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Economic Similarities and their Application to Inflation

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Abstract

An economic similarity between two economies is defined, which is a correspondence between futures of two economies in which all, or a certain subset of all acts of production and consumption, are the same. Prices and inflation can be different in similar economies so an economic similarity is the ideal tool for studying inflation and for expressing *monetary neutrality*, as one aspect of money. A formula for an economic similarity from an arbitrary economy to an economy with a different rate of inflation is stated and proved. An economic similarity is used to formulate and prove a rigorous expression of the quantity theory of money and a different economic similarity is used to elucidate how raising interest rates reduces inflation. The distortion of economic activity caused by raising interest rates to control inflation is identified and quantified. A strategy for managing inflation which avoids distortion of economic activity by focusing on loan repayments rather than on interest rates is identified. The existence of a “painless” inflation management strategy, which adjusts repayment rates as well as interest rates, is deduced by means of the more precise understanding of the neutrality aspect of money that is obtained by means of economic similarity. Although it is unexpected to discover a purely “technical” mechanism for managing inflation, the theoretical justification, provided by means of economic similarity, is strong. The practical implications of such a mechanism are considerable.

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1. Introduction

A natural objective of any rigorous discipline, including economics, is to be able to predict the consequences of actions which modify, or *intervene*, in a system such as an economy. This is made more difficult in economics by the inherent randomness, and therefore unpredictability, of the future.

This paper introduces a method by which a deterministic response to economic interventions can be discerned, without neglecting or minimizing the inherent randomness of an economy, by *comparing* two alternative economic futures. In order to be able to rigorously make comparisons of futures of stochastic economies, a specific mathematical model of an economy is adopted – a vector stochastic differential equation.

An economy, with its key components, is depicted in Figure 1.

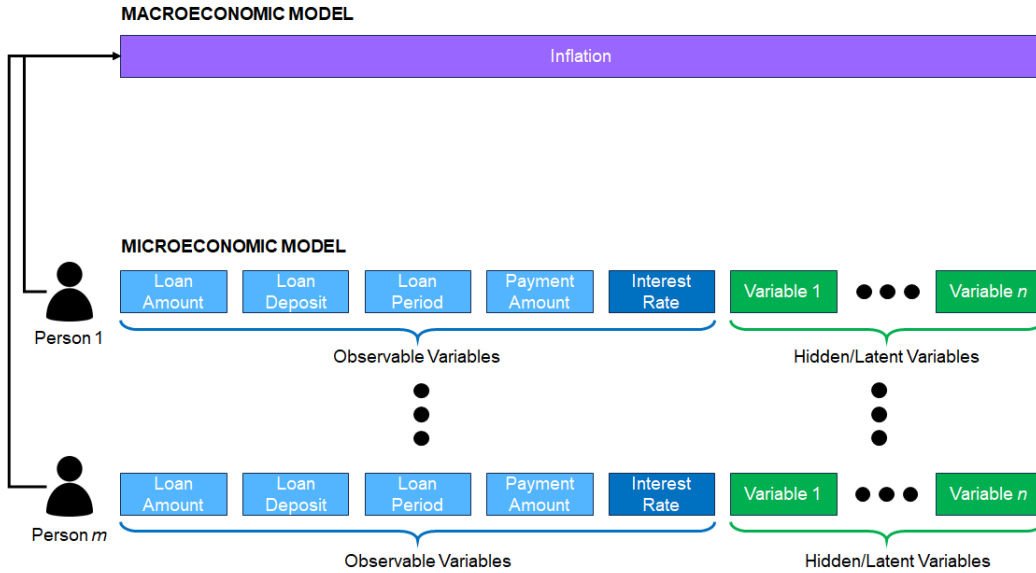


Figure 1: An Economic Model with Micro and Macro Parameters

At the microeconomic level, each individual person and/or firm has a zero or more loans, each with a specific interest rate, as one of the observable parameters. There are other hidden or latent parameters which we do not need to concern ourselves with. In a structural equations modelling (SEM) approach, much of the detail is lost. Instead, we model the economy by a vector stochastic differential equation (VSDE), which enables the dynamics of the economy to be modeled in the manner of a continuous time differential equation, but includes randomness in a consistent manner.

Figure 1 illustrates how constraining or restricting the parameters which affect the experience of individual people or organisations, such as interest rates, has an impact at the macroeconomic level on the inflation of the economy as a whole. The precise mechanism by which the individual decisions of people, organisations, or, in general, agents in an economy are able to create a consistent aggregate effect, for example, on inflation, is analysed mathematically in the experiments in Section 4.

1.1. *Economic Similarity*

Given two economies with the same structure, an economic similarity is a mapping between the space of all possible paths through the future of one economy to the space of all possible paths of the other economy. This maps all events of consumption or production of goods or services in one economy, to similar events in a path through time in the other economy. We call such a mapping a *similarity* if a certain subset of the possible types of events are mapped to identical (an exact similarity), or approximately the same (an approximate similarity), events.

Example 1. Icecream Economy

Suppose the original economy is called E_1 and the second one, to which E_1 is mapped, is called E_2 . If Joe Biggins consumes a chocolate icecream from Scheherazade Icecreams in Agog St, Newtown, for \$2.00, at 1.00pm on April 21, 2024 in a specific path in the space of paths of E_1 , an exact economic similarity maps this path to one in the space of possible paths in E_2 where Joe Biggins also consumes a chocolate icecream from Scheherazade at 1.00pm on April 21, 2024, but not necessarily for the same price, the same currency, or paid by the same method. In general, which events are mapped identically (or approximately) depends on the specific similarity. \square

The space of possible paths in an economy can be equipped with a probability measure, which records the probability of certain sets of paths, in order to provide a quantitative characterization of the economy. For example, the set of paths in which Joe Biggins has an icecream on April 21 might be assigned probability 0.75. The VSDE model which is introduced in Subsection 3.4 defines, in principle, all such probabilities. Such probabilities can be estimated, for example, by simulation. However, the key conclusions, expressed in Theorems 1–3 and

Corollaries 1–2 below, can be expressed and applied, to economic management, without constructing a probability model of an economy.

To apply the concept of economic similarity to economic management, it should be thought of as an alternative future which can be forced to occur by certain interventions, such as actions by a government or central bank with the purpose, for example, of managing the rate of inflation.

1.1.1. Motivation for Economic Similarity

Example 2. Two Economists

Martin and Marvin are two economists contemplating how to manage their economy. Marvin says to Martin: “I would like inflation to be 2%”. Martin: “What is it now?” Marvin: “4%” Martin: “I know how to reduce it by 2%, but it won’t necessarily be 2% afterwards.” Marvin: “Why not?” Martin: “Because a war might start, nuclear fusion might take off, or a global pandemic might break out, suppressing travel and commerce worldwide. What I am suggesting will reduce it by 2% relative to the rate of inflation in the unknown future, if I did not take any action, but it won’t guarantee any specific outcome.”

Martin continues: “I would prefer to be able to predict and control inflation, but since I can’t predict or control politics, science, or biology, I am willing to settle for something less. I *am* able to predict the *relative* effect of certain economic interventions, by comparison with what would happen if the intervention did not occur, so I am willing to settle for that.” □

What Martin is contemplating here is an *economic similarity* as defined and analysed here. The economy Martin is contemplating is different from the original economy in a systematic way. Although we can’t be certain about what is going to happen in the future in either the original economy, or the transformed economy, we can be certain about the changes between the two equally unknown futures.

1.2. Main Contributions

The relevant literature is reviewed in Section 2. The concept of an *economic similarity* which is a mapping between two economies, in which the actual production and consumption or financial transactions in a subset of observed categories is identical, is introduced in Section 3. Specific economic similarities are introduced

and used in this paper to develop a rigorous theory of inflation and to show how it is affected by the parameters of loans, in particular, the official interest rate and the schedules of repayments of loans.

Starting with an arbitrary economy, E_1 say, economic similarities with a different economy, E_2 say, are constructed in which the only changes between the two economies are inflation, interest rates, and repayment rates, are explored in Section 4. The conclusions which can be drawn from this theory of economically similar economies, with different inflation rates and interest rates, are discussed in Section 5.

Most notably, an additional non-neutral effect of changing interest rates while maintaining orthodox loan repayment schedules is identified. This demonstrates the *non-similarity* of economies with different inflation rates and the same *real* interest rate, when orthodox loan repayment schedules are adopted.

In Section 6, the money supply (in four different forms) in a similar economy is shown to increase at a rate lower or higher than the original economy by precisely the difference in the inflation rate between the two economies, providing a precise, rigorous expression of the quantity theory of money. In a corollary to this result it is shown that under the assumption that an economy seeks to be similar, inflation can be controlled by joint fiscal and monetary control of the money supply *without* “*economic pain*”.

Current practice of using interest rates to control inflation is analysed in Section 7 and an explanation for why it works, to reduce inflation, is provided. The explanation is not the conventional one that it cools an over-heated economy, but rather that the orthodox repayment schedule when interest rates are higher forces loans to be repaid more rapidly, at the start, leading to a lowering of the rate of growth of the money supply. It is also shown that this economic intervention involves a *distortion* of the original economy.

The key results are encapsulated in three theorems and their corollaries: Theorem 1 says that given an economy E_1 , with inflation rate $\alpha_1(t)$ over the future, there is a similar economy, E_2 , with inflation rate $\alpha_2(t)$, for any function $\alpha_2(t)$. The economy and the similarity are given in the theorem. Theorem 2 and its corollary say that the loan repayment schedules in the economy E_2 of the previous theorem must differ from those in E_1 in a very specific way which has the effect of front-loading repayments when the inflation rate is to be lowered. Theorem 3 and its corollary say that the growth of the money supply in a similar economy must grow at a greater or less rate exactly to the extent that the inflation rate will be greater

or less, and as a consequence, to change the inflation rate by a certain quantity, the rate of growth of the money supply, in several different senses, must be adjusted by the same quantity.

2. Literature Review

2.1. Neutrality of Money

The concept that money, as primarily a *means of exchange*, should not, or does not, influence the *real activity* of producing and consuming goods and services has been a preoccupation of economists from the beginnings of economics (Lucas Jr, 1996). Paradoxically, the fact that money influences real activity significantly (is non-neutral) has been observed by the same philosopher to whom is attributed the origin of the quantity theory of money (Hume, 1875) and the economist most strongly associated with real economic effects of government budget deficits begins the book (Keynes, 1923) with the following definition of monetary neutrality:

Money is only important for what it will procure. Thus a change in the monetary unit, which is uniform in its operation and affects all transactions equally, has no consequences. If, by a change in the established standard of value, a man received and owned twice as much money as he did before in payment for all rights and for all efforts, and if he also paid out twice as much money for all acquisitions and for all satisfactions, he would be wholly unaffected.

There is no contradiction between the non-neutrality of money and the quantity theory of money, or monetary neutrality as expressed here by Keynes. There is, however, difficulty in expressing these concepts in a way that shows that they both present different, valid aspects of the theory and practice of money. That money is non-neutral is clear. Subrick (2010) lists six mechanisms by means of which money influences the real economy. Gokan and Turnovsky (2021) analyses the impact of monetary change on wealth and income inequality.

One imperfect mechanism for separating the non-neutral and the neutral effects is by time scale. In the short term money is non-neutral and in the long term it is neutral. This has been famously ridiculed by Keynes (1923) with the quip “In the long run, we are all dead”. More critically, the character of neutrality that money exhibits in the long run is also present in the short term, although it is likely to be overshadowed, and hidden, by the non-neutral characteristics of money.

We argue that money *can* be neutral. Changes in the supply or management of money will usually influence the real economy, but there may be *some* schemes of supply or management of money, or *some aspects* of such schemes, that have no such influence, i.e. are neutral. It is not so much *money* itself that is or is not neutral, but a particular transformation of the money of an economy. In order to be able to discuss these aspects of schemes of management of money, it is necessary to introduce some new concepts. In particular, it is necessary to be able to *compare* one economy with another, and for this reason the concept of *an economy*, as an abstract but well-defined notion, is introduced. The neutrality aspect of money can then be captured by the notion that two economies *can be* economically similar.

2.2. *Inherent Stochastic Nature of Economic Models*

Economists, mathematicians, and computer scientists are experienced and comfortable with experiments in which the actions, and the effects, are all deterministic. But, if the degree of randomness is so great that it dominates the impact of deliberate actions, then it is hard to see how to conduct such experiments systematically.

One approach to address this difficulty is to ignore, or only consider at the margin, the effect of randomness. For example, the book (Osborne and Rubinstein, 2020) on economic models in general, and the specific model (Reserve Bank of Australia, 2018) of the Australian economy take this approach.

Another approach is to embrace the concept that an economy is stochastic as its key feature and apply general methods for observation, analysis, and control of time series, or stochastic processes (Crump et al., 2021), such as vector autoregressive (VAR) models, autoregressive integrated moving average (ARIMA) models (Petrică et al., 2016) and generalized autoregressive conditionally heteroskedastic (GARCH) models (Chávez et al., 2022).

Whereas macroeconomic and microeconomic models include structure which is able to *explain* the behaviour of economies, in particular, their response to economic interventions, general stochastic models include little in the way of explanatory structure, but are well adapted to modeling the randomness of economic phenomena.

The paper (Bhattarai, 2011) suggests that these two approaches are complimentary (overlooking the difficulty that real economies are both stochastic and structured). Keen (1995, 2011) suggests that modeling randomness by means of the chaotic behaviour of dynamic systems might resolve this problem.

Since the introduction of the landmark Black-Scholes-Merton option pricing model (Black and Scholes, 1972) in finance, stochastic differential equations (SDEs) and partial differential equations (PDEs) have been increasingly adopted in finance and economics (Achdou et al., 2014; Boucekkine et al., 2013; Fontanela et al., 2021). Despite their sophistication, they don't necessarily accurately model the character of the randomness inherent in these markets and economies, in particular, the heavy-tailed nature of certain critical random phenomena. The credit default of Scholes's and Merton's firm Long Term Capital Management (LTCM) was a forceful warning that details of a financial model can prove to be more important than expected when they are applied over an extended period.

Economists have also begun adopting SDEs or vector SDEs (VSDEs), such as (Ivanov and Swishchuk, 2008). Modeling both the randomness and the dynamics of an economy by means of a VSDE (Grinols and Turnovsky, 1998; Turnovsky, 1993, 2000; Turnovsky and Monteiro, 2007; Zhu, 1992) appears to form a more complete theoretical foundation for economic modeling and analysis than structural models on the one hand, and VAR models, or more complex time series (ARIMA or GARCH) models, on the other.

VSDEs are able to express the dynamics, or structural interconnections, of an economy because the equations, and in particular, the coefficients they include, reflect the assumptions of the modeler. Although a VAR model is less prone to arbitrary assumptions, and therefore might appear to be more scientific, in all scientific theories it is important to be able to propose and test hypotheses, and when these hypotheses are supported by experiments, they can be used to guide optimal control or management, which is one of the goals of economics. It is therefore important to use a model which is capable of expressing realistic hypotheses concerning the dynamics of an economy.

Another advantage of SDEs relative to time series models is the use of continuous time and the differential calculus. In some cases, important features of a dynamical system are better expressed in continuous time than in discrete time, and this advantage can also apply in stochastic systems. In particular, the dynamics of loans and other financial instruments (bonds and options for example) is well modeled by SDEs (and VSDEs) partly because of their use of continuous time.

