

Bonanno, Giacomo

Working Paper

A Kripke-Stalnaker-Lewis semantics for AGM belief revision

Working Paper, No. 354

Provided in Cooperation with:

University of California Davis, Department of Economics

Suggested Citation: Bonanno, Giacomo (2023) : A Kripke-Stalnaker-Lewis semantics for AGM belief revision, Working Paper, No. 354, University of California, Department of Economics, Davis, CA

This Version is available at:

<https://hdl.handle.net/10419/283197>

Standard-Nutzungsbedingungen:

Die Dokumente auf EconStor dürfen zu eigenen wissenschaftlichen Zwecken und zum Privatgebrauch gespeichert und kopiert werden.

Sie dürfen die Dokumente nicht für öffentliche oder kommerzielle Zwecke vervielfältigen, öffentlich ausstellen, öffentlich zugänglich machen, vertreiben oder anderweitig nutzen.

Sofern die Verfasser die Dokumente unter Open-Content-Lizenzen (insbesondere CC-Lizenzen) zur Verfügung gestellt haben sollten, gelten abweichend von diesen Nutzungsbedingungen die in der dort genannten Lizenz gewährten Nutzungsrechte.

Terms of use:

Documents in EconStor may be saved and copied for your personal and scholarly purposes.

You are not to copy documents for public or commercial purposes, to exhibit the documents publicly, to make them publicly available on the internet, or to distribute or otherwise use the documents in public.

If the documents have been made available under an Open Content Licence (especially Creative Commons Licences), you may exercise further usage rights as specified in the indicated licence.

A Kripke-Stalnaker-Lewis semantics for AGM belief revision

Giacomo Bonanno

University of California, Davis, USA
gfbonanno@ucdavis.edu

April 21, 2023

Abstract

We provide a new characterization of AGM belief revision in terms of a Kripke-Stalnaker-Lewis semantics. We consider pointed frames consisting of a set of states, a distinguished state interpreted as the actual state, a Kripke belief relation and a Stalnaker-Lewis selection function. Adding a valuation to a frame yields a model. Given a model, we identify the initial belief set K with the set of formulas that are believed at the actual state and the revised belief set $K*\phi$ (prompted by the input represented by formula ϕ) as the set of formulas that are the consequent of conditionals that (1) are believed at the actual state and (2) have ϕ as antecedent. We show that this class of models characterizes the set of AGM belief revision functions, in the following sense: (1) each model gives rise to a (possibly partial) belief revision function that can be completed into a full AGM belief revision function, and (2) for every AGM belief revision function there is a model whose associated belief revision function coincides with it.

Keywords: belief revision, conditional, belief relation, selection function, supposition, information.

1 Introduction

The dominant theory of belief revision is the so-called AGM theory, named after the authors who started this research program (Alchourrón et al. (1985)). It has spawned a vast literature, which is surveyed in Fermé and Hansson (2011; 2018). The AGM theory is a syntactic theory that represents the agent's initial beliefs as a consistent and deductively closed set of propositional formulas K and the stimulus for belief revision is represented by a propositional formula ϕ , typically interpreted as a new and reliable item of information. $K * \phi$ represents the revised belief set after the receipt of input ϕ . A number of postulates, or axioms, were proposed by Alchourrón et al. (1985) to capture the notion of "rational" belief change. From a semantic point of view, AGM belief revision was shown to correspond to the existence of a "plausibility" order on the set of possible worlds W (where a possible world is a maximally consistent set of propositional formulas), with the initial belief set K corresponding to the most plausible worlds and the revised belief set $K * \phi$ corresponding to the set of most plausible ϕ -worlds (a ϕ -world is a world at which ϕ is true) (Grove (1988), Katsuno and Mendelzon (1991)).

We provide an alternative semantic characterization of the AGM theory, in terms of belief in Stalnaker-Lewis conditionals (Stalnaker (1968), Lewis (1973)), where belief is modeled as a Kripke relation \mathcal{B} on a set of states S and conditionals are modeled via a Stalnaker-Lewis selection function f . We consider a modal language containing a unary modal operator \mathbb{B} (corresponding to the belief relation \mathcal{B}) and a binary conditional operator $\phi > \psi$, interpreted as "if ϕ is (were) the case then ψ is (would be) the case". For the purpose of this paper it is sufficient to focus on the fragment of this language where the conditional $\phi > \psi$ is restricted to Boolean (or non-modal) formulas; the set of Boolean formulas is denoted by Φ_0 . We consider pointed frames $\langle S, s_@, \mathcal{B}, f \rangle$ where $s_@$ denotes the actual state. Adding a valuation to a pointed frame yields a model. Given a model, we identify the initial belief set K with the set of Boolean formulas believed at the actual state: $K = \{\phi \in \Phi_0 : s_@ \models \mathbb{B}\phi\}$. Furthermore, given a Boolean formula $\phi \in \Phi_0$, we identify the "revised" belief set $K * \phi$ with the set of Boolean formulas ψ such that, at the actual state, the agent believes that $\phi > \psi$: $K * \phi = \{\psi \in \Phi_0 : s_@ \models \mathbb{B}(\phi > \psi)\}$, with the usual Stalnaker-Lewis semantics for the conditional $\phi > \psi$: $s \models \phi > \psi$ if $f(s, \|\phi\|) \subseteq \|\psi\|$ (for every formula χ , $\|\chi\|$ denotes the truth set of χ : $\|\chi\| = \{s \in S : s \models \chi\}$). Since, typically, in a given model there may be formulas $\phi \in \Phi_0$ such that $\|\phi\| = \emptyset$, the function $*$ so defined might be only a partial function, that is, a function $*$: $\Psi \rightarrow 2^{\Phi_0}$ where

Ψ might be a proper subset of Φ_0 . We show (Proposition 1) that the class of models considered characterizes the set of AGM belief revision functions, in the following sense:

- Given an arbitrary model, the associated (possibly partial) belief revision function $*$ can be extended to an AGM belief revision function $*' : \Phi_0 \rightarrow 2^{\Phi_0}$ ($*$ is an extension of $*$ if it coincides with $*$ on the domain of $*$), and
- Given an arbitrary AGM belief revision function $*' : \Phi_0 \rightarrow 2^{\Phi_0}$ there exists a model whose associated function $*$ coincides with $*'$.

The paper is organized as follows. In Section 2 we review the AGM theory and the standard semantics for it based on the notion of plausibility order. In Section 3 we develop the alternative semantics described above and state the characterization result. In Section 4 we discuss the issue of whether the AGM theory is properly viewed as a theory of belief change. In Section 5 we discuss related literature and Section 6 concludes. The proofs are given in Appendix A.

2 AGM belief revision functions

Let At be a countable set of atomic formulas (denoted by p, q , possibly with subscripts). We denote by Φ_0 the set of Boolean formulas constructed from At as follows: $\text{At} \subset \Phi_0$ and if $\phi, \psi \in \Phi_0$ then $\neg\phi$ ("not ϕ ") and $\phi \vee \psi$ (" ϕ or ψ ") belong to Φ_0 . The material conditional "if ϕ then ψ ", denoted by $\phi \rightarrow \psi$, the conjunction " ϕ and ψ ", denoted by $\phi \wedge \psi$, and the bi-conditional " ϕ if and only if ψ ", denoted by $\phi \leftrightarrow \psi$, are defined in the usual way.

Given a subset $K \subseteq \Phi_0$, its PL-deductive closure $[K]^{PL}$ (where 'PL' stands for Propositional Logic) is defined as follows: $\psi \in [K]^{PL}$ if and only if there exist $\phi_1, \dots, \phi_n \in K$ (with $n \geq 0$) such that $(\phi_1 \wedge \dots \wedge \phi_n) \rightarrow \psi$ is a tautology (that is, a theorem of Propositional Logic). A set $K \subseteq \Phi_0$ is *consistent* if $[K]^{PL} \neq \Phi_0$ (equivalently, if there is no formula ϕ such that both ϕ and $\neg\phi$ belong to $[K]^{PL}$). A set $K \subseteq \Phi$ is *deductively closed* if $K = [K]^{PL}$.

Let K be a consistent and deductively closed set, representing the agent's initial beliefs, and let $\Psi \subseteq \Phi_0$ be a set of formulas representing possible "inputs". A *belief revision function* (based on K and Ψ) is a function $*_{\Psi} : \Psi \rightarrow 2^{\Phi_0}$ (where 2^{Φ_0} denotes the set of subsets of Φ_0) that associates with every formula $\phi \in \Psi$ a set $K *_{\Psi} \phi \subseteq \Phi_0$ (interpreted as the revised beliefs in response to input ϕ ; we adopt the common notation of writing $K *_{\Psi} \phi$ instead of $*_{K, \Psi}(\phi)$). If $\Psi \neq \Phi_0$ then

$*_{\Psi}$ is called a *partial* belief revision function, while if $\Psi = \Phi_0$ then $*_{\Phi_0}$ is called a *full-domain* belief revision function; in this case we simplify the notation and omit the subscript Φ_0 .

Definition 2.1. Let $*_{\Psi} : \Psi \rightarrow 2^{\Phi_0}$ be a partial belief revision function and $*' : \Phi_0 \rightarrow 2^{\Phi_0}$ a full-domain belief revision function. We say that $*'$ is an *extension* of $*_{\Psi}$ if, for every $\phi \in \Psi$, $K*\phi = K*\psi\phi$.

A *full-domain* belief revision function is called an *AGM function* if it satisfies the following properties, known as the AGM postulates:

- (AGM1) $K*\phi = [K*\phi]^{PL}$.
- (AGM2) $\phi \in K*\phi$.
- (AGM3) $K*\phi \subseteq [K \cup \{\phi\}]^{PL}$.
- (AGM4) if $\neg\phi \notin K$, then $K \subseteq K*\phi$.
- (AGM5) $K*\phi = \Phi_0$ if and only if ϕ is a contradiction.
- (AGM6) if $\phi \leftrightarrow \psi$ is a tautology then $K*\phi = K*\psi$.
- (AGM7) $K*(\phi \wedge \psi) \subseteq [(K*\phi) \cup \{\psi\}]^{PL}$.
- (AGM8) if $\neg\psi \notin K*\phi$, then $[(K*\phi) \cup \{\psi\}]^{PL} \subseteq K*(\phi \wedge \psi)$.

AGM1, called the *Closure* axiom, requires the revised belief set to be deductively closed.

