

Milchtaich, Igal

Working Paper

Quantum advantage in Bayesian games

Working Paper, No. 2023-05

Provided in Cooperation with:

Department of Economics, Bar-Ilan University

Suggested Citation: Milchtaich, Igal (2023) : Quantum advantage in Bayesian games, Working Paper, No. 2023-05, Bar-Ilan University, Department of Economics, Ramat-Gan

This Version is available at:

<https://hdl.handle.net/10419/279453>

Standard-Nutzungsbedingungen:

Die Dokumente auf EconStor dürfen zu eigenen wissenschaftlichen Zwecken und zum Privatgebrauch gespeichert und kopiert werden.

Sie dürfen die Dokumente nicht für öffentliche oder kommerzielle Zwecke vervielfältigen, öffentlich ausstellen, öffentlich zugänglich machen, vertreiben oder anderweitig nutzen.

Sofern die Verfasser die Dokumente unter Open-Content-Lizenzen (insbesondere CC-Lizenzen) zur Verfügung gestellt haben sollten, gelten abweichend von diesen Nutzungsbedingungen die in der dort genannten Lizenz gewährten Nutzungsrechte.

Terms of use:

Documents in EconStor may be saved and copied for your personal and scholarly purposes.

You are not to copy documents for public or commercial purposes, to exhibit the documents publicly, to make them publicly available on the internet, or to distribute or otherwise use the documents in public.

If the documents have been made available under an Open Content Licence (especially Creative Commons Licences), you may exercise further usage rights as specified in the indicated licence.

QUANTUM ADVANTAGE IN BAYESIAN GAMES

Igal Milchtaich, Department of Economics, Bar-Ilan University, Israel
igal.milchtaich@biu.ac.il <https://sites.google.com/view/milchtaich>

August 2023

Quantum advantage in Bayesian games, or games with incomplete information, refers to the larger set of correlated equilibrium outcomes that can be obtained by using quantum mechanisms rather than classical ones. Earlier examples of such advantage go under the title of quantum pseudo-telepathy. By using measurements of entangled particles, the players in the Mermin–Peres magic square game and similar games can obtain a common payoff that is higher than that afforded by any classical mechanism. However, these common-interest games are very special. In general games, where payoffs differ across players and player types, the implementation of specific correlated equilibrium outcomes may require limiting the information that different player types receive through the signals or messages they receive from a correlation device or mechanism. Because of the inherently destructive nature of measurements in quantum mechanics, it is well suited for this task. In a *quantum correlated equilibrium*, players choose what part of the information “encoded” in the quantum state to read, and choosing the part meant for their actual type is required to be incentive compatible. This requirement makes the choice of measurement analogous to the choice of report to the mediator in a communication equilibrium, and the measurement value is analogous to the message sent back from the mediator. A choice of action follows.

This paper systematically explores the advantage quantum mechanisms possess over comparable classical mechanisms in correlated and communication equilibria. It identifies the specific properties of quantum mechanisms responsible for these advantages. It then presents a classification of the equilibrium outcomes (both type-action distributions and equilibrium payoffs) in correlated and communication equilibria according to the kind of (classical or quantum) mechanism employed.

1 Introduction

Quantum advantage refers to the potential use of quantum mechanical phenomena for achieving goals that are not achievable by classical methods. In computer science, that goal may be solving a problem that no classical algorithm is practically capable of solving. More fundamentally, quantum physics may be used for, in a sense, defying probability theory itself. The mathematical theory of probability is based on the concept of *probability space* (Shiryaev 1996), a main ingredient of which is a *sample space*, which is a set of points Ω . The interpretation is that each point $\omega \in \Omega$ represents a possible “state of the world”. Knowing the state amounts to resolving all uncertainty concerning all – past and present – random occurrences. It may be theoretically impossible to ever know the state. For example, it is not possible to learn the outcomes of an infinite sequence of coin tosses. It is only possible to know, after observing the outcomes of finitely many tosses, that the actual state ω belongs to a certain subset of Ω , an event. However, a most remarkable aspect of quantum physics is that it denies even the conceptual, theoretical possibility of states of the world. Such states

correspond to what are known as “hidden variables”. As first shown by Bell (1964), the existence of hidden variables is inconsistent with the actual quantitative predications of quantum mechanics. (For an informal exposition of Bell’s findings and their significance, see Albert 1994. See also below.) As these predications were repeatedly verified by experiments,¹ the meaning of Bell’s theoretical result is that quantum phenomena involve a kind of uncertainty that is not captured by “classical” probability theory, where the employment of a sample space is standard and is never questioned.

This paper explores the possibilities offered by quantum phenomena when applied to Bayesian, or incomplete information, games. This application is part of what is known as *quantum game theory* (see, for example, Eisert et al. 1999, Khan et al. 2018). Specifically, the present subject matter is correlated equilibria in Bayesian games (Forges 1993, Lehrer et al. 2013, Milchtaich 2014, Bergemann and Morris 2016). Such equilibria employ a mechanism, or correlation device, that substitutes for direct communication between players. Coordination is achieved by each player observing a (generally, private) signal sent by the mechanism, and these signals are generally correlated. The use of a mechanism prevents the players from sharing any private information they possess. However, with a *quantum* mechanism, the outcome may be as if information was shared, a phenomenon dubbed quantum pseudo-telepathy. In the following subsections, I review two earlier notable applications of this phenomenon to Bayesian games and briefly explain the physics behind them. I then describe what I believe is a significant missing part in this domain of quantum game theory.

1.1 Mermin–Peres magic square game

A beautiful demonstration of how classically impossible equilibrium outcomes in Bayesian games can be achieved by using quantum pseudo-telepathy is provided by the Mermin–Peres magic square game (Mermin 1990a, Peres 1990, Xu et al. 2022). This is a two-player common-interest game where one player, Alice, is randomly assigned to one of the three rows of a 3×3 table and the other player, Bob, is assigned to one of the columns. All nine possible assignments are equally probable, which in particular means that the players’ assignments, or *types*, are independent. Alice, who only knows her own type, has to fill her row by placing either $+1$ or -1 in each of the cells, with the proviso that the product of the three numbers has to be $+1$. Similarly for Bob’s column, except that for him the product has to be -1 . The players’ common payoff is the product of the numbers they place in the single cell shared by their row and column.

Alice and Bob can achieve an expected payoff of $(8/9 \times (+1) + 1/9 \times (-1) =) 7/9$ by agreeing in advance on the numbers they will put in all cells but one, say the lower-right cell, for example as follows:

+1	+1	+1
-1	-1	+1
+1	+1	± 1

(The only case where the payoff is not $+1$ is when the common cell is the corner one, where

¹ The 2022 Nobel Prize in Physics was awarded to Aspect, Clauser and Zeilinger for their work on “Bell tests” and related experimental measurements that underline the distinction between the quantum and classical worlds (Billings 2022).

Alice puts +1 and Bob puts −1.) No pair of pure strategies gives a higher expected payoff. The same holds for any correlated strategy, which employs a mechanism that sends a pair of random, correlated signals which the players receive before they fill out the cells. This is because each possible realization of the random signals is interpreted by the players as instructions for playing a particular pair of pure strategies, which gets them at most 7/9. However, the players can achieve a perfect payoff of +1 *with certainty* by using as signals the measurement values of two pairs of entangled qubits.

A historically important example of a *qubit* is the spin state of a silver atom in the century-old Stern–Gerlach experiment (Peres 2002, Cohen-Tannoudji et al. 2020). All atoms in this experiment initially travel with the same velocity along the same line, the y axis. They then pass through an inhomogeneous magnetic field whose mean direction is the positive direction of the z axis. Because of the inhomogeneity of the field, the atoms are deflected. According to classical physics, the angle of the deflection along the z axis should be approximately proportional to the magnitude of the z component of each atom’s intrinsic magnetic moment, and thus to have some continuous distribution. What is found instead is the result predicted by quantum mechanics, which is that the magnitude of the deflection is the same for all atoms and the only difference between individual atoms is its direction: half the atoms are deflected in the positive direction of the z axis (state $|0\rangle$, “up”) and half in the opposite direction (state $|1\rangle$, “down”). Rotating the magnet 90° , so that the spin in the x direction is measured instead, would have given a similar result, with half the atoms deflected in the positive direction (state $|+\rangle$) and half in the negative direction (state $|-\rangle$). However, the first measurement S_z and the second one S_x are incompatible. According to quantum mechanics, measurement of the z component of the spin *puts* the atom in the state ($|0\rangle$ or $|1\rangle$) corresponding to the outcome, and thus changes it irreversibly. There is no sense in which the three components of the spin have definite, objective values at any moment in time, as the measurement of any one of them is a destructive process. Correspondingly, alternating between the measurements would yield inconsistent results, with S_x possibly giving a different outcome each time it follows S_z . Only consecutive identical measurements are guaranteed to be consistent.

An example of a pair of *entangled* qubits is two particles that are created as the result of the decay of a spinless particle, for example, the decay of a neutral pion to an electron and a positron (Peres 2002). If the z component of the spin is measured for both particles, the measurements are bound to give opposite results: either the first is up and the second down or vice versa. The same goes for the x and y components, indeed, for measurements of the spin in any common direction. This prediction of quantum theory is at the heart of the famous Einstein–Podolsky–Rosen paradox (Einstein et al. 1935). The two measurements can be instantaneous or nearly so,² and the two particles can be arbitrarily far from one another at the time of measurement, which excludes any transmission of information between them. In their 1935 paper, EPR argued that, since the measurement of one particle could not have changed in any way the other particle, the latter’s three components of the spin must have

² According to relativity theory, instantaneity is a relative term. Two events that are simultaneous for one observer are not so for an observer moving in a particular direction relative to the first one. Moreover, a third observer, who moves in the opposite direction, would perceive the *order* of the events to be the opposite of that perceived by the second observer. Such subjectivity is inconsistent with causality (that is, it cannot be that one event caused the other) and occurs whenever the two events are sufficiently close in time or sufficiently distant in space so that light could not have reached one from the other (and *this* criterion is objective, not relative).

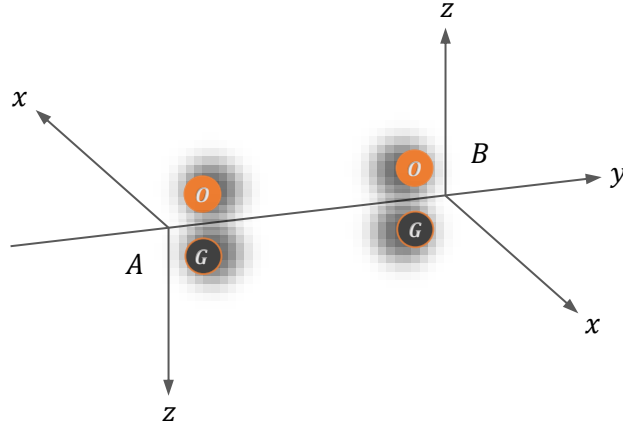


FIGURE 1 THE MECHANISM USED FOR THE MERMIN–PERES MAGIC SQUARE GAME. ALICE (A) AND BOB (B) MEASURE ONE COMPONENT OF THE SPIN OF EACH OF TWO PARTICLES. THE SPIN STATE OF EACH PARTICLE IS ENTANGLED WITH THAT OF THE SIMILARLY COLORED PARTICLE (O OR G) OF THE OTHER PLAYER.

physical reality. However, as mentioned above, Bell showed in 1964 that this idea of “hidden variables” is in fact unequivocally refuted by quantum mechanics itself (and his proof turns out to be highly relevant for the present work; more on this below).

