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Macroeconomic effects of a 10% cut in statutory marginal income tax rates on ordinary income *

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Abstract

This paper integrates a microsimulation (partial equilibrium) model of tax policy with a dynamic scoring approach to tax policy analysis using a dynamic general equilibrium macroeconomic model. Both approaches have strengths and weaknesses. Our integration of the two models combines the strength of both approaches to give tax revenue estimates based on the rich heterogeneity, realistic demographics, and many tax levers from the microsimulation model as well as dynamic model estimates that account for the effects of tax changes on macroeconomic variables.

*This research benefited from support from the Brigham Young University Macroeconomics and Computational Laboratory and from the Open Source Policy Center at the American Enterprise Institute. All Python code and documentation for the computational model is available at https://github.com/open-source-economics/OG-USA.

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1 Introduction

This note describes the macroeconomics effects of a simple change in tax policy—a 10% reduction in all statutory marginal tax rates on personal income. For the purposes of our simulations, we assume these changes in tax law are permanent and are instituted on January 1, 2015 with no anticipatory effects.

We present a static score of the revenue effects over the budget window. We then report the percentage changes in macroeconomic aggregates and prices in the budget window and in the model's long-run steady state equilibrium.

The rest of the paper is organized as follows. Section 2 provides background on the microsimulation model we use to obtain the static revenue estimate. The DGE model we use to compute the macroeconomic changes is detailed in Section 3. We describe our methodology for integrating the microsimulation and DGE models in Section 4. We present some the results of the policy proposal example in Section 5. Section 6 provides a discussion of the qualitative results, along with the caveats and limitations of the model.

2 Microsimulation Model: Tax-Calculator

The microsimulation model we use is called Tax-Caculator and is developed and maintained by a group of researchers at the Open Source Policy Center (OSPC).[1]. Here we outline the main structure of the Tax-Calculator microsimulation model, but encourage the interested reader to follow the links to the more detailed documentation.

The Tax Calculator uses as its base microdata on tax filers from the tax year 2009 Public Use Files (PUF) produced by the IRS. These data contain detailed records from the tax returns of about 200,000 tax filers who were selected from the population of filers through a stratified random sample of tax returns. These data come from IRS Form 1040 and a set of the associated forms and schedules. The PUF data are

[1]The documentation for using Tax-Calculator is available at http://taxcalc.readthedocs.org/en/latest/index.html A simple web application that provides an easy user interface for Tax-Calculator is available at http://www.ospc.org/taxbrain/. And all the source code is freely available at https://github.com/open-source-economics/Tax-Calculator.

then matched to the Current Population Survey (CPS) to get imputed values for filer demographics such as age, which are not included in the PUF, and to incorporate households from the population of non-filers. The PUF-CPS match includes 219,814 filers.

Since these data are for calendar year 2009, they must be "aged" to be representative of the potential tax paying population in the years of interest (e.g. the current year through the end of the budget window). To do this, macroeconomic forecasts of wages, interest rates, GDP, and other variables are used to grow the 2009 values to be representative of the values one might see in the years within the budget window. In addition to using macroeconomic variables to extrapolate the 2009 variables, a linear programming algorithm is used to re-weight the observations in each year in order to match target levels in the data such as total income and deduction amounts reported in more recent IRS data.

Using these microdata, the Tax Calculator is able to determine total tax liability and marginal tax rates by computing filer's tax reporting that minimize their total tax liability subject to the parameters describing the tax policy they face. Our determination of total tax liability from the microsimulation model includes federal income taxes and payroll taxes but excludes state income taxes and estate taxes.[2] The output of the microsimulation model is forecasts of the total tax liability in each year derived from marginal tax rates, and items from the filers' tax returns for each of the 219,814 filers in the microdata. Population sampling weights are determined through the extrapolation and targeting of the microsimulation model. These weights allow one to calculate population representative results from the model. One can determine changes in tax liability and marginal tax rates by doing the same simulation where the parameters describing the tax policy are updated to reflect the proposed policy rather than the baseline policy. Note that the baseline policy is a current-law baseline.

[2] As the microsimulation model is further developed, we will account for these.

3 Dynamic General Equilibrium Model

The DGE model in this paper is a close variant of the model used in DeBacker et al. (2015).[3] Our DGE model is comprised of heterogeneous individuals, perfectly competitive firms, and a government with a balanced budget requirement. A unit measure of identical firms make a static profit maximization decision in which they rent capital and hire labor to maximize profits given a Cobb-Douglas production function. The government levies taxes on individuals and makes lump sum transfers to individuals according to a balanced budget constraint. The model thus will present a relatively rich set of heterogeneity among households, but less in the production sector and a simple government sector. Further development of the model along these other dimensions is ongoing. But the household sector is the most relevant to how we integrate these two models, even if other elements of the model such as government financing are important determinants of the macroeconomic outcomes.

Individuals are assumed to live for a maximum of $E + S$ periods. We define an age-s individual as being in youth and out of the workforce during ages $1 \leq s \leq E$. We implement this dichotomy of being economically relevant by age in order to more easily match true population dynamics. Individuals enter the workforce at age $E + 1$ and remain in the workforce until they die or until the maximum age $E + S$. Because of mortality risk, individuals can leave both intentional bequests at the end of life ($s = E + S$) as well as accidental bequests if they die before the maximum age of $E + S$.

When individuals are born at age $s = 1$, they are randomly assigned to one of J lifetime income (ability) types. Individuals remain deterministically in their assigned lifetime income group throughout their lives. Related to hourly earnings, this process is calibrated to match the wage distribution by age in the United States, and labor is endogenously supplied by individuals. Our calibration of the hourly earnings process allows for a skewed distribution of earnings that fits U.S. life-cycle hourly earnings data. The economic environment is one of incomplete markets because

[3]This paper can be downloaded at http://mtweb.mtsu.edu/jdebacker/WealthTax.pdf.

the overlapping generations structure prevents households from perfectly smoothing consumption.

3.1 Population dynamics and lifetime earnings profiles

We define $\omega_{s,t}$ as the number of individuals of age s alive at time t. A measure $\omega_{1,t}$ of individuals with heterogeneous working ability is born in each period t and live for up to $E+S$ periods, with $S \geq 4$.[4] Individuals are termed "youth", and do not participate in market activity during ages $1 \leq s \leq E$. The individuals enter the workforce and economy in period $E+1$ and remain in the workforce until they unexpectedly die or live until age $s = E+S$.[5] The population of agents of each age in each period $\omega_{s,t}$ evolves according to the following function,

$$\begin{aligned} \omega_{1,t+1} &= \sum_{s=1}^{E+S} f_s \omega_{s,t} \quad \forall t \\ \omega_{s+1,t+1} &= (1 + i_s - \rho_s)\omega_{s,t} \quad \forall t \quad \text{and} \quad 1 \leq s \leq E+S-1 \end{aligned} \tag{1}$$

where $f_s \geq 0$ is an age-specific fertility rate, i_s is an age-specific net immigration rate, ρ_s is an age specific mortality hazard rate,[6] and $1 + i_s - \rho_s$ is constrained to be nonnegative. The total population in the economy N_t at any period is simply the sum of individuals in the economy, the population growth rate in any period t from the previous period $t-1$ is $g_{n,t}$, $\tilde{N}_t$ is the working age population, and $\tilde{g}_{n,t}$ is the working age population growth rate in any period t from the previous period $t-1$.[7] These parameters are defined as:

$$N_t \equiv \sum_{s=1}^{E+S} \omega_{s,t} \quad \forall t \tag{2}$$

[4]Theoretically, the model works without loss of generality for $S \geq 3$. However, because we are calibrating the ages outside of the economy to be one-fourth of S (e.g., ages 21 to 100 in the economy, and ages 1 to 20 outside of the economy), we need S to be at least 4.

[5]We model the population with individuals age $s \leq E$ outside of the workforce and economy in order most closely match the empirical population dynamics. Appendix A-1 gives more detail on the population process and its calibration.

[6]The parameter ρ_s is the probability that a individual of age s dies before age $s+1$.

[7]Appendix A-1 describes in detail the exogenous population dynamics.

$$g_{n,t+1} \equiv \frac{N_{t+1}}{N_t} - 1 \quad \forall t \tag{3}$$

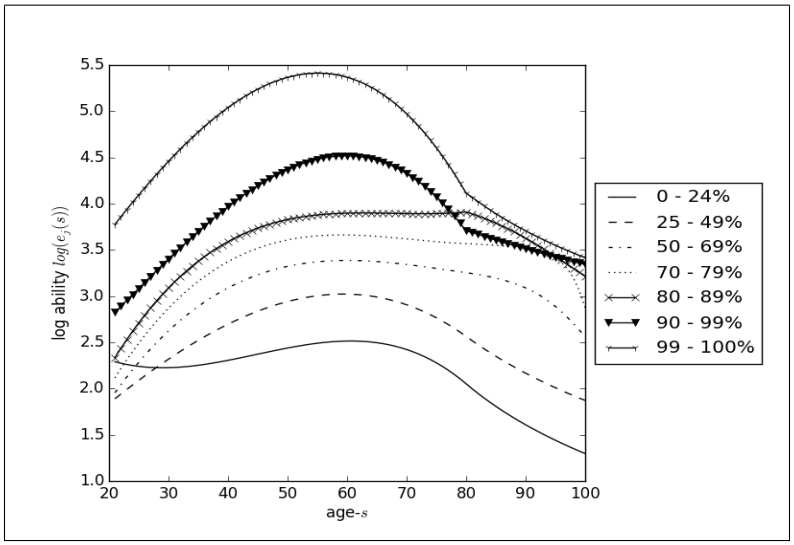
$$\tilde{N}_t \equiv \sum_{s=E+1}^{E+S} \omega_{s,t} \quad \forall t \tag{4}$$

$$\tilde{g}_{n,t+1} \equiv \frac{\tilde{N}_{t+1}}{\tilde{N}_t} - 1 \quad \forall t \tag{5}$$

At birth, a fraction λ_j of the $\omega_{1,t}$ measure of new agents is randomly assigned to each of the J lifetime income groups, indexed by $j = 1, 2, ...J$, such that $\sum_{j=1}^{J} \lambda_j = 1$. Note that lifetime income is endogenous in the model, therefore we define lifetime income groups by a particular path of earnings abilities. For each lifetime income group, the measure $\lambda_j \omega_{s,t}$ of individuals' effective labor units (which we also call ability) evolve deterministically according to $e_{j,s}$. This gives a different life cycle profile of earnings to each lifetime income group. An individual's working ability evolves over its working-age lifetime $E + 1 \leq s \leq E + S$ according to this age-dependent deterministic process. The processes for the evolution of the population weights $\omega_{s,t}$ as well as lifetime earnings are exogenous inputs to the model.

Figure 1 shows the calibrated trajectory of effective labor units (ability) $e_{j,s} \in \mathcal{E} \subset \mathbb{R}_{++}$ by age s for each type j for lifetime income distribution $\{\lambda_j\}_{j=1}^{7} = [0.25, 0.25, 0.20, 0.10, 0.10, 0.09, 0.01]$. We show effective labor units in logarithms because the difference in levels between the top one percent and the rest of the distribution is so large. These exogenous earnings processes are taken from DeBacker et al. (2015). All model individuals have the same time endowment and receive the same wage per effective labor unit, but some are endowed with more effective labor units. We utilize a measure of lifetime income, by using potential lifetime earnings, that allows us to define income groups in a way that accounts for the fact that earnings of individuals observed in the data are endogenous. It is in this way that we are able to calibrate the exogenous lifetime earnings profiles form the model with their data counterparts.

Figure 1: Exogenous life cycle income ability paths $\log(e_{j,s})$ with $S = 80$ and $J = 7$



3.2 Individual problem

Individuals are endowed with a measure of time $\tilde{l}$ in each period t, and they choose how much of that time to allocate between labor $n_{j,s,t}$ and leisure $l_{j,s,t}$ in each period. That is, an individual's labor and leisure choice is constrained by his total time endowment, which constraint is identical across all individuals.

$$n_{j,s,t} + l_{j,s,t} = \tilde{l} \tag{6}$$

At time t, all age-s individuals with ability $e_{j,s}$ know the real wage rate, w_t, and know the one-period real net interest rate, r_t, on bond holdings, $b_{j,s,t}$, that mature at the beginning of period t. They also receive accidental and intentional bequests. They choose how much to consume $c_{j,s,t}$, how much to save for the next period by loaning capital to firms in the form of a one-period bond $b_{j,s+1,t+1}$, and how much to work

$n_{j,s,t}$ in order to maximize expected lifetime utility of the following form,

$$
\begin{aligned}
U_{j,s,t} &= \sum_{u=0}^{E+S-s} \beta^u \left[\prod_{v=s}^{s+u-1} (1-\rho_v)\right] u\left(c_{j,s+u,t+u}, n_{j,s+u,t+u}, b_{j,s+u+1,t+u+1}\right) \\
\text{and} \quad u\left(c_{j,s,t}, n_{j,s,t}, b_{j,s+1,t+1}\right) &= \frac{\left(c_{j,s,t}\right)^{1-\sigma} - 1}{1-\sigma} \ldots \\
&\quad + e^{g_y t(1-\sigma)} \chi_s^n \left(b\left[1 - \left(\frac{n_{j,s,t}}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1}{\upsilon}} + k\right) + \rho_s \chi_j^b \frac{\left(b_{j,s+1,t+1}\right)^{1-\sigma} - 1}{1-\sigma} \\
&\qquad\qquad \forall j, t \quad \text{and } E+1 \leq s \leq E+S
\end{aligned}
\tag{7}
$$

where $\sigma \geq 1$ is the coefficient of relative risk aversion on consumption and on intended (precautionary) bequests, $\beta \in (0,1)$ is the agent's discount factor, and the product term in brackets depreciates the individual's discount factor by the cumulative mortality rate. The disutility of labor term in the period utility function looks nonstandard, but is simply the upper right quadrant of an ellipse that closely approximates the standard CRRA utility of leisure functional form.[8] The term χ_s^n is a constant term that varies by age s influencing the disutility of labor relative to the other arguments in the period utility function,[9] and g_y is a constant growth rate of labor augmenting technological progress, which we explain in Section 3.3.[10]

The last term in (7) incorporates a warm-glow bequest motive in which individuals value having savings to bequeath to the next generation in the chance they die before the next period. Including this term is essential to generating the positive wealth levels across the life cycle and across abilities that exist in the data. In addition, the term χ_j^b is a constant term that varies by lifetime income group j influencing the

[8] Appendix A-2 describes how the elliptical function closely matches the more standard utility of leisure of the form $\frac{(\tilde{l}-n_{j,s,t})^{1+\theta}}{1+\theta}$. This elliptical utility function forces an interior solution that automatically respects both the upper and lower bound of labor supply, which greatly simplifies the computation of equilibrium. In addition, the elliptical disutility of labor has a Frisch elasticity that asymptotes to a constant rather than increasing to infinity as it does in the CRRA case. For a more in-depth discussion see Evans and Phillips (2015)

[9] DeBacker et al. (2015) calibrate χ_s^n and χ_j^b to match average labor hours by age and some moments of the distribution of wealth.

[10] The term with the growth rate $e^{g_y t(1-\sigma)}$ must be included in the period utility function because consumption and bequests will be growing at rate g_y and this term stationarizes the individual Euler equation by making the marginal disutility of labor grow at the same rate as the marginal benefits of consumption and bequests. This is the same balanced growth technique as that used in Mertens and Ravn (2011).

marginal utility of bequests, $b_{j,s+1,t+1}$ relative to the other arguments in the period utility function. Allowing the χ^b_j scale parameter on the warm glow bequest motive vary by lifetime income group is critical for matching the distribution of wealth. As was mentioned in Section 3.1, individuals in the model have no income uncertainty because each lifetime earnings path $e_{j,s}$ deterministic, model agents thus hold no precautionary savings. Calibrating the χ^b_j for each income group j captures in a reduced form way some of the characteristics that individual income risk provides.