AGM2, called the *Success* axiom, postulates that the input be incorporated into the revised beliefs.

AGM3, called the *Inclusion* axiom, says that beliefs should be revised minimally, in the sense that no new formula should be added unless it can be deduced from the input received and the initial beliefs.¹

AGM4, called the *Preservation* axiom (see, for example, (Rott 2017, p.614)), says that if the input received is compatible with the initial beliefs, then any formula in the initial belief set should belong also to the revised belief set.²

AGM5 requires the revised beliefs to be consistent, unless the input ϕ is a contradiction (that is, unless $\neg\phi$ is a tautology).

AGM6 requires that if ϕ is propositionally equivalent to ψ then the result of revising by ϕ be identical to the result of revising by ψ .

¹Note that $\psi \in [K \cup \{\phi\}]^{PL}$ if and only if $(\phi \rightarrow \psi) \in K$ (since, by hypothesis, $K = [K]^{PL}$). $[K \cup \{\phi\}]^{PL}$ is called the *expansion* of K by ϕ and is often denoted by $K + \phi$.

²The original formulation of (AGM4) was as follows: if $\neg\phi \notin K$, then $[K \cup \{\phi\}]^{PL} \subseteq K*\phi$; however, this is implied by the stated axiom in conjunction with (AGM1) and (AGM2): since K is deductively closed, if $\psi \in [K \cup \{\phi\}]^{PL}$ then (see Footnote 1) $(\phi \rightarrow \psi) \in K$ and thus, by AGM4, $(\phi \rightarrow \psi) \in K*\phi$; by AGM1, $K*\phi = [K*\phi]^{PL}$ and, by AGM2, $\phi \in K*\phi$; thus $\psi \in [K*\phi]^{PL} = K*\phi$.

AGM1-AGM6 are called the *basic* AGM postulates, while AGM7 and AGM8 are called the *supplementary* AGM postulates.

AGM7 and AGM8 are a generalization of AGM3 and AGM4 that concern composite belief revisions of the form $K * (\phi \wedge \psi)$:

The idea is that, if K is to be changed minimally so as to include two sentences ϕ and ψ , such a change should be possible by first revising K with respect to ϕ and then expanding $K * \phi$ by ψ – provided that ψ does not contradict the beliefs in $K * \phi$ (Gärdenfors and Rott 1995, p. 54).

For an extensive discussion of the AGM postulates see Gärdenfors (1988), Gärdenfors and Rott (1995).

The standard semantics for AGM belief revision (Grove (1988), Katsuno and Mendelzon (1991)) is based on a total pre-order \lesssim on the set W of maximally consistent sets of formulas, called *possible worlds*, with the interpretation of $w \lesssim v$ as “the agent considers world w to be at least as plausible as world v ”.³ The initial belief set K consists of the formulas that are true at all the most plausible worlds in W (that is, at every world in $\{w \in W : w \lesssim v, \forall v \in W\}$) and the revised beliefs upon receiving input ϕ are given by the set of formulas that are true at all the most plausible ϕ -worlds (a world w is a ϕ -world if $\phi \in w$; $\|\phi\|$ denotes the set of ϕ -worlds): $K * \phi$ is the set of formulas that are true at all the worlds in $\{w \in \|\phi\| : w \lesssim v, \forall v \in \|\phi\|\}$. In the following section we provide an alternative semantics for AGM belief revision based on a belief relation together with a selection function. Such a semantics was also considered by other authors; see Section 5 for a discussion of related contributions.

3 A Kripke-Stalnaker-Lewis semantics for AGM

Given a binary relation $R \subseteq S \times S$ on a set S , for every $s \in S$ we define $R(s) = \{x \in S : (s, x) \in R\}$.

Definition 3.1. A *pointed frame* is a quadruple $\langle S, s_@, \mathcal{B}, f \rangle$ where

1. S is a set of *states*; subsets of S are called *events*.
2. $s_@ \in S$ is a distinguished element of S interpreted as the *actual state*.
3. $\mathcal{B} \subseteq S \times S$ is a binary *belief relation* on S which is serial: $\mathcal{B}(s) \neq \emptyset, \forall s \in S$.

³An equivalent semantics is in terms of Lewis’ systems of spheres (Lewis (1973)).

4. $f : \mathcal{B}(s_{@}) \times 2^S \setminus \emptyset \rightarrow 2^S$ is a *Stalnaker-Lewis selection function*⁴ that associates with every state-event pair (s, E) (with $s \in \mathcal{B}(s_{@})$ and $\emptyset \neq E \subseteq S$) a set of states $f(s, E) \subseteq S$ such that,
- (a) (Success) $\emptyset \neq f(s, E) \subseteq E$,
 - (b) (Weak Centering) if $s \in E$ then $s \in f(s, E)$,
 - (c) (Doxastic Priority 1) if $\mathcal{B}(s_{@}) \cap E \neq \emptyset$ then $f(s, E) \subseteq \mathcal{B}(s_{@}) \cap E$,
 - (d) (Intersection) $f(s, E) \cap F \subseteq f(s, E \cap F)$,
 - (e) (Doxastic Priority 2) Let $B_{EF} = \{s \in \mathcal{B}(s_{@}) : f(s, E) \cap F \neq \emptyset\}$. Then
 - (e.1) if $s \in B_{EF}$ then $f(s, E \cap F) \subseteq f(s, E) \cap F$,
 - (e.2) if $s \notin B_{EF}$ then $f(s, E \cap F) \subseteq f(\hat{s}, E \cap F)$ for some $\hat{s} \in B_{EF}$.

The set $\mathcal{B}(s)$ is the set of states that the agent considers possible at state s , so that $\mathcal{B}(s_{@})$ is the set of doxastic possibilities at the actual state $s_{@}$ and represents the agent's initial beliefs. $f(s, E)$ is the set of states that the agent considers closest, or most similar, to state s conditional on event E .

(4.a) of Definition 3.1 requires $f(s, E)$ to be non-empty and, furthermore, that every state in $f(s, E)$ be an E -state.

(4.b) postulates that if s is an E -state then it belongs to $f(s, E)$, that is, s itself is one of the E -states that are closest to s .

By (4.c) if there exists an E -state among those initially considered possible ($\mathcal{B}(s_{@}) \cap E \neq \emptyset$), then, for every $s \in \mathcal{B}(s_{@})$, the closest E -states to s must belong to $\mathcal{B}(s_{@}) \cap E$.

By (4.d), the closest E -states to s that are also F -states must belong to the set of closest $(E \cap F)$ -states to s .

(4.e) can be viewed as an extension of (4.c): it says that if, among the states initially considered possible, there is at least one state, call it s , that satisfies the property that among its closest E -states there is at least one that is also an F -state, then (1) the closest $(E \cap F)$ -states to s must belong to the intersection $f(s, E) \cap F$ and (2) for any other state that does not satisfy the property, the closest $(E \cap F)$ -states to it are contained in the set of closest $(E \cap F)$ -states to some state that does satisfy the property.

Adding a valuation to a pointed frame yields a model. Thus a *model* is a tuple $\langle S, s_{@}, \mathcal{B}, f, V \rangle$ where $\langle S, s_{@}, \mathcal{B}, f \rangle$ is a pointed frame and $V : \text{At} \rightarrow 2^S$ is a

⁴Note that, for the purpose of this paper, the domain of f can be taken to be $\mathcal{B}(s_{@}) \times 2^S \setminus \emptyset$ rather than $S \times 2^S \setminus \emptyset$. However, it can easily be extended to $S \times 2^S \setminus \emptyset$ as follows: first, fix an arbitrary function $g : S \setminus \mathcal{B}(s_{@}) \rightarrow \mathcal{B}(s_{@})$ and then define, for every $s \in S \setminus \mathcal{B}(s_{@})$ and every $\emptyset \neq E \subseteq S$, $f(s, E) = f(g(s), E)$.

valuation that assigns to every propositional letter $p \in \text{At}$ the set of states where p is true.

Recall that Φ_0 is the set of Boolean (or factual) formulas, that is, the set of formulas obtained from the set At of atomic formulas by closing with negation and disjunction. Let \mathbb{B} be the unary syntactic belief operator associated with the relation \mathcal{B} and $>$ the binary conditional operator associated with the selection function f . Let Φ_1 be the modal language constructed as follows.

- $\Phi_0 \subset \Phi_1$,
- if $\phi, \psi \in \Phi_0$ then $\phi > \psi \in \Phi_1$,
- all the Boolean combinations of formulas in Φ_1 .

Thus, for the purpose of this paper, the conditional $\phi > \psi$ is defined only for Boolean formulas.

Remark 1. Note that we allow for $\phi > \psi$ to be interpreted either as the indicative conditional 'if ϕ is the case then ψ is the case' (which is appropriate for the case where $\mathcal{B}(s_{@}) \cap \|\phi\| \neq \emptyset$) or as the subjunctive conditional 'if ϕ were the case then ψ would be the case' (which is appropriate for the case where $\mathcal{B}(s_{@}) \subseteq \|\neg\phi\|$).

Finally, let Φ be the modal language constructed as follows:

- $\Phi_1 \subset \Phi$,
- if $\phi \in \Phi_1$ then $\mathbb{B}\phi \in \Phi$,
- all the Boolean combinations of formulas in Φ .

Thus formulas in Φ are either Boolean or formulas of the form $\phi > \psi$, with ϕ and ψ Boolean, or of the form $\mathbb{B}\phi$ where ϕ is either Boolean or of the form $\psi > \chi$ with ψ and χ Boolean or a Boolean combination of such formulas.

Definition 3.2. Given a model $\langle S, s_{@}, \mathcal{B}, f, V \rangle$ define truth of a formula $\phi \in \Phi$ at a state $s \in S$, denoted by $s \models \phi$, as follows (recall that $\|\phi\|$ denotes the truth set of ϕ , that is, $\|\phi\| = \{s \in S : s \models \phi\}$):

1. if $p \in \text{At}$ then $s \models p$ if and only if $s \in V(p)$,
2. $s \models \neg\phi$ if and only if $s \not\models \phi$,
3. $s \models \phi \vee \psi$ if and only if $s \models \phi$ or $s \models \psi$ (or both),

4. $s \models \phi > \psi$ (with $\phi, \psi \in \Phi_0$) if and only if either $\|\phi\| = \emptyset$, or $\|\phi\| \neq \emptyset$ and $f(s, \|\phi\|) \subseteq \|\psi\|$,
5. $s \models \mathbb{B}\phi$ if and only if $\mathcal{B}(s) \subseteq \|\phi\|$.