And so the mechanism used for achieving the perfect outcome in the Mermin–Peres magic square game may take the following form (Figure 1). Pair *O* and pair *G* each consists of two spin-entangled particles, as described above. Alice and Bob can each measure a single component, *z* or *x*, of the spin of a different member of pair *O*, and the same for pair *G*. For convenience, each player’s positive directions of the *z* and *x* axes are chosen as the other player’s negative directions. This choice of signs means that if both players measure the *z* component of their particle *O* (measurement S_z^O) or both measure the *x* component (measurement S_x^O), they are guaranteed to get an equal outcome: either +1, corresponding to state $|0\rangle$ or $|+\rangle$, respectively, or -1 , corresponding to $|1\rangle$ or $|-\rangle$. The same goes for pair *G*. The actual measurements that the players perform depend on the cells they have to fill out, as detailed in the following table, which specifies for each cell a specific measurement, the negative of a measurement, or the product of two measurements, which can and should be performed as a single measurement:

S_z^O	S_z^G	$S_z^G S_z^O$
$-S_x^G$	$-S_x^O$	$S_x^O S_x^G$
$S_x^G S_z^O$	$S_x^O S_z^G$	

The players fill out the first two cells in their assigned row (Alice) or column (Bob) according to the outcomes of the measurements specified for these cells. The third cell does not actually require an additional measurement because, to conform with the sign requirements of the game, the number there must be the product (Alice) or the negative of the product (Bob) of the first two numbers. In particular, there is for each player only one legitimate way to complete the last row or column. The number Alice puts in the corner, blank cell must equal the outcome of the measurement $S_x^O S_z^G S_x^G S_z^O$ and for Bob it must be $-S_x^O S_x^G S_z^G S_z^O$. And, magically,

$$S_x^O S_z^G S_x^G S_z^O = -S_x^O S_x^G S_z^G S_z^O, \quad (1)$$

which means that in this cell, too, the players will put the same number. The identity (1)

would not of course hold if the symbols in it stood for numbers. However, the symbols here represent not numbers but measurements, which like all measurements in quantum mechanics are mathematically represented by linear operators in a suitable complex Hilbert space (which in the case of single spin-1/2 particles such as electrons or positrons is a two-dimensional space). The multiplication (composition) of operators, which can be represented as multiplication of (square) matrices, is not generally commutative. In fact, two operators commute if and only if they represent compatible physical quantities: measuring one quantity does not disturb the other. As indicated, this is not the case for the z and x components of the spin of a single particle (in particular, particle G of either player), whose measurements in fact anti-commute: $S_z^G S_x^G = -S_x^G S_z^G$. Hence the identity (1). Moreover, both sides of (1) turn up to be equal to $S_y^O S_y^G$, the product of the measurements of the y component of the spin of the O and G particles.

1.2 Greenberger–Horne–Zeilinger game

The mechanism described in Figure 1 is more complicated than required for refuting the existence of hidden variables. By only invoking a single pair of entangled particles, Bell proved that quantum mechanics predicts a particular *statistics* of particle measurements that is classically impossible (see Section 2.3). However, as first shown by Greenberger, Horne and Zeilinger (1989), the use of more than two particles enables a more dramatic demonstration of the “spookiness” of quantum mechanics, which concerns a *single* run of an experiment. Such a demonstration does not have to involve four particles, as in Figure 1, but can do with three, for example, three polarization-entangled photons (Pan et al. 2000). Brassard et al. (2005) recast the simplified three-particle account of the GHZ scenario provided by Mermin (1990b) into the framework of pseudo-telepathy in a three-player Bayesian game.

In the GHZ game, each of the three players is assigned a type, which can be I or II, such that the number of I’s is odd. The four possible type profiles are equally probable. (Equality is in fact not a crucial assumption. It would suffice to assume that the four probabilities are all positive. Either way, the players’ types are *not* independent.) The players, who only know their own type, have to choose between two actions, +1 or −1. Their common payoff is determined by the three players’ types and the *product* of their actions. If only one player has type I, then the payoff is 0 or 1 if the product is +1 or −1, respectively. If all types are I, then it is the opposite: payoff 1 if the product is +1 and 0 if it is −1.

The best the players can do classically is to choose some fixed three actions whose product is −1. This gets them a payoff of 1 with probability 0.75. However, they can get 1 with certainty by using, for example, three spin-entangled particles whose initial state is

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle).$$

Thus, if each player would measure the spin of one particle in the direction of that player’s z axis, the measurements would yield equal results – all particles up or all down – with both possibilities equally probable. What the players should measure, however, are not the z but the x or y component of their particle: x if their type is I and y if it is II. Each player then has to choose the action that matches the outcome of the measurement, +1 or −1. It can be shown that these strategies guarantee a perfect payoff. For example, if all types are I, the outcomes of the measurements (the x components of the spin, in this case) will all be +1 or only one of them will be so, and thus the product is guaranteed to be +1. This is because the

initial state can be presented also as

$$|\psi\rangle = \frac{1}{2}(|++\rangle + |+-\rangle + |-+\rangle + |--\rangle).$$

This representation shows that the only possible triplets of outcomes are those described above, and also that the four of them are equally probable.

Note that, for each player, the probability of $|+\rangle$ is $1/2$ (two out of four possibilities). This is true not only when all players measure the x components of the spin of their particle but also when the other players' type is II and they measure the y component. Thus, the outcome of the measurement does not give the player any information about the others' types. This fact is a manifestation of a fundamental no-signaling principle (see footnote 2). A measurement of an entangled particle cannot reveal *what* the other players are measuring (Albert 1994). It can only provide information about the *outcomes* they (would) get, which can be used for choosing coordinated actions. In this respect, the quantum mechanism is similar to a classical mechanism that is unaware of the players' types.

1.3 This work

Common-interest games, like those in the examples above, are somewhat marginal in game theory. This is because they do not give rise to what is arguably the most salient aspect of games: conflict. Game theory is all about (complete or partial) misalignment of interests, which is lacking here. This lack is inconsequential for examples as above because, as Section 3.1 shows, such examples arguably do not genuinely concern *games* but only *game forms*, which specify the players' types and possible actions but not the resulting payoffs. Payoff functions only play an incidental role in them. In other words, these examples only show what can and cannot be achieved *mechanically*, when there is no need to take into consideration possible conflicts of interest.

The particular significance of common interests in the context of Bayesian games (which are formally defined in Section 2) is that they entail that the mechanism used in a correlated equilibrium, whose goal is to coordinate the players' choice of actions, can only become more suitable for the job the more information it is able to pass around. There is no reason to hide information. But with non-aligned interests, a mechanism may be required to inform players *selectively*, with each type of each player only getting to know what it has to know in order to choose the intended action (Milchtaich 2014). Here, too, quantum mechanics may be useful.

An example demonstrating the usefulness of quantum mechanisms for "real" games, where the players' payoffs are not identical, is given in Section 3.2. It concerns a particular correlated equilibrium distribution that cannot be implemented by any classical mechanism that does not know the players' types. The reason for the impossibility is that one type of player should not be able to infer from the mechanism's message the action that the player's other type is supposed to play, because this additional information would also partially betray the *other player's* expected move and thus reveal that the action indicated by the message is in fact not optimal. A classical type-unaware mechanism cannot tell each type of player only its action. But, as shown in Section 3.3, a quantum mechanism can do that.

Section 4 presents a general description of quantum mechanisms and definitions of quantum correlated equilibrium (QCE) and equilibrium distribution. That section and the

following one also make the connection with communication equilibrium, where the players self-report their types to the mechanism before it send its messages to them. The equilibrium condition that truthful reporting be incentive compatible corresponds in a QCE to the incentive compatibility of choosing the measurement assigned to the player's actual type. Section 5.1 gives an example showing that, also in the domain of communication equilibria, the use of quantum mechanisms may extend the set of achievable equilibrium outcomes beyond what is classically possible.

The roots of quantum advantage in correlated and communication equilibria are discussed in Section 6. Theorem 1 shows that, in the case of communication equilibria, the advantage is wholly due to the potential incompatibility of each player's different possible measurements. Allowing only compatible physical measurements would eliminate the difference from (classical) communication equilibria where the players' type reports do not affect the messages the mechanism sends to other players. Theorem 2 shows that, in the case of correlated equilibria, the advantage is due to the multiplicity of possible measurements. Quantum mechanisms where each player is only allowed one measurement give rise to the same correlated equilibrium outcomes as classical type-unaware mechanisms.

Section 7 sheds further light on the connections between QCEs and classical correlated and communication equilibria by classifying the equilibrium outcomes according to the kind of mechanisms employed. This classification is similar to, and in a sense extends, the one previously obtained for classical mechanisms only (Milchtaich 2014). The seven classes considered here turn out to be linearly ordered by inclusion. The one corresponding to QCEs (quantum mechanisms) is situated somewhere in the middle, and is overshadowed only by classical correlated and communication equilibria where the mechanism's messages to players may depend on the other players' types (which the mechanism either knows or is informed about by the players themselves). It is shown that the same taxonomy applies both to the equilibrium distributions of types and actions (Section 7.1) and to the resulting payoff vectors (Section 7.2).

With the advent of quantum computing, the vision of putting the peculiar properties of entangled particles and similar quantum systems to everyday use rapidly becomes a reality. Soon, the intricate laboratory equipment needed for maintaining and making use of such systems may be replaced by handheld contraptions. This will make quantum mechanisms seem as mundane as stoplights or other classical coordination devices. It is hoped that this paper will contribute to laying the foundations for the use of such devices as part of new solution concepts: finding equilibria where, due to conflicts of interest, none existed before.

2 Bayesian games

A (finite) *Bayesian game*, or game with incomplete information, has a finite set of players $N = \{1, 2, \dots, n\}$ and, for each player i , a finite set of types T_i , a finite set of actions A_i and a payoff function $u_i = T \times A \rightarrow \mathbb{R}$, where $T = T_1 \times T_2 \times \dots \times T_n$ and $A = A_1 \times A_2 \times \dots \times A_n$ are respectively the collections of all type profiles and all action profiles. The actual type profile is random, and can be specified either as a random element $\mathbf{t} = (t_1, t_2, \dots, t_n)$ of T (throughout this paper, boldface symbols represent random elements) or as a probability vector $(\lambda_t)_{t \in T}$, the *common prior*. The two specifications are connected by

$$\lambda_t = \Pr(\mathbf{t} = t), \quad t \in T.$$

A *game form* is defined similarly except that the payoff functions are not specified.

2.1 Mechanisms and correlated strategies

A *mechanism* for a Bayesian game or game form is an extraneous source of messages. It sends to each player i a message m_i that is an element of some finite set M_i , the player's *message space*. The messages that the players receive may reflect to a lesser or greater extent the type profile. That is, the mechanism may be *type aware*. They generally combine also a random element that is independent of the types. Formally (Milchtaich 2014), a (classical) mechanism is defined as a random element \mathbf{m} that is (statistically) independent of \mathbf{t} , with values that are functions from T to $M = M_1 \times M_2 \times \dots \times M_n$. Thus, for each type profile $t \in T$, the messages that the players receive when their types are t are given by the corresponding *random message profile* $\mathbf{m}(t) = (\mathbf{m}_1(t), \mathbf{m}_2(t), \dots, \mathbf{m}_n(t))$. As the type profile is itself random, the actual messages are given by the random element $\mathbf{m}(\mathbf{t})$ of M .

