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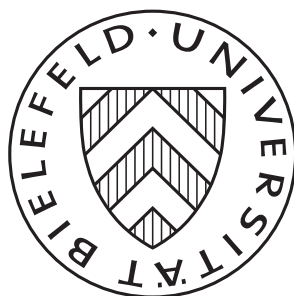


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Ergodic Mean-Field Games of Singular Control with Regime-Switching (extended version)

Jodi Dianetti, Giorgio Ferrari, and Ioannis Tzouanas



ERGODIC MEAN-FIELD GAMES OF SINGULAR CONTROL WITH REGIME-SWITCHING (EXTENDED VERSION)

JODI DIANETTI, GIORGIO FERRARI, AND IOANNIS TZOUANAS

ABSTRACT. This paper studies a class of stationary mean-field games of singular stochastic control with regime-switching. The representative agent adjusts the dynamics of a Markov-modulated Itô-diffusion via a two-sided singular stochastic control and faces a long-time-average expected profit criterion. The mean-field interaction is of scalar type and it is given through the stationary distribution of the population. Via a constructive approach, we prove the existence and uniqueness of the stationary mean-field equilibrium. Furthermore, we show that this realizes a symmetric ε_N -Nash equilibrium for a suitable ergodic N -player game with singular controls. The proof hinges on the characterization of the optimal solution to the representative player's ergodic singular stochastic control problem with regime switching, which is of independent interest and appears here for the first time.

Keywords: stationary mean-field games; singular control; regime-switching; ergodic criterion; ε -Nash equilibrium.

MSC subject classification: 49L20, 91A15, 91A16, 60G40, 35R35, 93C30.

1. INTRODUCTION

Mean-field games (MFGs in short) have been introduced independently in 2006 by Larsy-Lions [38] and Caines et al. [13] as limit models for symmetric N -player differential games, where the interaction is through the empirical distribution of the states (and possibly of the actions) of the players. In MFGs, a representative agent determines her best reply to a given flow of probability measures – e.g., representing the distribution of the states of the indistinguishable rivals – so that the counterpart to the Nash equilibrium concept arising in N -player games takes now the form of a consistency condition: The law of the optimally controlled state of the representative agent must agree with the given flow of probability measures. Since their introduction, because of their tractability, their relation to the theory of propagation of chaos and of forward-backward systems, and their ability to reproduce ε_N -Nash equilibria for suitably related symmetric N -player games, MFGs have attracted large attention in the mathematical and applied literature. We refer to the two-volume book by Carmona and Delarue [19] for a comprehensive presentation of results, approaches, and techniques, as well as to the paper by Carmona [18] for a review of applications of MFGs in Economics, Finance, and Engineering.

In stationary MFGs, the representative player interacts with the long-run distribution of the population. Such a concept has a long tradition in economic theory: Stationary equilibria appeared already in the 1980s in the context of games with a continuum of players (see [31] and [33]), and also play an important role in the analysis of competitive market models with heterogeneous agents (see, e.g., [1] and [41], amongst many others). Closely connected is also the concept of stationary oblivious equilibria, introduced by Adlakha et al. in [2]. Within the mathematical literature, stationary MFGs have been approached both via analytic and probabilistic methods. Among those papers adopting a partial differential equations (PDE) approach, we refer to the works of Bardi and Feleqi [7] for the study of the forward-backward system arising in stationary MFGs with regular controls, Gomes et al. [29] for extended stationary MFGs, Cardaliaguet and Porretta [17] for the study of the long-term behavior of the master equation arising in MFG theory, and to Bertucci [10] for the study of stationary mean-field

optimal stopping games. On the other hand, a probabilistic approach is followed in a series of recent contributions dealing with stationary MFGs with singular and impulsive controls, see Aïd et al. [3], Cao and Guo [16], Cao et al. [15], and Christensen et al. [20].

1.1. Our results. In this paper, we study a class of stationary MFGs where the underlying state process is a general singularly controlled one-dimensional diffusion whose coefficients are modulated by a continuous-time Markov chain with $d \geq 2$ states. More precisely, the representative agent optimally controls a Markov-modulated real-valued Itô-diffusion through a two-sided singular control in order to maximize an ergodic reward functional. This is given by the long-time-average of the time-integral of a running profit function, net of the proportional costs of actions. The mean-field interaction is of scalar type and comes through a real-valued parameter denoted by θ , which, at equilibrium, has to identify with a suitable generalized moment of the stationary distribution of the optimally controlled state process. From the economic point of view, θ can be thought of as a stationary price index arising from the aggregate productivity through an isoelastic demand function à la Spence-Dixit-Stiglitz (see pp. 7-8 in [1]), or of as a stationary demand due to aggregate advertising (see pp. 595-596 in [39]).

Under suitable assumptions on the problem's data, by employing mainly probabilistic means, we prove the existence and uniqueness of a stationary MFG equilibrium for the considered game. Furthermore, we show that this realizes an ε -Nash equilibrium for a related symmetric ergodic N -player game with singular controls.

Our first contribution consists in studying the representative player's optimal control problem and thus in providing, for the first time in the literature, the complete solution to a two-sided ergodic singular stochastic control problem with regime-switching (cf. Proposition 3.5 below). This is accomplished through the study of an auxiliary optimal stopping game, which is then tackled through probabilistic arguments similarly to Ferrari and Rodosthenous in [25] (where, however, the considered Dynkin game was related to a discounted singular stochastic control problem). Through a novel verification argument (see Propositions 3.4 and 3.5 below), we show that, for a given and fixed mean-field parameter θ , the optimal control is of barrier-type. That is, the optimal control uniquely solves a Skorokhod reflection problem (see e.g. Burdzy et al. [12]) at endogenously determined barriers (free boundaries), which depend on the underlying Markov chain and on the given and fixed mean-field parameter θ . As a byproduct, we also show that the optimal upwards and downwards reflection policies satisfy a couple of functional equations resembling those in Theorem 2 of [35].

The next step deals with the construction of the MFG equilibrium and with the proof of its uniqueness. To that end, we first show that the joint process constituted by the optimally controlled Markov-modulated diffusion process and the Markov chain admits a stationary distribution (cf. Proposition 4.2 below). As a matter of fact, we prove that its cumulative distribution function is the unique classical solution to a weakly-coupled system of ordinary differential equations (cf. (4.10) below). This is in line with the result in the Brownian setting obtained by D'Auria and Kella in Theorem 1 of [23]. Clearly, the stationary distribution and its cumulative distribution depend on the fixed mean-field parameter θ , since the optimally controlled state does. In order to proceed with the equilibrium analysis, we thus study the stability of the stationary distribution with respect to θ and actually prove its continuity with respect to such a parameter (cf. Theorem 4.1). Further exploiting the connection to the aforementioned Dynkin game of optimal stopping, we are then able to determine an invariant compact set where any equilibrium value of θ (if one exists) should lie. Combining those continuity and compactness results, an application of the Schauder-Tychonof fixed point theorem allows us to prove that there exists a unique stationary equilibrium (cf. Theorem 4.1).

The analysis of the considered stationary MFG is finally justified by the fact that its unique stationary mean-field equilibrium is able to realize an ε_N -Nash equilibrium for an N -player symmetric game with singular controls in which each player faces an ergodic net profit functional. It is worth noticing that in the N -player game the interaction comes through a suitable time-dependent average

of the players' states (see Eqs. (5.1) and (5.2) below), and it is therefore given in terms of the empirical distribution of players' states at the current time.

1.2. Related literature. Ergodic singular stochastic control problems for one-dimensional diffusions have been treated in general settings, including state-dependent costs of actions, and with different applications, in [5], [40], [42] and [32], [37], among others. However, in all those papers, no regime switching is included.

Our paper is placed within the recent bunch of literature dealing with MFGs with singular controls by following a probabilistic approach; see Aïd et al. [3], Cao and Guo [16], Cao et al. [15], Campi et al. [14], Dianetti et al. [22], Fu [26], Fu and Horst [27], and Guo and Xu [30]. Amongst those, the works that most relate to ours are those by Aïd et al. [3] and by Cao et al. [15]. Cao et al. consider in [15] ergodic MFGs involving a one-dimensional singularly controlled Itô-diffusion. However, differently to us, the control in [15] can be exerted only upwards and no regime-switching process is considered therein. In our work, similarly to [3], we consider a stationary MFG involving a singularly controlled one-dimensional diffusion whose coefficients are modulated by a continuous-time Markov chain. However, differently to [3], here the control is two-sided, rather than only increasing, the performance criterion is of ergodic type, rather than of discounted type, the dynamics of the underlying state process are general, rather than geometric, and the Markov chain has $d \geq 2$ states, rather than only two regimes.

We also clearly relate to those works dealing with MFGs involving regime-switching regular control models. Wang and Zhang [50] consider social optima of mean-field linear-quadratic-Gaussian control models with Markov jump parameters, while distributed games for large-population multi-agent systems with random time-varying parameters are investigated in [49]. Bensoussan et al. [9] focus on MFGs of risk-sensitive type with jump-diffusions and regime-switching. Furthermore, due to the application in networks with switching mechanism, mean-field control problems with regime-switching became recently of particular interest: see, among others, Bayraktar et al. [8], Zhang et al. [52] and Nguyen et al. [43].

1.3. Organization of the paper. The rest of the paper is organized as follows. In Section 2, we introduce the probabilistic setting and the MFG under study. Next, in Section 3, for a given and fixed mean-field parameter, we solve the ergodic stochastic control problem faced by the representative player. In Section 4 we then prove the existence and uniqueness of the mean-field equilibrium, while in Section 5 we provide the approximation result for a related N -player symmetric game. Finally, technical proofs are collected in the Appendices A and B.

2. PROBLEM FORMULATION

2.1. Probabilistic Setting. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space which satisfies the usual conditions, on which it is defined a one-dimensional Brownian motion $\{W_t\}_{t \geq 0}$ and an independent irreducible continuous-time Markov chain $\{Y_t\}_{t \geq 0}$. Denote by $\mathbb{F} := \{\mathcal{F}_t^{W,Y}\}_{t \geq 0}$ the filtration which is generated by W and Y , as usual augmented by \mathbb{P} -null sets of \mathcal{F} . The Markov chain Y has state space $\mathbb{Y} := \{1, \dots, d\}$ and transition matrix $\mathbb{Q} := \{q_{ij}\}_{1 \leq i, j \leq d}$. The transition rates are such that $\kappa_i := -q_{ii} > 0$ and the condition $\sum_{j \in \mathbb{Y}} q_{ij} = 0$ holds for every $i \in \mathbb{Y}$. Accordingly, the transition probabilities of Y are defined as

$$\mathbb{P}(Y_{t+\Delta t} = j | Y_t = i) := \begin{cases} q_{ij}\Delta t + o(\Delta t), & j \neq i, \\ 1 + q_{ii}\Delta t + o(\Delta t), & j = i. \end{cases}$$

For future frequent use, we denote by $p(i) := \frac{\kappa_i}{\sum_{j=1}^d \kappa_j}$ the i -th component of the stationary distribution of Y . Let

$$(2.1) \quad \mathcal{A} := \{ \{ \xi_t \}_{t \geq 0}, \mathbb{F}\text{-adapted, with bounded-variation, left-continuous, } \xi_0 = 0, \text{ a.s. } \}$$

and notice that any $\xi \in \mathcal{A}$ admits the Jordan decomposition $\xi = \xi^+ - \xi^-$, for ξ^\pm nondecreasing. Also, let $\{|\xi|_t\}_{t \geq 0} := \xi^+ + \xi^-$ denote the variation of $\xi \in \mathcal{A}$. Then, for given $\xi \in \mathcal{A}$ and Borel-measurable functions $b : \mathbb{R} \times \mathbb{Y} \rightarrow \mathbb{R}$, $\sigma : \mathbb{R} \times \mathbb{Y} \rightarrow (0, \infty)$, we introduce the process X^ξ with state space $\mathcal{I} := (\underline{x}, \bar{x}) \subseteq \mathbb{R}$ and dynamics

$$(2.2) \quad dX_t^\xi = b(X_t^\xi, Y_t)dt + \sigma(X_t^\xi, Y_t)dW_t + d\xi_t^+ - d\xi_t^-, \quad (X_0^\xi, Y_0) = (x, i) \in \mathcal{I} \times \mathbb{Y}.$$

The following assumption in particular ensures that there exists a unique strong solution to (2.2), for every $\xi \in \mathcal{A}$ and $(x, i) \in \mathcal{I} \times \mathbb{Y}$ (see Theorem 7 Chapter V in [46]). In the following, we shall denote such a strong solution by $(X^{x,\xi}, Y^i)$, when needed.

Assumption 2.1. *The following hold:*

(1) *The functions $b(\cdot, i)$ and $\sigma(\cdot, i)$ are twice continuously differentiable, for every $i \in \mathbb{Y}$.*

(2) *There exists $C > 0$, such that*

$$|b(x, i)| + |\sigma(x, i)| \leq C(1 + |x|), \quad \text{for any } (x, i) \in \mathcal{I} \times \mathbb{Y}.$$

(3) *There exists $c > 0$, such that*

$$b_x(x, i) \leq -c, \quad \text{for any } (x, i) \in \mathcal{I} \times \mathbb{Y}.$$

(4) *For any $(x, i) \in \mathcal{I} \times \mathbb{Y}$,*

$$\sigma(x, i) > 0.$$

Denoting by (X^0, Y) the unique strong solution to (2.2) with $\xi \equiv 0$, Conditions (1) and (2) in Assumption 2.1 imply that (X^0, Y) is regular, meaning that there exists a sequence of stopping times $\{\beta_n\}_{n \geq 0}$, with $\beta_n := \inf\{t \geq 0 : |X_t^0| = n\}$, such that $\beta_\infty := \lim_{n \rightarrow \infty} \beta_n = \infty$, \mathbb{P} -a.s.; for further details see Section 2.3 in [51]. For our subsequent analysis, we introduce the \mathbb{F} -adapted process $\{\widehat{X}_t\}_{t \geq 0}$, which evolves as

$$(2.3) \quad d\widehat{X}_t = (b(\widehat{X}_t, Y_t) + \sigma\sigma_x(\widehat{X}_t, Y_t))dt + \sigma(\widehat{X}_t, Y_t)d\widehat{W}_t, \quad (\widehat{X}_0, Y_0) = (x, i) \in \mathcal{I} \times \mathbb{Y},$$

for an \mathbb{F} -adapted Brownian motion \widehat{W} . Notice that Equation (2.3) also admits a unique strong solution (\widehat{X}^x, Y^i) which is regular, due to Assumption 2.1.

For $f : \mathbb{R} \times \mathbb{Y} \rightarrow \mathbb{R}$ such that $f(\cdot, i) \in C^2(\mathbb{R})$, for any $i \in \mathbb{Y}$, the infinitesimal generator of the uncontrolled process (X^0, Y) is denoted by $\mathcal{L}_{(X,Y)}$ and it is such that

$$(2.4) \quad (\mathcal{L}_{(X,Y)}f)(x, i) = \frac{1}{2}\sigma^2(x, i)f_{xx}(x, i) + b(x, i)f_x(x, i) + \sum_{j \neq i} q_{ij}(f(x, j) - f(x, i)),$$

while the infinitesimal generator $\mathcal{L}_{(\widehat{X}, Y)}$ for the process (\widehat{X}, Y) is such that

$$(2.5) \quad (\mathcal{L}_{(\widehat{X}, Y)}f)(x, i) = \frac{1}{2}\sigma^2(x, i)f_{xx}(x, i) + (b(x, i) + \sigma\sigma_x(x, i))f_x(x, i) + \sum_{j \neq i} q_{ij}(f(x, j) - f(x, i)).$$

In the rest of the paper, we adopt the following notation: $\mathbb{P}_{(x,i)}[\cdot] := \mathbb{P}[\cdot | X_0^\xi = x, Y_0 = i]$ and $\mathbb{E}_{(x,i)}[\cdot] := \mathbb{E}^\mathbb{P}[\cdot | X_0^\xi = x, Y_0 = i]$ for the hybrid-diffusion process (X^ξ, Y) , and $\widehat{\mathbb{P}}_{(x,i)}[\cdot] := \widehat{\mathbb{P}}[\cdot | \widehat{X}_0 = x, Y_0 = i]$ and $\widehat{\mathbb{E}}_{(x,i)}[\cdot] := \mathbb{E}^{\widehat{\mathbb{P}}}[\cdot | \widehat{X}_0 = x, Y_0 = i]$ for (\widehat{X}, Y) . We also set $\mathbb{P}_i[\cdot] := \mathbb{P}[\cdot | Y_0 = i]$ and denote by \mathbb{E}_i the corresponding expectation.