Fix a model $M = \langle S, s_@, \mathcal{B}, f, V \rangle$ and let $K = \{\phi \in \Phi_0 : \mathcal{B}(s_@) \subseteq \|\phi\|\} = \{\phi \in \Phi_0 : s_@ \models \mathbb{B}\phi\}$ (to simplify the notation, we omit the subscript denoting the model and thus write K rather than K_M); thus $\phi \in K$ if and only if at the actual state $s_@$ the agent believes ϕ . It is shown in the Appendix (Lemma 1) that the set $K \subseteq \Phi_0$ so defined is deductively closed and consistent.

Next, given a model $M = \langle S, s_@, \mathcal{B}, f, V \rangle$ define, for every $\phi \in \Phi_0$ such that $\|\phi\| \neq \emptyset$, $K*\phi = \{\psi \in \Phi_0 : s_@ \models \mathbb{B}(\phi > \psi)\}$. Thus $\psi \in K*\phi$ if and only if at the actual state $s_@$ the agent believes that if ϕ is (were) the case then ψ is (would be) the case. Since, in general, not every $\phi \in \Phi_0$ is such that $\|\phi\| \neq \emptyset$, this definition typically gives rise to a *partial* belief revision function. The next proposition says that this partial belief revision function can be extended to a full-domain AGM revision function; conversely, given a full-domain AGM revision function based on a consistent and deductively closed set K , there exists a model $M = \langle S, s_@, \mathcal{B}, f, V \rangle$ such that $K = \{\phi \in \Phi_0 : s_@ \models \mathbb{B}\phi\}$ and, for every $\phi \in \Phi_0$, $K*\phi = \{\psi \in \Phi_0 : s_@ \models \mathbb{B}(\phi > \psi)\}$. Thus the proposed semantics provides a characterization of AGM revision functions. The proof of Proposition 1 is given in the Appendix.

Proposition 1.

- (A) *Given a model $\langle S, s_@, \mathcal{B}, f, V \rangle$ let $K = \{\phi \in \Phi_0 : s_@ \models \mathbb{B}\phi\}$ and, for every $\phi \in \Phi_0$ such that $\|\phi\| \neq \emptyset$, let $K*\phi = \{\psi \in \Phi_0 : s_@ \models \mathbb{B}(\phi > \psi)\}$. Then K is consistent and deductively closed and the partial belief revision function $*$ so defined can be extended to a full-domain AGM belief revision function.*
- (B) *Let $K \subset \Phi_0$ be consistent and deductively closed and let $*$: $\Phi_0 \rightarrow 2^{\Phi_0}$ be an AGM belief revision function based on K . Then there exists a model $\langle S, s_@, \mathcal{B}, f, V \rangle$ such that $K = \{\phi \in \Phi_0 : s_@ \models \mathbb{B}\phi\}$ and, for every $\phi \in \Phi_0$, $K*\phi = \{\psi \in \Phi_0 : s_@ \models \mathbb{B}(\phi > \psi)\}$.*

4 Is the AGM theory a theory of belief change?

The AGM theory is usually presented as a theory of belief change, or belief revision, *in response to new information*. For example, (Fermé and Hansson 2018, p. v) write: "Belief change (belief revision) is a research area in formal

philosophy that makes use of logic to produce models of how human and artificial agents change their beliefs in response to new information." Thus the belief set $K*\phi$ is thought of as the new belief set after the information represented by the formula ϕ has been made compatible with the initial belief set K . This interpretation is apparent in the way in which the Success Axiom (AGM2) is described or criticized in the literature. Below is a sample of passages from the literature:

"The 'success' postulate guarantees that the incoming information is given absolute priority over the beliefs originally entertained." (Gärdenfors and Rott 1995, p. 53)

Postulate (AGM2) says that the new information ϕ should always be included in the new belief set. (AGM2) places enormous faith on the reliability of ϕ . The new information is perceived to be so reliable that it prevails over all previous conflicting beliefs, no matter what these beliefs might be." (Peppas 2008, p. 319)

"In AGM revision, new information has primacy. This is mirrored in the success postulate for revision. At each stage the system has total trust in the input information, and previous beliefs are discarded whenever that is needed to consistently incorporate the new information. This is an unrealistic feature since in real life, cognitive agents sometimes do not accept the new information that they receive." (Fermé and Hansson 2018, p. 65)

"A system obeying [the Success axiom] is totally trusting at each stage about the input information; it is willing to give up whatever elements of the background theory must be abandoned to render it consistent with the new information. Once this information has been incorporated, however, it is at once as susceptible to revision as anything else in the current theory. Such a rule of revision seems to place an inordinate value on novelty, and its behaviour towards what it learns seems capricious." (Cross and Thomason 1992, p. 251)

By contrast, in the semantics provided in Section 3, the Success axiom (AGM2) is *entirely trivial*, in that it merely requires the agent to believe that "if ϕ is (or were) the case then ϕ is (or would be) the case", which follows naturally from any meaningful reading of the conditional $\phi > \phi$.

Our semantics suggests that it is possible to re-interpret the AGM theory in a way that does not appeal to the notion of belief *change*: the "revised" belief

set $K*\phi$ can be viewed merely as expression of the agent's current beliefs that incorporate beliefs in conditionals.

There has been some discussion in the literature about what kind of belief change is captured by the AGM axioms. There are many different types of belief "change":

1. The mental simulation of scenarios that are known to be contrary to actual reality. This type of belief change has been recently investigated under the name of "reality-oriented mental simulation" (Berto et al. (2022)) or "pretense imagination" (Özgün and Schoonen (2022)).
2. The reaction to information that is accepted as entirely reliable: as (Stalnaker 2009, p. 194) puts it, an "input proposition [that] represents an item of information that the subject takes himself to have come to know".
3. The mental exercise of entertaining a supposition and examining its consequences. As (Eva et al. 2021, p. 1) put it, this is "a form of 'provisional belief revision' in which a person temporarily accepts the supposition as true and makes some appropriate changes to her other opinions so as to accommodate their supposition".

Several authors have noted that there seems to be a significant difference between supposing that ϕ and learning that ϕ :⁵

"Merely suppositional change is essentially different from "genuine" change due to new information." (Rott 1999, p. 410)

"There seems to be a need to distinguish actual belief revision from belief revision that is merely hypothetical. [...] Ordinary theories of belief change do not seem suited to handle the sort of hypothetical belief change that goes on, for example, in debates where the participants agree, "for the sake of argument", on a certain common ground on which possibilities can be explored and disagreements can be aired. One need not actually believe what one accepts in this way." (Seegerberg 1998, p. 1)

⁵There is also empirical evidence that, even in the case where what is being supposed or learned is compatible with the initial beliefs, people treat supposition and information differently: Zhao et al. (2012) found "substantial differences between the conditional probability of an event A supposing an event B compared to the probability of A after having learned B. Specifically, supposing B appears to have less impact on the credibility of A than learning that B is true."

"In none of these contexts is supposing that ϕ equivalent to believing that ϕ ... Changing full beliefs calls for some sort of accounting or justification. Supposition does not..." (Levi 1996, p.5)

The following example, illustrated in Figure 1, highlights the difference between supposing that ϕ and learning that ϕ .

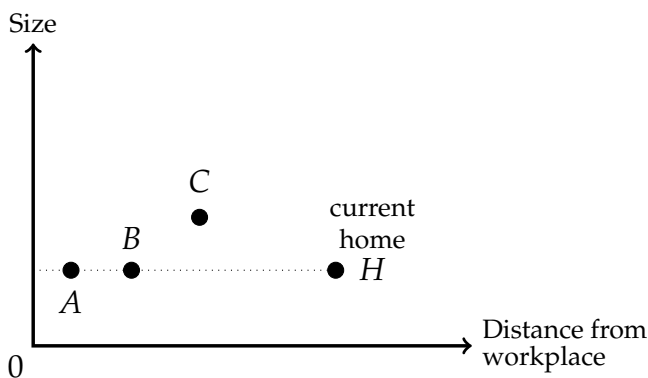


Figure 1: Four houses represented in terms of two attributes: distance from workplace and size.

My friend Bob has been complaining for years about his current house: it is too far from his workplace, it is too small, it is in a noisy neighborhood, etc. I accompanied Bob to view three houses: A, B and C, which differ, as shown in Figure 1, on the basis of two attributes: distance from workplace and size; H is Bob's current house. I later hear that Bob made an offer on one of those three houses. Since, in the past, Bob mainly complained about the long commute, I believe that distance from the workplace was the main attribute in Bob's mind and thus I believe that Bob made an offer on house A. If asked to reason on the *supposition* – "for the sake of argument" – that Bob did not make an offer on house A, it would be defensible for me to maintain my belief that Bob's main concern was distance to the work place and thus conclude (believe) that he made an offer on house B. On the other hand, if I were to be reliably *informed* that Bob did not make an offer on house A, then I could

infer that I was wrong in my belief that distance was the dominant attribute in Bob's mind: size might also have been an important factor and, on the basis of these considerations, I could come to the conclusion (belief) that he made an offer on house C.

In terms of our semantics, my initial beliefs could be such that $\mathcal{B}(s_{@}) = \{s_1\}$ with $s_1 \models A$, $f(s_1, \|\neg A\|) = \{s_2\}$ and $s_2 \models B$, so that my "belief on the supposition that $\neg A$ " is represented by my belief in the conditional $(\neg A > B)$ ($s_{@} \models \mathbb{B}(\neg A > B)$). On the other hand, if I come to know that $\neg A$, then I change my belief from \mathcal{B} to \mathcal{B}' with $\mathcal{B}'(s_{@}) = \{s_3\}$ and $s_3 \models C$.

Some authors have argued that the AGM axioms for belief revision are suitable for modeling suppositional beliefs but not for belief change in response to learning new information. For example, (Levi 1996, p. 117) writes "the contribution of Alchourrón, Gärdenfors and Makinson is best seen as a contribution to an account of reasoning for the sake of the argument and not as an account of the logic of belief change".

The semantics discussed in Section 3 shows the possibility of an interpretation of the AGM axioms that involves neither belief change nor belief revision:⁶ both the belief relation \mathcal{B} and the Stalnaker-Lewis selection function f remain fixed. On the other hand, belief change *can* be accounted for by means of a change in either \mathcal{B} or f , or both. The issue of belief change in response to reliable information will be explored in future work.