There are two senses in which the messages the mechanism sends may *not* reflect the players' types. The difference between them may seem subtle but is in fact highly consequential; each version gives rise to a different set of possible outcomes. With a *type-unaware* mechanism, the (random) message each player receives does not factor in the type profile:

$$\mathbf{m}_i(t) = \mathbf{m}_i(t'), \quad i \in N, t, t' \in T.$$

This means that the player's own type does not affect the signal,

$$\mathbf{m}_i(t) = \mathbf{m}_i(t'_i, t_{-i}), \quad i \in N, t, t' \in T \quad (S)$$

(where (t'_i, t_{-i}) is the type profile that takes player i 's type from t' and the other players' types from t), and neither do the types of the other players,

$$\mathbf{m}_i(t) = \mathbf{m}_i(t_i, t'_{-i}), \quad i \in N, t, t' \in T. \quad (O)$$

A corresponding pair of less restrictive conditions is that the message each player i receives does not provide any *information* about that player's type,

$$\mathbf{m}_i(t) \stackrel{d}{=} \mathbf{m}_i(t'_i, t_{-i}), \quad i \in N, t, t' \in T, \quad (\tilde{S})$$

or about the other players' types,

$$\mathbf{m}_i(t) \stackrel{d}{=} \mathbf{m}_i(t_i, t'_{-i}), \quad i \in N, t, t' \in T, \quad (\tilde{O})$$

where $\stackrel{d}{=}$ denotes equality in distribution. The last property \tilde{O} means that the message to each player i is conditionally independent of the other players' types, given the player's own type t_i . It corresponds to the quantum no-signaling property.

Properties S and \tilde{S} limit a mechanism's ability to provide different cues to different types of player. However, different types may still interpret the same message differently, with each type prompted to choose a different action. Thus, the action of player i is jointly determined by the message and by the player's *strategy* $\sigma_i: T_i \times M_i \rightarrow A_i$, which specifies an action a_i for each type t_i and received message m_i . A pair (\mathbf{m}, σ) consisting of a mechanism \mathbf{m} and a profile of strategies $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n)$ is a *correlated strategy*. The corresponding *random action profile* is the random vector $\mathbf{a} = (\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n)$ where

$$\mathbf{a}_i = \sigma_i(\mathbf{t}_i, \mathbf{m}_i(\mathbf{t})), \quad i \in N. \quad (2)$$

		Player 2			
		Type I		Type II	
Player 1	Type I	$\lambda_{I,I}$	+1	-1	
		+1	0	0.5	0.5
	-1	0.5	0	0.5	
		0.5	0.5		
Type II	$\lambda_{I,II}$	+1	-1		
	+1	0.5	0	0.5	
-1	0	0.5	0.5		
	0.5	0.5			

TABLE 1 A CORRELATED STRATEGY DISTRIBUTION IN A TWO-PLAYER BAYESIAN GAME. EACH OF THE 2×2 TABLES GIVES THE JOINT DISTRIBUTION (AND THE MARGINALS) OF THE PLAYERS' ACTIONS FOR A SINGLE TYPE PROFILE t , WHOSE PROBABILITY $\lambda_t > 0$ IS SPECIFIED BY THE COMMON PRIOR.

Properties O and \tilde{O} also limit the mechanism, and consequentially restrict the possible random action profiles. These restrictions are not the same: those resulting from O are more severe than for \tilde{O} (Milchtaich 2014). An example showing this is given in the next subsection.

2.2 Correlated strategy distributions

A *correlated strategy distribution* (CSD) in a Bayesian game or game form is any probability measure η on $T \times A$ whose marginal on T coincides with the common prior λ . As the latter specifies the probability of each type profile t , a CSD only adds the joint distribution of the players' action for every t with $\lambda_t > 0$. (For t lying outside the support of the common prior, any joint distribution of actions is consistent with any CSD.) The example shown in Table 1 refers to a two-player Bayesian game form where the two possible actions for each player are $+1$ and -1 and the common prior has full support, so that all type profiles are possible.

For every CSD η there exists a mechanism \mathbf{m} such that, for some strategy profile σ , η is the correlated strategy distribution of the correlated strategy (\mathbf{m}, σ) in the sense that it is the joint distribution of the random type profile \mathbf{t} and the random action profile \mathbf{a} given by (2). Any such mechanism is said to *implement* η . A simple mechanism implementing the CSD in Table 1 is one where the message to player 1 is determined by a coin toss between $+1$ and -1 and the message to player 2 is then determined by the type profile: identical to the message to 1 if at least one type is not I and the opposite message if both types are I. The corresponding players' strategies are simply to play according to their signal.

In the coin-toss mechanism just described, each player's message is equally likely to be $+1$ or -1 regardless of the type profile and it therefore gives no information about the other player's type; condition \tilde{O} is satisfied. However, the CSD in Table 1 is not implementable by any type-unaware mechanism or even by a mechanism satisfying O (Milchtaich 2004, 2014). The reason is that, for a correlated strategy (\mathbf{m}, σ) in which \mathbf{m} has property O , the joint distribution of the players' actions for every type profile $t = (t_1, t_2)$ coincides with the distribution of $(\mathbf{a}_1^{t_1}, \mathbf{a}_2^{t_2})$, where

$$\mathbf{a}_i^j = \sigma_i(j, \mathbf{m}_i(j, j)), \quad i = 1, 2, j = I, II.$$

The joint distribution of these four elements necessarily satisfies the following pair of *Bell inequalities* (see Brunner and Linden 2013):

$$-2 \leq -\rho_{I,I} + \rho_{I,II} + \rho_{II,I} + \rho_{II,II} \leq 2, \quad (3)$$

where

$$\rho_{I,II} = \Pr(\mathbf{a}_1^I = \mathbf{a}_2^{II}) - \Pr(\mathbf{a}_1^I \neq \mathbf{a}_2^{II}) = \mathbb{E}[\mathbf{a}_1^I \mathbf{a}_2^{II}]$$

and similarly for the other ρ 's. (When all actions are equally likely to be +1 and -1, as in Table 1, the ρ 's are actually the correlation coefficients between the actions.) The reason the pair of inequalities (3) must hold is that it is easily seen to be equivalent to the pair

$$\begin{aligned} \Pr(\mathbf{a}_1^I = \mathbf{a}_2^I) &\leq \Pr(\mathbf{a}_1^I = \mathbf{a}_2^{II}) + \Pr(\mathbf{a}_1^{II} = \mathbf{a}_2^I) + \Pr(\mathbf{a}_1^{II} = \mathbf{a}_2^{II}) \\ \Pr(\mathbf{a}_1^I \neq \mathbf{a}_2^I) &\leq \Pr(\mathbf{a}_1^I \neq \mathbf{a}_2^{II}) + \Pr(\mathbf{a}_1^{II} \neq \mathbf{a}_2^I) + \Pr(\mathbf{a}_1^{II} \neq \mathbf{a}_2^{II}), \end{aligned}$$

and these inequalities hold because every realization $(a_1^I, a_1^{II}, a_2^I, a_2^{II})$ necessarily satisfies an *even* number of the equalities in the first line. But the CSD described in Table 1 does not satisfy (3), as the expression in the middle equals 4. Therefore, this CSD is not implementable by any mechanism with property O , and in particular by any type-unaware mechanism.

The argument used for the last impossibility result can be generalized into a necessary and sufficient set of conditions for a CSD in a 2×2 Bayesian game with two types for each player to be implementable by a type-unaware mechanism. More precisely, these conditions, which are laid out below, concern given four joint distributions of players' actions, one for each type profile t . (As indicated, for t lying outside the support of the common prior, any joint distribution is consistent with any CSD. Therefore, specifying a joint distribution also for such t goes beyond specifying the CSD.) The same conditions are also necessary and sufficient for implementability by a mechanism with property O , which is therefore equivalent to implementability by a type-unaware mechanism.

As explained, every correlated strategy where the mechanism has property O induces a distribution over the 16 action quartets of the form $(a_1^I, a_1^{II}, a_2^I, a_2^{II})$. This distribution can be viewed as a probability vector lying in the unit simplex in \mathbb{R}^{16} . In the other direction, every distribution over action quartets corresponds to some correlated strategy with a type-unaware mechanism, for example, the correlated strategy where a quartet is randomly drawn according to the distribution, each player i is told (a_i^I, a_i^{II}) , and then chooses the first or second action depending on the player's actual type. The joint distribution of actions for each type profile is a marginal of the distribution over action quartets. Specifically, each entry in the 2×2 table describing it (as in Table 1) is the sum of four elements of the probability vector. For example, when player 1 has type I and player 2 has type II, the probability that the former plays **-1** and the latter plays **+1** is given by

$$\Pr(-1, +1, +1, +1) + \Pr(-1, +1, -1, +1) + \Pr(-1, -1, +1, +1) + \Pr(-1, -1, -1, +1).$$

The collection of all possible 16 numbers in these four 2×2 tables is therefore a linear transform of the unit simplex and is thus also a polytope in \mathbb{R}^{16} . As such, it can be described as the intersection of a finite number of half-spaces, in other words, as the collection of all points satisfying a particular set of linear equalities and (weak) inequalities. Some of these are obvious. The four numbers in each 2×2 table must be nonnegative and sum up to 1, and the marginals on the actions of each player type must be the same for both types of the other player: a "no-signaling" condition for actions. The non-obvious inequalities are called the Bell inequalities. They are obtained from (3) by moving the single minus sign in the middle expression to each of the four ρ 's in turn. No-signaling and the eight Bell inequalities

		Player 2			
		Type I		Type II	
Player 1	Type I	$\lambda_{I,I}$	+1	-1	
		$\lambda_{I,II}$	+1	-1	
	+1	0.25- α	0.25+ α	0.5	+1
	-1	0.25+ α	0.25- α	0.5	-1
		0.5	0.5		
Type II	$\lambda_{II,I}$	+1	-1		
		$\lambda_{II,II}$	+1	-1	
	+1	0.25+ α	0.25- α	0.5	+1
	-1	0.25- α	0.25+ α	0.5	-1
		0.5	0.5		

TABLE 2 QUANTUM ADVANTAGE IN CORRELATED STRATEGY DISTRIBUTIONS. THIS CSD, WHERE $\alpha = \sqrt{2}/8$, IS NOT IMPLEMENTABLE BY ANY CLASSICAL TYPE-UNAWARE MECHANISM BUT IS IMPLEMENTABLE BY A QUANTUM MECHANISM.

are therefore necessary and sufficient conditions for the existence of a distribution over action quartets such that the marginals on action pairs coincide with the given four joint distributions of players' actions.³ As shown, this "marginality" condition is necessary for the actions to be determined by a mechanism with property O and is sufficient for them to be determined by a type-unaware mechanism (that is, one satisfying both S and O). These logical relations mean that the three conditions are equivalent.

Observe that each of the three conditions is also equivalent to implementability by a mechanism with property O that directly tells the players how to play, with the corresponding players' strategies being to do as told. This is because, for any mechanism m with property O and a corresponding strategy profile, m can be modified by adding to it an element that "translates" each message according to the receiving player's strategy and directly instructs the player how to act. Obviously, the modified mechanism also has property O .