The parameter $\sigma \geq 1$ is the coefficient of relative risk aversion on bequests, and the mortality rate ρ_s appropriately discounts the value of this term.[11] Note that, because of this bequest motive, individuals in the last period of their lives ($s = S$) will die with positive savings $b > 0$. Also note that the CRRA utility of bequests term prohibits negative wealth holdings in the model, but is not a strong restriction since none of the wealth data for the lifetime income group j and age s cohorts is negative except for the lowest quartile.

The per-period budget constraints for each agent normalized by the price of consumption are the following,

$$c_{j,s,t} + b_{j,s+1,t+1} \leq (1 + r_t)\, b_{j,s,t} + w_t e_{j,s} n_{j,s,t} + \frac{BQ_{j,t}}{\lambda_j \tilde{N}_t} - T_{s,t} \qquad (8)$$
$$\text{where} \quad b_{j,E+1,t} = 0 \quad \text{for} \quad E+1 \leq s \leq E+S \quad \forall j, t$$

where $\tilde{N}_t$ is the total working age population at time t defined in (4) and $\lambda_j \tilde{N}_t$ is the number of the total working individuals of type j in period t. Note that the price of consumption is normalized to one, so w_t is the real wage and r_t is the net real interest rate. The term $BQ_{j,t}$ represents total bequests from individuals in income group j who died at the end of period $t-1$. $T_{s,t}$ is a function representing net taxes paid, which we specify more fully below in equation (10).

Implicit in the period budget constraint (8) is a strong assumption about the distribution of bequests. We assume that bequests are distributed evenly across all

[11] It is necessary for the coefficient of relative risk aversion σ to be the same on both the utility of consumption and the utility of bequests. If not, the resulting Euler equations are not stationarizable.

ages to those in the same lifetime income group. It is difficult to precisely calibrate the distribution of bequests from the data, both across income types j and across ages s. However, the assumptions about the bequest motive as well as how bequests are distributed are clearly important modeling decisions. Our current specification of bequests is the most persistent, which should make wealth inequality more persistent relative to other bequest specifications.[12] A large number of papers study the effects of different bequest motives and specifications on the distribution of wealth, though there is no consensus regarding the true bequest transmission process.[13]

Because the form of the period utility function in (7) ensures that $b_{j,s,t} > 0$ for all j, s, and t, total bequests will always be positive $BQ_{j,t} > 0$ for all j and t.

$$BQ_{j,t+1} = (1 + r_{t+1})\lambda_j \left(\sum_{s=E+1}^{E+S} \rho_s \omega_{s,t} b_{j,s+1,t+1} \right) \quad \forall j, t \tag{9}$$

In addition to each the budget constraint in each period, the utility function (7) imposes nonnegative consumption through infinite marginal utility, and the elliptical utility of leisure ensures individual labor and leisure must be strictly nonnegative $n_{j,s,t}, l_{j,s,t} > 0$. Because individual savings or wealth is always strictly positive, the aggregate capital stock is always positive.[14] An interior solution to the individual's problem (7) is assured.

In reality, each household is subject to many different taxes, all of which cannot be modeled in a DGE framework. It is the net tax liability function $T_{s,t}$ that we estimate from the microsimulation model output. This output includes information on all taxes paid in the American tax code. We also assume that every individual also receives an equal lump sum transfer T_t^H which is generated from a balanced budget constraint on the government. We represent the net tax liability function as

[12] Another allocation rule at the opposite extreme would be to equally divide all bequests among all surviving individuals. An intermediate rule would be some kind of distribution of bequests with most going to ones own type and a declining proportion going to the other types.

[13] See De Nardi and Yang (2014), De Nardi (2004), Nishiyama (2002), Laitner (2001), Gokhale et al. (2000), Gale and Scholz (1994), Hurd (1989), Venti and Wise (1988), Kotlikoff and Summers (1981), and Wolff (2015).

[14] An alternative would be to allow for individual borrowing as long as an aggregate capital constraint $K_t > 0$ for all t is satisfied.

an average effective tax rate times total income.

$$
\begin{aligned}
T_{s,t}(x,y) &= \tau_{s,t}(x,y)\big(x+y\big) - T_t^H \\
\text{where} \quad x &\equiv \frac{w_t e_{j,s} n_{j,s,t}}{e^{g_y t}} \quad \text{and} \quad y \equiv \frac{r_t b_{j,s,t}}{e^{g_y t}}
\end{aligned}
\tag{10}
$$

Note that the both the total tax liability function $T_{s,t}(x,y)$ and the average effective tax rate functions $\tau_{s,t}(x,y)$ are functions of stationarized labor income x and capital income y, separately. We detail the estimation of the tax functions $\tau_{s,t}(x,y)$ and $T_{s,t}(x,y)$ in Section 4.

The solution to the lifetime maximization problem (7) of individual with ability j subject to the per-period budget constraint (8) and the specification of taxes in (10) is a system of $2S$ Euler equations. The S static first order conditions for labor supply $n_{j,s,t}$ are the following,

$$
\begin{aligned}
(c_{j,s,t})^{-\sigma}\left(w_t e_{j,s} - \frac{\partial T_{s,t}}{\partial n_{j,s,t}}\right) &= e^{g_y t(1-\sigma)}\chi_s^n\left(\frac{b}{\tilde{l}}\right)\left(\frac{n_{j,s,t}}{\tilde{l}}\right)^{\upsilon-1}\left[1-\left(\frac{n_{j,s,t}}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1-\upsilon}{\upsilon}} \\
&\qquad\qquad \forall j,t, \quad \text{and} \quad E+1 \leq s \leq E+S \\
\text{where} \quad c_{j,s,t} &= (1+r_t)\,b_{j,s,t} + w_t e_{j,s} n_{j,s,t} + \frac{BQ_{j,t}}{\lambda_j \tilde{N}_t} - b_{j,s+1,t+1} - T_{s,t} \\
\text{and} \quad b_{j,E+1,t} &= 0 \quad \forall j,t
\end{aligned}
\tag{11}
$$

where the marginal tax rate with respect to labor supply $\frac{\partial T_{s,t}}{\partial n_{j,s,t}}$ is described in equation 35.[15]

An individual also has $S-1$ dynamic Euler equations that govern his saving decisions, $b_{j,s+1,t+1}$, with the included precautionary bequest saving in case of unexpected

[15] We also have to use a parameter H that multiplies the model labor income and the model capital income in the tax function in order to match their levels to the corresponding average levels in the microsimulation model data.

death. These are given by:

$$(c_{j,s,t})^{-\sigma} = \rho_s \chi^b_j \big(b_{j,s+1,t+1}\big)^{-\sigma} + \beta(1-\rho_s)(c_{j,s+1,t+1})^{-\sigma}\left[(1+r_{t+1}) - \frac{\partial T_{s+1,t+1}}{\partial b_{j,s+1,t+1}}\right]$$
$$\forall j,t, \quad \text{and} \quad E+1 \leq s \leq E+S-1 \tag{12}$$

where the marginal tax rate with respect to savings $\frac{\partial T_{s,t}}{\partial b_{j,s,t}}$ is described in equation 37. Lastly, Each individual also has one static first order condition for the last period of life $s = E+S$, which governs how much to bequeath to the following generation given that the individual will die with certainty. This condition is:

$$(c_{j,E+S,t})^{-\sigma} = \chi^b_j (b_{j,E+S+1,t+1})^{-\sigma} \quad \forall j,t \tag{13}$$

Define $\hat{\mathbf{\Gamma}}_t$ as the distribution of stationary individual savings across individuals at time t, including the intentional bequests of the oldest cohort.

$$\hat{\mathbf{\Gamma}}_t \equiv \left\{\{\hat{b}_{j,s,t}\}_{j=1}^{J}\right\}_{s=E+2}^{E+S+1} \quad \forall t \tag{14}$$

As will be shown in Section 3.5, the state in every period t for the entire equilibrium system described in the stationary, non-steady-state equilibrium characterized in Definition 2 is the stationary distribution of individual savings $\hat{\mathbf{\Gamma}}_t$ from (14). Because individuals must forecast wages, interest rates, and aggregate bequests received in every period in order to solve their optimal decisions and because each of those future variables depends on the entire distribution of savings in the future, we must assume some individual beliefs about how the entire distribution will evolve over time. Let general beliefs about the future distribution of capital in period $t+u$ be characterized by the operator $\Omega(\cdot)$ such that:

$$\hat{\mathbf{\Gamma}}^{\boldsymbol{e}}_{\boldsymbol{t+u}} = \Omega^u\left(\hat{\mathbf{\Gamma}}_{\boldsymbol{t}}\right) \quad \forall t, \quad u \geq 1 \tag{15}$$

where the e superscript signifies that $\hat{\Gamma}^e_{t+u}$ is the expected distribution of wealth at

time $t+u$ based on general beliefs $\Omega(\cdot)$ that are not constrained to be correct.[16]

3.3 Firm problem

A unit measure of identical, perfectly competitive firms exist in the economy. The representative firm is characterized by the following Cobb-Douglas production technology,

$$Y_t = ZK_t^{\alpha}\left(e^{g_y t}L_t\right)^{1-\alpha} \quad \forall t \tag{16}$$

where Z is the measure of total factor productivity, $\alpha \in (0,1)$ is the capital share of income, g_y is the constant growth rate of labor augmenting technological change, and L_t is aggregate labor measured in efficiency units. The firm uses this technology to produce a homogeneous output which is consumed by individuals and used in firm investment. The interest rate r_t paid to the owners of capital is the real interest rate net of depreciation. The real wage is w_t. The real profit function of the firm is the following.

$$\text{Real Profits} = ZK_t^{\alpha}\left(e^{g_y t}L_t\right)^{1-\alpha} - (r_t+\delta)K_t - w_t L_t \tag{17}$$

As in the individual budget constraint (8), note that the price output has been normalized to one.

Profit maximization results in the real wage, w_t, and the real rental rate of capital r_t being determined by the marginal products of labor and capital, respectively:

$$w_t = (1-\alpha)\frac{Y_t}{L_t} \quad \forall t \tag{18}$$

$$r_t = \alpha\frac{Y_t}{K_t} - \delta \quad \forall t \tag{19}$$

3.4 Government fiscal policy

The government is represented by a balanced budget constraint. The government collects taxes $\tau_{s,t}(x,y)(x+y)$ from all individuals and divides total revenues equally

[16] In Section 3.5 we will assume that beliefs are correct (rational expectations) for the stationary non-steady-state equilibrium in Definition 2.

among individuals in the economy to determine the lump-sum transfer.

$$T_t^H = \frac{1}{\tilde{N}_t} \sum_s \sum_j \omega_{s,t} \lambda_j \tau_{s,t}(w_t e_{j,s} n_{j,s,t}, r_t b_{j,s,t}) \big(w_t e_{j,s} n_{j,s,t} + r_t b_{j,s,t}\big) \tag{20}$$

Lump sum transfers have an impact on the distribution of income and wealth. However, if one constrains policy experiments to have the same steady-state revenue impact, the changes in inequality in economic outcomes due to changes in government transfers is equivalent in each policy experiment in the steady-state.

3.5 Market clearing and stationary equilibrium

Labor market clearing requires that aggregate labor demand L_t measured in efficiency units equal the sum of individual efficiency labor supplied $e_{j,s} n_{j,s,t}$. Capital market clearing requires that aggregate capital demand K_t equal the sum of capital investment by individuals $b_{j,s,t}$. Aggregate consumption C_t is defined as the sum of all individual consumptions, and aggregate investment is defined by the resource constraint $Y_t = C_t + I_t$ as shown in (23). That is, the following conditions must hold:

$$L_t = \sum_{s=E+1}^{E+S} \sum_{j=1}^{J} \omega_{s,t} \lambda_j e_{j,s} n_{j,s,t} \quad \forall t \tag{21}$$

$$K_t = \sum_{s=E+2}^{E+S+1} \sum_{j=1}^{J} \omega_{s-1,t-1} \lambda_j b_{j,s,t} \quad \forall t \tag{22}$$

$$\begin{aligned} Y_t &= C_t + K_{t+1} - (1-\delta) K_t \quad \forall t \\ \text{where} \quad C_t &\equiv \sum_{s=E+1}^{E+S} \sum_{j=1}^{J} \omega_{s,t} \lambda_j c_{j,s,t} \end{aligned} \tag{23}$$

The usual definition of equilibrium would be allocations and prices such that individuals optimize (11), (12), and (13), firms optimize (18) and (19), and markets clear (21) and (22). However, the variables in the equations characterizing the equilibrium are potentially non-stationary due to the growth rate in the total population $g_{n,t}$ each period coming from the cohort growth rates in (1) and from the deterministic growth

Table 1: Stationary variable definitions

Sources of growth			Not
$e^{g_y t}$	$\tilde{N}_t$	$e^{g_y t}\tilde{N}_t$	growing[a]
$\hat{c}_{j,s,t} \equiv \frac{c_{j,s,t}}{e^{g_y t}}$	$\hat{\omega}_{s,t} \equiv \frac{\omega_{s,t}}{\tilde{N}_t}$	$\hat{Y}_t \equiv \frac{Y_t}{e^{g_y t}\tilde{N}_t}$	$n_{j,s,t}$
$\hat{b}_{j,s,t} \equiv \frac{b_{j,s,t}}{e^{g_y t}}$	$\hat{L}_t \equiv \frac{L_t}{\tilde{N}_t}$	$\hat{K}_t \equiv \frac{K_t}{e^{g_y t}\tilde{N}_t}$	r_t
$\hat{w}_t \equiv \frac{w_t}{e^{g_y t}}$		$\hat{BQ}_{j,t} \equiv \frac{BQ_{j,t}}{e^{g_y t}\tilde{N}_t}$	
$\hat{y}_{j,s,t} \equiv \frac{y_{j,s,t}}{e^{g_y t}}$			
$\hat{T}_{s,t} \equiv \frac{T_{j,s,t}}{e^{g_y t}}$			

[a] The interest rate r_t in (19) is already stationary because Y_t and K_t grow at the same rate. Individual labor supply $n_{j,s,t}$ is stationary.

rate of labor augmenting technological change g_y in (16).

Table 1 characterizes the stationary versions of the variables of the model in terms of the variables that grow because of labor augmenting technological change, population growth, both, or none. With the definitions in Table 1, it can be shown that the equations characterizing the equilibrium can be written in stationary form in the following way. The static and intertemporal first-order conditions from the individual's optimization problem corresponding to (11), (12), and (13) are the following:

$$(\hat{c}_{j,s,t})^{-\sigma}\left(\hat{w}_t e_{j,s} - \frac{\partial \hat{T}_{s,t}}{\partial n_{j,s,t}}\right) = \chi^n_s\left(\frac{b}{\tilde{l}}\right)\left(\frac{n_{j,s,t}}{\tilde{l}}\right)^{\upsilon-1}\left[1-\left(\frac{n_{j,s,t}}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1-\upsilon}{\upsilon}}$$
$$\forall j,t, \quad \text{and} \quad E+1 \leq s \leq E+S \tag{24}$$
$$\text{where} \quad \hat{c}_{j,s,t} = (1+r_t)\,\hat{b}_{j,s,t} + \hat{w}_t e_{j,s} n_{j,s,t} + \frac{\hat{BQ}_{j,t}}{\lambda_j} - e^{g_y}\hat{b}_{j,s+1,t+1} - \hat{T}_{s,t}$$
$$\text{and} \quad \hat{b}_{j,E+1,t} = 0 \quad \forall j,t$$

$$(\hat{c}_{j,s,t})^{-\sigma} = \ldots$$
$$e^{-g_y\sigma}\left(\rho_s\chi^b_j\left(\hat{b}_{j,s+1,t+1}\right)^{-\sigma} + \beta(1-\rho_s)(\hat{c}_{j,s+1,t+1})^{-\sigma}\left[(1+r_{t+1}) - \frac{\partial \hat{T}_{s+1,t+1}}{\partial \hat{b}_{j,s+1,t+1}}\right]\right) \tag{25}$$
$$\forall j,t, \quad \text{and} \quad E+1 \leq s \leq E+S-1$$

$$(\hat{c}_{j,E+S,t})^{-\sigma} = \chi_j^b e^{-g_y \sigma} (\hat{b}_{j,E+S+1,t+1})^{-\sigma} \quad \forall j, t \tag{26}$$

The stationary firm first order conditions for optimal labor and capital demand corresponding to (18) and (19) are the following.

$$\hat{w}_t = (1-\alpha)\frac{\hat{Y}_t}{\hat{L}_t} \quad \forall t \tag{27}$$

$$r_t = \alpha \frac{\hat{Y}_t}{\hat{K}_t} - \delta = \alpha \frac{Y_t}{K_t} - \delta \quad \forall t \tag{19}$$

And the two stationary market clearing conditions corresponding to (21) and (22)—with the goods market clearing by Walras' Law—are the following.

$$\hat{L}_t = \sum_{s=E+1}^{E+S} \sum_{j=1}^{J} \hat{\omega}_{s,t} \lambda_j e_{j,s} n_{j,s,t} \quad \forall t \tag{28}$$

$$\hat{K}_t = \frac{1}{1+\tilde{g}_{n,t}} \left(\sum_{s=E+2}^{E+S+1} \sum_{j=1}^{J} \hat{\omega}_{s-1,t-1} \lambda_j \hat{b}_{j,s,t} \right) \quad \forall t \tag{29}$$

where $\tilde{g}_{n,t}$ is the growth rate in the working age population between periods $t-1$ and t described in (5). It is also important to note the stationary version of the characterization of total bequests $BQ_{j,t+1}$ from (9) and for the government budget constraint in (20).

$$\hat{BQ}_{j,t+1} = \frac{(1+r_{t+1})\lambda_j}{1+\tilde{g}_{n,t}} \left(\sum_{s=E+1}^{E+S} \rho_s \hat{\omega}_{s,t} \hat{b}_{j,s+1,t+1} \right) \quad \forall j, t \tag{30}$$

$$\hat{T}_t^H = \sum_s \sum_j \hat{\omega}_{s,t} \lambda_j \hat{T}_{s,t} \tag{31}$$

We can now define the stationary steady-state equilibrium for this economy in the following way.