5 Related literature

The characterization result of Proposition 1 is based on establishing a correspondence between $\psi \in K*\phi$ in the AGM framework (that is, believing ψ after revising by ϕ) and $s_{@} \models \mathbb{B}(\phi > \psi)$ in our framework (that is, believing – at the actual state – that "if ϕ is (were) the case then ψ is (would be) the case"). In the literature, the attempt to relate belief revision to conditionals of the form $\phi > \psi$ has led to Gärdenfors' well-known Triviality result (Gärdenfors (1986)). In his approach the Boolean propositional language Φ_0 is extended to the language that we called Φ_1 , which includes conditionals of the form $\phi > \psi$; furthermore, belief sets are allowed to contain such formulas and indeed it is postulated

⁶Unless one interprets belief in the conditional $\phi > \psi$ as "belief in ψ on the supposition that ϕ " and the latter is viewed as a form of "belief revision". For an investigation of the relationship between supposition and conditionals see Carter (2021).

(Gärdenfors 1986, p.84) that

$$(\phi > \psi) \in K \text{ if and only if } \psi \in K*\phi, \quad (\text{R})$$

which is intended to capture the “Ramsey test”.⁷ Gärdenfors proved that if (R) is added to the AGM postulates for belief revision, a logical contradiction ensues: only trivialized revision operators (or revision operators defined on a trivialized set of belief sets) are allowed.⁸ However, Gärdenfors’ triviality result does not apply to our framework because – although we considered a language containing modal formulas of the type $\phi > \psi$ and $\mathbb{B}(\phi > \psi)$ – for the purpose of establishing a link to the AGM theory we only considered a fragment of the language in which the initial belief set K and the revised belief sets $K*\phi$ are restricted, as in the original AGM theory, to the Boolean language Φ_0 and thus do not contain any modal formulas.

The semantics given in Section 3 – namely a Stalnaker-Lewis selection function with the addition of a Kripke belief relation – was also implicitly considered in Leitgeb (2007) who proposed a different interpretation of $\psi \in K*\phi$:

Let w be a possible world in which the agent’s actual belief set is K . Now consider the set W' of worlds w' in which our agent believes ϕ [...] we can thus reformulate the semantics of the revision operator as follows: “ $\psi \in K*\phi$ ” is true in w if and only if all those worlds w'' among the members of W' that are maximally similar to w in Lewis’ sense are worlds in which the agent believes ψ . [...] a rational agent has a conditional belief in ψ given ϕ if and only if: *if she believed ϕ , then she would believe ψ* . (Leitgeb 2007, p. 121) [For consistency with our previous notation, we used the symbols ϕ and ψ instead of A and B , respectively.]

In the notation of Section 3, we interpret Leitgeb’s suggestion as follows: $\psi \in K*\phi$ if and only if $s_{@} \models (\mathbb{B}\phi > \mathbb{B}\psi)$; thus a very different interpretation from ours.⁹ Leitgeb offers several arguments in favor of his suggested interpretation, but does not establish an exact correspondence between AGM belief revision

⁷The expression “Ramsey Test” refers to the following passage from (Ramsey 1950, p. 247): “If two people are arguing “If p will q ?” and are both in doubt as to p , they are adding p hypothetically to their stock of knowledge and arguing on that basis about q ”.

⁸Gärdenfors’ triviality result gave rise to a sizeable literature; see, for example, Leitgeb (2010), Lindström (1996), Lindström and Rabinowicz (1992), Lindström and Rabinowicz (1998).

⁹Leitgeb argues that “there are two different types of beliefs of “conditional character”: *beliefs in conditionals* and *conditional beliefs*.” (Leitgeb 2007, p. 115). We focused on the former while Leitgeb opted for the latter.

functions and the semantics he has in mind. Our objective is not to agree that our proposed interpretation is “the correct” one, but simply to show that it works, in the sense that it provides a semantic characterization of AGM belief revision that is different from the standard one based on plausibility orders. Proposition 1 shows that, in the AGM theory, $\psi \in K*\phi$ can be consistently interpreted as a belief in the conditional $\phi > \psi$: $\psi \in K*\phi$ if and only if $s_{@} \models \mathbb{B}(\phi > \psi)$.

The semantics consisting of a Stalnaker selection function¹⁰ augmented with a belief relation was also considered recently by Günther and Sisti (2022) who dubbed it “Stalnaker’s Ramsey Test”. However, the focus of Günther and Sisti (2022) is very different from ours: the authors do not establish a link to the AGM theory of belief revision and do not view the proposed semantics as an alternative characterization of AGM belief revision. The main purpose of Günther and Sisti (2022) is to argue that the “Stalnaker Ramsey Test” provides an alternative way of capturing Ramsey’s inferential account, which was framed in terms of variable hypotheticals.¹¹

We used a StalnakerLewis selection function to assign a truth value to the conditional $\phi > \psi$ at a state, without making a distinction between indicative and subjunctive conditionals (Remark 1). The view – a minority view, according to (Nolan 2003, pp. 215-6) – that a closest-world semantics is appropriate for both indicative and subjunctive conditionals has been defended by several authors (Davis (1979), Lycan (2001), Nolan (2003), Stalnaker (1975), Weatherson (2001)).

A common view concerning conditionals is that accepting a conditional involves supposing the antecedent and assessing the consequent in the resulting hypothetical state. A theory of conditionals which makes essential appeal to supposition has been defended (in various, closely related, forms) by a number of authors (see, for example, Barker (1995), Barnett (2006), Carter (2021), Ciardelli (2020), DeRose and Grandy (1999), Gillies (2004), Mackie (1972), Edgington (1995)). Whether *believing* a conditional is, or is not, the same as *accepting* a conditional and whether a suppositional interpretation of believing a conditional is, or is not, the “correct” interpretation, are issues that do not need to be settled for the purpose of the characterization result given in Proposition 1.

¹⁰The difference between a Stalnaker selection function and a Lewis selection function is that the former requires $f(s, E)$ to be a singleton, that is, that there be a unique E -state closest to s .

¹¹A variable hypothetical is a subjunctive rule that Ramsey expresses as “If I meet a ϕ I shall regard it as a ψ ” (Ramsey 1950, p. 241). (Günther and Sisti 2022, p.29) argue that the belief in the variable hypothetical $\forall x (\phi(x) \rightarrow \psi(x))$ can be faithfully translated into Stalnaker semantics as follows: for all worlds the agent cannot exclude to be the actual, the most similar ϕ -world is a ψ -world.

6 Conclusion

We provided a characterization of AGM belief revision in terms of a semantics that consists of a Stalnaker selection function with the addition of a belief relation. We have shown that the set of AGM belief revision functions corresponds to the set of functions that can be obtained from the class of models that we considered by identifying the initial belief set K with the set of formulas that the agent believes at the actual state ($K = \{\phi \in \Phi_0 : w_{@} \models \mathbb{B}\phi\}$) and the revised belief set $K*\phi$, in response to input ϕ , with the set of formulas that are the consequent of conditionals that (1) are believed at the actual state and (2) have ϕ as antecedent ($K*\phi = \{\psi \in \Phi_0 : s_{@} \models \mathbb{B}(\phi > \psi)\}$). We noted (Section 4) that this alternative semantics raises the question whether the AGM theory is necessarily a theory of belief change or revision. The proposed semantics also establishes a clear connection between AGM belief revision and conditionals which steers clear of the triviality result proved by Gärdenfors (1986). Our characterization of AGM belief revision made use of only a fragment of the modal language introduced in Section 3. A study of the full language and of its potential use in modeling belief change is left for future work.

A Proofs

Lemma 1. *Fix a model $M = \langle S, s_{@}, \mathcal{B}, f, V \rangle$ and let $K = \{\phi \in \Phi_0 : \mathcal{B}(s_{@}) \subseteq \|\phi\|\} = \{\phi \in \Phi_0 : s_{@} \models \mathbb{B}\phi\}$. Then K is deductively closed and consistent.*

Proof. First we show that K is deductively closed, that is, $K = [K]^{PL}$. If $\psi \in K$ then $\psi \in [K]^{PL}$, because $\psi \rightarrow \psi$ is a tautology; thus $K \subseteq [K]^{PL}$. To show that $[K]^{PL} \subseteq K$, let $\psi \in [K]^{PL}$, that is, there exist $\phi_1, \dots, \phi_n \in K$ ($n \geq 0$) such that $(\phi_1 \wedge \dots \wedge \phi_n) \rightarrow \psi$ is a tautology. Since $\|\phi_1 \wedge \dots \wedge \phi_n\| = \|\phi_1\| \cap \dots \cap \|\phi_n\|$ and, for all $i = 1, \dots, n$, $\phi_i \in K$ (that is, $\mathcal{B}(s_{@}) \subseteq \|\phi_i\|$), it follows that $\mathcal{B}(s_{@}) \subseteq \|\phi_1 \wedge \dots \wedge \phi_n\|$. Since $(\phi_1 \wedge \dots \wedge \phi_n) \rightarrow \psi$ is a tautology, $\|(\phi_1 \wedge \dots \wedge \phi_n) \rightarrow \psi\| = S$, that is, $\|\phi_1 \wedge \dots \wedge \phi_n\| \subseteq \|\psi\|$. Thus $\mathcal{B}(s_{@}) \subseteq \|\psi\|$, that is, $\psi \in K$. Next we show that $[K]^{PL} \neq \Phi_0$, that is, K is consistent. Let $p \in \text{At}$ be an atomic formula. Then $\|p \wedge \neg p\| = \emptyset$. By seriality of \mathcal{B} , $\mathcal{B}(s_{@}) \neq \emptyset$ so that $\mathcal{B}(s_{@}) \not\subseteq \|p \wedge \neg p\|$, that is, $(p \wedge \neg p) \notin K$ and hence, since $K = [K]^{PL}$, $(p \wedge \neg p) \notin [K]^{PL}$. \square

Lemma 2. *Fix a model $\langle S, s_{@}, \mathcal{B}, f, V \rangle$ and let $\phi \in \Phi_0$ be such that $\|\phi\| \neq \emptyset$. Then, $\forall \psi, \chi \in \Phi_0$*

- (A) $\forall s \in \mathcal{B}(s_{@})$, if $s \models (\phi \wedge \psi) > \chi$ then $s \models \phi > (\psi \rightarrow \chi)$,

(B) if $\mathcal{B}(s_{@}) \subseteq \|\phi > (\psi \rightarrow \chi)\|$ and, for some $s \in \mathcal{B}(s_{@})$, $f(s, \|\phi\|) \cap \|\psi\| \neq \emptyset$, then $\mathcal{B}(s_{@}) \subseteq \|(\phi \wedge \psi) > \chi\|$.