2.3 Quantum advantage in CSDs

A different correlated strategy distribution in the game form considered in Table 1 is shown in Table 2. Bell inequalities are still not satisfied, as

$$-\rho_{I,I} + \rho_{I,II} + \rho_{II,I} + \rho_{II,II} = -(-1/\sqrt{2}) + 1/\sqrt{2} + 1/\sqrt{2} + 1/\sqrt{2} = 2\sqrt{2} > 2.$$

Therefore, this CSD cannot be obtained by using any classical mechanism that does not know the players' types. Whoever, it can be obtained by using a quantum mechanism. A suitable quantum mechanism consists of a single pair of entangled particles, like either of the pairs in Figure 1, with each player measuring the spin of a different particle. Unlike in Figure 1, Alice's x and z directions are rotated with respect to Bob's, not 180° , but only 135° , so that her x direction forms this angle with both his x and z directions and her z direction is 135° from his z and 45° from his x . Suppose that Alice measures the z component of the spin of her particle if her type is I and the x component if it is II while Bob does the opposite: x for type I and z for II. The angle between the two directions of measurement is then 45° if both types are I and is 135° otherwise. In the former case, the correlation coefficient between the outcomes is $-\cos 45^\circ = -1/\sqrt{2}$, and in the latter, it is

³ The necessity of the Bell inequalities is proved by the argument presenting above (which concerns the particular pair (3)). Their sufficiency (when the no-signaling condition also holds) was first established by Fine (1982), whose proof uses an alternative form of the Bell inequalities, sometimes called the CH inequalities after Clauser and Horne (1974). This form assumes no-signaling.

– $\cos 135^\circ = 1/\sqrt{2}$ (Greenberger et al. 1990). These correlation coefficients and the fifty-fifty marginals mean that, if Alice and Bob always choose the action matching the outcome of their measurement, then (regardless of who is player 1 and who is 2) the joint distribution of their actions for every type profile is as in Table 2.

Significantly, even a quantum mechanism cannot implement the CSD specified by Table 1. This is because a value of 4 for the expression in the middle of (3) cannot be reached. In fact, the largest possible deviation from Bell inequalities is the last example’s value of $2\sqrt{2}$ (Tsirelson 1980).

3 Correlated equilibria

In the previous section, there is no reference in any of the subsections to the players’ payoff functions. The discussion only concerns the game form. This makes it only a first step in the analysis of a Bayesian game, albeit a useful one, in particular with regard to impossibilities. For if a counterexample shows that attaining a certain outcome with a certain kind of mechanism is *mechanically* impossible, then it also proves impossibility in the more complicated case where the players cannot be assigned arbitrary actions because their incentives need to be taken into account. The incentive-compatibility constraint is that, given the type \mathbf{t}_i of a player i and the message $\mathbf{m}_i(\mathbf{t})$ the player received, the conditional expectation of i ’s payoff cannot be increased by replacing the action \mathbf{a}_i specified by the player’s strategy σ_i (Eq. (2)) by any action \mathbf{a}'_i .

Definition 1 For a Bayesian game, a correlated strategy (\mathbf{m}, σ) is a *correlated equilibrium* if the corresponding random action profile \mathbf{a} is such that, for every player i and action \mathbf{a}'_i for that player,

$$\mathbb{E}[u_i(\mathbf{t}, \mathbf{a}) - u_i(\mathbf{t}, (\mathbf{a}'_i, \mathbf{a}_{-i})) \mid \mathbf{t}_i, \mathbf{m}_i(\mathbf{t})] \geq 0.$$

In this case, the correlated strategy distribution is said to be a *correlated equilibrium distribution* (CED) and the n -tuple $(\mathbb{E}[u_i(\mathbf{t}, \mathbf{a})])_{i=1}^n$ of the players’ expected payoffs is a *correlated equilibrium payoff vector* (CEP). In the special cases where the mechanism \mathbf{m} satisfies O or satisfies both S and O (type-unaware mechanism), the correlated equilibrium is called a *type* or *strategy* correlated equilibrium, respectively, and the same qualifiers are attached to the CED and CEP.

The qualifier ‘type’ (Cotter 1994) refers to the fact that the mechanism is allowed to send different messages to different types of the same player i , while ‘strategy’ (Cotter 1991) means that it is left for the player to interpret the mechanism’s message as a map from T_i to A_i , in other words, as a strategy in the Bayesian game.

Example 1 The CHSH game (Clauser et al. 1969) is the common-interest 2×2 Bayesian game obtained by completing the specification of the game form considered in Section 2.2 with $\lambda_t = 0.25$ for all t and defining the payoff functions in the following manner: If both players have type I, their payoffs are both -1 if they choose the same action and 1 if they choose different actions. For the other three type profiles, the payoffs are 1 in the first case and -1 in the second. Obviously, the CSD in Table 1 is a CED in this game, with the correlated equilibrium payoff 1 for both players. However, these are not *strategy* correlated equilibrium distribution and payoff. This is because, with any type-unaware mechanism, the

expected payoff for each player, which is easily seen to be given by one-quarter the expression in the middle of the Bell inequalities (3), can be at most $1/2$.⁴

As shown in Section 2.3, the quantum mechanism consisting of a pair of spin-entangled particles and the players' strategies of measuring the spin in the directions specified in that section and playing accordingly achieve the (quantum) highest possible expected payoff of $1/4 \times 2\sqrt{2} = 1/\sqrt{2}$. They therefore constitute a *quantum correlated equilibrium* in the CHSH game, with the CED in Table 2. Viewed in this light, Table 2 may be interpreted as demonstrating quantum advantage also in CEDs or, perhaps more significantly, in correlated equilibrium payoffs. Indeed, this is how this example is usually presented (Brunner and Linden 2013). Similarly, the examples of the Mermin–Peres magic square game and the GHZ game (Sections 1.1 and 1.2) also revolve around classically unachievable payoffs. However, as shown below, all these examples can also be viewed in a different light, which demotes payoff functions to a supporting role.

3.1 Separating hyperplanes

The correlated strategy distribution of a correlated strategy (\mathbf{m}, σ) is, by definition, a probability measure η on the (finite) product space $T \times A$, assigning a probability $\eta(\{(t, a)\})$ to every pair (t, a) consisting of a type profile and an action profile. It may therefore be viewed as a point in $\mathbb{R}^{|T \times A|}$. Of particular interest here is the subset of all points that correspond to some type-unaware mechanism. The next lemma shows that this set is a polytope.

Lemma 1 In every Bayesian game form, the collection of all CSDs implementable by type-unaware mechanisms is a polytope.

Proof. By Propositions 2 and 5 in Milchtaich (2014), this collection is a linear transform of a polytope. That polytope consists of all probability measures on $A_1^{T_1} \times A_2^{T_2} \times \dots \times A_n^{T_n}$, the set of strategy profiles in the game. (In the special case of a 2×2 game with two types for each player, the linear transformation is explicitly given in the last paragraph of Section 2.2.) ■

It follows from the lemma, by a standard separation theorem, that a CSD is *not* implementable by a type-unaware mechanism if and only if there is a hyperplane separating it from all CSDs that are thus implementable. In other words, there is, in this case, a linear functional f on $\mathbb{R}^{|T \times A|}$ whose value at that CSD is greater than the maximum of f in the polytope identified by Lemma 1. Now, the linearity of f means that the value it returns at every CSD η can be written as

$$f(\eta) = \sum_{(t,a) \in T \times A} b_{(t,a)} \eta(\{(t, a)\}),$$

where the b 's are constant coefficients. This expression can be naturally interpreted as an expected payoff. Specifically, it is the expected payoff for any correlated strategy with the CSD η when the players share the common payoff function u specified by the coefficients:

$$u(t, a) = b_{(t,a)}, \quad t \in T, a \in A.$$

⁴ The maximum of $1/2$ is achieved by the trivial strategy profile where both players always play $+1$. Therefore, this strategy profile and any type-unaware mechanism together constitute a strategy correlated equilibrium in the CHSH game.

		Player 2											
		Type I				Type II							
Player 1	Type I	0.25	+1	-1	+1	-1	0.25	+1	-1	+1	-1		
		+1	0	4	9β	25β	0.5	+1	15	0	32β	2β	0.5
		-1	4	0	25β	9β	0.5	-1	0	0	2β	32β	0.5
					0.5	0.5					0.5	0.5	
Player 1	Type II	0.25	+1	-1	+1	-1	0.25	+1	-1	+1	-1		
		+1	15	0	32β	2β	0.5	+1	0	4	9β	25β	0.5
		-1	0	0	2β	32β	0.5	-1	4	0	25β	9β	0.5
					0.5	0.5					0.5	0.5	

TABLE 3 QUANTUM ADVANTAGE IN CORRELATED EQUILIBRIUM DISTRIBUTIONS. FOR EACH TYPE PROFILE, THE LEFT 2×2 TABLE IS PLAYER 1'S PAYOFF MATRIX AND THE RIGHT ONE, WHERE $\beta = 1/68$, SPECIFIES THE JOINT DISTRIBUTION OF THE PLAYERS' ACTIONS. PLAYER 2 HAS A CONSTANT PAYOFF OF 0. SEE EXAMPLE 2 FOR MORE DETAILS.

It is thus a corollary of Lemma 1 that, for every Bayesian game form, a correlated strategy (\mathbf{m}, σ) produces (via (2)) a joint distribution of type and action profiles that is different from that produced by any correlated strategy (\mathbf{m}', σ') with type-unaware mechanism if and only if there is some common payoff function u for which (\mathbf{m}, σ) gives a higher expected payoff than every such (\mathbf{m}', σ') does. This corollary means that, as indicated, all the examples above, in which the players' payoffs are always equal, can be viewed as concerning game forms rather than games, and correlated strategy distributions rather than correlated equilibria.

3.2 Games with non-identical payoffs

A major difference between correlated equilibria in common-interest games as above and those in general Bayesian games is that the latter may require a mechanism that sends different messages to different types of some player i , which means that condition S does not hold. Specifically, the cue provided to some type t_i needs to be such that it only indicates the action assigned to that type and not those of the other types of player i . The reason the actions assigned to the other types should not be disclosed is that they may tell t_i too much about the actions of the *other* players (as they are correlated with i 's action), so that it would sometimes be possible for t_i to realize that the intended action is in fact not optimal: some alternative action can be expected to yield a higher payoff. The next example illustrates this possibility and demonstrates its significance.

Example 2 In the two-player Bayesian game described in Table 3, the specified correlated strategy distribution is a correlated equilibrium distribution. This is because, with a mechanism that simply randomizes according to the specified joint distribution of actions for each type profile and tells each player what action to take, heeding the mechanism always guarantees player 1 the highest expected payoff for the player's type (and the same is trivially true for player 2, who for simplicity is assumed to have a constant payoff function). For example, if player 1 has type I and the mechanism's recommendation is to play -1 , doing so yields the player an expected payoff of $25/68 \times 4$, whereas playing $+1$ would yield the lower payoff $9/68 \times 4 + 2/68 \times 15$. The correlated equilibrium payoff for player 1 is 5.