Definition 1 (**Stationary steady-state equilibrium**). A non-autarkic stationary steady-state equilibrium in the overlapping generations model with S-period lived agents and heterogeneous ability $e_{j,s}$ is defined as constant allocations $n_{j,s,t} = \bar{n}_{j,s}$

and $\hat{b}_{j,s+1,t+1} = \bar{b}_{j,s+1}$ and constant prices $\hat{w}_t = \bar{w}$ and $r_t = \bar{r}$ for all j, s, and t such that the following conditions hold:

i. individuals optimize according to (24), (25), and (26),

ii. Firms optimize according to (27) and (19),

iii. Markets clear according to (28) and (29), and

iv. The population has reached its stationary steady state distribution $\bar{\omega}_s$ for all ages s, characterized in Appendix A-1.

The steady-state equilibrium is characterized by the system of $2JS$ equations and $2JS$ unknowns $\bar{n}_{j,s}$ and $\bar{b}_{j,s+1}$. Appendix A-3 details how to solve for the steady-state equilibrium.

The non-steady state equilibrium is characterized by $2JST$ equations and $2JST$ unknowns, where T is the number of periods along the transition path from the current state to the steady state. The definition of the stationary non-steady-state equilibrium is similar to Definition 1, with the stationary steady-state equilibrium definition being a special case of the stationary non-steady-state equilibrium.

Definition 2 (**Stationary non-steady-state equilibrium**)**.** A non-autarkic stationary non-steady-state equilibrium in the overlapping generations model with S-period lived agents and heterogeneous ability $e_{j,s}$ is defined as allocations $n_{j,s,t}$ and $\hat{b}_{j,s+1,t+1}$ and prices $\hat{w}_t$ and r_t for all j, s, and t such that the following conditions hold:

i. individuals have symmetric beliefs $\Omega(\cdot)$ about the evolution of the distribution of savings, and those beliefs about the future distribution of savings equal the realized outcome (rational expectations),

$$\hat{\mathbf{\Gamma}}_{t+u} = \hat{\mathbf{\Gamma}}^{e}_{t+u} = \Omega^{u}\left(\hat{\mathbf{\Gamma}}_{t}\right) \quad \forall t, \quad u \geq 1$$

ii. individuals optimize according to (24), (25), and (26)

iii. Firms optimize according to (27) and (19), and

iv. Markets clear according to (28) and (29).

We describe the methodology to compute the solution to the non-steady-state equilibrium to Appendix A-4. We will use the time path solution to find effects of tax policies on macroeconomic variables over the budget window.

3.6 Calibration

Table 2 shows the calibrated values for the exogenous variables and parameters taken from DeBacker et al. (2015).

Note that the scale parameter χ_s^n takes on 80 values (one for each model age) that increase with age, representing an increasing disutility of labor that is not modeled anywhere else in the utility function. An hour of labor for an older person becomes more costly due to biological reasons related to aging. Such a parametrization helps to fit fact that hours worked decline much more sharply later in life than do hourly earnings.

Heterogeneity in the scale parameter multiplying useful in having the model generate a distribution of wealth similar to that observed in the data. Note that without such heterogeneity in this parameter, individuals at the high end of the earnings distribution in our model would not save as much as their real world counterparts given the deterministic earnings process in our model. They have no precautionary savings motive, only the warm-glow bequest motive for savings. One can view the assumption of heterogeneous utility weights as not just variation in preference across households, but also as reflecting differences in family size, expectations of income growth, or other variations that are not explicitly modeled here. We thus allow $\{\chi_j^b\}_{j=1}^7$ to take on seven values, one for each lifetime income group.

Table 2: List of exogenous variables and baseline calibration values

Symbol	Description	Value
$\hat{\mathbf{\Gamma}}_1$	Initial distribution of savings	$\bar{\mathbf{\Gamma}}$
N_0	Initial population	1
$\{\omega_{s,0}\}_{s=1}^{S}$	Initial population by age	(see App. A-1)
$\{f_s\}_{s=1}^{S}$	Fertility rates by age	(see App. A-1)
$\{i_s\}_{s=1}^{S}$	Immigration rates by age	(see App. A-1)
$\{\rho_s\}_{s=1}^{S}$	Mortality rates by age	(see App. A-1)
$\{e_{j,s}\}_{j,s=1}^{J,S}$	Deterministic ability process	(see DeBacker et al., 2015)
$\{\lambda_j\}_{j=1}^{J}$	Lifetime income group percentages	$[0.25, 0.25, 0.20, 0.10, 0.10, 0.09, 0.01]$
J	Number of lifetime income groups	7
S	Maximum periods in economically active individual life	80
E	Number of periods of youth economically outside the model	round $\left(\frac{S}{4}\right)$
R	Retirement age (period)	round $\left(\frac{9}{16}S\right)$
$\tilde{l}$	Maximum hours of labor supply	1
β	Discount factor	$(0.96)^{\frac{80}{S}}$
σ	Coefficient of constant relative risk aversion	3
b	Scale parameter in utility of leisure	0.6701
υ	Shape parameter in utility of leisure	1.3499
k	constant parameter in utility of leisure	-0.6548
χ_s^n	Disutility of labor level parameters	[1.53, 43.22]
χ_j^b	Utility of bequests level parameters	$[9.264 \times 10^{-5}, 118, 648.915]$
Z	Level parameter in production function	1
α	Capital share of income	0.35
δ	Capital depreciation rate	$1 - (1 - 0.05)^{\frac{80}{S}}$
g_y	Growth rate of labor augmenting technological progress	$(1 + 0.03)^{\frac{80}{S}} - 1$
T	Number of periods to steady state	160
ν	Dampening parameter for TPI	0.2

4 Integration of Microsimulation Model with DGE Model

We interface between the microsimulation and DGE model through the tax function $T_{s,t}$ in the household budget constraint (8). In particular, the tax calculator will produce microdata that can be used to calibrate these functions in a way that is consistent with the tax law parameters entered into the microsimulation model. Note that our estimated net tax liability function will be an approximation of the true underlying tax structure, incentives, and behavior, but our estimated parameters will fit smooth functions to the microsimulation model data that can be entered into the DGE model.

We begin our discussion of the integration by describing the functional form we use approximate the effective tax rate function. We then explain how these functional forms are estimated using data generated by the microsimulation model and provide some measures of the fit of the functional form under the baseline tax policies. Finally, we discuss how the tax parameters affecting payroll taxes are linked between the microsimulation and DGE models.

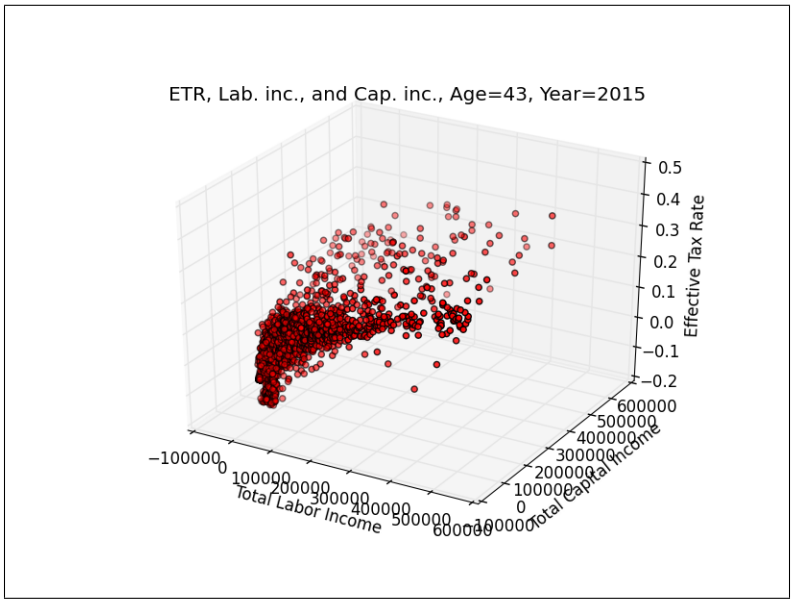
4.1 Modeling microsimulation tax structure

We model the average effective tax rate as a ratio of polynomials in labor and capital income. Important properties of this functional form are that it allows for negative average and marginal tax rates and that it produces marginal tax rate functions that are function of both labor and capital income. While we could have fit functions of total tax liability or marginal tax rates, fitting the average effective tax rate function has the benefit that the average effective tax rate is bounded above by one when the income concept used is total economic income.

Let x be total labor income $x \equiv wen$, and let y be total capital income $y \equiv rb$. Also, let the labor income share of total income be $\phi \equiv \frac{x}{x+y} = \frac{wen}{rb+wen}$. We then write our average effective tax rate functions as follows:

$$
\begin{aligned}
\tau(x,y) =& \Phi\Lambda + K \\
& \text{where} \quad \Phi \equiv \Big[\phi(max_x - min_x) + (1-\phi)(max_y - min_y)\Big] \\
& \text{and} \quad \Lambda \equiv \left(\frac{Ax^2 + By^2 + Cxy + Dx + Ey}{Ax^2 + By^2 + Cxy + Dx + Ey + F}\right) \\
& \text{and} \quad K \equiv \phi min_x + (1-\phi) min_y \\
& \text{where} \quad A, B, C, D, E, F, max_x, max_y > 0 \\
& \text{and} \quad max_x > min_x \quad \text{and} \quad max_y > min_y
\end{aligned}
\tag{32}
$$

Figure 2: Scatter plot of effective tax rates, labor income, and capital income from microsimulation model: $t = 2015$ and $s = 43$



This functional form with the ratio of polynomials $\Lambda = \frac{z}{z+k}$ is a nice structure because it is bounded between 0 and 1 for positive polynomial coefficients and positive labor and capital income. This class of functions also seems to robustly fit the shape of the tax data. Figure 2 shows a 3 dimensional scatterplot of the effective tax rate data from the microsimulation model as a function of total labor income x and total capital income y with both incomes truncated at \$500,000 in order to better see

the data. These data clearly have the negative exponential shape delivered by the functional form Λ.

The parameter max_x is the maximum average tax rate and min_x is the minimum average tax rate for all labor income x when capital income is zero $y = 0$. Conversely, the parameter max_y is the maximum average tax rate and min_y is the minimum average tax rate for all capital income y when labor income is zero $x = 0$. The parameters in the Φ and K term allow for some simple slope changes from labor income to capital income to better fit the data. These parameters are summarized in Table 3.

Table 3: Description of tax function $\tau(x, y)$ parameters

Symbol	Description
A	Coefficient on squared $x$term in $\tau(\cdot)$
B	Coefficient on squared y term in $\tau(\cdot)$
C	Coefficient on interaction term in $\tau(\cdot)$
D	Coefficient on linear x term in $\tau(\cdot)$
E	Coefficient on linear y term in $\tau(\cdot)$
F	Constant term in $\tau(\cdot)$
max_x	Maximum AETR on labor income x given $y = 0$
min_x	Minimum AETR on labor income x given $y = 0$
max_y	Maximum AETR on capital income y given $x = 0$
max_y	Minimum AETR on capital income y given $x = 0$

The total tax liability function is simply the average tax function times total income $\tau(x, y)(x + y)$.

$$T(x, y) \equiv \tau(x, y)\big(x + y\big) = \big(\Phi\Lambda + K\big)\big(x + y\big) \tag{33}$$

For the marginal tax rates in the next section, it will be helpful to rearrange terms in (33) in the following way, which cancels out the total income term $(x + y)$ with the

total income term in the denominator of ϕ, which appears in both Φ and K.

$$\begin{aligned} T(x,y) &= (\Phi\Lambda + K)(x+y) \\ &= \big[x(max_x - min_x) + y(max_y - min_y)\big]\Lambda + x\,min_x + y\,min_y \end{aligned} \tag{34}$$

Using the expression in (34), we can solve for the marginal tax rates with respect to both total labor income x and total capital income y and show that they are both strictly positive.

$$\begin{aligned} MTR_x &\equiv \frac{\partial T(x,y)}{\partial x} \\ &= (max_x - min_x)\Lambda + \big[x(max_x - min_x) + y(max_y - min_y)\big]\frac{\partial \Lambda}{\partial x} + min_x \\ \text{where} \quad \frac{\partial \Lambda}{\partial x} &= \frac{(2Ax + Cy + D)F}{(Ax^2 + By^2 + Cxy + Dx + Ey + F)^2} > 0 \end{aligned} \tag{35}$$

The marginal tax rate with respect to labor income x is not everywhere positive. In particular, $MTR_x > 0$ only when the following condition holds.

$$MTR_x > 0 \Leftrightarrow \Lambda\, max_x + (1-\Lambda) min_x > \big[x(max_x - min_x) + y(max_y - min_y)\big]\frac{\partial \Lambda}{\partial x} \tag{36}$$

This expression holds true more often when Λ is close to 1, which is when x and y are large. Marginal tax rates are more likely to be negative when minimum tax rates are negative and when x and y are small (Λ close to 0). Intuitively, if the minimum tax rate on labor income is negative and an individual shifts more income to labor income x ($\phi \uparrow$), this could reduce their tax liability.

The marginal tax rate with respect to capital income y is the following.

$$\begin{aligned} MTR_y &\equiv \frac{\partial T(x,y)}{\partial y} \\ &= (max_y - min_y)\Lambda + \big[x(max_x - min_x) + y(max_y - min_y)\big]\frac{\partial \Lambda}{\partial y} + min_y \\ \text{where} \quad \frac{\partial \Lambda}{\partial y} &= \frac{(2By + Cx + E)F}{(Ax^2 + By^2 + Cxy + Dx + Ey + F)^2} > 0 \end{aligned} \tag{37}$$

The marginal tax rate with respect to capital income y is not everywhere positive. In particular, $MTR_x > 0$ only when the following condition holds.

$$MTR_y > 0 \Leftrightarrow \Lambda\, max_y + (1-\Lambda) min_y > \left[x(max_x - min_x) + y(max_y - min_y)\right] \frac{\partial \Lambda}{\partial y} \quad (38)$$

The same intuition as described regarding MTR_x holds true here.