Proof. (A) Assume that $\|\phi\| \neq \emptyset$. Let $s \in \mathcal{B}(s_{@})$ be such that $s \models (\phi \wedge \psi) > \chi$, that is, either $\|\phi \wedge \psi\| = \emptyset$ or $\|\phi \wedge \psi\| \neq \emptyset$ and $f(s, \|\phi \wedge \psi\|) \subseteq \|\chi\|$. We need to show that $s \models \phi > (\psi \rightarrow \chi)$, that is (since $\|\phi\| \neq \emptyset$), that $f(s, \|\phi\|) \subseteq \|\psi \rightarrow \chi\|$.

Assume first that $\|\phi \wedge \psi\| = \emptyset$. Then – since $\|\phi \wedge \psi\| = \|\phi\| \cap \|\psi\|$ and, by 4(a) of Definition 3.1, $f(s, \|\phi\|) \subseteq \|\phi\|$ – it must be that $f(s, \|\phi\|) \cap \|\psi\| = \emptyset$, that is, $f(s, \|\phi\|) \subseteq \|\neg\psi\|$, so that (since $\|\neg\psi\| \subseteq \|\neg\psi\| \cup \|\chi\| = \|\psi \rightarrow \chi\|$) $f(s, \|\phi\|) \subseteq \|\psi \rightarrow \chi\|$, that is, $s \models \phi > (\psi \rightarrow \chi)$.

Assume now that $\|\phi \wedge \psi\| \neq \emptyset$ so that

$$f(s, \|\phi \wedge \psi\|) \subseteq \|\chi\|. \quad (1)$$

By (4d) of Definition 3.1 (with $E = \|\phi\|$ and $F = \|\psi\|$), $f(s, \|\phi\|) \cap \|\psi\| \subseteq f(s, \|\phi \wedge \psi\|)$. It follows from this and (1) that $f(s, \|\phi\|) \cap \|\psi\| \subseteq \|\chi\|$ and thus, since $\|\chi\| \subseteq \|\neg\psi\| \cup \|\chi\| = \|\neg\psi \vee \chi\| = \|\psi \rightarrow \chi\|$,

$$f(s, \|\phi\|) \cap \|\psi\| \subseteq \|\psi \rightarrow \chi\|. \quad (2)$$

On the other hand,

$$f(s, \|\phi\|) \cap \|\neg\psi\| \subseteq \|\neg\psi\| \subseteq \|\psi \rightarrow \chi\|. \quad (3)$$

Thus, by (2) and (3) (since $\|\neg\psi\| = S \setminus \|\psi\|$), $f(s, \|\phi\|) \subseteq \|\psi \rightarrow \chi\|$, that is, $s \models \phi > (\psi \rightarrow \chi)$.

(B) Assume that $\mathcal{B}(s_{@}) \subseteq \|\phi > (\psi \rightarrow \chi)\|$, that is,

$$\forall s \in \mathcal{B}(s_{@}), f(s, \|\phi\|) \subseteq \|\psi \rightarrow \chi\|. \quad (4)$$

Let $\hat{s} \in \mathcal{B}(s_{@})$ be such that $f(\hat{s}, \|\phi\|) \cap \|\psi\| \neq \emptyset$ (such an \hat{s} exists by hypothesis). Then, by (4),

$$f(\hat{s}, \|\phi\|) \cap \|\psi\| \subseteq \|\psi \rightarrow \chi\| \cap \|\psi\| \subseteq \|\chi\|. \quad (5)$$

Since $\|\phi\| \cap \|\psi\| = \|\phi \wedge \psi\|$, by (e.1) of Definition 3.1 (with $E = \|\phi\|$ and $F = \|\psi\|$), $f(\hat{s}, \|\phi \wedge \psi\|) \subseteq f(\hat{s}, \|\phi\|) \cap \|\psi\|$ and thus, by (5), $f(\hat{s}, \|\phi \wedge \psi\|) \subseteq \|\chi\|$. Thus we have shown that, letting $B_{\phi\psi} = \{s \in \mathcal{B}(s_{@}) : f(s, \|\phi\|) \cap \|\psi\| \neq \emptyset\}$,

$$\forall s \in B_{\phi\psi}, f(s, \|\phi \wedge \psi\|) \subseteq \|\chi\|. \quad (6)$$

If $s' \notin B_{\phi\psi}$ then, by (e.2) of Definition 3.1, $f(s', \|\phi \wedge \psi\|) \subseteq f(s, \|\phi \wedge \psi\|)$ for some $s \in B_{\phi\psi}$ and thus, by (6), $f(s', \|\phi \wedge \psi\|) \subseteq \|\chi\|$. Thus $\mathcal{B}(s_{@}) \subseteq \|(\phi \wedge \psi) > \chi\|$. \square

We restate Proposition 1 below.

Proposition.

- (A) Given a model $\langle S, s_{@}, \mathcal{B}, f, V \rangle$ let $K = \{\phi \in \Phi_0 : s_{@} \models \mathbb{B}\phi\}$ and, for every $\phi \in \Phi_0$ such that $\|\phi\| \neq \emptyset$, let $K*\phi = \{\psi \in \Phi_0 : s_{@} \models \mathbb{B}(\phi > \psi)\}$. Then K is consistent and deductively closed and the partial belief revision function $*$ so defined can be extended to a full-domain AGM belief revision function.
- (B) Let $K \subset \Phi_0$ be consistent and deductively closed and let $*$: $\Phi_0 \rightarrow 2^{\Phi_0}$ be an AGM belief revision function based on K . Then there exists a model $\langle S, s_{@}, \mathcal{B}, f, V \rangle$ such that $K = \{\phi \in \Phi_0 : s_{@} \models \mathbb{B}\phi\}$ and, for every $\phi \in \Phi_0$, $K*\phi = \{\psi \in \Phi_0 : s_{@} \models \mathbb{B}(\phi > \psi)\}$.

Proof. (A) Fix a model $\langle S, s_{@}, \mathcal{B}, f, V \rangle$ and let $K = \{\phi \in \Phi_0 : s_{@} \models \mathbb{B}\phi\}$. By Lemma 1, K is consistent and deductively closed. For every $\phi \in \Phi_0$ such that $\|\phi\| \neq \emptyset$ define

$$\begin{aligned} K*\phi &= \{\psi \in \Phi_0 : s_{@} \models \mathbb{B}(\phi > \psi)\} \\ &= \{\psi \in \Phi_0 : \forall s \in \mathcal{B}(s_{@}), f(s, \|\phi\|) \subseteq \|\psi\|\}. \end{aligned} \quad (7)$$

Define $*' : \Phi_0 \rightarrow 2^{\Phi_0}$ as follows:

$$K*\phi = \begin{cases} [\phi]^{PL} & \text{if } \|\phi\| = \emptyset \\ K*\phi & \text{if } \|\phi\| \neq \emptyset. \end{cases} \quad (8)$$

We want to show that the full-domain belief revision function (8) satisfies the AGM axioms.

(AGM1). We need to show that $K*\phi$ is deductively closed. If $\|\phi\| = \emptyset$ then this is true by construction since, by (8), $K*\phi = [\phi]^{PL}$. Assume, therefore, that $\|\phi\| \neq \emptyset$; we need to show that $K*\phi = [K*\phi]^{PL}$, where $K*\phi$ is given by (7). The inclusion $K*\phi \subseteq [K*\phi]^{PL}$ follows from the fact that, for every $\psi \in K*\phi$, $\psi \rightarrow \psi$ is a tautology. Next we show that $[K*\phi]^{PL} \subseteq K*\phi$. Since $\|\phi\| \neq \emptyset$, $f(s, \|\phi\|)$ is defined for every $s \in \mathcal{B}(s_{@})$. Fix an arbitrary $\psi \in [K*\phi]^{PL}$; then there exist $\phi_1, \dots, \phi_n \in K*\phi$ ($n \geq 0$) such that $(\phi_1 \wedge \dots \wedge \phi_n) \rightarrow \psi$ is a tautology, so that $\|(\phi_1 \wedge \dots \wedge \phi_n) \rightarrow \psi\| = S$, that is, $\|\phi_1 \wedge \dots \wedge \phi_n\| \subseteq \|\psi\|$. Fix an arbitrary $s \in \mathcal{B}(s_{@})$ and an arbitrary $i = 1, \dots, n$. Then, since $\phi_i \in K*\phi$, $f(s, \|\phi\|) \subseteq \|\phi_i\|$. Hence $f(s, \|\phi\|) \subseteq \|\phi_1\| \cap \dots \cap \|\phi_n\| = \|\phi_1 \wedge \dots \wedge \phi_n\|$. Since $\|\phi_1 \wedge \dots \wedge \phi_n\| \subseteq \|\psi\|$ it follows that $f(s, \|\phi\|) \subseteq \|\psi\|$, that is, $s \models (\phi > \psi)$. Thus, since $s \in \mathcal{B}(s_{@})$ was chosen arbitrarily, $\mathcal{B}(s_{@}) \subseteq \|\phi > \psi\|$, that is, $\psi \in K*\phi$.

(AGM2). We need to show that $\phi \in K*\phi$. If $\|\phi\| = \emptyset$ then this is true by construction, since, by (8), $K*\phi = [\phi]^{PL}$. Suppose, therefore, that $\|\phi\| \neq \emptyset$, so

that $f(s, \|\phi\|)$ is defined for every $s \in \mathcal{B}(s_{@})$. We need to show that $\mathcal{B}(s_{@}) \subseteq \|\phi > \phi\|$. Fix an arbitrary $s \in \mathcal{B}(s_{@})$. Then, by 4(a) of Definition 3.1 (Success), $f(s, \|\phi\|) \subseteq \|\phi\|$ so that $s \models \phi > \phi$. Since $s \in \mathcal{B}(s_{@})$ was chosen arbitrarily, $\mathcal{B}(s_{@}) \subseteq \|\phi > \phi\|$.

