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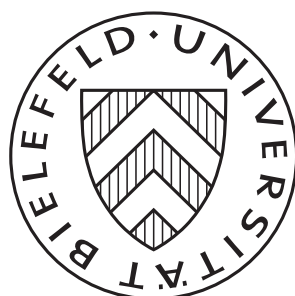


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Giorgio Ferrari and Shihao Zhu



OPTIMAL RETIREMENT CHOICE UNDER AGE-DEPENDENT FORCE OF MORTALITY

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ABSTRACT. This paper examines the retirement decision, optimal investment, and consumption strategies under an age-dependent force of mortality. We formulate the optimization problem as a combined stochastic control and optimal stopping problem with a random time horizon, featuring three state variables: wealth, labor income, and force of mortality. To address this problem, we transform it into its dual form, which is a finite time horizon, three-dimensional degenerate optimal stopping problem with interconnected dynamics. We establish the existence of an optimal retirement boundary that splits the state space into continuation and stopping regions. Regularity of the optimal stopping value function is derived and the boundary is proved to be Lipschitz continuous, and it is characterized as the unique solution to a nonlinear integral equation, which we compute numerically. In the original coordinates, the agent thus retires whenever her wealth exceeds an age-, labor income- and mortality-dependent transformed version of the optimal stopping boundary. We also provide numerical illustrations of the optimal strategies, including the sensitivities of the optimal retirement boundary concerning the relevant model's parameters.

Keywords: Optimal retirement time; Optimal consumption; Optimal portfolio choice; Duality; Optimal stopping; Free boundary; Stochastic control.

MSC Classification: 91B70, 93E20, 60G40.

JEL Classification: G11, E21, I13.

1. INTRODUCTION

The timing of retirement is a crucial financial decision that individuals must face as they approach later stages of life. However, determining the appropriate moment to retire is a complex undertaking, as it is influenced by numerous factors. One of the most prominent and evident predictors of retirement decisions is an individual's chronological age, as highlighted in

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empirical studies such as Wang and Shi [37]. Additionally, research indicates that individuals typically postpone retirement until they possess the necessary financial means to do so, with retirement timing being linked to both wealth and labor income, as explored in Honig [21]. In line with rational decision-making principles and the life-cycle model in economics, subjective life expectancy (how long one expects to live or expected mortality) also plays a significant role in retirement timing. Szinovacz et al. [36] used data from the Health and Retirement Study and found that expected mortality risk influenced retirement plans, and the effects were especially strong for expectations to work beyond age 65. Furthermore, factors like health status and familiar caregiving responsibilities contribute to the complexity of retirement decision-making. For a comprehensive examination of these factors, please refer to Fisher et al. [17]’s insightful review.

In addition to the aforementioned empirical studies, the field of optimal retirement time decision-making has been enriched by various theoretical models from a financial perspective. The historical origins of the optimal retirement time problem can be traced back to seminal works such as Jin Choi and Shim [26], Farhi and Panageas [16], Choi et al. [7], Dybvig and Liu [14], and Dybvig and Liu [15]. These foundational contributions paved the way for investigating the optimal investment and consumption behavior of individuals facing retirement decisions, resulting in significant advancements across various contexts. Notable advancements include the introduction of features like mandatory retirement dates and early retirement options Yang and Koo [38], the consideration of consumption ratcheting Jeon and Park [25], the exploration of partial information Chen et al. [6], the incorporation of habit persistence Chen et al. [5], Guan et al. [20], the examination of return ambiguity in risky asset prices Park and Wong [32], the analysis of non-Markovian environments Yang et al. [39], and the study of heterogeneous consumption patterns involving basic and luxury goods Jang et al. [23].

However, an important characteristic shared by much of the existing literature is the absence of consideration for mortality risk or the assumption of a constant force of mortality. This requirement, although common, can impose significant restrictions on the modeling framework. In reality, mortality has long been studied in the fields of mathematics and demography. Abraham De Moivre, in 1725, proposed perhaps the earliest mathematical mortality model, suggesting that the probability of surviving from birth to age x follows a linear function of age. Moreover, the most successful and influential mortality law is known as Gompertz’s law, formulated by Gompertz [19]. This law describes a function with exponentially increasing mortality rates and has found application in insurance economics as well (see, e.g., Milevsky and Young [30] and

De Angelis and Stabile [13] for the optimal annuitization problem with a Gompertz force of mortality).

In this paper, we aim to quantitatively examine the interplay between age, wealth, labor income, and mortality on the optimal retirement time. We extend the existing model by incorporating an age-dependent force of mortality, specifically the Gompertz model, to comprehensively analyze these influences. Our study focuses on an individual who confronts a mandatory retirement date but possesses the option to retire earlier, assuming irreversibility in retirement decisions. The individual can allocate her wealth to consumption and invest in a risky asset. Additionally, the labor income of the individual is stochastic until retirement, and post-retirement utility increases due to the availability of more leisure time.

As highlighted by Chen et al. [4], “the optimal retirement problem of a sophisticated individual with an age-dependent force of mortality has not been solved analytically.” Consequently, the primary objectives of our paper are twofold: first, to determine optimal consumption and portfolio-choice strategies considering realistic longevity risks stemming from an age-dependent force of mortality, and second, to fully characterize the optimal retirement time. By addressing these objectives, our study intends to fill the existing gap in the literature.

Notably, our findings, which are collected in Section 5 and Section 6, are somewhat intuitively convincing. For example, we show that it is optimal to retire when the agent’s wealth first reaches an endogenously determined boundary surface, which depends on the agent’s age, labor income and mortality. Intuitively, if the agent is sufficiently rich (her wealth exceeds the corresponding boundary), then retirement should be performed immediately; otherwise, it is optimal to wait for an increase of the wealth. Moreover, we show that both consumption and portfolio choices jump at the endogenous retirement time, which is consistent with Dybvig and Liu [14] and Chen et al. [5]. We also provide some interesting economic implications of the optimal retirement boundary through a numerical study in Section 6.

1.1. Overview of the mathematical analysis. From a mathematical point of view, we model the previous problem as a *random time horizon, three-dimensional stochastic control problem with discretionary stopping*. The three coordinates of the state process are the wealth process W , the force of mortality process M , and the stochastic labor income Y . The dominant feature of the wealth process W is that it is not the same before and after the retirement time τ . The agent’s aim is to choose consumption rate c , portfolio π , and the retirement time τ in order to maximize the total expected utility of consumption c , up to the random death time η .

In the literature, the derivation of the Hamilton-Jacobi-Bellman (HJB) equation is a frequently employed approach when analyzing stochastic optimization problem with Markovian

processes. In our specific case, the corresponding HJB equation (cf. (A.32)) turns out to be an involved combination of an HJB-type equation (reflecting the investment and consumption optimization) with a variational inequality (reflecting the retirement optimization). However, studying the properties of the value function and the optimal strategies, particularly the optimal stopping boundary, proves to be challenging when directly studying the equation. This difficulty arises due to the presence of a three-dimensional state process, rendering the classical “guess and verify” method ineffective.

In order to tame the intricate mathematical structure of our problem, where the consumption and portfolio choices nontrivially interact with the retirement decision, we combine a duality and a free-boundary approach, and proceed in our analysis as it follows.

Step 1. First, we conduct successive transformations (see Section 3) that connect the original stochastic control-stopping problem (with value function V) with its dual problem (with value function J) by martingale and duality methods (similar to Karatzas and Wang [29] or Yang and Koo [38]). The dual problem (with value function J) is a *finite time-horizon, three-dimensional optimal stopping problem with interconnected dynamics*, which is difficult to tackle directly. Due to the intrinsic homogeneous structure of the considered power utility function, we reduce the dimensionality by applying a measure change method, and reach a reduced-version dual stopping problem (with value function \tilde{J}).

Step 2. We then study the reduced-version dual problem, which is a *finite time-horizon, two-dimensional optimal stopping problem with interconnected dynamics*. The newly introduced state variable X (depending on the dual process Z and the labour income process Y) evolves as a geometric Brownian motion, where the drift depends on the mortality force M . The coupling between the two components of the state process makes the optimal stopping problem quite intricate.

It is also worth pointing out that the force of mortality process M does not possess any diffusive term, which leads to a novel analysis of the regularity of \hat{J} (given by the difference of \tilde{J} and the smooth payoff of immediate stopping; see (4.12)). As a matter of fact, the process (X, M) is a degenerate diffusion process (in the sense that the differential operator of (X, M) is a degenerate parabolic operator) so that the study of the regularity of \hat{J} in the interior of its continuation region cannot hinge on classical results for parabolic PDEs (see, e.g., Friedman [18]).

Additional technical difficulties arise when trying to infer properties of the optimal stopping boundary b . In fact, since the regularity of \hat{J} in the interior of its continuation region \mathcal{W} cannot be established a priori via classical PDE results, we were unable to establish continuity for the

mapping $(t, m) \mapsto b(t, m)$, even if monotonicity properties of the latter could be easily proved. It is indeed well known in optimal stopping and free-boundary theory that interior regularity of \widehat{J} in \mathcal{W} together with the monotonicity of b are the key ingredients for a rigorous study of the continuity of the boundary (for a deeper discussion please refer to De Angelis [10]).

We overcome those major technical hurdles by proving that the optimal boundary is in fact a locally Lipschitz-continuous function of time t and force of mortality m , without employing neither monotonicity of the boundary nor classical results on interior regularity for parabolic PDEs. In order to achieve this goal, we rely only upon probabilistic methods borrowed from De Angelis and Stabile [12], which are then specifically adjusted to tackle our problem.

As a matter of fact, we first prove that \widehat{J} is locally Lipschitz continuous and obtain probabilistic representations of its weak-derivatives (cf. De Angelis and Stabile [12]). Then, through a suitable application of the method developed in De Angelis and Stabile [13], by means of a version of the implicit function theorem for Lipschitz mappings (cf. Papi [31]), we can show that the free boundary surface $(t, m) \mapsto b(t, m)$ is locally-Lipschitz continuous. This enables us to prove that the optimal stopping time $(t, x, m) \mapsto \tau^*(t, x, m)$ is continuous, which in turn gives that \widehat{J} is a continuously differentiable functions of its three variables. Being that the process X is the only diffusive one, the C^1 -property of \widehat{J} implies that \widehat{J}_{xx} admits a continuous extension to the closure of the continuation region. Notice that it is in fact this regularity that could had not been derived from standard results on PDEs nor from Peskir [34], and it is in fact this regularity that allows (via an application of a weak version of Dynkin's formula) to derive an integral equation which is uniquely solved by the free boundary.

Step 3. After proving the strict convexity of J , we can come back to the original coordinates' system and via the duality relations we obtain the optimal consumption and portfolio policies, as well as the optimal retirement time, in terms of the optimal stopping boundary and value function (cf. Section 5).

Overall, we believe that the contributions of this paper are the following. As we discussed before, even though the literature on optimal retirement time problems is extensive from different perspectives, the introduction of the age-dependent force of mortality constitutes a novelty. From a mathematical point of view, we provide a rigorous theoretical analysis of the optimal retirement time in terms of the optimal boundary \widehat{b} . To the best of our knowledge, ours is the first work providing a complete analytical characterization of the value function and of the optimal strategies in the optimal retirement time problem with age-dependent force of mortality. Furthermore, we believe that the dual optimal stopping problem with the degenerate parabolic operator, studied as a device to characterize the optimal solution of the optimal retirement time

problem, is of interest of its own. By performing a thorough analysis on the regularity of \widehat{J} (a transformed version of the dual value function) and of the free boundary, we are able to provide a complete characterization of the optimal retirement time strategy through a nonlinear integral equation. The analysis in this paper is completed by solving numerically the integral equation and studying its sensitivity to variations in the model's parameters, thus extending the results of the related optimal retirement time model.

1.2. Organization of the paper. The rest of the paper is organized as follows. In Section 2, we introduce the optimal retirement time decision model with an age-dependent force of mortality. We transform the original stochastic control-stopping problem into a pure stopping problem in Section 3, while in Section 4 we study the optimal stopping problem. In Section 5, we provide the optimal retirement time boundary, optimal consumption plan and optimal portfolio in primal variables, and in Section 6 we present a numerical study and provide some economic implications. Appendix A collects the proofs of some results of Sections 4 and 5, whereas Appendix B includes some auxiliary results needed in the paper. In Appendix C, we give the details of the numerical method used in Section 6.

2. SETTING AND PROBLEM FORMULATION

2.1. The age-dependent mortality rate. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, endowed with a filtration $\mathbb{F} := \{\mathcal{F}_s, s \geq t\}$ satisfying the usual conditions, where $t \geq 0$ is a fixed initial time. The remaining lifetime of an agent η is an exogenous non-negative continuous random variable and is independent of \mathcal{F}_∞ . Moreover, η has the cumulative distribution function F , density function f , so that, for any $s \geq t$,

$$\mathbb{P}(\eta \geq s) = 1 - F(s) = \int_s^\infty f(u)du$$

is the probability that the agent is living at least another s years. The force of mortality (also called the hazard function) represents the instantaneous death rate for the agent surviving to time s , and it is defined by

$$M_s := \lim_{\Delta s \rightarrow 0} \frac{\mathbb{P}(s \leq \eta < s + \Delta s | \eta \geq s)}{\Delta s}.$$

Then the force of mortality M is expressed by

$$M_s = \frac{f(s)}{1 - F(s)},$$

which means that $F(s) = 1 - \exp\{-\int_t^s M_u du\}$, for $s \geq t$.

Consider an agent whose force of mortality rate evolves according to the standard Gompertz model (see, e.g., Gompertz [19]). The process $M := \{M_s, s \geq t\}$ thus follows the dynamics

$$(2.1) \quad dM_s = aM_s ds, \quad M_t = m > 0,$$

with $a > 0$. Notice that the usual form for the Gompertz model is $M_s = a_0^{-1}e^{(x_0+s-m_0)/a_0}$, hence we are using $a = \frac{1}{a_0}$ and $m = a_0^{-1}e^{(x_0-m_0)/a_0}$. Here, x_0 denotes the agent's age at initial time t , m_0 is called the modal value, and a_0 is the dispersion coefficient for the Gompertz model. This model is simple, and takes advantage of a long experience of calibration to real populations.

2.2. The financial market and labour income. We assume that the agent invests in a financial market with two assets. One of them is a risk-free bond, whose price $S^0 := \{S_s^0, s \geq t\}$ evolves as

$$dS_s^0 = rS_s^0 ds, \quad S_t^0 = s^0 > 0,$$

where $r > 0$ is a constant risk-free rate. The second one is a stock, whose price is denoted by $S := \{S_s, s \geq t\}$ and it satisfies the stochastic differential equation

$$dS_s = \mu S_s dt + \sigma S_s dB_s, \quad S_t = s > 0,$$

where $\mu \in \mathbb{R}$ and $\sigma > 0$ are given constants. Here, $B := \{B_s, s \geq t\}$ is an \mathbb{F} -adapted standard Brownian motion under \mathbb{P} .

For the agent, there is a mandatory retirement time $T \in [t, \infty)$. Let $\tau \in [t, T]$ be an \mathbb{F} -stopping time representing the time at which the agent chooses to retire. The agent receives a stochastic labor income $Y := \{Y_s, t \leq s \leq \tau\}$ as long as she is not retired. We assume the labor income to be spanned by the market and that we can express it as a geometric Brownian motion, similar to e.g. Bodie et al. [3] and Dybvig and Liu [14],

$$(2.2) \quad Y_s = ye^{(\mu_y - \frac{1}{2}\sigma_y^2)(s-t) + \sigma_y(B_s - B_t)}, \quad \text{for } t < s \leq \tau, \quad Y_t = y > 0,$$

with $Y_s = 0$ for $s > \tau$. Here $\mu_y \in \mathbb{R}$ and $\sigma_y > 0$ are constants, representing the instantaneous growth rate and the volatility of the labor income, respectively.

We define the market price of risk $\theta := \frac{\mu - r}{\sigma}$ and the state-price-density process $\xi_{s,t} := e^{-r(s-t) - \theta(B_s - B_t) - \frac{1}{2}\theta^2(s-t)}$. Since the market is complete and the labor income is perfectly correlated with the market, the present value of the future labour income $g(s)$ at time $s < T$ (also called ‘‘human capital’’ as in Dybvig and Liu [14]), under the assumption that the agent

does not choose early retirement and the agent is always alive, is given by

$$\begin{aligned}
g(s) &= \mathbb{E} \left[\int_s^T \xi_{u,s} Y_u du \middle| \mathcal{F}_s \right] = Y_s \mathbb{E} \left[\int_s^T \xi_{u,s} \frac{Y_u}{Y_s} du \middle| \mathcal{F}_s \right] = Y_s \left[\int_s^T e^{(-r+\mu_y-\sigma_y\theta)(u-s)} du \right] \\
(2.3) \quad &= \begin{cases} Y_s \frac{1 - e^{-\kappa(T-s)}}{\kappa}, & \text{if } \kappa \neq 0, \\ Y_s(T-s), & \text{if } \kappa = 0, \end{cases}
\end{aligned}$$

where $\kappa := r - \mu_y + \sigma_y\theta$ is the effective discount rate for labor income, which is assumed to be positive (see also Dybvig and Liu [15] and Guan et al. [20] for a similar requirement). Then, for $s \leq T$, we define

$$q(s) := \begin{cases} \frac{1 - e^{-\kappa(T-s)}}{\kappa}, & \text{if } \kappa \neq 0, \\ T - s, & \text{if } \kappa = 0, \end{cases}$$

so that $g(s) = q(s)Y_s$.

The agent also consumes from her wealth, while investing in the financial market. Denoting by π_s the amount of wealth invested in the stock at time s , the agent then also chooses the rate of spending in consumption c_s at time s . Therefore, the agent's wealth $W := \{W_s^{c,\pi,\tau}, s \geq t\}$ evolves as

$$(2.4) \quad dW_s^{c,\pi,\tau} = [\pi_s(\mu - r) + rW_s^{c,\pi,\tau} - c_s + Y_s \mathbf{1}_{\{s \leq \tau\}}] ds + \pi_s \sigma dB_s, \quad W_t^{c,\pi,\tau} = w.$$

In the following, we shall simply write W to denote $W^{c,\pi,\tau}$, where needed.

2.3. The optimization problem. Here and in the sequel, we write $\mathcal{O} := [t, T] \times \mathbb{R}_+^3$, $\mathcal{O}' := [t, T] \times (-g(t), +\infty) \times \mathbb{R}_+^2$ and $\mathcal{U} := [0, T] \times \mathbb{R}_+^2$, with $\mathbb{R}_+ := (0, \infty)$. We denote by $\mathcal{S}_{t,s}$ the class of \mathbb{F} -stopping times $\tau : \Omega \rightarrow [t, s]$ for $t \leq s \leq T$, and let $\mathcal{S} := \mathcal{S}_{t,T}$. Then we introduce the set of admissible strategies as it follows.

Definition 2.1. Let $(t, w, m, y) \in \mathcal{O}'$ be given and fixed. The triplet of choices (c, π, τ) is called an **admissible strategy** for (t, w, m, y) , and we write $(c, \pi, \tau) \in \mathcal{A}(t, w, m, y)$, if it satisfies the following conditions:

- (i) c and π are progressively measurable with respect to \mathbb{F} , $\tau \in \mathcal{S}$;
- (ii) $c_s \geq 0$ for all $s \geq t$ and $\int_0^\Theta (c_s + |\pi_s|^2) ds < \infty$ \mathbb{P} -a.s., for any $\Theta > 0$;
- (iii) $W_s^{c,\pi,\tau} > -g(s) \mathbf{1}_{\{s < \tau\}}$ for all $s \geq t$, where $g(s)$ is defined in (2.3) if $s < T$ and $g(s) = 0$ if $s \geq T$.

Condition (iii) means that the agent is allowed to borrow money and hold a negative wealth level, while the amount of money she borrows cannot exceed the total value of future labour income. Once the agent retires, she is prohibited from any borrowing.

The preferences of the agent are described through a power utility function. Due to a larger amount of free time when retired, the agent assigns a higher utility value to consumption during retirement, represented by a constant weighting $K > 1$. The utility of the agent for consumption c_s at time s is

$$U(c_s) = \frac{1}{1-\gamma} \left[(\mathbf{1}_{\{s < \tau\}} + \mathbf{1}_{\{\tau \leq s\}} K) \cdot c_s \right]^{1-\gamma}, \quad \gamma \neq 1, \quad \gamma > 0.$$

Specifically, we call U_1 the utility function before retirement, i.e., for $s < \tau$,

$$U_1(c_s) := \frac{1}{1-\gamma} c_s^{1-\gamma};$$

on the other hand, we call U_2 the utility function after retirement, i.e., for $s \geq \tau$,

$$U_2(c_s) := \frac{1}{1-\gamma} (K \cdot c_s)^{1-\gamma}.$$

From the perspective of time t , the agent's aim is then to maximize over all $(c, \pi, \tau) \in \mathcal{A}(t, w, m, y)$ the expected intertemporal utility functional

$$\mathbb{E} \left[\int_t^\eta e^{-\beta(s-t)} U(c_s) ds \middle| \mathcal{F}_t \right],$$

where $\beta > 0$ is a constant representing the discount rate. Thanks to Fubini's Theorem and independence between η and \mathcal{F}_∞ , we can disentangle the market risk and mortality risk and write

$$\mathbb{E} \left[\int_t^\eta e^{-\beta(s-t)} U(c_s) ds \middle| \mathcal{F}_t \right] = \mathbb{E} \left[\int_t^\infty e^{-\int_t^s (\beta + M_u) du} U(c_s) ds \middle| \mathcal{F}_t \right].$$

Hence, given the Markovian setting, the agent aims at determining

$$(2.5) \quad V(t, w, m, y) = \sup_{(c, \pi, \tau) \in \mathcal{A}(t, w, m, y)} \mathbb{E}_{t, w, m, y} \left[\int_t^\infty e^{-\int_t^s (\beta + M_u) du} U(c_s) ds \right],$$

where $\mathbb{E}_{t, w, m, y}$ denote the expectation under \mathbb{P} conditioned on $W_t = w, M_t = m$ and $Y_t = y$. In the rest of the paper, we shall focus on (2.5).

3. FROM CONTROL-STOPPING TO PURE STOPPING

3.1. The static budget constraint. An application of Itô's formula yields

$$\begin{aligned}
 & e^{-rs-\theta B_s-\frac{1}{2}\theta^2 s}(W_s + g(s)) + \int_t^s e^{-ru-\theta B_u-\frac{1}{2}\theta^2 u} c_u du \\
 (3.1) \quad & = e^{-rt-\theta B_t-\frac{1}{2}\theta^2 t}(w + g(t)) + \int_t^s e^{-ru-\theta B_u-\frac{1}{2}\theta^2 u} (\pi_u \sigma - (W_u + g(u))\theta + g(u)\sigma_y) dB_u, \quad s \leq \tau,
 \end{aligned}$$

and

$$\begin{aligned}
 & e^{-rs-\theta B_s-\frac{1}{2}\theta^2 s} W_s + \int_t^s e^{-ru-\theta B_u-\frac{1}{2}\theta^2 u} c_u du \\
 (3.2) \quad & = e^{-rt-\theta B_t-\frac{1}{2}\theta^2 t} w + \int_t^s e^{-ru-\theta B_u-\frac{1}{2}\theta^2 u} (\pi_u \sigma - W_u \theta) dB_u, \quad s > \tau.
 \end{aligned}$$

Since $W_s + g(s)\mathbf{1}_{\{s < \tau\}} > 0$ for any $s \geq t$, we can deduce that $W_s + g(s)\mathbf{1}_{\{s \leq \tau\}} > 0$ for any $s \geq t$ from the continuity of $W + g(\cdot)$. For an admissible plan $(c, \pi, \tau) \in \mathcal{A}(t, w, m, y)$, the left-hand side of (3.1) is nonnegative for $s \leq \tau$, so that the Itô's integral on the right-side is not only a continuous \mathbb{P} -local martingale, but it is also a supermartingale by Fatou's Lemma. Thus, recalling that $\xi_{s,t} = e^{-r(s-t)-\theta(B_s-B_t)-\frac{1}{2}\theta^2(s-t)}$, the optional sampling theorem implies the so-called budget constraint:

$$(3.3) \quad \mathbb{E}_{t,w,m,y}[\xi_{s,t}(W_s + g(s))] + \mathbb{E}_{t,w,m,y}\left[\int_t^s \xi_{u,t} c_u du\right] \leq w + g(t), \quad \text{if } t \leq s \leq \tau.$$

By similar arguments on (3.2) we also have

$$(3.4) \quad \mathbb{E}_{t,w,m}\left[\int_t^\infty \xi_{u,t} c_u du\right] \leq w, \quad \text{if } t = \tau \leq s.$$

3.2. The agent's optimization problem after retirement. In this subsection we will consider the agent's optimization problem after retirement, and over this time period only consumption and portfolio choice have to be determined. Formally, the model in the previous section accommodates this case if we let $\tau = t$, where t is the fixed starting time. That is, the agent chooses immediate retirement. Then, letting $\mathcal{A}_t(t, w, m) := \{(c, \pi) : (c, \pi, t) \in \mathcal{A}(t, w, m, 0)\}$, where the subscript t indicates that the retirement time is equal to t , the agent's value function after retirement is

$$(3.5) \quad \widehat{V}(t, w, m) := \sup_{(c,\pi) \in \mathcal{A}_t(t,w,m)} \mathbb{E}_{t,w,m}\left[\int_t^\infty e^{-\int_t^s (\beta + M_u) du} U_2(c_s) ds\right],$$

where M as defined in (2.1) and $\mathbb{E}_{t,w,m}$ denotes the expectation conditioned on $W_t = w$ and $M_t = m$.

