

Ceesay, Muhammed

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Secret vs Public Rings in Common Value Auctions

Ceesay Muhammed

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Abstract

For a single-object second-price common value auction with colluding bidders, and assuming an “almost all-inclusive ring”, we analyze whether an auctioneer who knows that a bidding ring is present at the auction should reveal their presence, and if so, whether to make the revelation publicly, or secretly to the non-ring bidder. We show that for a family of value functions, and assuming (where possible) that bidders use symmetric strategies, publicly revealing that a ring is present induces the non-ring bidder to submit a bid higher than the amount he bids when he (the non-ring bidder) is convinced that the auction is purely noncooperative. On one hand, this means that conditional on a ring operating at the auction, the auctioneer may improve his position by publicly announcing the ring presence, rather than keeping the ring concealed. On the other hand, this presents a new way that an auctioneer can cheat at the auction without having to employ skills, as even in the absence of colluding bidders, simply inducing bidders to believe that they are facing a ring causes them to bid higher than they would have.

JEL classification: D44.

Key Terms: *Almost-All-Inclusive Ring, Informational Structures.*

Department of Economics and Statistics, University of Naples Federico II. ceesay.muhammad@unina.it
Present Address: CEIS - Tor Vergata.
muhammed.ceesay@uniroma2.it

This paper is based on Chapter 3 of my PhD Thesis at the University of Naples Federico II. Parts of the introduction are reproduced word-for-word in a poster (available at https://siecon3-607788.c.cdn77.org/sites/siecon.org/files/media_wysiwyg/ceesay.pdf Accessed: 2021-08-03) at the 60th meeting of the Italian Economic Society in 2019 which analyzed the same questions under a private values framework. This paper initially included the private values part, which has since been excluded.

1 Introduction

The competition-inducing nature of auctions make them a widely-used allocation mechanism. Increased competition increases the likelihood that the auctioneer receives higher proceeds from sale. Furthermore, auctions by nature are viewed as “fair” mechanisms as they represent public platforms where all interested parties can compete for the item(s) on offer. In this light, auctions are favored by bidders as well (Rothkopf and Harstad, 1994).

Hence, the possibility that a group of bidders will seek to compromise competition is a primary consideration of an auctioneer (Klemperer, 2002). This has important theoretical ramifications, as several established theorems no longer hold when we account for the possibility that a subset of bidders can engage in bid rigging. In particular, with ex-ante symmetric bidders, collusion induces asymmetry and the revenue equivalence theorem no longer holds (Krishna, 2009). In addition, depending on the size and composition of bidders in the ring, certain auctions are more robust to collusion than others (Robinson, 1985), (Marshall and Marx, 2007).

Bid rigging is a criminal offence in most countries.¹ Individuals or firms who have been found to engage in anticompetitive practices are prosecuted and incur high fines, and in some cases prison sentences. For this reason bidding rings take measures to conceal their presence at auctions.

Early research on the issue of collusion in auctions focused on analysing which auctions were susceptible to manipulation by bidding rings and were mainly geared towards independent private values. Abstracting away from certain internal workings of the ring (incentive to reveal true value to other members), (Robinson, 1985) showed that if bidders were ex-ante symmetric, the second price auction was more susceptible to collusion. (Mailath and Zemsky, 1991) considered ex-ante heterogeneous bidders in second price auctions and showed that it was possible for rings to design incentive compatible mechanisms that deter ring members from shirking on agreed terms at the main auction. A similar approach was taken by (McAfee and McMillan, 1992) for first price auctions with ex-ante homogeneous bidders, analysing cases for which the ring was able to make side payments and otherwise. They showed that provided the ring could enforce members’ bidding strategies at the main auction,² it was possible to design efficient incentive-compatible mechanisms. (Graham and Marshall, 1987) examined bidding rings in Second price and English auctions. Incorporating a paradigm detailing the internal workings of rings, they showed the negative effects of ring presence on sellers revenue and showed that the auctioneer can offset the loss in revenue by imposing a higher reserve price.

Another strand in the literature focuses on measures to deter bidding rings from auctions. One such measure proposed by (Marshall and Marx, 2009) involves restricting information about bidder identity at auctions. This serves to make it difficult for rings to monitor members’ bids at the auction, and gives incentives to ring members to renege on agreed ring terms. Other measures involve the construction of collusive-proof mechanisms for which bidding as a ring is made infeasible (Pavlov, 2008; Che and Kim, 2006, 2009). These collusion-proof measures involve complexities that

¹In the United States, it is a violation of the Sherman Act of 1890.

²Basically employing a *Bid Submission Mechanism* as in (Marshall and Marx, 2007)

make them difficult to implement in practice (Marshall and Marx, 2012).

Given the proliferation of bidding rings, most bidders will go into an auction wary of the fact that they might be competing against bidding rings (Marshall and Marx, 2012).³ For instance, in England, the Office of Fair Trading (OFT) conducted investigations into allegations of bid rigging by 103 construction firms, reported in the decision OFT (2009), in which fines amounting to £192.2 million were imposed on the guilty parties. The decision noted that *Cover Pricing*⁴ was so pervasive a phenomenon that even textbooks on bidding⁵ referenced it and some went as far as providing guidance as to its implementation. Among the testimonies by the guilty parties was

*“Cover pricing is a universally accepted practice within the construction industry and has been so for as far back as most people can remember . . . cover pricing is so common as to be regarded as endemic within the construction industry, and has been practiced by all companies regardless of their size’.”*⁶

Most ring mechanisms involve a designated bidder representing the group at the main auction, while other members remain absent or submit phony bids (usually very close to the seller’s reserve price).

Assuming an auctioneer knows that a bidding ring is operating at the auction, is it in his interest to inform other non-ring bidders about the ring’s presence?

The motivation stems from the analysis of (Pagnozzi, 2011) in ascending auctions with affiliated values, which demonstrated that in addition to reducing competition among each other, bidding rings could also influence other non-ring bidders to bid lower than usual. The ring mechanism employed in this scenario involves all ring members being present at the main auction, with the designated representative bidding competitively, while other ring members drop out early. This premature exit sends misleading signals of a low value to other non-ring bidders, causing them to depress their bids as well. This has the combined effect of further reducing the seller’s revenue and further reducing the probability of winning of the other non-ring bidders, while the ring representative can afford to bid more aggressively as he suffers less winner’s curse. Given that this is the case, it is not too far-fetched to think that a non-ring bidder who is aware of this tactic will take note and will be wary when updating his estimate of the value of the item based on observed dropout prices. In such a case, it might be in the interest of both the non-ring bidder and the seller that ring presence is not concealed. This is because non-ring bidders who are aware of the ring’s tactic are not influenced to depress their bids. Hence the seller can as well expect higher profit compared to when non-ring bidders are oblivious of ring presence.

³Baldwin et al. (1997) examined forest service timber sales in the Pacific Northwest using a dataset spanning the periods 1975 - 1981 and found that “*winning bids are best explained by collusion*” Other empirical studies documenting collusion include Asker (2010), Porter (2005) and Pesendorfer (2000). These studies also allude to the fact that some bidding rings do not necessarily include all the bidders.

⁴A form of Bid rigging in reverse auctions involving a selected bidder who is chosen by the bidding ring to be the winner of the item at an agreed price (usually higher than the fair price) while other ring members submit significantly higher bids, to give a false impression of competition. This leads the principal to pay a higher price than he would have if the bidding process was uncompromised.

⁵OFT (2009) pp. 400 - 401.

⁶OFT (2009) pp. 398.

We argue that the potential benefits to the knowledge of ring presence applies to sealed-bid common value auctions as well. Typically, in standard second price common value auctions, the only information that a bidder believes he can elicit upon winning the auction is the signal of the highest competing bidder. Based on this inference, he makes an estimate of the value of the item. In the event that such a bidder is made aware that he is bidding against a ring, he knows that the ring representative bids based on the signals of the ring members. In essence, when the non-ring bidder wins in such a context, he has the possibility of inferring more information with regards to the signals of his competitors, leading to a better estimate of the value of the item. We show that knowing this may lead the non-ring bidder to adjust his bidding strategy compared to when he is unaware of the ring's presence.⁷ Whether this will be in favor of or against the auctioneer will depend on the nature of the value function. Specifying a family of value functions, we describe a common value auction environment in which the participants determine the value of the item on sale based on their signals and by computing an aggregate of their competitors' signals.