(AGM3). We need to show that $K*\phi \subseteq [K \cup \{\phi\}]^{PL}$. If $\|\phi\| = \emptyset$ (so that $\|\neg\phi\| = S$ and thus $\neg\phi \in K$) then $[K \cup \{\phi\}]^{PL} = \Phi_0$. Suppose, therefore, that $\|\phi\| \neq \emptyset$, so that $f(s, \|\phi\|)$ is defined for every $s \in \mathcal{B}(s_{@})$. We need to show that, if $\psi \in K*\phi$ (where $K*\phi$ is given by (7)) then $\psi \in [K \cup \{\phi\}]^{PL}$, that is (since, by Lemma 1, $K = [K]^{PL}$), that $(\phi \rightarrow \psi) \in K$, i.e. $\mathcal{B}(s_{@}) \subseteq \|\phi \rightarrow \psi\|$. Let $\psi \in K*\phi$ and fix an arbitrary $s \in \mathcal{B}(s_{@})$. Then, by (7), $s \models \phi > \psi$, that is,

$$f(s, \|\phi\|) \subseteq \|\psi\| \subseteq \|\neg\phi\| \cup \|\psi\| = \|\phi \rightarrow \psi\|. \quad (9)$$

If $s \in \|\phi\|$ then, by 4(b) of Definition 3.1 (Weak Centering), $s \in f(s, \|\phi\|)$ and thus, by (9), $s \in \|\phi \rightarrow \psi\|$. If $s \in \|\neg\phi\|$ then (since $\|\neg\phi\| \subseteq \|\neg\phi\| \cup \|\psi\| = \|\phi \rightarrow \psi\|$) $s \in \|\phi \rightarrow \psi\|$. Thus, since $s \in \mathcal{B}(s_{@})$ was chosen arbitrarily, $\mathcal{B}(s_{@}) \subseteq \|\phi \rightarrow \psi\|$.

(AGM4). We need to show that if $\neg\phi \notin K$ then $K \subseteq K*\phi$. Assume that $\neg\phi \notin K$, that is, $\mathcal{B}(s_{@}) \cap \|\phi\| \neq \emptyset$ so that $\|\phi\| \neq \emptyset$. Then $K*\phi = K*\phi$ and thus we have to show that if $\psi \in K$ (that is, $\mathcal{B}(s_{@}) \subseteq \|\psi\|$) then $\psi \in K*\phi$, that is, $\mathcal{B}(s_{@}) \subseteq \|\phi > \psi\|$. Fix an arbitrary $s \in \mathcal{B}(s_{@})$. Since $\mathcal{B}(s_{@}) \cap \|\phi\| \neq \emptyset$, by 4(c) of Definition 3.1 (Doxastic Priority 1), $f(s, \|\phi\|) \subseteq \mathcal{B}(s_{@}) \cap \|\phi\|$ so that (since $\mathcal{B}(s_{@}) \subseteq \|\psi\|$) $f(s, \|\phi\|) \subseteq \|\psi\|$, that is, $s \models \phi > \psi$. Thus, since $s \in \mathcal{B}(s_{@})$ was chosen arbitrarily, $\mathcal{B}(s_{@}) \subseteq \|\phi > \psi\|$.

(AGM5). We need to show that $K*\phi = \Phi_0$ if and only if $\neg\phi$ is a tautology (that is, ϕ is a contradiction). The ‘if’ part is true by construction, since, if $\neg\phi$ is a tautology, then $\|\neg\phi\| = S$ and thus $\|\phi\| = \emptyset$, so that, by (8), $K*\phi = [\phi]^{PL}$ and the deductive closure of a contradiction is equal to Φ_0 . Thus we only need to show that if ϕ is consistent (that is, $\neg\phi$ is not a tautology) then $K*\phi \neq \Phi_0$. If $\|\phi\| = \emptyset$ then this is true by construction, since the deductive closure of a consistent formula is not equal to Φ_0 . If $\|\phi\| \neq \emptyset$ then we need to show that there is a $\psi \in \Phi_0$ such that $\psi \notin K*\phi$, i.e. $s_{@} \not\models \mathbb{B}(\phi > \psi)$. Let ψ be $(p \wedge \neg p)$ where p is an atomic formula. Then $\|\psi\| = \emptyset$ so that, for every $s \in \mathcal{B}(s_{@})$, since by 4(a) of Definition 3.1 $f(s, \|\phi\|) \neq \emptyset$, $f(s, \|\phi\|) \not\subseteq \|\psi\|$, that is, $s \not\models (\phi > \psi)$ and thus (since, by seriality of \mathcal{B} , $\mathcal{B}(s_{@}) \neq \emptyset$) $s_{@} \not\models \mathbb{B}(\phi > \psi)$.

(AGM6). We need to show that if $\phi \leftrightarrow \psi$ is a tautology then $K*\phi = K*\psi$. If $\phi \leftrightarrow \psi$ is a tautology then $[\phi]^{PL} = [\psi]^{PL}$ and $\|\phi\| = \|\psi\|$ and thus if $\|\phi\| = \emptyset$ then, by (8), $[\phi]^{PL} = K*\phi = K*\psi = [\psi]^{PL}$. If $\|\phi\| \neq \emptyset$ then, since $\|\phi\| = \|\psi\|$, for every

$s \in \mathcal{B}(s_{@})$, $f(s, \|\phi\|) = f(s, \|\psi\|)$ and thus, for every $\chi \in \Phi_0$, $f(s, \|\phi\|) \subseteq \|\chi\|$ if and only if $f(s, \|\psi\|) \subseteq \|\chi\|$ so that $\mathcal{B}(s_{@}) \subseteq \|\phi > \chi\|$ if and only if $\mathcal{B}(s_{@}) \subseteq \|\psi > \chi\|$, that is, $\chi \in K*\phi$ if and only if $\chi \in K*\psi$.

(AGM7). We need to show that $K*'\phi \subseteq [(K*\phi) \cup \{\psi\}]^{PL}$. If $\|\phi\| = \emptyset$ then $\|(\phi \wedge \psi)\| = \emptyset$ so that, by construction, $K*'\phi \subseteq [(\phi \wedge \psi)]^{PL}$ and $[(K*\phi) \cup \{\psi\}]^{PL} = [(\phi \wedge \psi)]^{PL} \cup \{\psi\}^{PL}$, which is equal to $[(\phi \wedge \psi)]^{PL}$. Assume, therefore, that $\|\phi\| \neq \emptyset$ so that $K*\phi = K*\psi = \{\chi \in \Phi_0 : s_{@} \models \mathbb{B}(\phi > \chi)\}$. Since, by AGM1, $K*\phi$ is deductively closed,

$$\begin{aligned} \chi \in [(K*\phi) \cup \{\psi\}]^{PL} & \text{ if and only if } (\psi \rightarrow \chi) \in K*\phi \\ & \text{ if and only if } s_{@} \models \mathbb{B}(\phi > (\psi \rightarrow \chi)) \end{aligned} \quad (10)$$

Fix an arbitrary $\chi \in K*(\phi \wedge \psi)$, that is, a $\chi \in \Phi_0$ such that $s_{@} \models \mathbb{B}((\phi \wedge \psi) > \chi)$. Fix an arbitrary $s \in \mathcal{B}(s_{@})$. Then $s \models (\phi \wedge \psi) > \chi$. By (A) of Lemma 2, $s \models \phi > (\psi \rightarrow \chi)$. Hence $s_{@} \models \mathbb{B}(\phi > (\psi \rightarrow \chi))$, so that, by (10), $\chi \in [(K*\phi) \cup \{\psi\}]^{PL}$.

(AGM8). We need to show that if $\neg\psi \notin K*\phi$ then $[(K*\phi) \cup \{\psi\}]^{PL} \subseteq K*(\phi \wedge \psi)$. If $\|\phi\| = \emptyset$ then, as shown in the proof of (AGM7), $K*'\phi \subseteq [(K*\phi) \cup \{\psi\}]^{PL}$. Assume, therefore, that $\|\phi\| \neq \emptyset$, so that $f(s, \|\phi\|)$ is defined for every $s \in \mathcal{B}(s_{@})$ and $K*\phi = K*\psi = \{\chi \in \Phi_0 : s_{@} \models \mathbb{B}(\phi > \chi)\}$. Fix an arbitrary $\chi \in [(K*\phi) \cup \{\psi\}]^{PL}$. Then, since by (AGM1) $K*\phi$ is deductively closed, $(\psi \rightarrow \chi) \in K*\phi$, that is, $\mathcal{B}(s_{@}) \subseteq \|\phi > (\psi \rightarrow \chi)\|$. Since $\neg\psi \notin K*\phi$, there exists an $\hat{s} \in \mathcal{B}(s_{@})$ such that $f(\hat{s}, \|\phi\|) \not\subseteq \|\neg\psi\|$, that is, $f(\hat{s}, \|\phi\|) \cap \|\psi\| \neq \emptyset$. Thus, by (B) of Lemma 2, $\mathcal{B}(s_{@}) \subseteq \|(\phi \wedge \psi) > \chi\|$, that is, $\chi \in K*(\phi \wedge \psi)$.

(B) Next we show that if $K \subset \Phi_0$ is consistent and deductively closed and $* : \Phi_0 \rightarrow 2^{\Phi_0}$ is an AGM belief revision function based on K , then there exists a model $\langle S, s_{@}, \mathcal{B}, f, V \rangle$ such that $K = \{\phi \in \Phi_0 : \mathcal{B}(s_{@}) \subseteq \|\phi\|\} = \{\phi \in \Phi_0 : s_{@} \models \mathbb{B}\phi\}$ and, for all $\phi, \psi \in \Phi_0$, $\psi \in K*\phi$ if and only if $\mathcal{B}(s_{@}) \subseteq \|\phi > \psi\|$, that is, $\forall s \in \mathcal{B}(s_{@})$, $f(s, \|\phi\|) \subseteq \|\psi\|$. Define the following model $\langle S, s_{@}, \mathcal{B}, f, V \rangle$:

1. S is the set of maximally consistent sets of formulas in Φ_0 (recall that Φ_0 is the set of Boolean formulas).
2. The valuation $V : At \rightarrow S$ is defined by $V(p) = \{s \in S : p \in s\}$, so that, for every $\phi \in \Phi_0$, $\|\phi\| = \{s \in S : \phi \in s\}$. If $\Psi \subseteq \Phi_0$, define $\|\Psi\| = \{s \in S : \forall \phi \in \Psi, \phi \in s\}$.¹²

¹²Thus for $\phi \in \Phi_0$, $s \models \phi$ if and only if $\phi \in s$, while modal formulas of the type $\phi > \psi$ (with $\phi, \psi \in \Phi_0$), which are not in Φ_0 , do not belong to any $s \in S$ and the model validation rule is as before: $s \models \phi > \psi$ if and only if $f(s, \|\phi\|) \subseteq \|\psi\|$.

