial regression (3) $\tilde{u} = \tilde{Y} - \tilde{W}_2 \tilde{b}_2$, and we have the following moment conditions,

$$\tilde{Z}_2' (\tilde{Y} - \tilde{W}_2 \tilde{b}_2) = 0, \quad (11)$$

which can be rearranged to give the following system of k_3 linear equations for determining the $k_2 \times 1$ vector \tilde{b}_2 :

$$\tilde{Z}'_2 \tilde{W}_2 \tilde{b}_2 = \tilde{Z}'_2 \tilde{Y}. \quad (12)$$

3.3 Coefficients and residuals are equal

Theorem 1. *For the full regression model in (2) and the partial regression model (3), we have*

1. *the estimated coefficient vectors b_2 from the full model (2) and \tilde{b}_2 from the partial model (3) are identical, and*
2. *the residual vectors from the full model (2) and from the partial model (3) are identical when b_2 and \tilde{b}_2 are unique.*

The proof is very straightforward and just relies on showing that the equation systems whose solutions give us the estimator for the full and partial models are identical. In general, both are inconsistent, but when they have a solution, it will be the same. Details can be found in appendix A.

3.4 Special cases

There are three cases of interest, where we know, that (6), and hence (10), has a unique solution: the IV estimator for an exactly identified model (Greene, 2012, p. 225), the two stage least squared (2SLS) estimator for an overidentified model (Greene, 2012, p. 231), and, when all regressors are exogenous, the ordinary least squares estimator (Greene, 2012, p. 28).

3.4.1 IV estimator

In this case, the number of instruments is exactly equal to the number of endogenous variables, i.e. $k_3 = k_2$. Thus $Z = [W_1 : Z_2]$ is a $n \times (k_1 + k_2)$ matrix, and $Z'W$ is an invertible $(k_1 + k_2) \times (k_1 + k_2)$ matrix. Thus, the unique IV estimator of b for the full regression model (2) is given by (Greene, 2012, p. 225):

$$b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = (Z'W)^{-1} Z'Y, \quad (13)$$

where the inverse of $Z'W$ exists by assumption 2. Hence, in this case, b_2 is the (2, 1) block of the right hand side (RHS) of (13).

Turning to the partial model (3), we should first note that $\tilde{Z}_2 = M_{W_1} Z_2$ is a $n \times k_2$ matrix and $\tilde{W}_2 = M_{W_1} W_2$ is a $n \times k_2$ matrix as well. Hence, $\tilde{Z}_2' \tilde{W}_2$ is an invertible $k_2 \times k_2$ matrix. Thus, the IV estimator for the partial model (3) using \tilde{Z}_2 as the set of instrumental variables, is

$$\tilde{b}_2 = \left(\tilde{Z}_2' \tilde{W}_2 \right)^{-1} \tilde{Z}_2' \tilde{Y}, \quad (14)$$

and by the general result proved in theorem 1, we have $b_2 = \tilde{b}_2$.

3.4.2 2SLS estimator

In this case, the number of instrumental variables is strictly larger than the number of endogenous variables and we choose from among all possible linear combination of the set of instruments that one which minimizes the asymptotic covariance matrix of the instrumental variables estimator (Brundy and Jorgenson, 1971). In this case (Greene, 2012, p. 231), the unique 2SLS estimator for the full regression model (2) is given by

$$b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \left[W' Z (Z' Z)^{-1} Z' W \right]^{-1} W' Z (Z' Z)^{-1} Z' Y, \quad (15)$$

where $Z = [W_1 : Z_2]$ is the full set of instruments, including the exogenous variables, W_1 , and the inverse exists by assumption 2. But this is just a special case of the IV estimator in (13) with the set of instrumental variables given by the $n \times (k_1 + k_2)$ matrix

$$P_Z W = Z (Z' Z)^{-1} Z' W. \quad (16)$$

Thus, $(P_Z W)' W$ is an invertible $(k_1 + k_2) \times (k_1 + k_2)$ matrix and we have the 2SLS estimator

$$b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ (k_1 \times 1) \\ b_2 \\ (k_2 \times 1) \end{pmatrix} = \left[(P_Z W)' W \right]^{-1} (P_Z W)' Y, \quad (17)$$

so that b_2 is the (2, 1) block of the conformably partitioned right hand side (RHS) of (17).

Turning to the partial model (3), we estimate it with the instrumental variables formed by the residualized set of instrumental variables, $\tilde{Z}_2 = M_{W_1} Z_2$, which is a $n \times k_3$ matrix. Thus, the 2SLS estimator for the

partial regression model (3) is given by

$$\tilde{b}_2 = \left[\left(P_{\tilde{Z}_2} \tilde{W}_2 \right)' \tilde{W}_2 \right]^{-1} \left(P_{\tilde{Z}_2} \tilde{W}_2 \right)' Y, \quad (18)$$

and, once again, by the general result established in theorem 1, we have $b_2 = \tilde{b}_2$ for the 2SLS estimator.

3.4.3 OLS estimator

In this case, both sets of regressors in (1), W_1 and W_2 , are exogenous. Thus $Z = W = [W_1 : W_2]$ and the unique OLS estimator of the parameters in the full model is given by (Greene, 2012, p. 28):

$$b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = [W'W]^{-1} W'Y, \quad (19)$$

where the inverse of $W'W$ exists by assumption 1. Hence, b_2 is the (2, 1) block of the right hand side (RHS) of (19), and the OLS estimator from the partial model is

$$\tilde{b}_2 = \left(\tilde{W}_2' \tilde{W}_2 \right)^{-1} \tilde{W}_2' \tilde{Y}, \quad (20)$$

and, once again, by the general result established in theorem 1, we have $b_2 = \tilde{b}_2$ for the OLS estimator.

3.5 Relationship between estimated covariance matrices

I will now use the result in theorem 1 that the vector of residuals from the full and partial models are identical to investigate the relationship among the *estimators* of the covariance matrices of b_2 (instrumental variables estimator of β_2 in (1) from the full regression model) and \tilde{b}_2 (instrumental variables estimator of β_2 in (1) from the partial regression model).

3.5.1 Estimated covariance matrices are equal up to a degree of freedom correction

Using the result in theorem 9.4 in Greene (2012, p. 264), we see that the *estimator* of the asymptotic covariance matrix of $b' = (b_1', b_2')$, the instrumental variables estimator of (β_1', β_2') in (1) from the full

regression model in (2), is given by

$$\begin{aligned} \widehat{\text{Var}} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} &= \begin{pmatrix} \widehat{\text{Var}}(b_1) & \widehat{\text{E}}(b_1 - \beta_1, b'_2 - \beta'_2) \\ \widehat{\text{E}}(b_2 - \beta_2, b'_1 - \beta'_1) & \widehat{\text{Var}}(b_2) \end{pmatrix} \\ &= [(P_Z W)' (P_Z W)]^{-1} (P_Z W)' \widehat{\Omega}_f (P_Z W) [(P_Z W)' (P_Z W)]^{-1}, \end{aligned} \quad (21)$$

where $W = [W_1 : W_2]$, $Z = [W_1 : Z_2]$, $P_Z = Z(Z'Z)^{-1}Z'$ and $\widehat{\Omega}_f$ (subscript 'f' for identifying the full regression) is an estimate of the covariance matrix of the error term, ε , in the full regression model (2).

Hence, the estimated variance of b_2 is given by

$$\widehat{\text{Var}}(b_2) = \text{(2,2) block of } [(P_Z W)' (P_Z W)]^{-1} (P_Z W)' \widehat{\Omega}_f (P_Z W) [(P_Z W)' (P_Z W)]^{-1}. \quad (22)$$

Using a similar argument, we see that the estimator of the covariance matrix of \tilde{b}_2 , the instrumental variables estimator of $\tilde{\beta}_2$ in the partial regression model (3), is given by

$$\widehat{\text{Var}}(\tilde{b}_2) = \left[\left(P_{\tilde{Z}_2} \tilde{W}_2 \right)' \left(P_{\tilde{Z}_2} \tilde{W}_2 \right) \right]^{-1} \left(P_{\tilde{Z}_2} \tilde{W}_2 \right)' \widehat{\Omega}_p \left(P_{\tilde{Z}_2} \tilde{W}_2 \right) \left[\left(P_{\tilde{Z}_2} \tilde{W}_2 \right)' \left(P_{\tilde{Z}_2} \tilde{W}_2 \right) \right]^{-1}, \quad (23)$$

where $\widehat{\Omega}_p$ (subscript 'p' for identifying the partial regression) is an estimate of the covariance matrix of the error term, $\tilde{\varepsilon}$, in the partial regression model (3).