Although VSDEs provide a satisfactory theoretical model of an economy, their complexity remains an impediment to their application by a broad range of economists, and the researchers who make use of them often restrict consideration to quite

simplified economies (for example, including a single, or a very small number, of commodities).

The type of economic model which predominates in studies of monetary aspects of economics assumes that all households and organisations in the economy have well defined preferences, or a *utility function*, and that they act consistently with these preferences (Ben-Gad, 2003), (He et al., 2023), (Ortigueira, 1998). This enables the model to describe the dynamics of an economy, and predict its behaviour. Random shocks and decisions can be included which enables this to be a stochastic model. The utility function can be stochastic (Blavatsky, 2008). By adopting abstract preferences, or an abstract, non-specific utility function, with some general characteristics such as transitivity, a convenient degree of abstraction can be adopted.

Although this is basically a microeconomic model it can be used for modeling major features of economies, as in, for example (Friedman and Bordo, 2017a), (Woodford, 2001a), (Ducoudré, 2005). It provides a stochastic model for an economy, although the way this method is applied tends to make stochastic effects easy to ignore and it readily slips into making predictions which can't be interpreted literally. The concepts of the present paper provide an alternative mathematical basis for this type of economic reasoning. Modeling the choices of households and organisations is not done by modeling their preferences but instead by assuming that they make the *same* choices in the *same* circumstances. Section 4.1, for example, can be viewed as an alternative, more rigorous, way to present the quantity theory of money argument given in (Friedman and Bordo, 2017a, §1.v).

In this paper we adopt a vector stochastic different equation (VSDE) model of an economy in such a way that restricting the number of products and services that is modeled is not necessary. Because we are primarily concerned with analysing economic similarities, the complexity of the VSDE will not trouble us. Except in the case of loans, the details of how products and services are created, consumed, or transferred will not enter our experiments. Instead of adopting a VSDE model, we could have used a micro-economic model of the sort used by Friedman and Bordo (2017a), and the arguments in the remainder of the paper would be similar. The most significant difference arises in the treatment of loans, and, specifically, in the repayment strategies for loans. The ease with which VSDEs give rise to differential equations is convenient for modeling loans and is used in Section 4.2.1 to produce the most surprising result of the paper, Corollary 1, which is that the repayments for the corresponding loan in a similar economy are, in most cases,

“front loaded” relative to the original economy. The implications of this result flow through to Subsection 4.2 and Section 7.

2.3. *Economic Drivers of Inflation*

We will be examining inflation in relation to the underlying repayment rate and also the interest rate, whilst excluding any variables that don’t have a structurally significant impact of inflation. Having said this, it is still worthwhile stating that the key economic drivers for inflation are; (1) production costs, (2) demand, (3) monetary policy, (4) net capital inflow, and (5) net exports (Reserve Bank of Australia, 2023). Since our paper is concerned with interest rates and repayment rates, we will focus on monetary policy as the primary driver and not concern ourselves with drivers such as imports and exports, wages and jobs.

Although there is a great deal of debate about the root causes of hyperinflation, it is generally accepted that it emerges when there is an unchecked increase in the money supply, and is often associated with wars (or their aftermath), economic depressions, and political or social upheavals (Makochekanwa, 2007). The school of thought which can be termed the *quantity theory of money* posits that inflation is determined solely by the change in the relative supply of money and goods (Friedman and Bordo, 2017a; Kwon et al., 2009). The conclusions of this paper do not contradict this thesis; in Section 6 the money supply in a similar economy is analysed, leading to a rigorous and precise version of the quantity theory of money, and in Section 7, this thesis is adopted while analysing the effect of raising interest rates.

It is also worthwhile to capture the following list of the top ten countries that have experienced hyperinflation and stagflation. The conventional marker for hyperinflation, proposed in 1956 by economist Cagan (1956), is 50% inflation per month. The following texts and papers together with their references provide a valuable historical summary of these countries’ economies, and whether they resolved their hyperinflation (such as Yugoslavia), post-poned it (such as Germany in the 1910s) or are still in it (such as Argentina).

- Argentina (1975-1991, 2022-present) (Fudge, 2010),
- Brazil (1979-1990) (Pereira et al., 1991),
- Turkey (2018-present) (Kibritçiöğlu, 2002),
- Israel (1983-1985) (Bruno and Minford, 1986),

- Germany (1920-1923) (Graham, 1930),
- Hungary (1945-1946) (Siklos, 1989),
- Venezuela (1985-2023) (Pittaluga et al., 2021),
- Yugoslavia (1992-1994) (Petrović et al., 1999),
- Zimbabwe (2007-2008) (Coomer and Gstraunthaler, 2011),
- China (1948-1949) (Campbell and Tullock, 1954).

This paper proposes a rigorous technical model of inflation based on VSDEs that can form a sound foundation for economists and econometricians to tackle this important topic systematically.

3. Methodology

3.1. Modeling an Economy

Suppose that states of an economy, E , come from a vector space, $\mathcal{E} = \mathbb{R}^m$, for some $m > 0$. An economic future (to time $T > 0$) takes the form $X : [0, T] \rightarrow \mathcal{E}$ where, for all $t \in [0, T]$, $X(t) \in \mathcal{E}$. Each coordinate of an economic state describes an aspect of the economy by means of some numerical method of measurement. Different aspects of an economy are logically measured by different methods. Coffee consumption might be measured in cups, or litres. Many important economic parameters are naturally measured in currency. Prices, on the other hand, are logically measured in units of currency per quantity of material, for example \$/litre. Concrete is naturally measured in Kg, power in Watts, etc.

For greater flexibility, coordinates of $X(t)$ which refer to goods and services in their non-currency quantities will usually be expressed as the *cumulative* consumption or production of the quantity.

Example 3. States of an Economy

The space of states, \mathcal{E} , could be made up of a list of 10, or 1,000,000, or 10^{10} numerical quantities which describe a future record of the economy – details like “Reginald Axelrod purchased 3 cups of coffee between t_0 and t_1 ” – this particular measurement might be stored at coordinate 7354 in the vector $X(t)$, in which case $X_{7354}(t_1) - X_{7354}(t_0) = 3$. □

3.2. Economic Similarity

Definition 1. Economic Similarity. Suppose E_1 and E_2 are two economies, with states from $\mathcal{E}_e = \mathbb{R}^{m_e}$, $e = 1, 2$, and $F_e = \{X : X : [0, T] \rightarrow \mathcal{E}_e\}$ is the space of possible futures in E_e , $e = 1, 2$. Let $\mathcal{R} \subseteq \{1, \dots, \min(m_1, m_2)\}$, be a subset of the coordinates of \mathcal{E}_1 and \mathcal{E}_2 . These are the coordinates which represent real goods and services.

A mapping, $\mu : F_1 \rightarrow F_2$ with the property for all $X \in F_1$, for all $k \in \mathcal{R}$, $\mu(X)_k(t) = X_k(t)$, $t \geq 0$, is termed an economic similarity. In words, to each path through the future of economy E_1 there is a corresponding path through the future of economy E_2 in which the record, for coordinates in \mathcal{R} , is the same.

If $E_1 = E_2$ and $F_1 = F_2$, a similarity is termed a self-similarity.

If $E_1 \subseteq E_2$ a similarity is termed a sub-similarity. If $E_1 \supseteq E_2$ a similarity is termed a super-similarity. If $\mu : E_1 \rightarrow E_2$ is a sub-similarity, E_2 is referred to as an extension of E_1 .

Definition 2. Conservation of Money. Suppose E_1 is an economy with coordinates $1, \dots, m_1$, and $\mathcal{C} \subseteq \{1, \dots, m_1\}$, (view these as expressed in terms of a common currency). If

$$\sum_{i \in \mathcal{C}} X_i(t) = 0, \quad t \in [0, T].$$

we say the economy E_1 conserves money on \mathcal{C} .

Conservation of money will hold if the coordinates in \mathcal{C} represent the levels of a self-contained system of accounts in which double-entry bookkeeping is applied consistently, errors in account levels do not occur, there are no duplicate currency coordinates, and the central bank maintains an account which exactly balances all the fiat currency that it has created.

Definition 3. Money Conserving Similarity. Consider an economy E_1 with real coordinates \mathcal{R} and currency coordinates \mathcal{C} . An economic similarity, $\mu : E_1 \rightarrow E_2$ is said to be money conserving if whenever E_1 is money conserving on \mathcal{C} , E_2 is also money conserving on \mathcal{C} .

Intuitively, the changes brought about by the mapping μ , when it is a similarity, are not “really” significant. For example, they “only” constitute changes to levels

of certain accounts, or quantities of currency (perhaps of several different types of currency). A sub-similarity, on the other hand, adds details to the activity in an economy without changing anything in the original subset of coordinates, and a super-similarity removes details that are not mapped identically.

In our application of economic similarities to inflation, \mathcal{R} is the set of coordinates not denominated in currency but, we begin with an example where this is *not* the way the “real” coordinates of the economic similarity are selected.

Example 4. Management of a Pandemic

Suppose a global pandemic occurs which prevents the operation of restaurants for 12 months, but no other commodities or services are directly affected. The government decides to close all restaurants and provide financial compensation for all restaurant employees, suppliers, and owners for the period of the pandemic. The compensation for those in or affected by the changes in the restaurant industry is funded by a new tax which is targetted on restaurant *patrons*, which matches the cost of eating out which these patrons are saving by not going to restaurants.

This situation is modeled by setting E_1 to be the economy in which the global pandemic doesn’t occur, and E_2 is the economy in which the pandemic occurs but the new tax rebate, and tax, are applied. All the coordinates associated with restaurants, both the actual quantities and the money associated with these activities, are *not* in \mathcal{R} . The economic similarity maps non-restaurant activity in E_1 to the equivalent non-restaurant activity in E_2 . Restaurant production and consumption activity in E_1 is mapped to the financial events associated with taxes and rebates, a tax on the citizen or agent paying money, and a rebate for the citizen or agent receiving money. Every path in E_1 is mapped to a path in E_2 , and every event in each path in E_1 is mapped to an event in the path in E_2 , but only the coordinates in \mathcal{R} are mapped identically. The currency available to each citizen when they contemplate a non-restaurant activity is exactly the same in E_2 as in E_1 because of the taxes and rebates.

Because restaurant patrons are taxed at a level which matches what they would normally spend on patronising restaurants, their decisions concerning all other products, services, and activities, should be the same as they would have been if there had been no pandemic. In this way, an economic similarity is constructed which shows that inflation and unemployment will be the same in E_2 as in E_1 . \square

This example closely parallels the strategy which was used in many economies during the Covid-19 pandemic of 2020–2022. The main difference between the

narrative proposed here and what occurred in the real world is the use of a tax on restaurant clients in Example 4. Some sort of tax is not infeasible (although targeting restaurant patrons would be difficult), however, given that it would be quite unpopular, it is not surprising that such a tax was rarely if ever introduced. This example shows that the concept of economic similarity is already, in an informal manner, understood by economists. Examining the concept more explicitly is important on account of this existing informal recognition.

This example also introduces the concept of an *approximate* economic similarity which extends the concept to a greater range of applications.

Economic similarity and monetary neutrality are not synonymous. Economic similarity is defined in terms of the “real” parameters of an economy whereas monetary neutrality is defined in terms of the monetary parameters. For example, two economies which have more than one currency can be economically similar even though each of the currencies are not in a deterministic relationship with the corresponding currency in the other economy. Economic similarity can also exist between two economies with a single currency when certain categories of currency denominated parameters (eg investments) are not in the same deterministic relationship as the currency when it is used for commerce. Focusing on the real parameters rather than the monetary ones makes the concept of economic similarity more powerful for describing economic phenomena. For example it can be used to analyse economies in which two currencies are in simultaneous use, eg a fiat currency and a crypto-currency.

3.3. *Deterministic Changes*

For an economic similarity to be interesting and useful, we expect the coordinates not mapped identically (usually those representing money) to change. In some realistic economic examples the *change* that is caused between an original economy and a *modified* economy can be expressed as a deterministic function of the intervention.

By the *change* is meant the difference between the original economy, and the modified economy, due to the intervention, assuming that all the random effects, or choices, by the participants in the economy, are exactly the same. The most natural way to measure change is by subtraction of numerical values, but ratios of numerical values can also be used as a way to represent change, and other functions of two values could, in principle, also be used. The measure of difference, of

each component, might be different from one component to another of the vector of values being compared.

Careful selection of the comparison method used for each component can lead to the measure of *change* between the original economy and one where an intervention has occurred being a deterministic function of the intervention, even when the economy itself is stochastic. For example, comparisons between prices might use a ratio, whereas comparisons between quantities measured in product volumes (e.g. in kilograms), might use a difference.

In this paper, all quantities measured in currency will use a ratio to represent change. The change in the current inflation rate, will also be represented by a ratio.

3.4. The Stochastic Model

Let $X : [0, T] \rightarrow \mathcal{E}_1$ denote a possible future of an economy, E_1 . It is assumed that $X(t)$, and thereby the economy, E_1 , is modeled by a VSDE (Øksendal, 1998)

$$dX(t) = b_1(t, X)dt + \sigma_1(t, X)dW(t), \quad (1)$$

in which $b_1(t, X) \in \mathcal{E}_1$ is the *drift* term – representing the current trend, at any time, $\{W(t)\}_{t \in [0, T]}$ is an n -dimensional Wiener process (also known as Brownian motion), with values in \mathbb{R}^n , and $\sigma_1(t, X)$ is the *diffusion term*, a matrix with number of rows the same as the number of coordinates of the space \mathcal{E} and with n columns, to match the Wiener process.

Equation (1) as an Ito VSDE should be interpreted, more formally, as equivalent to an Ito stochastic integral:

$$X(t) = \int_0^t b_1(s, X)ds + \int_0^t \sigma_1(s, X)dW(s), \quad t \in [0, T]. \quad (2)$$

A mathematical explanation of this equation, with the assumptions adopted in regard to the coefficients b_1 and σ_1 , is provided in Appendix A.

Because each value, $X(t)$, of the process X in (1) is a vector of length m_1 , this single equation is equivalent to m_1 simultaneous scalar equations.

The term $\sigma_1(t, X)dW(t)$ models random shocks, or decisions, which are guided by the random movements of the process $W(t)$. We allow $b_1(t, X)$ and $\sigma_1(t, X)$ to depend on the past of X , so, in general, X is not a diffusion process. However, in

the case of loans, the dependence on X will be only on its most recent value, $X(t)$. A collection of possible evolutions (showing one coordinate only) is depicted in Figure 2.

Equation (1) expresses how the economy evolves through time, partly due to dynamic relationships between one part of the economy and others, and partly due to random choices, disturbances, or shocks. The shocks are provided by the multidimensional Wiener process, $W(t)$. When VSDEs are simulated (as we do in Figures 2 and 3) the Wiener process is provided by a random number generator in which each random number is independent from every other and they all have a Gaussian (normal) distribution.

When an economic similarity is applied and the changes from one economy to another are determined, the “choices and shocks” terms will always, when these equations are used in a further derivation, either cancel, or be small enough to neglect. The precise details of how this component of an economy are modeled is often, therefore, not relevant.