From the budget constraint (3.4), recalling that $\xi_{s,t} = e^{-r(s-t) - \theta(B_s - B_t) - \frac{1}{2}\theta^2(s-t)}$ and for a Lagrange multiplier $z > 0$, we have for any $(c, \pi) \in \mathcal{A}_t(t, w, m)$

$$\begin{aligned}
& \mathbb{E}_{t,w,m} \left[\int_t^\infty e^{-\int_t^s (\beta + M_u) du} U_2(c_s) ds \right] \\
& \leq \mathbb{E}_{t,w,m} \left[\int_t^\infty e^{-\int_t^s (\beta + M_u) du} U_2(c_s) ds \right] - z \mathbb{E}_{t,w,m} \left[\int_t^\infty \xi_{s,t} c_s ds \right] + zw \\
& = \mathbb{E}_{t,w,m} \left[\int_t^\infty e^{-\int_t^s (\beta + M_u) du} U_2(c_s) ds \right] - \mathbb{E}_{t,w,m} \left[\int_t^\infty e^{-\int_t^s (\beta + M_u) du} z P_s c_s ds \right] + zw \\
& = \mathbb{E}_{t,w,m} \left[\int_t^\infty e^{-\int_t^s (\beta + M_u) du} \left(U_2(c_s) - z P_s c_s \right) ds \right] + zw \\
(3.6) \quad & \leq \mathbb{E}_{t,w,m} \left[\int_t^\infty e^{-\int_t^s (\beta + M_u) du} U_2^*(z P_s) ds \right] + zw,
\end{aligned}$$

where

$$(3.7) \quad P_s := \xi_{s,t} e^{\int_t^s (\beta + M_u) du} \quad \text{and} \quad U_2^*(z) := \sup_{c \geq 0} [U_2(c) - cz].$$

Let then $Z_s := z P_s$. By Itô's formula, we obtain that the dual variable Z satisfies

$$(3.8) \quad dZ_s = (\beta - r + M_s) Z_s ds - \theta Z_s dB_s, \quad Z_t = z,$$

and we set

$$(3.9) \quad Q(t, z, m) := \mathbb{E}_{t,z,m} \left[\int_t^\infty e^{-\int_t^s (\beta + M_u) du} U_2^*(Z_s) ds \right].$$

Assumption 3.1. We assume $\beta \geq (1 - \gamma)(r + \frac{1}{2}\theta^2) + \frac{(\gamma-1)^2\theta^2}{2\gamma}$ throughout the paper.

Assumption 3.1 gives a sufficient condition to ensure the finiteness of Q defined in (3.9).

Proposition 3.1. Q is independent of time t , it is finite, and one has $Q \in C^{2,1}(\mathbb{R}_+^2)$. Moreover, Q satisfies

$$(3.10) \quad -\widehat{\mathcal{L}}Q = U_2^*,$$

where

$$\widehat{\mathcal{L}}Q := \frac{1}{2}\theta^2 z^2 Q_{zz} + (\beta - r + m)z Q_z + am Q_m - (\beta + m)Q.$$

Proof. First, we compute the convex dual of U_2 in (3.7); that is,

$$U_2^*(z) = K \frac{1-\gamma}{\gamma} \frac{\gamma}{1-\gamma} z^{\frac{\gamma-1}{\gamma}}, \quad z \geq 0.$$

From (2.1), we have

$$(3.11) \quad M_s = me^{a(s-t)}, \quad \forall s \geq t.$$

Therefore, by (3.9) and (3.11) we rewrite $Q(t, z, m)$ as follows

$$(3.12) \quad \begin{aligned} Q(t, z, m) &= \mathbb{E}_{t,z,m} \left[\int_t^\infty e^{-\int_t^s (\beta + M_u) du} K^{\frac{1-\gamma}{\gamma}} \frac{\gamma}{1-\gamma} Z_s^{\frac{\gamma-1}{\gamma}} ds \right] \\ &= z^{\frac{\gamma-1}{\gamma}} \frac{\gamma}{1-\gamma} K^{\frac{1-\gamma}{\gamma}} \int_t^\infty e^{-\frac{1}{\gamma} \int_t^s (\beta + me^{a(u-t)}) du} e^{[-\frac{\gamma-1}{\gamma}(r+\frac{1}{2}\theta^2) + \frac{1}{2} \frac{(\gamma-1)^2 \theta^2}{\gamma^2}] (s-t)} ds, \\ &= z^{\frac{\gamma-1}{\gamma}} \frac{\gamma}{1-\gamma} K^{\frac{1-\gamma}{\gamma}} \int_0^\infty e^{-\frac{1}{\gamma} \beta s' - \frac{1}{\gamma} \int_0^{s'} me^{au'} du'} e^{[-\frac{\gamma-1}{\gamma}(r+\frac{1}{2}\theta^2) + \frac{1}{2} \frac{(\gamma-1)^2 \theta^2}{\gamma^2}] s'} ds', \end{aligned}$$

where we have used the definition of P_s as in (3.7) and the fact that

$$\begin{aligned} \mathbb{E}[P_s^{\frac{\gamma-1}{\gamma}}] &= \mathbb{E}[(\xi_{s,t} e^{\int_t^s (\beta + M_u) du})^{\frac{\gamma-1}{\gamma}}] = e^{\frac{\gamma-1}{\gamma} \int_t^s (\beta + M_u) du} \mathbb{E}[e^{-r(s-t) - \theta(B_s - B_t) - \frac{1}{2}\theta^2(s-t)}]^{\frac{\gamma-1}{\gamma}} \\ &= e^{\int_t^s (\frac{\gamma-1}{\gamma}(\beta + M_u - r - \frac{1}{2}\theta^2) + \frac{1}{2} \frac{(\gamma-1)^2 \theta^2}{\gamma^2}) du}. \end{aligned}$$

This shows that Q is indeed time-independent.

Moreover, due to Assumption 3.1, we can verify that $Q(z, m) < \infty$ by (3.12), and that $Q \in C^{2,1}(\mathbb{R}_+^2)$. Finally it satisfies (3.10) by the well-known Feynman-Kac formula (see, e.g., Chapter 4 in Karatzas and Shreve [27]). \square

Given that Q is time-independent, with a slight abuse of notation in the sequel we then simply write $Q(z, m)$. It is possible to deduce properties of \widehat{V} through those of Q by the following duality relation.

Theorem 3.1. *The following dual relations holds:*

$$\widehat{V}(t, w, m) = \inf_{z>0} [Q(z, m) + zw], \quad Q(z, m) = \sup_{w>0} [\widehat{V}(t, w, m) - zw].$$

Proof. Since $(c, \pi) \in \mathcal{A}_t(t, w, m)$ is arbitrary, taking the supremum over $(c, \pi) \in \mathcal{A}_t(t, w, m)$ on the left-hand side in (3.6) and recalling (3.5), we get, for any $z > 0$,

$$\widehat{V}(t, w, m) \leq Q(z, m) + zw,$$

and thus

$$\widehat{V}(t, w, m) \leq \inf_{z>0} [Q(z, m) + zw] \quad \text{and} \quad Q(z, m) \geq \sup_{w>0} [\widehat{V}(t, w, m) - zw].$$

For the reverse inequalities, observe that the equality in (3.6) holds if and only if

$$(3.13) \quad c_s = \mathcal{I}_2^u(Z_s),$$

and

$$(3.14) \quad \mathbb{E}_{t,z,m} \left[\int_t^\infty \xi_{s,t} c_s ds \right] = w,$$

where we denote by \mathcal{I}_2^u the inverse of the marginal utility function $U_2'(\cdot)$.

Then, assuming (3.14) (we will prove its validity later), we define

$$\mathcal{X}(t, z, m) := \mathbb{E}_{t,z,m} \left[\int_t^\infty \xi_{s,t} \mathcal{I}_2^u(Z_s) ds \right], \quad \mathcal{Y}(t, w, m) := \mathbb{E}_{t,w,m} \left[\int_t^\infty e^{-\int_t^s (\beta + M_u) du} U_2(c_s) ds \right],$$

and notice that (3.6), (3.13) and (3.14) yield

$$\mathcal{Y}(t, \mathcal{X}(t, z, m), m) = Q(z, m) + z\mathcal{X}(t, z, m) \leq \widehat{V}(t, w, m),$$

where the last inequality is due to $\mathcal{Y}(t, \mathcal{X}(t, z, m), m) \leq \widehat{V}(t, w, m)$. The last display inequality thus provides

$$Q(z, m) \leq \sup_{w>0} [\widehat{V}(t, w, m) - zw] \quad \text{and} \quad \widehat{V}(t, w, m) \geq \inf_{z>0} [Q(z, m) + zw].$$

It thus remains only to show that equality (3.14) indeed holds. As a matter of fact, Lemma B.1 guarantees the existence of a candidate optimal portfolio process π^* such that $(c^*, \pi^*) \in \mathcal{A}_t(t, w, m)$ and (3.14) holds, where $c_s^* = \mathcal{I}_2^u(Z_s)$ is a candidate optimal consumption process. By Theorem 3.6.3 in Karatzas and Shreve [28] or Lemma 6.2 in Karatzas and Wang [29], one can then show that (c^*, π^*) is indeed optimal for the optimization problem \widehat{V} . \square

From Theorem 3.1 we see that \widehat{V} is time-independent, so that in the following, with a slight abuse of notation, we simply write $\widehat{V}(w, m)$.

3.3. The pure optimal stopping problem. From the agent's problem in (2.5), by the dynamic programming principle we can deduce that for any $(t, w, m, y) \in \mathcal{O}'$,

$$(3.15) \quad V(t, w, m, y) = \sup_{(c, \pi, \tau) \in \mathcal{A}(t, w, m, y)} \mathbb{E}_{t,w,m,y} \left[\int_t^\tau e^{-\int_t^s (\beta + M_u) du} U_1(c_s) ds + e^{-\int_t^\tau (\beta + M_u) du} \widehat{V}(W_\tau, M_\tau) \right].$$

Now, for any $(t, w, m, y) \in \mathcal{O}'$ and Lagrange multiplier $z > 0$, from the budget constraint (3.3) and (3.15), recalling P_s in (3.7), we have

$$\mathbb{E}_{t,w,m,y} \left[\int_t^\infty e^{-\int_t^s (\beta + M_u) du} U(c_s) ds \right]$$

$$\begin{aligned}
&\leq \sup_{(c,\pi,\tau)\in\mathcal{A}(t,w,m,y)} \mathbb{E}_{t,w,m,y} \left[\int_t^\tau e^{-\int_t^s(\beta+M_u)du} U_1(c_s) ds + e^{-\int_t^\tau(\beta+M_u)du} \widehat{V}(W_\tau, M_\tau) \right] \\
&- z \mathbb{E}_{t,w,m,y} \left[\xi_{\tau,t}(W_\tau + g(\tau)) + \int_t^\tau \xi_{s,t} c_s ds \right] + z(w + g(t)) \\
&= \sup_{(c,\pi,\tau)\in\mathcal{A}(t,w,m,y)} \mathbb{E}_{t,w,m,y} \left[\int_t^\tau e^{-\int_t^s(\beta+M_u)du} (U_1(c_s) - zP_s c_s) ds \right. \\
&+ \left. e^{-\int_t^\tau(\beta+M_u)du} \widehat{V}(W_\tau, M_\tau) - e^{-\int_t^\tau(\beta+M_u)du} zP_\tau(W_\tau + g(\tau)) \right] + z(w + g(t)) \\
&= \sup_{(c,\pi,\tau)\in\mathcal{A}(t,w,m,y)} \mathbb{E}_{t,w,m,y} \left[\int_t^\tau e^{-\int_t^s(\beta+M_u)du} (U_1(c_s) - zP_s c_s) ds \right. \\
&+ \left. e^{-\int_t^\tau(\beta+M_u)du} (\widehat{V}(W_\tau, M_\tau) - zP_\tau W_\tau - zP_\tau g(\tau)) \right] + z(w + g(t))
\end{aligned}
\tag{3.16}$$

$$\leq \sup_{\tau \in \mathcal{S}} \mathbb{E}_{t,z,m,y} \left[\int_t^\tau e^{-\int_t^s(\beta+M_u)du} U_1^*(Z_s) ds + e^{-\int_t^\tau(\beta+M_u)du} (Q(Z_\tau, M_\tau) - Z_\tau g(\tau)) \right] + z(w + g(t)),$$

where Z_s is defined in (3.8) and

$$U_1^*(z) := \sup_{c \geq 0} [U_1(c) - cz].
\tag{3.17}$$

Hence, defining

$$J(t, z, m, y) := \sup_{\tau \in \mathcal{S}} \mathbb{E}_{t,z,m,y} \left[\int_t^\tau e^{-\int_t^s(\beta+M_u)du} U_1^*(Z_s) ds + e^{-\int_t^\tau(\beta+M_u)du} (Q(Z_\tau, M_\tau) - Z_\tau g(\tau)) \right],
\tag{3.18}$$

we have

$$\mathbb{E}_{t,w,m,y} \left[\int_t^\infty e^{-\int_t^s(\beta+M_u)du} U(c_s) ds \right] \leq J(t, z, m, y) + z(w + g(t)).
\tag{3.19}$$

In the following sections, we perform a detailed probabilistic study of (3.18). Before doing that, we have the following theorem that establishes a dual relation between the original problem (2.5) and the optimal stopping problem (3.18).

Theorem 3.2. *The following duality relations holds:*

$$V(t, w, m, y) = \inf_{z > 0} [J(t, z, m, y) + z(w + g(t))], \quad J(t, z, m, y) = \sup_{w > -g(t)} [V(t, w, m, y) - z(w + g(t))].$$

Proof. Since $(c, \pi, \tau) \in \mathcal{A}(t, w, m, y)$ is arbitrary, taking the supremum over $(c, \pi, \tau) \in \mathcal{A}(t, w, m, y)$ on the left-hand side of (3.19), we get, for any $z > 0, w > -g(t)$,

$$V(t, w, m, y) \leq J(t, z, m, y) + z(w + g(t)),$$

so that $V(t, w, m, y) \leq \inf_{z>0}[J(t, z, m, y) + z(w + g(t))]$ and $J(t, z, m, y) \geq \sup_{w>-g(t)}[V(t, w, m, y) - z(w + g(t))]$.

For the reverse inequality, observe that equality holds in (3.19) if and only if (see also (3.16))

$$c_s = \mathcal{I}_1^u(Z_s), \quad Q(z, m) = \sup_{w>0}[\widehat{V}(w, m) - zw],$$

and

$$(3.20) \quad \mathbb{E}_{t,w,m,y} \left[\xi_{\tau,t} (W_\tau + g(\tau)) + \int_t^\tau \xi_{s,t} c_s ds \right] = w + g(t),$$

where \mathcal{I}_1^u denotes the inverse of the marginal utility function $U_1'(\cdot)$. From Lemma B.2, we know that there exists a portfolio process π^* such that (3.20) holds. From Theorem 3.1, we also know that $Q(z, m) = \sup_{w>0}[\widehat{V}(w, m) - zw]$. Next we define

$$\bar{\mathcal{X}}(t, z, m, y) := \mathbb{E}_{t,z,m,y} \left[\int_t^\tau \xi_{s,t} \mathcal{I}_1^u(Z_s) ds \right], \quad \bar{\mathcal{Z}}(t, w, m, y) := \mathbb{E}_{t,w,m,y}[\xi_{\tau,t}(W_\tau + g(\tau))],$$

and

$$\bar{\mathcal{Y}}(t, w, m, y) := \mathbb{E}_{t,w,m,y} \left[\int_t^\tau e^{-\int_t^s (\beta + M_u) du} U_1(c_s) ds + e^{-\int_t^\tau (\beta + M_u) du} \widehat{V}(W_\tau, M_\tau) \right].$$

Then by (3.19) and (3.20) we have

$$\begin{aligned} \bar{\mathcal{Y}}(t, \bar{\mathcal{X}}(t, z, m, y) + \bar{\mathcal{Z}}(t, w, m, y) - g(t), m, y) &= J(t, z, m, y) + z(\bar{\mathcal{X}}(t, z, m, y) + \bar{\mathcal{Z}}(t, w, m, y)) \\ &\leq V(t, w, m, y), \end{aligned}$$

where the last inequality is due to $\bar{\mathcal{Y}}(t, \bar{\mathcal{X}}(t, z, m, y) + \bar{\mathcal{Z}}(t, w, m, y) - g(t), m, y) \leq V(t, w, m, y)$.

This in turn gives

$$V(t, w, m, y) \geq \inf_{z>0}[J(t, z, m, y) + z(w + g(t))],$$

which completes the proof. □

4. STUDY OF THE DUAL OPTIMAL STOPPING PROBLEM

4.1. Dimensionality reduction. Notice that the dual optimal stopping problem in (3.18) has three state variables and a finite time-horizon. However, due to the homogeneous structure of the utility function U , we can apply a measure change method to reduce the dimensionality of the problem from three state variables (z, m, y) to two state variables (x, m) .

Lemma 4.1. *Consider the exponential martingale*

$$\zeta := \left\{ \zeta_s = \exp \left(\int_t^s -\frac{1}{2}(\sigma_y - \theta)^2 du + \int_t^s (\sigma_y - \theta) dB_u \right) : t \leq s \leq T \right\}.$$

Let $\widehat{\mathbb{P}}$ be the probability measure on (Ω, \mathcal{F}_T) such that $\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}} = \zeta_T$, and denote by $\widehat{\mathbb{E}}$ the expectation under $\widehat{\mathbb{P}}$. Then we have

$$(4.1) \quad J(t, z, m, y) = zy \widetilde{J}(t, z^{\frac{1}{1-\gamma}} y^{\frac{\gamma}{1-\gamma}}, m)$$

with

$$(4.2) \quad \widetilde{J}(t, x, m) := \sup_{t \leq \tau \leq T} \widehat{\mathbb{E}}_{t,x,m} \left[\int_t^\tau e^{-\kappa(s-t)} U_1^*(X_s) ds + e^{-\kappa(\tau-t)} \left(Q(X_\tau, M_\tau) - q(\tau) \right) \right],$$

and the underlying Markov process X being defined as $X := Z^{\frac{1}{1-\gamma}} Y^{\frac{\gamma}{1-\gamma}}$ (with Z and Y as in (3.8) and (2.2), respectively).

Proof. First, from (3.17) and (3.7) we compute the convex dual functions $U_1^*(z)$ and $U_2^*(z)$:

$$(4.3) \quad U_1^*(z) = \frac{\gamma}{1-\gamma} z^{\frac{\gamma-1}{\gamma}}, \quad U_2^*(z) = \frac{\gamma}{1-\gamma} K^{\frac{1-\gamma}{\gamma}} z^{\frac{\gamma-1}{\gamma}} = K^{\frac{1-\gamma}{\gamma}} U_1^*(z), \quad z \geq 0.$$

Plugging U_2^* into (3.9) we have

$$\begin{aligned} Q(z, m) &= \mathbb{E}_{z,m} \left[\int_t^\infty e^{-\int_t^s (\beta + M_u) du} \frac{\gamma}{1-\gamma} K^{\frac{1-\gamma}{\gamma}} Z_s^{\frac{\gamma-1}{\gamma}} ds \right] \\ &= z^{\frac{\gamma-1}{\gamma}} \frac{\gamma}{1-\gamma} K^{\frac{1-\gamma}{\gamma}} \mathbb{E}_{z,m} \left[\int_t^\infty e^{-\int_t^s (\beta + M_u) du} P_s^{\frac{\gamma-1}{\gamma}} ds \right] \end{aligned}$$

and

$$\begin{aligned} Q(z^{\frac{1}{1-\gamma}} y^{\frac{\gamma}{1-\gamma}}, m) &= (z(zy)^{\frac{\gamma}{1-\gamma}})^{\frac{\gamma-1}{\gamma}} \frac{\gamma}{1-\gamma} K^{\frac{1-\gamma}{\gamma}} \mathbb{E}_{z,m} \left[\int_t^\infty e^{-\int_t^s (\beta + M_u) du} P_s^{\frac{\gamma-1}{\gamma}} ds \right] \\ &= z^{\frac{\gamma-1}{\gamma}} (zy)^{-1} \frac{\gamma}{1-\gamma} K^{\frac{1-\gamma}{\gamma}} \mathbb{E}_{z,m} \left[\int_t^\infty e^{-\int_t^s (\beta + M_u) du} P_s^{\frac{\gamma-1}{\gamma}} ds \right] \\ (4.4) \quad &= \frac{Q(z, m)}{zy}. \end{aligned}$$

Direct calculations (cf. (3.8) and (2.2)) show

$$\begin{aligned} Z_s Y_s &= z P_s Y_s = z e^{\int_t^s (-r - \frac{1}{2}\theta^2 + \beta + M_u) du - \int_t^s \theta dB_u} y e^{(\mu_y - \frac{1}{2}\sigma_y^2)(s-t) + \sigma_y(B_s - B_t)} \\ &= zy e^{\int_t^s (-r - \frac{1}{2}\theta^2 + \beta + M_u + \mu_y - \frac{1}{2}\sigma_y^2) du - \int_t^s (\theta - \sigma_y) dB_u} \\ (4.5) \quad &= zy e^{\int_t^s (\beta + M_u - \kappa) du} \zeta_s. \end{aligned}$$

Then, by the continuous-time Bayes' rule (cf. Lemma 3.5.3 in Karatzas and Shreve [27]), we have for any fixed $\tau \in \mathcal{S}$, and for $x = z^{\frac{1}{1-\gamma}} y^{\frac{\gamma}{1-\gamma}}$,

$$\begin{aligned}
& \mathbb{E}_{t,z,m,y}[e^{-\int_t^\tau (\beta+M_u)du}(Q(Z_\tau, M_\tau) - Z_\tau g(\tau))] \\
&= \mathbb{E}_{t,z,m,y}[e^{-\int_t^\tau (\beta+M_u)du}(Q(Z_\tau, M_\tau) - Z_\tau Y_\tau q(\tau))] \\
&= \mathbb{E}_{t,z,m,y}[e^{-\int_t^\tau (\beta+M_u)du} Z_\tau Y_\tau (Q(Z_\tau, M_\tau)/(Z_\tau Y_\tau) - q(\tau))] \\
&= \mathbb{E}_{t,z,m,y}[e^{-\int_t^\tau (\beta+M_u)du} Z_\tau Y_\tau (Q(X_\tau, M_\tau) - q(\tau))] \\
&= zy \mathbb{E}_{t,x,m}[e^{-\int_t^\tau (\beta+M_u)du} e^{\int_t^\tau (\beta+M_u-\kappa)du} \zeta_\tau (Q(X_\tau, M_\tau) - q(\tau))] \\
(4.6) \quad &= zy \widehat{\mathbb{E}}_{t,x,m}[e^{-\kappa(\tau-t)}(Q(X_\tau, M_\tau) - q(\tau))],
\end{aligned}$$

where we have used (4.4), the definition of X and (4.5). Based on the same arguments, we also have for $x = z^{\frac{1}{1-\gamma}} y^{\frac{\gamma}{1-\gamma}}$

$$\begin{aligned}
\mathbb{E}_{t,z,m,y} \left[\int_t^\tau e^{-\int_t^s (\beta+M_u)du} U_1^*(Z_s) ds \right] &= \mathbb{E}_{t,z,m,y} \left[\int_t^\tau e^{-\int_t^s (\beta+M_u)du} U_1^*(X_s) Z_s Y_s ds \right] \\
&= zy \mathbb{E}_{t,z,m,y} \left[\int_t^\tau e^{-\int_t^s (\beta+M_u)du} U_1^*(X_s) e^{\int_t^s (\beta+M_u-\kappa)du} \zeta_s ds \right] \\
&= zy \zeta_t \widehat{\mathbb{E}}_{t,x,m} \left[\int_t^\tau e^{-\kappa(s-t)} U_1^*(X_s) ds \right] \\
(4.7) \quad &= zy \widehat{\mathbb{E}}_{t,x,m} \left[\int_t^\tau e^{-\kappa(s-t)} U_1^*(X_s) ds \right],
\end{aligned}$$

due to $\zeta_t = 1$. Here and in the sequel, $\widehat{\mathbb{E}}_{t,x,m}$ denotes the expectation under $\widehat{\mathbb{P}}$ conditioned on $X_t = x$ and $M_t = m$. Combining (4.6) and (4.7), together with (3.18), we have $J(t, z, m, y) = zy \widetilde{J}(t, z^{\frac{1}{1-\gamma}} y^{\frac{\gamma}{1-\gamma}}, m)$ with

$$\widetilde{J}(t, x, m) := \sup_{t \leq \tau \leq T} \widehat{\mathbb{E}}_{t,x,m} \left[\int_t^\tau e^{-\kappa(s-t)} U_1^*(X_s) ds + e^{-\kappa(\tau-t)} (Q(X_\tau, M_\tau) - q(\tau)) \right].$$

□

Before closing this subsection, we introduce the dynamic equation of X defined in Lemma 4.1. By Itô's formula and Girsanov's Theorem, $X := \{X_s, t \leq s \leq T\}$ under the new measure $\widehat{\mathbb{P}}$ evolves as

$$(4.8) \quad dX_s = X_s[(\rho+1)(\beta-r+M_s) + \mu_1] ds + X_s \sigma_1 d\widehat{B}_s, \quad X_t = x (= z^{\frac{1}{1-\gamma}} y^{\frac{\gamma}{1-\gamma}}),$$

where $\mu_1 := \frac{1}{2}(\rho+1)\rho\theta^2 + \rho\mu_y + \frac{1}{2}(\rho-1)\rho\sigma_y^2 - (\rho+1)\rho\theta\sigma_y + \sigma_1(\sigma_y - \theta)$, $\sigma_1 := \rho\sigma_y - (\rho+1)\theta$ with $\rho := \frac{\gamma}{1-\gamma}$, and \widehat{B} is a standard Brownian motion under the measure $\widehat{\mathbb{P}}$. Then, the solution

to (4.8) may be expressed as

$$X_s = x \exp \left(\int_t^s \left[(\rho + 1)(\beta - r + M_u) + \mu_1 - \frac{\sigma_1^2}{2} \right] du + \int_t^s \sigma_1 d\widehat{B}_u \right), \quad \text{for } s \geq t,$$

so that X depends on both initial values x and m .