Theorem 2. *Consider b_2 , the instrumental variables estimators of β_2 in (1) from the full regression model (2), and \tilde{b}_2 , the instrumental variables estimators of β_2 in (1) from the partial regression models in (3). If the estimators of the error vectors in the full and partial regression models are equal, $\widehat{\Omega}_f = \widehat{\Omega}_p$ (after making degrees of freedom corrections, if necessary), then*

$$\widehat{\text{Var}}(b_2) = \widehat{\text{Var}}(\tilde{b}_2) \text{ (after making degrees of freedom corrections)}. \quad (24)$$

Remark. The proof of the theorem can be found in appendix B. At this point, let us return to (12) to note an important point. Using the definitions of the residualized vectors and matrices, we see that (12) is

$$(M_{W_1} Z_2)' M_{W_1} W_2 \tilde{b}_2 = (M_{W_1} Z_2)' M_{W_1} Y, \quad (25)$$

which becomes

$$Z_2' M_{W_1} W_2 \tilde{b}_2 = Z_2' M_{W_1} Y, \quad (26)$$

because M_{W_1} is symmetric and idempotent. Thus, (12) is equivalent to

$$Z_2' \widetilde{W}_2 \tilde{b}_2 = Z_2' \tilde{Y}, \quad (27)$$

so that using Z_2 (instead of \tilde{Z}_2) as the set of instrumental variables for estimating the partial model (3) would give us an identical value of \tilde{b}_2 . Similarly, (12) is also equivalent to

$$\tilde{Z}_2' W_2 \tilde{b}_2 = \tilde{Z}_2' Y, \quad (28)$$

so that estimating a linear regression of Y on W_2 using \tilde{Z}_2 as the set of instrumental variables would also give us \tilde{b}_2 as the value of the coefficient vector on W_2 . I recommended that neither of these models, (27) or (28), be estimated in place of estimating (3) with \tilde{Z}_2 as the set of instrumental variables. This is because of reasons pertaining to the relationship between estimators of the covariance matrix of b_2 and \tilde{b}_2 .

If (27) is estimated, then the covariance matrix of the IV estimator would, instead of the expression in (23), become

$$\widehat{\text{Var}}(\tilde{b}_2) = \left[\left(P_{Z_2} \widetilde{W}_2 \right)' \left(P_{Z_2} \widetilde{W}_2 \right) \right]^{-1} \left(P_{Z_2} \widetilde{W}_2 \right)' \hat{\Omega}_p \left[\left(P_{Z_2} \widetilde{W}_2 \right)' \left(P_{Z_2} \widetilde{W}_2 \right) \right]^{-1}, \quad (29)$$

and the proof of theorem 2 would not go through. In fact, we would not be able to complete step 2 of the proof. This is because $P_Z \widetilde{W}_2 \neq P_{Z_2} \widetilde{W}_2$. If, on the other hand, (28) is estimated, then the residual vector from the partial regression becomes $Y - \tilde{b}_2 W_2$ and this is not equal to the residual from the full regression model. This can be seen from the argument in (45). Hence, $\hat{\Omega}_f \neq \hat{\Omega}_p$, and theorem 2 will not apply.

Let me summarize what we get from theorem 2: (a) if the estimate of the error covariance matrix depends only on the vector of regression residuals, then (22) and (23) are exactly equal; (b) if some degree of freedom correction is applied to generate the estimate of the error covariance matrix, then (22) and (23) are equal up to the relevant degree of freedom correction; (c) if the estimate of the error covariance matrix depends, in addition to the residual vector, on the elements of the matrix of regressors, then (22) and (23) are not, in general, equal even after making degree of freedom adjustments.

In particular, theorem 2 in this paper extends Ding (2021, Theorem 2, 3, 4) to linear instrumental

variables estimation and we have the following results:

1. In models with homoskedastic errors, the covariance matrices from the multiple and partial regressions are equal up to a degree of freedom adjustment, i.e. $(N - k_2)\widehat{\text{Var}}(\tilde{b}_2) = (N - k)\widehat{\text{Var}}(b_2)$. This is because $\widehat{\Omega}_p = 1/(N - k_2)\text{diag}[\tilde{u}_i^2]$ and $\widehat{\Omega}_f = 1/(N - k)\text{diag}[u_i^2]$, where ‘diag’ denotes a $N \times N$ diagonal matrix and $k = k_1 + k_2$.
2. In models with HC0 version of HCCM (MacKinnon and White, 1985), the covariance matrices from the multiple and partial regressions are exactly equal because $\widehat{\Omega}_p = \text{diag}[\tilde{u}_i^2]$ and $\widehat{\Omega}_f = \text{diag}[u_i^2]$.
3. In models with HC1 version of HCCM (MacKinnon and White, 1985), the covariance matrices from the multiple and partial regressions are equal up to a degree of freedom adjustment because $\widehat{\Omega}_p = N/(N - k_2)\text{diag}[\tilde{u}_i^2]$ and $\widehat{\Omega}_f = N/(N - k)\text{diag}[u_i^2]$. Hence, $(N - k_2)\widehat{\text{Var}}(\tilde{b}_2) = (N - k)\widehat{\text{Var}}(b_2)$.
4. In models using heteroskedasticity and autocorrelation (HAC) consistent covariance matrices (Newey and West, 1987), the covariance matrices from the multiple and partial regressions are exactly equal because $\widehat{\Omega}_p = (\omega_{|i-j|}\tilde{u}_i\tilde{u}_j')_{1 \leq i, j \leq N}$ and $\widehat{\Omega}_f = (\omega_{|i-j|}u_i u_j')_{1 \leq i, j \leq N}$, as long as the same weights are used in both regressions.
5. In models using the cluster robust estimate of the variance matrix (CRVE) (MacKinnon et al., 2023), the covariance matrices from the multiple and partial regressions are exactly equal if no degree of freedom correction is used or if $G/(G - 1)$ is used as the degree of freedom correction, where G is the number of clusters (Cameron and Miller, 2015, p. 8). If $G(N - 1)/(G - 1)(N - k)$ is used as the degree of freedom correction, then we have what MacKinnon et al. (2023) call the CV_1 feasible CRVE and for this CRVE, the following holds: $(N - k_2)\widehat{\text{Var}}(\tilde{b}_2) = (N - k)\widehat{\text{Var}}(b_2)$.

In all these cases, researchers can use estimated covariance matrices from the partial regression (3), with the relevant degree of freedom adjustment if necessary, to conduct proper statistical inference on the parameters from the multiple regression (2).

3.5.2 The cases that were left out

There are some cases where the estimate of the error covariance matrix depends, in addition to the residual vector, on elements of the covariate and/or instrumental variables matrix. In these cases, (22) and (23) will not be equal, even after making degrees of freedom corrections. Some common cases where this happens

are: (a) models using HC2, HC3, HC4, or HC5 forms of HCCM;³ (b) some variants of the HAC covariance matrix and the CV_2 and CV_3 forms of feasible CRVEs (MacKinnon et al., 2023). In these cases, it is not possible to use standard errors from the partial regression, with or without degrees of freedom correction, for inference about the coefficient vector in the full regression model. But, it is still possible to compute the correct covariance matrix for the coefficient vector in the full regression by using information that becomes available while estimating the partial regression. Thus, estimation of the full regression model can still be avoided.⁴

3.6 Limitation of the YFWL theorem for instrumental variables estimation

In the YFWL for instrumental variables estimation, I have assumed that the set of variables that are of interest are endogenous and the set of variables used for conditioning are exogenous. Can we switch the role of the endogenous and exogenous variables and still get the YFWL theorem? I would like to show that the answer is in the negative but then also argue that this negative result is not too restrictive.

Let us return to the full regression model (2) and specify that the set of variables of interest, W_2 , is exogenous, and the set of variables used for conditioning (or residualizing), W_1 , is endogenous. Suppose, now the researcher has a set of k_3 valid instrumental variables, Z_1 , satisfying the exogeneity and relevance conditions, where $k_3 \geq k_1$. The full set of instrumental variables is $Z = [Z_1 : W_2]$, and the instrumental variables estimator of the parameter vector (β'_1, β'_2) is now given by (b'_1, b'_2) , where,

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} Z'_1 W_1 & Z'_1 W_2 \\ W'_2 W_1 & W'_2 W_2 \end{pmatrix}^{-1} \begin{pmatrix} Z'_1 Y \\ W'_2 Y \end{pmatrix}, \quad (30)$$

so that, using results for the inverse of partitioned matrices (Greene, 2012, p. 994), we have,

$$b_2 = \left[W'_2 \left(I - W_1 (Z'_1 W_1)^{-1} Z'_1 \right) W_2 \right]^{-1} W'_2 \left(I - W_1 (Z'_1 W_1)^{-1} Z'_1 \right) Y. \quad (31)$$

As can be seen from the above expression, we do not get any projection matrices. Neither can the inverse on the left be opened up because W_2 is not a square matrix.

³HC2 and HC3 were introduced by MacKinnon and White (1985); HC4 was introduced by Cribari-Neto (2004); HC5 was introduced by Cribari-Neto et al. (2007). For a discussion of all the variants of HCCM, see MacKinnon (2013).

⁴See appendix E.