Only some transformations have the property that the random components exactly cancel: these transformations must typically be economic similarities. It is necessary, therefore, to ensure, when considering a transformation of an economy in which the random choices and shocks subtracted from each other, that they do cancel. For example, it is often necessary to divide an equation by a the level of a loan. Cancellation of these terms, from two different economies, will not be valid even when they come from economically similar economies unless the terms have been rescaled appropriately.

In Examples 4–7, an economic similarity is used to undertake economic experiments and the random choices and shocks must cancel approximately or exactly for the transformation to be an approximate or exact similarity. In Examples 4 and 7 the similarity is approximate and therefore the random terms in the VSDE cancel *approximately*, while in Examples 5 and 6, these random terms cancel exactly. In addition to these four examples, Theorem 1 provides a large class of exact economic similarities which apply to any initial economy.

Example 5. A Coffee Economy

Consider an economy consisting of a student with an allowance of \$ a per day. Suppose coffee costs \$ c per cup. The student’s only consumption is coffee, which is consumed randomly, only when the process $W_1(t) \bmod 2 < 1$ and when the

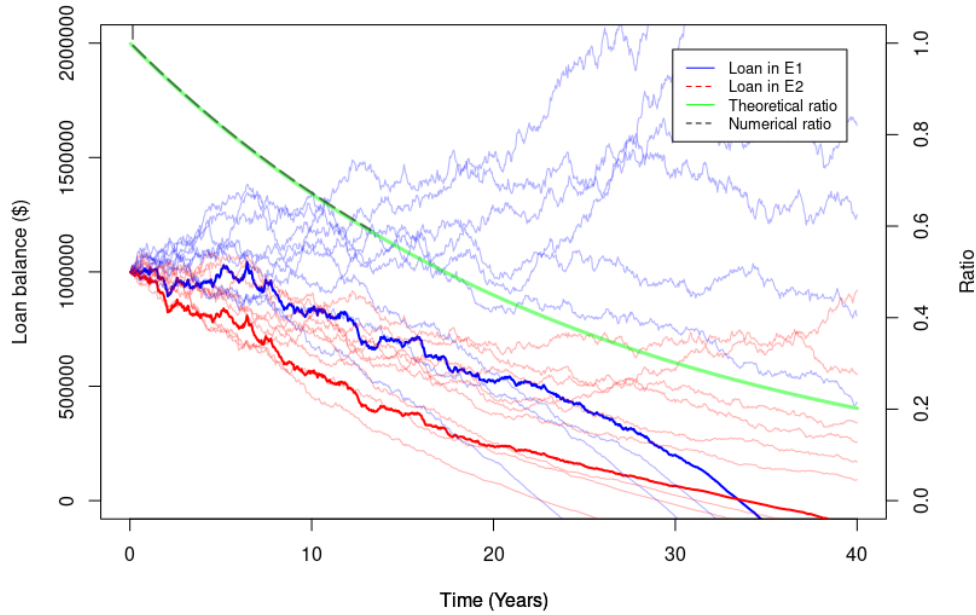


Figure 2: Multiple pairs of simulations compared

The blue paths are the simulated balance in a loan. The up movements correspond to someone refinancing their loan to a higher amount, whereas the downward movements correspond to someone paying down the loan earlier than expected. For each blue path there is a corresponding red path which plots the loan balance in the economically similar economy in which inflation is lower. The first pair of paths has been plotted with a bolder and thicker line so that the reader can easily see the pairing of simulations. The green path is the common ratio of the blue path to its red pair, or rather, the theoretical value of this ratio, under the assumption that discretization error in the simulations has been eliminated. The dashed black curve is the actual numerical ratio of the blue curve to the corresponding red curve, in a simulation, in one of the pairs. Until numerical error causes discrepancies, this numerical ratio is identical for all pairs of paths.

student has $\$c$ available cash. This simple economy can be modeled by the vector VSDE, for process X ,

$$dX_2(t) = \mathbb{1}_{\{t:X_1(t)>c\}} \mathbb{1}_{\{t:W_1(t) \bmod 2 < 1\}} dt, \quad (3)$$

$$dX_1(t) = (-c \mathbb{1}_{\{t:X_1(t)>c\}} \mathbb{1}_{\{t:W_1(t) \bmod 2 < 1\}} + a) dt, \quad t \in [0, T]. \quad (4)$$

The symbol $\mathbb{1}_A$ denotes the function taking the value 1 on A and 0 elsewhere. Because $W(t)$ and $X(t)$ are stochastic processes they are functions of $\omega \in \Omega$, and hence $\mathbb{1}_{\{t:X_1(t)>c\}}$ for example, is also a function of $\omega \in \Omega$. Suppose economy E_1 is modeled by (3)–(4) with the student's allowance set to $a = \$0.48$, and the cost of a cup of coffee to $c = \$1.00$.

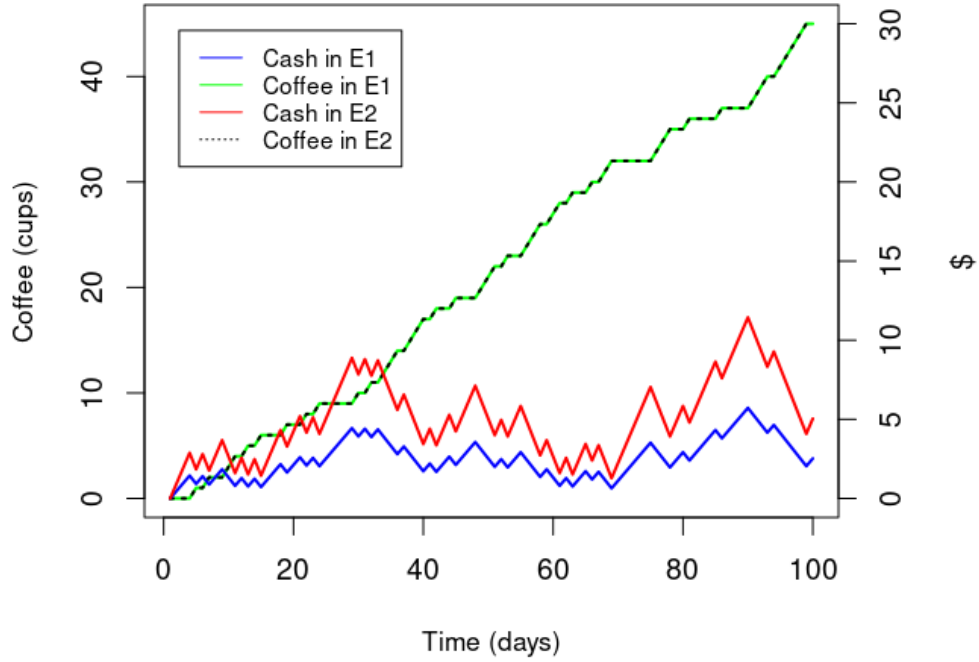


Figure 3: A Coffee Economy

In this example the first coordinate, in \mathcal{E}_1 , records how much coffee the student has consumed. The second coordinate, which is expressed in currency, is a record of the funds available for expences. The student is only able to consume coffee when the cash coordinate is $\geq \$c$. This is a realistic constraint which is likely to

arise in many forms in real economies. This is the sort of constraint which we need to include in an accurate economic model because it represents the dependence of events on the available resources.

An economically similar economy, E_2 say, is obtained by changing the price of coffee from \$1 to \$2 and changing the allowance of the student from 0.48 \$/day to 0.96 \$/day. The two economies are depicted in Figure 3. They are similar because the coffee consumption is the same, but they are not identical because the cash evolution is different. In this particular case, the cash in E_2 is exactly twice the cash in E_1 . The relationship between similar economies can be more complex, but the idea that this relationship can be non-random, even though each economy is random is well illustrated in this example and can remain true even when the relationship changes in a more complex way over time.

In order to strictly accord with the definition of an economy which has been given, there needs to be an additional equation, for each economy:

$$dX_3^{\{e\}}(t) = dW_1(t), \quad t \in [0, T], \quad (5)$$

for $e = 1, 2$, in the VSDEs. The references to $W_1(t)$ in (3)–(4) can then be replaced by references to $X_3^{\{e\}}(t)$. The superscript, $\{e\}$, refers to the economy.

Assuming Equation (5) has been added, and (3)–(4) have been modified to refer to $X_3^{\{e\}}(t)$, the random terms which cancel are just the term $dW_1(t)$, which appear in the two versions of (5), one for each economy. The equation which results from subtracting one equation from the other leads to the result, $X_3^{\{1\}}(t) = X_3^{\{2\}}(t)$. This equation can then be used with (3)–(4), for both economies, to deduce $X_1^{\{1\}}(t) = X_1^{\{2\}}(t)$ and $X_2^{\{1\}}(t) = 2X_2^{\{2\}}(t)$, $t \in [0, T]$. \square

Figure 2 shows the possible paths for a single coordinate of an economy over time, first in the original economy, E_1 , and then in a second economy, E_2 , which is the result of an intervention (a modification of E_1). The *ratio* between the loan balance in E_1 and in E_2 is also plotted, and it turns out to be a deterministic function, i.e. it does not depend on the specific path of the loan.

In the remainder of this section, we suppose that the quantities referenced in this economic model are *only* goods and services, *not including* financial services. In particular, these economies do not include debt, investment, or bank accounts. Loans will be added to the model in Section 4.2.

In general, modeling the motivations, forces, and influences which connect different aspects of an economy, is complex. One approach could be, for example, to encode these in the equations of the VSDE. This encompasses arbitrarily complex, and therefore realistic, dynamics. These equations will, in general, be difficult to estimate, or even interpret, however, simplified models of this type have been successfully employed in a number of papers (Grinols and Turnovsky, 1998; Turnovsky, 1993, 2000; Turnovsky and Monteiro, 2007; Zhu, 1992).

In this paper, we adopt an important specific assumption which affects the dynamics, namely that all the economic decisions, influences, forces, and interactions between different elements of one economy, expressed in quantitative terms (i.e. not in terms of currency) are *exactly the same* in a modified economy. In a sense, the differences between these two economies are unimportant. However, the two economies might have different prices, different quantities of money borrowed, and taxed, and different rates of inflation, so, if analysing or controlling inflation is a matter of interest, the difference between the economies is not *entirely* trivial. In other words, the economies under consideration are *economically similar*, in the sense of Definition 1.

In the present instance, given that our objective is to analyse the relationship between inflation and interest rates, this assumption is not so rigid as to rule out revealing and analysing the underlying dynamics by means of which interest rates, loan repayments, and inflation interact, which is the main goal of the paper.

In order for a similarity to represent inflation, the change of the VSDE needs to be explained, precisely, by the behaviour of all citizens and other actors being due to their changed perception of the value of money. Such similarities shall be referred to as inflation similarities. A precise form for such a similarity is given in (9)–(10) in the next section.

4. Experiments

4.1. First Experiment: Consumption and Inflation

In this experiment, we consider two economies with a mapping between them which is an economic similarity in which it is precisely the coordinates not denominated in currency which are included in \mathcal{R} . The consumption and production in these two economies is therefore exactly the same. We assume that both economies have just one currency.

4.1.1. Comparing Two Economies

Suppose all the paths of the economy E_1 satisfy (1). In addition, suppose we have a second economy, E_2 , which has states in $\mathcal{E}_2 = \mathcal{E}_1$, and the future of this economy is recorded in the process $\{Y(t)\}$, which satisfies the VSDE

$$dY(t) = b_2(t, Y)dt + \sigma_2(t, Y)dW(t), \quad t \in [0, T]. \quad (6)$$

Let F_e denote the space of possible paths in economy E_e , $e = 1, 2$, so $X \in F_1$ and $Y \in F_2$, and we now seek to find conditions on b_2 and σ_2 so that an economic similarity, $\mu : X \mapsto Y$, exists.

The Wiener process, $W(t)$, is the same in (1) and (6). This signifies the fact that the choices (decisions, shocks) which are made, or which occur, are the same in both economies. The VSDE coefficients $b_2(t, Y)$ and $\sigma_2(t, Y)$, which appear in this equation are different, but the differences between them and the corresponding coefficients in (1) are systematically made to ensure that an economic similarity holds, between the two economies. One way in which these systematic changes can be made is shown below, in (9)–(10).

Definition 4. Inflation Similarity

A similarity of critical interest in this example simply scales all quantities which are measured in currency, at time t , by a factor of the form

$$A(t) = \exp\left(\int_0^t a(s)ds\right), \quad t \in [0, T], \quad (7)$$

where $a(t)$ can be interpreted as the the difference between the rate of inflation in E_2 and in E_1 . It is assumed that $|a(t)|$ is bounded over all finite intervals.

For any process, Y , define $A[Y]$ to be a process, \tilde{Y} with the same form as Y such that at any coordinate, j , which is expressed in terms of currency,

$$\tilde{Y}_j(t) = A(t)Y_j(t), \quad t \in [0, T], \quad (8)$$

and at other coordinates, $\tilde{Y}_j(t) = Y_j(t)$.

Theorem 1. *Define the coefficients of the VSDE (6), as follows. At coordinates, j , which are expressed in terms of currency, for all $k > 0$, set*

$$b_2(t, Y)_j = A(t)b_1(t, A^{-1}[Y])_j, \quad (9)$$

and

$$\sigma_2(t, Y)_{jk} = A(t)\sigma_1(t, A^{-1}[Y])_{jk}, \quad (10)$$

for all $t \in [0, T]$, and at other coordinates set $\sigma_2(t, Y)_{jk}$ to be $\sigma_1(t, A^{-1}[Y])_{jk}$ and $b_2(t, Y)_j$ to be $b_1(t, A^{-1}[Y])_j$.

The mapping $\mu : E_1 \rightarrow E_2$, $X \mapsto Y = A[X]$, in which prices and all quantities measured by currency are multiplied by $A(t)$, is an economic similarity, with X a solution of (1) if and only if Y is a solution of (6).

Two proofs of this theorem are given in Appendix B.

Remark 1. *Theorem 1, and its proof, bears some resemblance to the argument presented in (Friedman and Bordo, 2017a, §III–VII), where a helicopter delivers money to all households in an economy and the result is to change prices. Whereas Friedman uses arguments based on human behaviour (by means of uncontroversial assumptions concerning their preferences), here, by taking advantage of the concept of economic similarity, the argument for Theorem 1 is mathematical, and assumes only that the choices of households and organisations are the same in the two economies being compared. The argument in (Friedman and Bordo, 2017a, §III) is, nevertheless, also based on ideas quite similar to economic similarity. For example, they write:*

“Once again, the helicopter has changed no real magnitude, added no real resources to the community, changed none of the physical opportunities available.”

Recall that in Subsection 3.4 we remarked that in order for a similarity to represent inflation, the change of the VSDE needs to be explained by the behaviour of all citizens and other actors being due to their changed perception of the value of money. This is the “meaning” of the $A^{-1}[Y]$ in (9)–(10). The term $A^{-1}[Y]$ in the definition of b_2 and σ_2 models the fact that the perception of citizens has been updated. Neither the theorem nor its proof *relies* on the fact that citizens’ perception of the rate of inflation is updated immediately. Instead, it is asserting that *if* perceptions *are* updated (after some time, for example), in this manner, the resulting economy will be similar. The issue of expectations concerning inflation is addressed again in the discussion after Theorem 3 and its corollary.