The tax functions above are specific to a given age and tax year. Thus, average effective tax rates in the DGE model will vary across the age of the model household, the model year, and amounts of capital and labor income. In this way, the tax function will incorporate some of the variation in statutory rates, as well as heterogeneity in income items, deductions items, credits, filing unit structure, and so forth into the DGE model that cannot account for this degree of heterogeneity. The effect of such heterogeneity on tax burdens will affect the average effective tax rate functions we fit to the output of the microsimulation model. For example, if filing units with a primary filer that is 65 years old and with total income of $60,000 have a higher proportion of income from tax exempt interest than do filing units who also have $60,000 in total income, but a primary filer who is 40 years old, then the tax functions will be able to account for these different portfolios by finding a lower average effective tax rate for the older filing unit than that younger filing unit, for a given amount of total income.

4.2 Estimating tax functions

Tax functions of the form above are estimated with output from the microsimulation model, a sample of which is shown in Figure 2. To map the output of the microsimulation model, which is based on income reported on tax returns, to the DGE model, where income is defined more broadly, we use the following definitions.

To calculate the average effective tax rates from the microsimulation model, we divided total tax liability by a measure of "adjusted total income". Adjusted total income is defined as total income (Form 1040, line 22) plus tax-exempt interest income, IRA distributions, pension income, and Social Security benefits (Form 1040, lines 8b, 15a, 16a, and 20a, respectively).

We consider adjusted total income from the microsimulation model to be the counterpart of total income in the DGE model. Total income in the DGE model is the sum of capital and labor income. We define labor income as earned income, which is the sum of wages and salaries (Form 1040, line 7) and self-employment income (Form 1040 lines 12 and 18) from the microsimulation model output. Capital income is defined as a residual.[17]

When we look at the raw output from the microsimulation model, we find that there are several observations with extreme values for their average effective tax rate. Since this is a ratio, such outliers are possible, for example when the denominator, adjusted total income, is very small. We omit such outliers by making the following restrictions on the raw output of the microsimulation model. First, we exclude observations with an average effective tax rate greater than 1.5 times the highest statutory marginal tax rate. Second, we exclude observations where the average effective tax rate is less than the lowest statutory marginal tax rate on income minus the maximum phase-in rate for the Earned Income Tax Credit (EITC). Finally, since total income cannot be negative in our DGE model, we drop observations from the microsimulation model where adjusted total income is less than \$5.[18]

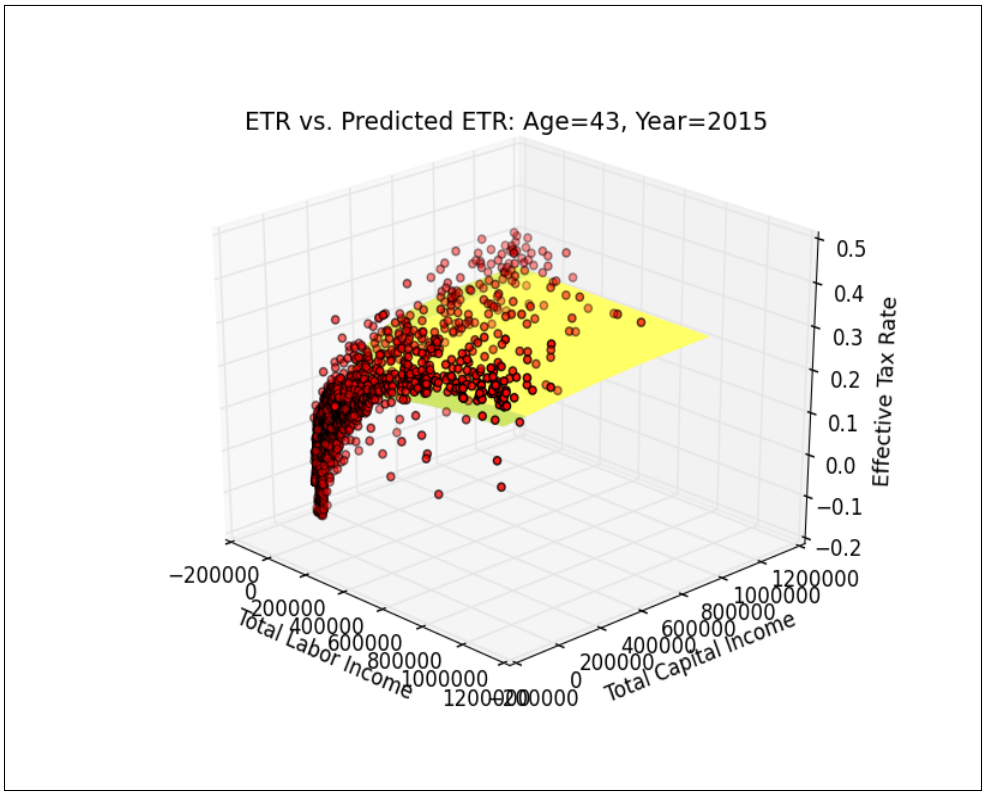
With the output of the microsimulation model cleaned, we move to our estimation. We estimate a transformation of the AETR function in Equation (32) separately for each tax year and each year of age of the primary filer. We estimate separate tax functions of capital income and labor income for each age tax filer in each year $\tau_{s,j}(x, y)$. We also estimate the version of (32) in which the labor income x and capital income y variables in the polynomials are transformed to percent deviations from their respective means. This helps with the scale of the variables in the optimization routine. The transformed AETR function is estimated using a constrained, weighted, non-linear least squares estimator. The weighting in this estimator come from the

[17]This is not an ideal definition of capital income, since it includes transfers between filers (e.g., alimony payments) and from the government (e.g., unemployment insurance), but we have chosen this definition for now in order to ensure that all of total income is classified as either capital or labor income. This accounting will be refined in the future.

[18]We choose \$5 rather than \$0 to provided additional assurance that small income values are not driving large AETRs.

CPS weights assigned to the filers in the microsimulation model.

Figure 3: Fitted tax function $\tau(x, y)$ versus microsimulation data τ_i: $t = 2015$ and $s = 43$



Let $\boldsymbol{\theta}_{s,t} = (A, B, C, D, E, F, max_x, min_x, max_y, min_y)$ be the vector of parameters of the tax function for a particular age of filers in a particular year. The estimate $\hat{\boldsymbol{\theta}}_{s,t}$ is the following,

$$\begin{aligned} \hat{\boldsymbol{\theta}}_{s,t} = \boldsymbol{\theta}_{s,t} : \quad & \min_{\boldsymbol{\theta}_{s,t}} \sum_{i=1}^{N} \Big[\tau_i - \tau\big(x_i, y_i | \boldsymbol{\theta}_{s,t}\big)\Big]^2 w_i, \\ & \text{subject to} \quad A, B, C, D, E, F, max_x, max_y > 0 \\ & \text{and} \quad max_x \geq min_x \quad \text{and} \quad max_y \geq min_y \end{aligned} \tag{39}$$

where τ_i is the average effective tax rate for observation i from the microsimulation output, $\tau(x_i, y_i|\boldsymbol{\theta}_{s,t})$ is the predicted average tax rate for individual i with x_i labor income and y_i capital income given parameters $\boldsymbol{\theta}_{s,t}$, and w_i is the CPS sampling weight of this observation. The number N is the total number of observations from

the microsimulation output for age s and year t. Figure 3 shows the typical fit of an estimated tax function $\tau_{s,t}\left(x, y|\boldsymbol{\theta}_{s,t}\right)$ to the data. The data in Figure 3 are the same age $s = 43$ and year $t = 2015$ as the data Figure 2. The only difference is that we have truncated both labor income x and capital income y in Figure 2 at \$1 million rather than \$500,000.

We use the microsimulation model data to estimate $S \times T$ sets of the 10 tax function parameters listed in Table 3. The total number of years of microsimulation data for which we estimate tax functions corresponds to the budget window of our desired revenue estimates, which is usually $T = 10$. Because the DGE model has a time horizon well beyond 10 years, we assume that the file year tax policy in year T is continued indefinitely from the end of the budget window forward. In addition, the data provide very few observations of primary filers with ages greater than 80 years old. The DGE model allows individuals to live for up to 100 periods. We apply the tax function estimated for 80 year-olds to those with model ages 80-100.

5 A 10% Reduction in Income Tax Rates

The policy change we consider is a reduction in the marginal statutory rates on individual filers ordinary income. The changes in tax policy are summaries in Table 4.

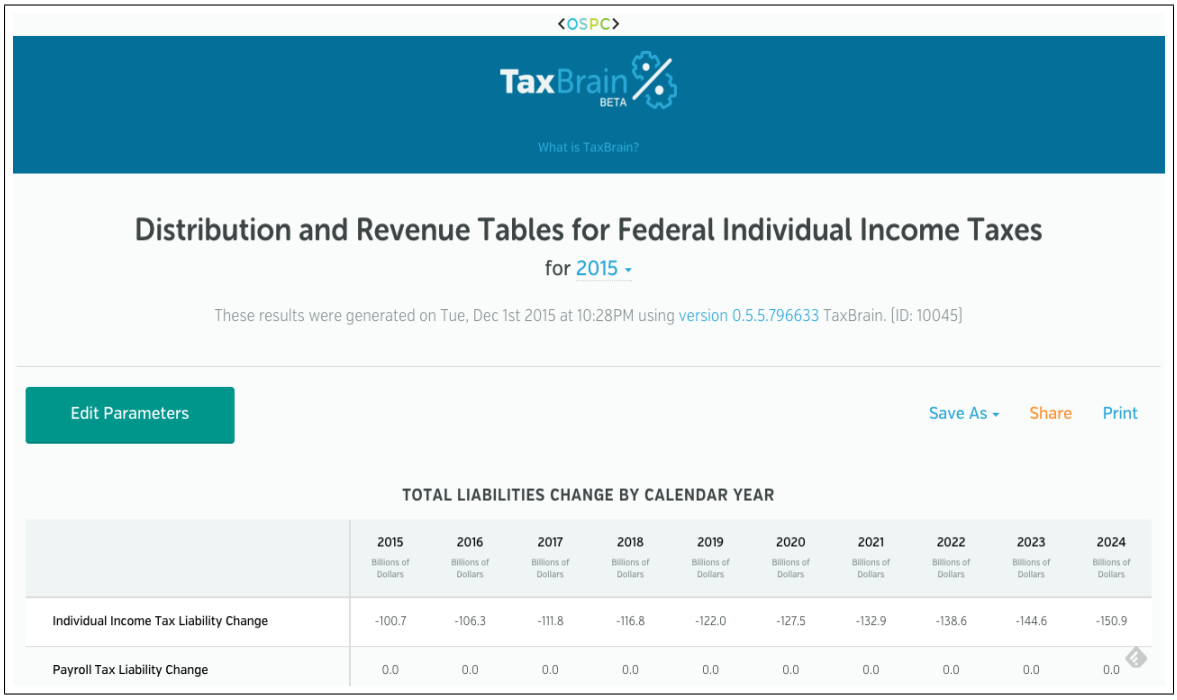
Table 4: Statutory Marginal Rates Under the Baseline and Policy Change

Bracket	Baseline	Policy
1	0.100	0.090
2	0.150	0.135
3	0.250	0.225
4	0.280	0.252
5	0.330	0.297
6	0.350	0.315
7	0.396	0.356

5.1 The Static Revenue Effects

Using the web application for the Tax-Calculator at www.ospc.org/TaxBrain, we calculate the revenue effects of the proposal to reduce the statutory marginal tax rate by 10% for each tax bracket. The static revenue effects as displaying on the Tax Brain webpage are about -$1.25 trillion over 10 years (2015-2024).

Figure 4: Static Revenue Estimate from the Tax Calculator



5.2 The Macroeconomic Effects

Table 6 displays the percentage changes in aggregate quantities and prices over the budget window and in the long-run steady state. These are computed through the DGE model described above, using the estimated tax functions for the baseline and policy tax parameters. The parameters of the estimated tax functions are given in Table 5.

Table 5: Tax Function Parameters

Parameter	Baseline	Policy
A	3.10E-10	9.45E-11
B	2.26E-10	1.57E-10
C	6.37E-12	7.29E-12
D	3.45E-04	1.99E-04
E	1.54E-04	7.94E-05
F	29.0441	18.3836
max_x	0.2910	0.2835
min_x	-0.0544	-0.0487
max_y	0.3037	0.2987
min_y	-0.0240	-0.0199

Table 6: Macroeconomic Effects of the Policy Change (% Change)

	2015	2016	2017	2018	2019	2020	2021	2022	2023	2024	2015-2024	SS
GDP	1.64	1.59	1.58	1.57	1.56	1.56	1.56	1.56	1.56	1.55	1.57	1.55
Consumption	1.68	1.67	1.68	1.67	1.67	1.67	1.68	1.64	1.64	1.58	1.66	1.33
Investment	3.66	3.22	2.68	3.48	2.33	3.24	3.06	2.62	2.85	2.22	2.92	2.04
Hours Worked	1.94	1.73	1.53	1.83	1.37	1.74	1.66	1.44	1.53	1.23	1.60	1.18
Avg. Wage	0.19	0.29	0.37	0.27	0.46	0.32	0.36	0.45	0.41	0.52	0.36	0.36
Interest Rate	-0.61	-0.93	-1.19	-0.86	-1.43	-0.99	-1.13	-1.39	-1.28	-1.62	-1.15	-1.23
Total Taxes	-2.73	-2.83	-2.69	-2.88	-2.57	-2.78	-2.81	-2.55	-2.72	-2.23	-2.68	-2.20

6 Discussion

Qualitatively, the results of the macro model are consistent with economic theory. The reduction in marginal tax rates increases the incentives to work and save. We subsequently see increases aggregate hours worked and investment. As a result, both GDP and consumption increase. That the increases in the capital stock are larger than the increase in hours worked, means that the marginal product of capital falls, while the marginal product of labor rises. Given our assumption of a competitive production sector, this results in lower real interest rates and higher wages.

Total tax revenue decreases due to the lower tax rates, but the increase in aggregate income that results from the additional investment and labor supply offset the revenue losses to some extent. Consider that the static score suggests revenue losses on the order of 4% of total tax revenue. These losses are between 2% and 3% when accounting for the macroeconomic effects of the tax policy.

A few caveats about the limitations of the model are in order. Of first order importance in determining the macroeconomic effects of changes in tax policy are the assumptions about how such tax changes are financed. The DGE model used here has a simple balanced budget requirement for the government. This means that tax cuts, as we consider here, are financed by immediate reductions in the lump sum transfers the government makes to all households. The assumption is the most conducive to reductions in taxes providing positive macroeconomic effects. If these tax cuts were temporary and financed by future tax increases, the stimulative effects of such cuts would be substantially reduced.

Not considered in this model but also important in determining the macroeconomic effects of fiscal policy are the policy responses of the central bank. Implicit in the results presented here is that the central bank does not respond to fiscal policy. If, for example, the central bank responded by holding interest rates constant, the supply side effects would be smaller and their would be less of a change in the macroeconomic aggregates.

Finally, one should note that while the levels of the macroeconomic aggregates

change in the steady state as a result of tax policy, that long run growth rates do not. These long run growth rates are governed by exogenous changes in population growth and factor productivity. Thus these long run growth rates, in this framework, are not dependent on tax policy.