For future frequent use, we also introduce here the new probability measure $\widetilde{\mathbb{P}}$ on (Ω, \mathcal{F}_T) such that

$$(4.9) \quad H_T := \frac{d\widetilde{\mathbb{P}}}{d\widehat{\mathbb{P}}}\Big|_{\mathcal{F}_T} = \exp \left\{ \int_t^T \frac{\gamma - 1}{\gamma} \sigma_1 d\widehat{B}_u - \int_t^T \frac{1}{2} \left(\frac{\gamma - 1}{\gamma} \right)^2 \sigma_1^2 du \right\}$$

and notice that

$$(4.10) \quad (X_s)^{\frac{\gamma-1}{\gamma}} = x^{\frac{\gamma-1}{\gamma}} H_s N(s, m),$$

where

$$(4.11) \quad N(s, m) := \exp \left(\int_t^s \frac{\gamma - 1}{\gamma} \left[(\rho + 1)(\beta - r + M_u^m) + \mu_1 - \frac{\sigma_1^2}{2} \right] du + \int_t^s \frac{1}{2} \left(\frac{\gamma - 1}{\gamma} \right)^2 \sigma_1^2 du \right).$$

By Girsanov's Theorem, the process $\widetilde{B} := \{\widehat{B}_s - \frac{\gamma-1}{\gamma} \sigma_1 s, s \in [t, T]\}$ is a new standard Brownian motion under the measure $\widetilde{\mathbb{P}}$. Moreover, it is easy to check that $N(s, m)$ is uniformly bounded in time, i.e., there exist $\underline{L}, \bar{L} : \mathbb{R}_+ \mapsto (0, \infty)$ such that $0 < \underline{L}(m) \leq \sup_{s \in [0, T]} N(s, m) \leq \bar{L}(m) < \infty$. In fact, $N(s, m) = \exp\left[(-\frac{1}{\gamma}(\beta - r) + \frac{\gamma-1}{\gamma}(\mu_1 - \frac{\sigma_1^2}{2}) + \frac{1}{2}(\frac{\gamma-1}{\gamma})^2 \sigma_1^2)(s - t)\right] \exp(-\frac{1}{\gamma} \int_t^s m e^{a(u-t)} du)$ is continuously differentiable in $[t, T] \times \mathbb{R}_+, t \in [0, T]$.

4.2. Preliminary properties of the value function. To study the optimal stopping problem (4.2), we find it convenient to introduce the function

$$(4.12) \quad \widehat{J}(t, x, m) := \widetilde{J}(t, x, m) - \widehat{W}(t, x, m)$$

with

$$(4.13) \quad \widehat{W}(t, x, m) := Q(x, m) - q(t).$$

Applying Itô's formula to $\{e^{-\kappa(s-t)}[Q(X_s, M_s) - q(s)], s \in [t, \tau]\}$, and taking conditional expectations we have

$$\begin{aligned} \widehat{\mathbb{E}}_{t,x,m} \left[e^{-\kappa(\tau-t)} (Q(X_\tau, M_\tau) - q(\tau)) \right] &= Q(x, m) - q(t) + \\ \widehat{\mathbb{E}}_{t,x,m} \left[\int_t^\tau e^{-\kappa(s-t)} \mathcal{L} \left(Q(X_s, M_s) - q(s) \right) ds \right], \end{aligned}$$

where, for any $F \in C^{1,2,1}(\mathcal{U})$, the second order differential operator \mathcal{L} is such that

$$\mathcal{L}F := F_t + \frac{1}{2}\sigma_1^2 x^2 F_{xx} + [(\rho + 1)(\beta - r + m) + \mu_1]x F_x + am F_m - \kappa F.$$

Combining (4.2), (4.12) and (4.13), we have

$$\begin{aligned} \widehat{J}(t, x, m) &= \sup_{t \leq \tau \leq T} \widehat{\mathbb{E}}_{t,x,m} \left[\int_t^\tau e^{-\kappa(s-t)} U_1^*(X_s) ds + e^{-\kappa(\tau-t)} (Q(X_\tau, M_\tau) - q(\tau)) \right] - \widehat{W}(t, x, m) \\ &= \sup_{t \leq \tau \leq T} \widehat{\mathbb{E}}_{t,x,m} \left[\int_t^\tau e^{-\kappa(s-t)} U_1^*(X_s) ds \right] + \widehat{\mathbb{E}}_{t,x,m} \left[\int_t^\tau e^{-\kappa(s-t)} \mathcal{L}(Q(X_s, M_s) - q(s)) ds \right] \\ &= \sup_{t \leq \tau \leq T} \widehat{\mathbb{E}}_{t,x,m} \left[\int_t^\tau e^{-\kappa(s-t)} (U_1^*(X_s) - U_2^*(X_s) + 1) ds \right] \\ (4.14) \quad &= \sup_{t \leq \tau \leq T} \widehat{\mathbb{E}}_{t,x,m} \left[\int_t^\tau e^{-\kappa(s-t)} \left[(1 - K^{\frac{1-\gamma}{\gamma}}) U_1^*(X_s) + 1 \right] ds \right], \end{aligned}$$

where we have used the fact that

$$\mathcal{L}(Q(X_s, M_s) - q(s)) = \mathcal{L}Q(X_s, M_s) - \mathcal{L}(q(s)) = -U_2^*(X_s) + 1.$$

Indeed, arguing similarly to the proof of Lemma 4.1 (cf. (4.4) and (4.7)), one finds (recalling that $x = x(z, y) = z^{\frac{1}{1-\gamma}} y^{\frac{\gamma}{1-\gamma}}$)

$$(4.15) \quad Q(z, m) = \mathbb{E}_{z,m} \left[\int_t^\infty e^{-\int_t^s (\beta + M_u) du} U_2^*(Z_s) ds \right] = zy \widehat{\mathbb{E}}_{x,m} \left[\int_t^\infty e^{-\kappa(s-t)} U_2^*(X_s) ds \right] = zyQ(x, m),$$

and $\mathcal{L}Q(x, m) = -U_2^*(x)$. Moreover, $\mathcal{L}(q(s)) = -e^{\kappa(T-s)} - \kappa \left(\frac{1 - e^{-\kappa(T-s)}}{\kappa} \right) = -1$.

Notice now that the process (X_t, M_t) is time-homogeneous, so that

$$\text{Law}[(u, X_u, M_u)_{u \geq t} | X_t = x, M_t = m] = \text{Law}[(t + s, X_s, M_s)_{s \geq 0} | X_0 = x, M_0 = m].$$

Let $\widehat{\mathbb{E}}_{x,m}$ be the expectation under $\widehat{\mathbb{P}}$ conditional on $X_0 = x, M_0 = m$. Hence, from (4.14),

$$(4.16) \quad \widehat{J}(t, x, m) = \sup_{0 \leq \tau \leq T-t} \widehat{\mathbb{E}}_{x,m} \left[\int_0^\tau e^{-\kappa s} \left[(1 - K^{\frac{1-\gamma}{\gamma}}) U_1^*(X_s) + 1 \right] ds \right].$$

As usual in optimal stopping theory, we let

$$(4.17) \quad \mathcal{W} := \{(t, x, m) \in \mathcal{U} : \widehat{J}(t, x, m) > 0\}, \quad \mathcal{I} := \{(t, x, m) \in \mathcal{U} : \widehat{J}(t, x, m) = 0\}$$

be the so-called continuation (waiting) and stopping (retiring) regions, respectively. We denote by $\partial\mathcal{W}$ the boundary of the set \mathcal{W} .

Since, for any stopping time τ , the mapping $(x, m) \mapsto \widehat{\mathbb{E}}_{x,m}[\int_0^\tau e^{-\kappa s} [(1 - K^{\frac{1-\gamma}{\gamma}}) U_1^*(X_s) + 1] ds]$ is continuous, \widehat{J} is lower-semicontinuous on \mathcal{U} . Hence, \mathcal{W} is open, \mathcal{I} is closed, and introducing

the stopping time

$$\tau^*(t, x, m) := \inf\{s \geq 0 : (t + s, X_s, M_s) \in \mathcal{I}\} \wedge (T - t), \quad \widehat{\mathbb{P}}_{x, m} - a.s.,$$

with $\inf \emptyset = +\infty$, one has that $\tau^*(t, x, m)$ is optimal for $\widehat{J}(t, x, m)$ (see e.g. Corollary I.2.9 in Peskir and Shiryaev [35]).

Proposition 4.1. *The value function \widehat{J} is such that $0 \leq \widehat{J}(t, x, m) \leq \frac{1}{\kappa}(1 - e^{-\kappa(T-t)})$ for all $(t, x, m) \in \mathcal{U}$ and it satisfies the following properties:*

- (i) *When $0 < \gamma < 1$, $x \mapsto \widehat{J}(t, x, m)$ is non-decreasing for all $(t, m) \in [0, T] \times \mathbb{R}_+$; when $\gamma > 1$, $x \mapsto \widehat{J}(t, x, m)$ is non-increasing for all $(t, m) \in [0, T] \times \mathbb{R}_+$;*
- (ii) *$t \mapsto \widehat{J}(t, x, m)$ is non-increasing for all $(x, m) \in \mathbb{R}_+^2$;*
- (iii) *$m \mapsto \widehat{J}(t, x, m)$ is non-decreasing for all $(t, x) \in [0, T] \times \mathbb{R}_+$.*

Proof. The proof is given in Appendix A.1. □

The next lemma shows that \mathcal{I} as in (4.17) is nonempty.

Lemma 4.2. *One has $\mathcal{I} \neq \emptyset$.*

Proof. Suppose that $\mathcal{I} = \emptyset$, then for all $(t, x, m) \in \mathcal{U}$ we have

$$0 \leq \widehat{J}(t, x, m) = \widehat{\mathbb{E}}_{x, m} \left[\int_0^{T-t} e^{-\kappa s} \left[(1 - K^{\frac{1-\gamma}{\gamma}}) U_1^*(X_s) + 1 \right] ds \right].$$

However, when $\gamma < 1$, taking $x \downarrow 0$, the term $(1 - K^{\frac{1-\gamma}{\gamma}}) U_1^*(X_s)$ converges monotonically to $-\infty$, leading to a contraction. Similarly, when $\gamma > 1$, taking $x \uparrow \infty$, the term $(1 - K^{\frac{1-\gamma}{\gamma}}) U_1^*(X_s)$ converges monotonically to $-\infty$, which brings to a contradiction. □

The next technical result states some properties of \widehat{J} that will be useful in the study of the regularity of the boundary $\partial\mathcal{W}$.

Proposition 4.2. *Recall $\kappa = r - \mu_y + \sigma_y \theta$, (4.9) and (4.11). The function \widehat{J} is locally Lipschitz-continuous on \mathcal{U} and for a.e. $(t, x, m) \in \mathcal{U}$ we have the following probabilistic representation formulae:*

$$(4.18) \quad \widehat{J}_x(t, x, m) = (K^{\frac{1-\gamma}{\gamma}} - 1) x^{-\frac{1}{\gamma}} \widehat{\mathbb{E}}_{x, m} \left[\int_0^{\tau^*} e^{-\kappa s} H_s N(s, m) ds \right],$$

$$(4.19) \quad \widehat{J}_m(t, x, m) = (K^{\frac{1-\gamma}{\gamma}} - 1) \frac{x^{\frac{\gamma-1}{\gamma}}}{1-\gamma} \widehat{\mathbb{E}}_{x, m} \left[\int_0^{\tau^*} e^{-\kappa s} e^{as} H_s N(s, m) ds \right].$$

Moreover, there exists a constant $C > 0$, independent of (t, x, m) , such that

$$(4.20) \quad -\frac{1}{T-t} (Cx^{\frac{\gamma-1}{\gamma}} \widetilde{\mathbb{E}}[\tau^*] + \widehat{\mathbb{E}}[\tau^*]) \leq \widehat{J}_t(t, x, m) \leq 0.$$

where $\tau^* := \tau^*(t, x, m)$ is the optimal stopping time for the problem with initial data (t, x, m) .

Proof. The proof is given in Appendix A.2. □

We conclude with asymptotic limits of \widehat{J} .

Proposition 4.3. *When $\gamma < 1$ we have*

$$\lim_{x \rightarrow 0} \widehat{J}(t, x, m) = 0, \quad \lim_{x \rightarrow \infty} \widehat{J}(t, x, m) = \frac{1}{\kappa} (1 - e^{-\kappa(T-t)}), \quad \text{for all } (t, m) \in [0, T] \times \mathbb{R}_+;$$

When $\gamma > 1$ we have

$$\lim_{x \rightarrow 0} \widehat{J}(t, x, m) = \frac{1}{\kappa} (1 - e^{-\kappa(T-t)}), \quad \lim_{x \rightarrow \infty} \widehat{J}(t, x, m) = 0, \quad \text{for all } (t, m) \in [0, T] \times \mathbb{R}_+.$$

Proof. The proof is given in Appendix A.3. □

4.3. Properties of the optimal boundary. In this section, we show that the boundary $\partial\mathcal{W}$ can be represented by a function $b(t, m)$. We establish connectedness of the sets \mathcal{W} and \mathcal{I} with respect to the x -variable and give some preliminary properties of the optimal boundary.

For our subsequent analysis, it is convenient to introduce the auxiliary infinite time-horizon optimal stopping problem

$$\widehat{J}_\infty(x, m) := \sup_{\tau \geq 0} \widehat{\mathbb{E}}_{x, m} \left[\int_0^\tau e^{-\kappa s} \left[(1 - K^{\frac{1-\gamma}{\gamma}}) U_1^*(X_s) + 1 \right] ds \right].$$

Lemma 4.3. *When $\gamma < 1$, then there exists a function $b : [0, T] \times \mathbb{R}_+ \mapsto (0, \infty)$, such that*

$$(4.21) \quad \mathcal{I} = \{(t, x, m) \in [0, T] \times \mathbb{R}_+^2 : 0 < x \leq b(t, m)\}.$$

Moreover, the function b has following properties:

- (i) $t \rightarrow b(t, m)$ is non-decreasing for any $m \in \mathbb{R}_+$;
- (ii) $m \rightarrow b(t, m)$ is non-increasing for any $t \in [0, T]$;
- (iii) one has $0 < b(t, m) \leq L := [(K^{\frac{1-\gamma}{\gamma}} - 1) \frac{\gamma}{1-\gamma}]^{\frac{\gamma}{1-\gamma}}$.

When $\gamma > 1$, then there exists a function $b : [0, T] \times \mathbb{R}_+ \mapsto (0, \infty)$, such that

$$(4.22) \quad \mathcal{I} = \{(t, x, m) \in [0, T] \times \mathbb{R}_+^2 : x \geq b(t, m)\}.$$

Moreover, the function b has following properties:

- (i) $t \rightarrow b(t, m)$ is non-increasing for any $m \in \mathbb{R}_+$;
- (ii) $m \rightarrow b(t, m)$ is non-decreasing for any $t \in [0, T]$;
- (iii) one has $L \leq b(t, m) \leq b_\infty(m) < \infty$, where $b_\infty(m) := \sup\{x > 0 : \widehat{J}_\infty(x, m) > 0\}$, with $m \mapsto b_\infty(m)$ being non-decreasing.

Proof. The proof is given in Appendix A.4. □

The local Lipschitz-continuity of the boundary that we prove in the next theorem has important consequences regarding the regularity of the value function \widehat{J} , as we will see in Proposition 4.5 below.

Theorem 4.1. *The free boundary b is locally Lipschitz-continuous on $[0, T] \times \mathbb{R}_+$.*

Proof. The proof is given in Appendix A.5. □

4.4. Characterization of the free boundary and the value function.

Assumption 4.1. *The model's parameters are such that $\sigma_y \gamma < \theta$.*

This assumption is sufficient to ensure the validity of the next lemma, which, given that b is locally-Lipschitz, is proven through a suitable application of the law of the iterated logarithm.

Lemma 4.4. *When $\gamma < 1$, let $(t, x, m) \in \mathcal{U}$ and set*

$$\hat{\tau}(t, x, m) := \inf\{s \geq 0 : X_s < b(t + s, M_s)\} \wedge (T - t).$$

Then $\hat{\tau}(t, x, m) = \tau^(t, x, m)$ a.s., where $\tau^*(t, x, m) = \inf\{s \geq 0 : X_s \leq b(t + s, M_s)\} \wedge (T - t)$.*

Similarly, when $\gamma > 1$, let $(t, x, m) \in \mathcal{U}$ and set

$$\hat{\tau}(t, x, m) := \inf\{s \geq 0 : X_s > b(t + s, M_s)\} \wedge (T - t).$$

Then $\hat{\tau}(t, x, m) = \tau^(t, x, m)$ a.s., where $\tau^*(t, x, m) = \inf\{s \geq 0 : X_s \geq b(t + s, M_s)\} \wedge (T - t)$.*

Proof. The proof is given in Appendix A.6. □

The previous lemma in turn yields the following continuity property of τ^* , which will then be fundamental in the proof of Proposition 4.5 below.

Proposition 4.4. *One has that $\mathcal{U} \ni (t, x, m) \mapsto \tau^*(t, x, m) \in [0, T - t]$ is continuous.*

Proof. The proof exploits Lemma 4.4 and arguments completely analogous to those employed in the proof of Proposition 5.2 in De Angelis and Ekström [11] and it is therefore omitted. \square

Proposition 4.5. *The value function $\widehat{J} \in C^{1,2,1}(\overline{\mathcal{W}}) \cap C^{1,1,1}(\mathcal{U})$ and solves the boundary value problem*

$$\left\{ \begin{array}{l} \mathcal{L}\widehat{J}(t, x, m) = -(1 - K^{\frac{1-\gamma}{\gamma}})U_1^*(x) - 1, \quad (t, x, m) \in \mathcal{W}, \\ \widehat{J}(t, x, m) = 0, \quad (t, x, m) \in \mathcal{I} \cap \{t < T\}, \\ \widehat{J}(T, x, m) = 0, \quad (x, m) \in \mathbb{R}_+^2, \\ \widehat{J}_t(t, x, m) = \widehat{J}_x(t, x, m) = \widehat{J}_m(t, x, m) = 0 \text{ on } \partial\mathcal{W} \cap \{t < T\}. \end{array} \right.$$

Proof. First we show that the function \widehat{J} is continuously differentiable over \mathcal{U} . From the representations of \widehat{J}_x , \widehat{J}_t and \widehat{J}_m in Proposition 4.2, and the continuity of $(t, x, m) \mapsto \tau^*(t, x, m)$ (cf. Proposition 4.4), we conclude that those weak derivatives are in fact continuous and therefore that $\widehat{J} \in C^{1,1,1}(\mathcal{W}) \cap C^{1,1,1}(\overset{\circ}{\mathcal{I}})$, where $\overset{\circ}{\mathcal{I}}$ denotes the interior of \mathcal{I} . In particular, $\widehat{J}_t = \widehat{J}_x = \widehat{J}_m = 0$ on $\overset{\circ}{\mathcal{I}}$. It thus remains to analyze the regularity of \widehat{J} across $\partial\mathcal{W}$.

Fix a point $(t_0, x_0, m_0) \in \partial\mathcal{W} \cap \{t < T\}$ and take a sequence $(t_n, x_n, m_n)_{n \geq 1} \subseteq \mathcal{W}$ with $(t_n, x_n, m_n) \rightarrow (t_0, x_0, m_0)$ as $n \rightarrow \infty$. Continuity of $(t, x, m) \mapsto \tau^*(t, x, m)$ implies that $\tau^*(t_n, x_n, m_n) \rightarrow \tau^*(t_0, x_0, m_0) = 0$, \mathbb{P} -a.s. as $n \rightarrow \infty$. Again, from Proposition 4.2, dominated convergence yields that $\widehat{J}_m(t_n, x_n, m_n) \rightarrow 0$, $\widehat{J}_x(t_n, x_n, m_n) \rightarrow 0$ and $\widehat{J}_t(t_n, x_n, m_n) \rightarrow 0$. Since (t_0, x_0, m_0) and the sequence (t_n, x_n, m_n) were arbitrary, we get $\widehat{J} \in C^{1,1,1}(\mathcal{U})$.

Let us now turn to study the regularity of \widehat{J} in \mathcal{W} . First of all notice that, because the solution to (4.8) is linear with respect to the initial datum x , we then have that $x \mapsto U_1^*(X_s)$ is convex, being U_1^* from (4.3) clearly convex. This fact implies that $x \mapsto \widehat{\mathbb{E}}_{t,x,m}[\int_t^\tau e^{-\kappa(s-t)} U_1^*(X_s) ds + e^{-\kappa(\tau-t)}(Q(X_\tau, M_\tau) - q(\tau))]$ is convex, as $Q(\cdot, m)$ is strictly convex. Hence, $x \mapsto \widetilde{J}(t, x, m)$ is convex. In particular, by Alexandrov's Theorem, \widetilde{J} admits second order derivatives with respect to x in a dense subset of \mathcal{U} . Further, since $Q(x, m)$ is C^2 with respect to x , we know that \widehat{J} admits second order derivatives with respect to x in a dense subset of \mathcal{U} (cf. (4.12) and (4.13)).

By Corollary 6 in Peskir [34], we also know that \widehat{J} solves in the sense of distributions

$$(4.23) \quad \mathcal{L}\widehat{J}(t, x, m) = -(1 - K^{\frac{1-\gamma}{\gamma}})U_1^*(x) - 1, \quad (t, x, m) \in \mathcal{W},$$

and it is such that

$$\begin{aligned} \widehat{J}(t, x, m) &= 0, \quad (t, x, m) \in \mathcal{I} \cap \{t < T\}, \\ \widehat{J}(T, x, m) &= 0, \quad (x, m) \in \mathbb{R}_+^2. \end{aligned}$$

By writing (4.23) as

$$\frac{\sigma_1^2 x^2}{2} \widehat{J}_{xx} = -\widehat{J}_t + \kappa \widehat{J} - am \widehat{J}_m - [(\rho + 1)(\beta - r + m) + \mu_1] x \widehat{J}_x - (1 - K^{\frac{1-\gamma}{\gamma}}) U_1^*(x) - 1,$$

we then see that \widehat{J}_{xx} admits a continuous extension to $\overline{\mathcal{W}}$, that we denote \bar{J}_{xx} . Then, by taking arbitrary $(t_0, x_0, m_0) \in \mathcal{W}$ we can write

$$\widehat{J}_x(t_0, x, m_0) = \widehat{J}_x(t_0, x_0, m_0) + \int_{x_0}^x \bar{J}_{xx}(t_0, u, m_0) du, \quad \forall (t_0, x, m_0) \in \mathcal{I},$$

and the latter yields that $\widehat{J}_x(t, \cdot, m)$ is continuous for any x such that $(t, x, m) \in \overline{\mathcal{W}}$. Since we already know that $\widehat{J} \in C^{1,1,1}(\mathcal{U})$, we conclude that $\widehat{J} \in C^{1,2,1}(\overline{\mathcal{W}})$. □

Corollary 4.1. *Recall (4.12). The function $\tilde{J} \in C^{1,2,1}(\overline{\mathcal{W}}) \cap C^{1,1,1}(\mathcal{U})$ and solves the boundary value problem*

$$\mathcal{L}\tilde{J}(t, x, m) = -U_1^*(x), \quad (t, x, m) \in \mathcal{W},$$

$$\tilde{J}(t, x, m) = \widehat{W}(t, x, m), \quad (t, x, m) \in \mathcal{I} \cap \{t < T\},$$

$$\tilde{J}(T, x, m) = \widehat{W}(T, x, m), \quad (x, m) \in \mathbb{R}_+^2,$$

$$\tilde{J}_t(t, x, m) = \widehat{W}_t(t, x, m), \quad \tilde{J}_x(t, x, m) = \widehat{W}_x(t, x, m), \quad \tilde{J}_m(t, x, m) = \widehat{W}_m(t, x, m) \quad \text{on } \partial\mathcal{W} \cap \{t < T\}.$$

Remark 4.1. *It is worth noting that standard PDE arguments typically require uniform ellipticity of the underlying second-order differential operator and thus could not be directly applied in the proof of Proposition 4.5, due to the fully degenerate diffusion process (X, M) . Therefore, we had to hinge on a novel series of intermediate results. First, we find the locally Lipschitz continuity of \widehat{J} (cf. Proposition 4.2) and then establish the locally-Lipschitz continuity of free boundary without relying upon the continuity of $\widehat{J}_x, \widehat{J}_m$ and \widehat{J}_t (cf. Theorem 4.1). Finally, we upgrade the regularity of \widehat{J} using the continuity of the optimal stopping time (cf. Propositions 4.4 and 4.5).*

We are now in the conditions of determining a nonlinear integral equation that characterizes uniquely the free boundary. Such a characterization results from an integral representation of the value function \widehat{J} . This is accomplished by the next theorem, which exploits the regularity properties of \widehat{J} proved so far.