Note that the function $A(t)$, or $A^{-1}(t)$, which is applied in (8) and (9)–(10), must be *consistent* for all quantities measured in a specific currency in order to be regarded as an *inflation* similarity. The concept that inflation might be different for

different sectors of an economy is feasible, for example if these sectors operate at quite different time scales, or if there is a significant transaction cost for entering the sector. If different currency-valued coordinates of an economy were to inflate at different rates, the economy described by (9)–(10) would not be consistent with the economic constraints of the original economy. Such a transformation would not be money conserving, as in Definition 3. Constraints which originally expressed the fact that agents must have enough money to fund their purchases, for example, would no longer say that in the “similar” economy unless the function $A(t)$ is consistent for all such coordinates.

This economic similarity has some important and unexpected consequences.

In particular, it shows that if an economy is stable with inflation rate α_1 , then it will also be stable with inflation rate $\alpha_2 \neq \alpha_1$ and identical consumption and production. Every detail of production and consumption in the original and modified economies, will be exactly the same. It is not necessary to assume that the inflation rate is constant. The rate of inflation can be an arbitrary real valued function of time, without invalidating this similarity. Since *cash* does not earn interest, this similarity assumes that there are no holdings of conventional cash. Although this is not strictly true of a modern economy, because most rational citizens take care to store their financial reserves in such a way that they increase at the highest rate compatible with prudent risk management, it is not an unrealistic assumption, at moderate levels of inflation. For example, Subrick (2010) does not include this issue in the six reasons given for non-neutrality of money. It seems likely that this is not due to accidental omission, but rather to the unimportance of this effect, relative to others, such as the effect on relative income due to inhomogeneous supply of money (referenced as Cantillon effects).

If the change of inflation rate is very high, on the other hand, this assumption is not valid, and the fact that cash deflates rapidly becomes highly significant.

The VSDE model of an economy (1) can include cash, which loses value (purchasing power) steadily at the rate of inflation, α_1 . However, if E_1 is such an economy, which contains cash, and E_2 is the economy modeled by (6), where $Y = A[X]$, the cash in E_2 will now lose purchasing power at the rate α_1 , not at the rate α_2 , as might be expected. This is not a defect of the similarity. It reveals, quantitatively, the important side effect of high levels of inflation, which is that the value of uninvested cash erodes quickly.

Another discrepancy of this formal concept of inflation from the real world phenomenon is that if the price for a certain product jumps at a discrete sequence of

times in E_1 , and in E_2 the inflation rate is higher, consistently, by 1%, for example, although prices for the product will still jump at the same sequence of times, in E_2 , they will also inflate continuously, by an additional 1%, in E_2 . Continuous price increases of this sort are not a natural phenomenon in most economies. However, a continuous price increase of this sort can be approximated to arbitrary accuracy by a process with discrete price jumps, so this discrepancy can be regarded as acceptable modeling error.

4.2. Second Experiment: Adding Finance

4.2.1. Modeling Loans

Now let us add financial instruments to the economy E_1 . Let us examine the specific equations which appear in the VSDE (1) which relate to loans.

We assume both economies include L loans. Each loan has an index (or an identity number), which will be denoted, in all the equations which refer to loans, by ℓ . We do not assume the interest rate, $i_\ell(t)$, on loan ℓ , is the same for all loans, and the interest rate for each loan might change over time.

It will be assumed that interventions affect all loans similarly, for example, if interest rates are increased, the increase is the same for all loans. Since this is true for both existing loans and new loans, this assumption implies that we assume all loans adopt a variable rate, i.e. the rate of an existing loan can be changed when the central bank changes its default rate.

The dynamics of a loan can be modeled by the following equations, (11)–(15). We denote the level in the loan account ℓ (the debt), by $\lambda_e^{\{\ell\}}(t)$, where $e = 1$ or 2 denotes the economy, and the cash available to the borrower due to loan ℓ at time t by $\rho_e^{\{\ell\}}(t)$. The spending of this cash is also part of the economic model, but we do not need to concern ourselves with it, since, by similarity, whether there is sufficient cash for a transaction or not will be the same in both economies, so long as we make sure that $\rho_e^{\{\ell\}}$ has the same real value in both economies for all loans. The interest rate in loan ℓ , in economy E_e , is denoted by $i_e^{\{\ell\}}(t)$, the rate of inflation at time t in E_e is denoted by $\alpha_e(t)$ and the rate of repayment, relative to the debt, is denoted by $r_e^{\{\ell\}}(t)$. The start time of loan ℓ is denoted by t_ℓ and its duration by T_ℓ . These will usually be the same in all the economies being compared. Lastly, the amount borrowed, at time t_ℓ is denoted by $P_e^{\{\ell\}}$. This will usually vary from one economy to the other.

In the full model, (1) or (6), the level of debt and the available cash associated with a certain loan will be a coordinate of the vector function $Y(t)$, but to focus attention on just these coordinates, we can express these parts of (1) or (6) which apply to $\lambda_e^{\{\ell\}}(t)$ and $\rho_e^{\{\ell\}}(t)$ thus:

$$\lambda_e^{\{\ell\}}(t) = 0, \quad t < t_\ell, \quad (11)$$

$$\lambda_e^{\{\ell\}}(t_\ell) = P_e^{\{\ell\}}, \quad (12)$$

$$\begin{aligned} d\lambda_e^{\{\ell\}}(t) &= b_{1,\lambda}(t, X)dt + \sigma_{e,\lambda}^{\{\ell\}}(t, X)dW(t), \\ &= \left(i_e^{\{\ell\}}(t) - r_e^{\{\ell\}}(t)\right)\lambda_e^{\{\ell\}}(t)dt + \sigma_{e,\lambda}^{\{\ell\}}(t, X)dW(t), \quad t > t_\ell \end{aligned} \quad (13)$$

$$\begin{aligned} \rho_e^{\{\ell\}}(t) &= 0, \quad t < t_\ell, \\ \rho_e^{\{\ell\}}(t_\ell) &= P_e^{\{\ell\}}, \end{aligned} \quad (14)$$

$$\begin{aligned} d\rho_e^{\{\ell\}}(t) &= b_{e,\rho}^{\{\ell\}}(t, X)dt + \sigma_{e,\rho}^{\{\ell\}}(t, X)dW(t). \\ &= -r_e^{\{\ell\}}(t)\lambda_e^{\{\ell\}}(t)dt + \sigma_{e,\rho}^{\{\ell\}}(t, X)dW(t), \quad t > t_\ell. \end{aligned} \quad (15)$$

In order that (12) holds, $b_{e,\lambda}(t, X)$ must have an impulse (a δ -function) of magnitude $P_e^{\{\ell\}}$ as its “value” at $t = t_\ell$. A similar observation applies to $b_{e,\rho}(t, X)$. Once this has been recognised it is clear how the three equations (11)–(13) are equivalent to one row of (1) or (6).

It is natural to assume that $\alpha_e(t)$ and $i_e^{\{\ell\}}(t)$ are constant over time, or vary slowly over time, but this assumption is not appropriate for $r_e(t)$, which varies between 0 and 1 and in most cases will increase over time, eventually reaching 1.

An alternative to (13), which expresses repayments as a rate in \$/year, is as follows:

$$d\lambda_e^{\{\ell\}}(t) = \left(i_e^{\{\ell\}}(t)\lambda_e^{\{\ell\}}(t) - \hat{r}_e^{\{\ell\}}(t)\right)dt + \sigma_{e,\lambda}^{\{\ell\}}(t)dW(t), \quad (16)$$

in which

$$\hat{r}_e^{\{\ell\}}(t) = r_e^{\{\ell\}}(t)\lambda_e^{\{\ell\}}(t) \quad (17)$$

is the rate of repayments expressed in current currency per year. Since (13) has certain advantages which (16) lacks, and conversely, we shall make use of both models. The rate $\hat{r}_e^{\{\ell\}}(t)$ will be referred to as the rate of repayment of loan ℓ in current currency (or cash), while the rate $r_e^{\{\ell\}}(t)$ will be referred to as the rate of repayment of loan ℓ relative to the loan balance. As with $\alpha_e(t)$ and $i_e(t)$, it is natural to assume that $\hat{r}_e(t)$ is constant over time, as a default choice.

4.3. *Reflections on Loan Modeling*

In a middle sized economy there are millions of individual loans and therefore, in the VSDE which models the economy there needs to be millions of loan modeling SDEs all of which hold simultaneously.

These equations do not need to be written down, or solved, or individually manipulated, in order for their role in modeling loans to be useful. An economy typically includes millions of simultaneous loans which are playing out, and influencing peoples' behaviour.

If these equations are not written down explicitly, how they can be of any use? The answer lies in the widely accepted proposition that “money is debt” (Lawson, 2018; Ryan-Collins et al., 2012). Strictly speaking, money can't be *identical* to debt because we use these terms in different senses. Nevertheless, debt and money are linked in such a way that money can be managed by the simultaneous regulation of the loans in an economy. In fact, this is already a fair description of the way central banks operate.

By means of the VSDEs which model the loans of an economy, without ever writing them down, we can determine the type of simultaneous transformations of all loans that can exist in a similar economy, with different inflation. What this shows is surprising and unorthodox, but once seen, makes sense. It is that, as well as changing the interest rates of loans, we need to change their repayment schedules. This result is theoretically sound, original, and important. It is also totally consistent with monetarism: we need to control the growth rate of money, so why not do it directly?

Once it is acknowledged that debt is the source of money, it follows that monetary regulation can and should be achieved by consistent management of the parameters of loans. These parameters include their principal, intensity of occurrence, duration, interest rate, and repayment rate. Of these parameters, the one which is currently employed, as the key strategy in management of inflation, is the interest rate. We have identified that regulation of the repayment rate should also be employed because doing so enables a transition to a similar economy and failing to do so *prevents* the resulting economy from being similar. The current default practice in regard to repayment rates does not leave repayment rates unchanged, in real terms. The proposal to regulate these repayment rates is not, therefore, an innovation, but rather an extension of rational control to another aspect of debt.

4.4. Similarity of Loans

In order to achieve the economic similarity between the original economy, E_1 , and the changed economy, E_2 , it is sufficient to assume, as discussed above, in Subsubsection 4.1.1, that, at coordinates, j , which are expressed in currency,

$$b_{2,\lambda}^{\{\ell\}}(t, Y)_j = A(t)b_{1,\lambda}^{\{\ell\}}(t, A^{-1}[Y])_j \quad (18)$$

$$b_{2,p}^{\{\ell\}}(t, Y)_j = A(t)b_{1,p}^{\{\ell\}}(t, A^{-1}[Y])_j \quad (19)$$

$$\sigma_{2,\lambda}^{\{\ell\}}(t, Y)_j = A(t)\sigma_{1,\lambda}^{\{\ell\}}(t, A^{-1}[Y])_j \quad (20)$$

and

$$\sigma_{2,p}^{\{\ell\}}(t, Y)_j = A(t)\sigma_{1,p}^{\{\ell\}}(t, A^{-1}[Y])_j \quad (21)$$

and at other coordinates, $b_{2,\lambda}^{\{\ell\}}(t, Y)_j = b_{1,\lambda}^{\{\ell\}}(t, A^{-1}[Y])_j$, ..., and $\sigma_{2,p}^{\{\ell\}}(t, Y)_j = \sigma_{1,p}^{\{\ell\}}(t, A^{-1}[Y])_j$.

This is not necessarily the only way to achieve economic similarity of E_2 with E_1 . For example, if the economy E_1 contains two different currencies which are exchangeable, and some transactions can use either currency, the inflation of these currencies could be changed independently from each other leading to an economically similar economy. It is also feasible that property values or equity values might experience inflation at a different rate from currency in an economic similarity. However, for simplicity, we restrict consideration to the case of one currency with (18)–(21) determining the transformation of the VSDE.

4.4.1. Economic Similarity of Loans

A first step required to ensure that two economies, E_e with interest rates $i_e^{\{\ell\}}(t)$, $\ell = 1, \dots, L$, loan repayment rates $r_e^{\{\ell\}}(t)$, $\ell = 1, \dots, L$ and inflation rates $\alpha_e(t)$, $e = 1, 2$, to be similar is for the differences $i_e^{\{\ell\}}(t) - r_e^{\{\ell\}}(t) - \alpha_e(t)$ to be a certain common constant for all loans. If this numerical difference is $\Delta_1(t)$, say, in the case of first economy, then in the second economy (in which the actual consumption and production is exactly the same), the differences $\Delta_2(t) = i_2^{\{\ell\}}(t) - r_2^{\{\ell\}}(t) - \alpha_2(t)$ could also be $\Delta_1(t)$. This is equivalent to (18), when applied to the component (13) of the VSDE.

In these two economies, not only are the goods and services exchanged exactly the same, but also the level of debt, in each loan, is the same in terms of real value (i.e. currency values when rescaled to a fixed, common standard).

As in Section 4.1, our goal is that in the second economy, all consumption, and all production (including financial transactions), are exactly the same as in the first economy, when financial quantities are normalized to a common reference currency. Thus, not only levels of consumption and production, but also levels of inflation discounted currency, loan repayments, and interest payments should be identical in the two economies, reflecting that the underlying dynamics, as determined by the needs and wants of the human participants are the fundamental drivers of their decisions.

In this experiment we have shown that there is an economy (a complete record of its future), which exhibits exactly the same consumption and production, but not the same inflation, repayment rates, and interest rates.

The actual price of any item will not, in general, steadily increase at exactly the rate $\alpha(t)$, even if this is the true inflation rate of the economy, calculated relative to a broad, well-selected basket of goods and services. Inflation is not necessarily the same for different sectors of an economy at any one time. However, this does not prevent us from analysing precisely, and rigorously, the inflation due to the interest rate setting chosen for an economy. This is because we do not seek to understand the dynamics of the model of an economy itself, but only the *changes* or *differences* between the economy E_1 and E_2 .

4.4.2. Transformations

Let us now investigate the possible transformations of an economy, E_1 , by changes in α , i and r which produce a similar economy. Each of these transformations can potentially be used to *change the future* of an economy. When a transformation is an economic similarity, we have a good reason to interpret the transformation as having such an effect, because all the rational actors in the economy are acting in exactly the same way, in the effect on the goods they produce and consume. Transformations which are not economic similarities can't be interpreted in this way because actors will experience changed patterns of consumption and production, and therefore can't be expected to act in the same way.

Single Currency Assumption

We assume, in this section, that all loans are denominated in the same currency, and the transformations under consideration are applied uniformly across all loans.

Since the rate of growth (as measured by the first factor on the right in (13)) of the level of the loan must be exactly the same in E_2 as in E_1 , when measured in normalized (year 1) currency,

$$-i_2^{\{\ell\}}(t) + r_2^{\{\ell\}}(t) + \alpha_2(t) = -i_1^{\{\ell\}}(t) + r_1^{\{\ell\}}(t) + \alpha_1(t), \quad (22)$$

so

$$r_2^{\{\ell\}}(t) - r_1^{\{\ell\}}(t) = i_2^{\{\ell\}}(t) - \alpha_2(t) + \alpha_1(t) - i_1^{\{\ell\}}(t), \quad (23)$$

and

$$\alpha_2(t) - \alpha_1(t) = r_1^{\{\ell\}}(t) - i_1^{\{\ell\}}(t) - r_2^{\{\ell\}}(t) + i_2^{\{\ell\}}(t), \quad t \geq 0. \quad (24)$$

This equation (these are all restatements of the same equation) ensures that the net debt associated with a loan, in $t = 0$ currency, is the same in the two economies, as required by their economic similarity. This is not sufficient, but it is necessary, to ensure economic similarity. Let us therefore explore the possible transformations which are consistent with (22)–(24).