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APPENDIX

A-1 Characteristics of exogenous population growth assumptions

In this appendix, we describe in detail the exogenous population growth assumptions in the model and their implications. In Section 3.1, we define the laws of motion for the population of each cohort $\omega_{s,t}$ to be the following.

$$\begin{aligned}\omega_{1,t+1} &= \sum_{s=1}^{E+S} f_s \omega_{s,t} \quad \forall t \\ \omega_{s+1,t+1} &= (1 + i_s - \rho_s)\omega_{s,t} \quad \forall t \quad \text{and} \quad 1 \leq s \leq E+S-1\end{aligned} \tag{1}$$

We can transform the nonstationary equations in (1) into stationary laws of motion by dividing both sides by the total populations N_t and N_{t+1} in both periods,

$$\begin{aligned}\hat{\omega}_{1,t+1} &= \frac{\sum_{s=1}^{E+S} f_s \hat{\omega}_{s,t}}{1 + g_{n,t+1}} \quad \forall t \\ \hat{\omega}_{s+1,t+1} &= \frac{(1 + \phi_s - \rho_s)\hat{\omega}_{s,t}}{1 + g_{n,t+1}} \quad \forall t \quad \text{and} \quad 1 \leq s \leq E+S-1\end{aligned} \tag{A.1.1}$$

where $\hat{\omega}_{s,t}$ is the percent of the total population in age cohort s and the population growth rate $g_{n,t+1}$ between periods t and $t+1$ is defined in (3),

$$\begin{bmatrix} \hat{\omega}_{1,t+1} \\ \hat{\omega}_{2,t+1} \\ \hat{\omega}_{2,t+1} \\ \vdots \\ \hat{\omega}_{E+S-1,t+1} \\ \hat{\omega}_{E+S,t+1} \end{bmatrix} = \frac{1}{1 + g_{n,t+1}} \times \ldots$$
$$\begin{bmatrix} f_1 & f_2 & f_3 & \ldots & f_{E+S-1} & f_{E+S} \\ 1 + i_1 - \rho_1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 + i_2 - \rho_2 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 + i_3 - \rho_3 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & \ldots & 1 + i_{E+S-1} - \rho_{E+S-1} & 0 \end{bmatrix} \begin{bmatrix} \hat{\omega}_{1,t} \\ \hat{\omega}_{2,t} \\ \hat{\omega}_{2,t} \\ \vdots \\ \hat{\omega}_{E+S-1,t} \\ \hat{\omega}_{E+S,t} \end{bmatrix} \tag{A.1.2}$$

where we restrict $1 + i_s - \rho_s \geq 0$ for all s.

We write (A.1.2) in matrix notation as the following.

$$\hat{\boldsymbol{\omega}}_{t+1} = \frac{1}{1 + g_{n,t+1}} \boldsymbol{\Omega} \hat{\boldsymbol{\omega}}_t \quad \forall t \tag{A.1.3}$$

The stationary steady state population distribution $\bar{\boldsymbol{\omega}}$ is the eigenvector $\boldsymbol{\omega}$ with eigenvalue $(1+\bar{g}_n)$ of the matrix $\boldsymbol{\Omega}$ that satisfies the following version of (A.1.3).

$$(1+\bar{g}_n)\bar{\boldsymbol{\omega}} = \boldsymbol{\Omega}\bar{\boldsymbol{\omega}} \tag{A.1.4}$$

Proposition 1. There exists a unique positive real eigenvector $\bar{\omega}$ of the matrix $\boldsymbol{\Omega}$, and it is a stable equilibrium.

Proof. First, note that the matrix $\boldsymbol{\Omega}$ is square and non-negative. This is enough for a general version of the Perron-Frobenius Theorem to state that a positive real eigenvector exists with a positive real eigenvalue. This is not yet enough for uniqueness. For it to be unique by a version of the Perron-Fobenius Theorem, we need to know that the matrix is irreducible. This can be easily shown. The matrix is of the form

$$\boldsymbol{\Omega} = \begin{bmatrix} * & * & * & \dots & * & * & * \\ * & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & * & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & * & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & * & 0 \end{bmatrix}$$

Where each * is strictly positive. It is clear to see that taking powers of the matrix causes the sub-diagonal positive elements to be moved down a row and another row of positive entries is added at the top. None of these go to zero since the elements were all non-negative to begin with.

$$\boldsymbol{\Omega^2} = \begin{bmatrix} * & * & * & \dots & * & * & * \\ * & * & * & \dots & * & * & * \\ * & 0 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & * & 0 & 0 \end{bmatrix}; \quad \boldsymbol{\Omega^{S+E-1}} = \begin{bmatrix} * & * & * & \dots & * & * & * \\ * & * & * & \dots & * & * & * \\ * & * & * & \dots & * & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ * & * & * & \dots & * & * & * \\ * & 0 & 0 & \dots & 0 & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{\Omega^{S+E}} = \begin{bmatrix} * & * & * & \dots & * & * & * \\ * & * & * & \dots & * & * & * \\ * & * & * & \dots & * & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ * & * & * & \dots & * & * & * \\ * & * & * & \dots & * & * & * \end{bmatrix}$$

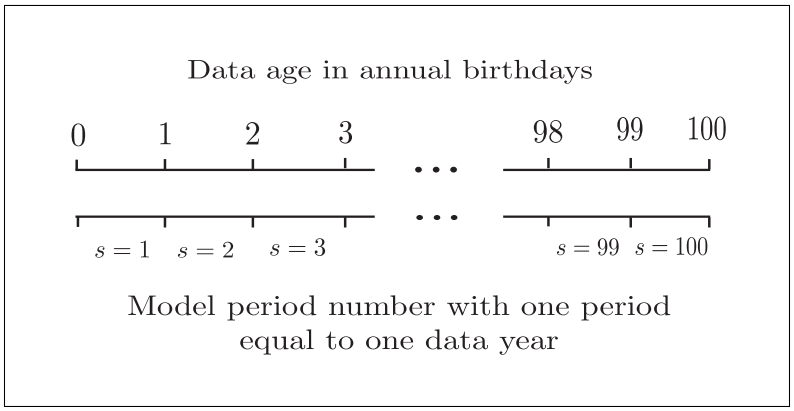
Existence of an $m \in \mathbb{N}$ such that $\left(\boldsymbol{\Omega^m}\right)_{ij} \neq 0 \quad (> 0)$ is one of the definitions of an irreducible (primitive) matrix. It is equivalent to saying that the directed graph associated with the matrix is strongly connected. Now the Perron-Frobenius Theorem for irreducible matrices gives us that the equilibrium vector is unique.

We also know from that theorem that the eigenvalue associated with the positive real eigenvector will be real and positive. This eigenvalue, p, is the Perron eigenvalue

and it is the steady state population growth rate of the model. By the PF Theorem for irreducible matrices, $|\lambda_i| \leq p$ for all eigenvalues λ_i and there will be exactly h eigenvalues that are equal, where h is the period of the matrix. Since our matrix $\mathbf{\Omega}$ is aperiodic, the steady state growth rate is the unique largest eigenvalue in magnitude. This implies that almost all initial vectors will converge to this eigenvector under iteration. □

For a full treatment and proof of the Perron-Frobenius Theorem, see Suzumura (1983). Because the population growth process is exogenous to the model, we calibrate it to annual age data for age years $s = 1$ to $s = 100$. As is shown in Figure 5, period $s = 1$ corresponds to the first year of life between birth and when an individual turns one year old.

Figure 5: Correspondence of model timing to data timing for model periods of one year



Our initial population distribution $\{\omega_{s,1}\}_{s=1}^{100}$ in Figure 6 comes from Census Bureau (2014) population estimates for both sexes for 2013. The fertility rates $\{f_s\}_{s=1}^{100}$ in Figure 7 come from Center for Disease Control (2010, Table 1). The mortality rates $\{\rho_s\}_{s=1}^{99}$ in Figure 8 come from the 2010 death probabilities in Social Security Administration (2010). We enforce a strict maximum age mortality rate of $\rho_{100} = 1$ in our model.

The immigration rates $\{i_s\}_{s=1}^{99}$ in Figure 9 are essentially residuals. We take total population for two consecutive years N_t and N_{t+1} and the population distribution by age in both of those years $\boldsymbol{\omega}_t$ and $\boldsymbol{\omega}_{t+1}$from the Census Bureau (2014) data. We then deduce the immigration rates $\{i_s\}_{s=1}^{99}$ using equation (A.1.1). We do this for three consecutive sets of years, so that our calibrated immigration rates by age are the average of our three years of deduced rates from the data for each age.

Figure 6: Initial population distribution $\omega_{s,1}$ by year, $1 \leq s \leq 100$

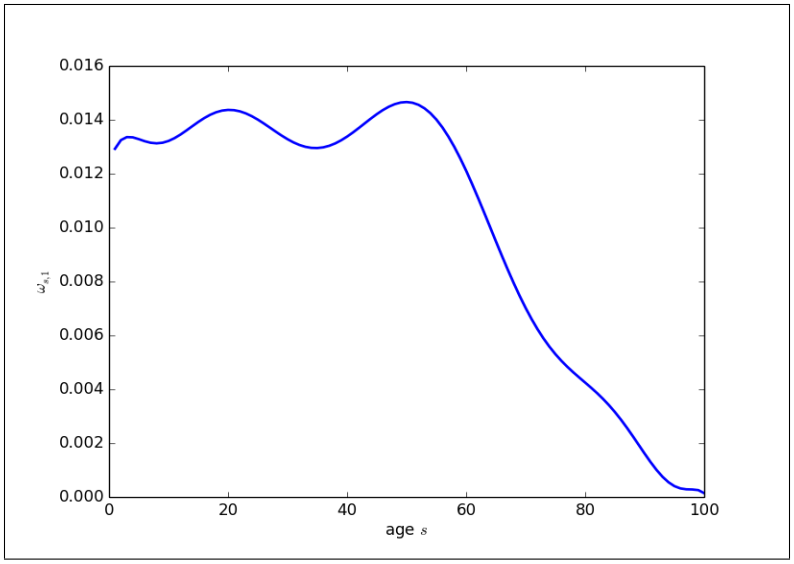


Figure 7: Fertility rates f_s by year, $1 \leq s \leq 100$

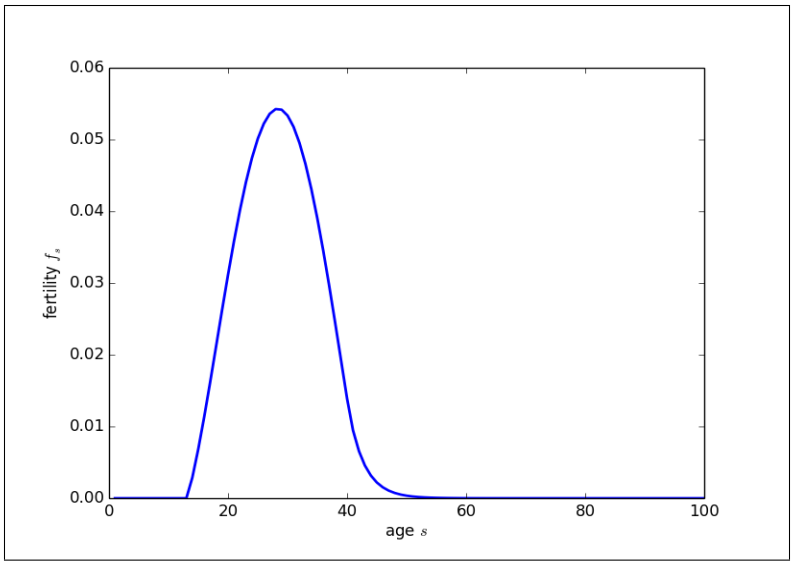


Figure 8: Mortality rates ρ_s by year, $1 \leq s \leq 100$

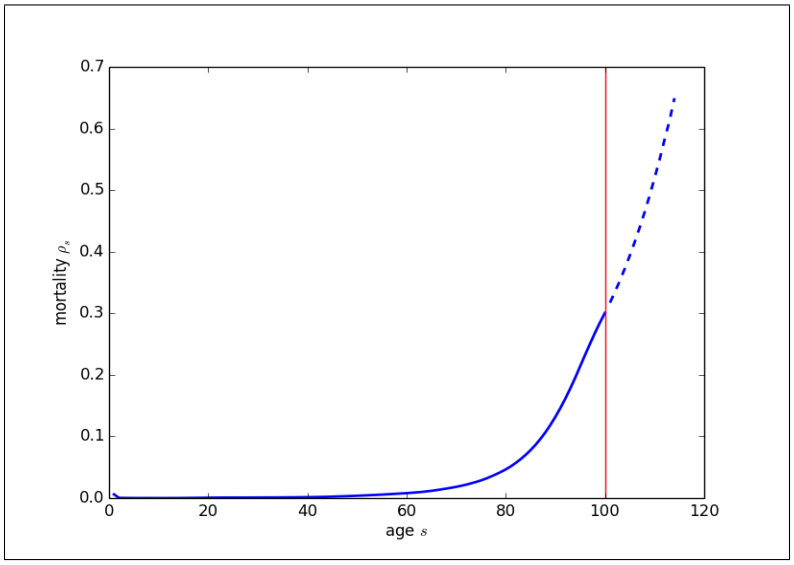


Figure 9: Immigration rates i_s by year, $1 \leq s \leq 100$

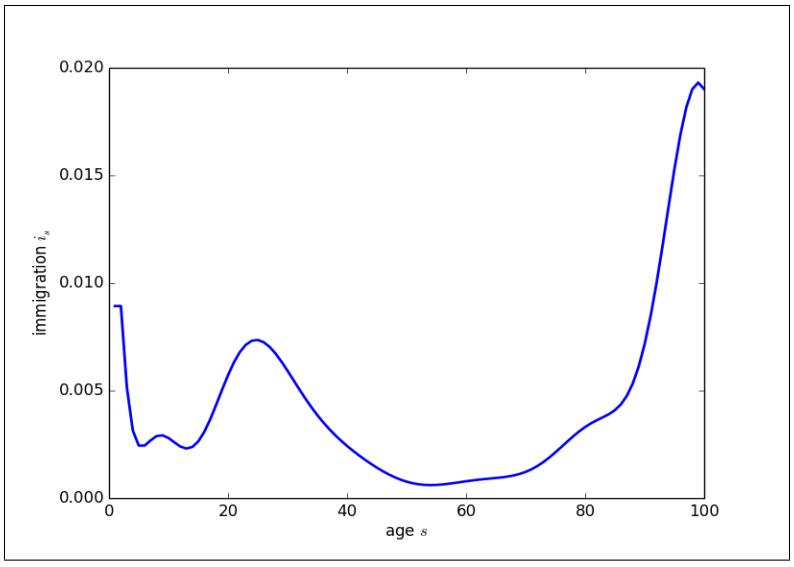


Figure 10 shows the predicted time path of the total population N_t given $\omega_{s,1}$ f_s, i_s, and ρ_s. Notice that the population approaches a constant growth rate. This is a result of the stationary population percent distribution $\bar{\boldsymbol{\omega}}$ eventually being reached. Figure 11 shows the steady-state population percent distribution by age $\bar{\boldsymbol{\omega}}$.

Figure 10: Forecast time path of population growth rate $g_{n,t}$

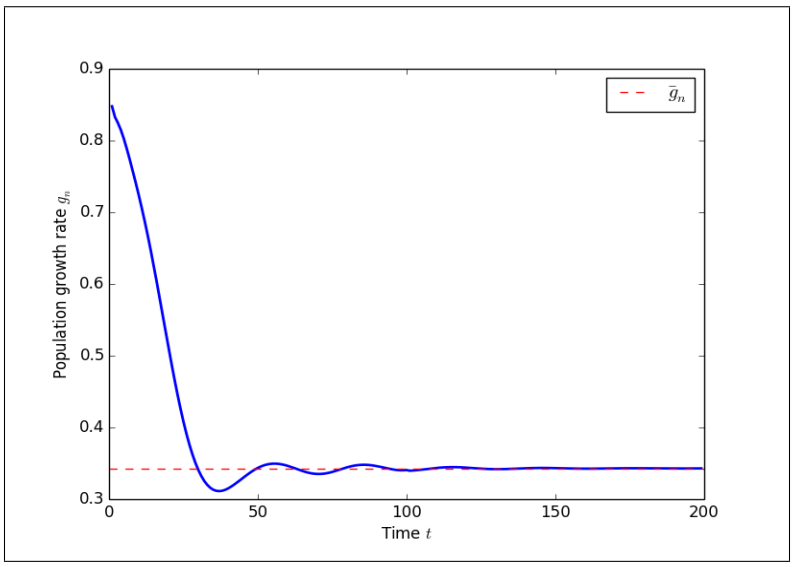
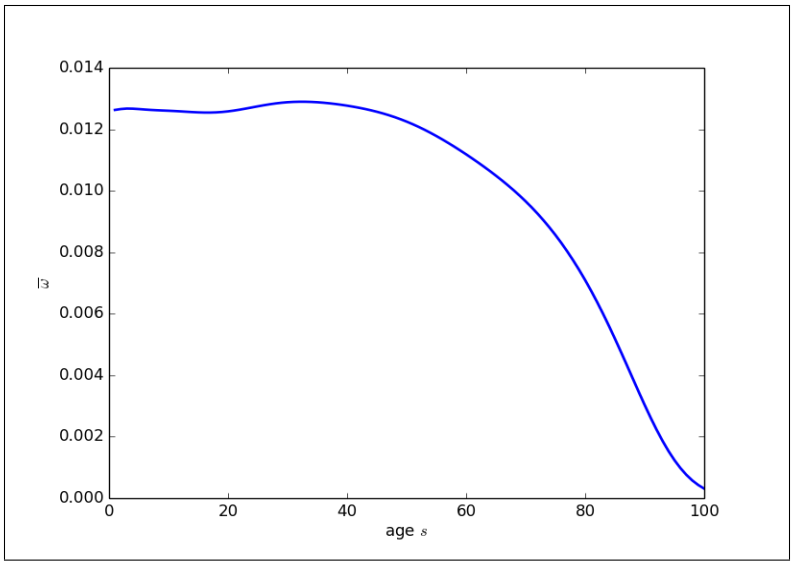


Figure 11: Steady-state population percent distribution by age $\bar{\omega}$



A-2 Derivation of elliptical disutility of labor supply

Evans and Phillips (2015) provide an exposition of the value of using elliptical disutility of labor specification as well as its relative properties to such standard disutility of labor functions such as constant relative risk aversion (CRRA) and constant Frisch elasticity (CFE). A standard specification of additively separable period utility in consumption and labor supply first used in King et al. (1988) is the following,

$$u(c,n) = \frac{c^{1-\sigma}-1}{1-\sigma} + \chi^n \frac{\left(\tilde{l}-n\right)^{1+\theta}}{1+\theta} \tag{A.2.1}$$

where $\sigma \geq 1$ is the coefficient of relative risk aversion on consumption, $\theta \geq 0$ is proportional to the inverse of the Frisch elasticity of labor supply, and $\tilde{l}$ is the time endowment or the maximum labor supply possible. The constant χ^n is a scale parameter influencing the relative disutility of labor to the utility of consumption.