Now let us turn to the partial regression, which, in this case will be

$$\tilde{Y} = \tilde{W}_2 \tilde{b}_2 + \tilde{u}, \quad (32)$$

where $\tilde{Y} = M_{W_1} Y$ and $\tilde{W}_2 = M_{W_1} W_2$. Since \tilde{W}_2 is orthogonal to W_1 (the endogenous variables), it is itself exogenous to the error of the full regression model. Hence, (32) can be estimated by the method of ordinary least squares, giving us

$$\tilde{b}_{2,OLS} = \left(\tilde{W}_2' \tilde{W}_2 \right)^{-1} \tilde{W}_2' \tilde{Y}. \quad (33)$$

If, instead instrumental variables estimation is conducted with the residualized instrumental variables, $\tilde{Z}_1 = M_{W_1} Z_1 = Z_1$ (because Z_1 is orthogonal to W_1), we will have

$$\tilde{b}_{2,IV} = \left(Z_1' \tilde{W}_2 \right)^{-1} Z_1' \tilde{Y}. \quad (34)$$

In general, neither (33) nor (34) is equal to (31). Hence, the neat results of the YFWL theorem cannot be derived in this case, i.e. for instrumental variables estimation when the set of conditioning variables are endogenous. This negative result is less restrictive than might appear, as I discuss below by way of summarizing the YFWL for instrumental variables estimation.

3.7 Summary

Let me summarize the facts about the YFWL theorem and highlight when it is applicable in terms of properties of the regressors and how they can be partitioned into two subsets. In the case of least squares estimation, ordinary or generalized, the YFWL theorem is applicable without any restrictions, i.e. the set of regressors can be partitioned into two subsets any which way. For instrumental variables estimation, there are some restrictions that need to be kept in mind. These restrictions arise because two different distinctions are at play that might not coincide: exogenous variables versus endogenous variables; and variables of interest versus those not of direct interest (and hence relegated to the conditioning set).

Let S_W , S_X and S_I denote, respectively, the full set of covariates, the subset of exogenous variables and the subset of variables of interest. Let $S_N = S_W \setminus S_X$ denote the set of endogenous variables, and let $S_D = S_W \setminus S_I$ denote the set of conditioning variables (variables not of direct interest but necessary for conditioning).

1. When $S_I = S_N$, then the YFWL theorem is applicable for instrumental variables estimation (using

the residualized set of instrumental variables) as shown in theorem 1 and theorem 2.

2. If $S_I \supset S_N$, then the YFWL theorem is applicable for instrumental variables estimation. Since the conditioning set does not contain any endogenous variables, the results of theorem 1 and theorem 2 apply. The only change that is required is to expand W_1 (the set of endogenous variables) in section 3.1 to include the exogenous variables that are of interest and to then use these variables as their own instrumental variables.
3. If $S_I \subset S_N$, then the YFWL theorem can be made applicable for instrumental variables estimation in the following way: take the set of variables of interest as being equal to the set of endogenous variables (or, equivalently, take the conditioning set to be the set of all the exogenous variables only) and apply the result of case 1 (so that results of theorem 1 and theorem 2 apply); finally, extract coefficients of interest and their covariance matrix.
4. If $S_I \supseteq S_X$, then the the YFWL theorem for instrumental variables estimation cannot be applied, as demonstrated in section 3.6 (for the case when $S_I = S_X$ or equivalently $S_D = S_N$). Note that this is the only case when the YFWL theorem is not applicable for instrumental variables estimation cannot be applied. Hence, the negative result is less restrictive than it might have appeared at first sight. After all, it only restricts one out of four possibilities

4 Does the FWL theorem apply to K-class and linear GMM estimators?

The LIML estimator (a member of the K-class estimators) and the linear GMM estimator are closely related to the 2SLS estimator (another member of the K-class of estimators) discussed above. Hence, it is natural to ask whether the FWL theorem, which I have derived above for the 2SLS estimator, applies to the K-class of estimators more generally and to linear GMM estimators as well.⁵ *The answer is that it applies, in large samples, to all consistent K-class estimators, but it does not apply to linear GMM estimators.* The clue to arriving at this answer is offered by trying to understand the common structure of the 2SLS, K-class and linear GMM estimators, and then using a crucial step from the proof of theorem 2.⁶

⁵The popular `ivreg2` suite of functions in STATA seems to allow for partialling for all these estimators. The paper that explains the technical details of `ivreg2` claims, by relying on the FWL theorem, that: “The invariance of the estimation results to partialling-out applies to one- and two-step estimators such as OLS, IV, LIML, and two-step GMM, but not to CUE or to GMM iterated more than two steps.” (Baum et al., 2007, p. 484). I show that the theorem does not apply to two-step GMM estimators of overidentified models.

⁶For a discussion of the K-class of estimators, see Davidson and MacKinnon (1993, p. 649).

4.1 The theory

To get started, let us recall from (17) that the 2SLS estimator of b_2 in the full model (2) is given by

$$\text{the (2,1) block of } [W'P_ZW]^{-1}W'P_ZY, \quad (35)$$

and the 2SLS estimator of \tilde{b}_2 in the partial model (3) is given by

$$\left[\widetilde{W}'_2P_{\widetilde{Z}_2}\widetilde{W}_2\right]^{-1}\widetilde{W}'_2P_{\widetilde{Z}_2}\widetilde{Y}.$$

From (28), we know that the above expression is equivalent to

$$\left[\widetilde{W}'_2P_{\widetilde{Z}_2}\widetilde{W}_2\right]^{-1}\widetilde{W}'_2P_{\widetilde{Z}_2}Y. \quad (36)$$

In the proof of theorem 2, I showed that the

$$\text{the (2,1) block of } [W'P_ZW]^{-1}W'P_Z,$$

was equal to

$$\left[\widetilde{W}'_2P_{\widetilde{Z}_2}\widetilde{W}_2\right]^{-1}\widetilde{W}'_2P_{\widetilde{Z}_2}.$$

Now using this in the context of (35) and (36), we see that this provides another way to see that (35) and (36) are equal. What is important is to note that the proof of theorem 2 relies crucially on P_Z being a projection matrix, i.e. a symmetric and idempotent $n \times n$ matrix (Strang, 2006, p. 164). This is important because when P_Z is a projection matrix, we can decompose it into two orthogonal matrices, P_{W_1} and $P_{\widetilde{Z}_2}$ (Rao et al., 2008, p. 323). It is because of this decomposition that I could show the crucial equality in theorem 2. With these preliminary comments, let us turn to the K-class and linear GMM estimators.

The K-class of estimators for b in the full model (2) is given by (Davidson and MacKinnon, 1993, p. 649):

$$b = [W'(I - KM_Z)W]^{-1}W'(I - KM_Z)Y, \quad (37)$$

where K is a scalar and $M_Z = I - Z(Z'Z)^{-1}Z'$ is the residual maker matrix with respect to the set of instrumental variables. When $K = 0$, we get the OLS estimator; when $K = 1$, we get the 2SLS estimator; when $K = \hat{\kappa}$, where $\hat{\kappa}$ is the smallest eigenvalue of $(W'M_ZW)^{-1/2}W'M_ZW(W'M_ZW)^{-1/2}$. On the other

hand, the the linear GMM estimator for b in the full model (2) is given by (Cameron and Trivedi, 2005, p. 185):

$$b = [W'ZW_NZ'W]^{-1}W'ZW_NZ'Y, \quad (38)$$

where W_N is some symmetric $k_3 \times k_3$ weighting matrix of full rank.

With regard to the K-class of estimators in (37), note that $(I - KM_Z)$ is symmetric because M_Z is symmetric, and

$$(I - KM_Z)(I - KM_Z) = I - KM_Z + (K^2 - K)M_Z \approx I - M_Z$$

where the last step holds, in large samples, for those K-class estimators where K tends to 1 at an appropriately fast rate.⁷ Thus, for consistent K-class estimators—which includes 2SLS, LIML, Fuller, but not OLS—in relatively large samples, $K \approx 1$, and $(I - KM_Z)$ behaves approximately like the projection matrix $P_Z = (I - M_Z)$. Thus, in large samples, consistent K-class estimators have the same structure as the 2SLS estimator. That is why the FWL theorem *can be applied approximately in large samples* for the whole class of consistent K-class of estimators, including the LIML estimator.⁸

Turning to the linear GMM estimator in (38), we see that, in general, while ZW_NZ' is symmetric, it is not idempotent.⁹ This implies that in the linear GMM estimator in (38), ZW_NZ' is not a projection matrix, even in large samples and not even approximately. Hence, the argument that establishes equality of the coefficient vectors from the full and partial regressions will not work. Hence, the FWL theorem cannot be applied to linear GMM estimators, including to the popular 2SGMM estimator.¹⁰

4.2 An empirical example

As an empirical example, I return to David Card's 1995 study of returns to schooling using geographical proximity as instrumental variable (Card, 1995). The model is

$$\log y = \beta_0 + \beta_1 * \text{SCHOOLING} + X\gamma + u \quad (39)$$

⁷K-class estimators, where K tends to 1 at a rate faster than $n^{-1/2}$, are consistent (Davidson and MacKinnon, 1993, p. 649).

⁸An alternative argument might be as follows: since the LIML is the limit of an iterative process involving OLS estimators, and since the FWL theorem applies to OLS estimators, the FWL theorem is also applicable to the LIML estimator. This argument is incomplete unless one shows that the set of estimators that form the sequence in the iterative process belong to a closed set. If it is an open set, then the limit (LIML) will not necessarily inherit the properties of the estimators that are sequences in the iterative process.

⁹As a counterexample, consider $W_N = I$ (the identity matrix of the correct size), which is symmetric and of full rank. Hence, $ZW_NZ' = ZZ'$. Now note that, in general, $ZZ'ZZ'$ is not equal to ZZ' . Thus, ZW_NZ' is not idempotent. Hence, it cannot be a projection matrix.