The mechanism just described has properties \tilde{S} and \tilde{O} . The message it sends to each player does not provide any information about either player's type, as it is equally likely to be $+1$ or -1 regardless of the type profile. But the mechanism does base its recommendations on the types and it is thus not type-unaware (properties S and O). However, as the correlated

strategy distribution satisfies the Bell inequalities (in particular, the middle expression in (3) is $120/68$), it is also implementable by a type-unaware mechanism. An example of such a mechanism is the following one. Messages to player i have the form $m_i = (a_i^I, a_i^{II})$, where the first and second components are the actions recommended for the player's types I and II, respectively. Pairs of messages (m_1, m_2) can therefore be written as $(a_1^I, a_1^{II}, a_2^I, a_2^{II})$. This quartet has the value $(-1, -1, -1, -1)$, $(-1, +1, +1, -1)$ or $(-1, -1, +1, +1)$ with probability $9/68$, $23/68$ and $2/68$, respectively, and the same probabilities apply to the "opposite" three quartets, where all the signs are inverted. It is easy to check that the correlated strategy consisting of this mechanism and the players' strategies of following the recommendations has the above correlated strategy distribution. However, although, as shown, this CSD is a correlated equilibrium distribution, the correlated strategy just described is *not* a correlated equilibrium. If player 1 receives the message $m_1 = (-1, -1)$, he can conclude that m_2 is either $(-1, -1)$ or $(+1, +1)$. In the former case, both types of player 1 would increase their expected payoff from 0 to $1/2 \times 4$ by playing $+1$ rather than the assigned action -1 . In the latter case, they would increase the payoff from $1/2 \times 4$ to $1/2 \times 15$. Thus, playing according to the message is not optimal for both types of player 1.

The incentive-incompatibility of the last mechanism's recommendations points to the problem with trying to implement the CED in Table 3 with a type-unaware mechanism. However, it is not a definite proof that the problem cannot be overcome, in other words, that the CED in question is not a strategy CED. What would prove this is a demonstration that the indicated deviation is beneficial for some type of player 1 under *any* distribution of the quartets $(a_1^I, a_1^{II}, a_2^I, a_2^{II})$ that gives the four marginals on action pairs specified in Table 3. (There are infinitely many such distributions.) This goal can be formulated as a linear programming problem and tackled by a standard mathematical software package. I used Wolfram Mathematica for this and found that the above deviation is indeed always profitable, so that no type-unaware mechanism is fit for the job. It turns out, however, that a quantum mechanism is so.

3.3 Quantum advantage in CEDs

A quantum correlated equilibrium with the distribution in Table 3 can be constructed by again using a mechanism employing one of the two pairs of spin-entangled particles in Figure 1. A strategy for each player specifies two directions. The first direction indicates the spin component of the player's particle measured by type I and the second direction indicates this for type II. The player then chooses the action ($+1$ or -1) coinciding with the outcome of the measurement. For player 2, the two directions will be Bob's x and z directions, respectively, in Figure 1. As player 2 is indifferent about the outcome, from that player's perspective this choice is as good as any other. Player 1's first direction may be described by the angle θ it forms with Bob's x direction; the angle with Bob's z direction is $90^\circ - \theta$.⁵ It remains to find the optimal θ , that is, the best response (with respect to the payoffs in Table 3) of type I of player 1 to player 2's strategy. By symmetry, player 1's optimal second direction, which is that employed by type II, is at the same angle of θ with Bob's z direction; the angle with Bob's x direction is $90^\circ - \theta$.

⁵ This description assumes that player 1's two directions also lay in Bob's xz plane. For simplicity, this could be made an explicit restriction on the choice of strategies. However, such a restriction is in fact unnecessary, because no direction outside the plane outperforms all those in it. Intuitively, the latter are the most informative directions.

For every $-180^\circ < \theta \leq 180^\circ$, the correlation coefficient between the outcome of player 1's measurement and that of the same type of player 2 is $-\cos \theta$ (Greenberger et al. 1990). As both measurement values are equally likely to be $+1$ or -1 , their joint distribution is

$$\begin{array}{cc|cc|c} & +1 & -1 & & \\ +1 & 0.25(1 - \cos \theta) & 0.25(1 + \cos \theta) & & 0.5 \\ -1 & 0.25(1 + \cos \theta) & 0.25(1 - \cos \theta) & & 0.5 \\ & 0.5 & 0.5 & & \end{array}$$

For the other type of player 2, θ in the table is replaced with $90^\circ - \theta$, equivalently, \cos is replaced with \sin . Using these two conditional distributions and the fact that player 2's two types are equally probable, player 1 can compute, upon getting a particular measurement value, the conditional expectation of the payoff for each of the two actions, $+1$ and -1 , and choose the action giving the higher payoff. The player's expected payoff is the average of these maximal payoffs for the two possible outcomes of measurement, an expression which simplifies to

$$\frac{23}{8} + \max\left\{\frac{15}{8}, \left|\frac{15}{8}\sin \theta - \cos \theta\right|\right\}.$$

The optimal θ is therefore determined by the first-order condition $(15/8)\cos \theta + \sin \theta = 0$. It follows that a best-response strategy for player 1 is to measure in the direction $\theta = \arctan(-15/8) \approx -62^\circ$ and to play according to the outcome. Plugging this θ into the above tables gives that the joint distributions of actions are as in Table 3. The conclusion proves that these quantum mechanism and strategies constitute a quantum correlated equilibrium having the CED in the table. This contrasts with the classical case, where, as shown, no type-unaware mechanism can give such a correlated equilibrium.

4 General quantum correlated equilibria

A general *quantum state* is a unit vector $|\psi\rangle$ in a finite-dimensional complex Hilbert space, the *state space*. It may be specified either abstractly or as a concrete state of a physical system.⁶ An *observable* is a Hermitian, or self-adjoint, linear operator on the state space. In the physical case, it represents a measurable physical quantity. An *n*-player *quantum mechanism* is a quantum state together with a nonempty set of observables for each player *i* such that each of *i*'s observables is compatible with every observable of every other player *j*. Mathematically, compatibility means that the two observables are commuting operators. Physically, it means that the players' measurements do not interfere with one another, and so the outcome for player *i* is statistically the same for all choices of measurements by the other players and there is thus no way for *i* to know what the others are measuring. The set of possible measurement values for an operator is its spectrum, the set of all eigenvalues. The probability of obtaining each outcome is determined by projecting $|\psi\rangle$ on the corresponding eigenspace: the probability is the square of the length of the projection (Cohen-Tannoudji et al. 2020). In the following, the set M_i of all possible measurement

⁶ A possible extension is to allow also *mixed*, or "random", quantum states, which are probability distributions over (pure) states. However, such a formal extension would not translate to a substantial one. This is because a mixed state can always be purified by considering a higher-dimensional state space (see Hughston et al. 1993). In other words, probabilistic uncertainty about the quantum state of a system can always be viewed as reflecting the quantum uncertainty embodied in the (pure) quantum state of a larger system, of which the one under consideration is a part.

values for a player i (which is the union over all the player's observables, a potentially infinite set) is assumed finite.

For an n -player Bayesian game (or game form) Γ , an n -player quantum mechanism determines a new game (respectively, a game form), where a *strategy* for a player i has two components. One component specifies the (admissible) observable each type t_i chooses. The other is a “classical” strategy as in Section 2, that is, a mapping $\sigma_i: T_i \times M_i \rightarrow A_i$ that specifies, for each type t_i and measurement value m_i , the action that player i takes. A quantum mechanism and a corresponding profile of strategies together constitute a *quantum correlated strategy* (QCS) in Γ . A *quantum CSD* is any correlated strategy distribution (see Section 2.2) that is the joint distribution of the players' type and action profiles in some QCS. Given the payoff function u_i of a player i , a quantum CSD determines the distribution, and in particular the expectation, of the player's payoff.

Definition 2 For a Bayesian game, a quantum correlated strategy where the expected payoff for every player i cannot be increased by changing only i 's strategy is a *quantum correlated equilibrium* (QCE), its correlated strategy distribution is a *quantum CED*, and the n -tuple of the players' expected payoffs is a *quantum CEP*.

It is shown below that a quantum CED or CEP is also a CED or CEP, respectively, according to Definition 1. That is, it coincides with the correlated strategy distribution or payoff vector of some classical correlated equilibrium (\mathbf{m}, σ) as in Section 3.

For a given QCS, consider any classical mechanism \mathbf{m} such that, for every type profile $t = (t_1, t_2, \dots, t_n)$, the distribution of $\mathbf{m}(t) = (\mathbf{m}_1(t), \mathbf{m}_2(t), \dots, \mathbf{m}_n(t))$ coincides with that of the (random) profile of measurement values that results when each player i chooses the observable that the QCS specifies for player type t_i . It is easy to see that the CSD of the classical correlated strategy (\mathbf{m}, σ) , where $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n)$ is the profile of classical strategies specified by the QCS, coincides with the quantum CSD. If the QCS is a QCE, then in particular no player i can increase his expected payoff by choosing, when the player's type is t_i , the observable specified for another type t'_i and/or switching to a different classical strategy $\sigma'_i: T_i \times M_i \rightarrow A_i$. Thus, a QCE necessarily satisfies the condition that

$$\mathbb{E}[u_i(\mathbf{t}, \mathbf{a}) - u_i(\mathbf{t}, \mathbf{a}') \mid \mathbf{t}_i] \geq 0, \quad (4)$$

where \mathbf{a} is the random action profile defined by (2) and $\mathbf{a}' = (\mathbf{a}'_1, \mathbf{a}'_2, \dots, \mathbf{a}'_n)$ is given by

$$\begin{aligned} \mathbf{a}'_i &= \sigma'_i(\mathbf{t}_i, \mathbf{m}_i(t'_i, \mathbf{t}_{-i})), \\ \mathbf{a}'_j &= \sigma_j(\mathbf{t}_j, \mathbf{m}_j(t'_i, \mathbf{t}_{-i})), \quad j \neq i. \end{aligned} \quad (5)$$

This necessary condition for QCE is also sufficient if the following “tightness” condition holds: every observable admissible to a player i is assigned by the QCS to at least one of the player's types. Obviously, this can be so only if the set of admissible observables is not larger than T_i . This is not the case for the example in Section 3.3, where the set has the cardinality of the continuum. However, restricting attention to QCSs where the observables satisfy the tightness condition would not affect the sets of quantum CSDs and CEDs, because eliminating possibilities (that is, observables) that are left unused can obviously only reinforce the optimality of those that are being used. A QCS satisfying the tightness condition is a QCE if and only if condition (4), which is formulated entirely in terms of a corresponding classical correlated strategy (\mathbf{m}, σ) , holds.

5 Communication equilibria

A fundamental property of quantum mechanisms is that, as discussed in the Introduction, they exclude any kind of communication between players. Yet, somewhat paradoxically, condition (4) is precisely the definition of communication equilibrium (Milchtaich 2014). Such an equilibrium differs from correlated equilibrium in that the message exchange between the mechanism and the players is two-way. First, each player sends a message to the mechanism, which without loss of generality is assumed to be a type report.⁷ Then, based on the profile of reported types, messages are sent in the opposite direction. The mechanism in a communication equilibrium thus serves as a mediator, or is just a (possibly, noisy) communication protocol.

Definition 3 For a Bayesian game, a correlated strategy (\mathbf{m}, σ) that satisfies condition (4) (with \mathbf{a} and \mathbf{a}' given by (2) and (5)) for every player i , type t'_i and strategy $\sigma'_i: T_i \times M_i \rightarrow A_i$ is a *communication equilibrium*, its correlated strategy distribution is a *communication equilibrium distribution* (MED), and the n -tuple of the players' expected payoffs is a *communication equilibrium payoff vector* (MEP).

A communication equilibrium has the same formal structure (\mathbf{m}, σ) as a correlated equilibrium (Section 2) but there is an interpretation difference. Here, the type profile fed into the mechanism is that of the reported types, not the true ones. Correspondingly, the incentive-compatibility constraint for communication equilibrium, which is expressed by (4), differs from that for correlated equilibrium in also requiring the type reports to be truthful. As the proof of the following proposition shows, Definition 1 is equivalent to Definition 3 with truthfulness assumed.