Although labor supply is only defined for $n \in [0, \tilde{l}]$, the marginal utility of leisure at $n = \tilde{l}$ is infinity and is not defined for $n > \tilde{l}$. However, utility of labor in this functional form is defined for $n < 0$. To avoid the well known and significant computational difficulty of computing the solution to the complementary slackness conditions in the Karush, Kuhn, Tucker constrained optimization problem, we impose an approximating utility function that has properties bounding the solution for n away from both $n = \tilde{l}$ and $n = 0$. The upper right quadrant of an ellipse has exactly this property and also has many of the properties of the original utility function. Figure 12 shows how our estimated elliptical utility function compares to the utility of labor from (A.2.1) over the allowed support of n.

The general equation for an ellipse in x and y space with centroid at coordinates (h, k), horizontal radius of a, vertical radius of b, and curvature υ is the following.

$$\left(\frac{x-h}{a}\right)^{\upsilon} + \left(\frac{y-k}{b}\right)^{\upsilon} = 1 \tag{A.2.2}$$

Figure 13 shows an ellipse with the parameterization $[h, k, a, b, \upsilon] = [1, -1, 1, 2, 2]$.

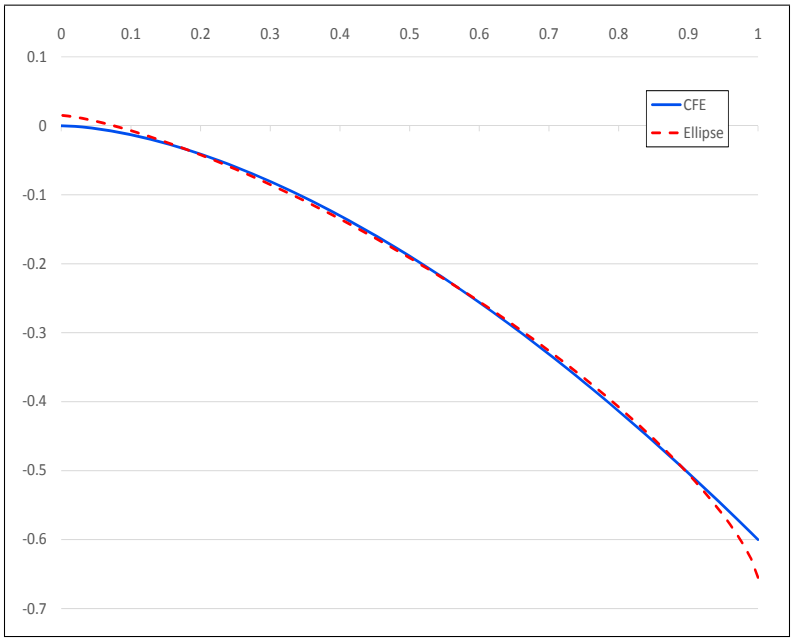
The graph of the ellipse in the upper-right quadrant of Figure 13 ($x \in [1, 2]$ and $y \in [-1, 1]$) has similar properties to the utility of labor term in (A.2.1). If we let the x variable be labor supply n, the utility of labor supply be $g(n)$, the x-coordinate of the centroid be zero $h = 0$, and the horizontal radius of the ellipse be $a = \tilde{l}$, then the equation for the ellipse corresponding to the standard utility specification is the following.

$$\left(\frac{n}{\tilde{l}}\right)^{\upsilon} + \left(\frac{g-k}{b}\right)^{\upsilon} = 1 \tag{A.2.3}$$

Solving the equation for g as a function of n, we get the following.

$$g(n) = b\left[1 - \left(\frac{n}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1}{\upsilon}} + k \tag{A.2.4}$$

Figure 12: Comparison of standard utility of labor n to elliptical utility



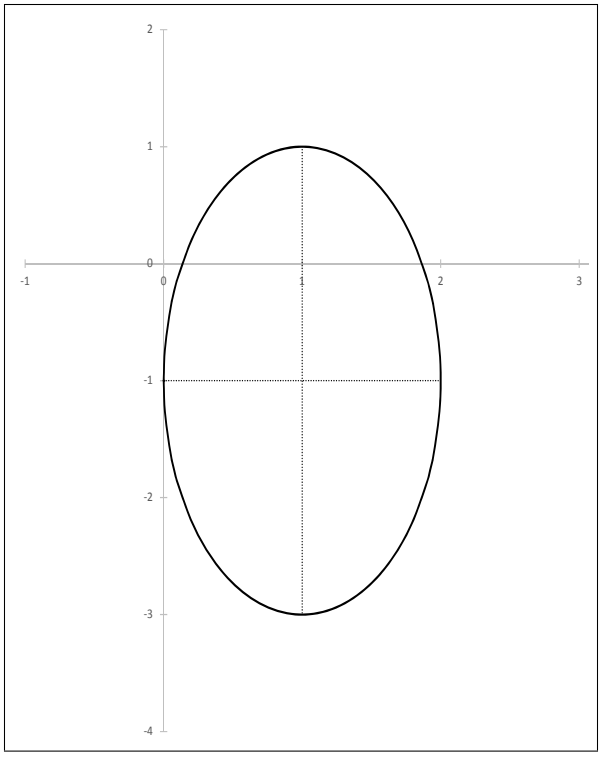
The υ parameter acts like a constant elasticity of substitution, and the parameter b is a shape parameter similar to χ^n in (A.2.1).

We use the upper-right quadrant of the elliptical utility function because the utility of n is strictly decreasing on $n \in (0, \tilde{l})$, because the slope of the utility function goes to negative infinity as n approaches its maximum of $\tilde{l}$ and because the slope of the utility function goes to zero as n approaches its minimum of 0. This creates interior solutions for all optimal labor supply choices $n^* \in (0, \tilde{l})$. Although it is more realistic to allow optimal labor supply to sometimes be zero, the complexity and dimensionality of our model requires this approximating assumption to render the solution method tractable.

Figure 12 shows how closely the estimated elliptical utility function matches the original utility of labor function in (A.2.1) with a Frisch elasticity of 1.5[19] . We choose the ellipse parameters b, k, and υ to best match the points on the original utility of labor function for $n \in [0, 1]$. We minimize the sum of absolute errors for 101 evenly spaced points on this domain. The estimated values of the parameters for the elliptical utility shown in Figure 12 and represented in equation (A.2.4) are $[b, k, \upsilon] = [.6701, -.6548, 1.3499]$.

[19]See Chetty et al. (2011), Keane and Rogerson (2012) and Peterman (2014) for discussion of this choice.

Figure 13: Ellipse with $[h, k, a, b, v] = [1, -1, 1, 2, 2]$



A-3 Solving for stationary steady-state equilibrium

This section describes the solution method for the stationary steady-state equilibrium described in Definition 1.

1. Use the techniques in Appendix A-1 to solve for the steady-state population distribution vector $\bar{\boldsymbol{\omega}}$ of the exogenous population process.

2. Choose an initial guess for the stationary steady-state distribution of capital $\bar{b}_{j,s+1}$ for all j and $s = E+2, E+3, ...E+S+1$ and labor supply $\bar{n}_{j,s}$ for all j and s.

   - A good first guess is a large positive number for all the $\bar{n}_{j,s}$ that is slightly less than $\tilde{l}$ and to choose some small positive number for $\bar{b}_{j,s+1}$ that is small enough to be less than the minimum income that an individual might have $\bar{w}e_{j,s}\bar{n}_{j,s}$.

3. Perform an unconstrained root finder that chooses $\bar{n}_{j,s}$ and $\bar{b}_{j,s+1}$ that solves the $2JS$ stationary steady-state Euler equations.

4. Make sure none of the implied steady-state consumptions $\bar{c}_{j,s}$ is less-than-or-equal-to zero.

   - If one consumption is less-than-or-equal-to zero $\bar{c}_{j,s} \leq 0$, then try different starting values.

5. Make sure that none of the Euler errors is too large in absolute value for interior stationary steady-state values. A steady-state Euler error is the following, which is supposed to be close to zero for all j and s:

$$\frac{\chi^n_s \left(\frac{b}{\tilde{l}}\right)\left(\frac{\bar{n}_{j,s}}{\tilde{l}}\right)^{\upsilon-1}\left[1-\left(\frac{\bar{n}_{j,s}}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1-\upsilon}{\upsilon}}}{(\bar{c}_{j,s})^{-\sigma}\left(\bar{w}e_{j,s}-\frac{\partial \bar{T}_s}{\partial \bar{n}_{j,s}}\right)} - 1 \qquad \forall j \quad \text{and} \quad E+1 \leq s \leq E+S \tag{A.3.1}$$

$$\frac{e^{-g_y\sigma}\left(\rho_s\chi^b_j\left(\bar{b}_{j,s+1}\right)^{-\sigma} + \beta(1-\rho_s)(\bar{c}_{j,s+1})^{-\sigma}\left[(1+\bar{r})-\frac{\partial \bar{T}_{s+1}}{\partial \bar{b}_{j,s+1}}\right]\right)}{(\bar{c}_{j,s})^{-\sigma}} - 1 \qquad \forall j \quad \text{and} \quad E+1 \leq s \leq E+S-1 \tag{A.3.2}$$

$$\frac{\chi^b_j e^{-g_y\sigma}(\bar{b}_{j,E+S+1})^{-\sigma}}{(\bar{c}_{j,E+S})^{-\sigma}} - 1 \quad \forall j \tag{A.3.3}$$

A-4 Solving for stationary non-steady-state equilibrium by time path iteration

This section describes the solution to the non-steady-state transition path equilibrium of the model and outlines the benchmark time path iteration (TPI) method of Auerbach and Kotlikoff (1987) for solving the stationary non-steady-state equilibrium transition path of the distribution of savings.

Taken together, the individual labor-leisure and intended bequest decisions in the last period of life show that the optimal labor supply and optimal intended bequests for age $s = E + S$ are each functions of individual savings, total bequests received, and the prices in that period: $n_{j,E+S,t} = \phi\big(\hat{b}_{j,E+S,t}, \hat{BQ}_{j,t}, \hat{w}_t, r_t\big)$ and $\hat{b}_{j,E+S+1,t+1} = \psi\big(\hat{b}_{j,E+S,t}, \hat{BQ}_{j,t}, \hat{w}_t, r_t\big)$. These two decisions are characterized by final-age version of the static labor supply Euler equation (24) and the static intended bequests Euler equation (26). individuals in their second-to-last period of life in period t have four decisions to make. They must choose how much to work this period, $n_{j,E+S-1,t}$, and next period, $n_{j,E+S,t+1}$, how much to save this period for next period, $\hat{b}_{j,E+S,t+1}$, and how much to bequeath next period, $\hat{b}_{j,E+S+1,t+2}$. The optimal responses for this individual are characterized by the $s = E+S-1$ and $s = E+S$ versions of the static Euler equations (24), the $s = E + S - 1$ version of the intertemporal Euler equation (25), and the $s = E + S$ static bequest Euler equation (26), respectively.

Optimal savings in the second-to-last period of life $s = E+S-1$ is a function of the current savings as well as the total bequests received and prices in the current period and in the next period $\hat{b}_{j,E+S,t+1} = \psi\big(\hat{b}_{j,E+S-1,t}, \hat{BQ}_{j,t}, \hat{w}_t, r_t, \hat{BQ}_{j,t+1}, \hat{w}_{t+1}, r_{t+1}|\Omega\big)$ given beliefs Ω. As before, the optimal labor supply at age $s = E + S$ is a function of the next period's savings, bequests received, and prices.

$$n_{j,E+S,t+1} = \phi\big(\hat{b}_{j,E+S,t+1}, \hat{BQ}_{j,t+1}, \hat{w}_{t+1}, r_{t+1}\big)$$

But the optimal labor supply at age $s = E + S - 1$ is a function of the current savings, current bequests received, and the current prices as well as the future bequests received and future prices because of the dependence on the savings decision in that same period $n_{j,E+S-1,t} = \phi\big(\hat{b}_{j,E+S-1,t}, \hat{BQ}_{j,t}, \hat{w}_t, r_t, \hat{BQ}_{j,t+1}, \hat{w}_{t+1}, r_{t+1}|\Omega\big)$ given beliefs Ω. By induction, we can show that the optimal labor supply, savings, and intended bequests functions for any individual with ability j, age s, and in period t is a function of current holdings of savings and the lifetime path of total bequests received and prices given beliefs Ω.

$$n_{j,s,t} = \phi\Big(\hat{b}_{j,s,t}, \big(\hat{BQ}_{j,v}, \hat{w}_v, r_v\big)_{v=t}^{t+S-s}|\Omega\Big) \quad \forall j, s, t \tag{A.4.1}$$

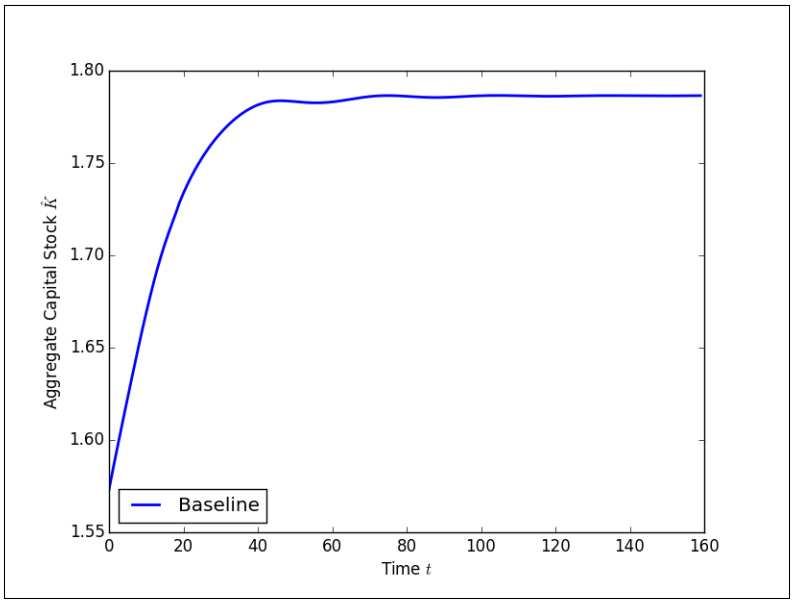
$$\hat{b}_{j,s+1,t+1} = \psi\Big(\hat{b}_{j,s,t}, \big(\hat{BQ}_{j,v}, \hat{w}_v, r_v\big)_{v=t}^{t+S-s}|\Omega\Big) \quad \forall j, t \quad \text{and} \quad E+1 \leq s \leq E+S \tag{A.4.2}$$

If one knows the current distribution of individuals savings and intended bequests, $\hat{\mathbf{\Gamma}}_t$, and beliefs about $\hat{\mathbf{\Gamma}}_t$, then one can predict time series for total bequests received

$\hat{BQ}_{j,t}$, real wages $\hat{w}_t$ and real interest rates r_t necessary for solving each individual's optimal decisions. Characteristic (i) in equilibrium definition 2 implies that individuals be able to forecast prices with perfect foresight over their lifetimes implies that each individual has correct information and beliefs about all the other individuals optimization problems and information. It also implies that the equilibrium allocations and prices are really just functions of the entire distribution of savings at a particular period, as well as a law of motion for that distribution of savings.