¹⁰This negative result for linear GMM estimators pertain to overidentified models. For exactly identified models, the linear GMM estimator reduces to the IV estimator, for which we can apply the FWL theorem, as I have shown in theorem 1 and theorem 2.

where y is earnings (wage in cents per hour in 1976), $SCHOOLING$ is years of schooling, and X is a set of exogenous controls like experience (and its square), race (whether black), whether lived in the South, whether lived in an SMSA, and regional dummy variables. The model (39) is estimated with OLS, 2SLS, LIML, Fuller’s K-class estimator and 2SGMM. Other than OLS, estimation uses two instrumental variables: (a) proximity to 2 year college; (b) proximity to 4 year college.¹¹ The source of the data is the National Longitudinal Survey of Young Men (NLSYM) and the sample size is 3101.¹²

Two sets of results are presented in Table 1. Row 1 reports results of estimating the full model; row 2 presents estimates of the partial model, where the latter model partials out the full set of exogenous variables from the outcome variable (log earnings), from the endogenous variable (years of schooling) and from the two instrumental variables (proximity to 2 year and 4 year college). From Table 1, we see that the parameter estimates from the full and partial models are identical for the K-class of estimators (OLS, 2SLS, Fuller and LIML), but they are different from the 2SGMM estimator. For the 2SGMM, the estimate from the full model is 0.33094, and from the partial model is 0.46638. This confirms the theoretical argument presented in the previous sub-section as to why the FWL applies to the K-class of estimators but does *not* apply to linear GMM estimators.

Table 1: Estimates of Returns to Schooling from Full and Partial Models

	OLS	2SLS	Fuller	LIML	2SGMM
Full model	0.07469	0.15706	0.15826	0.16403	0.33094
Partial model	0.07469	0.15706	0.15828	0.16403	0.46638

Notes: The table presents estimates of the return to schooling in a regression of log earnings on schooling, where two instrumental variables are used for estimation: proximity to 2 year college; proximity to 4 year college. The partial model partials out all the exogenous variables from the outcome, endogenous and instrumental variables. OLS = ordinary least squares; 2SLS = two stage least squares; Fuller = Fuller’s modified LIML; LIML = limited information maximum likelihood; 2SGMM = two step GMM.

¹¹I use `ivmodel` in R to get the data and implement the K-class of estimators. I write my own code for the 2SGMM estimator. In the first step, I estimate the model by 2SLS. Using the residuals from the first stage, I construct the optimal weighting matrix. In the second step, I use the optimal weighting matrix (from the first step) and solve the linear system of equations whose solution appears as the 2SGMM estimator in (38). R code for this example is available here: <https://drive.google.com/file/d/1ug0SZVwuW5vuxBHZbsVQWlaNXoxnNBW/view?usp=sharing>.

¹²“The NLSYM began in 1966 with 5525 men age 14–24 and continued with follow-up surveys through 1981.” (Card, 1995).

5 Historical evolution of the YFWL theorem

5.1 Pioneering contribution of G. Udny Yule

5.1.1 Novel notation

Yule (1907) considers the relationship among n random variables X_1, X_2, \dots , and X_n using a sample which has N observations on each of the variables. In particular, suppose this relationship is captured by a linear regression of the first variable, X_1 , on the other variables, X_2, \dots, X_n , and a constant. This linear regression is what Yule called a *multiple regression*. With reference to (1), therefore Yule (1907) uses: $k = n$, $X_1 = Y$, X_2 the second column of W , X_3 the third column of W , and so on.

Subtracting the sample mean from all variables, the multiple regression equation can be equivalently expressed in ‘deviation from mean’ form in which the constant drops out. Let x_1, x_2, \dots, x_n denote the same random variables, but now expressed as deviations from their respective sample means. Introducing a novel notation to express the coefficients and the residual, Yule (1907) writes the sample analogue of the multiple regression function as follows:

$$x_1 = \underbrace{b_{12.34\dots n}x_2 + b_{13.24\dots n}x_3 + \dots + b_{1n.234\dots n-1}x_n}_{\text{predicted value}} + \underbrace{x_{1.234\dots n}}_{\text{residual}}. \quad (40)$$

In (40), the subscripts of the coefficient on each regressor, as also the residual, is divided into the primary and secondary subscripts. The primary subscripts come before the period; the secondary subscripts come after the period. For the coefficients of regressors, the primary subscripts identify the dependent variable and that particular regressor, in that order; and the secondary subscripts identify the other regressors in the model. For the residual in (40), $x_{1.234\dots n}$, the primary subscript identifies the dependent variable and the secondary subscripts identify all the regressors.

It is useful to note three features of the subscripts. First, the order of the primary subscripts is important but the order among the secondary ones are not because the order in which the regressors are arranged is immaterial for estimation and inference. For instance, $b_{12.34\dots n}$ denotes the coefficient on x_2 for a regression of x_1 on x_2, x_3, \dots, x_n . Similarly, $b_{1j.234\dots n}$ denotes the coefficient on x_j for a regression of x_1 on x_2, x_3, \dots, x_n , and note that the secondary subscripts excludes j . Second, elements of the subscripts cannot be repeated because it is not meaningful to include the dependent variable as a regressor or repeat a regressor itself. Third, since the coefficients of the multiple regression function are related to partial correlations, the notation was

also used for denoting relevant partial correlations too.¹³

5.1.2 The theorem

Now consider, with reference to the multiple regression (40), what Yule called *partial regressions*. These refer to regressions with partialled out variables. For instance, with reference to (40), the partial regression of x_1 on x_2 would be constructed as follows: (a) first run a regression of x_1 on all the variables other than x_2 , i.e. x_3, x_4, \dots, x_n , and collect the residuals; (b) run a regression of x_2 on all the other regressors, i.e. x_3, x_4, \dots, x_n , and collect the residuals; (c) run a regression of the residuals from the first step (partialled out outcome variable) on the residuals from the second step (partialled out first regressor).

Yule (1907, p. 184) showed that, if parameters of the multiple regression and the partial regression(s) were estimated by the method of least squares then the corresponding coefficients would be numerically the same. For instance, considering the first regressor, x_2 , in (40), he demonstrated that

$$\frac{\sum (x_{1.34\dots n} \times x_{2.34\dots n})}{\sum (x_{2.34\dots n})^2} = b_{12.34\dots n} \quad (41)$$

where the summation runs over all observations in the sample.¹⁴

To see why (41) gives the claimed result, recall that $x_{1.34\dots n}$ is the residual from a regression of x_1 on x_3, \dots, x_n and $x_{2.34\dots n}$ is the residual from the regression of x_2 on x_3, \dots, x_n . By construction, both have zero mean in the sample. Recall, further, that in a regression of a zero-mean random y on another zero-mean random variable z , the coefficient on the latter is $\sum yz / \sum z^2$. Hence, the left hand side of (41) is the coefficient in a regression of $x_{1.34\dots n}$ on $x_{2.34\dots n}$. The right hand side is, of course, the coefficient on x_2 in (40). Hence, the result.

The above argument, of course, can be applied to any of the regressors in (40) and thus Yule (1907) proved the following general result: the coefficient on any regressor in (40) is the same as the coefficient in a bivariate regression of the residualized dependent variable (the vector of residual from a regression of the dependent variable on the other regressors) and the residualized regressor (the vector of residuals from a regression of the regressor of interest on the other regressors).

With reference to the question posed in section 2, Yule (1907), therefore, showed that a researcher could work with partial regressions if she were only interested in a subset of the coefficients in the multiple regression. She would arrive at the same estimate of the parameters by estimating partial regressions as she

¹³See, for instance, equation (2) in Yule (1907).

¹⁴For a proof see appendix C.

would if she had estimated the full model. In particular, when W_1 contained only one variable (other than the column of 1s), [Yule \(1907\)](#) provided the following answer: (a) run a regression of y (demeaned Y) on the variables in w_2 (demeaned variables in W_2), and collect the residuals; (b) run a regression of w_1 (demeaned variable in W_1 , i.e. excluding the constant) on the variables in w_2 , and collect the residuals; (c) regress the first set of residuals on the second to get the desired coefficient.

There are two things to note about Yule’s answer. First, he did not provide an answer to the question posed in section 2 when W_1 contained more than one random variable (excluding the constant). Of course, partial regressions could be estimated for each independent variable, but they had to be separately estimated as bivariate regressions with *all* the other variables used for the partial regressions. Second, [Yule \(1907\)](#) did not investigate the relationship among variances of the parameters estimated from multiple and partial regressions. Is the *estimated* standard error of a coefficient also identical from the full and partial regressions? Yule did not pose, and hence did not provide any answers to, this question.

5.1.3 Application to a regression with a linear time trend

We can immediately apply the result from Yule’s theorem given above to a regression with a linear time trend. With (40), let x_n denote the demeaned linear time trend variable. Applying (41), we will have the following result: the coefficient on any of the regressors, x_2, x_3, x_{n-1} in (40) is the same as the coefficient in a bivariate regression of residualized x_1 on the relevant regressor. This is of course the exact same result that was presented 26 later in [Frisch and Waugh \(1933\)](#).