Inflation and Interest: Let us suppose we change α and i in such a way that (22)–(24) are respected, while leaving r fixed. It follows from (24) that, for some *shift*, $\Delta_{\alpha,i}(t)$,

$$\alpha_2(t) = \alpha_1(t) + \Delta_{\alpha,i}(t), \quad (25)$$

and

$$i_2^{\{\ell\}}(t) = i_1^{\{\ell\}}(t) + \Delta_{\alpha,i}(t). \quad (26)$$

Inflation and Repayments: Let us now suppose we change α and r in such a way that (22)–(24) are respected, while leaving i fixed. It follows from (23) that, for some *shift*, $\Delta_{\alpha,r}(t)$,

$$\alpha_2(t) = \alpha_1(t) - \Delta_{\alpha,r}(t), \quad (27)$$

and

$$r_2^{\{\ell\}}(t) = r_1^{\{\ell\}}(t) + \Delta_{\alpha,r}(t). \quad (28)$$

Interest and Repayments: Finally, let us suppose we change i and r in such a way that (22)–(24) are respected, while leaving α fixed. It follows from (23) that, for some *shift*, $\Delta_{i,r}(t)$,

$$i_2^{\{\ell\}}(t) = i_1^{\{\ell\}}(t) + \Delta_{i,r}(t), \quad (29)$$

and

$$r_2^{\{\ell\}}(t) = r_1^{\{\ell\}}(t) + \Delta_{i,r}(t). \quad (30)$$

Using any two of the transformations (25)–(29), we can explore all possible solutions of (22)–(24). It is not obvious that if all loans are transformed in exactly the same way by transformation (25), the resulting economy will be economically similar (because the cash available to a borrower or a lender might be different).

Cash Flow: Unless neither the borrower nor the lender make use of it, the cash flow associated with a loan, as measured by the first factor on the left in (15), must be exactly the same in E_2 as in E_1 , when measured in normalized (year 1) currency. Assuming either the borrower or the lender makes use of this cash flow, there will be some paths in E_1 where a consumption event which occurs which relies on it in E_1 but can't occur in E_2 , or conversely. Thus, necessarily,

$$P_2^{\{\ell\}} = A(t_\ell)P_1^{\{\ell\}} \quad (31)$$

and

$$r_2^{\{\ell\}}(t) = r_1^{\{\ell\}}(t) \quad (32)$$

$\ell = 1, \dots, L$. Equations (31)–(32) ensure that the cash flow due to a loan, in year 1 currency, is the same in the two economies, as required by their economic similarity. If both (22) and (31)–(32) hold, the two economies E_1 and E_2 are economically similar, so, the economic choices made by all individuals and organisations in E_1 must be exactly the same in E_2 . In effect, (31)–(32) is the form taken by (19) when interpreted in the context of (15).

Under transformations (25)–(26) the transformed economy will be economically similar, because the loan and its rate of change are similar, and since (31)–(32) holds, the cash flow is also similar (from the point of view of the borrower and the lender). However, under transformation (27)–(28) or (29)–(30), the transformed economy is *not* economically similar unless $r_1(t) = r_2(t)$, $t > 0$, and if $r_1 \equiv r_2$, both these transformations are the identity, and therefore of no interest.

4.4.3. Characterization of Inflation Similarity

Theorem 2. (Required in Similarity). *Any inflation similarity, $\mu : E_1 \rightarrow E_2$ with the property that for any extension of $E'_1 \supseteq E_1$, there is an extension $E'_2 \supseteq E_2$ and an extension $\mu' : E'_1 \rightarrow E'_2$ of μ which is also an inflation similarity must satisfy (22)–(24) and (31)–(32).*

The proof is given in Appendix C.

As discussed in Subsubsection 4.2.1, it is useful to express repayment schedules in current currency. The repayments in a similar economy can readily be expressed in this way, relative to the cash repayment rate in the original economy:

Corollary 1. *If E_1 and E_2 are economically similar according to the rule (9)–(10), by an inflation similarity, then*

$$\hat{r}_2^{\{\ell\}}(t) = \hat{r}_1^{\{\ell\}}(t) \exp \left(\int_0^t (\alpha_2(s) - \alpha_1(s)) ds \right) \quad (33)$$

for all loans, $\ell = 1, \dots, L$.

Proof All quantities measured in currency which appear in the vector of state of the economy must be transformed by (8), since the transformation is an inflation similarity. In particular, this is true of the quantity $\lambda_e^{\{\ell\}}(t)$. Hence, by (17),

$$\begin{aligned} \hat{r}_2^{\{\ell\}}(t) &= r_2^{\{\ell\}}(t) \lambda_2^{\{\ell\}}(t) \\ &= r_2^{\{\ell\}}(t) A(t) \lambda_1^{\{\ell\}}(t) \\ &= A(t) \hat{r}_1(t), \end{aligned}$$

as asserted. □

It was already shown in Subsubsection 4.1.1, that an economically similar economy is always obtained by means of the transformation of a VSDE defined by (9)–(10). This is checked, by simulation, in Subsection 5.3. The assumptions adopted in the simulations in Subsection 5.3 are more specific than those of Subsubsection 4.1.1. For example, in the simulations each loan is totally independent of each other loan, but this has not been assumed in Subsubsection 4.1.1.

This result applies to government loans as well as private loans, and therefore places strict limits on the growth of government spending. Failing to act in accordance with this guidance is not rational for a government or a central bank, because continuing to increase spending at a faster rate will devalue the currency, and therefore not achieve anything. On the other hand, strict adherence to government debt growth target, precisely as dictated by Theorem 2, will lead to precisely the same activity in the economy, while reducing inflation. This completely agrees with monetarist common sense, and therefore should not come as a surprise.

4.4.4. The Phillips Curve

The hypothesis expressed by the Phillips curve is that conventional economic management is able to trade off inflation against unemployment. The Phillips curve is a plot of the possible combinations of inflation (on the y axis), and unemployment (on the x axis). It was hypothesised by Phillips (1958) that this curve has negative slope. Evidence of this relationship in the form of unemployment and inflation rates in the U.K. in the period 1861-1957 were provided. Friedman (1977), however, argues that the relationship between unemployment and inflation can vary depending on the time scale, and has given examples where the slope of the Phillips curve is positive rather than negative.

The economic similarity (9)–(10), which also satisfies (25)–(26) and (31)–(32), shows that there is no reason to believe there should be any relationship between unemployment and inflation. Starting with an economy E_1 with unemployment $u\%$ and inflation $a\%$, picking any number $b > 0$, this transformation can give us a second economy, E_2 with identical unemployment, and inflation $b\%$.

However, this economic similarity requires, because of (23) and (32), that all the similar economies provided by the transformation have the same value for $i - \alpha$. So, it remains feasible that a negative relationship might exist between inflation and unemployment if the variation of α and unemployment occurs in the context of fixed i . Alternatively, the negative relationship between inflation and unemployment might better be recast as a positive relationship between the effective interest rate, $i - \alpha$, and unemployment.

5. Inferences from Experiments

5.1. An Optimal Policy

Of the three financial parameters, $\alpha_2(t)$, $i_2^{\{\ell\}}(t)$, and $r_2^{\{\ell\}}(t)$ which feature in the model of the dynamics of loans presented above, it is sometimes assumed that it is possible to directly change $i_2^{\{\ell\}}(t)$, and that it is *not* possible to *directly* change $\alpha_2(t)$. Control of $r_2^{\{\ell\}}(t)$ by governments or central banks is feasible but not orthodox; control of $r_2^{\{\ell\}}(t)$ is discussed further in Subsection ??.

The *goal* of monetary policy is often to change $\alpha_2(t)$ – to force it to take values in a certain desired range. Achieving satisfactory levels of unemployment may also be a consideration.

Here is how we can use (22)–(24) to achieve this goal of controlling $\alpha_2(t)$ by means of changes to $i_2^{\{\ell\}}(t)$ and $r_2^{\{\ell\}}(t)$. Suppose inflation is too high in E_1 and E_2 is the economy that we want E_1 to become, due to our intervention. Let us suppose, for example, that we seek to reduce $\alpha_2(t)$ by 1%.

To achieve this, according to (25), which is the only transformation which is consistent with economic similarity, we should decrease $i_2(t)$, i.e., in E_2 , we set $i_2^{\{\ell\}} = i_1^{\{\ell\}} - 0.01$ while keeping $r_2^{\{\ell\}}(t) = r_1^{\{\ell\}}(t)$. By (24), assuming all actors in the economy make identical choices in E_2 , to the choices they made in E_1 , which is rational because of the economic similarity of these two economies, we find $\alpha_2(t) = \alpha_1(t) - 0.01$, $0 \leq t \leq T$. So, we have achieved our goal of revising the inflation in E_2 by means of changes to the key parameters of this economy. Furthermore, amongst all possible transformations which achieve the desired level of inflation, the *distortion* (change) in economic choices and actions in this economy which have to be adopted is the minimum possible (zero).

Observe that this appears to directly contradict current best practice, which is not to *decrease* interest rates, but instead to *increase* them. However, the proposed action is not *only* decreasing interest rates. The proposed action decreases interest rates while *maintaining the loan's rate of repayment as a proportion of debt*, $r_2^{\{\ell\}}(t)$, (i.e. keeping it fixed). If interest rates are decreased, in current practice, it is unlikely that the rate of repayment relative to the level of debt, as referenced here, would remain fixed. Thus, the proposed strategy requires control, or management, of $r_2^{\{\ell\}}(t)$, as well as control of $i_2^{\{\ell\}}(t)$. Note that $r_1^{\{\ell\}}(t)$ is unlikely to be constant even for short periods so the same is necessarily true for $r_2^{\{\ell\}}(t)$, that is to say although the proposed strategy does *not* change the rate of repayment relative to debt, the rate typically does change over time. Also, when $r_1^{\{\ell\}}(t) = r_2^{\{\ell\}}(t)$, $t > 0$, it will usually *not* be the case that $\hat{r}_1^{\{\ell\}}(t) = \hat{r}_2^{\{\ell\}}(t)$, $t > 0$, as already observed in Corollary 1.

5.2. Repayments in an Example Loan

The orthodox scheme of repayments in most nations of the world, at the moment, and over many years in the past, is for a loan to be repaid in installments of a fixed amount, at regular intervals. In the present context, it is reasonable to model this as a continuous system of repayment, at a rate $\hat{r}_1(t)$, which is fixed, $\hat{r}_1(t) = \hat{r}_1(0)$, $t \geq 0$, when expressed in current currency.

As a test and illustration of the transformation (25)–(26), it is therefore important to compare the rate of repayment in cash, $\widehat{r}_2(t)$, in E_2 , when the rate of repayment in cash in E_1 takes this orthodox form, i.e. a constant rate, and transformation (25)–(26) is used to define the repayments, rather than the orthodox approach of adopting a different constant cash rate. We do not assume that repayment schedules *must* take the form of a fixed rate, in current currency. However, since this is an important special case, we now investigate how this loan repayment strategy changes under an inflation similarity.

Applying (13) in E_1 , with $\sigma_{\lambda}^{\{\ell\}} = 0$, because of our assumption that the cash repayment rate is constant, (13) becomes

$$d\lambda_1^{\{\ell\}}(t) = \left(i_1^{\{\ell\}}(t) \lambda_1^{\{\ell\}}(t) - \widehat{r}_1(0) \right) dt. \quad (34)$$

For simplicity of notation, let us denote $i_1 = i_1(0)$, $\lambda_1(t) = \lambda_1^{\{\ell\}}(t)$, and $\widehat{r}_1 = \widehat{r}_1(0)$. Then (34) becomes

$$d\lambda_1(t) = \left(i_1 \lambda_1(t) - \widehat{r}_1 \right) dt, \quad (35)$$

which has the solution

$$\lambda_1(t) = \frac{P_{\ell}(1 - e^{i_1(t-T_{\ell})})}{1 - e^{-i_1 T_{\ell}}}, \quad 0 \leq t \leq T_{\ell}. \quad (36)$$

In order for the loan duration to be T_{ℓ} , we require

$$\widehat{r}_1 = \frac{P_{\ell} i_1}{1 - e^{-i_1 T_{\ell}}} \quad (37)$$

It follows that the cash rate of the loan under consideration, in E_2 , is

$$\widehat{r}_2(t) = \widehat{r}_1(t) \exp \left(\int_0^t (\alpha_2(s) - \alpha_1(s)) ds \right), \quad t \geq 0, \quad (38)$$

which was already shown in Corollary 1.

The last of the repayment functions which is of some interest is $\widetilde{r}_2(t)$, which is the constant cash repayment rate which would apply in E_2 if the repayments were made according to orthodox loan repayment strategy. It is given by

$$\widetilde{r}_2(t) = \frac{P_{\ell} i_2}{1 - e^{-i_2 T_{\ell}}}, \quad (39)$$

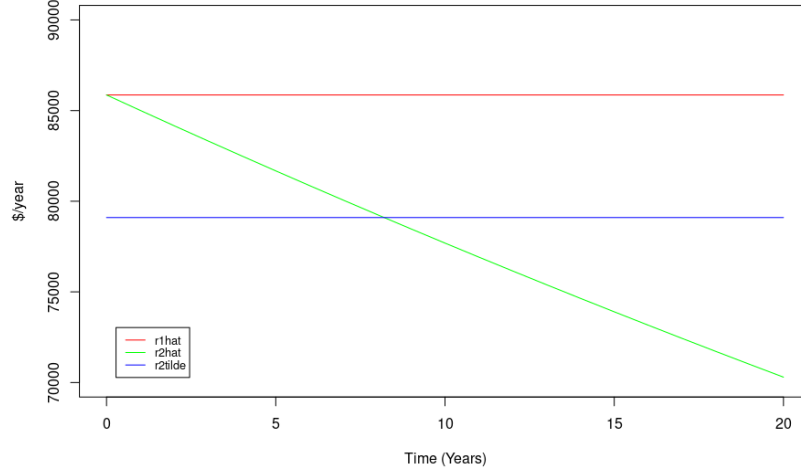


Figure 4: $\hat{r}_1(t)$, $\hat{r}_2(t)$ and $\tilde{r}_2(t)$
 $\hat{r}_1(t)$, $\hat{r}_2(t)$ and $\tilde{r}_2(t)$ depend on the interest rates i_1 and i_2 . $\hat{r}_2(t)$ depends also on α_1 and α_2 . This figure was plotted using $i_1 = 0.06$ and $i_2 = i_1 + \alpha_2 - \alpha_1 = 0.06 + 0.07 - 0.08 = 0.05$.

$0 \leq t \leq T$, which is just the same as (37), which was expressed in relation to E_1 , but now with the parameters of E_2 in place of those for E_1 .

The critical parameters of interest in this experiment, which are $\hat{r}_1(t)$, $\hat{r}_2(t)$, and $\tilde{r}_2(t)$ are plotted in Figure 4. The key observation which emerges in this example is that while the cash repayment rate is constant in E_1 , it is not constant in E_2 . If $\alpha_2 < \alpha_1$, the repayment rate in E_2 is decreasing, and when $\alpha_2 > \alpha_1$, the repayment rate in E_2 is increasing.