2.2. The Ergodic Mean-Field Game. Within the previous probabilistic setting, we now introduce the ergodic mean-field game (ergodic MFG for short) which will be the main object of our study. For $(x, i) \in \mathcal{I} \times \mathbb{Y} =: \mathcal{O}$, $\xi \in \mathcal{A}$ and $\theta \in \mathbb{R}_+$, we introduce the ergodic profit functional

$$(2.6) \quad J(x, i; \xi, \theta) := \limsup_{T \uparrow \infty} \frac{1}{T} \mathbb{E}_{(x,i)} \left[\int_0^T \pi(X_t^\xi, \theta) dt - k_1 \xi_T^+ + k_2 \xi_T^- \right],$$

where $0 < k_2 < k_1$. In (2.6), π is the instantaneous profit function satisfying Assumption 2.2 below, and θ is (for the moment) a fixed nonnegative number. We will see later, that θ drives the mean-field interaction (cf. Definition 2.1 below).

The instantaneous profit function π fulfills the following conditions.

Assumption 2.2. *The function $\pi : \mathbb{R} \times \mathbb{R}_+ \mapsto \mathbb{R}_+$ is such that:*

- (1) $\pi(\cdot, \theta) \in C^2(\mathcal{I})$, for any $\theta \in \mathbb{R}_+$;
- (2) $\pi(\cdot, \theta)$ is non-decreasing and concave, for any $\theta \in \mathbb{R}_+$;
- (3) $\pi_{x\theta}$ is continuous and it is such that $\pi_{x\theta}(x, \theta) < 0$, for any $(x, \theta) \in \mathcal{I} \times \mathbb{R}_+$;
- (4) for every $(x, i, \theta) \in \mathcal{O} \times \mathbb{R}_+$,

$$\lim_{x \downarrow \underline{x}} \widehat{\mathbb{E}}_{(x,i)} \left[\int_0^\infty e^{\int_0^t b_x(\widehat{X}_s, Y_s) ds} \pi_x(\widehat{X}_t, \theta) dt \right] = \infty,$$

and,

$$\lim_{x \uparrow \bar{x}} \widehat{\mathbb{E}}_{(x,i)} \left[\int_0^\infty e^{\int_0^t b_x(\widehat{X}_s, Y_s) ds} \pi_x(\widehat{X}_t, \theta) dt \right] = 0;$$

- (5) for every $(x, i, \theta) \in \mathcal{O} \times \mathbb{R}_+$,

$$\widehat{\mathbb{E}}_{(x,i)} \left[\int_0^\infty e^{-ct} |\pi_x(\widehat{X}_t, \theta)| dt \right] < \infty,$$

and, for some $\epsilon_0 := \epsilon_0(x) \in (0, 1)$,

$$\begin{aligned} & \widehat{\mathbb{E}} \left[\int_0^\infty e^{-ct} \sup_{x' \in (x, x+\epsilon_0)} \left(|\pi_{xx}(\widehat{X}_t^{x'}, \theta)| |\partial_x \widehat{X}_s^x|_{x=x'} \right) dt \right] + \\ & + \widehat{\mathbb{E}} \left[\int_0^\infty e^{-ct} |\pi_x(\widehat{X}_t^x, \theta)| \sup_{x' \in (x, x+\epsilon_0)} \left(\int_0^t |b_{xx}(\widehat{X}_s^{x'}, Y_s^i)| |\partial_x \widehat{X}_s^x|_{x=x'} ds \right) dt \right] < \infty, \end{aligned}$$

where $\partial_x \widehat{X}_t^x := \exp \left(\int_0^t (b_x + \partial_x(\sigma\sigma_x) - \frac{1}{2}\sigma_x^2)(\widehat{X}_s^x, Y_s) ds + \int_0^t \sigma_x(\widehat{X}_s^x, Y_s) d\widehat{W}_s \right)$ and $c > 0$ is the same constant as in Assumption 2.1-(3),

- (6) for every $(x, i) \in \mathcal{O}$, and for some $\epsilon_0 := \epsilon_0(x) \in (0, 1)$, it holds that

$$(2.7) \quad \widehat{\mathbb{E}} \left[\int_0^\infty e^{-ct} \sup_{x' \in (x, x+\epsilon_0)} \left(|b_{xx}(\widehat{X}_t^{x'}, Y_t^i)| |\partial_x \widehat{X}_t^x|_{x=x'} \right) dt \right] < \infty.$$

We are now ready to introduce the notion of stationary mean-field equilibrium. To that end, we provide the following assumption for the functions F and f that will appear in Definition 2.1 below. Furthermore, we restrict to those $\xi \in \mathcal{A}$ belonging to

$$(2.8) \quad \mathcal{A}_e := \left\{ \xi \in \mathcal{A} : \mathbb{E}[|\xi|_T] < \infty \text{ for any } T < \infty, \limsup_{T \uparrow \infty} \frac{1}{T} \mathbb{E}[|X_T^\xi|] = 0 \right\}.$$

Assumption 2.3. *$F : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, $f : \mathcal{I} \rightarrow \mathbb{R}_+$ are such that:*

- (1) F and f are strictly increasing continuously differentiable functions;
- (2) for $\beta \in (0, 1)$, there exists $C > 0$ such that:
 - (a) $|f(x)| \leq C(1 + |x|^\beta)$, $|F(x)| \leq C(1 + |x|^{\frac{1}{\beta}})$,
 - (b) $|F(x) - F(y)| \leq C(1 + |x| + |y|)^{\frac{1}{\beta}-1} |x - y|$;
- (3) $\lim_{x \uparrow \infty} F(x) = \lim_{x \uparrow \bar{x}} f(x) = \infty$.

Definition 2.1 (Ergodic MFG Equilibrium). *For $(x, i) \in \mathcal{O}$, a couple $(\xi^*(\theta^*), \theta^*) \in \mathcal{A}_e \times \mathbb{R}_+$ is said to be an **equilibrium of the ergodic MFG** for the initial condition (x, i) if*

- (1) $J(x, i; \xi^*(\theta^*), \theta^*) \geq J(x, i; \xi, \theta^*)$, for any $\xi \in \mathcal{A}_e$.

- (2) *The optimally controlled state process $(X^{\xi^*(\theta^*)}, Y)$ admits a limiting stationary distribution μ^{θ^*} and $\theta^* = F(\sum_{i=1}^d \int_{\mathcal{I}} f(x) \mu^{\theta^*}(dx, i))$.*

In the sequel, our solution plan will be as following:

- (1) For a fixed mean-field parameter $\theta \in \mathbb{R}_+$, we solve the ergodic control problem aiming at maximizing (2.6) over \mathcal{A}_e .
- (2) We then impose the consistency condition (2) in Definition 2.1 and we prove the existence and uniqueness of the mean-field parameter θ^* via a fixed-point argument.
- (3) We finally show that the ergodic MFG equilibrium realizes an ϵ_N -Nash equilibrium for a suitable ergodic N -player game of singular control with regime-switching.

3. THE ERGODIC OPTIMAL CONTROL PROBLEM

Recalling (2.6), in this section we fix $\theta \in \mathbb{R}_+$ and solve the ergodic control problem. In particular, we want to find

$$(3.1) \quad \bar{\lambda}(\theta) := \sup_{\xi \in \mathcal{A}_e} J(x, i; \xi, \theta).$$

Notice that the value $\bar{\lambda}$ is independent of $(x, i) \in \mathcal{O}$, since an initial jump does not alter the value of the limit in (2.6). In order to solve (3.1), we let $V : \mathcal{O} \times \mathbb{R}_+ \rightarrow \mathbb{R}$ and $\lambda : \mathbb{Y} \times \mathbb{R}_+ \rightarrow \mathbb{R}$ to be determined such that $V(\cdot, i; \theta) \in C^2(\mathcal{I})$, for any $(i, \theta) \in \mathbb{Y} \times \mathbb{R}_+$ and the pair (V, λ) solves the variational inequality

$$(3.2) \quad \max \{ \mathcal{L}_{(X, Y)} V(x, i; \theta) + \pi(x, \theta) - \lambda(\theta, i), V_x(x, i; \theta) - k_1, k_2 - V_x(x, i; \theta) \} = 0.$$

It will be shown in Proposition 3.4 below that a solution (V, λ) to (3.2) allows to obtain $\bar{\lambda}$ in the sense that

$$(3.3) \quad \bar{\lambda}(\theta) := \sum_{i=1}^d p(i) \lambda(\theta, i).$$

The following additional assumption on π holds throughout the rest of this paper.

Assumption 3.1. *For $x_-(\theta) := (x_-(1, \theta), \dots, x_-(d, \theta))$, $x_+(\theta) := (x_+(1, \theta), \dots, x_+(d, \theta))$, with $x_-(i, \theta) < x_+(i, \theta)$, $i \in \mathbb{Y}$, it holds:*

$$\pi_x(x, \theta) + k_1 b_x(x, i) \begin{cases} > 0, & x \in (\underline{x}, x_-(i, \theta)), \\ = 0, & x = x_-(i, \theta), \\ < 0, & x \in (x_-(i, \theta), \bar{x}), \end{cases}$$

$$\pi_x(x, \theta) + k_2 b_x(x, i) \begin{cases} > 0, & x \in (\underline{x}, x_+(i, \theta)), \\ = 0, & x = x_+(i, \theta), \\ < 0, & x \in (x_+(i, \theta), \bar{x}). \end{cases}$$

Assumption 3.1 together with Assumption 2.2-(4) guarantee that a solution to (3.2) will be of threshold type, meaning that there shall exist $\alpha(i, \theta) < \beta(i, \theta)$, $i \in \mathbb{Y}$, such that $V_x(x, i; \theta) < k_1$ on $\{(x, i) \in \mathcal{O} : x < \alpha(i, \theta)\}$ and $V_x(x, i; \theta) > k_2$ on $\{(x, i) \in \mathcal{O} : x > \beta(i, \theta)\}$.

3.1. An auxiliary optimal stopping game. To deal with (3.2), we introduce the auxiliary optimal stopping game (Dynkin game) with stopping functional

$$(3.4) \quad \widehat{J}(x, i; \tau, \sigma, \theta) := \widehat{\mathbb{E}}_{(x, i)} \left[\int_0^{\tau \wedge \sigma} e^{\int_0^t b_x(\widehat{X}_s, Y_s) ds} \pi_x(\widehat{X}_t, \theta) dt + k_1 e^{\int_0^\tau b_x(\widehat{X}_t, Y_t) dt} \mathbf{1}_{\{\tau < \sigma\}} \right. \\ \left. + k_2 e^{\int_0^\sigma b_x(\widehat{X}_t, Y_t) dt} \mathbf{1}_{\{\sigma < \tau\}} \right],$$

where $(\widehat{X}_t, Y_t)_{t \geq 0}$ is the unique strong solution to (2.3) and $\tau, \sigma \in \mathcal{T}$, where $\mathcal{T} := \{\rho : \Omega \rightarrow [0, \infty] : \rho \text{ is an } \mathbb{F}\text{-stopping time}\}$. In (3.4), we use the convention $e^{\int_0^\rho b_x(\widehat{X}_s, Y_s^i) ds} = 0$ on $\{\rho = \infty\}$. The Dynkin game is such that Player 1 aims at minimizing (3.4) over $\tau \in \mathcal{T}$, while Player 2 at maximizing (3.4) over $\sigma \in \mathcal{T}$. Theorem 2.1 in [44] allows to show that such a game indeed admits a value.

Theorem 3.1. *Let $(x, i, \theta) \in \mathcal{O} \times \mathbb{R}_+$. Then,*

$$(3.5) \quad \inf_{\tau \in \mathcal{T}} \sup_{\sigma \in \mathcal{T}} \widehat{J}(x, i; \tau, \sigma, \theta) = \sup_{\sigma \in \mathcal{T}} \inf_{\tau \in \mathcal{T}} \widehat{J}(x, i; \tau, \sigma, \theta),$$

and we define the value function

$$(3.6) \quad v(x, i; \theta) := \inf_{\tau \in \mathcal{T}} \sup_{\sigma \in \mathcal{T}} \widehat{J}(x, i; \tau, \sigma, \theta) = \sup_{\sigma \in \mathcal{T}} \inf_{\tau \in \mathcal{T}} \widehat{J}(x, i; \tau, \sigma, \theta).$$

Moreover, letting the continuation region be

$$\mathcal{C}^\theta := \{(x, i) \in \mathcal{O} : k_2 < v(x, i; \theta) < k_1\},$$

and the stopping regions be

$$\mathcal{S}_{\text{inf}}^\theta := \{(x, i) \in \mathcal{O} : v(x, i; \theta) \geq k_1\}, \quad \mathcal{S}_{\text{sup}}^\theta := \{(x, i) \in \mathcal{O} : v(x, i; \theta) \leq k_2\},$$

the stopping times

$\tau^*(x, i; \theta) := \inf\{t \geq 0 : (\widehat{X}_t, Y_t) \in \mathcal{S}_{\text{inf}}^\theta\}$, $\sigma^*(x, i; \theta) := \inf\{t \geq 0 : (\widehat{X}_t, Y_t) \in \mathcal{S}_{\text{sup}}^\theta\}$, $\widehat{\mathbb{P}}_{(x, i)}\text{-a.s.}$, realize a saddle-point. In particular,

$$v(x, i; \theta) = \widehat{J}(x, i; \tau^*(x, i; \theta), \sigma^*(x, i; \theta), \theta).$$

Proof. See Appendix A. □

The following proposition is easily proved thanks to Conditions (2) and (3) in Assumption 2.2.

Proposition 3.1. *It holds:*

- (1) *The mapping $x \mapsto v(x, i; \theta)$ is decreasing for every $(i, \theta) \in \mathbb{Y} \times \mathbb{R}_+$.*
- (2) *The mapping $\theta \mapsto v(x, i; \theta)$ is decreasing for every $(x, i) \in \mathcal{O}$.*

The next result excludes the possibility of empty stopping regions.