Proposition 1 Every communication equilibrium is a correlated equilibrium, and therefore every MED is a CED and every MEP is a CEP.

Proof With truthful reports ($t'_i = t_i$), inequality (4) becomes

$$\mathbb{E}[u_i(\mathbf{t}, \mathbf{a}) - u_i(\mathbf{t}, (\sigma'_i(t_i, \mathbf{m}_i(\mathbf{t})), \mathbf{a}_{-i})) \mid \mathbf{t}_i] \geq 0. \quad (6)$$

Thus, a necessary condition for communication equilibrium is that (6) holds for all i and σ'_i . For a strategy of the form

$$\sigma'_i(t_i, m_i) = \begin{cases} a'_i, & t_i = t_i^*, m_i = m_i^* \\ \sigma_i(t_i, m_i), & \text{otherwise} \end{cases},$$

with t_i^* and m_i^* such that $\Pr(\mathbf{t}_i = t_i^*, \mathbf{m}_i(\mathbf{t}) = m_i^*) > 0$ and any action a'_i , this requirement is equivalent to

$$\mathbb{E}[u_i(\mathbf{t}, \mathbf{a}) - u_i(\mathbf{t}, (a'_i, \mathbf{a}_{-i})) \mid \mathbf{t}_i = t_i^*, \mathbf{m}_i(\mathbf{t}) = m_i^*] \geq 0.$$

The latter is the requirement in Definition 1. ■

The significance of the requirement that truthful type reports be incentive compatible is demonstrated by the following simple example (Milchtaich 2014).

⁷ Since the messages players send can only depend on their types, each player could in principle use a gadget that takes type as input and outputs any required message. Such gadgets can be viewed as part of the mechanism.

Example 3 A correlated equilibrium distribution that is not a communication equilibrium distribution. Two players play a 2×2 coordination game where their common payoff is 1 if they choose the same action and -1 otherwise. Player 1 has the single type I and player 2 has two, equally likely, types, I and II. The following CSD is a CED but is not a MED:

		Player 2							
		Type I			Type II				
Player 1	Type I	0.5	+1	-1	0.5	+1	-1	0.5	
		+1	0.5	0	0.5	+1	0.25	0.25	0.5
		-1	0	0.5	0.5	-1	0.25	0.25	0.5
		0.5	0.5			0.5	0.5		

A correlated equilibrium with this distribution is one where two fair coins are flipped, a player of type I observes the outcome for the first coin and type II observes the second coin, and the players play $+1$ if and only if they observe heads. The reason there is no communication equilibrium with this distribution is that type II of player 2 would always be able to increase his expected payoff from 0 to 0.5 by mimicking the action of type I. That is, for $i = 2$, the inequality in (4) does not hold when $t_2 = \text{II}$ if $t_2' = \text{I}$ and σ_2' is a strategy such that $\sigma_2'(\text{II}, \cdot) = \sigma_2(\text{I}, \cdot)$.

As shown at the end of the previous section, the truthfulness requirement of communication equilibrium corresponds in a quantum correlated equilibrium to the requirement that no player type has an incentive to choose a different observable than that assigned to it by the QCE. The arguments presented there thus establish the following result.

Proposition 2 Every quantum CED is a MED, and therefore every quantum CEP is a MEP.

It follows from Proposition 2 (in conjunction with 1) that a quantum CED or CEP is also a (classical) CED or CEP, respectively, and that any CED that, like the one in Example 3, is not a MED is also not a quantum CED.

5.1 Quantum advantage in MEDs

The meaning of quantum advantage in MEDs is not as straightforward as in the cases of CSDs and CEDs. There, the mechanisms viewed as the classical counterparts of the quantum ones are the type-unaware mechanisms, that is, those satisfying S and O . But in the context of communication equilibria, such mechanisms are very special, in that their messages are totally unaffected, not (only) by the players' true types but (also) by the reported types.

A more compelling identification of the classical counterparts of quantum mechanisms in the present context uses the analogy pointed to above between the type a player chooses to report to a classical mechanism and the player's choice of observable in a quantum mechanism. In the latter case, the necessity of making a choice may stem from the destructive nature of measurements in quantum mechanics. However, a cleverly built classical mechanism may also be capable of encapsulating several alternative messages in a single object in such a way that only a single message can be read. For example, decoding any of several linked encoded messages may automatically render the others unreadable. Consider, then, a classical mechanism that – rather than waiting to the players' type reports and then sending its messages – sends to each player i an object as above and only indicates which of the encapsulated messages is addressed to each player type. If the messages differ, then the mechanism does not have property S . However, O still holds, as the other players' types have no effect on the messages to i 's different types.

		Player 2						
		Type I		Type II				
Player 1	Type I	0	+1	-1	0.3	+1	-1	0.5
		+1			+1	$0.25(1 + \rho_{I,II})$	$0.25(1 - \rho_{I,II})$	
		-1			-1	$0.25(1 - \rho_{I,II})$	$0.25(1 + \rho_{I,II})$	
						0.5	0.5	
Player 1	Type II	0.3	+1	-1	0.4	+1	-1	0.5
		+1	$0.25(1 + \rho_{II,I})$	$0.25(1 - \rho_{II,I})$	+1	$0.25(1 + \rho_{II,II})$	$0.25(1 - \rho_{II,II})$	
		-1	$0.25(1 - \rho_{II,I})$	$0.25(1 + \rho_{II,I})$	-1	$0.25(1 - \rho_{II,II})$	$0.25(1 + \rho_{II,II})$	
			0.5	0.5		0.5	0.5	

TABLE 4 QUANTUM ADVANTAGE IN COMMUNICATION EQUILIBRIUM DISTRIBUTIONS. THE 2×2 TABLES, WHERE $\rho_{I,II} = \rho_{II,I} = -\cos 150^\circ = \sqrt{3}/2$ AND $\rho_{II,II} = -\cos 60^\circ = -1/2$, SPECIFY THE JOINT DISTRIBUTION OF ACTIONS FOR TYPE PROFILES t WITH $\lambda_t > 0$. SEE EXAMPLE 4 FOR MORE DETAILS.

Similarly to the terminology used for correlated equilibria (Definition 1), the CSD of a communication equilibrium where the mechanism has property O may be called a *type MED*. What may reasonably be interpreted as quantum advantage in MEDs is the fact that there are type CEDs that are not type MEDs but are quantum CEDs. Such CEDs are implementable by a classical mechanism with property O if the signal each player receives is completely determined by the player's type (correlated equilibrium) but not if it can effectively be chosen by the player (communication equilibrium), yet they are implementable by a quantum mechanism, where the players may choose among the *observables* delivering the messages.

Example 4 A type CED that is not a type MED but is a quantum CED. In a 2×2 coordination game with payoffs as in Example 3 and the common prior λ shown in Table 4, the CSD in that table is a type CED and is also a quantum CED (hence, by Proposition 2, a MED) but is not a type MED. These properties of the CSD hinge on the impossibility of the type profile (I, I), which means that if both players report their type as I, at least one report is false. The strongest possible deterrent against such misreporting is a payoff of -1 , brought about by complete miscoordination of actions. A quantum mechanism is capable of implementing such a perfectly negative correlation for (I, I) while also producing the joint distributions specified by the table for the other three type profiles. For example, this is so for a pair of spin-entangled particles whose spin both players can measure either in the z direction, which they do when their type is I (hence, $\rho_{I,I} = -1$), or in an angle of 150° with respect to the z direction, which they do when their type is II, such that the angle between the two players' second directions is $(360^\circ - 2 \times 150^\circ) = 60^\circ$. Note that these correlation coefficients do not satisfy the Bell inequalities (3), which require $\rho_{I,I} \geq \rho_{II,I} + \rho_{I,II} - 2 = \sqrt{3} - 5/2 = -0.768$. Such high values of $\rho_{I,I}$ are insufficient for preventing type II from misreporting his type as I and changing his expected payoff as a result from $0.3\sqrt{3}/2 + 0.4(-1/2)$ to $0.3\rho_{I,I} + 0.4\sqrt{3}/2$, as this change is nonpositive if and only if $\rho_{I,I} \leq -0.955$.

As shown in Section 2.2, the impossibility of satisfying the Bell inequalities means that incentive compatibility cannot hold with a mechanism satisfying O . Thus, the above quantum CED is not a type MED. It is, however, a type CED. This is because with $\rho_{I,I} = \sqrt{3} - 5/2$, for example, all eight Bell equalities hold. As shown, this means that there is a mechanism with property O that tells the players how to act, which together with the players' strategies of doing as told gives the joint distributions of actions in Table 4 and also $\rho_{I,I} = \sqrt{3} - 5/2$. (The latter does not actually matter, and it does not materialize, as there is

no type reporting – or misreporting – in a correlated equilibrium.) These mechanism and strategies constitute a correlated equilibrium (with the CED in Table 4), as it is easy to check that heeding the mechanism always gives a player positive expected payoff while not doing so would result in negative expected payoff.

6 The roots of quantum advantage

What are the specific features of quantum mechanisms responsible for their advantage over comparable classical mechanisms? Is it possible to pinpoint these characteristics? The two theorems in this section address these questions. Theorem 1 concerns the quantum advantage in communication equilibrium distributions and Theorem 2 concerns the advantage in correlated equilibrium distributions.

As shown in Section 5.1, quantum correlated equilibria may allow for outcomes that cannot be obtained in any classical communication equilibrium in which the players' reported types do not influence the signals sent to other players (property O). The following theorem identifies the uniquely quantum phenomenon of incomparable observables as responsible for this advantage. However, somewhat unintuitively, it is the incompatibility of a player's *own* alternative observables that is responsible for this, as the observables of different players are always compatible by definition of quantum mechanism.

Theorem 1 A quantum CED is a type MED if and only if it is implementable by a quantum mechanism where all the observables (and not only those of different players) are mutually compatible.

Proof To prove that the compatibility condition is sufficient, it suffices to show that for every quantum correlated equilibrium in which the quantum mechanism has compatible observables, there is a corresponding classical mechanism \mathbf{m} (see Section 4) with property O . The compatibility of the observables means that the operators commute, and it is therefore possible to express the quantum state $|\psi\rangle$ specified by the quantum mechanism in an orthonormal basis $\{|\psi_l\rangle\}_{l=1}^L$ whose elements are eigenvectors common to all the operators. Thus,

$$|\psi\rangle = \sum_{l=1}^L \langle\psi_l|\psi\rangle |\psi_l\rangle.$$

Each basis element $|\psi_l\rangle$ defines a mapping from T to M , $(t_1, t_2, \dots, t_n) \mapsto (m_1, m_2, \dots, m_n)$. Specifically, each m_i is the eigenvalue corresponding to the eigenvector $|\psi_l\rangle$ of the observable specified by the QCE for player type t_i , which is also the measurement value obtained in state $|\psi_l\rangle$ when player i chooses that observable. Assigning to each $|\psi_l\rangle$ the probability $|\langle\psi_l|\psi\rangle|^2$ makes this a *random* mapping \mathbf{m} . The latter is easily seen to be a classical mechanism corresponding to the quantum one. The mechanism \mathbf{m} has property O because a player's chosen observable uniquely determines the outcome (i.e., the corresponding eigenvalue) for every $|\psi_l\rangle$; the other players' observables have no effect.