5.2 Frisch and Waugh’s result

5.2.1 Notation and result

[Frisch and Waugh \(1933, p. 389\)](#) study the relationship among $n + 1$ variables, one of which is a linear time trend. In reference to (1), therefore [Frisch and Waugh \(1933\)](#) use: $k = n$, $X_0 = Y$, X_1 the second column of W , X_2 the third column of W , and so on. They use the same convention of considering variables in the form of deviations from their respective sample means.¹⁵ Using the *exact same notation* as used by [Yule \(1907\)](#) and postulating an *a priori* true linear relationship among these variables, they write,

$$x_0 = \beta_{01.34\dots n}x_1 + \beta_{02.24\dots n}x_2 + \dots + \beta_{0n.234\dots n-1}x_n, \tag{42}$$

¹⁵“Consider now the n variables x_0, \dots, x_{n-1} and let time be an $(n+1)$ th variable x_n . Let all the variables be measured from their means so that $\sum x_i = 0$ ($i = 0, \dots, n$) where \sum denotes a summation over all the observations.” ([Frisch and Waugh, 1933, p. 394](#)).

and consider estimating the parameters of (42) in two ways. First, they consider the multiple regression of x_0 on x_1, \dots, x_n :

$$x_0 = b_{01.34\dots n}x_1 + b_{02.24\dots n}x_2 + \dots + b_{0n.234\dots n-1}x_n. \quad (43)$$

Second, they consider the multiple regression of x'_0 on x'_1, \dots, x'_{n-1} (note that x_n has been excluded from the set of regressors),

$$x'_0 = b'_{01.34\dots n}x'_1 + b'_{02.24\dots n}x'_2 + \dots + b'_{0,n-1.234\dots n-1}x'_{n-1}, \quad (44)$$

where the primed variables are the corresponding time-demeaned original variables, i.e., for $j = 0, 1, \dots, n-1$, x'_j is the residual in the regression of x_j on x_n .¹⁶ Using the basic rules of determinants and Cramer's rule for solving linear equation systems, Frisch and Waugh (1933, p. 394–396) demonstrated that the coefficients denoted by b in (43) are numerically equal to the corresponding coefficients denoted by b' in (44).¹⁷

With regard to the question posed in section 2, Frisch and Waugh provide the same answer as Yule: coefficients are the same whether they are estimated from the multiple or from partial regressions. There are both similarities and differences between Yule (1907) and Frisch and Waugh (1933). First, whereas in Yule (1907), only one variable could be included in W_1 (the subset of covariates that was of interest to the researcher), in Frisch and Waugh (1933) only one random variable could be included in W_2 (the subset of covariates that was *not* of interest to the researcher). Second, much like Yule (1907) before them, Frisch and Waugh (1933) did not investigate the relationship among estimated variances of the parameters of multiple and partial regressions. The question that is relevant for statistical inference, i.e. standard errors, had still not been posed.

5.3 Lovell extends the OLS analysis

Lovell (1963) extended the reach of the theorem significantly and addressed both questions that had been left unanswered by Yule (1907) and Frisch and Waugh (1933). On the one hand, Lovell (1963) partitioned the set of regressors, W , into two subsets without any restrictions on the number of variables in each subset; on the other, he laid the groundwork for thinking about the *estimated* covariance matrices of the coefficient vectors.

¹⁶Note that (42), (43) and (44) are just the predicted regressions functions. These equations exclude the residuals.

¹⁷For a proof see appendix D. In his presentation of the Frisch-Waugh theorem, Chipman (1998, p. 84–86) argues as if Frisch and Waugh had used projection matrices in their proof. That is not correct. Frisch and Waugh (1933) did not use projection matrices in their proof.

In the context of OLS estimation, [Lovell \(1963\)](#) demonstrated two important results: (a) the coefficient vectors are numerically the same whether they are estimated from the multiple or the partial regressions, and (b) the vector of residuals from the multiple and partial regressions are numerically the same.¹⁸ The first result completed the YFWL so far as the estimate of the coefficient is concerned because the partitioning of the set of regressors was completely general; the second result laid the groundwork for comparing estimated variances of the coefficient vectors from multiple and partial regressions.¹⁹

5.4 C. Giles extends the theorem to IV estimation

So far, the YFWL theorem was always posed in the context of least squares, ordinary or generalized, estimation. To the best of my knowledge, [Giles \(1984\)](#) was the first scholar to extend the theorem to instrumental variables estimation of an exactly identified model, i.e. in the setting where the number of instruments was exactly equal to the number of endogenous variables.²⁰ [Giles \(1984\)](#) demonstrated that the YFWL theorem holds for the IV estimator (instrumental variables estimator when the model is exactly identified) under 13 combinations of partialling. He shows that the coefficient vectors from the full and partial regression models are identical and that their asymptotic covariance matrices are the same. [Giles \(1984\)](#) did not comment on the *estimators* of asymptotic covariance matrices of the coefficient vector and, therefore, did not discuss the need for degrees of freedom corrections.

5.5 P. Ding considers standard errors for OLS estimation

For more than 100 years, the evolution of the theorem had primarily focused on comparing the coefficient vectors from multiple and partial regressions. The YFWL theorem had established that they are numerically the same. This was a very useful result but still did not answer the question that would be relevant if a researcher were interested in conducting statistical inference about the coefficient vector. To conduct inference about b_2 using results of estimating \tilde{b}_2 , we also need to be able to compare their *estimated* covariance matrices. While [Lovell \(1963\)](#) had commented briefly on this issue, to the best of my knowledge, [Ding \(2021, Theorem 2, 3, 4\)](#) provided the first systematic treatment. [Ding \(2021\)](#) demonstrated that, in the context of

¹⁸I omit the proofs because they are just special cases of the proofs offered in section 3.3 in this paper.

¹⁹It is straightforward to extend the YFWL from ordinary least squares to generalized least squares estimation, as Lovell had noted when commenting on autocorrelated errors ([Lovell, 1963](#), p. 1004). Other scholars have worked on variations of Lovell's results; see, for instance, [Fiebig and Bartels \(1996\)](#); [Krishnakumar \(2006\)](#).

²⁰"In the following discussion the necessary inverse matrices are assumed to exist, and we consider only the case of equal numbers of regressors and instruments as this includes such common estimators as two-stage least squares and other members of the k-class." ([Giles, 1984](#), p. 340). The last part of the sentence uses non-standard terminology. In current terminology, the two-stage least squares estimator refers to the case where the number of instruments is larger than the number of endogenous variables.

ordinary least squares estimation, covariance matrices of coefficient vectors from partial and full regressions would, in many contexts, be exactly equal or equal up to a degrees of freedom correction. I have extended these sets of results to instrumental variables estimation and have commented on the different cases in section 3.5.1 above.

6 Discussion and conclusion

By way of concluding this paper, I would like to offer two sets of comments. My first set of comments is about the substance and usefulness of the YFWL theorem, and the second set is about a puzzle in the history of the YFWL theorem.

There are two reasons that make the YFWL theorem so useful, the first computational and the second conceptual. Let me start with the computational usefulness. For a linear regression model estimated by the method of least squares, the YFWL theorem shows that if a researcher is interested in estimating only a subset of the parameters of the model, she need not estimate the full model. She can partition the set of regressors into two subsets, those that are of interest and those that are not of direct interest (but needs to be used for conditioning). She can regress the outcome variable on the conditioning set to create the residualized outcome variable. She can regress each of the variables of interest on the conditioning set to create corresponding residualized covariates. Finally, she can regress the residualized outcome variable on the residualized covariates of interest to get the estimated parameter of interest.

If, in addition, the variables of interest are endogenous and she has a set of valid instrumental variables, she can carry out the same procedure as above for instrumental variable estimation. In addition to creating the residualized outcome variable and the residualized covariates of interest (the endogenous variables), she must create residualized instrumental variables with respect to the exogenous variables in the model. Now, if she estimates an instrumental variables regression of the residualized outcome variable on the residualized covariates of interest using the residualized instrumental variables, then she will get consistent estimates of the parameters of interest.

In large samples, the same procedure can be justified for consistent K-class estimators (LIML being an important member of this class) because the YFWL theorem applies approximately to this class of estimators in large samples. It is important to note that the same procedure cannot be used for linear GMM estimators, including the popular two step (or optimal) GMM estimator. This is because the YFWL theorem does not apply to linear GMM estimators.

For statistical inference, she can work with a partial regression in many, but not all, cases. If the errors are homoskedastic, then the estimated covariance matrix from the partial regression can be used for inference once a degree of freedom adjustment is made. If the errors are heteroskedastic, as is often the case in *cross sectional* data sets, and the researcher wishes to use the HC0 or HC1 variant of HCCM, then she can use the estimated covariance matrix from the partial regression without any adjustment in the case of HC0 and with a degree of freedom adjustment in the case of HC1. If the researcher is using a *time series* data set and wishes to use a HAC covariance matrix, then she can use the estimated covariance matrix from the partial regression as is. If the researcher is using a *panel data* set and wishes to use the standard cluster robust covariance matrix, then she can use the estimated covariance matrix from the partial regression without any adjustment; if a degree of freedom correction is used to compute the cluster robust covariance matrix, then she will need to apply a corresponding degree of freedom adjustment.