We analyze the case of an *almost all-inclusive ring* - a ring consisting of all-but-one of the bidders- and a valuation structure such that the common value is completely specified by the bidders' signals.⁸ Under this framework, it becomes easy to see how the non-ring bidder changes his behavior with respect to knowing that a ring is present compared to bidding in the noncooperative case. This is due to the fact that with this specification, when the non-ring bidder is oblivious of ring presence, the most he believes he can infer about the value of the item upon winning is the value conditional on his signal and the inferred signal of the highest competitor - based on the winning price. However, when he is aware that he is bidding against the ring, given that the ring representative bids based on the signals of all ring members, upon winning the non-ring bidder in principle can infer the aggregate of his competitors' signals from the price he pays, and hence can infer the actual value of the item. We show that based on how this aggregate is computed, whether or not the non-ring bidder will alter his bidding strategy based on knowledge of ring presence depends on whether the ring presence is publicly known, or is privately known to him. This corresponds to saying that if a seller who knows that a ring is present decides to reveal this information, whether the non-ring bidder will alter his bidding strategy compared to his strategy in the noncooperative case depends on whether the seller reveals this information publicly or secretly

⁷An objection can be raised that when an auctioneer identifies a ring, he should either report this illegal activity to the relevant authorities (not to the other non-ring bidders). However, it is difficult in practice for an auctioneer to perfectly identify the group of conspiring bidders as the ring will take measures to keep their presence and identity concealed to avoid prosecution if identified.

Therefore, in most cases, the best the auctioneer can do is speculate about the ring's presence. This makes reporting to authorities a bit difficult as it requires providing concrete evidence and might involve costly legal proceedings. However, informing supposed non-ring bidders about this speculation does not mandate such requirements.

The idea for our work is to analyze how and whether the auctioneer should reveal this speculation to supposed non-ring bidders. Interpreting the speculation as *knowing that a ring is present* represents an extreme case which assumes (albeit implicitly) that this speculation is correct. Furthermore, the fact that collusion is widespread means that bidders themselves go into an auction suspecting that they may be facing a ring. We try to capture this idea by reasoning that saying that *a bidder suspects ring presence* is equivalent to saying that the auctioneer *informs* bidders of the possible presence of the ring.

⁸That is, the exact value of the item can be inferred based on all bidders' signals

to the non-ring bidder.

We phrase our main questions thus:

- If an auctioneer knows that a ring is present at the auction, should she reveal the ring presence to bidders?
- If yes, how? Should she publicly announce it or reveal it secretly to the non-ring bidders?
- What would be bidders’ responses to these announcements compared to the noncooperative case?

Our work closely parallels that of (Mares and Harstad, 2003), who consider the role of privately revealed information by the seller on expected revenue. However, in contrast to their work, our specification asserts that the actual value of the item is completely determined by bidder signals. Furthermore, the role of auctioneer information in our context is to aid the winner of the auction in estimating the ex-post value via the price paid.

If the computed aggregate of competitors’ signals is less than the arithmetic average of their signals, and signals are uniformly distributed, conditional on ring presence, we show that publicly announcing that a ring is present leads the non-ring bidder to bid higher compared to when the auction is noncooperative. Given that the auctioneer is a revenue maximizer, we also show via simulated estimates of expected revenues that whether or not the auctioneer wants to publicly reveal the ring presence depends on the number of bidders in the auction. In particular, we find that for $N = 5$ bidders, the auctioneer is better-off publicly revealing the ring presence, while for $N = 15$, the auctioneer is better-off keeping the ring concealed. In addition, since publicly announcing that a ring is present leads non-ring bidders to bid higher compared to when they believe that the auction is noncooperative, this gives the auctioneer an incentive to induce bidders to believe a ring is present even when no ring exists, presenting a novel way that an auctioneer may cheat that does not involve shill bidding.

It is worth noting that the conclusion that public revelation may be a good strategy is not a consequence of the analysis of the impact of public information revelation in (Milgrom and Weber, 1982). This is because their analysis rests on the specification that the auctioneer’s revealed information affects the valuation of bidders ex-ante. Our case concerns the scenario for which the exact specification of the value depends only on bidders signals. The impact of auctioneer information in this case only concerns the inferred value upon winning. That is, ex-post.

The rest of the article proceeds as follows: In Section 2, we specify the framework for our analysis. Section 3 is devoted to the characterization of bidding strategies under the various informational structures. We introduce the additional specification that signals are uniformly distributed in Section 4, and we compare expected revenues and profits for the auctioneer and bidders across informational structures. Section 5 shows that the results of section 4 do not necessarily hold under a different valuation structure. Section 6 concludes.

2 Framework

2.1 Pure Common Values

Assume a pure common values framework with $N \geq 3$ risk-neutral bidders $\mathcal{N} = \{1, 2, \dots, N\}$ each receiving an iid signal $X_i \in [0, 1]$ drawn from an atomless distribution $F(\cdot)$ with corresponding density $f(\cdot)$ strictly positive on $[0, 1]$. The common value is V defined by

$$V = \pi(X_1, \dots, X_N)$$

Denote $Y_{\mathcal{M}}^1$ as the largest of X_i for bidders in the set $\mathcal{M} \subset \mathcal{N}$. Also denote $\mathbf{X}_{-i} = \{X_j\}_{j \neq i}$. A salient feature of second price common value auctions is the multiplicity of equilibria (Bikhchandani and Riley, 1991; Krishna, 2009; Milgrom, 1981). Criteria for equilibrium selection in this context usually involves choosing equilibria that survive some form of perturbation of the value function (Larson, 2009). Where applicable, we impose that bidders use continuous symmetric strategies.

2.2 Informational Structures

The following are the informational structures⁹ we will consider

Purely Non-Cooperative Case

This is the scenario when there is no ring at the auction, and everyone is bidding independently. Everyone is an active bidder in this case.

Concealed Case

This is the case for which the ring is present at the auction, but is able to conceal its presence from the Non-Ring Bidder. This corresponds to the auctioneer being silent about his knowledge of ring presence.

Public Knowledge

This is the case for which the ring is present at the auction, and its presence is commonly known to all participants at the auction. This corresponds to the auctioneer publicly revealing his knowledge of ring presence.

Private Knowledge

This is the case for which the ring is present at the auction, and they think they are operating in secret, but the presence of the ring is privately known to the Non-ring bidder. This corresponds to the auctioneer revealing his knowledge of ring presence privately to the non-ring bidder.

⁹The term *Informational Structures* has been used by (Cheng and Tan, 2010) in a similar context.

2.3 Description of the Ring

Let a subset ($\mathcal{K} \subset \mathcal{N}$, $\mathcal{K} = \{1, 2, \dots, k\}$) of bidders decide to form a ring. Suppose for the sake of analysis that the ring is able to design a stable incentive compatible mechanism such that the ring member with the highest signal bids on behalf of the ring, and the other members submit phony bids.

When bidding for the ring, the ring representative knows the signals x_1, x_2, \dots, x_k . When $|\mathcal{K}| = N - 1$ we say that we have an “almost all-inclusive ring”. We use feminine pronouns for the ring representative, and masculine for the non-ring bidder.

2.4 Special family of Valuation Functions

We consider a family of value functions of the form

$$V(X_i, \mathbf{X}_{-i}) = X_i \cdot g(\mathbf{X}_{-i}) = X_j \cdot g(\mathbf{X}_{-j})$$

where

$$g : [0, 1]^{N-1} \longrightarrow [0, 1]$$

is a Borel-Measurable function, increasing and symmetric in all its arguments, and $\mathbf{X}_{-i} \equiv \{X_j\}_{j \neq i}$. We refer to $g(\cdot)$ as the *Auxiliary Signal*.¹⁰

This structure ensures that when the ring is competing against the non-ring bidder, the resulting environment can be interpreted as a two-bidder auction, where the non-ring bidder has signal X_N , and the ring representative has signal $g(\mathbf{X}_{-N})$. This allows us to explore the dynamics in bidding behavior across informational structures, in particular for the non-ring bidder, as in each case, he only has to make inference concerning the auxiliary signal $g(\mathbf{X}_{-N})$.

It would seem natural that the auxiliary signal be computed as the arithmetic average of competitors’ signals. However, we additionally assume that

$$g(\mathbf{X}_{-i}) \leq \left(\frac{1}{N-1}\right) \sum_{j \neq i} X_j \tag{1}$$

This can be interpreted as representing “caution” on the part of each bidder regarding estimating the value of the item based on fellow competitors’ signals. In other words, it reflects some kind of “pessimism” on the part of bidders as far as estimating the value via other signals.