To prove necessity, consider any communication equilibrium (\mathbf{m}, σ) such that \mathbf{m} has property O . This property implies that, for arbitrary, fixed type profile t^* ,

$$\mathbf{m}(t) = (\mathbf{m}_1(t_1, t_{-1}^*), \mathbf{m}_2(t_2, t_{-2}^*), \dots, \mathbf{m}_n(t_n, t_{-n}^*)), \quad t \in T.$$

Suppose, without loss of generality, that the messages to every player i are binary representations of nonnegative integers, specifically, $M_i = \{0,1\}^{K_i}$ for some $K_i \geq 1$. Construct a quantum mechanism employing $\sum_{i=1}^n |T_i| K_i$ numbered entangled qubits in a state that emulates \mathbf{m} , as follows. For each player i and type t_i , a K_i -tuple of consecutive qubits gives the message to that player type. Only player i is allowed to read this message, by measuring the state of each of the K_i bits, which is either $|0\rangle$ or $|1\rangle$. A player's observables thus correspond one-to-one with the player's types, and the observables are compatible because they address disjoint subsets of qubits. Any quantum state of the entire system can be written as

$$|\psi\rangle = \sum_{m_1^1=0}^{2^{K_1}} \sum_{m_1^2=0}^{2^{K_2}} \cdots \sum_{m_i^{j_i}=0}^{2^{K_i}} \cdots \sum_{m_n^{|T_n|}=0}^{2^{K_n}} a_{11\dots i\dots n}^{12\dots j_i\dots |T_n|} |m_1^1 m_1^2 \cdots m_i^{j_i} \cdots m_n^{|T_n|}\rangle,$$

where $|m_i^{j_i}\rangle$ is the state of the K_i -tuple corresponding to the j_i th type of player i . The emulation of \mathbf{m} is achieved by choosing the coefficients according to

$$\begin{aligned} & \left| a_{11\dots i\dots n}^{12\dots j_i\dots |T_n|} \right|^2 = \\ & = \Pr \left((\mathbf{m}_1(1, t_{-1}^*), \mathbf{m}_1(2, t_{-1}^*), \dots, \mathbf{m}_i(j_i, t_{-i}^*), \dots, \mathbf{m}_n(|T_n|, t_{-n}^*)) = (m_1^1, m_1^2, \dots, m_i^{j_i}, \dots, m_n^{|T_n|}) \right). \end{aligned} \quad (7)$$

Consider the quantum correlated strategy with the quantum mechanism constructed above where the players' strategies are to choose the observables corresponding to their true types and then to act according to σ . This QCS emulates the communication equilibrium (\mathbf{m}, σ) in that \mathbf{m} is a corresponding classical mechanism and σ is the profile of classical strategies specified by the QCS. As shown in Section 4, the communication equilibrium condition (4) (together with the tightness condition) entails that this QCS is actually a QCE. Thus, the correlated strategy distribution of (\mathbf{m}, σ) is a quantum CED that is implementable by a quantum mechanism with compatible observables. ■

As shown in Section 3.3, quantum correlated equilibria have an advantage over strategy correlated equilibria (which employ type-unaware classical mechanisms). The next theorem identifies the greater generality afforded by the quantum mechanisms as responsible for this advantage. Specifically, it shows that type-unaware classical mechanisms correspond in their effect to a special kind of quantum mechanisms: those where players have no choice of observables.

Theorem 2 A quantum CED is a strategy CED if and only if it is implementable by a quantum mechanism where every player has only one observable.

Proof Sufficiency follows from the observation that, if all types of each player must use the same observable, then the classical mechanism \mathbf{m} constructed in the first part of the proof of Theorem 1 has property S as well as O . Necessity follows from the observation that, if the classical mechanism \mathbf{m} in the second part of that proof has both properties, then the probability in (7) is different from zero only if $m_i^1 = m_i^1 = \cdots = m_i^{|T_i|}$ for all i . The conclusion means that it is possible to modify the construction in the proof so that the quantum mechanism allocates a common observable to all types of player i – a single K_i -tuple of consecutive qubits. ■

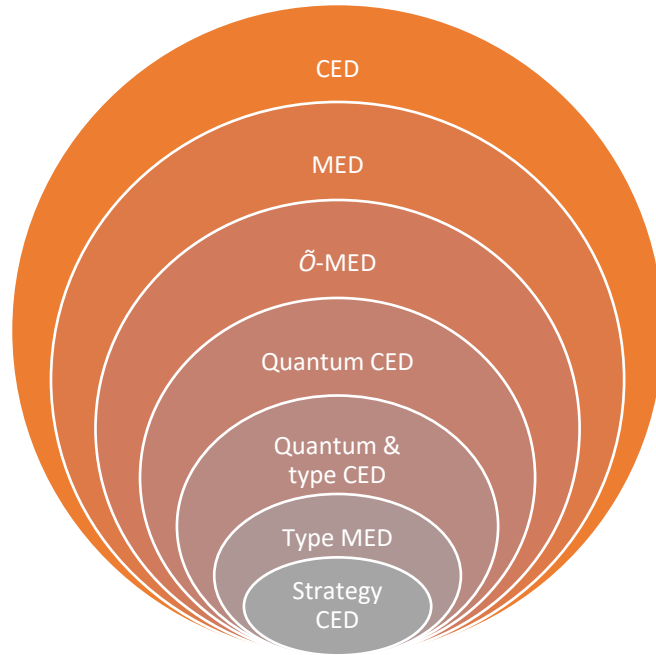


FIGURE 2 THE POSITION OF QUANTUM CED IN A CLASSIFICATION OF CEDS IN BAYESIAN GAMES. A CED OR MED IS THE JOINT DISTRIBUTION OF THE PLAYERS' TYPES AND ACTIONS IN SOME CORRELATED OR COMMUNICATION EQUILIBRIUM, RESPECTIVELY. THE PROPERTIES OF THE (CLASSICAL OR QUANTUM) MECHANISM EMPLOYED IN SOME SUCH EQUILIBRIUM DEFINE EACH OF THE SEVEN ATTRIBUTES.

7 Classifications

CSDs, CEDs and MEDs can be classified, and the classes partially ordered, according to the properties of the mechanisms capable of implementing them. With classical mechanisms, these classifications take the form of two-dimensional lattices (Milchtaich 2014, Fig. 1, 2 and 3). The question is, where do quantum CEDs fit in?

For a finite collection \mathcal{P} of properties of classical mechanisms, a CSD, CED or MED is *\mathcal{P} -implementable* (a \mathcal{P} -CSD, \mathcal{P} -CED or \mathcal{P} -MED for short) if it is the CSD of some correlated strategy, correlated equilibrium or communication equilibrium, respectively, where the mechanism has all the properties in \mathcal{P} . Allowing for quantum mechanisms extends the notion of implementability to also cover quantum implementability. This extension yields the concepts of quantum CSD and quantum CED introduced in this paper. Each kind of (classical or quantum) implementability defines an *attribute* of CSDs, CEDs or MEDs. It is the quality of being implementable by some mechanism with a particular collection of properties. For each of the three kinds of distributions, the various attributes are partially ordered according to the logical relations between them, that is, whether a particular attribute implies another. If the implication holds but the reverse implication does not hold (that is, the two attributes are not equivalent), then the second attribute is *weaker* than the first. In every Bayesian game, the class of all CSDs, CEDs or MEDs having the weaker attribute includes the class of those having the other, stronger attribute, while the reverse inclusion does not hold in general.

This section describes a simplified, unified classification of CEDs and MEDs (with the unification justified by Proposition 1). It considers only the seven attributes deemed most relevant for this paper, and of the properties of classical mechanisms considered in Milchtaich (2014) uses only \tilde{O} , O and S (see Section 2.1). The classification, shown in Figure

2, turns out to be one-dimensional. That is, the seven attributes are all comparable. The weakest attribute of CEDs is simply being one and the strongest attribute is being a strategy CED. These logical relations are proved in the following subsection. The second subsection shows that the same classification that applies to CEDs or MEDs (that is, to type-action distributions) also applies to the corresponding payoff vectors.

7.1 Classification of distributions

It is a simple observation that a MED can be a quantum CED only if it is \tilde{O} -implementable. This is because the compatibility of different players' observables in a quantum mechanism entails that players do not get from their measurements any information about the others' types, which for a corresponding classical mechanism \mathbf{m} spells the no-signaling property \tilde{O} . Thus, we have the following extension of Proposition 2:

Proposition 3 Every quantum CED is an \tilde{O} -MED.

The converse is false; not every \tilde{O} -MED is a quantum CED. For example, the correlated strategy described in Section 2.2, which uses a coin-toss mechanism that has property \tilde{O} , is obviously a communication equilibrium in the common-interest CHSH game (Example 1). But the MED, which is shown in Table 1, is not a quantum CED because it violates Tsirelson's bound (see Section 2.3).

The next, one-step more exclusive, class in the classifications of classical CEDs and MEDs (Milchtaich 2014) is the O -CEDs and O -MEDs, in other words, the type CEDs and MEDs. Property O goes beyond \tilde{O} by requiring the messages to the players to be completely unaffected by the (reported) types of the other players. The next result illustrates the significance of this difference.

Proposition 4 In a Bayesian 2×2 game with two types for every player and a common prior with full support, an \tilde{O} -implementable CSD, CED or MED is O -implementable if and only if it satisfies the Bell inequalities.

Proof Consider any \tilde{O} -CSD, \tilde{O} -CED or \tilde{O} -MED η . As shown in Milchtaich (2014, Sections 4.4 and 4.5), by a version of the revelation principle, there exists, respectively, a correlated strategy, correlated equilibrium or communication equilibrium (\mathbf{m}, σ) with the distribution η such that (i) the players' message spaces coincide with their action spaces, and (ii) the strategies are to act according to the received message, that is,

$$\sigma_i(j, m_i) = m_i, \quad i = 1, 2, j = I, II.$$

With such strategies, the joint distribution of actions for each type profile t coincides with the distribution of $\mathbf{m}(t)$. Now, inspection of the correlated and communication equilibrium conditions shows that they are left unaffected by a replacement of \mathbf{m} with any other mechanism $\tilde{\mathbf{m}}$ that has the same message distributions, that is,

$$\tilde{\mathbf{m}}(t) \stackrel{d}{=} \mathbf{m}(t), \quad t \in T.$$

What remains to be determined is whether some such $\tilde{\mathbf{m}}$ has property O . As shown in Section 2.2, this is so if and only if η satisfies the Bell inequalities and the no-signaling condition. The latter condition is automatically implied by the assumed \tilde{O} -implementability of η . This leaves the Bell inequalities as a necessary and sufficient condition for O -implementability. ■

A quantum CED is not necessarily a type CED, and vice versa. For example, as shown in Section 3, the CSD in Table 2 is a quantum CED in the CHSH game, but it follows from Proposition 4 that is not even an O -CSD. The CED in Example 3 is a type CED as the indicated two-coin mechanism has property O , but is not even a MED. These examples point to the existence of an additional, distinct class: quantum CEDs that are also type CEDs. Example 4 shows this class does not coincide with the class of type MEDs. In fact, it (properly) includes the latter class. This fact follows from Proposition 1, which shows that every type MED is a type CED, and from the next proposition, which shows that it is also a quantum CED.

Proposition 5 Every type MED (= O -MED) is a quantum CED.