If, on the other hand, the researcher wishes to use the HC2, HC3, HC4 or HC5 variants of heteroskedasticity consistent covariance matrix, or if she wishes to use other variants of the cluster robust covariance matrix or HAC covariance matrices, then she cannot use above results, i.e. the covariance matrices from multiple and partial regressions are no longer equal, even after degrees of freedom adjustments. Instead, the researcher can use the four-step computational method proposed in appendix E if she wishes to use information from the partial regression to conduct proper statistical inference about parameters in the multiple regression.

While the question of computation is certainly important, the YFWL theorem is perhaps even more relevant because of its indispensability for an intuitive understanding of multiple regression and partial correlations. In non-experimental settings, whenever a researcher has to analyze the relationship among three or more variables, she has to address a question that G. Udny Yule, a pioneer of regression analysis, raised more than a century ago.

[F]or instance, it might be found that changes in pauperism were highly correlated (positively) with changes in the out-relief ratio, and also with changes in the proportion of old; and the question might arise how far the first correlation was due merely to a tendency to give out-relief more freely to the old than the young, i.e. to a correlation between changes in out-relief and changes in the proportion of the old. (Yule, 1911, p. 225).

In other words, how would a researcher distinguish between the true, or partial, correlation between changes in pauperism and out-relief ratio from the multiple (or total) correlation that was being observed between these two variables merely because both were correlated with a third variable, the proportion of the old.

To address this question in the most general setting, [Yule \(1911\)](#) goes on to explain, the researcher must posit a linear relationship between the n variables of interest, $X_1, X_2, X_3, \dots, X_n$. Such a relationship, Yule, notes

... will be of the form

$$X_1 = a + b_2X_2 + b_3X_3 + \dots + b_nX_n.$$

If in such a generalized *regression* or *characteristic equation* we find a sensible positive value for any one coefficient such as b_2 , we know that there must be a positive correlation between X_1 and X_2 that cannot be accounted for by mere correlations of X_1 and X_2 with X_3, X_4 , or X_n for the effects of changes in these variables are allowed for in the remaining terms on the right. The magnitude of b_2 gives, in fact, the mean change in X_1 associated with a unit change in X_2 when all the remaining variables are kept constant. ([Yule, 1911](#), p. 226, emphasis in the original).

But how do we know that the “magnitude of b_2 gives, in fact, the mean change in X_1 associated with a unit change in X_2 when all the remaining variables are kept constant?” We know this precisely because of the YFWL theorem, which has rigorously established that b_2 is the correlation coefficient (normalized by the ratio of their standard deviations) between residualized version of X_1 and X_2 . By residualizing both X_1 and X_2 with respect to X_3, X_4, \dots, X_n , we have removed from the picture the possible influence these other variables might have had on X_1 and X_2 . Thus, we are now able to measure the correlation between X_1 and X_2 after removing the possible influence of the other variables out of the way, i.e. holding “all the remaining variables ... constant.” This essential conceptual meaning of ‘holding variables constant’ in a multiple regression rests completely on the YFWL theorem.²¹

My second comment relates to a puzzle in the intellectual history of the YFWL theorem. The puzzle arises from noting that [Yule \(1907\)](#)’s result is essentially the same as the result proved by [Frisch and Waugh \(1933\)](#), as I have demonstrated. What is puzzling, therefore, is that [Frisch and Waugh \(1933\)](#) do not cite [Yule \(1907\)](#). This omission seems puzzling to me given two fact. First, [Frisch and Waugh \(1933\)](#) refer to a previous paper by one of the authors: [Frisch \(1929\)](#). It is also to be noted that Frisch had published a related work in 1931 ([Frisch and Mudgett, 1931](#)). What is interesting is that in both these papers, there is explicit

²¹Yule was a leading figure behind the development of regression analysis. The substantive issue he seems to have analyzed in many studies is the effect of the out-relief ratio (proportion of poor provided monetary, food or other assistance but not required to be in an institution, compared to those who were required to enter an institution like a poor house to receive support) on pauperism. In [Yule \(1899\)](#), he demonstrated that out-relief is positive correlated with, perhaps causes and increase in, pauperism. Of course, in substantive terms, Yule’s regression analysis suffers from simultaneity bias. After all pauperism might also be a cause of administrative measures that change the out-relief ratio.

reference to the partial correlation formula introduced by Yule.²² Recall that Yule had introduced a novel notation for representing partial correlations and coefficients in a multiple regression equation in his 1907 paper that I have discussed. The notation was then introduced to a wider readership in statistics with his 1911 book (Yule, 1911). This book was extremely popular and ran several editions, later ones with Kendall (Yule and Kendall, 1948). Second, Frisch and Waugh (1933) use the exact same notation to represent the regression function that Yule had introduced in his 1907 paper. as I have highlighted.

These two facts suggest that Ragnar Frisch was familiar with Yule’s work. In fact, Yule’s work on partial correlations and multiple regression was the standard approach that was widely used and accepted by statisticians in the early part of the twentieth century (Griffin, 1931). Therefore, it is a puzzle of intellectual history as to why Frisch and Waugh (1933) did not cite Yule’s 1907 result about the equality of multiple and partial regression coefficients which was, in substantive terms, exactly what they went on to prove in their paper. Of course, Frisch and Waugh (1933) used a different method of proof from Yule (1907), as I have demonstrated in this paper. But in substantive terms, Frisch and Waugh (1933) proved the same result that Yule had established more than two decades ago. No matter what the reason for the omission in history, it seems that now is the right time to acknowledge Yule’s pioneering contribution—both in terms of computation and in terms of interpretation—by attaching his name to a theorem that is so widely used in econometrics.

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²²For instance, see Frisch (1929, p. 75, 76) and Frisch and Mudgett (1931, p. 389)

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Appendix A Proof of Theorem 1

Proof. Coefficients are identical. The instrumental variables estimator of β_2 in the full regression model (2) satisfies the linear equation system (10), and the instrumental variables estimator of $\tilde{\beta}_2$ in the partial regression model (3) satisfies the system of linear equations in (12). Since $\tilde{Y} = M_{W_1}Y$, $\tilde{W}_2 = M_{W_1}W_2$, $\tilde{Z}_2 = M_{W_1}Z_2$, and M_{W_1} is idempotent, we can immediately see that (10) and (12) are the *same* equation systems. Hence, the solutions, when they exist, will also be identically the same.

Note that a linear system of equations: (a) either has a unique solution (Strang (2006, chapter 1) or (b) has no solution or an infinite number of solutions (Strang, 2006, chapter 2). Both (10) and (12) consist of k_3 equations in k_2 unknowns. Since $k_3 \geq k_2$, there are more equations than unknowns. Hence, in general, there will be no solutions for (10) and (12), i.e. both these systems of equations are inconsistent. If, and only if, $Z_2'\tilde{Y}$ lies in the column space of $Z_2'\tilde{W}_2$ will there exist a (unique) solution. In such as situation, we will have $b_2 = \tilde{b}_2$, i.e. coefficients from the full and partial regression models will be identical.

Residuals are identical. Let u_M and u_P denote the vectors of residuals from the instrumental variables estimation of the multiple and partial regression models (2) and (3), respectively. Note two properties of the residual maker matrix M_{W_1} : first, that $M_{W_1}u_M = \left(I - W_1(W_1'W_1)^{-1}W_1'\right)u_M = u_M$ because $W_1'u_M = 0$ (by the exogeneity of W_1); and, second, that $M_{W_1}W_1 = \left(I - W_1(W_1'W_1)^{-1}W_1'\right)W_1 = 0$ (because the vector of residuals from projecting each column of W_1 onto the column space of W_1 is the vector of zeros).

Consider the case when the system of equations (10) and (12) have unique solutions, so that $b_2 = \tilde{b}_2$. We can see immediately that residuals from the instrumental variables estimation of the multiple and partial regression models are the same because,

$$u_M = M_{W_1}u_M = M_{W_1}(Y - W_1b_1 - W_2b_2) = \tilde{Y} - 0 - \tilde{W}_2b_2 = \tilde{Y} - \tilde{W}_2\tilde{b}_2 = u_P. \quad (45)$$

□

Appendix B Proof of Theorem 2

Proof. Comparing (22) and (23), we see that the proof will be completed if I can show that

$$\text{the (2,1) block of } [(P_ZW)'](P_ZW)]^{-1}(P_ZW)' \text{ is equal to } \left[\left(P_{\tilde{Z}_2} \tilde{W}_2 \right)' \left(P_{\tilde{Z}_2} \tilde{W}_2 \right) \right]^{-1} \left(P_{\tilde{Z}_2} \tilde{W}_2 \right)'.$$

We know from the argument in [Ding \(2021, p. 6\)](#), that if $X = [X_1 : X_2]$, then

$$(X'X)^{-1}X' = \begin{pmatrix} * \\ (\tilde{X}'_2\tilde{X}_2)^{-1}\tilde{X}'_2 \end{pmatrix}$$

where $\tilde{X}_2 = [I - X_1(X'_1X_1)^{-1}X'_1]X_2$, and the (1, 1) block on the RHS, $*$, is left unspecified because it is not needed for the computations. Now, let $X_1 = P_ZW_1$ and $X_2 = P_ZW_2$, so that $P_ZW = [P_ZW_1 : P_ZW_2]$. I will now show that $\tilde{X}_2 = P_{\tilde{Z}_2}\tilde{W}_2$ in two steps and that will complete the proof. In the first step, I will show that $\tilde{X}_2 = P_Z\tilde{W}_2$; in the second step I will show that $P_Z\tilde{W}_2 = P_{\tilde{Z}_2}\tilde{W}_2$.