We give the following incident as a motivation for a scenario under which such a value function may apply.

The figure below shows some of the offshore blocks for exploration in The Gambia.

¹⁰Technically, $g(\cdot)$ is also a random variable since we have assumed that it is Borel-Measurable

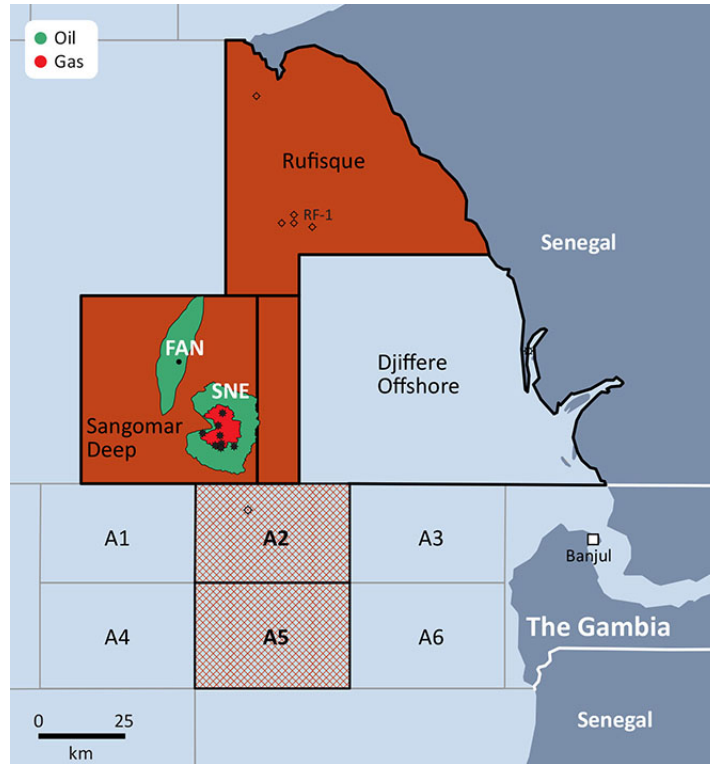


Figure 1: Gambia Oil Tract Adjacent Blocks

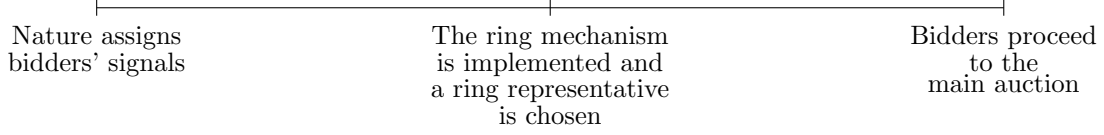
From - Offshore Energy, Accessed July 26 2021, <https://www.offshore-energy.biz/far-suspends-gambian-drilling-plans-as-govt-closes-borders/>

In March 2017, Australian exploration company *Far Limited* secured an 80 percent working interest in Blocks A2 and A5, originally licensed to Erin Energy, while licenses for blocks A4, A3 and A6 were yet to be assigned.¹¹ Blocks A2 and A5 were estimated to contain just over a million barrels of oil. In October 2018, drilling commenced on block A2, and the first well was found to contain “more water than oil” and was abandoned.¹² This development is unfavorable not only for *Far Limited* (who will likely have lower revised estimates of the amount of oil reserve in the blocks), but also for potential investors interested in the remaining unlicensed blocks. If we suppose that all potential investors - future bidders - use the same geoscientific techniques with regards to estimating the amount of oil, then going into the competition stages for the remaining blocks, they will be less confident with regards to estimating the actual amount of oil in the reserve based on their and other competitors’ initial estimates as the remaining blocks are in close proximity with blocks A2 and A5. It is this kind of a scenario that we capture with the assumption $g(\mathbf{X}_{-i}) < \frac{1}{N-1} \sum_{j \neq i} X_j$ in the sense that the final aggregate is less than the average.

¹¹Block A1 has since been assigned to *British Petroleum*

¹²<https://www.ogj.com/exploration-development/discoveries/article/17296333/fars-samo1-well-off-gambia-comes-up-dry> Accessed: 2021-07-26.

2.5 Timing



2.6 Equilibrium

By “Equilibrium” we refer to the fact that based on the information and belief of the bidders, they have no incentive to deviate. This is regardless of the fact that their beliefs may be inaccurate. For instance, in the concealed case, the Non-ring bidder’s belief of the auction being competitive is inaccurate. However, we analyze his strategy based on this incorrect belief.

3 The Second Price Auction

This section is devoted to the characterization of bidders’ equilibrium bidding strategies under the various informational structures. We start by analysing the equilibrium strategy adopted under the noncooperative case, using it as a benchmark. Then we proceed to strategies adopted under the Public Knowledge, Concealed and Private Knowledge Cases respectively. The analysis will center on the value function $X_i \cdot g(\mathbf{X}_{-i})$ and assumes that the ring is *almost all-inclusive*.

3.1 The Purely Non-Cooperative Equilibrium

When all bidders bid individually, there exists a symmetric increasing equilibrium strategy characterized below. Define

$$v(x, y) = \mathbb{E}[V|X_i = x_i, Y_{\mathcal{N}\setminus\{i\}}^1 = y]$$

Theorem 3.1. (Theorem 6, [Milgrom and Weber \(1982\)](#)) *The unique symmetric equilibrium strategy $\beta(\cdot)$ is given by*

$$\beta(x_i) = v(x_i, x_i) = \mathbb{E}[V|X_i = x_i, Y_{\mathcal{N}\setminus\{i\}}^1 = x_i] \tag{2}$$

$$\forall i \in \mathcal{N}$$

The intuition for the theorem is as follows. Since all bidders other than bidder i use $\beta(\cdot)$, suppose bidder i wins when he submits a bid b and he pays a price $\bar{\theta}$. He infers that $\beta(Y_{\mathcal{N}\setminus\{i\}}^1) = \bar{\theta} \iff Y_{\mathcal{N}\setminus\{i\}}^1 = \beta^{-1}(\bar{\theta})$, and his expected value for the item conditional on winning is

$$\mathbb{E}[V|X_i = x_i, Y_{\mathcal{N}\setminus\{i\}}^1 = \beta^{-1}(\bar{\theta})]$$

and he is happy to win so long as he makes a profit. That is, as long as

$$\bar{\theta} \leq \mathbb{E}[V|X_i = x_i, Y_{\mathcal{N}\setminus\{i\}}^1 = \beta^{-1}(\bar{\theta})]$$

So the maximum amount he will bid is a θ such that he breaks even upon winning. That is θ satisfying

$$\theta = \mathbb{E}[V|X_i = x_i, Y_{\mathcal{N}\setminus\{i\}}^1 = \beta^{-1}(\theta)] \quad (3)$$

$\beta(\cdot)$ being an equilibrium bidding strategy requires $\theta = \beta(x_i)$. Substituting $\theta = \beta(x_i)$ into (3), we get

$$\beta(x_i) = \mathbb{E}[V|X_i = x_i, Y_{\mathcal{N}\setminus\{i\}}^1 = x_i] = v(x_i, x_i)$$

With respect to $V = X_i \cdot g(\mathbf{X}_{-i})$, (2) becomes

Under Noncooperation, the symmetric equilibrium bidding strategy is

$$\beta(x_i) = x_i \cdot \mathbb{E}[g(\mathbf{X}_{-i})|Y_{\mathcal{N}\setminus\{i\}}^1 = x_i] \quad (4)$$

And the auctioneer's expected profit is

$$\mathbb{E}[\mathbb{R}_{\text{No Ring}}] = \mathbb{E}_{\text{2nd Max}}\{\{\beta(x_i)\}_{i \in \mathcal{N}}\}$$

$\mathbb{E}_{\text{2nd Max}}$ is short-hand for *Expected 2nd-Highest Value*.

3.2 Equilibrium Behavior under Public Knowledge

Suppose that the presence of the ring is common knowledge among bidders. In this case, the auction can be interpreted as a two-bidder auction with asymmetric bidders, with the non-ring bidder (without loss of generality) having signal x_N and the ring representative having signal $\mathbf{x}_R = g(\mathbf{x}_{-N})$.