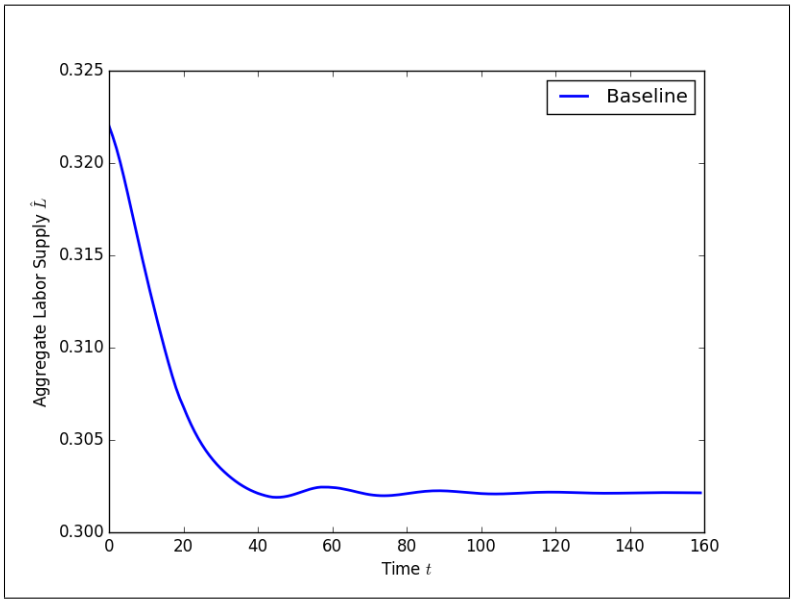
In equilibrium, the steady-state individual labor supply, $\bar{n}_{j,s}$, for all j and s, the steady-state savings, $\bar{b}_{j,E+S+1}$, the steady-state real wage, $\bar{w}$, and the steady-state real rental rate, $\bar{r}$, are simply functions of the steady-state distribution of savings $\bar{\Gamma}$. This is clear from the steady-state version of the capital market clearing condition (29) and the fact that aggregate labor supply is a function of the sum of exogenous efficiency units of labor in the labor market clearing condition (28). The two firm first order conditions for the real wage $\hat{w}_t$ (27) and real rental rate r_t (19) are only functions of the stationary aggregate capital stock $\hat{K}_t$ and aggregate labor $\hat{L}_t$.

Figure 14: Equilibrium time path of K_t for $S = 80$ and $J = 7$ in baseline model



To solve for any stationary non-steady-state equilibrium time path of the economy from an arbitrary current state to the steady state, we follow the time path iteration (TPI) method of Auerbach and Kotlikoff (1987). The approach is to choose an arbitrary time path for the stationary aggregate capital stock $\hat{K}_t$, stationary aggregate labor $\hat{L}_t$, and total bequests received $\hat{BQ}_{j,t}$ for each type j. This initial guess of a path implies arbitrary beliefs that violate the rational expectations requirement. We then solve for individuals' optimal decisions given the time paths of those variables, which decisions imply new time paths of those variables. We then update the time

Figure 15: Equilibrium time path of L_t for $S = 80$ and $J = 7$ in baseline model



path as a convex combination of the initial guess and the new implied path. Figures 14 and 15 show the equilibrium time paths of the aggregate capital stock and aggregate labor, respectively, for the calibration described in Table 2 for $T = 160$ periods starting from an initial distribution of savings in which $b_{j,s,1} = \bar{\mathbf{\Gamma}}$ for all j and s in the case that no policy experiment takes place. The initial capital stock $\hat{K}_1$ is not at the steady state $\bar{K}$ because the initial population distribution is not at the steady-state.

The computational approach to solving for the non-steady-state transition path equilibrium is the time path iteration (TPI) method of Auerbach and Kotlikoff (1987). TPI finds a fixed point for the transition path of the distribution of capital for a given initial state of the distribution of capital. The idea is that the economy is infinitely lived, even though the agents that make up the economy are not. Rather than recursively solving for equilibrium policy functions by iterating on individual value functions, one must recursively solve for the policy functions by iterating on the entire transition path of the endogenous objects in the economy (see Stokey and Lucas (1989, ch. 17)).

The key assumption is that the economy will reach the steady-state equilibrium described in Definition 1 in a finite number of periods $T < \infty$ regardless of the initial state. Let $\hat{\mathbf{\Gamma}}_t$ represent the distribution of stationary savings at time t.

$$\hat{\mathbf{\Gamma}}_t \equiv \left\{ \left\{ \hat{b}_{j,s,t} \right\}_{j=1}^{J} \right\}_{s=E+2}^{E+S+1}, \quad \forall t \tag{14}$$

In Section 3.5, we describe how the stationary non-steady-state equilibrium time path of allocations and price is characterized by functions of the state $\hat{\mathbf{\Gamma}}_t$ and its law of motion. TPI starts the economy at any initial distribution of savings $\hat{\mathbf{\Gamma}}_1$ and solves for its equilibrium time path over T periods to the steady-state distribution $\bar{\mathbf{\Gamma}}_T$.

The first step is to assume an initial transition path for aggregate stationary capital $\hat{\boldsymbol{K}}^i = \left\{\hat{K}_1^i, \hat{K}_2^i, ...\hat{K}_T^i\right\}$, aggregate stationary labor $\hat{\boldsymbol{L}}^i = \left\{\hat{L}_1^i, \hat{L}_2^i, ...\hat{L}_T^i\right\}$, and total bequests received $\hat{\boldsymbol{BQ}}_j^i = \{\hat{BQ}_{j,1}^i, \hat{BQ}_{j,2}^i, ...\hat{BQ}_{j,T}^i\}$ for each ability type j such that T is sufficiently large to ensure that $\hat{\boldsymbol{\Gamma}}_T = \bar{\boldsymbol{\Gamma}}$, $\hat{K}_T^i\left(\boldsymbol{\Gamma}_T\right)$, $\hat{L}_T^i\left(\boldsymbol{\Gamma}_T\right) = \bar{L}\left(\bar{\boldsymbol{\Gamma}}\right)$, and $\hat{BQ}_{j,T}^i\left(\boldsymbol{\Gamma}_T\right) = \bar{BQ}_j\left(\bar{\boldsymbol{\Gamma}}\right)$ for all $t \geq T$. The superscript i is an index for the iteration number. The transition paths for aggregate capital and aggregate labor determine the transition paths for both the real wage $\hat{\boldsymbol{w}}^i = \{\hat{w}_1^i, \hat{w}_2^i, ...\hat{w}_T^i\}$ and the real return on investment $\boldsymbol{r}^i = \{r_1^i, r_2^i, ...r_T^i\}$. The time paths for the total bequests received also figure in each period's budget constraint and are determined by the distribution of savings and intended bequests.

The exact initial distribution of capital in the first period $\hat{\boldsymbol{\Gamma}}_1$ can be arbitrarily chosen as long as it satisfies the stationary capital market clearing condition (29).

$$\hat{K}_1 = \frac{1}{1+\tilde{g}_{n,1}} \sum_{s=E+2}^{E+S+1} \sum_{j=1}^{J} \hat{\omega}_{s-1,0} \lambda_j \hat{b}_{j,s,1} \tag{A.4.3}$$

Similarly, each initial value of total bequests received $\hat{BQ}_{j,1}^i$ must be consistent with the initial distribution of capital through the stationary version of (9).

$$\hat{BQ}_{j,1} = \frac{(1+r_1)\lambda_j}{1+\tilde{g}_{n,1}} \sum_{s=E+1}^{E+S} \rho_s \hat{\omega}_{s,0} \hat{b}_{j,s+1,1} \quad \forall j \tag{A.4.4}$$

However, this is not the case with $\hat{L}_1^i$. Its value will be endogenously determined in the same way the K_2^i is. For this reason, a logical initial guess for the time path of aggregate labor is the steady state in every period $L_t^1 = \bar{L}$ for all $1 \leq t \leq T$.

It is easiest to first choose the initial distribution of savings $\hat{\boldsymbol{\Gamma}}_1$ and then choose an initial aggregate capital stock $\hat{K}_1^i$ and initial total bequests received $\hat{BQ}_{j,1}^i$ that correspond to that distribution. As mentioned earlier, the only other restrictions on the initial transition paths for aggregate capital, aggregate labor, and total bequests received is that they equal their steady-state levels $\hat{K}_T^i = \bar{K}\left(\bar{\boldsymbol{\Gamma}}\right)$, $\hat{L}_T^i = \bar{L}\left(\bar{\boldsymbol{\Gamma}}\right)$, and $\hat{BQ}_{j,T}^i = \bar{BQ}_j\left(\bar{\boldsymbol{\Gamma}}\right)$ by period T. Evans and Phillips (2014) have shown that the initial guess for the aggregate capital stocks $\hat{K}_t^i$ for periods $1 < t < T$ can take on almost any positive values satisfying the constraints above and still have the time path iteration converge.

Given the initial savings distribution $\hat{\boldsymbol{\Gamma}}_1$ and the transition paths of aggregate capital $\hat{\boldsymbol{K}}^i = \left\{\hat{K}_1^i, \hat{K}_2^i, ...\hat{K}_T^i\right\}$, aggregate labor $\hat{\boldsymbol{L}}^i = \left\{\hat{L}_1^i, \hat{L}_2^i, ...\hat{L}_T^i\right\}$, and total bequests received $\hat{\boldsymbol{BQ}}_j^i = \left\{\hat{BQ}_{j,1}^i, \hat{BQ}_{j,2}^i, ...\hat{BQ}_{j,T}^i\right\}$, as well as the resulting real wage $\hat{\boldsymbol{w}}^i = \{\hat{w}_1^i, \hat{w}_2^i, ...\hat{w}_T^i\}$, and real return to savings $\boldsymbol{r}^i = \{r_1^i, r_2^i, ...r_T^i\}$, one can solve for the period-1 optimal labor supply and intended bequests for each type j of $s = E+S$-aged agents in the last period of their lives $n_{j,E+S,1} = \phi_{j,E+S}(\hat{b}_{j,E+S,1}, \hat{BQ}_{j,E+S,1}, \hat{w}_1, r_1)$ and $\hat{b}_{j,E+S+1,2} = \psi_{j,E+S}(\hat{b}_{j,E+S,1}, \hat{BQ}_{j,E+S,1}, \hat{w}_1, r_1)$ using his two $s = E+S$ static Euler

equations (24) and (26).

$$(\hat{c}_{j,E+S,1})^{-\sigma}\left(\hat{w}_1^i e_{j,E+S} - \frac{\partial \hat{T}_{E+S,1}}{\partial n_{j,E+S,1}}\right) = ...$$
$$\chi^n_{E+S}\left(\frac{b}{\tilde{l}}\right)\left(\frac{n_{j,E+S,1}}{\tilde{l}}\right)^{\upsilon-1}\left[1-\left(\frac{n_{j,E+S,1}}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1-\upsilon}{\upsilon}} \quad \forall j$$

where $\hat{c}_{j,E+S,1} = ...$

$$\left(1+r_1^i\right)\hat{b}_{j,E+S,1} + \hat{w}_1^i e_{j,E+S} n_{j,E+S,1} + \frac{\hat{BQ}_{j,1}}{\lambda_j} - e^{g_y}\hat{b}_{j,E+S+1,2} - \hat{T}_{E+S,1} \tag{A.4.5}$$

$$(\hat{c}_{j,E+S,1})^{-\sigma} = \chi^b_j e^{-g_y\sigma}(\hat{b}_{j,E+S+1,2})^{-\sigma} \quad \forall j \tag{A.4.6}$$

Note that this is simply two equations (A.4.5) and (A.4.6) and two unknowns $n_{j,E+S,1}$ and $\hat{b}_{j,E+S+1,2}$.

We then solve the problem for all j types of $E+S-1$-aged individuals in period $t=1$, each of which entails labor supply decisions in the current period $n_{j,E+S-1,1}$ and in the next period $n_{j,E+S,2}$, a savings decision in the current period for the next period $\hat{b}_{j,E+S,2}$ and an intended bequest decision in the last period $\hat{b}_{j,E+S+1,3}$. The labor supply decision in the initial period and the savings period in the initial period for the next period for each type j of $E+S-1$-aged individuals are policy functions of the current savings and the total bequests received and prices in this period and the next $\hat{b}_{j,E+S,2} = \psi_{j,E+S-1}(\hat{b}_{j,E+S-1,1}, \{\hat{BQ}_{j,t}, \hat{w}_t, r_t\}_{t=1}^2)$ and $\hat{n}_{j,E+S-1,1} = \phi_{j,E+S-1}(\hat{b}_{j,E+S-1,1}, \{\hat{BQ}_{j,t}, \hat{w}_t, r_t\}_{t=1}^2)$. The labor supply and intended bequests decisions in the next period are simply functions of the savings, total bequests received, and prices in that period $\hat{n}_{j,E+S,2} = \phi_{j,E+S}(\hat{b}_{j,E+S,2}, \hat{BQ}_{j,2}, \hat{w}_2, r_2)$ and $\hat{b}_{j,E+S+1,3} = \psi_{j,E+S}(\hat{b}_{j,E+S,2}, \hat{BQ}_{j,2}, \hat{w}_2, r_2)$. These four functions are characterized by the following versions of equations (24), (25), and (26).

$$(\hat{c}_{j,E+S-1,1})^{-\sigma}\left(\hat{w}_1^i e_{j,E+S-1} - \frac{\partial \hat{T}_{E+S-1,1}}{\partial n_{j,E+S-1,1}}\right) = ...$$
$$\chi^n_{E+S-1}\left(\frac{b}{\tilde{l}}\right)\left(\frac{n_{j,E+S-1,1}}{\tilde{l}}\right)^{\upsilon-1}\left[1-\left(\frac{n_{j,E+S-1,1}}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1-\upsilon}{\upsilon}} \quad \forall j \tag{A.4.7}$$

$$(\hat{c}_{j,E+S-1,1})^{-\sigma} = ...$$
$$e^{-g_y\sigma}\left(\rho_{E+S-1}\chi^b_j\left(\hat{b}_{j,E+S,2}\right)^{-\sigma} + \beta(1-\rho_{E+S-1})(\hat{c}_{j,E+S,2})^{-\sigma}\left[\left(1+r_2^i\right) - \frac{\partial \hat{T}_{E+S,2}}{\partial b_{j,E+S,2}}\right]\right) \quad \forall j \tag{A.4.8}$$

$$(\hat{c}_{j,E+S,2})^{-\sigma}\left(\hat{w}_2^i e_{j,E+S} - \frac{\partial \hat{T}_{E+S,2}}{\partial n_{j,E+S,2}}\right) = ...$$
$$\chi^n_{E+S}\left(\frac{b}{\tilde{l}}\right)\left(\frac{n_{j,E+S,2}}{\tilde{l}}\right)^{\upsilon-1}\left[1-\left(\frac{n_{j,E+S,2}}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1-\upsilon}{\upsilon}} \quad \forall j \tag{A.4.9}$$

$$(\hat{c}_{j,E+S,2})^{-\sigma} = \chi^b_j e^{-g_y\sigma}(\hat{b}_{j,E+S+1,3})^{-\sigma} \quad \forall j \tag{A.4.10}$$

Note that this is four equations (A.4.7), (A.4.8), (A.4.9), and (A.4.10) and four unknowns $n_{j,E+S-1,1}$, $\hat{b}_{j,E+S,2}$, $n_{j,E+S,2}$, and $\hat{b}_{j,E+S+1,3}$.