Theorem 4.2. *If $\gamma < 1$, \widehat{J} from (4.16) has the representation*

(4.24)

$$\widehat{J}(t, x, m) = \widehat{\mathbb{E}}_{x, m} \left[\int_0^{T-t} e^{-\kappa s} \left[(1 - K^{\frac{1-\gamma}{\gamma}}) U_1^*(X_s) + 1 \right] \mathbf{1}_{\{X_s \geq b(t+s, M_s)\}} ds \right], \quad (t, x, m) \in \mathcal{U}.$$

Moreover, recalling $L = [(K^{\frac{1-\gamma}{\gamma}} - 1) \frac{\gamma}{1-\gamma}]^{\frac{\gamma}{1-\gamma}}$, the optimal boundary b is the unique continuous solution bounded from above by L to the following nonlinear integral equation: For all $(t, m) \in [0, T] \times \mathbb{R}_+$,

$$(4.25) \quad 0 = \widehat{\mathbb{E}}_{b(t, m), m} \left[\int_0^{T-t} e^{-\kappa s} \left[(1 - K^{\frac{1-\gamma}{\gamma}}) U_1^*(X_s) + 1 \right] \mathbf{1}_{\{X_s \geq b(t+s, M_s)\}} ds \right],$$

with $\lim_{t \uparrow T} b(t, m) = L$.

If $\gamma > 1$, \widehat{J} from (4.16) has the representation

(4.26)

$$\widehat{J}(t, x, m) = \widehat{\mathbb{E}}_{x, m} \left[\int_0^{T-t} e^{-\kappa s} \left[(1 - K^{\frac{1-\gamma}{\gamma}}) U_1^*(X_s) + 1 \right] \mathbf{1}_{\{X_s \leq b(t+s, M_s)\}} ds \right], \quad (t, x, m) \in \mathcal{U}.$$

Moreover, the optimal boundary b is the unique continuous solution bounded from below by L to the following nonlinear integral equation: For all $(t, m) \in [0, T] \times \mathbb{R}_+$,

$$(4.27) \quad 0 = \widehat{\mathbb{E}}_{b(t, m), m} \left[\int_0^{T-t} e^{-\kappa s} \left[(1 - K^{\frac{1-\gamma}{\gamma}}) U_1^*(X_s) + 1 \right] \mathbf{1}_{\{X_s \leq b(t+s, M_s)\}} ds \right],$$

with $\lim_{t \uparrow T} b(t, m) = L$.

Proof. Step 1. In this step, we only need to prove (4.24), since the derivation in the case of $\gamma > 1$ is similar. Let $(t, x, m) \in \mathcal{U}$ be given and fixed, let $(K_n)_{n \geq 0}$ be a sequence of compact sets increasing to $[0, T] \times \mathbb{R}_+^2$ and define

$$\tau_n := \inf\{s \geq 0 : (t + s, X_s^x, M_s^m) \notin K_n\} \wedge (T - t), \quad n \geq 0.$$

Since $\widehat{J} \in C^{1,1,1}(\mathcal{U})$, $\widehat{J}_{xx} \in L_{loc}^\infty(\mathcal{U})$, and $\widehat{\mathbb{P}}[(t + s, X_s^x, M_s^m) \in \partial\mathcal{W}] = 0$ for all $s \in [0, T - t)$, we can apply a weak version of Dynkin's formula (see, e.g., Bensoussan and Lions [2], Lemma 8.1 and Th. 8.5, pp. 183-186) so to obtain

$$\widehat{J}(t, x, m) = \widehat{\mathbb{E}}_{x, m} \left[e^{-\kappa \tau_n} \widehat{J}(t + \tau_n, X_{\tau_n}^x, M_{\tau_n}^m) - \int_0^{\tau_n} e^{-\kappa s} \mathcal{L} \widehat{J}(t + s, X_s^x, M_s^m) ds \right].$$

Therefore, using (4.23), we also find

$$\widehat{J}(t, x, m) = \widehat{\mathbb{E}}_{x, m} \left[e^{-\kappa \tau_n} \widehat{J}(t + \tau_n, X_{\tau_n}^x, M_{\tau_n}^m) \right]$$

$$+ \int_0^{\tau_n} e^{-\kappa s} \left((1 - K^{\frac{1-\gamma}{\gamma}}) U_1^*(X_u^x) + 1 \right) \mathbf{1}_{\{X_s^x \geq b(t+s, M_s^m)\}} ds \Big],$$

where we have used again that $\widehat{\mathbb{P}}[(t+s, X_s^x, M_s^m) \in \partial\mathcal{W}] = 0$.

Finally, we take $n \uparrow \infty$, apply the dominated convergence theorem, and use that $\tau_n \uparrow (T-t)$ and $\widehat{J}(T, x, m) = 0$ (cf. Proposition 4.5) to obtain (4.24).

Step 2. Next, for $\gamma < 1$, we find the limit value of $b(t, m)$ when $t \rightarrow T$. The argument is inspired by the proof of Proposition 4.10 in De Angelis and Stabile [13]. The case of $\gamma > 1$ can be treated similarly and we thus omit it. Firstly, the limit $b(T-, m) := \lim_{t \rightarrow T} b(t, m)$ exists, since b is monotone on $[0, T] \times \mathbb{R}_+$. Notice that $b(t, m) \leq L$ for all $(t, m) \in [0, T] \times \mathbb{R}_+$ (cf. Lemma 4.3) and therefore $b(T-, m) \leq L$. Arguing by contradiction, we assume $b(T-, m) < L$. Then we pick a, c such that $b(T-, m) < a < c < L$ and $t' < T$ such that $(a, c) \times [t', T] \times \mathbb{R}_+ \subseteq \mathcal{W}$. Let $\varphi \in C^\infty(a, c)$ with $\varphi \geq 0$ such that $\int_a^c \varphi(y) dy = 1$. Recall (4.2) and define $F_\varphi(s) := \int_a^c \widehat{J}_t(s, y, m) \varphi(y) dy$. Now, denoting by \mathbb{L}^* the adjoint of the operator \mathbb{L} , where $\mathbb{L}f := \frac{1}{2} \sigma_1^2 x^2 f_{xx} + [(\rho+1)(\beta-r+m) + \mu_1] x f_x + a m f_m$. Therefore, we have $\mathcal{L}\widehat{J} = \widehat{J}_t + \mathbb{L}\widehat{J} - \kappa\widehat{J} = -(1 - K^{\frac{1-\gamma}{\gamma}}) U_1^*(x) - 1$ on \mathcal{W} . Further,

$$\begin{aligned} \lim_{s \uparrow T} F_\varphi(s) &= \lim_{s \uparrow T} \int_a^c \left((K^{\frac{1-\gamma}{\gamma}} - 1) U_1^*(y) - 1 - \mathbb{L}\widehat{J} + \kappa\widehat{J} \right) \varphi(y) dy \\ &= \lim_{s \uparrow T} \int_a^c \left(\left((K^{\frac{1-\gamma}{\gamma}} - 1) U_1^*(y) - 1 \right) \varphi(y) + \widehat{J}(s, y, m) (-\mathbb{L}^* + \kappa) \varphi(y) \right) dy \\ &= \int_a^c \left((K^{\frac{1-\gamma}{\gamma}} - 1) U_1^*(y) - 1 \right) \varphi(y) dy > 0, \end{aligned}$$

since $\widehat{J}(T, y, m) = 0$ and $(K^{\frac{1-\gamma}{\gamma}} - 1) U_1^*(y) - 1 > (K^{\frac{1-\gamma}{\gamma}} - 1) U_1^*(L) - 1 = 0$ for any $y \in (a, c)$ when $\gamma < 1$. From the above, we also deduce that $F_\varphi(\cdot)$ is continuous up to T ; thus there exists $\delta > 0$ such that $F_\varphi(s) > 0$ for $s \in [T - \delta, T]$ and we obtain

$$\begin{aligned} 0 &< \int_{T-\delta}^T F_\varphi(s) ds = \int_a^c \left(\widehat{J}(T, y, m) - \widehat{J}(T - \delta, y, m) \right) \varphi(y) dy \\ &= \int_a^c \left(-\widehat{J}(T - \delta, y, m) \right) \varphi(y) dy < 0 \end{aligned}$$

since $\widehat{J}(T, y, m) = 0$ and $\widehat{J}(T - \delta, y, m) > 0$ for $y \in (a, c)$. This is a contradiction.

Step 3. Given that (4.24) holds for any $(t, x, m) \in \mathcal{U}$, we can take $x = b(t, m)$ in (4.24), which leads to (4.25), upon using that $\widehat{J}(t, b(t, m), m) = 0$ (cf. Proposition 4.5). The fact that b is the unique continuous solution to (4.25) bounded from above (resp. below) by L can be proved by following the four-step procedure from the proof of uniqueness provided in Theorem

3.1 of Peskir [33]. Since the present setting does not create additional difficulties we omit further details. □

5. OPTIMAL BOUNDARIES AND STRATEGIES IN TERMS OF PRIMAL VARIABLES

In the previous section, we studied the properties of the dual value function $J(t, z, m, y)$ and used (t, z, m, y) where t denotes time, z denotes marginal utility, m denotes the force of mortality and y denotes the labour income as the coordinate system for the study. In this section we will come back to study the value function $V(t, w, m, y)$ in the original coordinate system (t, w, m, y) , where w denotes the wealth of the agent.

Proposition 5.1. *The function J in (3.18) is strictly decreasing and strictly convex with respect to z .*

Proof. The proof is inspired by Lemma 8.1 in Karatzas and Wang [29]. Firstly, defining

$$(5.1) \quad \bar{J}(t, z, m, y; \tau) := \mathbb{E}_{t, z, m, y} \left[\int_t^\tau e^{-\int_t^s (\beta + M_u) du} U_1^*(Z_s) ds + e^{-\int_t^\tau (\beta + M_u) du} \left(Q(Z_\tau, M_\tau) - Z_\tau g(\tau) \right) \right],$$

we have $J(t, z, m, y) = \sup_{\tau \in \mathcal{S}} \bar{J}(t, z, m, y; \tau)$. From (3.12), it is easy to check that Q is strictly convex and strictly decreasing with respect to z . Moreover, U_1^* is strictly convex and strictly decreasing with respect to z (cf. (4.3)). Therefore, \bar{J} is strictly convex and strictly decreasing with respect to z . We denote by $\bar{\mathcal{S}}_{z_i}$ the set of stopping times that attain the supremum in (5.1) for every given $(t, z_i, m, y) \in \mathcal{O}$. For any $0 < z_1 < z_2 < \infty$, $0 < \alpha < 1$, and $z_0 := \alpha z_1 + (1 - \alpha) z_2$, we have

$$J(t, z_2, m, y) = \bar{J}(t, z_2, m, y; \bar{\tau}_2) < \bar{J}(t, z_1, m, y; \bar{\tau}_2) \leq J(t, z_1, m, y),$$

where $\bar{\tau}_i \in \bar{\mathcal{S}}_{z_i}$, $i = 0, 1, 2$ are optimal stopping times. Further we have

$$\begin{aligned} J(t, z_0, m, y) &= \bar{J}(t, z_0, m, y; \bar{\tau}_0) < \alpha \bar{J}(t, z_1, m, y; \bar{\tau}_0) + (1 - \alpha) \bar{J}(t, z_2, m, y; \bar{\tau}_0) \\ &\leq \alpha J(t, z_1, m, y) + (1 - \alpha) J(t, z_2, m, y), \end{aligned}$$

which completes the proof. □

From Theorem 3.2, for any $(t, w, m, y) \in \mathcal{O}'$, we know that $V(t, w, m, y) = \inf_{z > 0} [J(t, z, m, y) + z(w + g(t))]$. Since $z \mapsto J(t, z, m, y) + z(w + g(t))$ is strictly convex (cf. Proposition 5.1), then

there exists an unique $z^*(t, w, m, y) > 0$ such that

$$(5.2) \quad V(t, w, m, y) = J(t, z^*(t, w, m, y), m, y) + z^*(t, w, m, y)(w + g(t)),$$

where $z^*(t, w, m, y) := \mathcal{I}^J(t, -w - g(t), m, y)$, with \mathcal{I}^J being the inverse function of J_z . Moreover, $z^* \in C(\mathcal{O}')$ and, for any $(t, m, y) \in [0, T] \times \mathbb{R}_+^2$, $z^*(t, w, m, y)$ is strictly decreasing with respect to w . Hence, for any $(t, m, y) \in [0, T] \times \mathbb{R}_+^2$, $z^*(t, \cdot, m, y)$ is a bijection and therefore has an inverse function $w^*(t, \cdot, m, y)$, which is continuous, strictly decreasing, and maps \mathbb{R}_+ to $(-g(t), \infty)$.

Let us now define

$$(5.3) \quad \left\{ \begin{array}{l} \widehat{b}(t, m, y) := w^*(t, b(t, m)^{1-\gamma} y^{-\gamma}, m, y), \\ \mathcal{W}_z := \{(t, z, m, y) \in \mathcal{O} : (t, z^{\frac{1}{1-\gamma}} y^{\frac{\gamma}{1-\gamma}}, m) \in \mathcal{W}\}, \\ \mathcal{I}_z := \{(t, z, m, y) \in \mathcal{O} : (t, z^{\frac{1}{1-\gamma}} y^{\frac{\gamma}{1-\gamma}}, m) \in \mathcal{I}\}, \\ \mathcal{W}_w := \{(t, w, m, y) \in \mathcal{O}' : (t, z^*(t, w, m, y), m, y) \in \mathcal{W}_z\}, \\ \mathcal{I}_w := \{(t, w, m, y) \in \mathcal{O}' : (t, z^*(t, w, m, y), m, y) \in \mathcal{I}_z\}. \end{array} \right.$$

Then, by Lemma 4.3 we have

$$(5.4) \quad \mathcal{W}_w := \{(t, w, m, y) \in \mathcal{O}' : -g(t) < w < \widehat{b}(t, m, y)\}, \quad \mathcal{I}_w := \{(t, w, m, y) \in \mathcal{O}' : w \geq \widehat{b}(t, m, y)\},$$

so that we can express the optimal retirement time in terms of the initial coordinates as:

$$(5.5) \quad \tau^*(t, w, m, y) = \inf\{s \geq 0 : W_s^w \geq \widehat{b}(t + s, M_s^m, Y_s^y)\} \wedge (T - t).$$

Theorem 5.1. *Let $(t, w, m, y) \in \mathcal{O}'$ and recall that $\mathcal{I}_1^u(\cdot)$ denotes the inverse of $U_1^u(\cdot)$. Then*

$$c^*(t, w, m, y) := \mathcal{I}_1^u(V_w(t, w, m, y)), \quad \pi^*(t, w, m, y) := \frac{-\theta V_w(t, w, m, y) - \sigma_y y V_{wy}(t, w, m, y)}{\sigma V_{ww}(t, w, m, y)}, \text{ a.e. on } \mathcal{O}'$$

define the optimal feedback maps, while

$$\tau^* = \inf\{s \geq 0 : V(t + s, W_s^w, M_s^m, Y_s^y) \leq \widehat{V}(W_s^w, M_s^m)\} \wedge (T - t)$$

is the optimal retirement time. Hence, $c_s^* = c^*(s, W_s^*, M_s, Y_s)$, $\pi_s^* = \pi^*(s, W_s^*, M_s, Y_s)$ and τ^* , $\mathbb{P}_{t, w, m, y}$ -a.s., provide an optimal control triple, where W^* is assumed as a strong solution to SDE (2.4), after substituting (c^*, π^*, τ^*) for (c, π, τ) . Further, we have $W_s^* = -J_z(s, Z_s, M_s, Y_s) - g(s)$, where Z_s is the solution of Equation (3.8) with the initial condition $Z_t = z^*$.

Proof. The proof is given in Appendix A.7. □

Thanks to (5.3) and Theorem 5.1 we can finally express the optimal retirement threshold \widehat{b} and the optimal portfolio π in terms of b and z^* , respectively.

Proposition 5.2. *One has that*

$$\widehat{b}(t, m, y) = -yQ(b(t, m), m) - \frac{b(t, m)y}{1 - \gamma}Q_x(b(t, m), m)$$

for any $(t, m, y) \in \mathcal{U}$, and

$$\begin{aligned} \pi^*(t, w, m, y) &= \frac{\theta}{\sigma}z^*[\widetilde{J}_x(t, x^*, m)\frac{x^*y}{z^*}\frac{2 - \gamma}{(1 - \gamma)^2} + \widetilde{J}_{xx}(t, x^*, m)\frac{(x^*)^2y}{z^*}\frac{1}{(1 - \gamma)^2}] \\ &\quad - \sigma_y y[q(t) + [\widetilde{J}(t, x^*, m) + \widetilde{J}_x(t, x^*, m)x^*\frac{\gamma - \gamma^2 + 1}{(1 - \gamma)^2} + (x^*)^2\frac{\gamma}{(1 - \gamma)^2}\widetilde{J}_{xx}(t, x^*, m)]] \end{aligned}$$

with $x^* := (z^*(t, w, m, y))^{\frac{1}{1-\gamma}}y^{\frac{\gamma}{1-\gamma}}$.

Proof. We know that $\widehat{b}(t, m, y) = w^*(t, b(t, m)^{1-\gamma}y^{-\gamma}, m, y)$, where $w^*(t, \cdot, m, y)$ is the inverse function of $z^*(t, \cdot, m, y)$. Since $J_z(t, z^*(t, w, m, y), m, y) = -w - g(t)$, by taking $w = w^*(t, z, m, y)$, computations show that

$$J_z(t, z, m, y) = J_z(t, z^*(t, w^*(t, z, m, y), m, y), m, y) = -w^*(t, z, m, y) - g(t).$$

Hence, from (5.3) and (A.33) we have

$$\begin{aligned} \widehat{b}(t, m, y) &= w^*(t, b(t, m)^{1-\gamma}y^{-\gamma}, m, y) = -J_z(t, b(t, m)^{1-\gamma}y^{-\gamma}, m, y) - g(t) \\ &= -y\widetilde{J}(t, b(t, m), m) - \frac{b(t, m)y}{1 - \gamma}\widetilde{J}_x(t, b(t, m), m) - g(t) \\ &= -y\left(Q(b(t, m), m) - q(t)\right) - \frac{b(t, m)y}{1 - \gamma}Q_x(b(t, m), m) - g(t) \\ &= -yQ(b(t, m), m) - \frac{b(t, m)y}{1 - \gamma}Q_x(b(t, m), m), \end{aligned}$$

where W is defined in (4.15).

To prove the second statement, we notice that (in the a.e. sense) $V_w(t, w, m, y) = z^*$, $V_{ww}(t, w, m, y) = -\frac{1}{J_{zz}(t, z^*, m, y)}$, $V_{wy} = \frac{-q(t) - J_{zy}(t, z^*, m, y)}{J_{zz}(t, z^*, m, y)}$ (cf. (A.33) and (A.34)), which then yield

$$\begin{aligned} \pi^*(t, w, m, y) &= \frac{-\theta V_w(t, w, m, y) - \sigma_y y V_{wy}(t, w, m, y)}{\sigma V_{ww}(t, w, m, y)} \\ &= \frac{\theta}{\sigma}z^*J_{zz}(t, z^*, m, y) - \sigma_y y\left(q(t) + J_{zy}(t, z^*, m, y)\right) \\ &= \frac{\theta}{\sigma}z^*\left[\widetilde{J}_x(t, x^*, m)\frac{x^*y}{z^*}\frac{2 - \gamma}{(1 - \gamma)^2} + \widetilde{J}_{xx}(t, x^*, m)\frac{(x^*)^2y}{z^*}\frac{1}{(1 - \gamma)^2}\right] \\ &\quad - \sigma_y y\left[q(t) + \left(\widetilde{J}(t, x^*, m) + \widetilde{J}_x(t, x^*, m)x^*\frac{\gamma - \gamma^2 + 1}{(1 - \gamma)^2} + (x^*)^2\frac{\gamma}{(1 - \gamma)^2}\widetilde{J}_{xx}(t, x^*, m)\right)\right], \text{ a.e. on } \mathcal{O}', \end{aligned}$$

where $x^* := x^*(t, w, m, y) = (z^*(t, w, m, y))^{1-\gamma} y^{\frac{\gamma}{1-\gamma}}$. \square

Remark 5.1. Given \widehat{b} as in Proposition 5.2, 5.5 rewrites as $\tau^*(t, w, m, y) = \inf\{s \geq 0 : \frac{W_s^w}{Y_s^y} \geq \Upsilon(t + s, M_s^m)\}$, for $\Upsilon(t, m) := -Q(b(t, m), m) - \frac{b(t, m)}{1-\gamma} Q_x(b(t, m), m)$; that is, there exists a critical wealth-to-wage ratio above which it is optimal to retire, which is consistent with the results of Dybvig and Liu [14].

6. NUMERICAL STUDY

In this section, we present numerical illustrations of optimal strategies. Firstly, we show the retirement boundary in the dual and primal variables, as described in Theorem 4.2 and Proposition 5.2, respectively. Moreover, we investigate the sensitivity of the optimal retirement boundaries with respect to the relevant parameters and analyze the consequent economic meaning. We use a recursive iteration method proposed by Huang et al. [22] in order to solve the integral equation (4.27) and provide a detailed explanation of the method in Appendix C. The numerics was performed using Mathematica 13.1.

TABLE 1. Basic parameters set in the numerical illustrations

μ	σ	r	β	γ	μ_y	σ_y	a	T	K
0.08	0.2	0.04	0.01	3	0.01	0.05	1/10.5	10	2

The basic parameters are listed in Table 1¹. Here, for the age-dependent force of mortality of x_0 -year old individual, we use the Gompertz-Makeham model from Milevsky and Young [30]: $M_s = a_0^{-1} e^{(x_0+s-m_0)/a_0}$, with corresponding modal value $m_0 = 88.18$, scale parameter $a_0 = 1/a = 10.5$. We then consider an agent who is 55 years old at the initial time t ($x_0 = 55$) and the mandatory retirement time is 65 years old ($T = 10$), hence $m = a_0^{-1} e^{(x_0-m_0)/a_0} \approx 0.004$.

6.1. Optimal retirement boundaries and sensitive analysis. The retirement boundary $b(t, m)$ of Theorem 4.2 is illustrated in Figure 1. As it has been proven in Lemma 4.3, we can observe that $b(t, m)$ is a decreasing function of time t . Intuitively, an elder agent will be more likely to retire and the retirement region \mathcal{I} expands with time. The dotted line represents L , such that $b(T, m) = L$. Moreover, from Figure 2, we find that an increasing of m induces larger $b(t, m)$, which is also consistent with the theoretical results provided in Lemma 4.3.

Next, fixing $m = 0.004$, we illustrate retirement boundary in the primal variables $\widehat{b}(t, m, y)$, which is displayed in Figure 3. Above the surface is the stopping region \mathcal{I}_w , which is defined in (5.4). Then we will study the sensitivity of $\widehat{b}(t, m, y)$ with respect to some of the model's parameters.