In the derivation of the strategy $\beta(\cdot)$ (the strategy under Noncooperation), there was no need to impose symmetry of distribution of signals. In essence, provided the value function remains the same, the main determinant - other than his own signal - of the strategy of any player is the highest competing bid, from which he can infer the signal of the highest competing bidder. This suggests that the equilibrium bids when the presence of the ring is common knowledge will have a similar structure, as the signals in this case are X_N and $g(\mathbf{X}_{-N}) = \mathbf{X}_R$. We derive the equilibrium strategies below

$$\text{Define } \tilde{v}(a, b) = \mathbb{E}[V|X_i = a, g(\mathbf{X}_{-i}) = b]$$

Theorem 3.2. *When ring presence is common knowledge, there exists a continuum of equilibria*

$$\begin{aligned}\tilde{\beta}_N(x_N) &= \tilde{v}(x_N, \phi(x_N)) \\ \tilde{\beta}_R(\mathbf{x}_R) &= \tilde{v}(\phi^{-1}(\mathbf{x}_R), \mathbf{x}_R)\end{aligned}$$

where

$$\phi : [0, 1] \longrightarrow [0, 1]$$

is increasing and surjective.

Proof. The approach is identical to (Krishna, 2009). First, since $g(\cdot)$ is Borel-Measurable, $g(\mathbf{X}_{-N})$ is also a random variable. In that sense, $g(\mathbf{X}_{-N})$ can be interpreted as the signal of the ring bidder. And the rest of the proof follows as in (Krishna, 2009), pg 118-119. □

The equilibrium strategies above are similar to that of (Milgrom, 1981) and (Bikhchandani and Riley, 1991), who prove the existence of a continuum of asymmetric equilibrium strategies in second-price auction with symmetric bidders.¹³ The focus of the analysis will be on symmetric strategies. In this regard, we restrict to equilibria for which $\phi(x) = x$, so that

Corollary 3.1. *The symmetric equilibrium strategies are*

$$\begin{aligned}\tilde{\beta}_N(x_N) &= \tilde{v}(x_N, x_N) \\ \tilde{\beta}_R(\mathbf{x}_R) &= \tilde{v}(\mathbf{x}_R, \mathbf{x}_R)\end{aligned}$$

With respect to $V = X_i \cdot g(\mathbf{X}_{-i})$,

Under Public Knowledge, the equilibrium bidding strategies adopted by the ring representative and the non-ring bidder are respectively

$$\tilde{\beta}_R(\mathbf{x}_R) = \mathbf{x}_R^2 \text{ and } \tilde{\beta}_N(x_N) = x_N^2$$

The auctioneer's expected revenue is

$$\mathbb{E}[\mathbb{R}_{\text{Public}}] = \mathbb{E}_{2\text{nd Max}}\{\tilde{\beta}_N(x_N), \tilde{\beta}_R(\mathbf{x}_R)\}$$

At this point, we can compare the bidding strategies employed in this case to those used in the noncooperative case. In particular,

Proposition 1. *Compared to the noncooperative case, the non-ring bidder submits a higher bid under common knowledge. That is,*

$$\beta(x_N) \leq \tilde{\beta}_N(x_N)$$

¹³(Milgrom, 1981) considered the case for a two-bidder auction. (Bikhchandani and Riley, 1991) generalized the result to the N -bidder case

Proof. By definition, $X_j \leq Y_{\mathcal{N} \setminus \{N\}}^1$, $\forall j \neq N$. That is,

$$X_1 \leq Y_{\mathcal{N} \setminus \{N\}}^1, X_2 \leq Y_{\mathcal{N} \setminus \{N\}}^1, \dots, X_{N-1} \leq Y_{\mathcal{N} \setminus \{N\}}^1$$

Adding the inequalities yields

$$\sum_{j \neq N} X_j \leq (N-1) \cdot Y_{\mathcal{N} \setminus \{N\}}^1 \implies \left(\frac{1}{N-1}\right) \sum_{j \neq N} X_j \leq Y_{\mathcal{N} \setminus \{N\}}^1 \quad (5)$$

By assumption, $g(\mathbf{X}_{-N}) \leq \left(\frac{1}{N-1}\right) \sum_{j \neq N} X_j$. Coupled with (5), we get

$$g(\mathbf{X}_{-N}) \leq Y_{\mathcal{N} \setminus \{N\}}^1$$

So

$$Y_{\mathcal{N} \setminus \{N\}}^1 = x_N \implies g(\mathbf{X}_{-N}) \leq x_N$$

Therefore,

$$\mathbb{E}[g(\mathbf{X}_{-N}) | Y_{\mathcal{N} \setminus \{N\}}^1 = x_N] \leq x_N \iff \underbrace{x_N \cdot \mathbb{E}[g(\mathbf{X}_{-N}) | Y_{\mathcal{N} \setminus \{N\}}^1 = x_N]}_{\beta(x_N)} \leq \underbrace{x_N^2}_{\tilde{\beta}_N(x_N)} \quad (6)$$

□

The intuition behind this result follows from the following observation: Under noncooperation, the only information that the non-ring bidder can infer upon winning the item is the highest signal of his competitors. However, under public knowledge, upon winning, the non-ring bidder is able to infer the entire aggregate. In this sense, the non-ring bidder's inferred estimate of the value of the item is more informative under public knowledge than under noncooperation. Hence, the non-ring bidder suffers less winner's curse compared to the noncooperative case, leading him to want to bid more aggressively.

3.3 Equilibrium Behavior under the Concealed Case

Now we consider the case under which the ring is able to conceal its presence from the non-ring bidder (and the auctioneer). In this scenario, the non-ring bidder still maintains the noncooperative bidding strategy $\beta(\cdot)$, as he believes he is bidding competitively against other $N-1$ identical bidders. The objective of the ring representative will be to best respond to $\beta(\cdot)$ conditional on the reported ring signals.

3.3.1 The General Case

We derive the equilibrium bidding strategy for the ring representative in a very general context involving an arbitrary set of ring members \mathcal{K} , and the general value function $V = \pi(X_1, X_2, \dots, X_N)$ specified in Section 2.1.

Theorem 3.3. *When the ring is able to operate undetected in the auction, the unique equilibrium bidding strategy for the ring representative is to bid a value ω , such that*

$$\omega = \mathbb{E}[V|X_1 = x_1, \dots, X_k = x_k, Y_{\mathcal{N} \setminus \mathcal{K}}^1 = \beta^{-1}(\omega)]$$

Proof. In the case where the ring representative wins the auction at price ρ , she can infer (via the price she pays) the signal of the highest non-ring bidder, since this bidder uses the equilibrium strategy $\beta(\cdot)$,

$$\beta(Y_{\mathcal{N} \setminus \mathcal{K}}^1) = \rho \iff Y_{\mathcal{N} \setminus \mathcal{K}}^1 = \beta^{-1}(\rho)$$

Therefore, the ring bidder's estimated valuation conditional on winning is

$$\mathbb{E}[V|X_1 = x_1, \dots, X_k = x_k, Y_{\mathcal{N} \setminus \mathcal{K}}^1 = \beta^{-1}(\rho)]$$

So she is happy to win so long as the price ρ is at most equal to her estimated value, otherwise she makes a loss.

That is, as long as

$$\rho \leq \mathbb{E}[V|X_1 = x_1, \dots, X_k = x_k, Y_{\mathcal{N} \setminus \mathcal{K}}^1 = \beta^{-1}(\rho)]$$

Therefore, the highest she will bid is a ρ such that she breakseven. That is, ρ such that

$$\rho = \mathbb{E}[V|X_1 = x_1, \dots, X_k = x_k, Y_{\mathcal{N} \setminus \mathcal{K}}^1 = \beta^{-1}(\rho)]$$

□

Corollary 3.2. *For the family of valuation functions $V = u(X_i, g(\mathbf{X}_{-i}))$ and $|\mathcal{K}| = N - 1$, the equilibrium bidding strategy for the ring representative when the ring believes it is concealed, $\bar{\beta}_R(\mathbf{x}_R)$ is characterised by*

$$\bar{\beta}_R(\mathbf{x}_R) = \omega = \mathbb{E}[V|g(\mathbf{X}_{-N}) = \mathbf{x}_R, X_N = \beta^{-1}(\omega)]$$