Proof This result is established by the second part of the proof of Theorem 1, which in fact proves more than the necessity of the condition in the theorem. It shows that *every* classical mechanism with property O can be emulated by a quantum mechanism with mutually compatible observables. It follows that the class of quantum CEDs implementable by a quantum mechanism where all the observables are compatible coincides with the class of *all* type MEDs. ■

The next rung in the classifications of CEDs and MEDs are the classes of S, O -CEDs and S, O -MEDs, that is, those implementable by type-unaware mechanisms. These two are actually the same class: the strategy CEDs. This is because, with a mechanism that totally ignores the players' types, there is no difference between communication and correlated equilibrium. The distinction of this class from O -CEDs and O -MEDs is proved by the next example (Milchtaich 2014).

Example 5 A type MED (and type CED) that is not a strategy CED (= S, O -CED). Consider the common-interest Bayesian game with the payoffs specified by the left 2×2 tables below.

		Player 2												
		Type I					Type II							
Player 1	Type I	0.25	+1	-1	+1	-1	0.5	+1	-1	0.25	+1	-1	0.5	
	+1	2	0	0.5	0	0.5	+1	3	0	0.25	0.25	0.25	0.5	
	-1	0	2	0	0.5	0.5	-1	0	0	0.25	0.25	0.5	0.5	
					0.5	0.5				0.5	0.5			
Type II	0.25	+1	-1	+1	-1	0.25	+1	-1	+1	-1	0.25	+1	-1	0.5
+1	3	0	0.25	0.25	0.5	+1	2	0	0.5	0	0	0.5	0.5	
-1	0	0	0.25	0.25	0.5	-1	0	2	0	0.5	0	0.5	0.5	
			0.5	0.5					0.5	0.5				

The CSD shown in the right 2×2 tables is a type MED (hence, a type CED) in this game. It is implementable by the two-coin mechanism described in Example 3, which has property O . Together with the strategies of playing +1 if and only if heads is observed, this mechanism constitutes a communication equilibrium, as it is easy to check that misreporting the type would reduce a player's payoff from $(1/2 \times 2 + 1/8 \times 3 =) 1.375$ to $(1/4 \times 2 + 1/4 \times 3 =) 1.25$.

This type MED is not a strategy CED; no implementing type-unaware mechanism exists. The reason is that, since the players' actions are perfectly correlated whenever they have the same type, any message m_1 that a type-unaware mechanism sends to player 1, thereby causing the player's two types to choose particular actions, must always be accompanied by a message to player 2 that elicit the same pair of actions from that player's types. Either the two types' actions are different or they are the same. The first possibility is excluded by the

equilibrium assumption, because the type of player 1 prompted by m_1 to play -1 will get $1/2 \times 2$ from doing so but $1/2 \times 3$ from choosing $+1$, a profitable deviation. This leaves only the second case, in which the two *players* always choose the same action. However, the conclusion contradicts the fact that there is probability 0.5 that the players' actions differ when their types differ.

7.2 Classification of payoff vectors

There exists a parallel classification of CEPs and MEPs, which pertains to the n -tuple of the players' expected payoffs rather than the joint distribution of their types and actions. In every Bayesian game, the joint distribution of types and actions uniquely determines the expected payoffs. However, this mapping is obviously not one-to-one, which is significant for the question of implementability. For example, the expected payoff of both players in the CED in Example 3 is 0.25. And although this CED is not a MED, hence not a quantum CED, the CEP *is* a MEP, a quantum CEP and even a strategy CEP. A type-unaware classical mechanism implementing the 0.25 payoffs is one where the players receive binary signals with this correlation coefficient. If the same were true for every CEP, then the move from distributions to payoff vectors would result in a merger of the seven classes in Figure 2. But in fact, as the following theorem shows, no two classes merge; the classification remains untucked.

Theorem 3 For every two classes of correlated (or communication) equilibrium distributions in Figure 2, the corresponding collections of correlated equilibrium payoff vectors are generally not identical. Thus, the classification of CEPs and MEPs mirrors that of CEDs and MEDs.

Proof The argument is similar to that in the proof of Theorem 3 in Milchtaich (2014). It shows that if a CED has one attribute \mathcal{A} of those in Figure 2 (say, it is an \tilde{O} -MED) but not another attribute \mathcal{B} (say, it is not a quantum CED *and* a type CED), then there is in some *other game* a CED with attribute \mathcal{A} whose payoff vector differs from that of every CED with attribute \mathcal{B} .

That other Bayesian game is the extension of the original game obtained by adding as "dummy players" all the elements of $T \times A$. Each such player (t, a) has only one possible type and a single action and thus cannot affect the true, original players but is only affected by them. Specifically, the dummy player's payoff function $u_{(t,a)}$ is defined as the indicator function $1_{(t,a)}$, which means that the expected payoff is the probability that the true players' type profile is t and their action profile is a . Incorporation of the unique types and actions of the dummy players extends every CSD η in the original game in a trivial manner to a CSD $\hat{\eta}$ in the extended game, and this extension is clearly a bijective mapping. The payoff vector of $\hat{\eta}$ extends that of η by adding the expected payoffs of the dummy players, which (being the probabilities of type-action pairs) coincide with η . It follows that no other CSD in the extended game has the same payoff vector as $\hat{\eta}$. To complete the proof, it remains to show that a CED η in the original game and its extension $\hat{\eta}$ have the same set of attributes (of those in Figure 2).

First, every attribute \mathcal{A} of η is also shared by $\hat{\eta}$. This is because every correlated strategy (\mathbf{m}, σ) whose CSD is η can be extended to a correlated strategy $(\hat{\mathbf{m}}, \hat{\sigma})$ with the CSD $\hat{\eta}$ by sending to the dummy players some constant messages and assigning them their single strategies. The two correlated strategies – one in the original game and the other in the

extended game – clearly share the same properties in $\{S, \tilde{S}, O, \tilde{O}\}$, and if one of them is a correlated or a communication equilibrium, then the other is also so. For quantum correlated strategies, an extension with a similar function is obtained by allowing every dummy player only the trivial observable, i.e., the identity operator on the state space. These players’ “measurements” then have no effect on the true players.

Second, every attribute \mathcal{B} of $\hat{\eta}$ is also shared by η . The argument is similar to that in the previous paragraph, except for the possible complication arising from interactions between the mechanism and the dummy players, which need to be nullified first. For a given classical correlated strategy $(\hat{m}, \hat{\sigma})$ whose CSD is $\hat{\eta}$, this means that any non-constant signals that the mechanism \hat{m} sends to dummy players need to be replaced with constant signals. This replacement is clearly inconsequential, and it turns the correlated strategy into one that is the extension, and thus shares all the properties, of a correlated strategy in the original game. For a given quantum correlated strategy whose CSD is $\hat{\eta}$, the possible complication is that the quantum mechanism may allow some dummy players non-trivial observables. Depending on the outcomes of these players’ measurements, the true players may be left with the system in several possible quantum states – a mixed state. However, as indicated in footnote 6, measurements performed on a system in a mixed state are functionally the same as (that is, their outcomes are statistically indistinguishable from) local measurements in some larger system that is in a pure state. That state and the possible measurements indicated by the true players’ observables constitute a quantum mechanism, which together with these players’ given strategies constitute a QCS in the original game. It is easy to see the latter is a QCE if and only if the given CQS in the extended game is so. ■

References

- Albert DZ (1994) Quantum Mechanics and Experience. Harvard University Press, Cambridge, MA
- Bell JS (1964) On the Einstein Podolsky Rosen paradox. *Physics* 1, 195–200
- Bergemann D, Morris S (2016) Bayes correlated equilibrium and the comparison of information structures in games. *Theoretical Economics* 11, 487–522
- Billings, L (2022) Explorers of quantum entanglement win 2022 Nobel Prize in Physics. *Scientific American* <https://www.scientificamerican.com/article/explorers-of-quantum-entanglement-win-2022-nobel-prize-in-physics1/>
- Brassard G, Broadbent A, Tapp A (2005) Recasting Mermin’s multi-player game into the framework of pseudo-telepathy. *Quantum Information & Computation* 5, 538–550
- Brunner N, Linden N (2013) Connection between Bell nonlocality and Bayesian game theory. *Nature Communications* 4, 2057
- Clauser JF, Horne MA (1974) Experimental consequences of objective local theories. *Physical review D* 10, 526–534
- Clauser JF, Horne MA, Shimony A, Holt RA (1969) Proposed experiment to test local hidden-variable theories. *Physical Review Letters* 23, 880–884
- Cohen-Tannoudji C, Diu B, Laloë, F (2020) Quantum Mechanics, Second Edition. Wiley-VCH
- Cotter KD (1991) Correlated equilibrium in games with type-dependent strategies. *Journal of Economic Theory* 54, 48–68

- Cotter KD (1994) Type correlated equilibria for games with payoff uncertainty. *Economic Theory* 4, 617–627
- Einstein A, Podolsky B, Rosen N (1935) Can quantum-mechanical description of physical reality be considered complete? *Physical Review* 47, 777–780
- Eisert J, Wilkens M, Lewenstein M (1999) Quantum games and quantum strategies. *Physical Review Letters* 83, 3077–3080
- Fine A (1982) Hidden variables, joint probability and the Bell inequalities. *Physical Review Letters* 48, 291–295
- Forges F (1993) Five legitimate definitions of correlated equilibrium in games with incomplete information. *Theory and Decision* 35, 277–310
- Greenberger DM, Horne MA, Zeilinger A (1989) Going beyond Bell's theorem. In: Kafatos M (ed) *Bell's Theorem, Quantum Theory and Conceptions of the Universe*, pp. 73–76. Springer, Dordrecht
- Greenberger DM, Horne MA, Shimony A, Zeilinger A (1990) Bell's theorem without inequalities. *American Journal of Physics* 58, 1131–1143
- Hughston LP, Jozsa R, Wootters WK (1993) A complete classification of quantum ensembles having a given density matrix. *Physics Letters A* 183, 14–18
- Khan FS, Solmeyer N, Balu R, Humble TS (2018) Quantum games: a review of the history, current state, and interpretation. *Quantum Information Processing* 17, 309
- Lehrer E, Rosenberg D, Shmaya E (2013) Garbling of signals and outcome equivalence. *Games and Economic Behavior* 81, 179–191
- Mermin ND (1990a) Simple unified form for the major no-hidden-variables theorems. *Physical Review Letters* 65, 3373
- Mermin ND (1990b) Quantum mysteries revisited. *American Journal of Physics* 58, 731–734
- Milchtaich I (2004) [Random-player games](#). *Games and Economic Behavior* 47, 353–388
- Milchtaich I (2014) [Implementability of correlated and communication equilibrium outcomes in incomplete information games](#). *International Journal of Game Theory* 43, 283–350
- Pan JW, Bouwmeester D, Daniell M, Weinfurter H, Zeilinger A (2000) Experimental test of quantum nonlocality in three-photon Greenberger–Horne–Zeilinger entanglement. *Nature* Vol. 403 Issue 6769, 515–519
- Peres A (1990) Incompatible results of quantum measurements. *Physics Letters A* 151, 107
- Peres A (2002) *Quantum Theory: Concepts and Method*. kluwer Academic Publishers, New York
- Shiryaev AN (1996) *Probability, 2nd Edition*. Springer, New York
- Tsirelson BS (1980) Quantum generalizations of Bell's inequality. *Letters in Mathematical Physics* 4, 93–100
- Xu JM, Zhen YZ, Yang YX, Cheng ZM, Ren ZC, Chen K, Wang XK, Wang HT (2022) Experimental Demonstration of Quantum Pseudotelepathy. *Physical Review Letters* 129, 050402