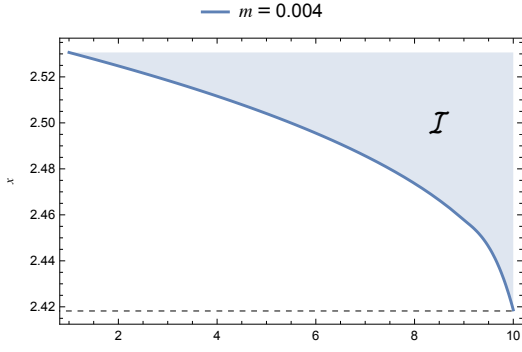


FIGURE 1. The retirement boundary $b(t, m)$ in dual variables (t, x, m)

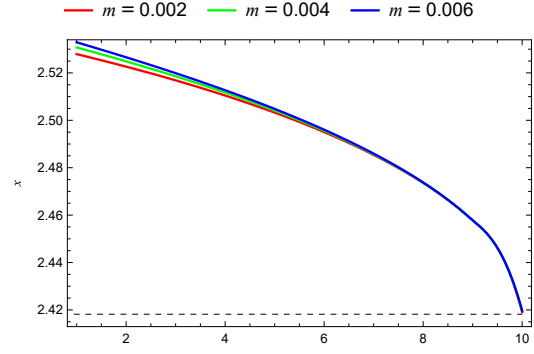


FIGURE 2. The retirement boundary $b(t, m)$ in dual variables (t, x, m) with different m values

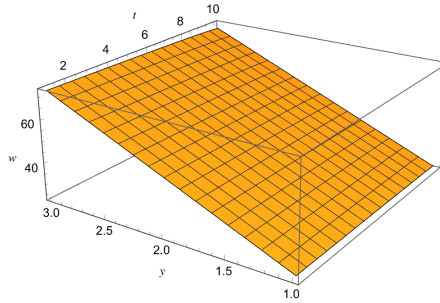


FIGURE 3. The retirement boundary $\hat{b}(t, m, y)$ in primal variables $(t, x, m = 0.004, y)$

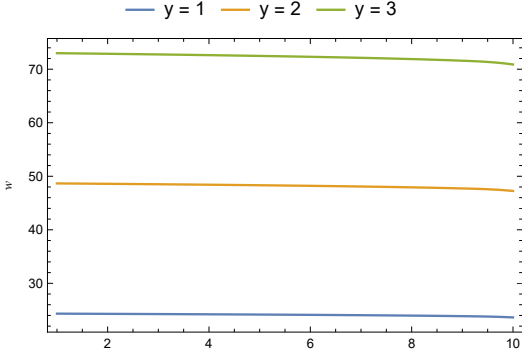


FIGURE 4. The retirement boundary $\hat{b}(t, m, y)$ in primal variables $(t, w, m = 0.004, y)$

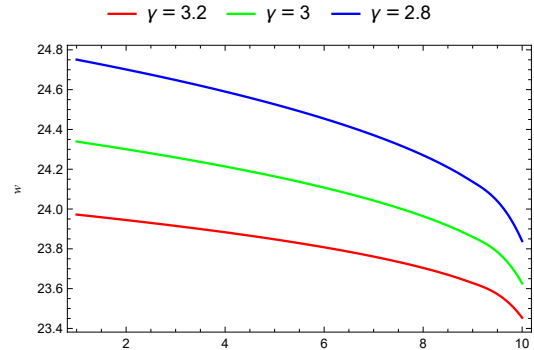


FIGURE 5. The retirement boundary $\hat{b}(t, m, y)$ in primal variables $(t, w, m = 0.004, y = 1)$

In Figure 4 we can observe the sensitivity of the optimal retirement boundary with respect to the initial labor income. Since an increase in y implies higher human capital (the present value of future labor income (cf. (2.3))), the agent delays her decision to retire. As time goes by, the agent is expected to receive more future labour income. Figure 5 shows that if the risk aversion level γ is larger, the agent is more likely to retire later. This is an intuitive result, since a more risk averse agent invests less in the stock market, thus relies more on the income from labor and needs to work longer to accumulate enough wealth to be able to finance her retirement.

¹We just consider the case $\gamma > 1$ in the numerical study, the case $\gamma < 1$ can be treated using the same method.

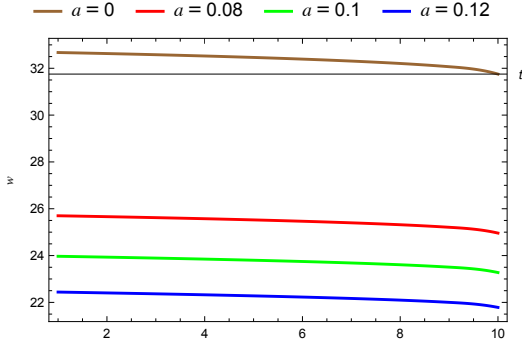


FIGURE 6. The retirement boundary $\hat{b}(t, m, y)$ in primal variables ($t, w, m = 0.004, y = 1$)

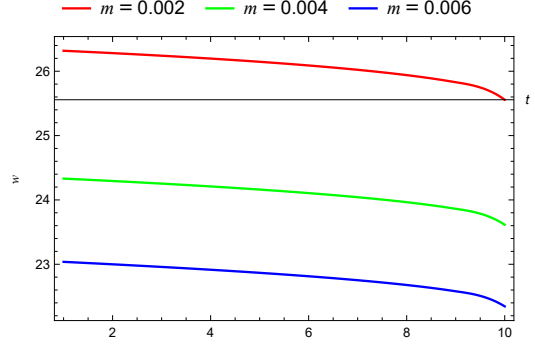


FIGURE 7. The retirement boundary $\hat{b}(t, m, y)$ in primal variables ($t, w, m = 0.004, y = 1$)

Figure 6 shows the effect of a change in a on the boundary. If a increases, growth rate of the agent's force of mortality becomes larger. Thus, the agent is willing to retire earlier. If we consider a constant force of mortality (i.e., $a = 0$) as that in existing literature, we will get a higher retirement boundary. It means the agent will retire later because the age-dependent force of mortality is ignored. Similarly, increasing the initial force of mortality m , makes the agent retire earlier in order to profit earlier from the leisure. This is shown in Figure 7.

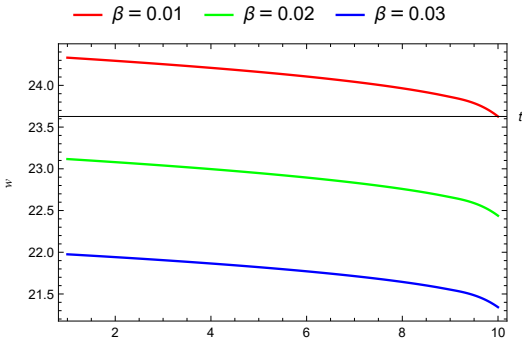


FIGURE 8. The retirement boundary $\hat{b}(t, m, y)$ in primal variables ($t, w, m = 0.004, y = 1$)

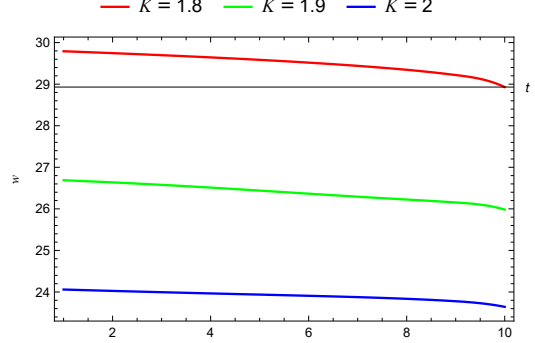


FIGURE 9. The retirement boundary $\hat{b}(t, m, y)$ in primal variables ($t, w, m = 0.004, y = 1$)

Figure 8 illustrates that as the discount rate β becomes larger, the boundary becomes lower. As β can be interpreted as the subjective impatience of the agent, increasing the value of β makes the agent more impatient, with the result of an heavier discount of future utility. Consequently, the agent is willing to retire earlier. Figure 9 shows the effect of a change in K on the boundary \hat{b} . It is clear that if K is larger, the agent assigns higher utility value to consumption during retirement, so that the agent will choose to retire earlier.

6.2. Optimal consumption and portfolio. Figure 10 illustrates the optimal consumption $c^*(t, w, m, y)$. We find that the consumption jumps down at retirement if the relative risk aversion coefficient is greater than 1, where the dotted line is the critical wealth level for retirement.

This is the so-called retirement consumption puzzle, which states that consumption drops at retirement (cf. Banks et al. [1]). Similar to Dybvig and Liu [14], consumption jumps on the retirement date because the marginal utility per unit of consumption changes after retirement.

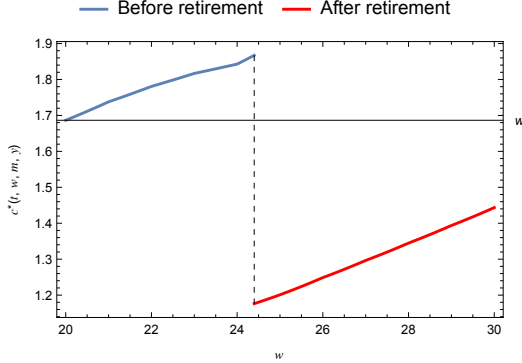


FIGURE 10. The optimal consumption $c^*(t, w, m, y)$ in primal variables ($t = 0, w, m = 0.004, y = 1$)

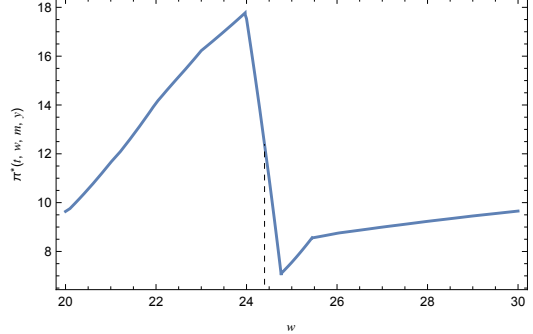


FIGURE 11. The optimal portfolio $\pi^*(t, w, m, y)$ in primal variables ($t = 0, w, m = 0.004, y = 1$)

Figure 11 shows the optimal proportion of risky investment $\pi^*(t, w, m, y)$. We observe a substantial decline of proportion in stock, as it is observed by the empirical evidence in Coile and Milligan [9]. Hence, people tend to withdraw their investment in risky asset upon retirement in order to hedge the risk of unemployment; this is the so-called “saving for retirement”.

APPENDIX A. PROOFS

A.1. Proof of Proposition 4.1.

Proof. Finiteness. It is clear that $\widehat{J}(t, x, m) \geq 0$ for all $(t, x, m) \in \mathcal{U}$. For $\gamma \in (0, 1) \cup (1, \infty)$, we have $(1 - K^{\frac{1-\gamma}{\gamma}})U_1^*(X_s) < 0$, so that

$$0 \leq \sup_{0 \leq \tau \leq T-t} \widehat{\mathbb{E}} \left[\int_0^\tau e^{-\kappa s} \left[(1 - K^{\frac{1-\gamma}{\gamma}})U_1^*(X_s) + 1 \right] ds \right] \leq \widehat{\mathbb{E}} \left[\int_0^{T-t} e^{-\kappa s} ds \right] \leq \frac{1}{\kappa} (1 - e^{-\kappa(T-t)}).$$

Monotonicity in x . When $\gamma < 1$, we find that $(1 - K^{\frac{1-\gamma}{\gamma}}) \frac{\gamma}{1-\gamma} < 0$ and $\frac{\gamma-1}{\gamma} < 0$. Therefore, $x \mapsto \widehat{J}(t, x, m)$ is non-decreasing for all $(t, m) \in [0, T] \times \mathbb{R}_+$. Analogously, when $\gamma > 1$, $(1 - K^{\frac{1-\gamma}{\gamma}}) \frac{\gamma}{1-\gamma} < 0$ and $\frac{\gamma-1}{\gamma} > 0$. Therefore, $x \mapsto \widehat{J}(t, x, m)$ is non-increasing for all $(t, m) \in [0, T] \times \mathbb{R}_+$.

Monotonicity in t . Monotonicity in t is a trivial consequence of the fact that the $\int_0^T e^{-\kappa s} [(1 - K^{\frac{1-\gamma}{\gamma}})U_1^*(X_s) + 1] ds$ is independent of t .

Monotonicity in m . When $\gamma < 1$, we find that $\rho + 1 = \frac{1}{1-\gamma} > 0$, thus $m \mapsto X_t$ is increasing $\widehat{\mathbb{P}}$ -a.s. for all $t \in [0, T]$ by uniqueness of the trajectories. As a consequence, $m \mapsto \widehat{J}(t, x, m)$ is non-decreasing for all $(t, x) \in [0, T] \times \mathbb{R}_+$.

When $\gamma > 1$, we find that $\rho + 1 = \frac{1}{1-\gamma} < 0$, thus $m \mapsto X_t$ is decreasing $\widehat{\mathbb{P}}$ -a.s. for all $t \in [0, T]$ by uniqueness of the trajectories, and $m \mapsto \widehat{J}(t, x, m)$ is therefore non-decreasing for all $(t, x) \in [0, T] \times \mathbb{R}_+$. □

A.2. Proof of Proposition 4.2.

Proof. The proof is divided into two steps.

Step 1: Here we show that $\widehat{J}(t, x, \cdot)$ is locally-Lipschitz and (4.19) holds for a.e. $m \in \mathbb{R}_+$ and each given $(t, x) \in [0, T] \times \mathbb{R}_+$ (with the null set where $\widehat{J}(t, x, \cdot)$ is not differentiable being a priori dependent on (t, x)). Similar arguments, that we omit for brevity, also show that \widehat{J} is locally Lipschitz in x .

First we obtain bounds for the left and right derivatives of $\widehat{J}(t, x, \cdot)$. Fix $(t, x, m) \in \mathcal{U}$, pick $\epsilon > 0$, and notice that $\tau^* := \tau^*(t, x, m)$ is suboptimal for $\widehat{J}(t, x, m + \epsilon)$ (and independent of ϵ). Then, recalling (4.10) and (4.11) we obtain

$$\begin{aligned}
& \widehat{J}(t, x, m + \epsilon) - \widehat{J}(t, x, m) \\
& \geq \widehat{\mathbb{E}} \left[\int_0^{\tau^*} e^{-\kappa s} (1 - K^{\frac{1-\gamma}{\gamma}}) \frac{\gamma}{1-\gamma} \left[(X_s^{x, m+\epsilon})^{\frac{\gamma-1}{\gamma}} - (X_s^{x, m})^{\frac{\gamma-1}{\gamma}} \right] ds \right] \\
& = \epsilon \widehat{\mathbb{E}} \left[\int_0^{\tau^*} e^{-\kappa s} (1 - K^{\frac{1-\gamma}{\gamma}}) \frac{\gamma}{1-\gamma} \left[\frac{(X_s^{x, m+\epsilon})^{\frac{\gamma-1}{\gamma}} - (X_s^{x, m})^{\frac{\gamma-1}{\gamma}}}{\epsilon} \right] ds \right] \\
& = \epsilon \widehat{\mathbb{E}} \left[\int_0^{\tau^*} e^{-\kappa s} (1 - K^{\frac{1-\gamma}{\gamma}}) \frac{\gamma}{1-\gamma} H_s x^{\frac{\gamma-1}{\gamma}} \left[\frac{N(s, m + \epsilon) - N(s, m)}{\epsilon} \right] ds \right] \\
\text{(A.1)} \quad & = \epsilon \widehat{\mathbb{E}} \left[\int_0^{\tau^*} e^{-\kappa s} (1 - K^{\frac{1-\gamma}{\gamma}}) \frac{\gamma}{1-\gamma} H_s x^{\frac{\gamma-1}{\gamma}} \frac{\partial N(s, m)}{\partial m} \Big|_{m=m_\epsilon} ds \right],
\end{aligned}$$

for some $m_\epsilon \in (m, m + \epsilon)$, where the last step has used the mean value theorem. Dividing (A.1) by ϵ and taking limits as $\epsilon \downarrow 0$ gives

$$\liminf_{\epsilon \rightarrow 0} \frac{\widehat{J}(t, x, m + \epsilon) - \widehat{J}(t, x, m)}{\epsilon} = \widehat{\mathbb{E}} \left[\int_0^{\tau^*} e^{-\kappa s} (1 - K^{\frac{1-\gamma}{\gamma}}) \frac{\gamma}{1-\gamma} H_s x^{\frac{\gamma-1}{\gamma}} \frac{\partial N(s, m)}{\partial m} ds \right].$$

Since symmetric arguments applied to $\widehat{J}(t, x, m) - \widehat{J}(t, x, m - \epsilon)$ lead to the reverse inequality, we obtain

$$\limsup_{\epsilon \rightarrow 0} \frac{\widehat{J}(t, x, m) - \widehat{J}(t, x, m - \epsilon)}{\epsilon} = \widehat{\mathbb{E}} \left[\int_0^{\tau^*} e^{-\kappa s} (1 - K^{\frac{1-\gamma}{\gamma}}) \frac{\gamma}{1-\gamma} H_s x^{\frac{\gamma-1}{\gamma}} \frac{\partial N(s, m)}{\partial m} ds \right].$$

It now remains to show that $\widehat{J}(t, x, \cdot)$ is locally Lipschitz, so that a.e. $m \in \mathbb{R}_+$ is a point of differentiability. With the same notation as above, let $\tau_\epsilon^* := \tau^*(t, x, m + \epsilon)$ be optimal for the problem with initial data $(t, x, m + \epsilon)$. By arguments analogous to those used previously we

find

$$\begin{aligned}
& \widehat{J}(t, x, m + \epsilon) - \widehat{J}(t, x, m) \\
& \leq \widehat{\mathbb{E}} \left[\int_0^{\tau_\epsilon^*} e^{-\kappa s} (1 - K^{\frac{1-\gamma}{\gamma}}) \frac{\gamma}{1-\gamma} H_s x^{\frac{\gamma-1}{\gamma}} \left[N(s, m + \epsilon) - N(s, m) \right] ds \right] \\
\text{(A.2)} \quad & = \widetilde{\mathbb{E}} \left[\int_0^{\tau_\epsilon^*} e^{-\kappa s} (1 - K^{\frac{1-\gamma}{\gamma}}) \frac{\gamma}{1-\gamma} x^{\frac{\gamma-1}{\gamma}} \left[N(s, m + \epsilon) - N(s, m) \right] ds \right].
\end{aligned}$$

Then, by the Hölder inequality, we can write from (A.2)

$$\begin{aligned}
& \widehat{J}(t, x, m + \epsilon) - \widehat{J}(t, x, m) \\
\text{(A.3)} \quad & \leq \widetilde{\mathbb{E}} \left[\int_0^T \left| e^{-\kappa s} (1 - K^{\frac{1-\gamma}{\gamma}}) \frac{\gamma}{1-\gamma} x^{\frac{\gamma-1}{\gamma}} \right|^2 ds \right]^{\frac{1}{2}} \widetilde{\mathbb{E}} \left[\int_0^T \left| N(s, m + \epsilon) - N(s, m) \right|^2 ds \right]^{\frac{1}{2}}
\end{aligned}$$

Clearly, $\widetilde{\mathbb{E}}[\int_0^T |e^{-\kappa s} (1 - K^{\frac{1-\gamma}{\gamma}}) \frac{\gamma}{1-\gamma} x^{\frac{\gamma-1}{\gamma}}|^2 ds]^{\frac{1}{2}} \leq c(x)$, and because $N(t, \cdot)$ is continuously differentiable (cf. (4.11)), there exists a positive function $c(t, m)$ such that, for any $\epsilon > 0$, $|N(t, m + \epsilon) - N(t, m)| \leq c(t, m)\epsilon$.

Therefore, from (A.3) we have

$$\widehat{J}(t, x, m + \epsilon) - \widehat{J}(t, x, m) \leq c(t, x, m)\epsilon.$$

The estimate in (A.2) implies $|\widehat{J}(t, x, m + \epsilon) - \widehat{J}(t, x, m)| \leq \widehat{c}(t, x, m)\epsilon$, for some other constant $\widehat{c}(t, x, m) > 0$ which can be taken uniform over compact sets. Symmetric arguments allow also to prove that $|\widehat{J}(t, x, m) - \widehat{J}(t, x, m - \epsilon)| \leq \widehat{c}(t, x, m)\epsilon$. Therefore, $\widehat{J}(t, x, \cdot)$ is locally-Lipschitz and (4.19) holds for almost all $(t, x, m) \in \mathcal{U}$.

Step 2: We borrow arguments from the proof of Proposition 3.4 in De Angelis and Stabile [13]. From the monotonicity of \widehat{J} in t (cf. Proposition 4.2), we know that $\widehat{J}(t + \epsilon, x, m) - \widehat{J}(t, x, m) \leq 0$, for all $\epsilon > 0$ such that $t + \epsilon \leq T$. Next we show that for all $\delta > 0$, $(x, m) \in \mathbb{R}_+^2$ and any $t \in [0, T - \delta]$, we have

$$\text{(A.4)} \quad |\widehat{J}(t \pm \epsilon, x, m) - \widehat{J}(t, x, m)| \leq c_\delta \epsilon$$

for some $c_\delta > 0$ only depending on δ , and for all $\epsilon \leq T - t$. Let $\tau^* = \tau^*(t, x, m)$ be optimal in $\widehat{J}(t, x, m)$ and define $\nu_\epsilon := \tau^* \wedge (T - t - \epsilon)$ for $\epsilon > 0$. Since ν_ϵ is admissible and suboptimal for $\widehat{J}(t + \epsilon, x, m)$, we get

$$\begin{aligned}
& \widehat{J}(t + \epsilon, x, m) - \widehat{J}(t, x, m) \\
& \geq \widehat{\mathbb{E}}_{x, m} \left[\int_0^{\nu_\epsilon} e^{-\kappa s} \left[(1 - K^{\frac{1-\gamma}{\gamma}}) U_1^*(X_s) + 1 \right] ds \right] - \widehat{\mathbb{E}}_{x, m} \left[\int_0^{\tau^*} e^{-\kappa s} \left[(1 - K^{\frac{1-\gamma}{\gamma}}) U_1^*(X_s) + 1 \right] ds \right]
\end{aligned}$$

$$\begin{aligned}
&= -\widehat{\mathbb{E}}_{x,m} \left[\int_{\nu_\epsilon}^{\tau^*} e^{-\kappa s} \left[(1 - K^{\frac{1-\gamma}{\gamma}}) U_1^*(X_s) + 1 \right] ds \right] \\
\text{(A.5)} \quad &= -\widehat{\mathbb{E}}_{x,m} \left[e^{-\kappa \xi} \left[(1 - K^{\frac{1-\gamma}{\gamma}}) U_1^*(X_\xi) + 1 \right] (\tau^* - \nu_\epsilon) \right]
\end{aligned}$$

for $\xi(\omega) \in (\nu_\epsilon(\omega), \tau^*(\omega))$, $\omega \in \Omega$, as we have used the mean value theorem in the last equality. Furthermore,

$$\left| e^{-\kappa s} \left[(1 - K^{\frac{1-\gamma}{\gamma}}) U_1^*(X_s) + 1 \right] \right| = \left| e^{-\kappa s} \left((1 - K^{\frac{1-\gamma}{\gamma}}) \frac{\gamma}{1-\gamma} X_s^{\frac{\gamma-1}{\gamma}} + 1 \right) \right| \leq C X_s^{\frac{\gamma-1}{\gamma}} + 1$$

with a uniform constant $C > 0$, and

$$0 \leq \tau^* - \nu_\epsilon \leq \epsilon \mathbf{1}_{\{\tau^* \geq T-t-\epsilon\}}.$$

Hence, exploiting the last two display equations in (A.5), one obtains the bound

$$\left| e^{-\kappa \xi} \left[(1 - K^{\frac{1-\gamma}{\gamma}}) U_1^*(X_\xi) + 1 \right] (\tau^* - \nu_\epsilon) \right| \leq (C X_\xi^{\frac{\gamma-1}{\gamma}} + 1) \epsilon \mathbf{1}_{\{\tau^* \geq T-t-\epsilon\}}.$$

In conclusion, noticing that $\nu_\epsilon \leq \tau^*$, and recalling (4.9), (4.10) and (4.11) we have

$$\begin{aligned}
\widehat{J}(t+\epsilon, x, m) - \widehat{J}(t, x, m) &\geq -\widehat{\mathbb{E}}_{x,m} \left[(C X_\xi^{\frac{\gamma-1}{\gamma}} + 1) \epsilon \mathbf{1}_{\{\tau^* \geq T-t-\epsilon\}} \right] \\
&= -\widehat{\mathbb{E}}_{x,m} \left[C x^{\frac{\gamma-1}{\gamma}} H_\xi N(\xi, m) \epsilon \mathbf{1}_{\{\tau^* \geq T-t-\epsilon\}} \right] - \widehat{\mathbb{E}}_{x,m} \left[\epsilon \mathbf{1}_{\{\tau^* \geq T-t-\epsilon\}} \right] \\
&\geq -C x^{\frac{\gamma-1}{\gamma}} \widetilde{\mathbb{E}}_{x,m} \left[\epsilon \mathbf{1}_{\{\tau^* \geq T-t-\epsilon\}} \right] - \epsilon \widehat{\mathbb{P}}[\tau^* \geq T-t-\epsilon] \\
\text{(A.6)} \quad &= -C x^{\frac{\gamma-1}{\gamma}} \widetilde{\mathbb{P}}[\tau^* \geq T-t-\epsilon] - \epsilon \widehat{\mathbb{P}}[\tau^* \geq T-t-\epsilon]
\end{aligned}$$

for a different constant $C > 0$ since $N(s, m)$ is bounded (cf. (4.11)). Using the Markov inequality, we obtain

$$\widehat{\mathbb{P}}[\tau^* \geq T-t-\epsilon] \leq \frac{\widehat{\mathbb{E}}[\tau^*]}{T-t-\epsilon}, \quad \widetilde{\mathbb{P}}[\tau^* \geq T-t-\epsilon] \leq \frac{\widetilde{\mathbb{E}}[\tau^*]}{T-t-\epsilon},$$

which plugged back into (A.6) give

$$\text{(A.7)} \quad \widehat{J}(t+\epsilon, x, m) - \widehat{J}(t, x, m) \geq -C x^{\frac{\gamma-1}{\gamma}} \epsilon \frac{\widetilde{\mathbb{E}}[\tau^*]}{T-t-\epsilon} - \epsilon \frac{\widehat{\mathbb{E}}[\tau^*]}{T-t-\epsilon}.$$

Equation (A.7) implies (A.4) and, moreover, allow us to conclude that \widehat{J} is locally Lipschitz a.e. in $[0, T)$. Let $(t, x, m) \in [0, T) \times \mathbb{R}_+^2$ be a point of differentiability of \widehat{J} . Dividing (A.7) by ϵ and letting $\epsilon \rightarrow 0$, we obtain the lower bound in (4.20). □

A.3. Proof of Proposition 4.3.