With respect to $V = X_i \cdot g(\mathbf{X}_{-i})$,

When the ring is concealed, the equilibrium bidding strategies for the ring representative and the non-ring bidder are respectively

$$\bar{\beta}_R(\mathbf{x}_R) = \omega = \mathbb{E}[V|g(\mathbf{X}_{-N}) = \mathbf{x}_R, X_N = \beta^{-1}(\omega)] \equiv \omega = x_R \cdot \beta^{-1}(\omega)$$

and

$$\bar{\beta}_N(x_N) = \mathbb{E}[V|X_N = x_N, Y_{N \setminus \{N\}}^1 = x_N] \equiv x_N \cdot \mathbb{E}[g(x_{-N})|Y_{N \setminus \{N\}}^1 = x_N]$$

and the auctioneer's expected revenue is

$$\mathbb{E}[\mathbb{R}_{\text{Concealed}}] = \mathbb{E}_{2^{\text{nd Max}}} \{\bar{\beta}_R(\mathbf{x}_R), \bar{\beta}_N(x_N)\}$$

3.4 Equilibrium Behavior under Private Knowledge

Now we consider the situation where the non-ring bidder is aware of the ring, but the ring operates with the belief that their presence is concealed. In other words, when the auctioneer reveals the presence of the ring secretly to the non-ring bidder. Denote the bidding strategies adopted by the ring representative and the non-ring bidder as $\hat{\beta}_R(\mathbf{x}_R)$ and $\hat{\beta}_N(x_N)$ respectively. Since the ring is unaware that their presence is known, the ring representative employs the same strategy as in the concealed case. That is, $\hat{\beta}_R(\mathbf{x}_R) = \bar{\beta}_R(\mathbf{x}_R)$. The objective of the non-ring bidder will be to best-respond to $\hat{\beta}_R(\mathbf{x}_R)$.

Theorem 3.4. *Under private knowledge, the non-ring bidder does not change his bid relative to the noncooperative case. That is*

$$\hat{\beta}_N(x_N) = \beta(x_N)$$

Proof. If he wins at a price ω , the non-ring bidder can infer the realized value of $g(\mathbf{X}_{-N})$ from the amount he pays. His inferred value conditional on winning is

$$\mathbb{E}[V|X_N = x_N, g(\mathbf{X}_{-N}) = \mathbf{x}_R]$$

He makes profit provided that

$$\underbrace{x_N \cdot \mathbf{x}_R}_{\substack{\text{His inferred} \\ \text{value} \\ \text{conditional} \\ \text{on winning}}} > \underbrace{\beta^{-1}(\omega) \cdot \mathbf{x}_R}_{\text{The amount He pays}} \quad (7)$$

and this is satisfied when $x_N > \beta^{-1}(\omega) \implies \beta(x_N) > \omega$. Therefore, conditional on winning,

- Gaining corresponds to the condition $\beta(x_N) > \omega$.
- Making losses corresponds to the condition $\beta(x_N) < \omega$.
- Being indifferent to winning or losing corresponds to the condition $\beta(x_N) = \omega$.

We show that the non-ring bidder does not benefit from bidding higher or lower than $\beta(x_N)$.

Suppose he bids higher. That is, he submits a bid $\bar{b} > \beta(x_N)$. There are three possible cases that may arise:

- Case 1: $\omega < \beta(x_N) < \bar{b}$
 - In this case, the non ring bidder wins, and makes profit since $\omega < \beta(x_N)$. Therefore, bidding \bar{b} and $\beta(x_N)$ yield the same outcome.
- Case 2: $\beta(x_N) < \omega < \bar{b}$
 - In this case, the non ring bidder wins, but loses money as $\beta(x_N) < \omega$. Bidding $\beta(x_N)$ is better as the non-ring bidder loses nothing.
- Case 3: $\beta(x_N) < \bar{b} < \omega$
 - In this case, the non ring bidder loses, and is happy to lose, since $\beta(x_N) < \omega$. Therefore, bidding \bar{b} and $\beta(x_N)$ yields the same outcome.

$$\beta(x_N) \text{ weakly dominates any } \bar{b} > \beta(x_N)$$

I follow a similar argument to show that he does not profit from bidding lower.

Suppose he bids lower. That is, he submits a bid $\underline{b} < \beta(x_N)$. Similarly, we can expect three possible scenarios:

- Case 1: $\omega < \underline{b} < \beta(x_N)$
 - In this case, the non ring bidder wins, and makes profit since $\omega < \beta(x_N)$. In essence he is not hurt by depressing his bid. Bidding $\beta(x_N)$ and \underline{b} yield the same outcome.
- Case 2: $\underline{b} < \omega < \beta(x_N)$
 - In this case, the non ring bidder loses, but could have won by bidding $\beta(x_N)$, and made a positive profit. Hence in this case, depressing his bid lost him potential profit. Bidding $\beta(x_N)$ is better as the non-ring bidder would have won and made profit.
- Case 3: $\underline{b} < \beta(x_N) < \omega$
 - In this case, the non ring bidder loses, and is happy to lose, since $\beta(x_N) < \omega$. Bidding $\beta(x_N)$ and \underline{b} yield the same outcome.

$$\beta(x_N) \text{ weakly dominates any } \underline{b} < \beta(x_N)$$

Altogether, we conclude that the best response for the non-ring bidder is $\beta(x_N)$. □

With respect to $V = X_i \cdot g(\mathbf{X}_{-i})$,

Under Private Knowledge, the equilibrium strategy profile for the ring representative and the non-ring bidder are respectively:

$$\hat{\beta}_R(\mathbf{x}_R) = \omega = \mathbb{E}[V|g(\mathbf{X}_{-N}) = \mathbf{x}_R, X_N = \beta^{-1}(\omega)] \equiv \omega = \mathbf{x}_R \cdot \beta^{-1}(\omega)$$

and

$$\hat{\beta}_N(x_N) = \mathbb{E}[V|X_N = x_N, Y_{N \setminus \{N\}}^1 = x_N] \equiv x_N \cdot \mathbb{E}[g(x_{-N})|Y_{N \setminus \{N\}}^1 = x_N]$$

and the auctioneer's expected revenue is

$$\mathbb{E}[\mathbb{R}_{\text{Private}}] = \mathbb{E}_{2\text{nd Max}}\{\hat{\beta}_R(\mathbf{x}_R), \hat{\beta}_N(x_N)\}$$

The auctioneer's expected profit is

$$\mathbb{E}[\mathbb{R}_{\text{Private}}] = \mathbb{E}_{2\text{nd Max}}\{\omega, \beta(x_N)\}$$

We can hence state the following proposition

Proposition 2. *When the presence of the ring is privately revealed to the non-ring bidder, he does not change his bidding strategy with respect to the noncooperative case.*

What this result emphasizes is the fact that since aggregation of other bidders' information is characterized by caution, only knowing that there is a ring seems not a sufficient reason for the non-ring bidder to change his bidding strategy. Rather, he needs in addition that the ring representative be aware that he (the non-ring bidder) knows about the ring presence.

Corollary 3.3. *The auctioneer does not benefit from private revelation compared to no revelation*

Remark 1: The result above indicates that the non-ring bidder has the same best response to $N - 1$ noncooperating bidders as he does against the ring representative under private knowledge.

Remark 2: This implies that the pair of private knowledge strategies correspond to an asymmetric equilibrium of the public knowledge case. Hence our analysis of the distinction in bidder behavior between the private and public knowledge cases is contingent on our restriction that equilibrium strategies are symmetric in the public knowledge case.

Remark 3: Furthermore, it implies that the concealed and the private knowledge cases are equivalent.

3.4.1 A New Way of Cheating?

We can analyze the benefits (costs) to the auctioneer when bidders suspect the presence of a ring, when there is none. The results vary depending on whether they expect a public or private ring.

When they expect a private knowledge scenario, they use the strategy $\beta(x_i)$, and the auctioneer's expected profit is

$$\mathbb{E}[\mathbb{R}_{\text{Suspect}}^{\text{Private}}] = \mathbb{E}_{2\text{nd Max}}\{\beta(x_i)\}$$

which is the same expected revenue as when bidders expect that there is no ring. That is

$$\mathbb{E}[\mathbb{R}_{\text{No Ring}}] = \mathbb{E}[\mathbb{R}_{\text{Suspect}}^{\text{Private}}]$$

while when they expect a public knowledge scenario, they use the strategy $\tilde{\beta}_i(x_i)$, and the auctioneer's expected profit is

$$\mathbb{E}[\mathbb{R}_{\text{Suspect}}^{\text{Public}}] = \mathbb{E}_{2\text{nd Max}}\{\tilde{\beta}_i(x_i)\}$$

Since $\beta(x_i) < \tilde{\beta}_i(x_i)$,

$$\underbrace{\mathbb{E}_{2\text{nd Max}}\{\beta(x_i)\}}_{\mathbb{E}[\mathbb{R}_{\text{No Ring}}]} < \underbrace{\mathbb{E}_{2\text{nd Max}}\{\tilde{\beta}_i(x_i)\}}_{\mathbb{E}[\mathbb{R}_{\text{Suspect}}^{\text{Public}}]}$$

and we can state the following proposition.