Proposition 3.2. *The following hold:*

$$\mathcal{S}_{\text{inf}}^\theta \neq \emptyset \text{ and } \mathcal{S}_{\text{sup}}^\theta \neq \emptyset.$$

Proof. See Appendix A. □

We define the free-boundaries

$$(3.7) \quad \alpha_i(\theta) := \sup\{x \in \mathcal{I} : v(x, i; \theta) \geq k_1\}, \quad \beta_i(\theta) := \inf\{x \in \mathcal{I} : v(x, i; \theta) \leq k_2\},$$

for any $(i, \theta) \in \mathbb{Y} \times \mathbb{R}_+$, with the conventions $\sup \emptyset = \underline{x}$ and $\inf \emptyset = \bar{x}$. Then, thanks to the monotonicity of $v(\cdot, i; \theta)$, $(i, \theta) \in \mathbb{Y} \times \mathbb{R}_+$, we have

$$(3.8) \quad \mathcal{C}^\theta = \{(x, i) \in \mathcal{O} : x \in (\alpha_i(\theta), \beta_i(\theta))\},$$

$$(3.9) \quad \mathcal{S}_{\text{inf}}^\theta = \{(x, i) \in \mathcal{O} : x \in (\underline{x}, \alpha_i(\theta))\} \text{ and } \mathcal{S}_{\text{sup}}^\theta = \{(x, i) \in \mathcal{O} : x \in [\beta_i(\theta), \bar{x}]\}.$$

Notice that, due to Assumption 3.1, $x_-(i, \theta) < \alpha(i, \theta)$ and $\beta(i, \theta) < x_+(i, \theta)$, for any $(i, \theta) \in \mathbb{Y} \times \mathbb{R}_+$. The next proposition provides regularity results about the value of the Dynkin game (3.6). The proof is in the same spirit as that of Theorem 4.3 in [25].

Proposition 3.3. *For any $(i, \theta) \in \mathbb{Y} \times \mathbb{R}_+$ we have $v(\cdot, i; \theta) \in C^1(\mathcal{I}) \cap C^2(\mathcal{I} \setminus \{\alpha_i(\theta), \beta_i(\theta)\})$.*

Proof. See Appendix A. □

By Proposition 3.3 and arguments based on the semi-harmonic characterization of v (see [44]), we are allowed to conclude that $(v(x, i; \theta), \alpha_i(\theta), \beta_i(\theta))_{i \in \mathbb{Y}}$ solves the free-boundary problem:

$$(3.10) \quad \begin{cases} \mathcal{L}_{(\widehat{X}, Y)} v(x, i; \theta) + b_x(x, i)v(x, i; \theta) + \pi_x(x, \theta) = 0, & x \in (\alpha_i(\theta), \beta_i(\theta)), & i \in \mathbb{Y}, \\ \mathcal{L}_{(\widehat{X}, Y)} v(x, i; \theta) + b_x(x, i)v(x, i; \theta) + \pi_x(x, \theta) \leq 0, & x \in (\alpha_i(\theta), \bar{x}), & i \in \mathbb{Y}, \\ \mathcal{L}_{(\widehat{X}, Y)} v(x, i; \theta) + b_x(x, i)v(x, i; \theta) + \pi_x(x, \theta) \geq 0, & x \in (\underline{x}, \beta_i(\theta)), & i \in \mathbb{Y}, \\ v(x, i; \theta) = k_1, & x \in (\underline{x}, \alpha_i(\theta)], & i \in \mathbb{Y}, \\ v(x, i; \theta) = k_2, & x \in [\beta_i(\theta), \bar{x}), & i \in \mathbb{Y}, \\ v_x(x, i; \theta) = 0, & x \in (\underline{x}, \alpha_i(\theta)], & i \in \mathbb{Y}, \\ v_x(x, i; \theta) = 0, & x \in [\beta_i(\theta), \bar{x}), & i \in \mathbb{Y}. \end{cases}$$

3.2. The solution to the ergodic singular control problem. In this section we establish the relation between the Dynkin game of Section 3.1 and the ergodic singular control problem (3.1).

Proposition 3.4. *Recall v as in (3.6), the free boundaries $(\alpha_i(\theta))_{i \in \mathbb{Y}}$ and $(\beta_i(\theta))_{i \in \mathbb{Y}}$ as in (3.7), and define*

$$(3.11) \quad U(x, i; \theta) := K_i(\theta) + \int_{\alpha_i(\theta)}^x v(y, i; \theta) dy, \quad (x, i, \theta) \in \mathcal{O} \times \mathbb{R}_+,$$

with $K_i(\theta) := - \int_{\alpha_i(\theta)}^{\beta_i(\theta)} v(y, i; \theta) dy$. Then we have

$$(3.12) \quad \begin{cases} \mathcal{L}_{(X, Y)} U(x, i; \theta) + \pi(x, \theta) = \lambda(\theta, i), & x \in (\alpha_i(\theta), \beta_i(\theta)), & i \in \mathbb{Y}, \\ \mathcal{L}_{(X, Y)} U(x, i; \theta) + \pi(x, \theta) \leq \lambda(\theta, i), & x \notin (\alpha_i(\theta), \beta_i(\theta)), & i \in \mathbb{Y}, \\ U_x(x, i; \theta) = k_1, & x \in (\underline{x}, \alpha_i(\theta)], & i \in \mathbb{Y}, \\ U_x(x, i; \theta) \leq k_1, & x \in (\alpha_i(\theta), \beta_i(\theta)), & i \in \mathbb{Y}, \\ U_x(x, i; \theta) = k_2, & x \in [\beta_i(\theta), \bar{x}), & i \in \mathbb{Y}, \\ U_x(x, i; \theta) \geq k_2, & x \in (\alpha_i(\theta), \beta_i(\theta)), & i \in \mathbb{Y}, \end{cases}$$

where

$$(3.13) \quad \lambda(\theta, i) := \sum_{j \neq i} q_{ij} \left(\int_{\alpha_j(\theta)}^{\alpha_i(\theta)} v(y, j; \theta) dy + K_i(\theta) - K_j(\theta) \right) + (b(\alpha_i(\theta), i)k_1 + \pi(\alpha_i(\theta), \theta)).$$

Proof. By Proposition 3.3 and the definition of U , we obtain that $U(\cdot, i; \theta) \in C^2(\mathcal{I})$ for $i \in \mathbb{Y}$. Then,

$$\begin{aligned} \mathcal{L}_{(X, Y)} U(x, i; \theta) + \pi(x, \theta) &= \frac{1}{2} \sigma^2(x, i) U_{xx}(x, i; \theta) + b(x, i) U_x(x, i; \theta) \\ &\quad + \sum_{j \neq i} q_{ij} (U(x, j; \theta) - U(x, i; \theta)) + \pi(x, \theta), \end{aligned}$$

so that, by (3.11),

$$(3.14) \quad \begin{aligned} \mathcal{L}_{(X, Y)} U(x, i; \theta) + \pi(x, \theta) &= \underbrace{\left(\frac{1}{2} \sigma^2(x, i) v_x(x, i; \theta) + b(x, i) v(x, i; \theta) + \pi(x, \theta) \right)}_{(1)} \\ &\quad + \sum_{j \neq i} q_{ij} \left(K_j(\theta) + \int_{\alpha_j(\theta)}^x v(y, j; \theta) dy - K_i(\theta) - \int_{\alpha_i(\theta)}^x v(y, i; \theta) dy \right). \end{aligned}$$

Using that v satisfies (3.10) (see in particular the fourth to seventh display equations), via an integration by parts we have that

$$(1) = \int_{\alpha_i(\theta)}^x \left(\frac{1}{2} \sigma^2(y, i) v_{xx}(y, i; \theta) + (b(y, i) + \sigma \sigma_x(y, i)) v_x(y, i; \theta) + b_x(y, i) v(y, i; \theta) + \pi_x(y, \theta) \right) dy \\ + (b(\alpha_i(\theta), i) k_1 + \pi(\alpha_i(\theta), \theta)).$$

Then, using the second equation in (3.10) we find

$$(1) \leq \int_{\alpha_i(\theta)}^x - \sum_{j \neq i} q_{ij} (v(y, j; \theta) - v(y, i; \theta)) dy + (b(\alpha_i(\theta), i) k_1 + \pi(\alpha_i(\theta), \theta)).$$

Substituting the latter inequality into (3.14) yields

$$\begin{aligned} \mathcal{L}_{(X, Y)} U(x, i; \theta) + \pi(x, \theta) &\leq \\ &\leq - \int_{\alpha_i}^x \sum_{j \neq i} q_{ij} (v(y, j; \theta) - v(y, i; \theta)) dy + (b(\alpha_i(\theta), i) k_1 + \pi_x(\alpha_i(\theta), \theta)) \\ &\quad + \sum_{j \neq i} q_{ij} \left(K_j(\theta) + \int_{\alpha_j(\theta)}^x v(y, j; \theta) dy - K_i(\theta) - \int_{\alpha_i(\theta)}^x v(y, i; \theta) dy \right) \\ &= \sum_{j \neq i} q_{ij} \left(\int_{\alpha_i(\theta)}^x v(y, j; \theta) dy - \int_{\alpha_j(\theta)}^x v(y, j; \theta) dy \right) + (b(\alpha_i(\theta), i) k_1 + \pi(\alpha_i(\theta), \theta)) \\ &= \sum_{j \neq i} q_{ij} \left(\int_{\alpha_j(\theta)}^{\alpha_i(\theta)} v(y, j; \theta) dy \right) + (b(\alpha_i(\theta), i) k_1 + \pi(\alpha_i(\theta), \theta)). \end{aligned}$$

Thus, defining

$$(3.15) \quad \begin{aligned} \lambda(\theta, i) &:= \sum_{j \neq i} q_{ij} \left(\int_{\alpha_j(\theta)}^{\alpha_i(\theta)} v(y, j; \theta) dy \right) + (b(\alpha_i(\theta), i) k_1 + \pi(\alpha_i(\theta), \theta)) \\ &\quad + \sum_{j \neq i} q_{ij} \left(\int_{\alpha_i(\theta)}^{\beta_j(\theta)} v(y, i; \theta) dy - \int_{\alpha_j(\theta)}^{\beta_j(\theta)} v(y, j; \theta) dy \right), \end{aligned}$$

we see that (U, λ) solves the variational inequality (3.12). \square

In order to derive the optimal control rule we introduce the following definition.

Definition 3.1 (Skorokhod reflection regime-switching problem). *Let $\mathcal{D}[0, \infty)$ be the space of càdlàg processes on $[0, \infty)$. Given $x \in \mathcal{I}$, $i \in \mathbb{Y}$, $X \in \mathcal{D}[0, \infty)$, and a vector $(\alpha, \beta) = (\alpha_j, \beta_j)_{j \in \mathbb{Y}} \in \mathbb{R}^{2d}$ with $\alpha_j < \beta_j$, for any $j \in \mathbb{Y}$, the process $(X^\xi, \xi) \in \mathcal{D}[0, \infty) \times \mathcal{A}$ is said to be a solution to the Skorokhod reflection problem $\mathbf{SP}(\alpha, \beta, i, x)$ for the noise (W, Y) if it satisfies the following properties:*

(1) *Letting $I : \mathcal{D}[0, \infty) \rightarrow \mathcal{D}[0, \infty)$ be such that*

$$(3.16) \quad (X)_{t \geq 0} \mapsto \left(x + \int_0^t b(X_s, Y_s) ds + \int_0^t \sigma(X_s, Y_s) dW_s \right)_{t \geq 0},$$

then

$$(3.17) \quad X_t^\xi = I(X^\xi)_t + \xi_t^+ - \xi_t^-, \quad \mathbb{P} \otimes dt\text{-a.s.};$$

(2) $X_t^\xi \in [\alpha_{Y_t}, \beta_{Y_t}]$, $\mathbb{P} \otimes dt\text{-a.s.}$;

(3) $\int_0^T \mathbf{1}_{\{X_t^\xi > \alpha_{Y_t}\}} d\xi_t^+ = \int_0^T \mathbf{1}_{\{X_t^\xi < \beta_{Y_t}\}} d\xi_t^- = 0$, for any $T > 0$, \mathbb{P} -a.s.

We then have the following result.

Theorem 3.2. *There exists $\xi^*(\theta) \in \mathcal{A}_e$ which is the unique solution to $\mathbf{SP}(\alpha(\theta), \beta(\theta), x, i)$ with decomposition $\xi^*(\theta) = \xi^{*,+}(\theta) - \xi^{*, -}(\theta)$, where*

$$(3.18) \quad \begin{aligned} \xi^{*,+}(\theta) &:= \sup_{0 \leq s \leq t} \left(\alpha_{Y_s}(\theta) - I(X^{\xi^*(\theta)})_s + \xi_s^{*, -}(\theta) \right)^+, \\ \xi_s^{*, -}(\theta) &:= \sup_{0 \leq s \leq t} \left(I(X^{\xi^*(\theta)})_s + \xi_s^{*,+}(\theta) - \beta_{Y_s}(\theta) \right)^+ \end{aligned}$$

Proof. See Appendix B. □

We now prove that $\xi^*(\theta)$ solving $\mathbf{SP}(\alpha(\theta), \beta(\theta), x, i)$ is optimal for (3.1).

Proposition 3.5. *For every $(x, i) \in \mathcal{O}$, $\xi^*(\theta) = \xi^{*,+}(\theta) - \xi^{*, -}(\theta)$ solving $\mathbf{SP}(\alpha(\theta), \beta(\theta), x, i)$ is optimal for (3.1). Moreover, (3.3) holds with λ as in (3.13).*

Proof. Let $T > 0$ and $(x, i) \in \mathcal{O}$. Recall that U as in (3.11) is such that $U(\cdot, i) \in C^2(\mathcal{I})$, $i \in \mathbb{Y}$, and (U, λ) solves (3.12), with λ as in (3.13). Let $\{\eta_n\}_{n \in \mathbb{N}}$ be an increasing sequence of \mathbb{F} -stopping times such that $\eta_n \uparrow \infty$, $n \rightarrow \infty$. Fixing $\xi \in \mathcal{A}_e$ and applying Itô-Meyer's formula (in the sense of [11]) to $(U(X_{T \wedge \eta_n}, Y_{T \wedge \eta_n}; \theta))_{T \geq 0}$, we have:

$$(3.19) \quad \begin{aligned} U(X_{T \wedge \eta_n}^\xi, Y_{T \wedge \eta_n}; \theta) &= U(x, i; \theta) + \int_0^{T \wedge \eta_n} \mathcal{L}_{(X, Y)} U(X_s^\xi, Y_s; \theta) ds \\ &+ \int_0^{T \wedge \eta_n} U_x(X_s^\xi, Y_s; \theta) \sigma(X_s^\xi, Y_s) dW_s + \int_0^{T \wedge \eta_n} U_x(X_s^\xi, Y_s; \theta) (d\xi_s^{+,c} - d\xi_s^{-,c}) \\ &+ \sum_{0 \leq s \leq T \wedge \eta_n, s \in \Lambda^\xi} (U(X_{s+}^\xi, Y_s; \theta) - U(X_s^\xi, Y_s; \theta)) + M_{T \wedge \eta_n} - M_0 \end{aligned}$$

where $\Lambda^\xi := \{t \geq 0 : \xi_{t+} \neq \xi_t\}$, $\xi^{\pm, c}$ denotes the continuous part of ξ^\pm , and

$$(3.20) \quad M_t := \int_0^t \int_{\mathbb{R}_+} (U(X_s^\xi, h(Y_s, z) + Y_0; \theta) - U(X_s^\xi, Y_s; \theta)) \nu(dt, dz)$$

is an \mathbb{F} -local martingale, where $\nu(dt, dz) := \varphi(dt, dz) - dt \otimes m(dz)$. In particular, $\varphi(dt, dz)$ denotes the Poisson random measure with intensity $dt \otimes m(dz)$ for Lebesgue measure m , and the function $h : \mathbb{Y} \times \mathbb{R}_+$ has the form

$$h(i, z) = \sum_{j=1}^d (j - i) \mathbf{1}_{\{z \in \Delta_{i,j}\}},$$

with $\Delta_{i,j}$ being the partition of the \mathbb{R}_+ where each interval has length q_{ij} (for further details see Chapter 2 in [51]). We then define the \mathbb{F} -local martingale $\{\tilde{M}_t\}_{t \geq 0}$ as

$$(3.21) \quad \tilde{M}_t := M_t + \int_0^t U_x(X_s^\xi, Y_s; \theta) \sigma(X_s^\xi, Y_s) dW_s$$

and notice that $\mathbb{E}_{(x,i)}[\tilde{M}_{T \wedge \eta_n}] = 0, \forall T \geq 0, n \in \mathbb{N}$. Then, taking expectations in (3.19), we obtain

$$\begin{aligned} \mathbb{E}_{(x,i)}[U(X_{T \wedge \eta_n}^\xi, Y_{T \wedge \eta_n}; \theta)] &= U(x, i; \theta) + \mathbb{E}_{(x,i)} \left[\int_0^{T \wedge \eta_n} \mathcal{L}_{(X, Y)} U(X_s^\xi, Y_s; \theta) ds \right] \\ &+ \mathbb{E}_{(x,i)} \left[\int_0^{T \wedge \eta_n} U_x(X_s^\xi, Y_s; \theta) (d\xi_s^{+,c} - d\xi_s^{-,c}) \right] \\ &+ \mathbb{E}_{(x,i)} \left[\sum_{0 \leq s \leq T \wedge \eta_n, s \in \Lambda^\xi} (U(X_{s+}^\xi, Y_s; \theta) - U(X_s^\xi, Y_s; \theta)) \right]. \end{aligned}$$