Proof. Taking $\tau = T - t$, for any $\gamma \neq 1$ we have,

$$\widehat{J}(t, x, m) \geq \widehat{\mathbb{E}}_{x, m} \left[\int_0^{T-t} e^{-\kappa s} \left[(1 - K^{\frac{1-\gamma}{\gamma}}) U_1^*(X_s) + 1 \right] ds \right].$$

On the other hand, we have $\widehat{J}(t, x, m) \leq \frac{1}{\kappa}(1 - e^{-\kappa(T-t)})$ from Proposition 4.1. Hence, overall,

$$\widehat{\mathbb{E}}_{x, m} \left[\int_0^{T-t} e^{-\kappa s} \left[(1 - K^{\frac{1-\gamma}{\gamma}}) U_1^*(X_s) + 1 \right] ds \right] \leq \widehat{J}(t, x, m) \leq \frac{1}{\kappa} (1 - e^{-\kappa(T-t)}).$$

Then, recalling that $U_1^*(x) = \frac{\gamma}{1-\gamma} x^{\frac{\gamma-1}{\gamma}}$ (cf. (4.3)), we have

$$\lim_{x \downarrow 0} \widehat{J}(t, x, m) = \frac{1}{\kappa} (1 - e^{-\kappa(T-t)}), \quad \forall \gamma > 1,$$

and

$$\lim_{x \rightarrow +\infty} \widehat{J}(t, x, m) = \frac{1}{\kappa} (1 - e^{-\kappa(T-t)}), \quad \forall \gamma < 1.$$

Next we prove $\lim_{x \rightarrow \infty} \widehat{J}(t, x, m) = 0$ when $\gamma > 1$. The proof of $\lim_{x \rightarrow 0} \widehat{J}(t, x, m) = 0$ when $\gamma < 1$ is similar, and we thus omit it. We notice that $\lim_{x \uparrow \infty} \tau^*(t, x, m)$ exists a.s. by monotonicity of $x \mapsto \tau^*(t, x, m)$, which is inherited by that of $x \mapsto \widehat{J}(t, x, m)$. We want to show that $\lim_{x \uparrow \infty} \tau^*(t, x, m) = 0$ a.s., which then implies $\lim_{x \uparrow \infty} \widehat{J}(t, x, m) = 0$. By arguing by contradiction, suppose that $A(t, m) := \{\omega \in \Omega : \lim_{x \uparrow \infty} \tau^*(t, x, m) > 0\}$ has positive probability. Let $\tau_\infty^*(t, m) := \lim_{x \rightarrow \infty} \tau^*(t, x, m)$. Then, if $\gamma > 1$, by dominated convergence

$$\begin{aligned} \lim_{x \rightarrow \infty} \widehat{J}(t, x, m) &= \lim_{x \rightarrow \infty} \widehat{\mathbb{E}}_{x, m} \left[\int_0^{\tau^*(t, x, m)} e^{-\kappa s} \left[(1 - K^{\frac{1-\gamma}{\gamma}}) U_1^*(X_s) + 1 \right] ds \right] \\ &\leq \lim_{x \rightarrow \infty} \widehat{\mathbb{E}}_{x, m} \left[\mathbf{1}_{A(t, m)} \int_0^{\tau^*(t, x, m)} e^{-\kappa s} \left[(1 - K^{\frac{1-\gamma}{\gamma}}) U_1^*(X_s) + 1 \right] ds \right] \\ &\quad + \lim_{x \rightarrow \infty} \widehat{\mathbb{E}}_{x, m} \left[\mathbf{1}_{A^c(t, m)} \int_0^{\tau^*(t, x, m)} e^{-\kappa s} \left[(1 - K^{\frac{1-\gamma}{\gamma}}) U_1^*(X_s) + 1 \right] ds \right] \\ &= -\infty, \end{aligned}$$

But $\widehat{J}(t, x, m) \geq 0$ (cf. Proposition 4.1), therefore it must be $\tau_\infty^*(t, m) = 0$ a.s. We then have $\lim_{x \rightarrow \infty} \widehat{J}(t, x, m) = 0$ when $\gamma > 1$. □

A.4. Proof of Lemma 4.3.

Proof. Step 1: Here we prove (4.21). Equation (4.22) can be proved by similar arguments. Since $x \mapsto \widehat{J}(t, x, m)$ is non-decreasing by Proposition 4.1, we can define $b(t, m) := \sup\{x > 0 :$

$\widehat{J}(t, x, m) \leq 0\}$ (with the convention $\sup \emptyset = 0$), so that $\mathcal{I} = \{(t, x, m) \in [0, T] \times \mathbb{R}_+^2 : 0 < x \leq b(t, m)\}$. Notice that actually $b > 0$ on $[0, T] \times \mathbb{R}_+$ since $\mathcal{I} \neq \emptyset$ by Lemma 4.2.

Step 2: Here we prove the property (i) for the case $\gamma < 1$. Analogous considerations allow to prove it when $\gamma > 1$. We know that $t \rightarrow \widehat{J}(t, x, m)$ is non-increasing by Proposition 4.1. Therefore, for $t \geq 0, s \leq T - t$, we have

$$b(t, m) = \sup\{x > 0 : \widehat{J}(t, x, m) \leq 0\} \leq \sup\{x > 0 : \widehat{J}(t + s, x, m) \leq 0\} = b(t + s, m).$$

Step 3: Here we prove property (ii), again only in the case $\gamma < 1$. Since $m \rightarrow \widehat{J}(t, x, m)$ is non-decreasing by Proposition 4.1, for $m > 0, h > 0$, we have

$$b(t, m) = \sup\{x > 0 : \widehat{J}(t, x, m) \leq 0\} \geq \sup\{x > 0 : \widehat{J}(t, x, m + h) \leq 0\} = b(t, m + h),$$

which implies that $m \mapsto b(t, m)$ is non-increasing when $\gamma < 1$.

Step 4: Here we provide the proof of property (iii), i.e., the bounds of the boundary. Noticing that, due to (4.16),

$$\mathcal{R} := \{(t, x, m) \in \mathcal{U} : (1 - K^{\frac{1-\gamma}{\gamma}})U_1^*(x) + 1 > 0\} \subseteq \mathcal{W},$$

we have

$$(A.8) \quad \mathcal{R}^C := \{(t, x, m) \in \mathcal{U} : (1 - K^{\frac{1-\gamma}{\gamma}})U_1^*(x) + 1 \leq 0\} \supseteq \mathcal{I}.$$

We first show $b(t, m) \leq L := \left[(K^{\frac{1-\gamma}{\gamma}} - 1) \frac{\gamma}{1-\gamma} \right]^{\frac{\gamma}{1-\gamma}}$ for $\gamma < 1$. Recalling $U_1^*(x)$ as in (4.3), one has

$$(1 - K^{\frac{1-\gamma}{\gamma}})U_1^*(x) + 1 \leq 0 \iff x^{\frac{\gamma-1}{\gamma}} \geq \frac{1}{(K^{\frac{1-\gamma}{\gamma}} - 1) \frac{\gamma}{1-\gamma}}$$

for all $(t, m) \in [0, T] \times \mathbb{R}_+$; that is,

$$(t, x, m) \in \mathcal{R}^C \iff x \leq \left[(K^{\frac{1-\gamma}{\gamma}} - 1) \frac{\gamma}{1-\gamma} \right]^{\frac{\gamma}{1-\gamma}}.$$

Because $\mathcal{I} = \{(t, x, m) \in \mathcal{U} : 0 < x \leq b(t, m)\}$ (cf. (4.21)), (A.8) and the latter equation imply

$$b(t, m) \leq \left[(K^{\frac{1-\gamma}{\gamma}} - 1) \frac{\gamma}{1-\gamma} \right]^{\frac{\gamma}{1-\gamma}}, \quad \text{for } (t, m) \in [0, T] \times \mathbb{R}_+.$$

Analogous arguments, now using $\gamma > 1$, allow to obtain

$$b(t, m) \geq \left[\left(K^{\frac{1-\gamma}{\gamma}} - 1 \right) \frac{\gamma}{1-\gamma} \right]^{\frac{\gamma}{1-\gamma}}, \quad \text{for } (t, m) \in [0, T] \times \mathbb{R}_+.$$

To prove that $b(t, m) \leq b_\infty(m) < \infty$ when $\gamma > 1$, we observe that

$$b(t, m) = \sup\{x > 0 : \widehat{J}(t, x, m) > 0\} \leq \sup\{x > 0 : \widehat{J}_\infty(x, m) > 0\} := b_\infty(m)$$

due to $\widehat{J}(t, x, m) \leq \widehat{J}_\infty(x, m)$. Moreover, $m \mapsto b_\infty(m)$ is non-decreasing because $m \mapsto \widehat{J}_\infty(x, m)$ is non-decreasing. Given that $m \mapsto b_\infty(m)$ is non-decreasing, in order to show that $b_\infty(m) < \infty$ for all $m \in \mathbb{R}_+$ we proceed as follows. Suppose that $\exists m_0 \in \mathbb{R}_+$ such that $b_\infty(m_0) = +\infty$. Then, by monotonicity of b_∞ , $b_\infty = +\infty$ on (m_0, ∞) . That is, $\forall (x, m) \in \mathbb{R}_+^2$ such that $m > m_0$ we have $\widehat{J}_\infty(x, m) > 0$ and no-stopping is optimal; that is,

$$0 < \widehat{\mathbb{E}}_{x, m} \left[\int_0^\infty e^{-\kappa s} \left[(1 - K^{\frac{1-\gamma}{\gamma}}) U_1^*(X_s) + 1 \right] ds \right],$$

which is equivalent to

$$\widehat{\mathbb{E}}_{x, m} \left[\int_0^\infty e^{-\kappa s} (1 - K^{\frac{1-\gamma}{\gamma}}) U_1^*(X_s) ds \right] > -\frac{1}{\kappa}.$$

However, taking $x \uparrow \infty$ the right-hand side of the previous inequalities converges monotonically to $-\infty$. Hence, a contradiction. \square

A.5. Proof of Theorem 4.1.

Proof. The proof is organized in five steps.

Step 1. For $\epsilon > 0$, define the function

$$F^\epsilon(t, x, m) := \widehat{J}(t, x, m) - \epsilon.$$

Let now $(t, x, m) \in \mathcal{W}$, $\lambda^\epsilon, L_1^\epsilon, L_2^\epsilon \geq 0$ (possibly depending on (t, x, m)), and, for $u \in \mathbb{R}$, denote by $B_\delta(u) := \{u' \in \mathbb{R} : |u' - u| < \delta\}$, $\delta > 0$. Since F^ϵ is locally-Lipschitz continuous in \mathcal{U} (cf. Proposition 4.2), if the following conditions are satisfied

- (i) $F^\epsilon(t, x, m) = 0$;
- (ii) $\|F_x^\epsilon(t, x, m)\|_\infty^{-1} < \lambda^\epsilon$;
- (iii) $\|F_t^\epsilon(B_\delta(t) \times B_\delta(x) \times B_\delta(m))\|_\infty \leq L_1^\epsilon$ and $\|F_m^\epsilon(B_\delta(t) \times B_\delta(x) \times B_\delta(m))\|_\infty \leq L_2^\epsilon$,

then a version of the implicit function theorem (see, e.g., the Corollary at p.256 in Clarke [8] or Theorem 3.1 in Papi [31]) implies that, for suitable $\delta' > 0$, there exists a unique continuous

function $b_\epsilon(t, m) : (t - \delta', t + \delta') \times (m - \delta', m + \delta') \mapsto (x - \delta', x + \delta')$ such that

$$\widehat{J}(t, b_\epsilon(t, m), m) = \epsilon \quad \text{in } (t - \delta', t + \delta') \times (m - \delta', m + \delta'),$$

and also

$$(A.9) \quad \begin{aligned} |b_\epsilon(t_1, m) - b_\epsilon(t_2, m)| &\leq \lambda^\epsilon L_1^\epsilon |t_1 - t_2|, \quad \forall t_1, t_2 \in (t - \delta', t + \delta'), \\ |b_\epsilon(t, m_1) - b_\epsilon(t, m_2)| &\leq \lambda^\epsilon L_2^\epsilon |m_1 - m_2|, \quad \forall m_1, m_2 \in (m - \delta', m + \delta'). \end{aligned}$$

According to Proposition 4.2, when $\gamma < 1$ we have $\widehat{J}_x(t, x, m) > 0$ for a.e. x inside \mathcal{W} . Then, by Propositions 4.1 and 4.3 it clearly follows that such a b_ϵ above indeed exists, and also $b_\epsilon(t, m) > b(t, m) > 0$. When $\gamma > 1$ we have $\widehat{J}_x(t, x, m) < 0$ for a.e. x inside \mathcal{W} . Then, by Propositions 4.1 and 4.3 it clearly follows that such a b_ϵ above indeed exists, and also $0 \leq b_\epsilon(t, m) < b(t, m)$.

Moreover, when $\gamma < 1$ the family $(b_\epsilon)_{\epsilon > 0}$ decreases as $\epsilon \rightarrow 0$, so that its limit b_0 exists. Such a limit is such that the mapping $(t, m) \mapsto b_0(t, m)$, is upper semicontinuous, as decreasing limit of continuous functions, and $b_0(t, m) \geq b(t, m)$. Since $\widehat{J}(t, b_\epsilon(t, m), m) = \epsilon$, it is clear that taking limits as $\epsilon \rightarrow 0$, we get $\widehat{J}(t, b_0(t, m), m) = 0$ by continuity of \widehat{J} (cf. Proposition 4.2), and therefore $b_0(t, m) \leq b(t, m)$ due to the definition of the stopping region \mathcal{I} in (4.21). Hence,

$$(A.10) \quad \lim_{\epsilon \rightarrow 0} b_\epsilon(t, m) = b(t, m), \quad \text{for all } (t, m) \in [0, T] \times \mathbb{R}_+.$$

Arguing symmetrically, we also have that (A.10) holds true for $\gamma > 1$.

Step 2. We here prove that $b_\epsilon(t, m)$ is bounded uniformly in ϵ . Clearly, we can restrict the attention to $\epsilon \in (0, \epsilon_0)$ for some $\epsilon_0 > 0$. From Lemma 4.3, we know that, when $\gamma < 1$, we have $b(t, m) \leq L$. Since now $\lim_{\epsilon \rightarrow 0} b_\epsilon(t, m) = b(t, m)$ (cf. (A.10)), we thus have that $0 \leq b_\epsilon(t, m) \leq 1 + L, \forall \epsilon \in (0, \epsilon_0)$, which provides the desired uniform bound. When $\gamma > 1$, we have $b(t, m) \leq b_\infty(m)$. Hence, similarly, we have that $L \leq b_\epsilon(t, m) \leq 1 + b_\infty(m), \forall \epsilon \in (0, \epsilon_0)$, which provides the desired uniform bound.

Step 3. Here we show that b_ϵ is locally-Lipshitz continuous in m , uniformly with respect to ϵ .

Step 3-(a). Here we determine an **upper bound for** $|\widehat{J}_m(t, b_\epsilon(t, m), m)|$. Recalling $\widehat{J}_m(t, x, m)$ as in Proposition 4.2, by Proposition 4.1 we have

$$0 \leq \widehat{J}_m(t, b_\epsilon(t, m), m) = (K^{\frac{1-\gamma}{\gamma}} - 1)(b_\epsilon(t, m))^{\frac{\gamma-1}{\gamma}} (\rho + 1) \widehat{\mathbb{E}}_{x, m} \left[\int_0^{T_\epsilon^*} e^{-\kappa s} e^{as} H_s N(s, m) ds \right]$$

where $\tau_\epsilon^* := \tau^*(t, b_\epsilon(t, m), m)$. Then,

$$\begin{aligned}
& |\widehat{J}_m(t, b_\epsilon(t, m), m)| \\
& \leq (K^{\frac{1-\gamma}{\gamma}} - 1)(\rho + 1)(b_\epsilon(t, m))^{\frac{\gamma-1}{\gamma}} \widehat{\mathbb{E}}_{x,m} \left[\int_0^{\tau_\epsilon^*} e^{-\kappa s} e^{as} H_s N(s, m) ds \right] \\
\text{(A.11)} \quad & \leq (K^{\frac{1-\gamma}{\gamma}} - 1)(\rho + 1)(b_\epsilon(t, m))^{\frac{\gamma-1}{\gamma}} e^{aT} \widehat{\mathbb{E}}_{x,m} \left[\int_0^{\tau_\epsilon^*} e^{-\kappa s} H_s N(s, m) ds \right] := L_2^\epsilon(t, m).
\end{aligned}$$

Step 3-(b). Here we determine a **lower bound for** $\widehat{J}_x(t, b_\epsilon(t, m), m)$. From (4.18), we have

$$\text{(A.12)} \quad \widehat{J}_x(t, b_\epsilon(t, m), m) = (K^{\frac{1-\gamma}{\gamma}} - 1)(b_\epsilon(t, m))^{\frac{-1}{\gamma}} \widehat{\mathbb{E}}_{x,m} \left[\int_0^{\tau_\epsilon^*} e^{-\kappa s} H_s N(s, m) ds \right],$$

so that from (A.12) we obtain

$$\text{(A.13)} \quad |\widehat{J}_x(t, b_\epsilon(t, m), m)| = |(K^{\frac{1-\gamma}{\gamma}} - 1)|(b_\epsilon(t, m))^{\frac{-1}{\gamma}} \widehat{\mathbb{E}}_{x,m} \left[\int_0^{\tau_\epsilon^*} H_s N(s, m) ds \right] =: \frac{1}{\lambda_1^\epsilon(t, m)}.$$

Step 3-(c). From (A.9) (with $\lambda^\epsilon = \lambda_1^\epsilon$), (A.11) and (A.13) we conclude that when $\gamma < 1$ the family of weak derivatives $(|\partial_m b_\epsilon(t, m)|)_{\epsilon \geq 0}$ is uniformly bounded; i.e.,

$$\begin{aligned}
& \sup_{\epsilon \geq 0} |\partial_m b_\epsilon(t, m)| \leq \sup_{\epsilon \geq 0} (\lambda_1^\epsilon(t, m) L_2^\epsilon(t, m)) \\
\text{(A.14)} \quad & = |\rho + 1| b_\epsilon(t, m) e^{aT} \leq |\rho + 1| (1 + L) e^{aT}.
\end{aligned}$$

Similarly, when $\gamma > 1$ the family of weak derivatives $(|\partial_m b_\epsilon(t, m)|)_{\epsilon \geq 0}$ is uniformly bounded; i.e.,

$$\text{(A.15)} \quad \sup_{\epsilon \geq 0} |\partial_m b_\epsilon(t, m)| \leq |\rho + 1| (1 + b_\infty(m)) e^{aT}$$

since $b_\epsilon(t, m) \leq 1 + b_\infty(m)$ by Step 2.

Step 4. We show that b_ϵ is locally-Lipschitz continuous in t , uniformly with respect to ϵ .

Step 4-(a). Here we find an **upper bound for** $|\widehat{J}_t(t, b_\epsilon(t, m), m)|$. Recalling that

$-\frac{1}{T-t}(Cx^{\frac{\gamma-1}{\gamma}} \widetilde{\mathbb{E}}[\tau^*] + \widehat{\mathbb{E}}[\tau^*]) \leq \widehat{J}_t(t, x, m) \leq 0$ by (4.20), one has

$$\text{(A.16)} \quad |\widehat{J}_t(t, b_\epsilon(t, m), m)| \leq \frac{1}{T-t} (Cb_\epsilon(t, m)^{\frac{\gamma-1}{\gamma}} \widetilde{\mathbb{E}}[\tau_\epsilon^*] + \widehat{\mathbb{E}}[\tau_\epsilon^*]) := L_1^\epsilon(t, m).$$

In the sequel a change of probability measures through a Girsanov argument will be needed in order to take care of the expectation on the right-hand side of (A.16). To this end, recall the

probability measure $\tilde{\mathbb{P}}$ on (Ω, \mathcal{F}_T) defined as

$$(A.17) \quad \frac{d\tilde{\mathbb{P}}}{d\widehat{\mathbb{P}}} = \exp \left\{ \frac{\gamma-1}{\gamma} \sigma_1 \widehat{B}_T - \frac{1}{2} \left(\frac{\gamma-1}{\gamma} \right)^2 \sigma_1^2 T \right\}.$$

Then, by Girsanov's Theorem, the process $\tilde{B} := \{\widehat{B}_s - \frac{\gamma-1}{\gamma} \sigma_1 s, s \in [0, T]\}$ is a standard Brownian motion under the new measure $\tilde{\mathbb{P}}$.

Step 4-(b). We here determine another **lower bound** for \widehat{J}_x . From (A.13) and recalling that $N(s, m) \geq \underline{L}(m) > 0$, we have by a change of measure (cf. (A.17))

$$(A.18) \quad \begin{aligned} |\widehat{J}_x(t, b_\epsilon(t, m), m)| &= |1 - K^{\frac{1-\gamma}{\gamma}} |(b_\epsilon(t, m))^{-\frac{1}{\gamma}} \widehat{\mathbb{E}}_{x,m} \left[\int_0^{\tau_\epsilon^*} e^{-\kappa s} H_s N(s, m) ds \right]| \\ &= |1 - K^{\frac{1-\gamma}{\gamma}} |(b_\epsilon(t, m))^{-\frac{1}{\gamma}} \tilde{\mathbb{E}}_{x,m} \left[\int_0^{\tau_\epsilon^*} e^{-\kappa s} N(s, m) ds \right]| \\ &\geq |1 - K^{\frac{1-\gamma}{\gamma}} |(b_\epsilon(t, m))^{-\frac{1}{\gamma}} \tilde{\mathbb{E}}_{x,m} \left[\int_0^{\tau_\epsilon^*} e^{-\kappa T} \underline{L}(m) ds \right]| \\ &\geq C_1 |1 - K^{\frac{1-\gamma}{\gamma}} |(b_\epsilon(t, m))^{-\frac{1}{\gamma}} \tilde{\mathbb{E}}_{x,m}[\tau_\epsilon^*] := \frac{1}{\lambda_2^\epsilon(t, m)}, \end{aligned}$$

where $C_1 := e^{-\kappa T} \underline{L}(m)$.

Step 4-(c). Our aim here is to find a bound of $\frac{\widehat{\mathbb{E}}[\tau_\epsilon^*]}{\widehat{\mathbb{E}}[\tau_\epsilon^*]}$, uniformly with respect to ϵ when $\gamma > 1$. This term arises from the ratio of the two expectations in (A.16) and (A.18).