3. Choose an arbitrary $s_{@} \in S$ and define $\mathcal{B}(s_{@}) = \|K\|$.
4. In order to define the selection function f , note first that if ϕ is a contradiction then, by (AGM5), $K*\phi = \Phi_0$ and, since $\|\phi\| = \emptyset$, by (5) of Definition 3.2, for every formula $\psi \in \Phi_0$, and for every $s \in S$, $s \models \phi > \psi$ so that $\{\psi \in \Phi_0 : \mathcal{B}(s_{@}) \subseteq \|\phi > \psi\|\} = \Phi_0$. Thus we only need to consider consistent formulas. Let $\Phi_{cn} \subseteq \Phi_0$ be the set of consistent Boolean formulas and let $\mathcal{E} = \{E \subseteq S : E = \|\phi\| \text{ for some } \phi \in \Phi_{cn}\}$. Define $f : \mathcal{B}(s_{@}) \times \mathcal{E} \rightarrow S$ as follows:¹³

$$\forall s \in \mathcal{B}(s_{@}), f(s, \|\phi\|) = \|K*\phi\|. \quad (11)$$

Lemma 3.

- (a) $\forall \phi \in \Phi_{cn}, \|K\| \cap \|\phi\| = \|[K \cup \{\phi\}]\|^{PL}$.
- (b) $\forall \phi, \psi \in \Phi_{cn}, \|K*\phi\| \cap \|\psi\| = \|[(K*\phi) \cup \{\psi\}]\|^{PL}$.

Proof. (a) By hypothesis, K is deductively closed. Thus, $\forall \chi \in \Phi_0$,

$$\chi \in [K \cup \{\phi\}]^{PL} \text{ if and only if } (\phi \rightarrow \chi) \in K. \quad (12)$$

First we show that

$$\|K\| \cap \|\phi\| \subseteq \|[K \cup \{\phi\}]\|^{PL}.$$

Fix an arbitrary $s \in \|K\| \cap \|\phi\|$; we need to show that $s \in \|[K \cup \{\phi\}]\|^{PL}$, that is, that $\forall \chi \in [K \cup \{\phi\}]^{PL}, \chi \in s$. Since $s \in \|\phi\|$, $\phi \in s$. Fix an arbitrary $\chi \in [K \cup \{\phi\}]^{PL}$; then, by (12), $(\phi \rightarrow \chi) \in K$; thus, since $s \in \|K\|$, $(\phi \rightarrow \chi) \in s$. Hence, since both ϕ and $\phi \rightarrow \chi$ belong to s and s is deductively closed, $\chi \in s$.

Next we show that

$$[K \cup \{\phi\}]^{PL} \subseteq \|K\| \cap \|\phi\|.$$

Let $s \in \|[K \cup \{\phi\}]\|^{PL}$. Then, since $\phi \in [K \cup \{\phi\}]^{PL}$, $\phi \in s$, that is, $s \in \|\phi\|$. It remains to show that $s \in \|K\|$, that is, that, for every $\chi \in K$, $\chi \in s$. Fix an arbitrary $\chi \in K$; then, since, by hypothesis, K is deductively closed, $(\phi \rightarrow \chi) \in K$. Thus, by (12), $\chi \in [K \cup \{\phi\}]^{PL}$ and thus (since $s \in \|[K \cup \{\phi\}]\|^{PL}$) $\chi \in s$.

- (b) By AGM1, $K*\phi$ is deductively closed. Thus, $\forall \chi \in \Phi_0$,

$$\chi \in [(K*\phi) \cup \{\psi\}]^{PL} \text{ if and only if } (\psi \rightarrow \chi) \in K*\phi. \quad (13)$$

First we show that

$$\|K*\phi\| \cap \|\psi\| \subseteq \|[(K*\phi) \cup \{\psi\}]\|^{PL}.$$

¹³The domain of the function f is $\mathcal{B}(s_{@}) \times \mathcal{E}$ rather than $\mathcal{B}(s_{@}) \times 2^S$, since the only events that matter are those in \mathcal{E} . However it is possible to extend the domain of f to $\mathcal{B}(s_{@}) \times 2^S$.

Fix an arbitrary $s \in \|K * \phi\| \cap \|\psi\|$; we need to show that $s \in \|[(K * \phi) \cup \{\psi\}]^{PL}\|$, that is, that $\forall \chi \in [(K * \phi) \cup \{\psi\}]^{PL}, \chi \in s$. Since $s \in \|\psi\|$, $\psi \in s$. Fix an arbitrary $\chi \in [(K * \phi) \cup \{\psi\}]^{PL}$; then, by (12), $(\psi \rightarrow \chi) \in K * \phi$; thus, since $s \in \|K * \phi\|$, $(\psi \rightarrow \chi) \in s$. Hence, since both ψ and $\psi \rightarrow \chi$ belong to s and s is deductively closed, $\chi \in s$.

Next we show that

$$\|[(K * \phi) \cup \{\psi\}]^{PL}\| \subseteq \|K * \phi\| \cap \|\psi\|.$$

Let $s \in \|[(K * \phi) \cup \{\psi\}]^{PL}\|$. Then, since $\psi \in [(K * \phi) \cup \{\psi\}]^{PL}$, $\psi \in s$, that is, $s \in \|\psi\|$. It remains to show that $s \in \|K * \phi\|$, that is, that, for every $\chi \in K * \phi$, $\chi \in s$. Fix an arbitrary $\chi \in K * \phi$; then, since, by AGM2, $K * \phi$ is deductively closed, $(\psi \rightarrow \chi) \in K * \phi$. Thus, by (13), $\chi \in [(K * \phi) \cup \{\psi\}]^{PL}$ and thus (since $s \in \|[(K * \phi) \cup \{\psi\}]^{PL}\|$) $\chi \in s$. \square

First we show that the selection function defined in (11) satisfies the properties of Part 4 of Definition 3.1.

4(a) of Definition 3.1 (Success). We need to show that if $s \in \mathcal{B}(s_{@})$ and $\phi \in \Phi_{cn}$ (so that $\|\phi\| \neq \emptyset$) then $\emptyset \neq f(s, \|\phi\|) \subseteq \|\phi\|$. By construction $f(s, \|\phi\|) = \|K * \phi\|$. By AGM5, since ϕ is consistent, $\|K * \phi\| \neq \emptyset$ and, by AGM2, $\phi \in K * \phi$, that is, $\|K * \phi\| \subseteq \|\phi\|$.

4(b) of Definition 3.1 (Weak Centering). Let $s \in \mathcal{B}(s_{@})$ and $\phi \in \Phi_{cn}$. We need to show that if $s \in \|\phi\|$ then $s \in f(s, \|\phi\|)$. Assume that $s \in \|\phi\|$. By construction, $\mathcal{B}(s_{@}) = \|K\|$ thus $s \in \|K\| \cap \|\phi\|$ so that, by (a) of Lemma 3, $s \in \| [K \cup \{\phi\}]^{PL} \|$. By AGM3, $K * \phi \subseteq [K \cup \{\phi\}]^{PL}$ so that $\| [K \cup \{\phi\}]^{PL} \| \subseteq \|K * \phi\|$. Thus $s \in \|K * \phi\|$. By (11), $\|K * \phi\| = f(s, \|\phi\|)$. Hence $s \in f(s, \|\phi\|)$.

4(c) of Definition 3.1 (Doxastic Priority 1). We have to show that if $\mathcal{B}(s_{@}) \cap \|\phi\| \neq \emptyset$ then, for every $s \in \mathcal{B}(s_{@})$, $f(s, \|\phi\|) \subseteq \mathcal{B}(s_{@}) \cap \|\phi\|$. By Part (a) of Lemma 3,

$$\|K\| \cap \|\phi\| = \| [K \cup \{\phi\}]^{PL} \|. \quad (14)$$

Suppose that $\mathcal{B}(s_{@}) \cap \|\phi\| \neq \emptyset$; then, since $\mathcal{B}(s_{@}) = \|K\|$, $\|K\| \cap \|\phi\| \neq \emptyset$ and thus $\neg\phi \notin K$, so that, by AGM1-AGM4 (see Footnote 2), $K * \phi = [K \cup \{\phi\}]^{PL}$. Hence $\|K * \phi\| = \| [K \cup \{\phi\}]^{PL} \|$. It follows from this and (14) that $\mathcal{B}(s_{@}) \cap \|\phi\| = \|K\| \cap \|\phi\| = \|K * \phi\|$. By (11), $\forall s \in \mathcal{B}(s_{@}), f(s, \|\phi\|) = \|K * \phi\|$.

4(d) of Definition 3.1 (Intersection). We have to show that, $\forall s \in \mathcal{B}(s_{@})$ and $\forall \phi, \psi \in \Phi_{cn}$, $f(s, \|\phi\|) \cap \|\psi\| \subseteq f(s, \|\phi\| \cap \|\psi\|)$. First of all, since $\|\phi\| \cap \|\psi\| = \|\phi \wedge \psi\|$, $f(s, \|\phi\| \cap \|\psi\|) = f(s, \|\phi \wedge \psi\|)$. Fix arbitrary $s \in \mathcal{B}(s_{@})$ and $\phi, \psi \in \Phi_{cn}$. By (11),

$f(s, \|\phi\|) = \|K * \phi\|$ and thus

$$f(s, \|\phi\|) \cap \|\psi\| = \|K * \phi\| \cap \|\psi\|. \quad (15)$$

By Part (b) of Lemma 3,

$$\|K * \phi\| \cap \|\psi\| = \|[(K * \phi) \cup \{\psi\}]^{PL}\| \quad (16)$$

By AGM7, $K * (\phi \wedge \psi) \subseteq [(K * \phi) \cup \{\psi\}]^{PL}$ so that

$$\|[(K * \phi) \cup \{\psi\}]^{PL}\| \subseteq \|K * (\phi \wedge \psi)\|. \quad (17)$$

By (11), $\|K * (\phi \wedge \psi)\| = f(s, \|\phi \wedge \psi\|)$. It follows from this and (15), (16) and (17) that $f(s, \|\phi\|) \cap \|\psi\| \subseteq f(s, \|\phi \wedge \psi\|)$.