Proposition 3. *In the absence of a ring, the auctioneer can improve his position by convincing bidders that they are facing a public ring.*

This presents a novel way that the auctioneer can induce bidders to bid higher than usual without having to resort to employing shill bidders.

3.4.2 A Brief Recap of The Main Ideas

The non-ring bidder bids higher under Public Knowledge compared to the noncooperative case.

Privately revealing the ring presence to the non-ring bidder does not cause him to change his behavior with respect to the noncooperative case.

This implies that the auctioneer can make bidders bid higher by convincing them that they are facing a public ring. Hence, even when there is no ring at the auction, an auctioneer can artificially raise prices in this way.

4 Characterization under the Uniform Distribution

We introduce the additional specification that the underlying signals are uniformly distributed. This is motivated by the fact that the implicit nature of the equilibrium bidding strategy of the ring representative under private knowledge means it is not immediately clear how to express the auctioneer's expected revenue in this case. We further simplify by fixing a functional form for the auxiliary signal $g(\cdot)$. This is done by assuming a value function of the form

$$V(X_1, X_2, \dots, X_n) = \prod_{i=1}^n X_i$$

with each X_i independently and uniformly distributed on $[0, 1]$.

In the event of ring presence,

- The non-ring bidder (Bidder N) knows only his signal x_N .
- The ring representative knows all signals but x_N .

We can rewrite the value function as

$$V(X_1, \dots, X_N) = \prod_i X_i = X_i \cdot \left(\prod_{j \neq i} X_j \right) = X_i \cdot g(\mathbf{X}_{-i})$$

where $g(\mathbf{X}_{-i}) = \prod_{j \neq i} X_j$. Note that $g(\mathbf{X}_{-i})$ is indeed lower than $\frac{1}{N-1} \sum_{j \neq i} X_j$.¹⁴

These additional modifications allow us to explicitly derive the equilibrium bidding strategies under the different informational structures and make relevant comparative static analyzes relating to auctioneer expected revenue and bidders' expected profit.

4.1 Equilibrium Strategies under Noncooperation

Under the purely Non-Cooperative case, each bidder uses the strategy¹⁵

$$\beta(x_i) = v(x_i, x_i) = \mathbb{E}[V | X_i = x_i, Y_{\mathcal{N} \setminus \{i\}}^1 = x_i] = x_i \cdot x_i \cdot \underbrace{\frac{\mathbb{E}[x_j | x_j < x_i]_{j \neq i}}{2} \dots \frac{x_i}{2}}_{N-2 \text{ times}} \quad (8)$$

¹⁴This follows from

$$g(\mathbf{X}_{-i}) = \underbrace{\prod_{j \neq i} X_j}_{\substack{\text{Since the} \\ X_j \text{'s are between} \\ 0 \text{ and } 1}} \leq \left(\prod_{j \neq i} X_j \right)^{\frac{1}{N-1}} \leq \underbrace{\frac{1}{N-1} \sum_{j \neq i} X_j}_{\substack{\text{Via the} \\ \text{Inequality of} \\ \text{Arithmetic} \\ \text{and Geometric} \\ \text{means}}}$$

¹⁵Note that in general, for $V = \prod_i X_i$ and any arbitrary distribution $F(\cdot)$ with density $f(\cdot)$, we have the following

$$v(x, x) = x^2 \int_0^x \int_0^{y_{\mathcal{N} \setminus \{i\}}^2} \dots \int_0^{y_{\mathcal{N} \setminus \{i\}}^{N-2}} \left(\prod_{s=2}^{N-2} y_{\mathcal{N} \setminus \{i\}}^s \right) \frac{(N-2)! \prod_{s=2}^{N-2} f(y_{\mathcal{N} \setminus \{i\}}^s)}{F(x)^{N-2}} dy_{\mathcal{N} \setminus \{i\}}^{N-2} \dots dy_{\mathcal{N} \setminus \{i\}}^2$$

where $y_{\mathcal{N} \setminus \{i\}}^s$ is the sth-highest signal from bidders in $\mathcal{N} \setminus \{i\}$.

The Symmetric Equilibrium Bidding Strategy under Noncooperation is

$$\beta(x_i) = \frac{x_i^N}{2^{N-2}}$$

4.2 Equilibrium Strategies under Public Knowledge

As in Theorem 3.2,

The symmetric equilibrium strategy profile under Public knowledge is

$$\begin{aligned}\tilde{\beta}_R(\mathbf{x}_R) &= (\mathbf{x}_R)^2 \\ \tilde{\beta}_N(x_N) &= (x_N)^2\end{aligned}$$

4.3 Equilibrium Strategies under the Concealed case

The non-ring bidder bids as in the noncooperative case. That is, he uses the strategy $\bar{\beta}_N(x_N) = \frac{x_N^N}{2^{N-2}}$. The strategy for the ring representative is given by Corollary 3.2 as $\omega = \mathbf{x}_R \cdot \beta^{-1}(\omega)$.

$\beta^{-1}(\omega) = (2^{N-2} \cdot \omega)^{\frac{1}{N}}$, which implies

$$\omega = \mathbf{x}_R \cdot (2^{N-2} \cdot \omega)^{\frac{1}{N}} \implies \bar{\beta}_R(\mathbf{x}_R) = \omega = (\mathbf{x}_R)^{\frac{N}{N-1}} 2^{\frac{N-2}{N-1}}$$

The equilibrium strategy profile under the Concealed case is

$$\begin{aligned}\bar{\beta}_R(\mathbf{x}_R) &= (\mathbf{x}_R)^{\frac{N}{N-1}} 2^{\frac{N-2}{N-1}} \\ \bar{\beta}_N(x_N) &= \frac{x_N^N}{2^{N-2}}\end{aligned}$$

4.4 Equilibrium Strategies under Private Knowledge

In the private knowledge case, the ring believes that it is concealed. Hence, the strategy for the ring representative is the same as in the Concealed case. The non-ring bidder has to best-respond to this strategy. Theorem 3.4 implies that the non-ring bidder does not bid differently from the concealed case. That is, the non-ring bidder maintains the strategy $\frac{x_N^N}{2^{N-2}}$. The fact that the non-ring bidder behaves the same seems counterintuitive. Indeed, to fix this idea, we can use a breakeven price argument.

Suppose the non-ring bidder wins at a price ρ . This implies that

$$\rho = (\mathbf{x}_R)^{\frac{N}{N-1}} 2^{\frac{N-2}{N-1}} \implies \mathbf{x}_R = 2^{\frac{2-N}{N}} \rho^{\frac{N-1}{N}}$$

Therefore, his inferred value for the item upon winning is

$$x_N \cdot 2^{\frac{2-N}{N}} \rho^{\frac{N-1}{N}}$$

and he is happy to win provided he makes a profit. That is, provided

$$\rho \leq x_N \cdot 2^{\frac{2-N}{N}} \rho^{\frac{N-1}{N}}$$

Therefore, the maximum he will bid is an amount such that he breaksevens. That is, a $\bar{\rho}$ such that

$$\bar{\rho} = x_N \cdot 2^{\frac{2-N}{N}} \bar{\rho}^{\frac{N-1}{N}} \implies \bar{\rho} = \frac{x_N^N}{2^{N-2}}$$

Essentially the result implied by Theorem 3.4.

The strategy profile under private knowledge is

$$\hat{\beta}_R(\mathbf{x}_R) = (\mathbf{x}_R)^{\frac{N}{N-1}} 2^{\frac{N-2}{N-1}}$$

$$\hat{\beta}_N(x_N) = \frac{x_N^N}{2^{N-2}}$$

4.5 Auctioneer Revenue Analysis

Now we consider the implications of the aforementioned information structures on seller's revenue, using the formulation $V = \prod_i X_i$, with $N = 5$ bidders (4 in the ring) and then with $N = 15$ bidders (14 in the ring).

We simulate the auctioneer's expected revenue and bidders' expected profit across the different informational structures. The results are presented in Table 1 below.