Since the dynamics of X_s^ϵ under $\widehat{\mathbb{P}}$ are

$$dX_s^\epsilon = X_s^\epsilon [(\rho + 1)(\beta - r + M_s) + \mu_1] ds + X_s^\epsilon \sigma_1 d\widehat{B}_s, \quad X_t^\epsilon = b_\epsilon(t, m),$$

those become under $\tilde{\mathbb{P}}$ (remember that $\tilde{B} := \{\widehat{B}_s - \frac{\gamma-1}{\gamma} \sigma_1 s, s \in [0, T]\}$)

$$dX_s^\epsilon = X_s^\epsilon [(\rho + 1)(\beta - r + M_s) + \mu_1 + \frac{\gamma-1}{\gamma} \sigma_1^2] ds + X_s^\epsilon \sigma_1 d\tilde{B}_s, \quad X_t^\epsilon = b_\epsilon(t, m).$$

Now, if on $(\Omega, \mathcal{F}, \widehat{\mathbb{P}})$ we define

$$d\tilde{X}_s^\epsilon = \tilde{X}_s^\epsilon [(\rho + 1)(\beta - r + M_s) + \mu_1 + \frac{\gamma-1}{\gamma} \sigma_1^2] ds + \tilde{X}_s^\epsilon \sigma_1 d\tilde{B}_s, \quad \tilde{X}_t^\epsilon = b_\epsilon(t, m).$$

and $\tilde{\tau}_\epsilon^* := \inf\{s \in [0, T-t] : (t+s, \tilde{X}_s^\epsilon, M_s) \in \mathcal{I}\}$, then we see that

$$\text{Law}(X_s^\epsilon | \widehat{\mathbb{P}}) = \text{Law}(\tilde{X}_s^\epsilon | \widehat{\mathbb{P}}), \quad \text{Law}(\tau_\epsilon^* | \widehat{\mathbb{P}}) = \text{Law}(\tilde{\tau}_\epsilon^* | \widehat{\mathbb{P}}),$$

where $\tau_\epsilon^* := \inf\{s \in [0, T-t] : (t+s, X_s^\epsilon, M_s) \in \mathcal{I}\}$. Moreover, by the comparison principles for SDEs, we have that $X_s^\epsilon \leq \tilde{X}_s^\epsilon, \widehat{\mathbb{P}}$ -a.s., for all $s \in [0, T-t]$ since $\gamma > 1$, and, therefore, we have

$\tau_\epsilon^* \leq \tilde{\tau}_\epsilon^*$, $\widehat{\mathbb{P}}$ -a.s., and

$$(A.19) \quad \tilde{\mathbb{E}}[\tau_\epsilon^*] = \widehat{\mathbb{E}}[\tilde{\tau}_\epsilon^*] \geq \widehat{\mathbb{E}}[\tau_\epsilon^*].$$

Hence, when $\gamma > 1$, combining (A.9) (with $\lambda^\epsilon = \lambda_2^\epsilon$), (A.16), (A.18) and (A.19) we find

$$(A.20) \quad \begin{aligned} \sup_{\epsilon \geq 0} |\partial_t b_\epsilon(t, m)| &\leq \sup_{\epsilon \geq 0} (\lambda_2^\epsilon(t, m) L_1^\epsilon(t, m)) \\ &\leq \frac{\frac{1}{T-t} (C b_\epsilon(t, m))^{\frac{\gamma-1}{\gamma}} \tilde{\mathbb{E}}[\tau_\epsilon^*] + \widehat{\mathbb{E}}[\tau_\epsilon^*]}{C_1 (1 - K^{\frac{1-\gamma}{\gamma}}) (b_\epsilon(t, m))^{\frac{-1}{\gamma}} \tilde{\mathbb{E}}_{x, m}[\tau_\epsilon^*]} \leq \frac{C b_\epsilon(t, m)}{(T-t) C_1 (1 - K^{\frac{1-\gamma}{\gamma}})} + \frac{(b_\epsilon(t, m))^{\frac{1}{\gamma}}}{(T-t) C_1 (1 - K^{\frac{1-\gamma}{\gamma}})} \\ &\leq \frac{C b_\infty(m) + (b_\infty(m))^{\frac{1}{\gamma}}}{(T-t) C_1 (1 - K^{\frac{1-\gamma}{\gamma}})}. \end{aligned}$$

Step 4-(d). Our next task is to find a bound for the ratio $\frac{\widehat{\mathbb{E}}[\tau_\epsilon^*]}{\tilde{\mathbb{E}}[\tau_\epsilon^]}$, uniformly with respect to ϵ , when $\gamma < 1$. The following proof arguments are borrowed from De Angelis and Stabile [13]. Due to Lemma 4.3, there exists $l > 0$ such that $2l \leq L^{\frac{\gamma-1}{\gamma}}$, and we denote

$$\tau_l := \inf\{t \geq 0 : X_t^{\frac{\gamma-1}{\gamma}} \leq l\}.$$

For this part of the proof, it is convenient to think of Ω as the canonical space of continuous paths $\omega = \{\omega(t), t \geq 0\}$ and denote by ϑ_s the shifting operator $\vartheta_s \omega = \{\omega(s+t), t \geq 0\}$. Moreover, we recall that $\widehat{\mathbb{E}}_{x, m}[\cdot] = \widehat{\mathbb{E}}[\cdot | X_0 = x, M_0 = m]$ and $\tilde{\mathbb{E}}_{x, m}[\cdot] = \tilde{\mathbb{E}}[\cdot | X_0 = x, M_0 = m]$. With this notation, for fixed (x_ϵ, m) and any $s \geq 0$, we have $\widehat{\mathbb{P}}[\tau_\epsilon^* > s] = \widehat{\mathbb{P}}_{x_\epsilon, m}[\tau^* > s]$ and $\tilde{\mathbb{P}}[\tau_\epsilon^* > s] = \tilde{\mathbb{P}}_{x_\epsilon, m}[\tau^* > s]$, because

$$\tau_\epsilon^* := \tau^*(t, x_\epsilon, m) = \inf\{s \geq 0 : (t+s, X_s^{x_\epsilon}, M_s^m) \in \mathcal{I}\} \wedge (T-t), \quad \widehat{\mathbb{P}}_{x_\epsilon, m} \text{ and } \tilde{\mathbb{P}}_{x_\epsilon, m} - a.s..$$

Our first estimate gives (cf. (A.17))

$$(A.21) \quad \begin{aligned} \tilde{\mathbb{E}}[\tau_\epsilon^*] &= \tilde{\mathbb{E}}_{x_\epsilon, m}[\tau^*] = x_\epsilon^{\frac{1-\gamma}{\gamma}} \widehat{\mathbb{E}}_{x_\epsilon, m}[X_{\tau^*}^{\frac{\gamma-1}{\gamma}} N(\tau^*, m)^{-1} \tau^*] \\ &\geq c_1 x_\epsilon^{\frac{1-\gamma}{\gamma}} \widehat{\mathbb{E}}_{x_\epsilon, m}[X_{\tau^*}^{\frac{\gamma-1}{\gamma}} \tau^*] \end{aligned}$$

for a suitable constant $c_1 > 0$, independent of (t, x_ϵ) . Next we obtain

$$(A.22) \quad \begin{aligned} \widehat{\mathbb{E}}_{x_\epsilon, m}[X_{\tau^*}^{\frac{\gamma-1}{\gamma}} \tau^*] &= \widehat{\mathbb{E}}_{x_\epsilon, m}[X_{\tau^*}^{\frac{\gamma-1}{\gamma}} \tau^* (\mathbf{1}_{\{\tau^* \leq \tau_l\}} + \mathbf{1}_{\{\tau^* > \tau_l\}})] \\ &\geq \widehat{\mathbb{E}}_{x_\epsilon, m}[\tau^* \mathbf{1}_{\{\tau^* \leq \tau_l\}}] + \widehat{\mathbb{E}}_{x_\epsilon, m}[X_{\tau^*}^{\frac{\gamma-1}{\gamma}} \tau^* \mathbf{1}_{\{\tau^* > \tau_l\}}]. \end{aligned}$$

The last term can be further estimated via

$$\begin{aligned}
& \widehat{\mathbb{E}}_{x_\epsilon, m} [X_{\tau^*}^{\frac{\gamma-1}{\gamma}} \tau^* \mathbf{1}_{\{\tau^* > \tau_l\}}] = \widehat{\mathbb{E}}_{x_\epsilon, m} [X_{\tau^*}^{\frac{\gamma-1}{\gamma}} \tau^* \mathbf{1}_{\{\tau^* > \tau_l\}} (\mathbf{1}_{\{\tau^* < T-t\}} + \mathbf{1}_{\{\tau^* = T-t\}})] \\
\text{(A.23)} \quad & \geq c_2 \widehat{\mathbb{E}}_{x_\epsilon, m} [\tau^* \mathbf{1}_{\{\tau^* > \tau_l\}} \mathbf{1}_{\{\tau^* < T-t\}}] + \widehat{\mathbb{E}}_{x_\epsilon, m} [X_{\tau^*}^{\frac{\gamma-1}{\gamma}} \tau^* \mathbf{1}_{\{\tau^* > \tau_l\}} \mathbf{1}_{\{\tau^* = T-t\}}],
\end{aligned}$$

where we have used that $\{\tau^* < T-t\} \subseteq \{X_{\tau^*} \leq L\} = \{X_{\tau^*}^{\frac{\gamma-1}{\gamma}} \geq L^{\frac{\gamma-1}{\gamma}} := c_2\}$ under $\widehat{\mathbb{P}}_{x_\epsilon, m}$.

The last term in the above expression may be controlled by using iterated conditioning and the strong Markov property as

$$\begin{aligned}
& \widehat{\mathbb{E}}_{x_\epsilon, m} [X_{\tau^*}^{\frac{\gamma-1}{\gamma}} \tau^* \mathbf{1}_{\{\tau^* > \tau_l\}} \mathbf{1}_{\{\tau^* = T-t\}}] \\
& = \widehat{\mathbb{E}}_{x_\epsilon, m} [(T-t) \mathbf{1}_{\{\tau^* > \tau_l\}} \widehat{\mathbb{E}}_{x_\epsilon, m} [X_{\tau^*}^{\frac{\gamma-1}{\gamma}} \mathbf{1}_{\{\tau^* = T-t\}} | \mathcal{F}_{\tau_l}]] \\
& = \widehat{\mathbb{E}}_{x_\epsilon, m} [(T-t) \mathbf{1}_{\{\tau^* > \tau_l\}} \widehat{\mathbb{E}}_{x_\epsilon, m} [X_{\tau_l + \tau^* \circ \vartheta_{\tau_l}}^{\frac{\gamma-1}{\gamma}} \mathbf{1}_{\{\tau^* \circ \vartheta_{\tau_l} = T-t-\tau_l\}} | \mathcal{F}_{\tau_l}]] \\
& = \widehat{\mathbb{E}}_{x_\epsilon, m} [(T-t) \mathbf{1}_{\{\tau^* > \tau_l\}} \widehat{\mathbb{E}}_{X_{\tau_l}, M_{\tau_l}} [X_{\tau^*}^{\frac{\gamma-1}{\gamma}} \mathbf{1}_{\{\tau^* = T-t\}}]] \\
& \geq \widehat{\mathbb{E}}_{x_\epsilon, m} [(T-t) \mathbf{1}_{\{\tau^* > \tau_l\}} \mathbf{1}_{\{\tau^* = T-t\}} \widehat{\mathbb{E}}_{X_{\tau_l}, M_{\tau_l}} [X_{\tau^*}^{\frac{\gamma-1}{\gamma}} \mathbf{1}_{\{\tau^* = T-t\}}]] \\
\text{(A.24)} \quad & \geq c_3 \widehat{\mathbb{E}}_{x_\epsilon, m} [(T-t) \mathbf{1}_{\{\tau^* > \tau_l\}} \mathbf{1}_{\{\tau^* = T-t\}}],
\end{aligned}$$

where

$$c_3 := \widehat{\mathbb{E}}_{X_{\tau_l}, M_{\tau_l}} [X_{T-t}^{\frac{\gamma-1}{\gamma}} \mathbf{1}_{\{\tau^* = T-t\}}] > 0.$$

Notice that the strictly positivity of c_3 may be verified by using the known joint law of the Brownian motion and its running supremum. Indeed, recalling that $l \leq \frac{L^{\frac{\gamma-1}{\gamma}}}{2}$, we have

$$\begin{aligned}
\widehat{\mathbb{E}}_{X_{\tau_l}, M_{\tau_l}} [X_{T-t}^{\frac{\gamma-1}{\gamma}} \mathbf{1}_{\{\tau^* = T-t\}}] & = \widehat{\mathbb{E}}_{X_{\tau_l}, M_{\tau_l}} [X_{T-t}^{\frac{\gamma-1}{\gamma}} \mathbf{1}_{\{\inf_{s \in [0, T-t]} X_s \geq b(t+s, M_s)\}}] \\
& = \widehat{\mathbb{E}}_{X_{\tau_l}, M_{\tau_l}} [X_{T-t}^{\frac{\gamma-1}{\gamma}} \mathbf{1}_{\{[\inf_{s \in [0, T-t]} X_s]^{\frac{\gamma-1}{\gamma}} \leq (b(t+s, M_s))^{\frac{\gamma-1}{\gamma}}\}}] \\
& = \widehat{\mathbb{E}}_{X_{\tau_l}, M_{\tau_l}} [X_{T-t}^{\frac{\gamma-1}{\gamma}} \mathbf{1}_{\{\sup_{s \in [0, T-t]} X_s^{\frac{\gamma-1}{\gamma}} \leq (b(t+s, M_s))^{\frac{\gamma-1}{\gamma}}\}}] \\
& \geq \widehat{\mathbb{E}}_{X_{\tau_l}, M_{\tau_l}} [X_{T-t}^{\frac{\gamma-1}{\gamma}} \mathbf{1}_{\{\sup_{s \in [0, T-t]} X_s^{\frac{\gamma-1}{\gamma}} \leq L^{\frac{\gamma-1}{\gamma}}\}}] \\
& \geq \widehat{\mathbb{E}}_{X_{\tau_l}, M_{\tau_l}} [X_{T-t}^{\frac{\gamma-1}{\gamma}} \mathbf{1}_{\{\sup_{s \in [0, T-t]} X_s^{\frac{\gamma-1}{\gamma}} \leq 2l\}}].
\end{aligned}$$

Hence, overall, from (A.22), (A.23), (A.24), we obtain

$$\begin{aligned}
\widehat{\mathbb{E}}_{x_\epsilon, m} [X_{\tau^*}^{\frac{\gamma-1}{\gamma}} \tau^*] & \geq l \widehat{\mathbb{E}}_{x_\epsilon, m} [\tau^* \mathbf{1}_{\{\tau^* \leq \tau_l\}}] + c_2 \widehat{\mathbb{E}}_{x_\epsilon, m} [\tau^* \mathbf{1}_{\{\tau^* > \tau_l\}} \mathbf{1}_{\{\tau^* < T\}}] + c_3 \widehat{\mathbb{E}}_{x_\epsilon, m} [T \mathbf{1}_{\{\tau^* > \tau_l\}} \mathbf{1}_{\{\tau^* = T\}}] \\
& \geq c_4 \widehat{\mathbb{E}}_{x_\epsilon, m} [\tau^*] = c_4 \widehat{\mathbb{E}}[\tau_\epsilon^*]
\end{aligned}$$

where $c_4 = l \wedge c_2 \wedge c_3$. Now we plug the latter into (A.21) and get

$$\frac{\widehat{\mathbb{E}}[\tau_\epsilon^*]}{\widetilde{\mathbb{E}}[\tau_\epsilon^*]} \leq \frac{1}{c_1 c_4} x_\epsilon^{\frac{\gamma-1}{\gamma}}.$$

If we now let $x_\epsilon = b_\epsilon(t, m)$, we have

$$\begin{aligned} \sup_{\epsilon \geq 0} |\partial_t b_\epsilon(t, m)| &\leq \sup_{\epsilon \geq 0} (\lambda_2^\epsilon(t, m) L_1^\epsilon(t, m)) \\ &\leq \frac{\frac{1}{T-t} (C b_\epsilon(t, m)^{\frac{\gamma-1}{\gamma}} \widetilde{\mathbb{E}}[\tau_\epsilon^*] + \widehat{\mathbb{E}}[\tau_\epsilon^*])}{C_1 (K^{\frac{1-\gamma}{\gamma}} - 1) (b_\epsilon(t, m))^{\frac{-1}{\gamma}} \widetilde{\mathbb{E}}_{x, m}[\tau_\epsilon^*]} \leq \frac{C b_\epsilon(t, m)}{(T-t) C_1 (K^{\frac{1-\gamma}{\gamma}} - 1)} + \frac{b_\epsilon(t, m)}{(T-t) C_1 (K^{\frac{1-\gamma}{\gamma}} - 1) c_1 c_4} \\ (A.25) \quad &\leq \frac{(C + \frac{1}{c_1 c_4})(1 + L)}{(T-t) C_1 (K^{\frac{1-\gamma}{\gamma}} - 1)}, \end{aligned}$$

which gives the required uniform bound.

Step 5. Combining the findings of the previous steps, by (A.9) we have that b_ϵ is locally-Lipschitz continuous, with Lipschitz constants that are independent of ϵ (see (A.11), (A.13) and (A.20)). Furthermore, the family $(b_\epsilon)_\epsilon$ is also uniformly bounded (cf. Step 2).

Hence, by Ascoli-Arzelà theorem we can extract a subsequence $(\epsilon_j)_{j \in \mathbb{N}}$ such that $b_{\epsilon_j} \rightarrow g$ uniformly, with g being Lipschitz continuous with the same Lipschitz constant of b_ϵ . However, b_{ϵ_j} converges to b (cf. Step 1), which, by uniqueness of the limit, is then locally-Lipschitz continuous. □

A.6. Proof of Lemma 4.4.

Proof. It is easy to check that $\hat{\tau}(t, x, m) \geq \tau^*(t, x, m)$ by their definitions. In order to show the reverse inequality, the rest of the proof is organized in two steps.

Step 1. We claim that

$$\hat{\tau}(t, b(t, m), m) = 0, \quad \widehat{\mathbb{P}} - a.s.$$

due to the Lipschitz continuity of $b(t, m)$ and the law of the iterated logarithm of Brownian motion. As a matter of fact, we fix a point $(t_0, x_0, m_0) \in \partial \mathcal{W} \cap \{t < T\}$ and take a sequence $(t_n, x_n, m_n)_{n \in \mathbb{N}} \subseteq \mathcal{W}$ with $(t_n, x_n, m_n) \rightarrow (t_0, x_0, m_0)$ as $n \rightarrow \infty$. We also fix $\omega \in \Omega_0$, with $\widehat{\mathbb{P}}(\Omega_0) > 0$, and assume that $\limsup_{n \rightarrow \infty} \hat{\tau}(t_n, x_n, m_n)(\omega) =: \lambda > 0$. Now we need to distinguish two cases: $\gamma < 1$ and $\gamma > 1$.

Step 1-(a). We start with the case $\gamma < 1$. The equality $\hat{\tau}(t, x, m) = \tau^*(t, x, m)$ is trivial for (t, x, m) such that $x < b(t, m)$, hence we fix $(t, x, m) \in \mathcal{U}$ with $x \geq b(t, m)$ in the subsequent proof. Then, if $\limsup_{n \rightarrow \infty} \hat{\tau}(t_n, x_n, m_n) = \lambda > 0$ on Ω_0 , for any $\omega \in \Omega_0$ we have,

$$X_s^{x_n}(\omega) \geq b(t_n + s, M_s^{m_n}), \quad \forall n \in \mathbb{N}, \forall s \in [0, \frac{\lambda}{2}].$$

Upon using that $(t, m) \mapsto b(t_n + s, M_s^{m_n})$ is Lipschitz continuous (cf. Theorem 4.1), we let $n \rightarrow \infty$ and obtain

$$\begin{aligned} X_s^{x_0}(\omega) &\geq b(t_0, m_0) + b(t_0 + s, M_s^{m_0}) - b(t_0, M_s^{m_0}) + b(t_0, M_s^{m_0}) - b(t_0, m_0) \\ (A.26) \quad &= b(t_0, m_0) + \int_0^s \partial_t b(t_0 + u, M_s^{m_0}) du + \int_{m_0}^{m_0 e^{as}} \partial_m b(t_0, u) du. \end{aligned}$$

However, from (A.14) and (A.25) we have

$$\begin{aligned} \partial_m b(t_0, u) &\geq -|\rho + 1|(1 + L)e^{aT}, \\ \partial_t b(t_0 + u, M_s^{m_0}) &\geq -\frac{(C + \frac{1}{c_1 c_4})(1 + L)}{(T - t_0 - u)(K^{\frac{1-\gamma}{\gamma}} - 1)}, \end{aligned}$$

which used in (A.26) give

$$\begin{aligned} X_s^{x_0}(\omega) &\geq b(t_0, m_0) - \int_0^s \frac{(C + \frac{1}{c_1 c_4})(1 + L)}{(T - t_0 - u)(K^{\frac{1-\gamma}{\gamma}} - 1)} du - \int_{m_0}^{m_0 e^{as}} |\rho + 1|(1 + L)e^{aT} du \\ &= b(t_0, m_0) - \frac{(C + \frac{1}{c_1 c_4})(1 + L)}{K^{\frac{1-\gamma}{\gamma}} - 1} \int_0^s \frac{1}{T - t_0 - u} du - (m_0 e^{as} - m_0)|\rho + 1|(1 + L)e^{aT} \\ &\geq b(t_0, m_0) + \frac{(C + \frac{1}{c_1 c_4})(1 + L)}{K^{\frac{1-\gamma}{\gamma}} - 1} \frac{s}{s + t_0} - (m_0 e^{as} - m_0)|\rho + 1|(1 + L)e^{aT} \\ &= b(t_0, m_0) + \bar{C}_1 \frac{s}{s + t_0} - \bar{C}_2 (e^{as} - 1), \end{aligned}$$

where $\bar{C}_1 := \frac{(C + \frac{1}{c_1 c_4})(1 + L)}{K^{\frac{1-\gamma}{\gamma}} - 1}$ and $\bar{C}_2 := m_0 |\rho + 1|(1 + L)e^{aT}$. Since $b(t_0, m_0) = x_0$, using the explicit representation for $X_s^{x_0}$ we find

$$(A.27) \quad x_0 \exp \left(\int_0^s \left[(\rho + 1)(\beta - r + M_u) + \mu_1 - \frac{\sigma_1^2}{2} \right] du + \left(\int_0^s \sigma_1 d\hat{B}_u \right) (\omega) \right) \geq x_0 + \bar{C}_1 \frac{s}{s + t_0} - \bar{C}_2 (e^{as} - 1).$$

By the law of the iterated logarithm (cf. Theorem 9.23 in Karatzas and Shreve [27]), for all $\epsilon > 0$ we have (along a sequence of times converging to zero)

$$\hat{B}_s(\omega) \geq (1 - \epsilon) \sqrt{2s \log(\log(\frac{1}{s}))},$$

which combined with (A.27) yields (with $\sigma_1 = \rho\sigma_y - (\rho + 1)\theta = \frac{\gamma\sigma_y - \theta}{1-\gamma} < 0$ due to $\gamma < 1$ and Assumption 4.1)

$$x_0 e^{((\rho+1)(\beta-r) + \mu_1 - \frac{\sigma_1^2}{2})s + (\rho+1) \int_0^s M_u du} e^{\sigma_1(1-\epsilon)\sqrt{2s \log(\log(\frac{1}{s}))}} \geq x_0 + \bar{C}_1 \frac{s}{s+t_0} - \bar{C}_2(e^{as} - 1).$$

On the other hand, since $e^x = 1 + x + O(x^2)$ when $x \approx 0$, the last display equation implies (for s small enough) that

$$\begin{aligned} x_0 \left[1 + \sigma_1(1-\epsilon)\sqrt{2s \log(\log(\frac{1}{s}))} + ((\rho+1)(\beta-r) + \mu_1 - \frac{\sigma_1^2}{2})s + (\rho+1) \int_0^s M_u du \right] \\ \geq x_0 + \bar{C}_1 \frac{s}{s+t_0} - \bar{C}_2(e^{as} - 1), \end{aligned}$$

which simplified gives

$$\begin{aligned} x_0(-\sigma_1)(1-\epsilon)\sqrt{2s \log(\log(\frac{1}{s}))} - x_0((\rho+1)(\beta-r) + \mu_1 - \frac{\sigma_1^2}{2})s - x_0(\rho+1) \int_0^s M_u du \\ \leq -\bar{C}_1 \frac{s}{s+t_0} + \bar{C}_2(e^{as} - 1). \end{aligned} \tag{A.28}$$

Then dividing by s and letting $s \downarrow 0$, we obtain that the left hand-side of the inequality in (A.28) is ∞ (since $\sqrt{2s \log(\log(\frac{1}{s}))}/s \rightarrow \infty$ for $s \downarrow 0$), but the right hand-side of the inequality in (A.28) is the constant $a\bar{C}_2 - \frac{\bar{C}_1}{t_0}$. Thus, we reach a contradiction and $\hat{\tau}(t, b(t, m), m) = 0, \hat{\mathbb{P}}$ -a.s.

Step 1-(b). Now we consider the case $\gamma > 1$. The equality $\hat{\tau}(t, x, m) = \tau^*(t, x, m)$ is trivial for (t, x, m) such that $x > b(t, m)$, hence we fix $(t, x, m) \in \mathcal{U}$ with $x \leq b(t, m)$ in the subsequent proof. Then we have

$$X_s^{x_n}(\omega) \leq b(t_n + s, M_s^{m_n}), \quad \forall n \in \mathbb{N}, \forall s \in [0, \frac{\lambda}{2}].$$

Upon using that $(t, m) \mapsto b(t_n + s, M_s^{m_n})$ is Lipschitz continuous (cf. Theorem 4.1), we let $n \rightarrow \infty$ and obtain

$$\begin{aligned} X_s^{x_0}(\omega) &\leq b(t_0, m_0) + b(t_0 + s, M_s^{m_0}) - b(t_0, M_s^{m_0}) + b(t_0, M_s^{m_0}) - b(t_0, m_0) \\ &= b(t_0, m_0) + \int_0^s \partial_t b(t_0 + u, M_s^{m_0}) du + \int_{m_0}^{m_0 e^{as}} \partial_m b(t_0, u) du. \end{aligned} \tag{A.29}$$

However, from (A.15) and (A.20) we have

$$\begin{aligned} \partial_m b(t_0, m) &\leq |\rho + 1|(1 + b_\infty(m))e^{aT}, \\ \partial_t b(t_0 + u, M_s^{m_0}) &\leq \frac{Cb_\infty(M_s^{m_0}) + (b_\infty(M_s^{m_0}))^{\frac{1}{\gamma}}}{(T - t_0 - u)C_1(1 - K^{\frac{1-\gamma}{\gamma}})}, \end{aligned}$$

which used in (A.29) give

$$\begin{aligned}
X_s^{x_0}(\omega) &\leq b(t_0, m_0) + \int_0^s \frac{Cb_\infty(M_s^{m_0}) + (b_\infty(M_s^{m_0}))^{\frac{1}{\gamma}}}{(T-t_0-u)C_1(1-K^{\frac{1-\gamma}{\gamma}})} du + \int_{m_0}^{m_0 e^{as}} |\rho+1|(1+b_\infty(u))e^{aT} du \\
&= b(t_0, m_0) + \frac{Cb_\infty(M_s^{m_0}) + (b_\infty(M_s^{m_0}))^{\frac{1}{\gamma}}}{C_1(1-K^{\frac{1-\gamma}{\gamma}})} \int_0^s \frac{1}{T-t_0-u} du + (m_0 e^{as} - m_0)|\rho+1|(1+b_\infty(m_0 e^{as}))e^{aT} \\
&\leq b(t_0, m_0) + \frac{Cb_\infty(M_s^{m_0}) + (b_\infty(M_s^{m_0}))^{\frac{1}{\gamma}}}{C_1(1-K^{\frac{1-\gamma}{\gamma}})} \frac{s}{T-t_0-s} + (m_0 e^{as} - m_0)|\rho+1|(1+b_\infty(m_0 e^{as}))e^{aT} \\
&= b(t_0, m_0) + F(s, m_0) \frac{s}{T-t_0-s} + (m_0 e^{as} - m_0)|\rho+1|(1+b_\infty(m_0 e^{as}))e^{aT},
\end{aligned}$$

where we have used the monotonicity of b_∞ in Lemma 4.3, and $F(s, m_0) := \frac{Cb_\infty(m_0 e^{as}) + (b_\infty(m_0 e^{as}))^{\frac{1}{\gamma}}}{C_1(1-K^{\frac{1-\gamma}{\gamma}})}$.