First step: Since $Z = [W_1 : Z_2]$, Lemma 2 in [Ding \(2021, p. 5\)](#) shows that $P_Z = P_{W_1} + P_{\tilde{Z}_2}$ and $P_{W_1}P_{\tilde{Z}_2} = 0$.²³ Hence, $P_ZW_1 = P_{W_1}W_1 + P_{\tilde{Z}_2}W_1 = P_{W_1}W_1 = W_1$, because the first term is zero,

$$P_{\tilde{Z}_2}W_1 = \tilde{Z}_2(\tilde{Z}'_2\tilde{Z}_2)^{-1}\tilde{Z}'_2W_1 = \tilde{Z}_2(\tilde{Z}'_2\tilde{Z}_2)^{-1}Z'_2[I - W_1(W'_1W_1)^{-1}W'_1]W_1 = 0,$$

and the second term is

$$P_{W_1}W_1 = [W_1(W'_1W_1)^{-1}W'_1]W_1 = W_1.$$

Hence,

$$\begin{aligned} \tilde{X}_2 &= [I - P_ZW_1(W'_1P_ZW_1)^{-1}W'_1P_Z]P_ZW_2 \\ &= [P_Z - P_ZW_1(W'_1P_ZW_1)^{-1}W'_1P_ZP_Z]W_2 \\ &= P_Z[I - W_1(W'_1P_ZW_1)^{-1}W'_1P_Z]W_2 = P_Z[I - W_1(W'_1W_1)^{-1}W'_1]W_2 = P_ZM_{W_1}W_2 = P_Z\tilde{W}_2, \end{aligned}$$

where I have used the idempotency of P_Z and the fact that $P_ZW_1 = W_1$, which was shown above.

Second step: Note that

$$P_Z\tilde{W}_2 = (P_{W_1} + P_{\tilde{Z}_2})\tilde{W}_2 = P_{W_1}\tilde{W}_2 + P_{\tilde{Z}_2}\tilde{W}_2 = P_{W_1}M_{W_1}W_2 + P_{\tilde{Z}_2}\tilde{W}_2 = 0 + P_{\tilde{Z}_2}\tilde{W}_2,$$

because $P_{W_1} \perp M_{W_1}$. □

²³This result on the decomposition of projection matrices is used in [Chipman \(1998, p. 85\)](#) and [Rao et al. \(2008, page 323\)](#).

Appendix C Proof of Yule's result

The first step of the proof is to derive the normal equations arising from the least squares method of estimation. Given the random variables x_1, x_2, \dots, x_n , which are demeaned versions of X_1, X_2, \dots, X_n , the method of least squares chooses constants b_1, b_2, \dots, b_n to minimize

$$\sum (x_1 - b_2x_2 - b_3x_3 - \dots - b_nx_n)^2,$$

where the sum runs over all observations in the sample. The first order conditions of this minimization problem are referred to as the 'normal equations'. Differentiating the above with respect to b_j , we have

$$\sum x_j (x_1 - b_2x_2 - b_3x_3 - \dots - b_nx_n) = 0 \quad (j = 2, 3, \dots, n), \quad (46)$$

so that, using Yule's notation, the normal equations are given by

$$\sum x_j x_{1.234\dots n} = 0 \quad (j = 2, 3, \dots, n), \quad (47)$$

which shows that the residual in the regression equation (40) is uncorrelated with all the regressors included in the model, a result that holds for any regression function whose coefficients are estimated with the method of least squares.

Now consider, $x_{2.34\dots n}$, the residuals from a regression of x_2 on x_3, x_4, \dots, x_n , and note that it is a linear function of $x_2, x_3, x_4, \dots, x_n$. Hence, using (47), we have

$$\sum x_{1.234\dots n} x_{2.34\dots n} = \sum x_{1.234\dots n} (x_2 - c_1x_3 - \dots - c_{n-2}x_n) = 0,$$

for some constants c_1, \dots, c_{n-2} .

We are now ready to prove the main result: coefficients from multiple and partial regressions are numer-

ically the same.

$$\begin{aligned}
0 &= \sum x_{2.34\dots n} x_{1.234\dots n} \\
&= \sum x_{2.34\dots n} (x_1 - b_{12.34\dots n} x_2 + b_{13.24\dots n} x_3 + \dots + b_{1n.234\dots n-1} x_n) \\
&= \sum x_{2.34\dots n} (x_1 - b_{12.34\dots n} x_2) \\
&= \sum x_{2.34\dots n} x_1 - b_{12.34\dots n} \sum x_{2.34\dots n} x_2 \\
&= \sum x_{2.34\dots n} (b_{13.4\dots n} x_3 + b_{14.3\dots n} x_4 + \dots + b_{1n.34\dots n} x_n + x_{1.34\dots n}) - b_{12.34\dots n} \sum x_{2.34\dots n} x_2 \\
&= \sum x_{2.34\dots n} x_{1.34\dots n} - b_{12.34\dots n} \sum x_{2.34\dots n} x_2 \\
&= \sum x_{2.34\dots n} x_{1.34\dots n} - b_{12.34\dots n} \sum x_{2.34\dots n} (b_{23.4\dots n} x_3 + b_{24.3\dots n} x_4 + \dots + b_{2n.34\dots n} x_n + x_{2.34\dots n}) \\
&= \sum x_{2.34\dots n} x_{1.34\dots n} - b_{12.34\dots n} \sum x_{2.34\dots n} x_{2.34\dots n}
\end{aligned}$$

Hence, we have (41):

$$\frac{\sum (x_{1.34\dots n} \times x_{2.34\dots n})}{\sum (x_{2.34\dots n})^2} = b_{12.34\dots n}.$$

Appendix D Proof of Frisch and Waugh's result

Consider (43) and let

$$m_{ij} = \sum x_i x_j \quad (i, j = 0, 1, 2, \dots, n), \tag{48}$$

where the sum runs over all observations, and x_0, x_1, \dots, x_n are demeaned versions of X_0, X_1, \dots, X_n . Then, [Frisch and Waugh \(1933, p. 394\)](#) assert that the regression equation in (43) can be expressed as the following equation that specifies the determinant of the relevant $(n + 1)$ -dimensional matrix to be zero:

$$\begin{vmatrix}
x_0 & x_1 & \cdots & x_n \\
m_{10} & m_{11} & \cdots & m_{1n} \\
\vdots & \vdots & \ddots & \vdots \\
m_{n0} & m_{n1} & \cdots & m_{nn}
\end{vmatrix} = 0. \tag{49}$$

To see this, which [Frisch and Waugh \(1933\)](#) do not explain, perhaps because it was obvious to them, we need to recall two things. First, if we expand the determinant in (49) using the first row of the matrix, we

will get an equation of the following form,

$$a_0x_0 + a_1x_1 + \cdots + a_nx_n = 0, \quad (50)$$

where a_0 is the determinant obtained by deleting the first row and first column (of the matrix whose determinant is being considered in (49)), a_1 is -1 times the determinant obtained by deleting the first row and second column, a_2 is 1 times the determinant obtained by deleting the first row and third column, and so on.²⁴ Assuming $a_0 \neq 0$, which is guaranteed as long as the regressors are not perfectly collinear, this gives

$$x_0 = -\frac{a_1}{a_0}x_1 - \cdots - \frac{a_n}{a_0}x_n. \quad (51)$$

This has the same form as (43) and all we need to do is to show that the coefficients in (43) are what appears in (51). To do so, we can use the normal equation and Cramer's rule.

Recall that the normal equations that had been written in (47) can, with reference to the least squares estimation (43), be written more compactly as $X'Xb = X'y$, where $X = [x_1 : x_2 : \cdots : x_n]$ is the matrix obtained by stacking the regressors column-wise, $y = x_0$ is the dependent variable, and b is the least squares coefficient vector:

$$b = [b_{01.34\dots n} \quad b_{02.24\dots n} \quad \cdots \quad b_{0n.234\dots n-1}].$$

Note that the (i, j) -th element of $X'X$ is m_{ij} as defined in (53), and the i -th element of $X'y$ is m_{i0} . Thus, the normal equations can be written as

$$\begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1j} & \cdots & m_{1n} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ m_{n1} & m_{n2} & \cdots & m_{nj} & \cdots & m_{nn} \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} m_{10} \\ \vdots \\ m_{n0} \end{bmatrix}. \quad (52)$$

Applying Cramer's rule (Strang, 2006, p. 221) to this equation system to solve the b vector, keeping track of how many columns are switched and recalling that switching rows (columns) of a matrix only changes the sign of the determinant (Strang, 2006, p. 203), we see that the coefficients in (43) and (51) are identical.