4(e) of Definition 3.1 (Doxastic Priority 2). For $\phi, \psi \in \Phi_{cn}$, let $B_{\phi\psi} = \{s \in \mathcal{B}(s_{@}) : f(s, \|\phi\|) \cap \|\psi\| \neq \emptyset\}$. We need to show that

(e.1) if $s \in B_{\phi\psi}$ then $f(s, \|\phi \wedge \psi\|) \subseteq f(s, \|\phi\|) \cap \|\psi\|$,

(e.2) if $s \notin B_{\phi\psi}$ then $f(s, \|\phi \wedge \psi\|) \subseteq f(\hat{s}, \|\phi \wedge \psi\|)$ for some $\hat{s} \in B_{\phi\psi}$.

Since, by (11), for all $s \in \mathcal{B}(s_{@})$, $f(s, \|\phi\|) = \|K * \phi\|$ and $f(s, \|\phi \wedge \psi\|) = \|K * (\phi \wedge \psi)\|$ it is sufficient to show that if $\|K * \phi\| \cap \|\psi\| \neq \emptyset$ then $\|K * (\phi \wedge \psi)\| \subseteq \|K * \phi\| \cap \|\psi\|$. Assume that $\|K * \phi\| \cap \|\psi\| \neq \emptyset$. Then $\neg\psi \notin K * \phi$.¹⁴ Thus, by AGM8, $[(K * \phi) \cup \{\psi\}]^{PL} \subseteq K * (\phi \wedge \psi)$ so that

$$\|K * (\phi \wedge \psi)\| \subseteq \|[(K * \phi) \cup \{\psi\}]^{PL}\|. \quad (18)$$

By Part (b) of Lemma 3, $\|[(K * \phi) \cup \{\psi\}]^{PL}\| = \|K * \phi\| \cap \|\psi\|$. It follows from this and (18) that $\|K * (\phi \wedge \psi)\| \subseteq \|K * \phi\| \cap \|\psi\|$.

To complete the proof it remains to show that, $\forall \phi, \psi \in \Phi_0$, $\psi \in K * \phi$ if and only if $\mathcal{B}(s_{@}) \subseteq \|\phi > \psi\|$, that is, $\forall s \in \mathcal{B}(s_{@})$, $f(s, \|\phi\|) \subseteq \|\psi\|$.

Let $\psi \in K * \phi$; then $\psi \in s$ for all $s \in \|K * \phi\|$, that is, $\|K * \phi\| \subseteq \|\psi\|$. By (11), $\forall s \in \mathcal{B}(s_{@})$, $f(s, \|\phi\|) = \|K * \phi\|$. Thus, $\forall s \in \mathcal{B}(s_{@})$, $f(s, \|\phi\|) \subseteq \|\psi\|$, that is, $\mathcal{B}(s_{@}) \subseteq \|\phi > \psi\|$.

Conversely, suppose that $\mathcal{B}(s_{@}) \subseteq \|\phi > \psi\|$, that is, $\forall s \in \mathcal{B}(s_{@})$, $f(s, \|\phi\|) \subseteq \|\psi\|$. By (11), $\forall s \in \mathcal{B}(s_{@})$, $f(s, \|\phi\|) = \|K * \phi\|$. Thus $\|K * \phi\| \subseteq \|\psi\|$ (recall that $\mathcal{B}(s_{@}) \neq \emptyset$), that is, $\forall s \in \|K * \phi\|$, $\psi \in s$. Hence $\psi \in K * \phi$. \square

¹⁴Proof. Suppose that $\neg\psi \in K * \phi$. Fix an arbitrary $s \in \|K * \phi\| \cap \|\psi\|$. Since $s \in \|K * \phi\|$, $\neg\psi \in s$ and since $s \in \|\psi\|$, $\psi \in s$, yielding a contradiction (since s is maximally consistent).

References

- C. Alchourrón, P. Gärdenfors, and D. Makinson. On the logic of theory change: partial meet contraction and revision functions. *The Journal of Symbolic Logic*, 50:510–530, 1985.
- S. J. Barker. Towards a pragmatic theory of ‘if’. *Philosophical Studies*, 79(2): 185–211, 1995.
- D. Barnett. Zif is if. *Mind*, 115(459):519–566, 2006. doi: 10.1093/mind/fzl519.
- F. Berto, I. Canavotto, and A. Giordani. Voluntary imagination: a fine-grained analysis. *The Review of Symbolic Logic*, 15(2):362–387, 2022. doi: 10.1017/S1755020320000039.
- S. Carter. A suppositional theory of conditionals. *Mind*, 130(520):1059–1086, 2021. doi: 10.1093/mind/fzaa071.
- I. Ciardelli. Indicative conditionals and graded information. *Journal of Philosophical Logic*, 49(3):509–549, 2020. doi: 10.1007/s10992-019-09528-6.
- C. Cross and R. Thomason. Conditionals and knowledge-base update. In P. Gärdenfors, editor, *Belief Revision*, volume 29 of *Cambridge Tracts in Theoretical Computer Science*, pages 247–275. Cambridge University Press, 1992.
- W. A. Davis. Indicative and subjunctive conditionals. *Philosophical Review*, 88(4):544–564, 1979. doi: 10.2307/2184844.
- K. DeRose and R. E. Grandy. Conditional assertions and “biscuit” conditionals. *Notus*, 33(3):405–420, 1999. doi: 10.1111/0029-4624.00161.
- D. Edgington. On conditionals. *Mind*, 104(414):235–329, 1995.
- B. Eva, T. Shear, and B. Fitelson. Four approaches to supposition. Technical report, PhilSci archive preprint, 2021.
- E. Fermé and S. O. Hansson. AGM 25 years. *Journal of Philosophical Logic*, 40: 295–331, 2011.
- E. Fermé and S. O. Hansson. *Belief change: introduction and overview*. Springer, 2018.
- P. Gärdenfors. Belief revisions and the Ramsey test for conditionals. *Philosophical Review*, 95(1):81–93, 1986. doi: 10.2307/2185133.
-

- P. Gärdenfors. *Knowledge in flux: modeling the dynamics of epistemic states*. MIT Press, 1988.
- P. Gärdenfors and H. Rott. Belief revision. In D. M. Gabbay, C. J. Hogger, and J. A. Robinson, editors, *Handbook of Logic in Artificial Intelligence and Logic Programming*, volume 4, pages 35–132. Oxford University Press, 1995.
- A. S. Gillies. Epistemic conditionals and conditional epistemics. *Noûs*, 38(4): 585–616, 2004.
- A. Grove. Two modellings for theory change. *Journal of Philosophical Logic*, 17: 157–170, 1988. doi: 10.1007/BF00247909.
- M. Günther and C. Sisti. Ramsey’s conditionals. *Synthese*, 200(2):165, Apr 2022. ISSN 1573-0964. doi: 10.1007/s11229-022-03586-1.
- H. Katsuno and A. O. Mendelzon. Propositional knowledge base revision and minimal change. *Artificial Intelligence*, 52(3):263 – 294, 1991. doi: 10.1016/0004-3702(91)90069-V.
- H. Leitgeb. Belief in conditionals vs. conditional beliefs. *Topoi*, 26:115–132, 2007.
- H. Leitgeb. On the Ramsey test without triviality. *Notre Dame Journal of Formal Logic*, 51(1):21–54, 2010.
- I. Levi. *For the sake of the argument*. Cambridge University Press, 1996.
- D. Lewis. *Counterfactuals*. Harvard University Press, 1973.
- S. Lindström. The Ramsey test and the indexicality of conditionals: A proposed resolution of Gärdenfors’ paradox. In A. Fuhrmann and H. Rott, editors, *Logic, Action and Information. Essays on Logic in Philosophy and Artificial Intelligence*, pages 208–228. de Gruyter, 1996.
- S. Lindström and W. Rabinowicz. Conditionals and the Ramsey test. In D. Dubois and H. Prade, editors, *Belief Change*, pages 147–188. Springer Netherlands, Dordrecht, 1998. doi: 10.1007/978-94-011-5054-5_4.
- S. Linström and W. Rabinowicz. The Ramsey test revisited*. *Theoria*, 58(2-3): 131–182, 1992. doi: 10.1111/j.1755-2567.1992.tb01138.x.
- W. G. Lycan. *Real Conditionals*. Oxford, England: Oxford University Press UK, 2001.
-

- J. L. Mackie. *Truth, Probability and Paradox: Studies in Philosophical Logic*. Clarendon Press, 1972.
- D. Nolan. Defending a possible-worlds account of indicative conditionals. *Philosophical Studies*, 116(3):215–269, 2003. doi: 10.1023/B:PHIL.0000007243.60727.d4.
- A. Özgün and T. Schoonen. The logical development of pretense imagination. *Erkenntnis*, Feb 2022. doi: 10.1007/s10670-021-00476-9.
- P. Peppas. Chapter 8 belief revision. In F. van Harmelen, V. Lifschitz, and B. Porter, editors, *Handbook of Knowledge Representation*, volume 3 of *Foundations of Artificial Intelligence*, pages 317–359. Elsevier, 2008. doi: 10.1016/S1574-6526(07)03008-8.
- F. P. Ramsey. General propositions and causality. In R. B. Braithwaite, editor, *The Foundations of Mathematics and other Logical Essays*, pages 237–257. Humanities Press, 1950.
- H. Rott. Coherence and conservatism in the dynamics of belief. *Erkenntnis*, 50: 387–412, 1999.
- H. Rott. Preservation and postulation: lessons from the new debate on the Ramsey test. *Mind*, 126(502):609–626, 2017. doi: 10.1093/mind/fzw028.
- K. Segerberg. Irrevocable belief revision in Dynamic Doxastic Logic. *Notre Dame Journal of Formal Logic*, 39(3):287–306, 1998.
- R. Stalnaker. A theory of conditionals. In N. Rescher, editor, *Studies in logical theory*, pages 98–112. Blackwell, 1968.
- R. Stalnaker. Indicative conditionals. *Philosophia*, 5(3):269–286, 1975. doi: 10.1007/bf02379021.
- R. Stalnaker. Iterated belief revision. *Erkenntnis*, 128:189–209, 2009.
- B. Weatherston. Indicative and subjunctive conditionals. *Philosophical Quarterly*, 51(203):200–216, 2001. doi: 10.1111/1467-9213.00224.
- J. Zhao, V. Crupi, K. Tentori, B. Fitelson, and D. Osherson. Updating: Learning versus supposing. *Cognition*, 124:373–378, 2012. doi: 10.1016/j.cognition.2012.05.001.
-