Informational Structure	Seller Expected Revenue	Ring Member Expected Profit	non-ring Bidder Expected Profit
NonCooperative	0.02782098	0.0006937737	0.0006937737
Secret / Private Ring	0.009461446	0.005287447	0.0006791732
Public Ring	0.009748861	0.0001632564	0.02088852
Convince of Public Ring	0.4763		

Table 1: Expected Revenue and Profits Across Informational Structures for $N = 5$

The numbers in blue are used to signify the best case scenario, while the ones in red indicate the worst case respectively. Secret Ring refers to the ring under the Concealed case. Private Ring refers to the ring under the Private Knowledge case, while Public Ring refers to the ring under

Public Knowledge.

We find that

- Compared to the Secret, Public and Private Ring cases, the auctioneer is most happy when every party bids noncooperatively. However, in the event that a ring is present at the auction, he prefers that their presence be made public.
- The best case for the non-ring bidder is under public knowledge. The rationale is pretty straightforward. By construction, the non-ring bidder starts as the *Stronger* bidder, in the sense that the signal of the non-ring bidder *First Order Stochastically Dominates* that of the ring representative.¹⁶ Under public knowledge, both the ring representative and the non-ring bidder use the same strategies. This implies that it is more likely that the non-ring bidder wins the item.
- For the ring representative, his best case is under the Concealed and Private Knowledge cases.
- In the absence of a ring, convincing bidders that they are facing a public ring significantly raises the auctioneer’s expected revenue compared to the Noncooperative case.

Basically, in terms of auctioneer expected revenue, for $N = 5$, we find that

$$\mathbb{E}[\mathbb{R}_{\text{No Ring}}] > \mathbb{E}[\mathbb{R}_{\text{Public}}] > \mathbb{E}[\mathbb{R}_{\text{Concealed / Private}}]$$

In addition, we compare the probability of winning across the informational structures

Informational Structure	Ring Representative	non-ring Bidder
NonCooperative	0.2	0.2
Secret / Private Ring	0.664689	0.335311
Public Ring	0.062264	0.937736

Table 2: Probability of Winning Across Informational Structures

In terms of chances of winning, the best case for the non-ring bidder is the public knowledge case. However, for the ring representative, the best case is private knowledge case. In fact, ring members are worse off under the public ring case than the noncooperative case. In essence, we are presented with a scenario for which the ring only benefits from operating when they believe their presence is concealed. That is, when the ring operates at the auction, they endeavor to conceal their presence.

However, this revenue ranking does not hold for all N . In particular, for $N = 15$, conditional on ring presence, the auctioneer expected revenues under the Concealed, Public Knowledge and Private Knowledge cases are

¹⁶Since the non-ring bidder’s signal is drawn from a Uniform $[0,1]$, while the ring representative’s is drawn from the product of $N - 1$ Uniform $[0,1]$ s.

Informational Structure	Seller Expected Revenue
Secret / Private Ring	1.635401e-06
Public Ring	2.156533e-07

Table 3: Expected Revenues for $N = 15$

We find that in contrast to the case with $N = 5$, conditional on ring presence, the auctioneer is better off keeping the presence of the ring secret.

That is, for $N = 15$,

$$\mathbb{E}[\mathbb{R}_{\text{Concealed / Private}}] > \mathbb{E}[\mathbb{R}_{\text{Public}}]$$

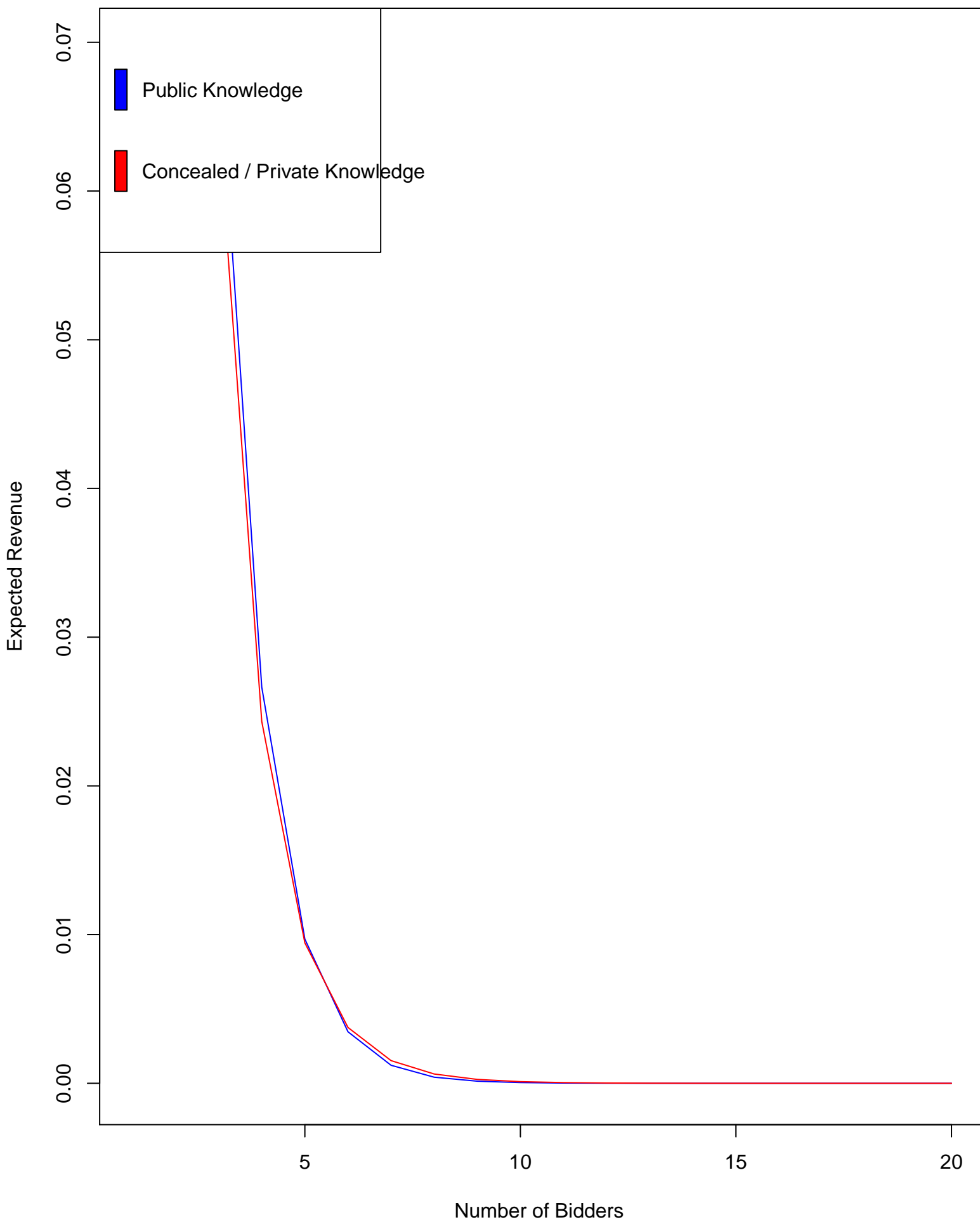
implying that the preference ranking between secret and public rings is contingent on the number of bidders.

This seeming reversal in revenue ordering can be explained via the following observation. Consider the fact that the equilibrium strategy for the ring representative under public knowledge and private knowledge are $\tilde{\beta}_R(\mathbf{x}_R) = (\mathbf{x}_R)^2$ and $\hat{\beta}_R(\mathbf{x}_R) = (\mathbf{x}_R)^{\frac{N}{N-1}} 2^{\frac{N-2}{N-1}}$ respectively. Note that for $N \geq 3$, $\hat{\beta}_R(\mathbf{x}_R) > \tilde{\beta}_R(\mathbf{x}_R)$ since $2 > \mathbf{x}_R$. Basically, despite the fact that the non-ring bidder bids higher under public knowledge compared to the private knowledge / concealed case, this is accompanied by lower bidding by the ring representative compared to the private knowledge case. This yields the implication that the revenue ranking between the Secret / Private Ring and the Public Ring cases is not universal with respect to the number of bidders.

Conditional on ring presence, the revenue ranking for the auctioneer between the Private Knowledge and the Public Knowledge cases depends on the number of bidders in the auction. If the number of bidders are sufficiently small, the auctioneer is better-off under public knowledge. Otherwise the auctioneer might be better-off keeping the ring concealed.