Since $b(t_0, m_0) = x_0$, then we have

$$\begin{aligned}
&x_0 \exp \left(\int_0^s \left[(\rho+1)(\beta-r+M_u) + \mu_1 - \frac{\sigma_1^2}{2} \right] du + \int_0^s \sigma_1 d\widehat{B}_u(\omega) \right) \\
\text{(A.30)} \quad &\leq x_0 + F(s, m_0) \frac{s}{T-t_0-s} + (m_0 e^{as} - m_0)|\rho+1|(1+b_\infty(m_0 e^{as}))e^{aT}.
\end{aligned}$$

By the law of the iterated logarithm (cf. Theorem 9.23 in Karatzas and Shreve [27]), for all $\epsilon > 0$ we have (along a sequence of times converging to zero)

$$\widehat{B}_s(\omega) \geq (1-\epsilon) \sqrt{2s \log(\log(\frac{1}{s}))},$$

which combined with (A.30) yields (with $\sigma_1 = \rho\sigma_y - (\rho+1)\theta = \frac{\gamma\sigma_y - \theta}{1-\gamma} > 0$ due to $\gamma > 1$ and Assumption 4.1)

$$\begin{aligned}
&x_0 e^{((\rho+1)(\beta-r) + \mu_1 - \frac{\sigma_1^2}{2})s + (\rho+1) \int_0^s M_u du} e^{\sigma_1(1-\epsilon) \sqrt{2s \log(\log(\frac{1}{s}))}} \\
&\leq x_0 + F(s, m_0) \frac{s}{T-t_0-s} + (m_0 e^{as} - m_0)|\rho+1|(1+b_\infty(m_0 e^{as}))e^{aT}.
\end{aligned}$$

On the other hand, since $e^x = 1 + x + O(x^2)$ when $x \approx 0$, the last display equation implies (for s small enough) that

$$\begin{aligned}
&x_0 \left[1 + \sigma_1(1-\epsilon) \sqrt{2s \log(\log(\frac{1}{s}))} + ((\rho+1)(\beta-r) + \mu_1 - \frac{\sigma_1^2}{2})s + (\rho+1) \int_0^s M_u du \right] \\
&\leq x_0 + F(s, m_0) \frac{s}{T-t_0-s} + (m_0 e^{as} - m_0)|\rho+1|(1+b_\infty(m_0 e^{as}))e^{aT},
\end{aligned}$$

which simplified gives

$$\begin{aligned}
&x_0 \sigma_1(1-\epsilon) \sqrt{2s \log(\log(\frac{1}{s}))} + x_0((\rho+1)(\beta-r) + \mu_1 - \frac{\sigma_1^2}{2})s + x_0(\rho+1) \int_0^s M_u du \\
\text{(A.31)} \quad &\leq F(s, m_0) \frac{s}{T-t_0-s} + (m_0 e^{as} - m_0)|\rho+1|(1+b_\infty(m_0 e^{as}))e^{aT}.
\end{aligned}$$

Then dividing by s and letting $s \downarrow 0$, we obtain that the left hand-side of the inequality in (A.31) is ∞ (since $\sqrt{2s \log(\log(\frac{1}{s}))}/s \rightarrow \infty$ for $s \downarrow 0$), but the right hand-side of the inequality in (A.28) is the constant $\frac{F(0, m_0)}{T-t_0} + m_0 a |\rho + 1| (1 + b_\infty(m_0)) e^{aT}$. Thus, we reach a contradiction and $\hat{\tau}(t, b(t, m), m) = 0, \widehat{\mathbb{P}}$ -a.s.

Step 2. In order to prove that $\hat{\tau}(t, x, m) \leq \tau^*(t, x, m)$, one can finally use arguments as in the proof of Lemma 5.1 in De Angelis and Ekström [11].

□

A.7. Proof of Theorem 5.1.

Proof. The proof is organized in two steps.

Step 1: We show that $V \in C^{1,1,1,1}(\mathcal{O}')$ and $V_{ww}, V_{yy}, V_{wy} \in L_{loc}^\infty(\mathcal{O}')$ and V is a solution in the a.e. sense to the HJB equation

$$(A.32) \quad 0 = \max \left\{ \widehat{V} - V, \sup_{c, \pi} \left[V_t + \frac{1}{2} \sigma^2 \pi^2 V_{ww} + (\pi(\mu - r) + r w - c + y) V_w + \sigma \pi \sigma_y y V_{wy} \right. \right. \\ \left. \left. + \frac{1}{2} \sigma_y^2 y^2 V_{yy} + \mu_y y V_y + a m V_m + U_1(c) - (\beta + m) V \right] \right\}.$$

Step 1-(a): Firstly we show the regularity of J . From (4.1) we know that $J(t, z, m, y) = zy \tilde{J}(t, z^{\frac{1}{1-\gamma}} y^{\frac{\gamma}{1-\gamma}}, m)$. Therefore,

$$(A.33) \quad \begin{aligned} J_t &= zy \tilde{J}_t, & J_m &= zy \tilde{J}_m, & J_z &= y \tilde{J} + z^{\frac{1}{1-\gamma}} y^{\frac{\gamma}{1-\gamma}} \left[\frac{y}{1-\gamma} \tilde{J}_x \right], & J_y &= z \tilde{J} + z^{\frac{1}{1-\gamma}} y^{\frac{\gamma}{1-\gamma}} \left[\frac{z\gamma}{1-\gamma} \tilde{J}_x \right], \\ J_z &= y \tilde{J} + z^{\frac{1}{1-\gamma}} y^{\frac{\gamma}{1-\gamma}} \left[\frac{y}{1-\gamma} \tilde{J}_x \right], & J_y &= z \tilde{J} + z^{\frac{1}{1-\gamma}} y^{\frac{\gamma}{1-\gamma}} \left[\frac{z\gamma}{1-\gamma} \tilde{J}_x \right], \\ J_{yy} &= z^{\frac{1}{1-\gamma}+1} y^{\frac{\gamma}{1-\gamma}-1} \frac{\gamma}{(1-\gamma)^2} \tilde{J}_x + z^{\frac{2}{1-\gamma}+1} y^{\frac{2\gamma}{1-\gamma}-1} \frac{\gamma^2}{(1-\gamma)^2} \tilde{J}_{xx}, & \text{in the a.e. sense,} \\ J_{zz} &= z^{\frac{1}{1-\gamma}-1} y^{\frac{\gamma}{1-\gamma}+1} \frac{2-\gamma}{(1-\gamma)^2} \tilde{J}_x + z^{\frac{2}{1-\gamma}-1} y^{\frac{2\gamma}{1-\gamma}+1} \frac{1}{(1-\gamma)^2} \tilde{J}_{xx}, & \text{in the a.e. sense,} \end{aligned}$$

$$J_{zy} = \tilde{J} + z^{\frac{1}{1-\gamma}} y^{\frac{\gamma}{1-\gamma}} \frac{\gamma - \gamma^2 + 1}{(1-\gamma)^2} \tilde{J}_x + z^{\frac{2}{1-\gamma}} y^{\frac{2\gamma}{1-\gamma}} \frac{\gamma}{(1-\gamma)^2} \tilde{J}_{xx}, \text{ in the a.e. sense.}$$

Then we conclude that $J \in C^{1,1,1,1}(\mathcal{O}) \cap C^{1,2,1,2}(\overline{\mathcal{W}})$ with $J_{yy}, J_{zz}, J_{zy} \in L_{loc}^\infty(\mathcal{O})$ due to Corollary 4.1.

Step 1-(b): Now we show the regularity of V . From (5.2), using that $J_z(t, z^*(t, w, m, y), m, y) = -w - g(t)$ and $J_{zy}(t, z^*(t, w, m, y), m, y) + J_{zz} z_y^* = -q(t)$, one has

$$(A.34) \quad V_t = J_t(t, z^*(t, w, m, y), m, y) + J_z(t, z^*(t, w, m, y), m, y) z_t^*(t, w, m, y)$$

$$\begin{aligned}
& + (w + g(t))z_t^*(t, w, m, y) + g'(t)z_t^*(t, w, m, y) = J_t(t, z^*(t, w, m, y), m, y) + g'(t)z_t^*(t, w, m, y), \\
V_m & = J_m(t, z^*(t, w, m, y), m, y) + J_z(t, z^*(t, w, m, y), m, y)z_m^*(t, w, m, y) \\
& + (w + g(t))z_m^*(t, w, m, y) = J_m(t, z^*(t, w, m, y), m, y), \\
V_y & = J_y(t, z^*(t, w, m, y), m, y) + J_z(t, z^*(t, w, m, y), m, y)z_y^*(t, w, m, y) \\
& + (w + g(t))z_y^*(t, w, m, y) + z^*(t, w, m, y)q(t) = J_y(t, z^*(t, w, m, y), m, y) + z^*(t, w, m, y)q(t), \\
V_{yy} & = J_{yy}(t, z^*(t, w, m, y), m, y) + J_{yz}(t, z^*(t, w, m, y), m, y)z_y^*(t, w, m, y) + z_y^*(t, w, m, y)q(t), \\
V_w & = J_z(t, z^*(t, w, m, y), m, y)z_w^*(t, w, m, y) + z^*(t, w, m, y) + (w + g(t))z_w^*(t, w, m, y) = z^*(t, w, m, y), \\
V_{wy} & = z_y^*(t, w, m, y) = \frac{-q(t) - J_{zy}(t, z^*(t, w, m, y), m, y)}{J_{zz}(t, z^*(t, w, m, y), m, y)}, \text{ in the a.e. sense,} \\
V_{ww} & = z_w^*(t, w, m, y) = -\frac{1}{J_{zz}(t, z^*(t, w, m, y), m, y)}, \text{ in the a.e. sense.}
\end{aligned}$$

The proof is then completed due to Step 1-(a).

Step 1-(c): Now we characterize the optimal retirement time in the primal variables and show that V is a solution in the a.e. sense to the HJB equation. Define $\widetilde{W}(t, z, m, y) := Q(z, m) - zg(t)$. Recalling that $J \geq \widetilde{W}$ on \mathcal{O}' by (3.18), we notice that if $J(t, z^*(t, w, m, y), m, y) = \widetilde{W}(t, z^*(t, w, m, y), m, y)$, then the function $z \mapsto (J - \widetilde{W})(t, z, m, y)$ attains its minimum value 0 at $(t, z^*(t, w, m, y), m, y)$. Hence,

$$J_z(t, z^*(t, w, m, y), m, y) = \widetilde{W}_z(t, z^*(t, w, m, y), m, y) = -w - g(t).$$

This means that $z^*(t, w, m, y)$ is a stationary point of the convex function $z \mapsto \widetilde{W}(t, z, m, y) + z(w + g(t))$, so that

$$\begin{aligned}
\widetilde{W}(t, z^*(t, w, m, y), m, y) + (w + g(t))z^*(t, w, m, y) & = \min_z (\widetilde{W}(t, z, m, y) + z(w + g(t))) \\
& = \min_z [Q(z, m) + zw] = \widehat{V}(w, m),
\end{aligned}$$

by Theorem 3.1. Together with (5.2), we obtain $V(t, w, m, y) = \widehat{V}(w, m)$.

On the other hand, if $V(t, w, m, y) = \widehat{V}(w, m)$, then by (5.2) and Theorem 3.1

$$\begin{aligned}
& J(t, z^*(t, w, m, y), m, y) + (w + g(t))z^*(t, w, m, y) \\
& = \inf_z (\widetilde{W}(t, z, m, y) + z(w + g(t))) \leq \widetilde{W}(t, z^*(t, w, m, y), m, y) + (w + g(t))z^*(t, w, m, y).
\end{aligned}$$

Hence, since $J \geq \widetilde{W}$ on \mathcal{O}' , $J(t, z^*(t, w, m, y), m, y) = \widetilde{W}(t, z^*(t, w, m, y), m, y)$.

Combining these two arguments we have that

$$\begin{aligned}
& \{(t, w, m, y) \in \mathcal{O}' : V(t, w, m, y) = \widehat{V}(w, m)\} \\
&= \{(t, w, m, y) \in \mathcal{O}' : J(t, z^*(t, w, m, y), m, y) = \widetilde{W}(t, z^*(t, w, m, y), m, y)\} \\
&= \{(t, x, m) \in \mathcal{U} : z^*y\widetilde{J}(t, x^*(t, w, m, y), m) = z^*yQ(t, x^*(t, w, m, y), m) - z^*yq(t)\} \\
&= \{(t, x, m) \in \mathcal{U} : \widetilde{J}(t, x^*(t, w, m, y), m) = \widehat{W}(x^*(t, w, m, y), m)\} \\
&= \{(t, x, m) \in \mathcal{U} : \widehat{J}(t, x^*(t, w, m, y), m) = 0\},
\end{aligned}$$

where $x^*(t, w, m, y) := (z^*(t, w, m, y))^{\frac{1}{1-\gamma}}y^{\frac{\gamma}{1-\gamma}}$. This, together with (5.4), leads to express the optimal retirement time in the original coordinates as

$$\left\{ \begin{aligned} \tau^*(t, w, m, y) &= \inf\{s \geq 0 : W_s^w \geq \widehat{b}(t+s, M_s^m, Y_s^y)\} \wedge (T-t) \\ &= \inf\{s \geq 0 : V(t+s, W_s^w, M_s^m, Y_s^y) = \widehat{V}(W_s^w, M_s^m)\} \wedge (T-t). \end{aligned} \right.$$

Due to the regularity of V and the dual relations between V and J (cf. Step 1-(b)), from Corollary 4.1 we then find that V is a solution in the a.e. sense to the HJB equation.

Step 2: Assuming that there exists a unique strong solution W^* to the SDE (2.4), when π, c and τ are replaced by π^*, c^* and τ^* , respectively. From (5.2) we have $W_s^* = -J_z(s, Z_s, M_s, Y_s) - g(s)$, where Z_s is the solution to Equation (3.8) with the initial condition $Z_t = z^*$. Since $J_z < 0$ by Proposition 5.1, it is easy to verify that $(c^*, \pi^*, \tau^*) \in \mathcal{A}(t, w, m, y)$, in particular, $W_s^* > -g(s)$. Finally, a standard verification argument leads to the result. \square

APPENDIX B. TWO AUXILIARY RESULTS

Lemma B.1. *Let $w > 0$ be given, let $c \geq 0$ be a consumption process satisfying*

$$\mathbb{E}_{w,m} \left[\int_t^\infty \xi_{s,t} c_s ds \right] = w.$$

Then, there exists a portfolio process π such that the pair (c, π) is admissible and

$$W_s^{c,\pi,\tau} > 0, \text{ for } s \geq \tau.$$

Proof. The proof is similar to Theorem 3.3.5 in Karatzas and Shreve [28], and we thus omit details. \square

Lemma B.2. For any $\tau \in \mathcal{S}$, let $w > -g(t)$ be given, let $c \geq 0$ be a consumption process. For any \mathcal{F}_τ -measurable random variable ϕ with $\mathbb{P}[\phi > -g(\tau)] = 1$ such that

$$\mathbb{E}_{t,w,m,y} \left[\xi_{\tau,t}(\phi + g(\tau)) + \int_t^\tau \xi_{s,t} c_s ds \right] = w + g(t),$$

there exists a portfolio process π such that the pair (c, π) is admissible and

$$W_s^{c,\pi,\tau} > -g(s), \text{ for } s \leq \tau, \phi = W_\tau^{c,\pi,\tau}.$$

Proof. The proof is similar to Lemma 6.3 in Karatzas and Wang [29], and we thus omit details. \square

APPENDIX C. RECURSIVE INTEGRATION METHOD

The numerical method is inspired by Huang et al. [22] and Jeon et al. [24]. Here we only consider the case $\gamma > 1$. We rewrite (4.27) as

$$(C.1) \quad 0 = \int_0^{T-t} e^{-\kappa s} \widehat{\mathbb{E}}_{\bar{b}(0),m} \left[\left((1 - K^{\frac{1-\gamma}{\gamma}}) U_1^*(X_s) + 1 \right) \mathbf{1}_{\{X_s \leq \bar{b}(s)\}} \right] ds,$$

where $\bar{b}(s) := b(t+s, M_s)$. Now we compute the expectation inside the integral in the right-hand side of (C.1),

$$\begin{aligned} & \widehat{\mathbb{E}}_{\bar{b}(0),m} \left[\left((1 - K^{\frac{1-\gamma}{\gamma}}) U_1^*(X_s) + 1 \right) \mathbf{1}_{\{X_s \leq \bar{b}(s)\}} \right] \\ &= (1 - K^{\frac{1-\gamma}{\gamma}}) \frac{\gamma}{1-\gamma} \widehat{\mathbb{E}}_{\bar{b}(0),m} \left[X_s^{\frac{\gamma-1}{\gamma}} \mathbf{1}_{\{X_s \leq \bar{b}(s)\}} \right] + \widehat{\mathbb{P}}_{\bar{b}(0),m} [X_s \leq \bar{b}(s)] \\ &= (1 - K^{\frac{1-\gamma}{\gamma}}) \frac{\gamma}{1-\gamma} \widehat{\mathbb{E}}_{\bar{b}(0),m} \left[\bar{b}(0)^{\frac{\gamma-1}{\gamma}} H_s N(s, m) \mathbf{1}_{\{X_s \leq \bar{b}(s)\}} \right] + \widehat{\mathbb{P}}_{\bar{b}(0),m} [X_s \leq \bar{b}(s)] \\ &= (1 - K^{\frac{1-\gamma}{\gamma}}) \frac{\gamma}{1-\gamma} \bar{b}(0)^{\frac{\gamma-1}{\gamma}} N(s, m) \widetilde{\mathbb{P}}_{\bar{b}(0),m} [X_s \leq \bar{b}(s)] + \widehat{\mathbb{P}}_{\bar{b}(0),m} [X_s \leq \bar{b}(s)]. \end{aligned}$$

By direct computations we have

$$\begin{aligned} \widehat{\mathbb{P}}_{\bar{b}(0),m} [X_s \leq \bar{b}(s)] &= \Phi \left(\frac{\log \frac{\bar{b}(s)}{\bar{b}(0)} - \int_0^s [(\rho + 1)(\beta - r + me^{au}) + \mu_1 - \frac{\sigma_1^2}{2}] du}{\sigma_1 \sqrt{s}} \right) = \Phi(d^1(s, \frac{\bar{b}(s)}{\bar{b}(0)})), \\ \widetilde{\mathbb{P}}_{\bar{b}(0),m} [X_s \leq \bar{b}(s)] &= \Phi \left(\frac{\log \frac{\bar{b}(s)}{\bar{b}(0)} - \int_0^s [(\rho + 1)(\beta - r + me^{au}) + \mu_1 + \frac{(\gamma-2)\sigma_1^2}{2\gamma}] du}{\sigma_1 \sqrt{s}} \right) = \Phi(d^2(s, \frac{\bar{b}(s)}{\bar{b}(0)})), \end{aligned}$$

where

$$d^1(s, y) := \frac{\log y - \int_0^s [(\rho + 1)(\beta - r + me^{au}) + \mu_1 - \frac{\sigma_1^2}{2}] du}{\sigma_1 \sqrt{s}},$$

$$d^2(s, y) := \frac{\log y - \int_0^s [(\rho + 1)(\beta - r + me^{au}) + \mu_1 + \frac{(\gamma-2)\sigma_1^2}{2\gamma}] du}{\sigma_1 \sqrt{s}},$$

with $\Phi(\cdot)$ being the cumulative distribution function of a standard normal random variable.

Then the integral equation (C.1) can be converted to

$$0 = \int_0^{T-t} e^{-\kappa s} \left((1 - K^{\frac{1-\gamma}{\gamma}}) \frac{\gamma}{1-\gamma} \bar{b}(0)^{\frac{\gamma-1}{\gamma}} N(s, m) \Phi\left(d^2\left(s, \frac{\bar{b}(s)}{\bar{b}(0)}\right)\right) + \Phi\left(d^1\left(s, \frac{\bar{b}(s)}{\bar{b}(0)}\right)\right) \right) ds.$$

Letting $\xi := T - t$ and $\bar{b}^*(\xi) = \bar{b}(T - t - \xi)$, we can show that $\bar{b}^*(\xi)$ satisfies the following integral equation:

$$(C.2) \quad 0 = \int_0^\xi G(\xi, s, \bar{b}^*(\xi), \bar{b}^*(\xi - s)) ds,$$

where

$$G(\xi, s, \bar{b}^*(\xi), \bar{b}^*(\xi - s)) := e^{-\kappa s} \left((1 - K^{\frac{1-\gamma}{\gamma}}) \frac{\gamma}{1-\gamma} \bar{b}^*(\xi)^{\frac{\gamma-1}{\gamma}} N(s, m) \Phi\left(d^2\left(s, \frac{\bar{b}^*(\xi - s)}{\bar{b}^*(\xi)}\right)\right) + \Phi\left(d^1\left(s, \frac{\bar{b}^*(\xi - s)}{\bar{b}^*(\xi)}\right)\right) \right).$$

In order to solve the above integral equation (C.2), the recursive iteration method proceeds as follows.

We divide the interval $[0, \xi]$ into n subintervals with end points $\xi_j, j = 0, 1, 2, \dots, n$ where $\xi_0 = 0, \xi_n = \xi$ and $\Delta\xi = \frac{\xi}{n}$. Let \bar{b}_j^* denote the numerical approximation to $\bar{b}^*(\xi_j), j = 0, 1, \dots, n$. For $\xi = \xi_1$, by the trapezoidal rule, integral equation (C.2) is approximated by

$$(C.3) \quad 0 = \frac{\Delta\xi}{2} [G(\xi_1, \xi_0, \bar{b}_1^*, \bar{b}_1^*) + G(\xi_1, \xi_1, \bar{b}_1^*, \bar{b}_0^*)].$$

Since $\bar{b}_0^* = b(T, M_{T-t}^m) = L$, the only unknown in (C.3) is \bar{b}_1^* . We can solve the algebraic equation (C.3) by applying the bisection method. Similarly, for $\xi = \xi_2$, we have

$$(C.4) \quad 0 = \frac{\Delta\xi}{2} [G(\xi_2, \xi_0, \bar{b}_2^*, \bar{b}_2^*) + 2G(\xi_2, \xi_1, \bar{b}_2^*, \bar{b}_1^*) + G(\xi_2, \xi_2, \bar{b}_2^*, \bar{b}_0^*)].$$

Since \bar{b}_1^* is known from previous step, equation (C.4) can be solved for \bar{b}_2^* by the same procedure.

Hence, for $\bar{b}_k^*, k = 2, 3, \dots, n$, we can obtain \bar{b}_k^* recursively as the solution of the following algebraic equation,

$$0 = \frac{\Delta\xi}{2} [G(\xi_k, \xi_0, \bar{b}_k^*, \bar{b}_k^*) + 2 \sum_{j=1}^{k-1} G(\xi_k, \xi_{k-j}, \bar{b}_k^*, \bar{b}_j^*) + G(\xi_k, \xi_k, \bar{b}_k^*, \bar{b}_0^*)].$$

Now from the values of $\{\bar{b}_i^*\}_{i=1}^n$, \hat{J} in (4.26) can be approximated by

$$\hat{J}(\xi, x, m) \approx \hat{J}_n(\xi, x, m) := \frac{\Delta\xi}{2} [G(\xi_n, \xi_0, x, \bar{b}_n^*) + 2 \sum_{j=1}^{n-1} G(\xi_n, \xi_{n-j}, x, \bar{b}_j^*) + G(\xi_n, \xi_n, x, \bar{b}_0^*)].$$

As shown by Huang et al. [22], for sufficiently large number of subintervals n , the approximated free boundary \bar{b}_n^* converges to $\bar{b}^*(\xi)$, and therefore, \hat{J}_n converges to \hat{J} as well.

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