When $\alpha_2 < \alpha_1$, the rate of repayment in E_2 , which is economically similar to E_1 , is initially the same as in E_1 , and higher than the rate which would be used under orthodox management of a loan. It is this “front-loading” of repayments which makes them different from the orthodox management which would normally be used after a change of interest rates. This causes the money supply to increase at the rate α_2 rather than α_1 , and leads, intuitively, to a change in the inflation rate. Rather than this occurring as a consequence of “economic pain”, it follows under the assumption that all producers and consumers behave exactly the same way in E_2 as in E_1 . The difference between these economies is the prices, interest rates, and repayment schedules, with the latter being the innovation.

5.3. A Simple Loan with Random Effects

Let us now re-introduce the random effects that normally occur in a loan due to early repayments and redraws and check if the conclusions we have drawn are unaffected. In this case, it is no longer appropriate to predict the entire future of the loan, but we can, according to the theory of economic similarity, predict the *changes* that will occur, between E_1 and E_2 .

Example 6. Similar Loans

Suppose economies E_1 and E_2 are modeled by (13)–(15), i.e. each economy has just one loan, and that these two economies are economically similar by an inflation similarity. Dividing each of the equations (15) by $\lambda_e^{\{\ell\}}(t)$, and then subtracting one equation from the other, by economic similarity, the random terms cancel and we obtain:

$$\begin{aligned} \frac{d\lambda_2^{\{\ell\}}(t)}{\lambda_2^{\{\ell\}}(t)} - \frac{d\lambda_1^{\{\ell\}}(t)}{\lambda_1^{\{\ell\}}(t)} &= \left(i_2^{\{\ell\}}(t) - r_2^{\{\ell\}}(t)\right) - \left(i_1^{\{\ell\}}(t) - r_1^{\{\ell\}}(t)\right)dt, \\ \implies d\log \lambda_2^{\{\ell\}}(t) - d\log \lambda_1^{\{\ell\}}(t) &= \left(i_2^{\{\ell\}}(t) - r_2^{\{\ell\}}(t)\right) - \left(i_1^{\{\ell\}}(t) - r_1^{\{\ell\}}(t)\right)dt, \end{aligned}$$

so

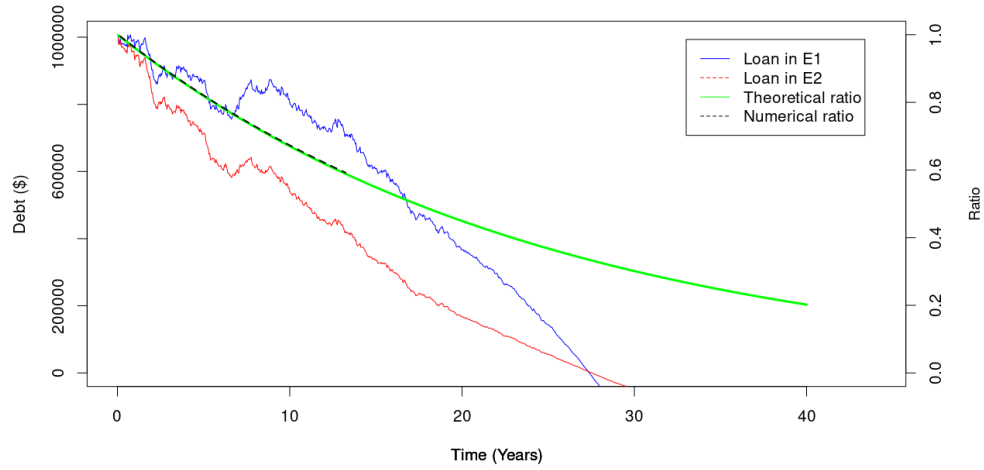
$$\lambda_2(t)/\lambda_1(t) = \exp \left(\int_0^t \left(i_2^{\{\ell\}}(s) - r_2^{\{\ell\}}(s) \right) - \left(i_1^{\{\ell\}}(s) - r_1^{\{\ell\}}(s) \right) ds \right). \quad (40)$$

A plot of a loan in E_1 , and the corresponding loan after transformation in E_2 is shown in Figure 5. Observe that although the loan is random, the *ratio* between the level of debt in E_2 to the level of debt in E_1 is deterministic, and is determined solely by the change in interest rate, from E_1 to E_2 . \square

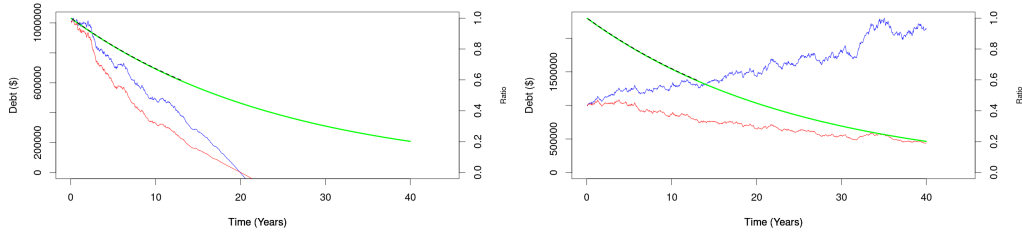
6. The Money Supply

6.1. Definitions of Money Supply

In this section we introduce four alternative definitions of money supply. The definitions given here are based on quantities of money associated with certain types of loans, namely: total loans; total fiat currency, from the central bank; total government debt; and, lastly, net cash available from current loans. This approach



Scenario 1



Scenario 2

Scenario 3

Figure 5: Loan in E_1 subject to random changes, and its corresponding loan in E_2

Scenario 1: Loan is payed off in both E_1 and E_2 , just *over* the 30 year loan term.

Scenario 2: Similar to Scenario 1, the loan is now payed off in both E_1 and E_2 , just *under* the 30 year loan term.

Scenario 3: In about a third of the simulations, we see that the two loans diverge in both E_1 and E_2 . The reason for this is that in some cases, the random changes introduced by the dispersion term cause the debt to stray into a region where interest payments are so large that the repayments aren't sufficient to reduce it. In this case, there is a good chance that the loan will never be repaid.

of defining the money supply in terms of certain types of loans, in several different ways, is appropriate in the present case because the model enables these specific quantities to be clearly identified.

Definition 5. Loans Money Supply. *Let*

$$M_e^{\{\lambda\}}(t) = \sum_{\ell=1}^L \lambda_e^{\{\ell\}}(t), \quad (41)$$

$e = 1, 2$, i.e. the total of all outstanding loans in economy e .

Definition 6. Central Bank Money Supply. *The set of indexes of the loans financed by the central bank is denoted by C , and*

$$M_e^{\{C\}}(t) = \sum_{\ell \in C} \lambda_e^{\{\ell\}}(t). \quad (42)$$

which is therefore the total of all money created by the central bank, in economy e .

Definition 7. Government Debt. *The set of indexes of the loans to the government is denoted by G and*

$$M_e^{\{G\}}(t) = \sum_{\ell \in G} \lambda_e^{\{\ell\}}(t), \quad (43)$$

which is therefore total of all money borrowed by the government in economy e .

Because loans have to be repaid, with interest, there comes a time in the life of every loan when the balance of money from the principle, less all the repayments, is less than or equal to zero. Usually this occurs before the end of the loan. It therefore makes sense to also define a concept of money supply which deducts the repayments. In some respects, this concept of money supply captures the concept of money supply more effectively than the previous definitions.

Definition 8. Spendable Money Supply. *The total of all loans, less all interest paid on those loans, in economy e , is denoted by $M_e^{\{S\}}$. Thus,*

$$M_e^{\{S\}}(t) = \sum_{\ell=1}^L \rho_e^{\{\ell\}}(t). \quad (44)$$

6.2. The Quantity of Money Theory of Inflation

Theorem 3. *If E_2 is similar to E_1 , by an inflation similarity (as in Definition 4), with $a(t) = -\Delta(t)$. Then*

$$\begin{aligned}
 M_2^{\{\lambda\}}(t)/M_1^{\{\lambda\}}(t) &= M_2^{\{C\}}(t)/M_1^{\{C\}}(t) \\
 &= M_2^{\{G\}}(t)/M_1^{\{G\}}(t) \\
 &= M_2^{\{S\}}(t)/M_1^{\{S\}}(t) \\
 &= \exp\left(\int_0^t -\Delta(s)ds\right). \tag{45}
 \end{aligned}$$

Proof Let $\mu: X \mapsto Y = A[X]$ be the inflation similarity between E_1 and E_2 . For each loan, ℓ , in E_1 , we have $\lambda_2^{\{\ell\}} = A[\lambda_1^{\{\ell\}}]$, and $\rho_2^{\{\ell\}} = A[\rho_1^{\{\ell\}}]$, hence $M_2^{\{\lambda\}} = A[M_1^{\{\lambda\}}]$, $M_2^{\{C\}} = A[M_1^{\{C\}}]$, $M_2^{\{G\}} = A[M_1^{\{G\}}]$, and $M_2^{\{S\}} = A[M_1^{\{S\}}]$, from which (45) follows. \square

This accords with the quantity theory of money (Friedman and Bordo, 2017b; Hussain and Zafar, 2018; Kwon et al., 2009). The result (45) does not assume or rely on a deterministic relationship between money supply and inflation. Instead, it is expressed in terms of a comparison between two different similar economies.

Example 2. Two Economists (continued)

Marvin: “So what happens to the money supply when interest rates are lowered and the repayment rates relative to debt are fixed?”

Martin: “I can’t actually predict what will happen to the money supply, because I can’t predict what will happen in science, technology, trade, politics, or the weather. But I can predict that if the lowering interest rates strategy is adopted, while keeping repayment rates relative to debt unchanged, even for the government, the money supply will grow at a rate less than it would without applying this strategy.” \square

Theorem 3 including both a definition of money supply as total loans and a definition as total government debt should not be interpreted to suggest that *either* monetary or fiscal measures can be used to manage inflation. If $A(\Delta)$ is the statement that the inflation rate is Δ lower in E_2 than in E_1 , and $MS(\lambda, \Delta)$, $MS(C, \Delta)$,

$MS(G, \Delta)$, $MS(S, \Delta)$, are the statements that the rate of growth of total loans is Δ lower, that growth of fiat currency is Δ lower, that the rate of growth of government debt is Δ lower, and that the rate of growth of spendable cash is Δ lower, respectively, the assertion of the theorem is:

$$A(\Delta) \Rightarrow MS(\lambda, \Delta) \wedge MS(C, \Delta) \wedge MS(G, \Delta) \wedge MS(S, \Delta), \quad (46)$$

not

$$(MS(\lambda, \Delta) \Rightarrow A(\Delta)) \wedge (MS(C, \Delta) \Rightarrow A(\Delta)) \wedge (MS(G, \Delta) \Rightarrow A(\Delta)) \\ \wedge (MS(S, \Delta) \Rightarrow A(\Delta)).$$

Consequently, this theorem implies that fiscal *and* monetary strategies must be applied *simultaneously*, in order to achieve a desirable adjustment to inflation. A similar observation has been made in (Woodford, 2001a). According to the theorem, to be effective, the rate of monetary expansion must be reduced in all areas at once: total loans, total quantity of central bank money creation, *and* government spending: i.e. monetary *and* fiscal controls should be applied.

The converse implication to (46) seems feasible and would be more useful than (46) because it justifies a realistic method for managing inflation. This is too much to expect. However, the following corollary to Theorem 3 comes close:

Corollary 2. *If E_2 is similar to E_1 , by an inflation similarity, and (45) holds, then the inflation rates, $\alpha_1(t)$ and $\alpha_2(t)$, of the two economies are related by*

$$\alpha_2(t) - \alpha_1(t) = -\Delta(t). \quad (47)$$

Proof Since the two economies are similar, by an inflation similarity, Theorem 3 applies for some $a(t)$. But (45) is assumed to be true, with $a(t) = -\Delta(t)$, so (47) is true, as asserted in the corollary. \square

This is almost as strong as the converse of (46) because if two possible future economies are different due to changed rate of growth of $M_e^{\{\lambda\}}$, $M_e^{\{C\}}$, $M_e^{\{G\}}$ and $M_e^{\{S\}}$, it is natural to suppose that all households and organisations in the two economies will make the same choices in both economies, in which case the two economies will be economically similar, and so Corollary 2 can be applied.

Because Theorem 3 and Corollary 2 assume E_2 is similar to E_1 , they do not exclude the possibility that *perceptions* of inflation do not *immediately* adjust to the

revised rate of growth of the money supply, and therefore they do not apply immediately after the intervention. It is reasonable to suppose, however, that these expectations will become less significant over time, and then the theorem and the corollary will be applicable, i.e. after a delay, during which expectations adjust to reflect the rate of growth of the money supply, Theorem 3 will apply.

Remark 2. *Given that this corollary apparently provides a painless mechanism for changing the rate of inflation of an economy, it is important to point out that not only does it rely on the money supply in all four definitions being uniformly regulated, by altering the interest rate for all loans, but it assumes also that the repayment schedules of all loans will be changed, uniformly, to repay loans more quickly at the earlier stages of their duration. Repayment schedules already do this, when evaluated in terms of the real value of their repayments. The concept of front-loading loan repayments is not so much a new idea as a more thorough exploitation of an existing idea.*

When we compare economies which are not exactly similar, as we do in the next subsection, $M_e^{\{S\}}$ is the preferred concept of money supply.

Remark 3. *In (Friedman and Bordo, 2017b), the quantity theory of money is presented in the form*

$$Y = vM \quad (48)$$

in which Y is economic output, v is the velocity of money, which depends on a number of other parameters of the “real” (i.e. not via prices) economy, such as the rate of return on bonds, the return on equities, utility preferences, real wealth, and so on, and M is the money supply. Since M is proportional to P , the price of a “standard” product, (48) determines P from the quantity of money and of goods, so that they “match”. In this way, it conveys similar implications, albeit as a prediction, to Theorem 3. The latter can be inferred from (48) if v is regarded as constant, but in this case (48) is a much stronger statement which relies for its interpretation on the unknown function, v .

Because of the fundamentally stochastic nature of all economies, equation (48) is difficult to use. All three of the related parameters can change significantly in unpredictable ways from moment to moment. A statement of the quantity theory in terms of a comparison between alternative futures avoids this problem of interpretation.

7. Raising Interest Rates

This section considers the economic intervention of raising (or lowering) the nominal interest rate (reserve bank interest rate, federal funds rate) while retaining an orthodox loan repayment schedule. In many nations this is currently a key strategy used by central banks to control inflation (Asso et al., 2010; Benhabib et al., 2001; Woodford, 2001b).

The original economy is denoted by E_1 and the economy with raised interest rates by E_* . Use of $*$ as the subscript in E_* is to remind us that E_* is not exactly similar to E_1 .

Since the transformed economy in this case is not exactly economically similar to the original one, because an orthodox repayment strategy is adopted, let us approach the question of inflation by estimating the rate of growth of the spendable money supply, $M_*^{\{S\}}$ say, in E_* . Theorem 3 shows that for similar economies, the change in the rate of growth of the money supply determines, precisely, the change in the rate of growth of inflation. So, even though the transformation considered here is not an exact similarity, we hypothesise that this effect also holds approximately in this situation. This accords with the quantity theory of money, which is widely accepted.