Employing the second display inequality in (3.12), we find

$$(3.22) \quad \mathbb{E}_{(x,i)} [U(X_{T \wedge \eta_n}^\xi, Y_{T \wedge \eta_n}; \theta)] \leq U(x, i; \theta) + \mathbb{E}_{(x,i)} \left[\int_0^{T \wedge \eta_n} (\lambda(\theta, Y_s) - \pi(X_s^\xi, \theta)) ds \right] \\ + \mathbb{E}_{(x,i)} \left[\int_0^{T \wedge \eta_n} U_x(X_s^\xi, Y_s; \theta) (d\xi_s^{+,c} - d\xi_s^{-,c}) \right] \\ + \mathbb{E}_{(x,i)} \left[\sum_{0 \leq s \leq T \wedge \eta_n, s \in \Lambda^\xi} (U(X_{s+}^\xi, Y_s; \theta) - U(X_s^\xi, Y_s; \theta)) \right].$$

Since

$$U(X_{s+}^\xi, Y_s; \theta) - U(X_s^\xi, Y_s; \theta) = \mathbf{1}_{\{\Delta \xi_s^+ > 0\}} \int_0^{\Delta \xi_s^+} U_x(X_s^\xi + z, Y_s; \theta) dz \\ - \mathbf{1}_{\{\Delta \xi_s^- > 0\}} \int_0^{\Delta \xi_s^-} U_x(X_s^\xi - z, Y_s; \theta) dz,$$

then (3.22), together with the fourth and sixth conditions in (3.12), give

$$(3.23) \quad \mathbb{E}_{(x,i)} [U(X_{T \wedge \eta_n}^\xi, Y_{T \wedge \eta_n}; \theta)] \leq U(x, i; \theta) + \mathbb{E}_{(x,i)} \left[\int_0^{T \wedge \eta_n} (\lambda(\theta, Y_s) - \pi(X_s^\xi, \theta)) ds \right] \\ + \mathbb{E}_{(x,i)} \left[\int_0^{T \wedge \eta_n} (k_1 d\xi_t^+ - k_2 d\xi_t^-) \right].$$

If now the dominated convergence theorem can be applied (we will verify this later), taking limits as $n \uparrow \infty$ in (3.23) we find

$$(3.24) \quad \mathbb{E}_{(x,i)} [U(X_T^\xi, Y_T; \theta)] \leq U(x, i; \theta) + \mathbb{E}_{(x,i)} \left[\int_0^T (\lambda(\theta, Y_s) - \pi(X_s^\xi, \theta)) ds + k_1 \xi_T^+ - k_2 \xi_T^- \right].$$

Rearranging the terms, dividing by T , and sending $T \uparrow \infty$, we conclude that

$$(3.25) \quad \limsup_{T \uparrow \infty} \left(\frac{1}{T} \mathbb{E}_{(x,i)} [U(X_T^\xi, Y_T; \theta)] + \frac{1}{T} \mathbb{E}_{(x,i)} \left[\int_0^T \pi(X_t^\xi, \theta) dt - k_1 \xi_T^+ + k_2 \xi_T^- \right] \right) \\ \leq \limsup_{T \uparrow \infty} \left(\frac{1}{T} U(x, i; \theta) + \frac{1}{T} \mathbb{E}_{(x,i)} \left[\int_0^T \lambda(\theta, Y_t) dt \right] \right),$$

which implies

$$(3.26) \quad \limsup_{T \uparrow \infty} \left(\frac{1}{T} \mathbb{E}_{(x,i)} [U(X_T^\xi, Y_T; \theta)] + \frac{1}{T} \mathbb{E}_{(x,i)} \left[\int_0^T \pi(X_s^\xi, \theta) ds - k_1 \xi_T^+ + k_2 \xi_T^- \right] \right) \\ \leq \limsup_{T \uparrow \infty} \frac{1}{T} U(x, i; \theta) + \limsup_{T \uparrow \infty} \frac{1}{T} \mathbb{E}_{(x,i)} \left[\int_0^T \lambda(\theta, Y_s) ds \right].$$

Hence, using the inequality

$$\liminf_{n \rightarrow \infty} (f_n) + \limsup_{n \rightarrow \infty} (g_n) \leq \limsup_{n \rightarrow \infty} (f_n + g_n),$$

on the left-hand side of (3.26), we find that

$$(3.27) \quad \liminf_{T \uparrow \infty} \frac{1}{T} \mathbb{E}_{(x,i)} [U(X_T^\xi, Y_T; \theta)] + \limsup_{T \uparrow \infty} \frac{1}{T} \mathbb{E}_{(x,i)} \left[\int_0^T \pi(X_s^\xi, \theta) ds - k_1 \xi_T^+ + k_2 \xi_T^- \right] \\ \leq \limsup_{T \uparrow \infty} \frac{1}{T} U(x, i; \theta) + \limsup_{T \uparrow \infty} \frac{1}{T} \mathbb{E}_{(x,i)} \left[\int_0^T \lambda(\theta, Y_s) ds \right].$$

Since $U_x \in [k_2, k_1]$ by (3.12), we have that exists $C > 0$ such that for every $(x, i, \theta) \in \mathcal{O} \times \mathbb{R}_+$ one has

$$(3.28) \quad |U(x, i, \theta)| \leq C(1 + |x|).$$

This fact gives

$$(3.29) \quad \lim_{T \uparrow \infty} \frac{1}{T} U(x, i, \theta) = 0,$$

and, thanks to the fact that $\xi \in \mathcal{A}_e$ (cf. (2.8)), we also find

$$(3.30) \quad \begin{aligned} \liminf_{T \uparrow \infty} \frac{1}{T} \mathbb{E}_{(x,i)} [U(X_T^\xi, Y_T; \theta)] &\geq \liminf_{T \uparrow \infty} \frac{1}{T} \mathbb{E}_{(x,i)} [-C(1 + |X_T^\xi|)] \\ &= -C \limsup_{T \uparrow \infty} \frac{1}{T} \mathbb{E}_{(x,i)} [|X_T^\xi|] = 0, \end{aligned}$$

as well as,

$$(3.31) \quad \begin{aligned} \limsup_{T \uparrow \infty} \frac{1}{T} \mathbb{E}_{(x,i)} [U(X_T^\xi, Y_T; \theta)] &\leq C \limsup_{T \uparrow \infty} \frac{1}{T} \mathbb{E}_{(x,i)} [C(1 + |X_T^\xi|)] \\ &\leq \limsup_{T \uparrow \infty} \frac{1}{T} \mathbb{E}_{(x,i)} [|X_T^\xi|] = 0. \end{aligned}$$

Hence, using (3.29) and (3.30) in (3.27), we obtain

$$(3.32) \quad \begin{aligned} \limsup_{T \uparrow \infty} \frac{1}{T} \mathbb{E}_{(x,i)} \left[\int_0^T \pi(X_s^\xi, \theta) ds - k_1 \xi_T^+ + k_2 \xi_T^- \right] \\ \leq \limsup_{T \uparrow \infty} \frac{1}{T} \mathbb{E}_{(x,i)} \left[\int_0^T \lambda(\theta, Y_s) ds \right] = \sum_{i \in \mathbb{Y}} \lambda(\theta, i) p(i). \end{aligned}$$

Here, the last equality is due to the ergodicity of Y (see, for instance, Corollary 25.9 in [34]). Since (3.32) holds for every $\xi \in \mathcal{A}_e$, we get

$$(3.33) \quad \bar{\lambda}(\theta) \leq \sum_{i \in \mathbb{Y}} \lambda(\theta, i) p(i).$$

Letting now $\xi^*(\theta)$ be the solution to $\mathbf{SP}(\alpha(\theta), \beta(\theta), x, i)$, we notice that in (3.23), and thus in (3.25), the inequality becomes an equality. Then, using that $\limsup_{n \uparrow \infty} (f_n + g_n) \leq \limsup_{n \uparrow \infty} (f_n) + \limsup_{n \uparrow \infty} (g_n)$ on the left-hand side of (3.25) (with equality) and $\liminf_{n \uparrow \infty} (f_n) + \limsup_{n \uparrow \infty} (g_n) \leq \limsup_{n \uparrow \infty} (f_n + g_n)$ on the right-hand side of (3.25) (with equality) we obtain

$$(3.34) \quad \begin{aligned} \limsup_{T \uparrow \infty} \frac{1}{T} \mathbb{E}_{(x,i)} [U(X_T^{\xi^*(\theta)}, Y_T; \theta)] + \limsup_{T \uparrow \infty} \frac{1}{T} \mathbb{E}_{(x,i)} \left[\int_0^T \pi(X_s^{\xi^*(\theta)}, \theta) ds - k_1 \xi_T^{*,+}(\theta) + k_2 \xi_T^{*, -}(\theta) \right] \\ \geq \liminf_{T \uparrow \infty} \frac{1}{T} U(x, i, \theta) + \limsup_{T \uparrow \infty} \frac{1}{T} \mathbb{E}_{(x,i)} \left[\int_0^T \lambda(\theta, Y_s) ds \right]. \end{aligned}$$

By means of (3.29), (3.31) and the fact that $\xi^*(\theta) \in \mathcal{A}_e$ we thus conclude from (3.34)

$$(3.35) \quad \limsup_{T \uparrow \infty} \frac{1}{T} \mathbb{E}_{(x,i)} \left[\int_0^T \pi(X_s^{\xi^*(\theta)}, \theta) ds - k_1 \xi_T^{*,+}(\theta) + k_2 \xi_T^{*, -}(\theta) \right] \geq \limsup_{T \uparrow \infty} \frac{1}{T} \mathbb{E}_{(x,i)} \left[\int_0^T \lambda(\theta, Y_s) ds \right].$$

This, together with (3.33), provides the claimed optimality of $\xi^*(\theta)$ and that

$$(3.36) \quad \bar{\lambda}(\theta) = \sum_{i \in \mathbb{Y}} \lambda(\theta, i) p(i).$$

In order to complete the proof, it remains to show that expectations and limits as $n \uparrow \infty$ can be interchanged in (3.23). By writing (3.23) as

$$\begin{aligned} & \mathbb{E}_{(x,i)} [U(X_{T \wedge \eta_n}^\xi, Y_{T \wedge \eta_n}; \theta)] + \mathbb{E}_{(x,i)} \left[\int_0^{T \wedge \eta_n} (\pi(X_s^\xi, \theta(Y_s)) ds + k_2 d\xi_t^-) \right] \\ & \leq U(x, i; \theta) + \mathbb{E}_{(x,i)} \left[\int_0^{T \wedge \eta_n} (\lambda(\theta, Y_s) ds + \kappa_1 d\xi_s^+) \right], \end{aligned}$$

it is easy to see that the limit as $n \uparrow \infty$ can be interchanged with the expectations of the integral terms by the monotone convergence theorem. We are thus left to show that

$$(3.37) \quad \lim_{n \uparrow \infty} \mathbb{E}_{(x,i)} [U(X_{T \wedge \eta_n}^\xi, Y_{T \wedge \eta_n}; \theta)] = \mathbb{E}_{(x,i)} [U(X_T^\xi, Y_T; \theta)].$$

By (3.28),

$$|U(X_{T \wedge \eta_n}^\xi, Y_{T \wedge \eta_n}; \theta)| \leq C(1 + |X_{T \wedge \eta_n}^\xi|) \leq C(1 + \sup_{t \leq T} |X_t^\xi|).$$

Hence, by standard estimates based on the Lipschitz property of $b(\cdot, i)$ and $\sigma(\cdot, i)$ and the fact that $\xi \in \mathcal{A}_e$, we conclude that $\mathbb{E}_{(x,i)} [\sup_{t \leq T} |X_t^\xi|] < \infty$, thus allowing the application of the dominated convergence theorem in order to derive (3.37). \square

4. MEAN-FIELD EQUILIBRIUM

In the following, we prove the existence and uniqueness of the mean-field equilibrium (cf. Definition 2.1) by an application of Schauder-Tychonof fixed-point theorem. Let $\mathcal{P}(\mathcal{O})$ be the space of probability measures on \mathcal{O} , endowed with the weak topology.

4.1. Continuity of the free-boundaries w.r.t. the mean-field term. In this subsection we establish some regularity of the free-boundaries with respect to the variable θ . For the subsequent analysis we introduce the following assumptions.

Assumption 4.1. *The following hold.*

- (1) *There exists $\kappa : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that*

$$\lim_{\theta \downarrow 0} \pi_x(x, \theta) = \infty, \quad \lim_{\theta \uparrow \infty} \pi_x(x, \theta) = \kappa(x), \quad \text{for any } x \in \mathcal{I}$$

and, for some $\hat{x}_- := (\hat{x}_-(1), \dots, \hat{x}_-(d))$ and $\hat{x}_+ := (\hat{x}_+(1), \dots, \hat{x}_+(d))$, we have

$$\kappa(x) + k_1 b_x(x, i) \begin{cases} > 0, & x < \hat{x}_-(i), \\ = 0, & x = \hat{x}_-(i), \\ < 0, & x > \hat{x}_-(i), \end{cases} \quad \kappa(x) + k_2 b_x(x, i) \begin{cases} > 0, & x < \hat{x}_+(i), \\ = 0, & x = \hat{x}_+(i), \\ < 0, & x > \hat{x}_+(i). \end{cases}$$

- (2) *It holds that $b_{xx}(x, i) \leq 0$ for any $(x, i) \in \mathcal{O}$, and, for any compact $\mathcal{X} \times \Theta \subset \mathcal{I} \times \mathbb{R}_+$, there exists a constant $C_{(\mathcal{X}, \Theta)} > 0$ such that*

$$\widehat{\mathbb{E}}_{(x,i)} \left[\int_0^\infty e^{-ct} |\pi_{x\theta}(\widehat{X}_t, \theta)| dt \right] < C_{(\mathcal{X}, \Theta)} < \infty,$$

for any $(x, i, \theta) \in \mathcal{X} \times \mathbb{Y} \times \Theta$, with c as in Assumption 2.1.