This process is repeated for every age of individual alive in $t = 1$ down to the age $s = E + 1$ individual at time $t = 1$. Each of these individuals j solves the full set of remaining $S - s + 1$ labor supply decisions, $S - s$ savings decisions, and one intended bequest decision at the end of life. After the full set of lifetime decisions has been solved for all the individuals alive at time $t = 1$, each ability j individual born in period $t \geq 2$ can be solved for, the solution to which is characterized by the following full set of Euler equations analogous to (24), (25), and (26).

$$(\hat{c}_{j,s,t})^{-\sigma}\left(\hat{w}^i_t e_{j,s} - \frac{\partial \hat{T}_{s,t}}{\partial n_{j,s,t}}\right) = \chi^n_s\left(\frac{b}{\tilde{l}}\right)\left(\frac{n_{j,s,t}}{\tilde{l}}\right)^{\upsilon-1}\left[1-\left(\frac{n_{j,s,t}}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1-\upsilon}{\upsilon}} \tag{A.4.11}$$
$$\forall j \quad \text{and} \quad E+1 \leq s \leq E+S \quad \text{and} \quad t \geq 2$$

$$(\hat{c}_{j,s,t})^{-\sigma} = ...$$
$$e^{-g_y\sigma}\left(\rho_s\chi^b_j(\hat{b}_{j,s+1,t+1})^{-\sigma} + \beta(1-\rho_s)(\hat{c}_{j,s+1,t+1})^{-\sigma}\left[(1+r^i_{t+1}) - \frac{\partial \hat{T}_{s+1,t+1}}{\partial b_{j,s+1,t+1}}\right]\right)$$
$$\forall j \quad \text{and} \quad E+1 \leq s \leq E+S-1 \quad \text{and} \quad t \geq 2 \tag{A.4.12}$$

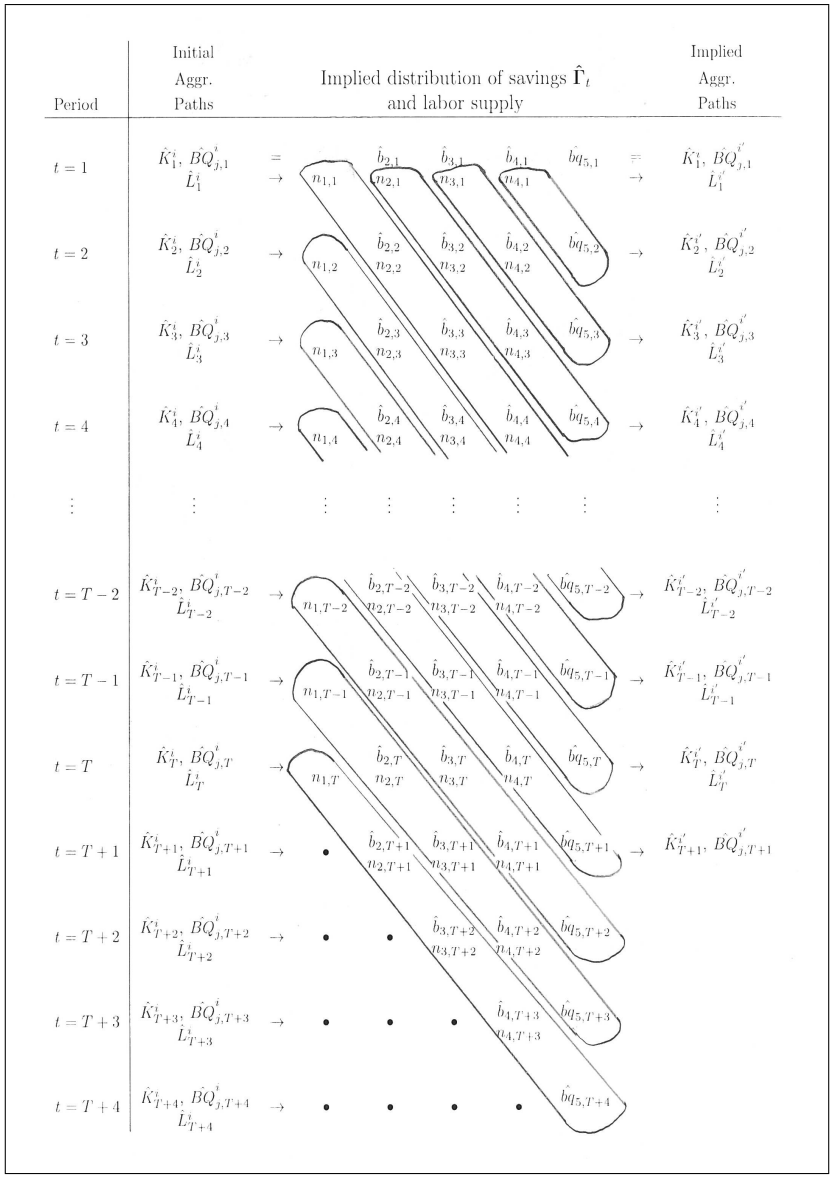
$$(\hat{c}_{j,E+S,t})^{-\sigma} = \chi^b_j e^{-g_y\sigma}(\hat{b}_{j,E+S+1,t+1})^{-\sigma} \quad \forall j \quad \text{and} \quad t \geq 2 \tag{A.4.13}$$

For each individual of ability type j entering the economy in period $t \geq 1$, the entire set of $2S$ lifetime decisions is characterized by the $2S$ equations represented in (A.4.11), (A.4.12), and (A.4.13).

We can then solve for the entire lifetime of savings and labor supply decisions for each age $s = 1$ individual in periods $t = 2, 3, ...T$. The central part of the schematic diagram in Figure 16 shows how this process is done in order to solve for the equilibrium time path of the economy from period $t = 1$ to T. Note that for each full lifetime savings and labor supply path solved for an individual born in period $t \geq 2$, we can solve for the aggregate capital stock and total bequests received implied by those savings decisions $\hat{\boldsymbol{K}}^{i'}$ and $\hat{\boldsymbol{BQ}}^{i'}_j$ and aggregate labor implied by those labor supply decisions $\hat{\boldsymbol{L}}^{i'}$.

Once the set of lifetime saving and labor supply decisions has been computed for all individuals alive in $1 \leq t \leq T$, we use the individual decisions to compute a new

Figure 16: Diagram of TPI solution method within each iteration for $S = 4$ and $J = 1$



implied time path of the aggregate capital stock and aggregate labor. The implied paths of the aggregate capital stock $\hat{\boldsymbol{K}}^{i'} = \{\hat{K}_1^i, \hat{K}_2^{i'}, ...\hat{K}_T^{i'}\}$, aggregate labor $\hat{\boldsymbol{L}}^{i'} = \{\hat{L}_1^i, \hat{L}_2^{i'}, ...\hat{L}_T^{i'}\}$, and total bequests received $\hat{\boldsymbol{BQ}}_j^{i'} = \{\hat{BQ}_{j,1}^i, \hat{BQ}_{j,2}^{i'}, ...\hat{BQ}_{j,T}^{i'}\}$ in general do not equal the initial guessed paths $\hat{\boldsymbol{K}}^i = \{\hat{K}_1^i, \hat{K}_2^i, ...\hat{K}_T^i\}$, $\hat{\boldsymbol{L}}^i = \{\hat{L}_1^i, \hat{L}_2^i, ...\hat{L}_T^i\}$, and $\hat{\boldsymbol{BQ}}_j^i = \{\hat{BQ}_{j,1}^i, \hat{BQ}_{j,2}^i, ...\hat{BQ}_{j,T}^i\}$ used to compute the individual savings and labor supply decisions $\hat{\boldsymbol{K}}^{i'} \neq \hat{\boldsymbol{K}}^i$, $\hat{\boldsymbol{L}}^{i'} \neq \hat{\boldsymbol{L}}^i$, and $\hat{\boldsymbol{BQ}}_j^{i'} \neq \hat{\boldsymbol{BQ}}_j^i$.

Let $\|\cdot\|$ be a norm on the space of time paths of the aggregate capital stock $\hat{\boldsymbol{K}} \in \mathcal{K} \subset \mathbb{R}_{++}^T$, aggregate labor supply $\hat{\boldsymbol{L}} \in \mathcal{L} \subset \mathbb{R}_{++}^T$, and J paths of total bequests received $\hat{\boldsymbol{BQ}}_j \in \mathcal{B} \subset \mathbb{R}_{++}^T$. Then the fixed point necessary for the equilibrium transition path from Definition 2 has been found when the distance between these $J+2$ paths is arbitrarily close to zero.

$$\left\| \left[\hat{\boldsymbol{K}}^{i'}, \hat{\boldsymbol{L}}^{i'}, \left\{\hat{\boldsymbol{BQ}}_j^{i'}\right\}_{j=1}^J\right] - \left[\hat{\boldsymbol{K}}^i, \hat{\boldsymbol{L}}^i, \left\{\hat{\boldsymbol{BQ}}_j^i\right\}_{j=1}^J\right] \right\| \leq \varepsilon \quad \text{for} \quad \varepsilon > 0 \tag{A.4.14}$$

If the fixed point has not been found $\left\| \left[\hat{\boldsymbol{K}}^{i'}, \hat{\boldsymbol{L}}^{i'}, \left\{\hat{\boldsymbol{BQ}}_j^{i'}\right\}_{j=1}^J\right] - \left[\hat{\boldsymbol{K}}^i, \hat{\boldsymbol{L}}^i, \left\{\hat{\boldsymbol{BQ}}_j^i\right\}_{j=1}^J\right] \right\| > \varepsilon$, then new transition paths for the aggregate capital stock and aggregate labor are generated as a convex combination of $\left[\hat{\boldsymbol{K}}^{i'}, \hat{\boldsymbol{L}}^{i'}, \left\{\hat{\boldsymbol{BQ}}_j^{i'}\right\}_{j=1}^J\right]$ and $\left[\hat{\boldsymbol{K}}^i, \hat{\boldsymbol{L}}^i, \left\{\hat{\boldsymbol{BQ}}_j^i\right\}_{j=1}^J\right]$.

$$\begin{aligned}
\hat{\boldsymbol{K}}^{i+1} &= \nu\hat{\boldsymbol{K}}^{i'} + (1-\nu)\hat{\boldsymbol{K}}^i \\
\hat{\boldsymbol{L}}^{i+1} &= \nu\hat{\boldsymbol{L}}^{i'} + (1-\nu)\hat{\boldsymbol{L}}^i \\
\hat{\boldsymbol{BQ}}_1^{i+1} &= \nu\hat{\boldsymbol{BQ}}_1^{i'} + (1-\nu)\hat{\boldsymbol{BQ}}_1^i \qquad \text{for} \quad \nu \in (0,1] \\
&\vdots \\
\hat{\boldsymbol{BQ}}_J^{i+1} &= \nu\hat{\boldsymbol{BQ}}_J^{i'} + (1-\nu)\hat{\boldsymbol{BQ}}_J^i
\end{aligned} \tag{A.4.15}$$

This process is repeated until the initial transition paths for the aggregate capital stock, aggregate labor, and total bequests received are consistent with the transition paths implied by those beliefs and individual and firm optimization.

In essence, the TPI method iterates on individual beliefs about the time path of prices represented by a time paths for the aggregate capital stock $\hat{\boldsymbol{K}}^i$, aggregate labor $\hat{\boldsymbol{L}}^i$, and total bequests received $\hat{\boldsymbol{BQ}}_j^i$ until a fixed point in beliefs is found that are consistent with the transition paths implied by optimization based on those beliefs.

The following are the steps for computing a stationary non-steady-state equilibrium time path for the economy.

1. Input all initial parameters. See Table 2.

    (a) The value for T at which the non-steady-state transition path should have converged to the steady state should be at least as large as the number of periods it takes the population to reach its steady state $\bar{\boldsymbol{\omega}}$ as described in Appendix A-1.

2. Choose an initial distribution of savings and intended bequests $\hat{\mathbf{\Gamma}}_1$ and then calculate the initial state of the stationarized aggregate capital stock $\hat{K}_1$ and total bequests received $\hat{BQ}_{j,1}$ consistent with $\hat{\mathbf{\Gamma}}_1$ according to (29) and (A.4.4).

   (a) Note that you must have the population weights from the previous period $\hat{\omega}_{s,0}$ and the growth rate between period 0 and period 1 $\tilde{g}_{n,1}$to calculate $\hat{BQ}_{j,1}$.

3. Conjecture transition paths for the stationarized aggregate capital stock $\hat{\boldsymbol{K}}^1 = \{\hat{K}_t^1\}_{t=1}^{\infty}$, stationarized aggregate labor $\hat{\boldsymbol{L}}^1 = \{\hat{L}_t^1\}_{t=1}^{\infty}$, and total bequests received $\hat{\boldsymbol{BQ}}_j^1 = \{\hat{BQ}_{j,t}^1\}_{t=1}^{\infty}$ where the only requirements are that $\hat{K}_1^i$ and $\hat{BQ}_{j,1}^i$ are functions of the initial distribution of savings $\hat{\mathbf{\Gamma}}_1$ for all i is your initial state and that $\hat{K}_t^i = \bar{K}$, $\hat{L}_t^i = \bar{L}$, and $\hat{BQ}_{j,t}^i = \bar{BQ}_j$ for all $t \geq T$. The conjectured transition paths of the aggregate capital stock $\hat{\boldsymbol{K}}^i$ and aggregate labor $\hat{\boldsymbol{L}}^i$ imply specific transition paths for the real wage $\hat{\boldsymbol{w}}^i = \{\hat{w}_t^i\}_{t=1}^{\infty}$ and the real interest rate $\boldsymbol{r}^i = \{r_t^i\}_{t=1}^{\infty}$ through expressions (27) and (19).

   (a) An intuitive choice for the time path of aggregate labor is the steady-state in every period $\hat{L}_t^1 = \bar{L}$ for all t.

4. With the conjectured transition paths $\hat{\boldsymbol{w}}^i$, $\boldsymbol{r}^i$, and $\hat{\boldsymbol{BQ}}_j^i$ one can solve for the lifetime policy functions of each individual alive at time $1 \leq t \leq T$ using the systems of Euler equations of the form (24), (25), and (26) and following the diagram in Figure 16.

5. Use the implied distribution of savings and labor supply in each period (each row of $\hat{b}_{j,s,t}$ and $n_{j,s,t}$ in Figure 16) to compute the new implied time paths for the aggregate capital stock $\hat{\boldsymbol{K}}^{i'} = \{\hat{K}_1^i, \hat{K}_2^{i'}, ...\hat{K}_T^{i'}\}$, aggregate labor supply $\hat{\boldsymbol{L}}^{i'} = \{\hat{L}_1^i, \hat{L}_2^{i'}, ...\hat{L}_T^{i'}\}$, and total bequests received $\hat{\boldsymbol{BQ}}_j^{i'} = \{\hat{BQ}_{j,1}^i, \hat{BQ}_{j,2}^{i'}, ...\hat{BQ}_{j,T}^{i'}\}$.

6. Check the distance between the two sets time paths.

$$\left\| \left[\hat{\boldsymbol{K}}^{i'}, \hat{\boldsymbol{L}}^{i'}, \{\hat{\boldsymbol{BQ}}_j^{i'}\}_{j=1}^{J} \right] - \left[\hat{\boldsymbol{K}}^{i}, \hat{\boldsymbol{L}}^{i}, \{\hat{\boldsymbol{BQ}}_j^{i}\}_{j=1}^{J} \right] \right\|$$

   (a) If the distance between the initial time paths and the implied time paths is less-than-or-equal-to some convergence criterion $\varepsilon > 0$, then the fixed point has been achieved and the equilibrium time path has been found (A.4.14).

   (b) If the distance between the initial time paths and the implied time paths is greater than some convergence criterion $\|\cdot\| > \varepsilon$, then update the guess for the time paths according to (A.4.15) and repeat steps (4) through (6) until a fixed point is reached.