7.1. Quantifying the Effect of Raising Interest Rates on Inflation

Suppose interest rates in E_* are raised to $i_* = i_1 + \delta$ say. An algebraic argument that the growth rate of $M_*^{\{S\}}$, the spendable money supply in E_* , will be less than that of $M_1^{\{S\}}$ by $K\delta$, for some $K > 0.5$ is presented in Appendix D. This is confirmed by simulation in the following example.

Example 7. Simulation of Raised Interest Rates

Simulation experiments comparing the two strategies of lowering interest rates, with an adjustment to repayments which makes the transformation a similarity, on the one hand, and raising interest rates, with orthodox adjustment of repayments, on the other hand, have been undertaken. Each simulation includes 20 loans, each of which contributes to $M_e^{\{S\}}$. Only the loans of each economy are modeled.

Simulations where interest rates are changed in the other direction (which makes inflation higher) are also undertaken for comparison. In the plots of each simulation, shown in Figure 6, the spendable money supply in E_1 is compared to either

the spendable money supply in E_* (in Figures 6a and 6c) or to the spendable money supply in E_2 (in Figures 6b and 6d). The ratio of $M_2^{\{S\}}$ or $M_*^{\{S\}}$ to $M_1^{\{S\}}$ is also plotted, as appropriately, in each plot. In the cases where the mapping is a similarity (Figures 6a and 6c), the theoretical value of this ratio is also plotted.

These simulations confirm that if orthodox repayments are adopted for loans, raising interest rates by δ has approximately the same effect as reducing them would have if economically similar repayments were adopted for loans. \square

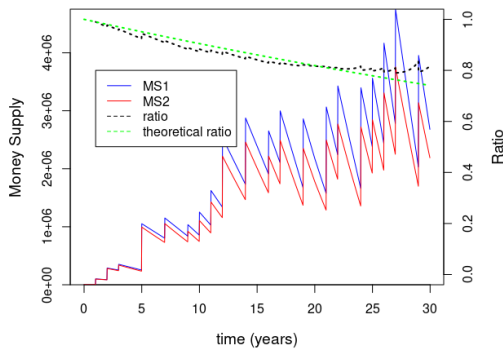
In general, only some loans will be modified, so the simulations of the previous example and the algebraic estimate, (D.6), of the appendix overestimate the modification of the growth of the money supply caused by both interventions. The effect on aggregate growth, in practice, will combine together the contributions of each loan, some of which have not actually been modified. Some loans, for example government bond issues, will naturally be much more significant than others.

7.2. Quantifying the Economic Distortion

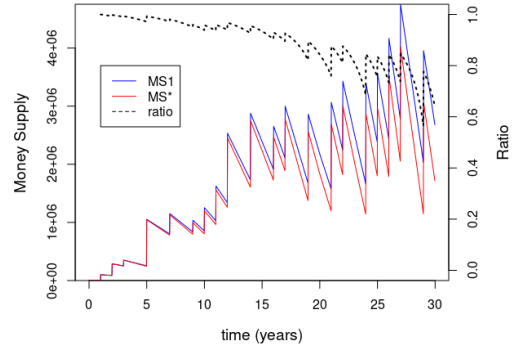
Let us compare the economy E_* to the one, E_2 say, analysed previously, which has inflation rate $\alpha_2 = \alpha_1 - \delta$ and is economically similar to E_1 . We know that E_2 has interest rate $i_2 = i_1 - \delta$ and its repayment rate is $r_2 = r_1$. The two economies E_2 and E_* then have approximately the same rate of increase of spendable money supply, $M_e^{\{S\}}$.

If we introduce a tax on debt, levied on all citizens and organisations that have debt, at the interest rate δ , and a *rebate* levied on credit, for all citizens and organisations that hold a credit, at the same rate, the total tax income will exactly match the total of the rebates. Introducing these taxes and rebates will exactly cancel the difference between increasing interest rates by δ on the one hand, and decreasing them by δ on the other.

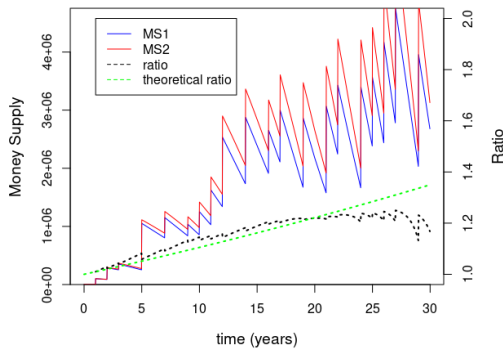
This economy, based on higher interest rates with additional taxes and rebates, is economically similar to E_2 , and therefore also to the original economy. In this way we can express, quantitatively, the distortion of the economy. The key difference between E_* and E_2 is the fact that interest rates in E_* have been increased to $i_* = i_1 + \delta$ whereas in E_2 , interest rates were reduced to $i_2 = i_1 - \delta$. The precise values of the taxes and rebates which represent the difference between E_* and E_2 therefore takes the form of a tax, at rate $\delta + \delta = 2\delta$ on individuals that hold a debt, and a rebate, at the same rate, on individuals that hold a credit.



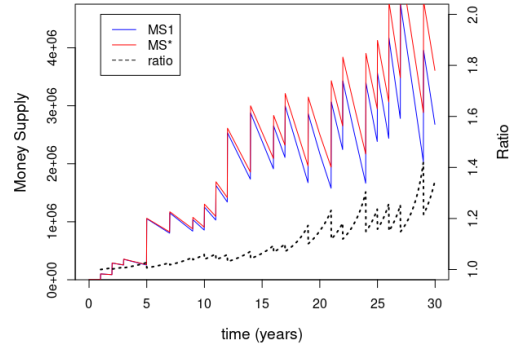
(a) Interest rates reduced by 1%, using a similarity



(b) Interest rates increased by 1%, using orthodox repayments



(c) Interest rates increased by 1%, using a similarity



(d) Interest rates decreased by 1%, using orthodox repayments

Figure 6: Experiments comparing the Money Supply, $M^{\{S\}}$, under the conditions of raising (or lowering) interest rates, with orthodox repayments, to economically similar lowering (or raising) of interest rates

Note: the taxes and rebates are not actually applied, so the affected individuals and organisations do not behave differently on account of them; we are using these taxes and rebates only as a method to quantify the distortion of the economy from the economy E_2 , which is economically similar to E_1 . The taxes and rebates are a measure of the *dissimilarity* of E_* from E_2 .

Another issue with traditional monetary interventions for controlling inflation is the tendency to overshoot the target, leading to cycles of “boom and bust” (Friedman, 1977). Explaining such cycles has been a goal of economics from its very origins, including contributions from Marx, Keynes, and Friedman.

Increasing interest rates as an economic intervention has this weakness, according to Friedman, because it’s short term and long term effects are different, which makes it difficult to make adjustments to interest rates leading to the desired outcome. From the point of view of here, the problem can be explained as an expected side-effect of the economic distortion.

The debate concerning use of interest rates to control inflation with or without linkage to government policy continues. The term *Ricardian* is used to signify the hypothesis that interest rates can be used to manage inflation (and to some extent unemployment) without coordination with government economic policy, and *non-Ricardian* describes the view that these two strategies must be coordinated (Woodford, 2001a). Theorem 3 asserts that if the economy sought after by economic policy is to be similar to the original economy, and hence distortion of economic activity is minimized, the non-Ricardian stance should be adopted, i.e. fiscal and monetary policy should be aligned.

8. Conclusion

The concept of *economic similarity* – a precisely defined generalization of the concept of monetary neutrality – has been introduced and applied to modeling economic interventions, especially those that are used by central banks to manage inflation. Four examples of economic similarities were given, two where the similarity is exact, and two where it is approximate. A general theorem which defines a large class of similarities between an arbitrary starting economy and a second economy was proved.

A special case of these similarities was identified – an inflation similarity. Since this similarity shows that essentially the same economic activity can occur with an arbitrary rate of inflation, it gives important insight into the concept of the

Phillips curve (which expresses the view that there is a natural negative correlation between inflation and unemployment); as also observed by Friedman (1977), unemployment is not inherently negatively correlated with the rate of inflation. A second theorem was presented which constrains the form that can be taken by an inflation similarity. For two economies to be similar, by an inflation similarity, the repayment schedules for all loans must be adjusted to shift the repayments forwards or backwards in time relative to their original evolution.

This observation concerning inflation similarities leads to a new explanation for why raising interest rates, while adopting an orthodox repayment schedule, causes the rate of inflation to reduce. It also introduces an alternative strategy for managing inflation, which is based on an unorthodox loan repayment strategy, one where the rate of repayment is relatively higher at the start of the loan. This alternative strategy for managing inflation is potentially much less disruptive and does not reduce economic activity as a side effect. This challenges the conventional belief that control of inflation necessitates “economic pain”.

Lastly, in Theorem 3, it is shown that the rate of growth of the money supply in an economically similar economy increases (or decreases), relative to the original economy, by exactly the same quantity as the increase (or decrease) in the rate of inflation, using four different definitions of money supply. This theorem is, in a sense, a precise statement of the quantity theory of money. A corollary shows that inflation can be adjusted by means of an economic intervention, but it must be applied *uniformly*, adjusting the rate of growth of the money supply in all senses simultaneously.

8.1. Old concepts re-interpreted

The concept of monetary neutrality has been presented in a form where it can be understood as a characteristic of money which co-exists with the contrary characteristic as an influence upon economic activity.

Some monetarists concepts have been re-discovered, or confirmed, by the new methods of this paper. The quantity theory of money has been presented as a comparison between two alternative futures of an economy. It was shown that if the two alternative futures are economically similar, the difference in the rate of growth of the money supply in the two economies (defined in four different ways) must be the same as the difference between the inflation rates of the two economies.

8.2. *New concepts*

The concept of economic similarity, the key innovation of this paper, is not limited to the applications considered in this paper. It has been used here to tackle some classical economic problems with rigour and precision. The most surprising discovery this has revealed is that orthodox loan repayment schedules for loans and bonds prevent the actions of raising or lowering interest rates from leading to a similar economy. This *distortion* of the economy has been identified as an unavoidable ingredient in the orthodox management of inflation. It is identified as the source of the “economic pain” which is widely regarded as the unavoidable accompaniment to monetary interventions for managing inflation.

However, we saw in Corollary 2, that there is a strategy for managing inflation which is more tightly aligned with the quantity theory of money – exponential weighting of loan and bond repayments – and since the economy this strategy leads to is economically similar, there is no need for “economic pain”. An alternative explanation of this strategy was provided, which is that real repayment rates are changed, by the actions of central banks, but this is not currently recognised as a consequence of their actions. If this is done, and the ideal that interventions for controlling inflation should lead to similar economies is adopted, the same concept that central banks should explicitly regulate repayment rates appears as a natural component of financial regulation.

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Appendix A. Formal Explanation of the VSDE Model

The process X is an \mathcal{E}_1 -valued stochastic process on the same probability space, (Ω, \mathcal{F}, P) say, on which the Wiener process, $\{W_t\}$ is defined. We require it to have expected integral norm squared, $E \left(\int_0^t \|X(s)\|^2 ds \right) < T$ for all $t > 0$, in which $\|\cdot\|$ denotes the Euclidean norm on \mathcal{E}_1 . Both sides of (2) are random processes, and the *equation* is in the sense that the expectation of the integral with respect to t of the squared difference between the two sides is zero.

Of the two integrals on the RHS of (2), the second is an Ito integral and the first is a Riemann integral. Both depend on the process X in a way that has yet to be made clear.

Let $\{\mathcal{F}_t\}_{t \geq 0}$ be an increasing sequence of sets of subsets of \mathcal{F} representing the history of process W_s up to time t , i.e. $\mathcal{F}_t = \sigma(\{W_s : s \leq t\})$ (including null sets). Let \mathcal{B} denote the Borel sets on $[0, T]$.

The formal definition of an Ito integral (Øksendal, 1998)

$$I = \int_S^T f(t, \omega) dW_t(\omega), \quad (\text{A.1})$$

where $0 \leq S < T < \infty$, requires that (i) $f : [0, T] \times \Omega \rightarrow \mathbb{R}$ is $\mathcal{B} \times \mathcal{F}$ measurable, (ii) f is adapted to the family of σ -algebras $\{\mathcal{F}_t\}_{0 \leq t \leq T}$ and (iii)

$$E \left[\int_S^T f(t, \omega)^2 dt \right] < \infty.$$

For this definition to be applied to the second integral in (2), we need to be able to interpret $\sigma_1(t, X)$ as a matrix of functions satisfying these requirements just stated for f . Since, as a stochastic process, X is, in effect, a matrix of functions of $\omega \in \Omega$ (although this dependence on ω is often suppressed), interpretation of $\sigma_1(t, X)$ in this form is reasonable. However, the conditions required of f need to be ensured by imposing additional conditions on σ_1 .

Let us suppose that for some $n \geq 0$, $0 \leq \delta_1 < \dots < \delta_n$, and some measurable function $\tilde{\sigma}_1 : [0, T] \times \mathbb{R} \times \dots \times \mathbb{R} \rightarrow \mathbb{R}$, σ_1 takes the form:

$$\sigma_1 : (t, X) \mapsto \tilde{\sigma}_1(t, X(t - \delta_1), \dots, X(t - \delta_n)). \quad (\text{A.2})$$

Since X is a stochastic process, the mapping $F : [0, T] \times \Omega \rightarrow \mathbb{R}^{m_1 \times n}$ defined by

$$F : (t, \omega) \mapsto \tilde{\sigma}_1(t, X_{t-\delta_1}(\omega), \dots, X_{t-\delta_n}(\omega)).$$

ensures that each of the elements of the matrix F meets the requirements expected of f in (A.1). The integral (A.1), with $f = F_{ij}$, for each $i = 1, \dots, m_1$, $j = 1, \dots, n$, therefore provides the formal interpretation of the second integral in (2). For this interpretation to be applicable, it is necessary that σ_1 is of the specific form considered here.

If σ_1 depends on only one past value, $X(t)$ for example, of the process X , rather than expressing it as a function of t and X in (1), it would be appropriate to express σ_1 directly as a function of t and $X(t)$. This is the conventional form in which SDEs are usually expressed. In the role of modeling an economy, however, it is potentially important to introduce a more complex dependence on the past of the process X . In addition, it seems appropriate that this dependence on the past is *strict*, i.e. $\delta_j > 0$, $j = 1, \dots, n$. This is both realistic and theoretically convenient. However, it is sometimes simpler to allow dependence of $\sigma_1(t, X)$ on $X(t)$, as we do, in Example 5.

It is feasible that the dependence of σ_1 on the past of the process X is only *approximately* of the form (A.2), to any degree of accuracy that is sought, however the intricacies of such a definition are not appropriate to consider in the present context.