- (3) *For any $i \in \mathbb{Y}$ we have*

$$\sum_{j \in \mathbb{Y}} q_{ji} \leq 0.$$

Lemma 4.1. *For any $i \in \mathbb{Y}$, the maps $\theta \mapsto \alpha_i(\theta)$ and $\theta \mapsto \beta_i(\theta)$ are nonincreasing.*

Proof. Take arbitrary $i \in \mathbb{Y}$ and let $\theta_1, \theta_2 \in \mathbb{R}_+$ such that $\theta_1 \leq \theta_2$. Then, by Proposition 3.1, we have that the map $\theta \mapsto v(x, i; \theta)$ is decreasing, for any $(x, i) \in \mathcal{O}$. Hence,

$$\{x \in \mathcal{I} : v(x, i; \theta_2) \geq k_1\} \subseteq \{x \in \mathcal{I} : v(x, i; \theta_1) \geq k_1\},$$

which, by (3.7), implies that

$$\alpha_i(\theta_2) \leq \alpha_i(\theta_1).$$

The monotonicity of $\theta \mapsto \beta_i(\theta)$, $i \in \mathbb{Y}$, can be proved similarly. \square

Lemma 4.2. *The following hold.*

- (1) *For any compact \mathcal{X} , the map $\theta \mapsto v(x, i; \theta)$ is locally Lipschitz continuous, uniformly for $(x, i) \in \mathcal{X} \times \mathbb{Y}$.*
- (2) *The function $v_x(x, i; \theta)$ is nonpositive for any $(x, i, \theta) \in \mathcal{O} \times \mathbb{R}_+$ and for fixed $\theta \in \mathbb{R}_+$ is strictly negative for any $(x, i) \in \mathcal{C}^\theta$, where \mathcal{C}^θ is the continuation region as in Theorem 3.1.*

Proof. Take a compact $\mathcal{X} \times \Theta \subset \mathcal{I} \times \mathbb{R}_+$ and generic $(x, i, \theta) \in \mathcal{X} \times \mathbb{Y} \times \Theta$. Following the proof of Proposition 3.3 (cf. Appendix A), for $\epsilon > 0$ small enough, by mean value theorem, we have that

$$\begin{aligned} |v(x, i; \theta + \epsilon) - v(x, i; \theta)| &\leq \widehat{\mathbb{E}}_{(x, i)} \left[\int_0^\infty e^{\int_0^t b_x(\widehat{X}_s, Y_s) ds} |\pi_x(\widehat{X}_t, \theta + \epsilon) - \pi_x(\widehat{X}_t, \theta)| dt \right] \\ &= \epsilon \widehat{\mathbb{E}}_{(x, i)} \left[\int_0^\infty e^{\int_0^t b_x(\widehat{X}_s, Y_s) ds} |\pi_{x\theta}(\widehat{X}_t, \tilde{\theta})| dt \right] \\ &\leq \epsilon \widehat{\mathbb{E}}_{(x, i)} \left[\int_0^\infty e^{-ct} |\pi_{x\theta}(\widehat{X}_t, \tilde{\theta})| dt \right] \leq \epsilon C_{(\mathcal{X}, \Theta^*)}, \end{aligned}$$

for $\tilde{\theta} \in (\theta, \theta + \epsilon)$ and a compact Θ^* such that $\theta + \epsilon \in \Theta^*$ for any $\theta \in \Theta$. Thus, since $C_{(\mathcal{X}, \Theta^*)} < \infty$ by Assumption 4.1-(2), we deduce that the map $\theta \mapsto v(x, i; \theta)$ is locally Lipschitz continuous, proving Claim (1).

We next prove Claim (2). Using the representation of v_x in Lemma A.1, Assumption 2.2-(2) and Assumption 4.1-(2), one can easily check (by observing that $\partial_x \widehat{X}_t^x > 0$, $\widehat{\mathbb{P}} \otimes dt$ -a.s., since $\partial_x \widehat{X}^x$ is a solution to a linear equation) that the function v_x is nonpositive for any $(x, i, \theta) \in \mathcal{O} \times \mathbb{R}_+$. For arbitrary fixed $\theta \in \mathbb{R}_+$ and for any $(x, i) \in \mathcal{C}^\theta$ we know that $\tau^*(x, i; \theta) \wedge \sigma^*(x, i; \theta) > 0$, $\widehat{\mathbb{P}}$ -a.s., hence by representation of v_x (cf. Lemma A.1) we conclude that $v_x(x, i; \theta) < 0$. \square

Proposition 4.1. *The map $\theta \mapsto (\alpha_i(\theta), \beta_i(\theta))_{i \in \mathbb{Y}}$ is continuous.*

Proof. For fixed $i \in \mathbb{Y}$ we, first prove that the map $\theta \mapsto \alpha_i(\theta)$ is left-continuous. Fix $\theta \in \mathbb{R}_+$, a sequence $\{\theta^n\}_{n \in \mathbb{N}}$, and a compact $\Theta \subset \mathbb{R}_+$ such that $\theta^n \nearrow \theta$, and $\theta, \theta^n \in \Theta$ for any $n \in \mathbb{N}$. Since the map $\theta \mapsto \alpha_i(\theta)$ is nonincreasing (cf. Lemma 4.1), arguing by contradiction, we assume that there exists $\delta > 0$ such that $\alpha_i(\theta) + \delta \leq \alpha_i(\theta^n)$ for any $n \in \mathbb{N}$. Observe that, since the function α_i is finite and monotone (cf. Lemma 4.1), one has

$$(4.1) \quad \alpha_i(\theta^n) \in \left[\inf_{y \in \Theta} \alpha_i(y), \sup_{y \in \Theta} \alpha_i(y) \right] =: \mathcal{K}, \quad \text{for any } n,$$

with \mathcal{K} being compact. Thanks to the fourth display equation in (3.10) we have that $v(\alpha_i(\theta), i; \theta) = v(\alpha_i(\theta^n), i; \theta^n) = k_1$ for any $n \in \mathbb{N}$. Hence, using Proposition 3.3 and Lemma 4.2 we can write

$$\begin{aligned} 0 &= v(\alpha_i(\theta^n), i; \theta^n) - v(\alpha_i(\theta), i; \theta) \\ &= (v(\alpha_i(\theta^n), i; \theta^n) - v(\alpha_i(\theta^n), i; \theta)) + (v(\alpha_i(\theta^n), i; \theta) - v(\alpha_i(\theta), i; \theta)) \\ &= - \int_{\theta^n}^{\theta} v_\theta(\alpha_i(\theta^n), i; y) dy + \int_{\alpha_i(\theta)}^{\alpha_i(\theta^n)} v_x(y, i; \theta) dy \end{aligned}$$

or, equivalently,

$$(4.2) \quad \int_{\alpha_i(\theta)}^{\alpha_i(\theta^n)} (-v_x(y, i; \theta)) dy = \int_{\theta^n}^{\theta} (-v_\theta(\alpha_i(\theta^n), i; y)) dy.$$

By Lemma 4.2 we have $v_x(x, i; \theta) < 0$ for fixed $\theta \in \Theta$ and for any $(x, i) \in \mathcal{C}^\theta$, then for $\epsilon \in (0, \delta)$ small enough we have that

$$(4.3) \quad \begin{aligned} \int_{\alpha_i(\theta)}^{\alpha_i(\theta^n)} (-v_x(y, i; \theta)) dy &\geq \int_{\alpha_i(\theta)+\epsilon}^{\alpha_i(\theta)+\delta} (-v_x(y, i; \theta)) dy \\ &\geq \min_{y \in [\alpha_i(\theta)+\epsilon, \alpha_i(\theta)+\delta]} |v_x(y, i; \theta)| (\delta - \epsilon) > 0, \end{aligned}$$

and

$$(4.4) \quad \int_{\theta^n}^{\theta} (-v_\theta(\alpha_i(\theta^n), i; y)) dy \leq \max_{y \in \Theta} |v_\theta(\alpha_i(\theta^n), i; y)| (\theta - \theta^n).$$

Combining (4.3) and (4.4) with (4.2) we arrive at the inequality,

$$0 < \min_{y \in [\alpha_i(\theta)+\epsilon, \alpha_i(\theta)+\delta]} |v_x(y, i; \theta)| (\delta - \epsilon) \leq \max_{(x, y) \in \mathcal{K} \times \Theta} |v_\theta(x, i; y)| (\theta - \theta^n) \leq C(\theta - \theta^n),$$

for a constant C not depending on n (thanks to Lemma 4.2). Thus, letting $n \rightarrow \infty$ leads to a contradiction.

We next show that the map $\theta \mapsto \alpha_i(\theta)$ is right-continuous. Fix a sequence $\{\theta^n\}_{n \in \mathbb{N}} \subseteq \Theta$ such that $\theta^n \searrow \theta$. Again by the monotonicity of the map $\theta \mapsto \alpha_i(\theta)$, arguing by contradiction we assume that there exists $\delta > 0$ such that $\alpha_i(\theta^n) \leq \alpha_i(\theta) - \delta$ for any $n \in \mathbb{N}$. Similarly to the first part of the proof, we find

$$\int_{\theta}^{\theta^n} (-v_\theta(\alpha_i(\theta^n), i; y)) dy = \int_{\alpha_i(\theta^n)}^{\alpha_i(\theta)} (-v_x(y, i; \theta^n)) dy.$$

which leads to the following inequality

$$0 < \min_{y \in [\alpha_i(\theta)-\delta, \alpha_i(\theta)-\epsilon]} |v_x(y, i; \theta)| (\delta - \epsilon) \leq \max_{(x, y) \in \mathcal{K} \times \Theta} |v_\theta(x, i; y)| (\theta^n - \theta) \leq C(\theta - \theta^n),$$

for \mathcal{K} as in (4.1) and a constant C not depending on n . Thus, letting $n \rightarrow \infty$, we obtain a contradiction.

By repeating the same argument, one can show that the map $\theta \mapsto \beta_i(\theta)$ is continuous, thus completing the proof. \square

4.2. Analysis of the stationary distribution. First of all, we show that a stationary distribution for the process $(X_t^{\xi^*(\theta)}, Y_t)_{t \geq 0}$ exists.

Proposition 4.2. *For any $\theta \in \mathbb{R}_+$, there exists a unique stationary distribution $\nu^\theta \in \mathcal{P}(\mathcal{O})$ for the hybrid-reflected process $(X_t^{\xi^*(\theta)}, Y_t)_{t \geq 0}$.*

Proof. In light of Theorem 2.1 in [47], in order to establish the existence and uniqueness of a stationary distribution, it is sufficient to show that $(X_t^{\xi^*(\theta)}, Y_t)_{t \geq 0}$ is a regenerative process (cf. Definition (D1) in [47]) with finite length regenerative epochs. In particular, for $(x, i) \in \mathcal{O}$, setting $\eta := \inf\{t > 0 : X_t^{\xi^*(\theta)} = \alpha_{Y_t}(\theta)\}$, $\mathbb{P}_{(x, i)}$ -a.s., by strong Markov property we have that the processes $(X_{t+\eta}^{\xi^*(\theta)}, Y_{t+\eta})_{t \geq 0}$ and the collection of random variables $((X_t^{\xi^*(\theta)}, Y_t)_{t < \eta}, \eta)$ are independent. Moreover, when starting from points in the set $\{(\alpha_i(\theta), i) | i \in \mathbb{Y}\}$, the process $(X_t^{\xi^*(\theta)}, Y_t)_{t \geq 0}$ has the same distribution as the process $(X_{t+\eta}^{\xi^*(\theta)}, Y_{t+\eta})_{t \geq 0}$. Thus, $(X_t^{\xi^*(\theta)}, Y_t)_{t \geq 0}$ is a regenerative process, with regenerative epoch η . In order to conclude the proof, it remains to show that $\mathbb{E}_{(x, i)}[\eta] < \infty$, for any $(x, i) \in \mathcal{O}$.

For $(x, i) \in \mathcal{O}$ and $y \in [\min_{j \in \mathbb{Y}} \alpha_j(\theta), \max_{j \in \mathbb{Y}} \beta_j(\theta)]$, we define the \mathbb{F} -hitting time $\eta(y) := \inf\{t > 0 : X_t^{\xi^*(\theta)} = y\}$. If $Y_0 = i$ by Proposition B.2 we have that $X_t^{x, \xi^*(\theta)} \leq X_t^{\beta_i(\theta), \xi^*(\theta)}$, \mathbb{P}_i -a.s.. Thus, we have

$$(4.5) \quad \mathbb{E}_{(x,i)}[\eta] \leq \mathbb{E}_{(\beta_i(\theta), i)}[\eta] \leq \mathbb{E}_{(\beta_i(\theta), i)}[\eta(\alpha_i(\theta))].$$

We now estimate $\mathbb{E}_{(\beta_i(\theta), i)}[\eta(\alpha_i(\theta))]$. Let τ be the first jump of the process Y and denoting by Z the solution to the SDE

$$(4.6) \quad dZ_t = b(Z_t, i)dt + \sigma(Z_t, i)dW_t, \quad Z_0 = \beta_i(\theta),$$

for $\tilde{\eta} := \inf\{t > 0 : Z_t = \alpha_i(\theta)\}$, by a comparison principle (a slightly different version of Proposition B.2) we also have $X_t^{\beta_i(\theta), \xi^*(\theta)} \leq Z_t$, for any $t \in [0, \tau \wedge \tilde{\eta}]$, \mathbb{P}_i -a.s., which in turn implies

$$\mathbb{P}_{(\beta_i(\theta), i)}(\eta(\alpha_i(\theta)) < \tau) \geq \mathbb{P}_{(\beta_i(\theta), i)}(\tilde{\eta} < \tau).$$

Therefore, we find

$$(4.7) \quad \mathbb{P}_{(\beta_i(\theta), i)}(\eta(\alpha_i(\theta)) < \tau) \geq \mathbb{P}_{(\beta_i(\theta), i)}(\tilde{\eta} < \tau) = \int_0^\infty \mathbb{E}[\mathbf{1}_{\{\tilde{\eta} < t\}}] F_\tau(dt) := \rho > 0,$$

where F_τ denotes the distribution function of τ and the last inequality follows from the fact that the SDE (4.6) induces a regular diffusion (cf. Assumption 2.1).

Using (4.7), we have

$$(4.8) \quad \begin{aligned} \mathbb{E}_{(x,i)}[\eta(\alpha_i(\theta))] &= \mathbb{E}_{(x,i)}[\eta(\alpha_i(\theta))\mathbf{1}_{\{\eta(\alpha_i(\theta)) < \tau\}}] + \mathbb{E}_{(x,i)}[\eta(\alpha_i(\theta))\mathbf{1}_{\{\eta(\alpha_i(\theta)) \geq \tau\}}] \\ &\leq \mathbb{E}_i[\tau] + \mathbb{E}_{(x,i)}[\mathbb{E}_{(X_\tau^{\xi^*(\theta)}, Y_\tau)}[\eta(\alpha_i(\theta))(1 - \mathbf{1}_{\{\eta(\alpha_i(\theta)) < \tau\}})]] \\ &= \mathbb{E}_i[\tau] + \mathbb{E}_{(x,i)}[(1 - \mathbf{1}_{\{\eta(\alpha_i(\theta)) < \tau\}})]\mathbb{E}_{(x,i)}[\mathbb{E}_{(X_\tau^{\xi^*(\theta)}, Y_\tau)}[\eta(\alpha_i(\theta))]] \\ &= \mathbb{E}_i[\tau] + (1 - \mathbb{P}_{(x,i)}(\eta(\alpha_i(\theta)) < \tau))\mathbb{E}_{(x,i)}[\eta(\alpha_i(\theta))], \\ &\leq \mathbb{E}_i[\tau] + (1 - \rho)\mathbb{E}_{(x,i)}[\eta(\alpha_i(\theta))], \end{aligned}$$

where in the second equality we use the strong Markov property of $(X_t^{\xi^*(\theta)}, Y_t)_{t \geq 0}$. Then, since (4.8) holds for any $(x, i) \in \mathcal{O}$, we obtain

$$\sup_{(x,i) \in \mathcal{O}} \mathbb{E}_{(x,i)}[\eta(\alpha_i(\theta))] \leq \mathbb{E}_i[\tau] + (1 - \rho) \sup_{(x,i) \in \mathcal{O}} \mathbb{E}_{(x,i)}[\eta(\alpha_i(\theta))].$$

Equivalently, we write

$$(4.9) \quad \sup_{(x,i) \in \mathcal{O}} \mathbb{E}_{(x,i)}[\eta(\alpha_i(\theta))] \leq \frac{1}{\rho} \mathbb{E}_i[\tau] < \infty,$$

where, in the last inequality, we have used that the process $(Y_t)_{t \geq 0}$ is irreducible and positive recurrent.