The figure below depicts simulated values of revenues for the Public Knowledge, Concealed and Private Knowledge cases.

Revenue Comparisons



5 Analysis with a Different Valuation Structure

In this section, we highlight the fact that some of the results under the valuation family with $g(\mathbf{X}_{-i}) \leq \frac{1}{N-1} \sum_{j \neq i} X_j$ do not hold for certain other valuation functions. In particular, the result that the non-ring bidder bids higher under public knowledge does not always hold.

Assume as in the previous section that bidders' signals are uniformly distributed, but the common value now takes the additive form

$$V(X_1, X_2, \dots, X_N) = \sum_{i=1}^N X_i$$

We can rewrite $V(X_1, X_2, \dots, X_N) = X_i + \underbrace{\sum_{j \neq i} X_j}_{g(\mathbf{X}_{-i})}$, where the auxiliary signal is not necessarily

between 0 and 1. Denote the non-ring bidder's signal as x_N . When the ring is formed, the ring representative bids using $g(\mathbf{X}_{-N})$, which we denote as \mathbf{x}_R . In this sense, $V = x_N + \mathbf{x}_R$. However, note that in this case, $g(\mathbf{X}_{-i}) \geq \frac{1}{N-1} \sum_{j \neq i} X_j$.

5.1 Noncooperative Equilibrium

Under noncooperation, the equilibrium bidding strategy according to (Milgrom and Weber, 1982) is

$$\beta(x_i) = \mathbb{E}[V | X_i = x_i, Y_{N/\{i\}}^1 = x_i] = x_i + \underbrace{x_i + \dots + x_i}_{N-2 \text{ times}} = \frac{N+2}{2} x_i$$

The noncooperative equilibrium strategy is

$$\beta(x_i) = \frac{N+2}{2} x_i$$

5.2 Equilibrium under the Concealed Case

When the ring believes they are operating in secret, the ring representative has to best-respond to $\beta(x_i)$.

When the ring representative wins at a price ρ , she infers that

$$\frac{N+2}{2} x_N = \rho \implies x_N = \frac{2}{N+2} \rho$$

Therefore, her inferred estimate of the value of the item upon winning is $\mathbf{x}_R + \frac{2}{N+2} \rho$, and he is happy to win as long as $\rho \leq \mathbf{x}_R + \frac{2}{N+2} \rho$. So he bids $\bar{\rho}$ such that

$$\bar{\rho} = \mathbf{x}_R + \frac{2}{N+2} \bar{\rho} \implies \bar{\rho} = \frac{N+2}{N} \mathbf{x}_R$$

Under the concealed case, the equilibrium bidding strategies for the non-ring bidder and the ring representative are respectively

$$\begin{aligned}\bar{\beta}_N(x_N) &= \frac{N+2}{2}x_N \\ &\text{and} \\ \bar{\beta}_R(\mathbf{x}_R) &= \frac{N+2}{N}\mathbf{x}_R\end{aligned}$$

5.3 Private Knowledge Equilibrium

Given that the ring believes it is concealed, the ring representative bids $\bar{\rho}$. The non-ring bidder best-responds to this strategy.

When the non-ring bidder wins at a price κ , he infers that $\frac{N+2}{N}\mathbf{x}_R = \kappa \implies \mathbf{x}_R = \frac{N}{N+2}\kappa$. Therefore his inferred value is $x_N + \frac{N}{N+2}\kappa$, and he is happy to win so long as $\kappa \leq x_N + \frac{N}{N+2}\kappa$. So he bids

$$\hat{\kappa} = x_N + \frac{N}{N+2}\hat{\kappa} \implies \hat{\kappa} = \frac{N+2}{2}x_N$$

The equilibrium strategies under private knowledge are

$$\begin{aligned}\hat{\beta}_N(x_N) &= \frac{N+2}{2}x_N \\ &\text{and} \\ \hat{\beta}_R(\mathbf{x}_R) &= \frac{N+2}{N}\mathbf{x}_R\end{aligned}$$

As in the case with $g(\mathbf{X}_{-i}) = \prod_{j \neq i} X_j$, under private knowledge, the non-ring bidder does not change his bid with respect to the noncooperative case.

5.4 Symmetric Public Knowledge Equilibrium

It is easy to show that in the public knowledge case, the pair

$$\begin{aligned}\tilde{\beta}_R(\mathbf{x}_R) &= \alpha \cdot \mathbf{x}_R \\ \tilde{\beta}_N(x_N) &= \frac{\alpha}{\alpha-1}x_N\end{aligned}$$

are equilibrium strategies. We show that $\alpha \cdot \mathbf{x}_R$ and $\frac{\alpha}{\alpha-1}x_N$ are mutual best responses.

Suppose the ring representative uses $\alpha \cdot \mathbf{x}_R$.

When the non-ring bidder wins at a price ρ , he infers that $\alpha \cdot \mathbf{x}_R = \rho \implies \mathbf{x}_R = \frac{\rho}{\alpha}$. Therefore, he is happy to win until a price $\bar{\rho}$ such that

$$\bar{\rho} = x_N + \frac{\bar{\rho}}{\alpha} \implies \bar{\rho} = \frac{\alpha}{\alpha-1}x_N$$

Essentially, the non-ring bidder uses the strategy $\frac{\alpha}{\alpha-1}x_N$ as a best response to $\alpha \cdot \mathbf{x}_R$.

Suppose now, that the non-ring bidder uses $\frac{\alpha}{\alpha-1}x_N$.

When the ring representative wins at ρ , she infers that $\frac{\alpha}{\alpha-1}x_N = \rho \implies x_N = \frac{\alpha-1}{\alpha}\rho$, and he is happy to win up to a $\hat{\rho}$ such that

$$\hat{\rho} = \mathbf{x}_R + \frac{\alpha-1}{\alpha}\hat{\rho} \implies \hat{\rho} = \alpha \cdot \mathbf{x}_R$$

So $\alpha \cdot \mathbf{x}_R$ and $\frac{\alpha}{\alpha-1}x_N$ are mutual best responses. Furthermore, symmetry requires

$$\alpha = \frac{\alpha}{\alpha-1} \implies \alpha = 2 .$$

The symmetric equilibrium strategies under public knowledge are

$$\tilde{\beta}_N(x_N) = 2x_N$$

and

$$\tilde{\beta}_R(\mathbf{x}_R) = 2\mathbf{x}_R$$

Comparing the equilibrium strategy of the non-ring bidder under private knowledge ($\frac{N+2}{2}x_N$) and under public knowledge ($2x_N$), since $N \geq 3$,

$$\frac{N+2}{2} > 2 \implies \frac{N+2}{2}x_N > 2x_N$$

implying that the non-ring bidder bids higher under private knowledge.

This conclusion is in contrast to the result under the multiplicative valuation structure.

A corollary of this observation is also the conclusion that the fact that the auctioneer can cheat by convincing bidders that they are facing a public ring is also contingent on the valuation structure, as bidders bid less when the valuation structure is additive.

We summarize this fact in the following proposition.

Proposition 4. *When the valuation structure is additive, non-ring bidders bid higher under private knowledge compared to the noncooperative, concealed and public knowledge cases.*

6 Conclusion

We have analyzed the question of whether in a pure common value second price auction, an auctioneer who suspects that a ring might be present at the auction should reveal this information, and if so, whether this revelation should be made publicly to all participants or privately to other

non-ring bidders. We showed that for a particular family of value functions with an underlying uniform distribution, publicly announcing that a ring is present leads a non-ring bidder to bid higher than he would have when the auction is purely noncooperative. Furthermore, we showed that a consequence of this result is that an auctioneer might always want to convince bidders publicly that a ring exists even if a ring is not present, as the higher bids by bidders upon receiving this information means the auctioneer earns higher expected revenue compared to the noncooperative case. In addition, conditional on a ring operating at the auction, we find that the decision to conceal or reveal the ring presence is contingent on the number of bidders present at the auction. In particular, we found that for $N = 5$ bidders, the auctioneer is better-off revealing the ring presence publicly, while the reverse is true for $N = 15$. Finally, we illustrated that the conclusions above do not necessarily hold when the valuation structure is different.

A limitation of this study has to do with the assumption of an *almost all-inclusive* ring and the assumption that the underlying signals are uniformly distributed. A further issue is the fact that the results also rest on the assumption that bidders use symmetric equilibrium strategies where possible, in particular for the Public Knowledge case. Future work will be geared towards relaxing these assumptions.

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