Now let us consider the first integral in (2). In order to be able to model loans, we require, rather than b_1 being a measurable function, that, for any X , $B_1 : t \mapsto \int_0^t b_1(s, X) ds : [0, T] \times \mathcal{F} \rightarrow \mathbb{R}^{m_1}$ is locally bounded, measurable and adapted to $\{\mathcal{F}_t\}$. This allows the function b_1 to include instantaneous jumps of the sort that is needed to be able to model the start of a loan. Since our only assumption concerning X is that it is a stochastic process which is a solution of (1), in order to define the first integral in (2) unambiguously, as with σ_1 , we require that B_1 takes the form:

$$B_1 : (t, X) \mapsto \tilde{B}_1(t, X(t - \delta_1), \dots, X(t - \delta_n)). \quad (\text{A.3})$$

for some $n \geq 0$, $0 < \delta_1 \leq \dots \leq \delta_n \leq T$, and some measurable $\tilde{B}_1 : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$. Then the first integral at (2) must take the value

$$\omega \mapsto \tilde{B}_1(t, X_{t-\delta_1}(\omega), \dots, X_{t-\delta_n}(\omega)). \quad (\text{A.4})$$

In addition, we require that

$$E \left(\int_0^T \|B_1(t, X)\|^2 \right) < \infty. \quad (\text{A.5})$$

The requirement that X , σ_1 and b_1 are adapted to the family of σ -algebras $\{\mathcal{F}_t\}_{t \geq 0}$ ensures that the equations which guide the development of the economy can be influenced by the past, but can not be influenced by the future (or the present).

Appendix B. Proof of Theorem 1

Proof First let us suppose that the events and changes in X_t and Y_t occur at a discrete series of times. Suppose, as an inductive hypothesis, that we have proved economic similarity up to time t_0 . Consider the first event occurring at a time $t > t_0$. By the hypothesis, the values given to $\sigma_1(t, A^{-1}[Y])_{jk} dW_t$ and $b_1(t, A^{-1}[Y])_j dt$ will be identical to the corresponding values for the process X , at t , and hence, when multiplied by $A(t)$, if the coordinate j is measured in currency, the equation $Y(t) = A[X](t)$ holds also at t . That completes the proof, assuming the economy jumps from state to state at discrete moments in time.

Since the processes $W(t)$ and $X(t)$ change continuously, the proof requires consideration of cases where X does not change at discrete series on times. In such a case, using the argument in (Øksendal, 1998, §3.1), there exists a sequence of elementary (piecewise constant, over time) approximations which converges in mean square to X .

As discussed in Appendix A, the VSDEs (1) and (6) are interpreted as statements about an expected mean square difference, for example, (1) is interpreted as

$$\int_0^T \left\| X(t) - \left(B_1(t, X) + \int_0^t \sigma_1(s, X) dW(s) \right) \right\|^2 dt = 0, \quad (\text{B.1})$$

in which B_1 satisfies the conditions set out in Appendix A (for example, (A.5)) and the integral is an Ito stochastic integral. An Ito integral is formed as a limit of integrals of *elementary* functions (Øksendal, 1998, Chapter 3). Each elementary function changes with time at a finite set of times earlier than t . There is a sequence, $\{\phi_n\}_{n > 0}$ say, of elementary functions, which converges in mean square to Y for which both $Y - \phi_n$ and the LHS of (B.1) are $< 1/n$. Let $B = \sup_{0 \leq t \leq T} A(t)$. This implies

$$\int_0^T \left\| A[\phi_n](t) - \left(B_2(t, A[\phi_n]) + \int_0^t \sigma_2(s, A[\phi_n]) dW(s) \right) \right\|^2 dt < B^2/n. \quad (\text{B.2})$$

Thus, $Y = \lim_{n \rightarrow \infty} A[\phi_n] = A[X]$ is well defined and is a solution of (6). \square

A second algebraic proof of Theorem 1 is as follows:

Proof With some operator notation, (B.1), which is the equation we seek to show holds, can be rewritten,

$$Y = b_2[Y] + \sigma_2[Y] \quad (\text{B.3})$$

and (9)-(10), which are the equations which define b_2 and σ_2 , can be rewritten

$$b_2[Y] = A[b_1[A^{-1}[Y]]], \quad (\text{B.4})$$

and

$$\sigma_2[Y] = A[\sigma_1[A^{-1}[Y]]]. \quad (\text{B.5})$$

Substituting these expressions and $Y = A[X]$ into the RHS of (B.3) we find,

$$\begin{aligned} b_2[Y] + \sigma_2[Y] &= A[b_1[A^{-1}[A[X]]] + A[\sigma_1[A^{-1}[A[X]]]] \\ &= A[b_1[X] + \sigma_1[X]] \\ &= A[X] = Y, \end{aligned}$$

confirming that (B.3) holds, when $Y = A[X]$, which is what we sought to show. \square

Remark 4. *The first proof of this theorem relies on the fact that an economy is modeled by an Ito stochastic differential equation (or, more specifically, the equation defined in Appendix B). However, the second proof does not make any use of this assumption. If the Ito SDE was replaced by a different type of SDE, or by many other types of stochastic models (for example, a micro-economic model of the type used by Friedman and Bordo (2017a)), the proof would still be valid. Essentially, the concept that there are many economically similar economies, and that these can be found systematically, does not rely on the specific model of an economy that is adopted.*

Appendix C. Proof of Theorem 2

Proof The mapping (9)–(10) ensures that in the similar economy, all flows of cash which are referenced in the VSDE, (6). are interpreted by citizens, and economic actors in general, as having the same value in E_2 as in E_1 . If the cash flow (15) appears in the VSDE (6), we must therefore have $r_2^{\{\ell\}}(t) = r_1^{\{\ell\}}(t)$, for all

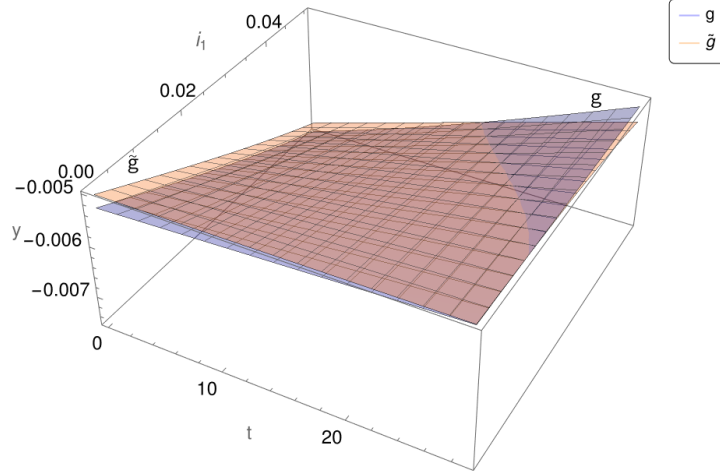


Figure D.7: $\tilde{g}(t)$ vs $g(t)$, with $T_\ell = 30$, $\delta = 0.01$, $t = 0-30$, and $i_1 = 0.0 - 0.05$

loans, ℓ . If $\rho_2^{\{\ell\}}(t)$ doesn't appear anywhere in (6), it will, nevertheless, appear in the VSDE for one of the extensions of E_1 , hence, in this case also, for all loans, $\ell = 1, \dots, L$, $r_2^{\{\ell\}}(t) = r_1^{\{\ell\}}(t)$.

That is to say, this cash flow, even if it doesn't appear in the VSDE (1) from the borrower to the lender must also be transformed according to (8). Thus, $r_2^{\{\ell\}}(t) = r_1^{\{\ell\}}(t)$, $t \geq 0$, even for loans when the repayment cash flow doesn't appear in (1). \square

Appendix D. Algebraic Estimate of the Effect of Increased Interest Rates

In this appendix, the rate of growth of the spendable money supply, $M_*^{\{S\}}$, in an economy, E_* , which differs from the original economy, E_1 , by the uniform raising of interest rates, by $\delta > 0$, together with an orthodox change to the repayments of loans, is shown to be lower than in the economy E_1 by at least $K\delta$, where $K \geq 0.5$.

Divide (15), for E_* and E_1 , by the level of the loan, in each case. Using the general

identity $\left(\frac{df(t)}{dt}\right)/f(t) = \frac{d \log |f(t)|}{dt}$, we find:

$$\left(\frac{\rho_*^{\{\ell\}}(t)}{\lambda_*^{\{\ell\}}(t)}\right) \frac{d \log |\rho_*^{\{\ell\}}(t)|}{dt} = \frac{\hat{r}_*^{\{\ell\}}(t)}{\lambda_*^{\{\ell\}}(t)} + \frac{\sigma_{*,\rho}^{\{\ell\}}(t, X^{\{*\}})}{\lambda_*^{\{\ell\}}(t)} dW(t) \quad (\text{D.1})$$

and

$$\left(\frac{\rho_1^{\{\ell\}}(t)}{\lambda_1^{\{\ell\}}(t)}\right) \frac{d \log |\rho_1^{\{\ell\}}(t)|}{dt} = \frac{\hat{r}_1^{\{\ell\}}(t)}{\lambda_1^{\{\ell\}}(t)} + \frac{\sigma_{1,\rho}^{\{\ell\}}(t, X^{\{1\}})}{\lambda_*^{\{\ell\}}(t)} dW(t). \quad (\text{D.2})$$

Subtracting (D.2) from (D.1), and cancelling random terms due to approximate economic similarity, we find (assuming $\delta \geq 0$):

$$\left(\frac{\rho_*^{\{\ell\}}(t)}{\lambda_*^{\{\ell\}}(t)}\right) \frac{d \log |\rho_*^{\{\ell\}}(t)|}{dt} - \left(\frac{\rho_1^{\{\ell\}}(t)}{\lambda_1^{\{\ell\}}(t)}\right) \frac{d \log |\rho_1^{\{\ell\}}(t)|}{dt} = \frac{\hat{r}_1^{\{\ell\}}(t)}{\lambda_1^{\{\ell\}}(t)} - \frac{\hat{r}_*^{\{\ell\}}(t)}{\lambda_*^{\{\ell\}}(t)}$$

which, using (37) and (36),

$$\begin{aligned} &= \left(\frac{i_1}{1 - e^{i_1(t-T_\ell)}} - \frac{i_*}{1 - e^{i_*(t-T_\ell)}} \right) \\ &= -\delta \left(\frac{1 + e^{i_1(t-T_\ell)}(i_1(t-T_\ell) - 1)}{(1 - e^{i_1(t-T_\ell)})^2} \right) + O(\delta^2) \\ &\approx -\delta(a + bt) \end{aligned} \quad (\text{D.3})$$

$$\begin{aligned} \text{where } a &= \frac{e^{i_1 T_\ell}(-i_1 T_\ell + e^{i_1 T_\ell} - 1)}{(e^{i_1 T_\ell} - 1)^2} \text{ and } b = -\frac{i_1 e^{i_1 T_\ell}(i_1 T_\ell e^{i_1 T_\ell} + i_1 T_\ell - 2e^{i_1 T_\ell} + 2)}{(e^{i_1 T_\ell} - 1)^3} \\ &\approx -K\delta, \end{aligned} \quad (\text{D.4})$$

where $0.5 \leq K \leq 1$. For example, when $i_1 = 0.05$ and $T_\ell = 30$, $K \approx 0.73$. A plot of $g : t \mapsto \left(\frac{i_1}{1 - e^{i_1(t-T_\ell)}} - \frac{i_*}{1 - e^{i_*(t-T_\ell)}}\right)$ vs $\tilde{g} : t \mapsto -\delta(a + bt)$ is shown in Figure D.7. This figure spans a wide range of possible values for i_1 and t , with the specific choices $\delta = 0.01$ and $T_\ell = 30$, and over the entire range the error in the approximation (D.3) is almost insignificant, which justifies, in turn, (D.4). The maximum relative error in (D.3) over the parameter range $i_1 \in (0.01, 0.1)$, $\delta \in (0.01, 0.1)$, $T_\ell \in (1, 50)$, $t \in (0, T_\ell)$, has been numerically evaluated as 16%.

If the loans in E_1 and E_* proceed without random redraws and early payments, as explored in Subsection 5.2, the following inequalities are known: $\lambda_*^{\{\ell\}}(t) \geq \lambda_1^{\{\ell\}}(t)$

and $\rho_*^{\{\ell\}}(t) \leq \rho_1^{\{\ell\}}(t)$, $0 \leq t \leq T_\ell$, from which it follows that $\frac{\rho_*^{\{\ell\}}(t)}{\lambda_*^{\{\ell\}}(t)} \leq \frac{\rho_1^{\{\ell\}}(t)}{\lambda_1^{\{\ell\}}(t)}$, $0 \leq t \leq T_\ell$. Also, $\frac{d \log |\rho_*^{\{\ell\}}(t)|}{dt} \leq 0$. Assuming these inequalities hold in the present instance (which would be true, for example, if all loans proceeded strictly according to the orthodox scheme of interest and capital payments), it follows from (D.4) that

$$\left(\frac{\rho_1^{\{\ell\}}(t)}{\lambda_1^{\{\ell\}}(t)} \right) \frac{d \log |\rho_*^{\{\ell\}}(t)|}{dt} - \left(\frac{\rho_1^{\{\ell\}}(t)}{\lambda_1^{\{\ell\}}(t)} \right) \frac{d \log |\rho_1^{\{\ell\}}(t)|}{dt} \leq -K\delta, \quad (\text{D.5})$$

and therefore, by $\frac{\rho_1^{\{\ell\}}(t)}{\lambda_1^{\{\ell\}}(t)} \leq 1$, when $\rho_*^{\{\ell\}}$ is positive,

$$\begin{aligned} \frac{d \log |\rho_*^{\{\ell\}}(t)/\rho_1^{\{\ell\}}(t)|}{dt} &\leq -K\delta \\ \implies \rho_*^{\{\ell\}}(t) &\leq e^{-K\delta t} \rho_1^{\{\ell\}}(t), \quad t \geq 0. \end{aligned} \quad (\text{D.6})$$

The bound (D.6) has only been shown to be true for the loans with positive values of $\rho_*^{\{\ell\}}(t)$. However, it has already been shown that whether or not this is true, it is always the case that

$$\rho_*^{\{\ell\}}(t) \leq \rho_1^{\{\ell\}}(t),$$

and if $\rho_*^{\{\ell\}}(t) < 0$, (D.6) follows from this.

Since (D.6) is therefore true for *all* loans, if all the loans start at time 0, the rate of growth of their total, $M_*^{\{S\}}$, is also lower than that of $M_1^{\{S\}}$ by at least $K\delta$, when $\delta > 0$ (and conversely will be larger by $-K\delta$ when $\delta < 0$).

In order for this conclusion to also hold if the loans do not all start at time 0, it is necessary to add an assumption that loans starting *after* time 0 have their principle values also reduced, relative to those in E_1 , by the ratio e^{-Kt_ℓ} , where t_ℓ is the starting time of loan ℓ . The loans in Example 7 were simulated in both ways, with and without this modification to the principles, and it made little difference. Figure 6 shows the results when the principles were not modified. The reason the modification does not make a significant difference in these simulations is because it's effect only becomes noticeable after a period of time approaching the lengths of the loans.

There are two reasons, in any case, why a reduction of the level of the principles of loans is expected in an economy where interest rates have been increased, and

orthodox adjustment of repayments applies: first, since $M_*^{\{S\}}(t)$ is increasing at a lower rate, due to the accelerated repayments of existing loans, in the early part of evolution of the economy, it is natural that the rate of growth of loan principles should also adopt this lower rate of growth; secondly, most loans are financed by other, larger loans; these larger loans are subject to an accelerated repayments schedule; the principles of future loans are therefore subjected to downward pressure on account of the repayment schedules of the loans from which they are financed.