Finally, plugging (4.9) into (4.5), we conclude that $\mathbb{E}_{(x,i)}[\eta] < \infty$, which in turn implies that $(X_t^{\xi^*(\theta)}, Y_t)_{t \geq 0}$ is a regenerative process with finite mean regenerative epochs. Hence, by Theorem 2.1 in [47], there exists a unique stationary distribution, concluding the proof. \square

Next, we characterize the stationary distribution and we study its stability with respect to changes of the boundary.

Theorem 4.1. *Let $\nu^\theta \in \mathcal{P}(\mathcal{O})$ be the stationary distribution of $(X_t^{\xi^*(\theta)}, Y_t)_{t \geq 0}$. The following hold:*

- (1) *For any $i \in \mathbb{Y}$, the cumulative distribution function $\mu^\theta(x, i) := \nu^\theta((x, x], i)$, $x \in \mathcal{I}$ has regularity $\mu^\theta(\cdot, i) \in C^1([\alpha_i(\theta), \beta_i(\theta)]) \cap C^2((\alpha_i(\theta), \beta_i(\theta)) \setminus \bigcup_{j \in \mathbb{Y}} \{\alpha_j(\theta), \beta_j(\theta)\})$ and it is the unique nondecreasing solution of the equation*

$$(4.10) \quad \frac{1}{2} \sigma^2(x, i) \mu_{xx}^\theta(x, i) - (b(x, i) - \sigma \sigma_x(x, i)) \mu_x^\theta(x, i) + \sum_{j \in \mathbb{Y}} q_{ji} \mu^\theta(x, j) = 0, \quad x \in (\alpha_i(\theta), \beta_i(\theta)),$$

satisfying the boundary conditions $\mu^\theta(x, i) = 0$, $x \leq \alpha_i(\theta)$ and $\mu^\theta(x, i) = p(i)$, $x \geq \beta_i(\theta)$.
 (2) The map $\nu : \mathbb{R}_+ \rightarrow \mathcal{P}(\mathcal{O})$, $\theta \mapsto \nu^\theta$ is continuous.

Proof. The proof of the first item follows closely the proof of Theorem 1 in [23], and it is proved in Appendix C. Therefore, we only provide a proof of Claim 2.

For any $(\alpha, \beta) = (\alpha_i, \beta_i)_{i \in \mathbb{Y}}$, we denote by $\mu(\cdot, \cdot; \alpha, \beta)$ the solution of (4.10) with $(\alpha(\theta), \beta(\theta)) = (\alpha, \beta)$. Then for $\Theta \subset \mathbb{R}_+$ compact such that $\theta \in \Theta$ and for $\{\theta^n\}_{n \in \mathbb{N}} \subset \Theta$ such that $\theta^n \rightarrow \theta$, as $n \uparrow \infty$, we know by Proposition 4.1 that $\alpha_i(\theta^n) \rightarrow \alpha_i(\theta) = \alpha_i$ and $\beta_i(\theta^n) \rightarrow \beta_i(\theta) = \beta_i$ for any $i \in \mathbb{Y}$ as $n \uparrow \infty$. For simplicity we set $(\alpha_i^n, \beta_i^n)_{i \in \mathbb{Y}} := (\alpha_i(\theta^n), \beta_i(\theta^n))_{i \in \mathbb{Y}}$ for any $n \in \mathbb{N}$. Unless repeating the subsequent argument with minor modifications, we assume that there exists $n_0 \in \mathbb{N}$ such that, for any $n \geq n_0$ and for any $i \in \mathbb{Y}$, we have $(\alpha_i^n, \beta_i^n) \subseteq (\alpha_i, \beta_i)$. For any $n \in \mathbb{N}$, let $\mu^n(x, i) := \nu^n(\underline{x}, x, i)$, $(x, i) \in \mathcal{O}$ be the cumulative distribution function of the stationary distribution of the solution to $\mathbf{SP}(\alpha^n, \beta^n, x, i)$ (cf. Proposition 4.2). According to Claim 1, μ^n satisfies (4.10) with boundary condition on (α^n, β^n) . Next, for any $n \in \mathbb{N}$ and for any $(x, i) \in \bigcup_{i=1}^d (\alpha_i^n, \beta_i^n) \times \{i\}$, define $v^n(x, i) := \mu(x, i) - \mu^n(x, i)$. Since μ and μ^n are solutions to the linear system (4.10), for each $n \in \mathbb{N}$, v^n is a solution to the boundary value problem

$$(4.11) \quad \begin{cases} \mathcal{L}_{(\widehat{X}, i)}^* v^n(x, i) + \sum_{j \in \mathbb{Y}} q_{ji} v^n(x, j) = 0, & x \in (\alpha_i^n, \beta_i^n), i \in \mathbb{Y}, \\ v^n(\alpha_i^n, i) = \mu(\alpha_i^n, i), & v^n(\beta_i^n, i) = \mu(\beta_i^n, i) - p(i), \end{cases}$$

where $\mathcal{L}_{(\widehat{X}, i)}^* v^n(x, i) := \frac{1}{2} \sigma^2(x, i) v_{xx}^n(x, i) - (b(x, i) - \sigma \sigma_x(x, i)) v_x^n(x, i)$. Now, thanks to Assumption 2.1 and Assumption 4.1-(3), we can apply Theorem 1 in [48] to (4.11) which gives us

$$\sup_{x \in (\alpha_i^n, \beta_i^n)} |v^n(x, i)| \leq C \max \{ |\mu(\alpha_i^n)|, |\mu(\beta_i^n, i) - p(i)| \}.$$

Therefore, for any $x \in (\alpha_i, \beta_i)$, we deduce that $\mu^n(x, i) \rightarrow \mu(x, i)$ as $n \rightarrow \infty$. Hence, by an application of Theorem 5.25 in [34] we obtain that $\nu^n \rightarrow \nu^\theta$ as $n \rightarrow \infty$ thus completing the proof of Claim 2. \square

4.3. Existence and uniqueness of the MFG equilibrium. Next, we turn our attention to the main result of this section. To this end, we introduce the operator $\mathcal{T} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, as

$$(4.12) \quad \mathcal{T}\theta := F \left(\sum_{i=1}^d \int_{\mathcal{I}} f(x) \nu^\theta(dx, i) \right) = F(\langle f, \nu^\theta \rangle), \quad \theta \in \mathbb{R}_+$$

where $\langle f, \nu^\theta \rangle := \sum_{i \in \mathbb{Y}} \int_{\mathcal{I}} f(x) \nu^\theta(dx, i)$. Thanks to the previous results, we can now prove the existence and uniqueness of a stationary mean-field equilibrium.

Theorem 4.2. *There exists a unique MFG equilibrium θ^* ; i.e., a unique $\theta^* \in \mathbb{R}_+$ such that $\mathcal{T}\theta^* = \theta^*$.*

Proof. We divide the proof in three steps.

Step 1: Set of relevant θ . We introduce the following auxiliary Dynkin game

$$(4.13) \quad \underline{v}(x, i) := \inf_{\tau \in \mathcal{T}} \sup_{\sigma \in \mathcal{T}} \widehat{\mathbb{E}}_{(x, i)} \left[\int_0^{\tau \wedge \sigma} e^{\int_0^t b_x(\widehat{X}_s, Y_s) ds} \kappa(\widehat{X}_s) ds + k_1 e^{\int_0^\tau b_x(\widehat{X}_s, Y_s) ds} \mathbf{1}_{\{\tau < \sigma\}} + k_2 e^{\int_0^\sigma b_x(\widehat{X}_s, Y_s) ds} \mathbf{1}_{\{\sigma < \tau\}} \right].$$

Arguing as in Section 3.1, the Dynkin game (4.13) admits a value and a saddle point. In particular, we can define $\underline{\alpha}_i := \sup\{x \in \mathcal{I} : \underline{v}(x, i) \geq k_1\}$ and $\underline{\beta}_i := \inf\{x \in \mathcal{I} : \underline{v}(x, i) \leq k_2\}$, as well as the stopping regions $\underline{\mathcal{S}}_{\text{inf}} := \{(x, i) \in \mathcal{O} : \underline{v}(x, i) \geq k_1\}$ and $\underline{\mathcal{S}}_{\text{sup}} := \{(x, i) \in \mathcal{O} : \underline{v}(x, i) \leq k_2\}$. Thus, we have that the stopping times $\underline{\tau}^*(x, i) := \inf\{t \geq 0 : (\widehat{X}_t, Y_t) \in \underline{\mathcal{S}}_{\text{inf}}\}$ and $\underline{\sigma}^*(x, i) := \inf\{t \geq 0 : (\widehat{X}_t, Y_t) \in \underline{\mathcal{S}}_{\text{sup}}\}$ realize a saddle-point for (4.13). Also by Condition (3) in Assumption 2.2 and

Condition (1) in Assumption 4.1 we conclude that $\pi_x(x, \theta) \geq \kappa(x)$, for any $\theta \in \mathbb{R}_+$, so that (cf. (3.4) and (3.6))

$$(4.14) \quad v(x, i) \leq v(x, i; \theta), \quad \text{for any } (x, i; \theta) \in \mathcal{O} \times \mathbb{R}_+.$$

Then, for any $(i, \theta) \in \mathbb{Y} \times \mathbb{R}_+$, we have

$$(4.15) \quad \alpha_i(\theta) = \sup\{x \in \mathcal{I} : v(x, i; \theta) \geq k_1\} \geq \sup\{x \in \mathcal{I} : \underline{v}(x, i) \geq k_1\} = \underline{\alpha}_i.$$

Analogously, we find

$$(4.16) \quad \beta_i(\theta) \geq \underline{\beta}_i, \quad \text{for any } (i, \theta) \in \mathbb{Y} \times \mathbb{R}_+.$$

We then define

$$(4.17) \quad \underline{\theta} := F\left(\sum_{i=1}^d \int_{\mathcal{I}} f(x) \underline{\nu}(dx, i)\right)$$

where $\underline{\nu}$ is the unique stationary distribution of the process $(\underline{X}_t, Y_t)_{t \geq 0}$ and \underline{X} is the solution to the SP $((\underline{\alpha}_i, \underline{\beta}_i)_{i \in \mathbb{Y}}; \underline{X}_0, Y_0)$. Then, by Proposition B.2, since f is increasing, we obtain

$$(4.18) \quad \mathbb{E}[f(\underline{X}_t)] \leq \mathbb{E}[f(X_t^{\xi^*(\theta)})], \quad \text{for any } t \geq 0.$$

Hence, by Proposition 4.2, we can employ the ergodic theorem (see pg.274 in [47]), in order to deduce that

$$\langle f, \underline{\nu} \rangle = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \mathbb{E}[f(\underline{X}_t)] dt \leq \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \mathbb{E}[f(X_t^{\xi^*(\theta)})] dt = \langle f, \nu^\theta \rangle,$$

which in turn, by monotonicity of F , leads to

$$(4.19) \quad \underline{\theta} \leq \mathcal{T}\theta.$$

To find an upper bound for $\mathcal{T}\theta$, we proceed as follows. Since, $X_t^{\xi^*(\theta)} \in [\alpha_{Y_t}(\theta), \beta_{Y_t}(\theta)]$, for any $t \geq 0$, \mathbb{P} -a.s. for any $\theta \in \mathbb{R}_+$, thanks to the monotonicity of F and f (see Condition (1) Assumption 2.3), we have

$$\mathcal{T}\theta = F\left(\sum_{i=1}^d \int_{\mathcal{I}} f(x) \nu^\theta(dx, i)\right) \leq F\left(\sum_{i=1}^d p(i) f(\beta_i(\theta))\right) \leq F\left(\sum_{i=1}^d p(i) f(\beta_i(\underline{\theta}))\right),$$

for any $\theta \geq \underline{\theta}$. In the last inequality above we have used the fact that the map $\theta \mapsto (\beta_i(\theta))_{i \in \mathbb{Y}}$ is nonincreasing (cf. Lemma 4.1). Hence, we define $\bar{\theta} := F\left(\sum_{i=1}^d p(i) f(\beta_i(\underline{\theta}))\right)$ and for any $\underline{\theta} \leq \theta$ we find that

$$(4.20) \quad \mathcal{T}\theta \leq \bar{\theta}.$$

Thus, combining (4.19) and (4.20), we conclude that any potential fixed point of \mathcal{T} must lie in the convex, compact set

$$(4.21) \quad K := [\underline{\theta}, \bar{\theta}] \subset \mathbb{R}_+.$$

Step 2: Continuity of \mathcal{T} . We begin by observing that the barriers related to $\theta \in K$, belong to a compact \widehat{K} , defined in terms of the compact set K in (4.21). Indeed, by Step 1 we know that $\underline{\alpha}_i \leq \alpha_i(\theta)$ for any $(i, \theta) \in \mathbb{Y} \times K$, and, by monotonicity of the map $\theta \mapsto \beta_i(\theta)$ for $i \in \mathbb{Y}$ we have $\beta_i(\theta) \leq \beta_i(\underline{\theta})$ for $i \in \mathbb{Y}$. Thus, for each $\theta \in K$ we have,

$$(4.22) \quad \alpha_i(\theta), \beta_i(\theta) \in \left[\min_{j \in \mathbb{Y}} \alpha_j, \max_{j \in \mathbb{Y}} \beta_j(\underline{\theta}) \right] =: \widehat{K}, \quad \text{for any } i \in \mathbb{Y}.$$

Define now the map $\mathcal{T}_1 : K \rightarrow \mathcal{P}(\widehat{K} \times \mathbb{Y})$ by

$$\mathcal{T}_1(\theta) := \nu^\theta, \quad \theta \in K.$$

By (4.22), such a map is well defined. Moreover, thanks to Claim 2 in Theorem 4.1, the map \mathcal{T}_1 is continuous. Next, we denote by $\mathcal{T}_2 : \mathcal{P}(\widehat{K} \times \mathbb{Y}) \rightarrow K$ the map

$$\mathcal{T}_2(\nu) := F\left(\sum_{i=1}^d \int_{\mathcal{I}} f(x)\nu(dx, i)\right).$$

Since the functions f and F are continuous and the probability measures have compact support, the map \mathcal{T}_2 is clearly continuous. Concluding, the map $\mathcal{T} := \mathcal{T}_2 \circ \mathcal{T}_1 : K \rightarrow K$ is continuous in the convex compact set K and, by Schauder-Tychonof fixed point theorem (Corollary 17.56 in [4]), there exist $\theta^* \in K$, such that $\mathcal{T}\theta^* = \theta^*$.

Step 3: Uniqueness. Let $\theta^* \in K$ be the fixed-point of \mathcal{T} and let $\theta' \in K$ another fixed-point of \mathcal{T} such that $\theta^* \neq \theta'$. Without loss of generality we can assume that $\theta^* < \theta'$ (the opposite inequality can be treated similarly). Following the same arguments as in Step 1 we conclude that

$$\alpha_i(\theta^*) \geq \alpha_i(\theta'), \quad \beta_i(\theta^*) \geq \beta_i(\theta'), \quad \forall i \in \mathbb{Y},$$

so that $\theta' = \mathcal{T}\theta' \leq \mathcal{T}\theta^* = \theta^*$, which leads to a contradiction. \square

5. N-PLAYER GAME AND APPROXIMATE NASH EQUILIBRIA

In this section we establish the classical connection between MFG and N -player game, by constructing approximate Nash equilibria starting from the MFG equilibrium.