Thus, for $j = 1, 2, \dots, n$

$$b_j = \frac{|B_j|}{|A|},$$

²⁴The signs alternate because the determinant obtained by deleting the first row and the j -th column is multiplied by $-1^{(1+j)}$, where $j = 1, 2, \dots, n + 1$.

where

$$A = \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1j} & \cdots & m_{1n} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ m_{n1} & m_{n2} & \cdots & m_{nj} & \cdots & m_{nn} \end{bmatrix}$$

and

$$B_j = \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{10} & \cdots & m_{1n} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ m_{n1} & m_{n2} & \cdots & m_{n0} & \cdots & m_{nn} \end{bmatrix}$$

is obtained from A by replacing the j -th column with $(m_{10} \ \dots \ m_{n0})'$.

Now consider (44) and let

$$m'_{ij} = \sum x'_i x'_j \quad (i, j = 0, 1, 2, \dots, n), \quad (53)$$

where the sum runs, once again, over all observations. Using the same logic as above, we will be able to see that the regression equation in (44) can be expressed as

$$\begin{vmatrix} x'_0 & x'_1 & \cdots & x'_{n-1} \\ m'_{10} & m'_{11} & \cdots & m'_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ m'_{n-1,0} & m'_{n-1,1} & \cdots & m'_{n-1,n-1} \end{vmatrix} = 0. \quad (54)$$

The strategy will now be to use (49) and (54) to show that the first $n - 1$ coefficients in (43) are equal to the corresponding coefficients in (44). The first thing is to relate m_{ij} and m'_{ij} . This is easy to do by noting that

$$x'_j = x_j - \frac{m_{jn}}{m_{nn}} x_n \quad (j = 0, 1, 2, \dots, n - 1), \quad (55)$$

because x'_j is the residual from a regression of x_j on x_n . Multiplying x_i on both sides and summing over all observations for x_i and x_j , we get

$$m'_{ij} = m_{ij} - \frac{m_{in} m_{jn}}{m_{nn}}. \quad (56)$$

The second step is to return to (49) and carry out a series of elementary row operations that converts elements in the second through the penultimate row from m_{ij} to m'_{ij} . From the row beginning with m_{10} in (49) subtract $-m_{1,n}/m_{n,n}$ times the last row; from the row beginning with m_{20} subtract $-m_{2,n}/m_{n,n}$ times the last row; and so on. Note that these row operations do not change the determinant of the matrix

(Strang, 2006, p. 204). Hence, these series of elementary row operations will convert (49) to

$$\begin{vmatrix} x_0 & x_1 & \cdots & x_{n-1} & x_n \\ m'_{10} & m'_{11} & \cdots & m'_{1,n-1} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ m'_{n-1,0} & m'_{n-1,1} & \cdots & m'_{n-1,n-1} & 0 \\ m_{n0} & m_{n1} & \cdots & m_{n,n-1} & m_{nn} \end{vmatrix} = 0. \quad (57)$$

because of (56). Now consider the following determinant equation

$$\begin{vmatrix} x'_0 & x'_1 & \cdots & x'_{n-1} & 0 \\ m'_{10} & m'_{11} & \cdots & m'_{1,n-1} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ m'_{n-1,0} & m'_{n-1,1} & \cdots & m'_{n-1,n-1} & 0 \\ c_0 & c_1 & \cdots & c_{n-1} & 1 \end{vmatrix} = 0, \quad (58)$$

for some arbitrary constants c_0, \dots, c_{n-1} , and by expanding the determinant by the last column, note that (58) is equivalent to (54).

Expanding (57) by the first row, we get

$$a_0 x_0 + a_1 x_1 + \cdots + a_{n-1} x_{n-1} + a_n x_n = 0 \quad (59)$$

and expanding (58) by the first row, we get

$$a'_0 x'_0 + a'_1 x'_1 + \cdots + a'_{n-1} x'_{n-1} = 0, \quad (60)$$

where

$$a_i = a'_i m_{nn} \quad (i = 0, 1, 2, \dots, n-1). \quad (61)$$

Rearranging (59) as

$$x_0 = -\frac{a_1}{a_0} x_1 - \cdots - \frac{a_{n-1}}{a_0} x_{n-1} - \frac{a_n}{a_0} x_n, \quad (62)$$

and rearranging (60) as

$$x'_0 = -\frac{a'_1}{a'_0} x'_1 - \cdots - \frac{a'_{n-1}}{a'_0} x'_{n-1}, \quad (63)$$

and noting that (57) is a re-expression of (43) while (58) is a re-expression of (44), we have

$$b_{0i.34\dots n} = b'_{0i.34\dots n} \quad (i = 1, 2, \dots, n-1). \quad (64)$$

Appendix E Computational extension of Ding's results

Let me discuss the HC2 case in detail; the other cases can be dealt with in a similar manner. Let $h_{ii} = W'_i (W'W)^{-1} W_i$, where W_i is the i -th column of $W = [W_1 : W_2]$ with reference to (2). The HC2 estimated covariance matrix of b_2 in (2) is given by

$$\text{the (2, 2) block of } (W'W)^{-1} W' \frac{\text{diag}[u_i^2]}{1 - h_{ii}} W (W'W)^{-1}, \quad (65)$$

which, using the argument about block matrices in Ding (2021, p. 6), is

$$\text{the (2, 2) block of } \begin{pmatrix} * & \\ (\widetilde{W}'_2 \widetilde{W}_2)^{-1} \widetilde{W}'_2 & \end{pmatrix} \frac{\text{diag}[u_i^2]}{1 - h_{ii}} \begin{pmatrix} \widetilde{W}_2 (\widetilde{W}'_2 \widetilde{W}_2)^{-1} & * \\ & \end{pmatrix}, \quad (66)$$

and, therefore, is given by

$$(\widetilde{W}'_2 \widetilde{W}_2)^{-1} \widetilde{W}'_2 \frac{\text{diag}[u_i^2]}{1 - h_{ii}} \widetilde{W}_2 (\widetilde{W}'_2 \widetilde{W}_2)^{-1}, \quad (67)$$

which, since u_i^2 can be replaced with \tilde{u}_i^2 , is

$$(\widetilde{W}'_2 \widetilde{W}_2)^{-1} \widetilde{W}'_2 \frac{\text{diag}[\tilde{u}_i^2]}{1 - h_{ii}} \widetilde{W}_2 (\widetilde{W}'_2 \widetilde{W}_2)^{-1}. \quad (68)$$

If we can estimate h_{ii} using information available after estimating the partial regression (3), we can use (68) to compute the covariance matrix that is necessary to conduct proper statistical inference on the coefficient vector b_2 in the multiple regression (2). In fact, using results on the inverse of partitioned matrices, we can do so. Hence, the following procedure for estimation and inference coefficient vector b_1 in the multiple regression (2) based on the estimation of the partial regression (3) can be suggested.

1. Compute $\tilde{Y} = [I - \widetilde{W}_1 (\widetilde{W}'_1 \widetilde{W}_1)^{-1} \widetilde{W}'_1] Y$, and $\widetilde{W}_2 = [I - \widetilde{W}_1 (\widetilde{W}'_1 \widetilde{W}_1)^{-1} \widetilde{W}'_1] W_2$.
2. Estimate the partial regression (3) by regressing \tilde{Y} on \widetilde{W}_2 and get the coefficient vector \tilde{b}_2 .
3. For $i = 1, 2, \dots, N$, compute $h_{ii} = W'_i (W'W)^{-1} W_i$ for the matrix of covariates, $W = [W_1 : W_2]$, in

the multiple regression (2). Let $\widetilde{W}_2 = [I - \widetilde{W}_1(\widetilde{W}_1'\widetilde{W}_1)^{-1}\widetilde{W}_1'] W_2$, and note that, using results for the inverse of partitioned matrices (Greene, 2012, p. 994), we have

$$(W'W)^{-1} = \begin{pmatrix} W^{11} & W^{12} \\ W^{21} & W^{22} \end{pmatrix}, \quad (69)$$

where

$$W^{11} = (W_1'W_1)^{-1} + (W_1'W_1)^{-1} W_1'W_2 \left(\widetilde{W}_2'\widetilde{W}_2\right)^{-1} W_2'W_1 (W_1'W_1)^{-1},$$

$$W^{12} = W^{11'} = - \left(\widetilde{W}_2'\widetilde{W}_2\right)^{-1} W_2'W_1 (W_1'W_1)^{-1}$$

and

$$W^{22} = \left(\widetilde{W}_2'\widetilde{W}_2\right)^{-1}.$$

Thus, we only compute W^{11} , W^{12} and W^{22} , and then stack them to get $(W'W)^{-1}$. This avoids computing inverse of the $k \times k$ matrix $W'W$, and instead only involves computing inverses of $k_1 \times k_1$ or $k_2 \times k_2$ matrices.

4. Use the matrix of covariates, \widetilde{W}_2 and the vector of residuals, \tilde{u} , from the partial regression (3), to compute the covariance matrix of \tilde{b}_1 using (68).

The four step procedure outlined above can be used for HC3, HC4 variants of HCCM, and might even be applied to different variants of cluster-robust covariance matrices for which Theorem 3 and 4 in Ding (2021) do not hold. The first step would remain unchanged; the second step would only change if any quantity other than, or in addition to, h_{ii} is needed; only the third step would change significantly, where the researcher would need to use the correct expression for the relevant estimated covariance matrix, in place of (68).