5.1. N-player game. Let W and Y be as in Section 2 and assume the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ to be large enough to accommodate a sequence of independent and identically distributed \mathbb{F} -adapted processes $\{(W^n, Y^n)\}_{n \in \mathbb{N}}$ as well as independent and identically distributed $\mathcal{I} \times \mathbb{Y}$ -random variables $\{(x_0^n, i_0^n)\}_{n \in \mathbb{N}}$, (x_0, i_0) . Each (W^n, Y^n) has the same distribution as (W, Y) and the random variables (W^n, Y^n) , (W, Y) , $\{(x_0^n, i_0^n)\}_{n \in \mathbb{N}}$ and (x_0, i_0) are assumed to be independent.

When player n does not intervene, its state process X^n evolves accordingly to the SDE

$$dX_t^n = b(X_t^n, Y_t^n)dt + \sigma(X_t^n, Y_t^n)dW_t^n, \quad (X_0^n, Y_0^n) = (x_0^n, i_0^n).$$

Given a boundary vector $(\alpha^n, \beta^n) = (\alpha_j^n, \beta_j^n)_{j \in \mathbb{Y}} \in \mathbb{R}^{2d}$, the (α^n, β^n) -barrier-type strategy for player n is the reflection process for the state of player n in the regime-switching domain $(\alpha_{Y_t^n}^n, \beta_{Y_t^n}^n)_{t \geq 0}$; that is, the ξ^n -component of the solution (X^n, ξ^n) to the Skorokhod problem $\mathbf{SP}(\alpha^n, \beta^n, x, i)$ for the noise (W^n, Y^n) , accordingly to Definition 3.1. Without carrying the dependence on the index of the player, we denote by \mathcal{A}_b the set of barrier-type strategies. Thus, when player n choose a strategy $\xi^n \in \mathcal{A}_b$, its state process X^{n, ξ^n} evolves following the regime-switching (reflected) SDE

$$dX_t^{n, \xi^n} = b(X_t^{n, \xi^n}, Y_t^n)dt + \sigma(X_t^{n, \xi^n}, Y_t^n)dW_t^n + d\xi_t^{n,+} - d\xi_t^{n,-}, \quad (X_{0-}^{n, \xi^n}, Y_0^n) = (x_0^n, i_0^n).$$

The (generic) vector $\xi := (\xi^1, \dots, \xi^N) \in \mathcal{A}_b^N$ represents a tuple of control policies of the N players. We denote by $\xi^{-n} = (\xi^\ell)_{\ell \neq n}$ a vector of strategies of the opponents of player n , and we use the notation $\xi = (\xi^n, \xi^{-n})$. The profit functional of player n is defined as

$$(5.1) \quad J^n(\xi^n, \xi^{-n}) := \limsup_{T \uparrow \infty} \frac{1}{T} \mathbb{E} \left[\int_0^T \pi(X_t^{n, \xi^n}, \theta_{\xi^{-n}}^N(t)) dt - k_1 \xi_T^{n,+} + k_2 \xi_T^{n,-} \right],$$

where $\theta_{\xi^{-n}}^N$ denotes the mean-field interaction term between the players and has the form

$$(5.2) \quad \theta_{\xi^{-n}}^N(t) := F\left(\frac{1}{N-1} \sum_{\ell \neq n} f(X_t^{\ell, \xi^\ell})\right), \quad t \geq 0.$$

Our aim is to use the solution of the ergodic MFG as an approximating solution of the N -player game. In particular, for \widehat{K} being the compact set as in (4.22), defining the set of restricted barrier strategies

$$(5.3) \quad \widehat{\mathcal{A}}_b := \{(\alpha^n, \beta^n)\text{-barrier-type strategy with } (\alpha_j^n, \beta_j^n)_{j \in \mathbb{Y}} \in \widehat{K}^{2d}\},$$

we give the following definition of ϵ -Nash equilibrium.

Definition 5.1 (ϵ -Nash equilibrium). *For $\epsilon > 0$, $\bar{\xi} = (\bar{\xi}^1, \dots, \bar{\xi}^N) \in \widehat{\mathcal{A}}_b^N$ is called ϵ -Nash equilibrium for the N -player game if, for any $n = 1, \dots, N$, one has*

$$(5.4) \quad J^n(\bar{\xi}^n, \bar{\xi}^{-n}) \geq J^n(\xi^n, \bar{\xi}^{-n}) - \epsilon, \quad \forall \xi^n \in \widehat{\mathcal{A}}_b.$$

Let θ^* be the unique MFG equilibrium as in Theorem 4.2 and $(\alpha(\theta^*), \beta(\theta^*)) = (\alpha_i(\theta^*), \beta_i(\theta^*))_{i \in \mathbb{Y}}$ be the related boundary (see (3.7)). We now use the MFG equilibrium θ^* in order to construct profile strategies for the N -player games. For any $N \geq 1$ and $n = 1, \dots, N$, define the strategy $\bar{\xi}^n$ as the $(\alpha(\theta^*), \beta(\theta^*))$ -barrier strategy for the noise (W^n, Y^n) .

Remark 5.1 (On the initial distribution). *We point out that all the results in the previous sections hold true also if the deterministic initial condition $(X_{0-}, Y_0) = (x, i) \in \mathcal{I} \times \mathbb{Y}$ is replaced by the random initial condition $(X_{0-}, Y_0) = (x_0, i_0)$. Indeed, by considering (with slight abuse of notation) the payoff*

$$J(x_0, i_0; \xi, \theta) := \limsup_{T \uparrow \infty} \frac{1}{T} \mathbb{E} \left[\mathbb{E}_{(x_0, i_0)} \left[\int_0^T \pi(X_t^\xi, \theta) dt - k_1 \xi_T^+ + k_2 \xi_T^- \right] \right],$$

by the Markov property of the solution to the reflected Skorokhod problem the MFG equilibrium $(\xi(\theta^*), \theta^*)$ is still given by Theorem 4.2, with $\xi(\theta^*)$ characterized as in Proposition 3.5.

In light of the definition of \widehat{K} in (4.22), we have $\bar{\xi}^n \in \widehat{\mathcal{A}}_b$. Let $\bar{\xi}$ be the related profile strategy; i.e., set $\bar{\xi} := (\bar{\xi}^1, \dots, \bar{\xi}^N)$.

We first show the following preliminary result.

Proposition 5.1. *For any $N \geq 1$ and $\{\xi^n\}_{n \leq N} \in \widehat{\mathcal{A}}_b^N$, the ergodic limit*

$$\lim_{T \uparrow \infty} \frac{1}{T} \int_0^T G(X_t^{1, \xi^1}, Y_t^1, \dots, X_t^{N, \xi^N}, Y_t^N) dt = \int_{\mathcal{O}^N} G(x_1, i_1, \dots, x_N, i_N) \otimes_{\ell=1}^N \nu^\ell(dx_\ell, i_\ell), \quad \mathbb{P}\text{-a.s.}$$

holds for any bounded function $G : \mathcal{O}^N \rightarrow \mathbb{R}$.

Proof. Since $\{(W_t^n, Y_t^n)_{t \geq 0}\}_{n \in \mathbb{N}}$ is i.i.d., the family $\{(X_t^{n, \xi^n}, Y_t^n)_{t \geq 0}\}_{n \leq N}$ is i.i.d. For any fixed $1 \leq n \leq N$, by the proof of Proposition 4.2 we know that the process $(X_t^{n, \xi^n}, Y_t^n)_{t \geq 0}$ has a unique stationary distribution ν^n . Thus, by independence of the processes $(X_t^{n, \xi^n}, Y_t^n)_{t \geq 0}$, the stationary distribution for $(X^{1, \xi^1}, Y^1, \dots, X^{N, \xi^N}, Y^N)$, denoted by $\bar{\nu}^N \in \mathcal{P}(\mathcal{O}^N)$, exists and it is given by

$$\bar{\nu}^N(dx_1, i_1, \dots, dx_N, i_N) = \otimes_{\ell=1}^N \nu^\ell(dx_\ell, i_\ell).$$

Clearly, this distribution is unique. Thus, by Theorems 3.2.6 and 3.3.1 in [21], the ergodic limit holds. \square

5.2. Approximation result. We enforce the following condition.

Assumption 5.1. *There exists $C > 0$ such that,*

$$|\pi(x, \theta_1) - \pi(x, \theta_2)| \leq C(1 + |x|^\beta)|\theta_1 - \theta_2|,$$

for any $\theta_1, \theta_2 \in \mathbb{R}_+$ and $x \in \mathcal{I}$.

In the spirit of [15] and [16], we can now state and prove the main result of this section.

Theorem 5.1. *The profile strategy $\bar{\xi} := (\bar{\xi}^1, \dots, \bar{\xi}^N) \in \widehat{\mathcal{A}}_b^N$ is an ϵ_N -Nash equilibrium for the ergodic N -player game, with $\epsilon_N \rightarrow 0$ as $N \rightarrow \infty$.*

Proof. We first introduce some notation. Take barriers $(\alpha, \beta) = (\alpha_i, \beta_i)_{i \in \mathbb{Y}} \in \widehat{K}^{2d}$, with \widehat{K} as in (4.22). Denote by (X^{n, ξ^n}, ξ^n) the solution to $\mathbf{SP}(\alpha, \beta, x, i)$ for the noise (W^n, Y^n) of player n , and denote by $(\tilde{X}, \tilde{\xi})$ the solution to $\mathbf{SP}(\alpha, \beta, x, i)$ for the noise (W, Y) of the MFG. By uniqueness in law of the solution to the Skorokhod problem, the processes (X^{n, ξ^n}, ξ^n) and $(\tilde{X}^{n, \xi^n}, \tilde{\xi}^n)$ have the same law, hence

$$(5.5) \quad \begin{aligned} J^n(\xi^n, \theta^*) &:= \limsup_{T \uparrow \infty} \frac{1}{T} \mathbb{E} \left[\int_0^T \pi(X_t^{n, \xi^n}, \theta^*) dt - k_1 \xi_T^{n,+} + k_2 \xi_T^{n,-} \right] \\ &= \limsup_{T \uparrow \infty} \frac{1}{T} \mathbb{E} \left[\int_0^T \pi(\tilde{X}_t^{n, \xi^n}, \theta^*) dt - k_1 \tilde{\xi}_T^{n,+} + k_2 \tilde{\xi}_T^{n,-} \right] =: J(\tilde{\xi}, \theta^*). \end{aligned}$$

Next, introduce the error estimate between the functionals of the N -player game and the MFG. For $N \in \mathbb{N}$ and $1 \leq n \leq N$, set

$$R_N(\xi^n) := J^n(\xi^n, \bar{\xi}^{-n}) - J^n(\xi^n, \theta^*), \quad \xi^n \in \widehat{\mathcal{A}}_b.$$

Since the control $\xi(\theta^*)$ is optimal for θ^* (cf. Proposition 3.5), using (5.5), we have that

$$\begin{aligned} J^n(\bar{\xi}^n, \bar{\xi}^{-n}) - J^n(\xi^n, \bar{\xi}^{-n}) &= R_N(\bar{\xi}^n) - R_N(\xi^n) + J^n(\bar{\xi}^n, \theta^*) - J^n(\xi^n, \theta^*) \\ &= R_N(\bar{\xi}^n) - R_N(\xi^n) + J(\xi(\theta^*), \theta^*) - J(\tilde{\xi}^n, \theta^*) \\ &\geq R_N(\bar{\xi}^n) - R_N(\xi^n), \quad \text{for any } \xi^n \in \widehat{\mathcal{A}}_b. \end{aligned}$$

Therefore, in order to complete the proof it is sufficient to show that

$$(5.6) \quad \lim_{N \uparrow \infty} \sup_{\xi^n \in \widehat{\mathcal{A}}_b} R_N(\xi^n) = 0.$$

By using that $\overline{\lim}_{T \uparrow \infty} (\alpha_T + \beta_T) \geq \overline{\lim}_{T \uparrow \infty} (\alpha_T) + \underline{\lim}_{T \uparrow \infty} (\beta_T)$, write

$$(5.7) \quad \begin{aligned} R_N(\xi^n) &= \limsup_{T \uparrow \infty} \frac{1}{T} \mathbb{E} \left[\int_0^T \pi(X_t^{n, \xi^n}, \theta_{\bar{\xi}^{-n}}^N(t)) dt - k_1 \xi_T^{n,+} + k_2 \xi_T^{n,-} \right] \\ &\quad - \limsup_{T \uparrow \infty} \frac{1}{T} \mathbb{E} \left[\int_0^T \pi(X_t^{n, \xi^n}, \theta^*) dt - k_1 \xi_T^{n,+} + k_2 \xi_T^{n,-} \right] \\ &\leq \limsup_{T \uparrow \infty} \frac{1}{T} \mathbb{E} \left[\int_0^T \left(\pi(X_t^{n, \xi^n}, \theta_{\bar{\xi}^{-n}}^N(t)) - \pi(X_t^{n, \xi^n}, \theta^*) \right) dt \right]. \end{aligned}$$

Next, by Proposition 4.2, there exists a stationary distribution ν^{ξ^n} for the process $(X_t^{n, \xi^n}, Y_t)_{t \geq 0}$. Moreover, Proposition 5.1 ensures that the process $((X^{n, \xi^n}, Y^n), \{(X^\ell, \bar{\xi}^\ell, Y^\ell)_{t \geq 0}\}_{\ell \neq n})$ admits an ergodic distribution $\otimes_{\ell \neq n} \nu^\ell \otimes \nu^{\xi^n}$ with support in the compact set $(\widehat{K} \times \mathbb{Y})^N$. Thus, by continuity of the functions f, F and π , Proposition 5.1 also allows to rewrite (5.7) as

$$R_N(\xi^n) \leq \int_{\mathcal{O}} \int_{\mathcal{O}^{N-1}} \left(\pi \left(x_n, F \left(\sum_{\ell \neq n} \frac{f(x_\ell)}{N-1} \right) \right) - \pi(x_n, \theta^*) \right) \otimes_{\ell \neq n} \nu^\ell(dx_\ell, i_\ell) \otimes \nu^{\xi^n}(dx_n, i_n).$$

We proceed by further estimating $\sup_{\xi^n \in \widehat{A}_b} R_N(\xi^n)$. From the latter inequality, since $\otimes_{\ell \neq n} \nu^\ell \otimes \nu^{\xi^n}$ has compact support, we can use Assumption 5.1 to obtain

$$\begin{aligned} R_N(\xi^n) &\leq C \int_{\mathcal{O}^N} (1 + |x_n|^\beta) \left| F \left(\sum_{\ell \neq n} \frac{f(x_\ell)}{N-1} \right) - F(\langle f, \nu^{\theta^*} \rangle) \right| \otimes_{\ell \neq n} \nu^\ell(dx_\ell, i_\ell) \otimes \nu^{\xi^n}(dx_n, i_n) \\ &\leq C \left(\int_{\mathcal{O}} (1 + |x_n|^\beta) \nu^{\xi^n}(dx_n, i_n) \right) \\ &\quad \times \left(\int_{\mathcal{O}^{N-1}} \left| F \left(\sum_{\ell \neq n} \frac{f(x_\ell)}{N-1} \right) - F(\langle f, \nu^{\theta^*} \rangle) \right| \otimes_{\ell \neq n} \nu^\ell(dx_\ell, i_\ell) \right) \\ &\leq C_{\widehat{K}} \left(\int_{\mathcal{O}^{N-1}} \left| F \left(\sum_{\ell \neq n} \frac{f(x_\ell)}{N-1} \right) - F(\langle f, \nu^{\theta^*} \rangle) \right| \otimes_{\ell \neq n} \nu^\ell(dx_\ell, i_\ell) \right), \end{aligned}$$

where the constant $C_{\widehat{K}}$ depends only on the compact \widehat{K} , since ν^{ξ^n} has support in the compact \widehat{K} ($\xi^n \in \widehat{A}_b$ as in (5.3)). Next, using the local Lipschitz property of F (cf. Assumption (2b)-(2.3)), and again the fact that ν^ℓ are supported in the compact $\widehat{K} \times \mathbb{Y}$, we obtain

$$\begin{aligned} &\sup_{\xi^n \in \widehat{A}_b} R_N(\xi) \\ &\leq C_{\widehat{K}} \int_{\mathcal{O}^{N-1}} \left(1 + \sum_{\ell \neq n} \frac{|f(x_\ell)|}{N-1} + \langle f, \nu^{\theta^*} \rangle \right)^{\frac{1}{\beta}-1} \left| \sum_{\ell \neq n} \frac{f(x_\ell)}{N-1} - \langle f, \nu^{\theta^*} \rangle \right| \otimes_{\ell \neq n} \nu^\ell(dx_\ell, i_\ell) \\ &\leq C_{\widehat{K}} \int_{\mathcal{O}^{N-1}} \left| \frac{1}{N-1} \sum_{\ell \neq n} f(x_\ell) - \langle f, \nu^{\theta^*} \rangle \right| \otimes_{\ell \neq n} \nu^\ell(dx_\ell, i_\ell). \end{aligned}$$

Finally, by using the strong law of large numbers (cf. Theorem 4.23 in [34]) and the dominated convergence theorem, a limit as $N \rightarrow \infty$ in the latter inequality leads to (5.6). This completes the proof of the theorem. \square

APPENDIX A. RESULTS ON THE OPTIMAL STOPPING GAME

Proof of Theorem 3.1. Let $(x, i, \theta) \in \mathcal{O} \times \mathbb{R}_+$ and $\tau, \sigma \in \mathcal{T}$ given and fixed. We aim at applying Theorem 2.1 in [44], and for this we first notice that

$$\begin{aligned} \widehat{J}(x, i; \tau, \sigma, \theta) &= \widehat{\mathbb{E}}_{(x,i)} \left[\int_0^{\tau \wedge \sigma} e^{\int_0^t b_x(\widehat{X}_s, Y_s) ds} \pi_x(\widehat{X}_t, \theta) dt + k_1 e^{\int_0^\tau b_x(\widehat{X}_s, Y_s) ds} \mathbf{1}_{\{\tau \leq \sigma\}} \right. \\ &\quad \left. + k_2 e^{\int_0^\sigma b_x(\widehat{X}_s, Y_s) ds} \mathbf{1}_{\{\sigma < \tau\}} \right] = \widehat{\mathbb{E}}_{(x,i)} \left[\int_0^\infty e^{\int_0^t b_x(\widehat{X}_s, Y_s) ds} \pi_x(\widehat{X}_t, \theta) dt \right. \\ &\quad \left. - \int_{\tau \wedge \sigma}^\infty e^{\int_0^t b_x(\widehat{X}_s, Y_s) ds} \pi_x(\widehat{X}_t, \theta) dt + k_1 e^{\int_0^\tau b_x(\widehat{X}_s, Y_s) ds} \mathbf{1}_{\{\tau \leq \sigma\}} + k_2 e^{\int_0^\sigma b_x(\widehat{X}_s, Y_s) ds} \mathbf{1}_{\{\sigma < \tau\}} \right] \\ &= \widehat{\mathbb{E}}_{(x,i)} \left[\int_0^\infty e^{\int_0^t b_x(\widehat{X}_s, Y_s) ds} \pi_x(\widehat{X}_t, \theta) dt \right] - \widehat{\mathbb{E}}_{(x,i)} \left[\widehat{\mathbb{E}}_{(x,i)} \left[\int_{\tau \wedge \sigma}^\infty e^{\int_0^t b_x(\widehat{X}_s, Y_s) ds} \pi_x(\widehat{X}_t, \theta) dt \middle| \mathcal{F}_{\tau \wedge \sigma} \right] \right] \\ &\quad + \widehat{\mathbb{E}}_{(x,i)} \left[k_1 e^{\int_0^\tau b_x(\widehat{X}_s, Y_s) ds} \mathbf{1}_{\{\tau \leq \sigma\}} + k_2 e^{\int_0^\sigma b_x(\widehat{X}_s, Y_s) ds} \mathbf{1}_{\{\sigma < \tau\}} \right]. \end{aligned}$$

Then, defining

$$G_0(x, i; \theta) := \widehat{\mathbb{E}}_{(x,i)} \left[\int_0^\infty e^{\int_0^t b_x(\widehat{X}_s, Y_s) ds} \pi_x(\widehat{X}_t, \theta) dt \right],$$

by the strong Markov property of $(\widehat{X}_t, Y_t)_{t \geq 0}$ we obtain

$$(A.1) \quad \widehat{J}(x, i; \tau, \sigma, \theta) = G_0(x, i; \theta) - \widehat{\mathbb{E}}_{(x, i)} \left[e^{\int_0^{\tau \wedge \sigma} b_x(\widehat{X}_s, Y_s) ds} G_0(\widehat{X}_{\tau \wedge \sigma}, Y_{\tau \wedge \sigma}; \theta) \right] \\ + \widehat{\mathbb{E}}_{(x, i)} \left[k_1 e^{\int_0^\tau b_x(\widehat{X}_s, Y_s) ds} \mathbf{1}_{\{\tau \leq \sigma\}} + k_2 e^{\int_0^\sigma b_x(\widehat{X}_s, Y_s) ds} \mathbf{1}_{\{\sigma < \tau\}} \right].$$

Therefore,

$$(A.2) \quad \widehat{J}(x, i; \tau, \sigma, \theta) = G_0(x, i; \theta) + \widehat{\mathbb{E}}_{(x, i)} \left[e^{\int_0^\tau b_x(\widehat{X}_s, Y_s) ds} (k_1 - G_0(\widehat{X}_\tau, Y_\tau; \theta)) \mathbf{1}_{\{\tau \leq \sigma\}} \right] \\ + \widehat{\mathbb{E}}_{(x, i)} \left[e^{\int_0^\sigma b_x(\widehat{X}_s, Y_s) ds} (k_2 - G_0(\widehat{X}_\sigma, Y_\sigma; \theta)) \mathbf{1}_{\{\sigma < \tau\}} \right];$$

that is,

$$\widehat{J}(x, i; \tau, \sigma, \theta) = G_0(x, i; \theta) + \widehat{\mathbb{E}}_{(x, i, 1)} [G_1(\bar{X}_t; \theta) \mathbf{1}_{\{\tau \leq \sigma\}} + G_2(\bar{X}_t; \theta) \mathbf{1}_{\{\sigma < \tau\}}],$$

where we have set $G_1(x, i, z; \theta) := z(k_1 - G_0(x, i; \theta))$, $G_2(x, i, z; \theta) := z(k_2 - G_0(x, i; \theta))$, we have introduced the 3-dimensional right-continuous strong Markov process

$$(A.3) \quad \bar{X}_t := (\widehat{X}_t, Y_t, Z_t),$$

with $Z_t := z \cdot e^{\int_0^t b_x(\widehat{X}_s, Y_s) ds}$, and $\widehat{\mathbb{E}}_{(x, i, 1)}$ is the expectation with the respect to $\widehat{\mathbb{P}}_{(x, i, 1)}[\cdot] := \widehat{\mathbb{P}}[\cdot | \widehat{X}_0 = x, Y_0 = i, Z_0 = 1]$. We observe that by Condition (3) in Assumption 2.1 and Condition (5) in Assumption 2.2,

$$\widehat{\mathbb{E}}_{(x, i, 1)} \left[\sup_{t \geq 0} |G_i(\bar{X}_t; \theta)| \right] < \infty, \quad i = 0, 1, 2,$$

where as well as, $G_2(\bar{X}_t; \theta) \leq G_1(\bar{X}_t; \theta)$ for any $t \geq 0$. Also,

$$\lim_{t \uparrow \infty} G_1(\bar{X}_t; \theta) = 0 \quad \text{and} \quad \lim_{t \uparrow \infty} G_2(\bar{X}_t; \theta) = 0, \quad \widehat{\mathbb{P}}_{(x, i, 1)}\text{-a.s.}$$

Hence, we can apply Theorem 2.1 from [44] and complete the proof. \square

Proof of Proposition 3.2. We argue by contradiction and we suppose that $\mathcal{S}_{\text{sup}}^\theta = \emptyset$. This implies that $\sigma^* = \infty$, $\widehat{\mathbb{P}}_{(x, i)}$ -a.s. and for any $(x, i) \in \mathcal{O}$, and we thus obtain that

$$k_2 < v(x, i; \theta) = \inf_{\tau \in \mathcal{T}} \widehat{\mathbb{E}}_{(x, i)} \left[\int_0^\tau e^{\int_0^t b_x(\widehat{X}_s, Y_s) ds} \pi_x(\widehat{X}_t, \theta) dt + k_2 e^{\int_0^\tau b_x(\widehat{X}_s, Y_s) ds} \right] \\ \leq \widehat{\mathbb{E}}_{(x, i)} \left[\int_0^\infty e^{\int_0^t b_x(\widehat{X}_s, Y_s) ds} \pi_x(\widehat{X}_t, \theta) dt \right].$$

By Condition (4) and Condition (5) in Assumption 2.2 we reach a contradiction with $k_2 > 0$ by taking $x \uparrow \bar{x}$. The proof of $\mathcal{S}_{\text{inf}}^\theta \neq \emptyset$ follows by similar arguments. \square

The next result, which will be used in the proof of Proposition 3.3. below, provides the so-called *semiharmonic characterization* of v (see [44]). Its proof is direct consequence of Theorem 2.1 in [44].

Proposition A.1. *For any $(x, i) \in \mathcal{O}$, we have under $\widehat{\mathbb{P}}_{(x, i)}$ that*

- (1) $\left(\int_0^{t \wedge \tau^*(\theta)} \pi_x(\widehat{X}_t, \theta) dt + e^{\int_0^{t \wedge \tau^*(\theta)} b_x(\widehat{X}_t, Y_t) dt} v(\widehat{X}_{t \wedge \tau^*(\theta)}, Y_{t \wedge \tau^*(\theta)}) \right)_{t \geq 0}$ is an \mathbb{F} -submartingale;
- (2) $\left(\int_0^{t \wedge \sigma^*(\theta)} \pi_x(\widehat{X}_t, \theta) dt + e^{\int_0^{t \wedge \sigma^*(\theta)} b_x(\widehat{X}_t, Y_t) dt} v(\widehat{X}_{t \wedge \sigma^*(\theta)}, Y_{t \wedge \sigma^*(\theta)}) \right)_{t \geq 0}$ is an \mathbb{F} -supermartingale;
- (3) $\left(\int_0^{t \wedge \tau^*(\theta) \wedge \sigma^*(\theta)} \pi_x(\widehat{X}_t, \theta) dt + e^{\int_0^{t \wedge \tau^*(\theta) \wedge \sigma^*(\theta)} b_x(\widehat{X}_t, Y_t) dt} v(\widehat{X}_{t \wedge \tau^*(\theta) \wedge \sigma^*(\theta)}, Y_{t \wedge \tau^*(\theta) \wedge \sigma^*(\theta)}) \right)_{t \geq 0}$ is an \mathbb{F} -martingale.

Proof of Proposition 3.3. The proof is organized in several steps.

Step 1: We start by proving that $v(\cdot, i; \theta) \in C^0(\mathcal{I})$, for any $(i, \theta) \in \mathbb{Y} \times \mathbb{R}_+$. Letting $(x, i, \theta) \in \mathcal{O} \times \mathbb{R}_+$ and $\epsilon \in (0, 1)$, we recall the optimal stopping times $\tau^*(\theta) = \inf\{t \geq 0 : \widehat{X}_t \leq \alpha_{Y_t}(\theta)\}$, $\sigma^*(\theta) = \inf\{t \geq 0 : \widehat{X}_t \geq \beta_{Y_t}(\theta)\}$ $\widehat{\mathbb{P}}_{(x,i)}$ -a.s. and define $\sigma_\epsilon^*(\theta) := \inf\{t \geq 0 : \widehat{X}_t \geq \beta_{Y_t}(\theta)\}$ $\widehat{\mathbb{P}}_{(x+\epsilon,i)}$ -a.s., analogously $\tau_\epsilon^*(\theta)$. Then we have that

$$\begin{aligned}
& |v(x + \epsilon, i; \theta) - v(x, i; \theta)| \\
& \leq \left| \widehat{J}(x + \epsilon, i; \tau^*(\theta), \sigma_\epsilon^*(\theta), \theta) - \widehat{J}(x, i; \tau^*(\theta), \sigma_\epsilon^*(\theta), \theta) \right| \\
& \leq \left| \widehat{\mathbb{E}} \left[\int_0^{\tau^*(\theta) \wedge \sigma_\epsilon^*(\theta)} \left(e^{\int_0^t b_x(\widehat{X}_s^{x+\epsilon}, Y_s^i) ds} \pi_x(\widehat{X}_t^{x+\epsilon}, \theta) - e^{\int_0^t b_x(\widehat{X}_s^x, Y_s^i) ds} \pi_x(\widehat{X}_t^x, \theta) \right) ds \right] \right| \\
& \quad + k_1 \left| \widehat{\mathbb{E}} \left[\left(e^{\int_0^{\tau^*(\theta)} b_x(\widehat{X}_s^{x+\epsilon}, Y_s^i) ds} - e^{\int_0^{\tau^*(\theta)} b_x(\widehat{X}_s^x, Y_s^i) ds} \right) \mathbf{1}_{\{\tau^*(\theta) < \sigma_\epsilon^*(\theta)\}} \right] \right| \\
& \quad + k_2 \left| \widehat{\mathbb{E}} \left[\left(e^{\int_0^{\sigma_\epsilon^*(\theta)} b_x(\widehat{X}_s^{x+\epsilon}, Y_s^i) ds} - e^{\int_0^{\sigma_\epsilon^*(\theta)} b_x(\widehat{X}_s^x, Y_s^i) ds} \right) \mathbf{1}_{\{\sigma_\epsilon^*(\theta) < \tau^*(\theta)\}} \right] \right| \\
\text{(A.4)} \quad & =: A^\epsilon + k_1 B^\epsilon(\tau^*(\theta)) + k_2 B^\epsilon(\sigma_\epsilon^*(\theta)).
\end{aligned}$$

We bound the first term as follows,

$$\begin{aligned}
\text{(A.5)} \quad A^\epsilon & \leq \widehat{\mathbb{E}} \left[\int_0^\infty e^{\int_0^t b_x(\widehat{X}_s^{x+\epsilon}, Y_s^i) ds} \left| \pi_x(\widehat{X}_t^{x+\epsilon}, \theta) - \pi_x(\widehat{X}_t^x, \theta) \right| ds \right] \\
& \quad + \widehat{\mathbb{E}} \left[\int_0^\infty \left| e^{\int_0^t b_x(\widehat{X}_s^{x+\epsilon}, Y_s^i) ds} - e^{\int_0^t b_x(\widehat{X}_s^x, Y_s^i) ds} \right| \cdot \left| \pi_x(\widehat{X}_t^x, \theta) \right| dt \right].
\end{aligned}$$

Thanks to Condition (3) in Assumption 2.1, (A.5) gives

$$\begin{aligned}
\text{(A.6)} \quad A^\epsilon & \leq \widehat{\mathbb{E}} \left[\int_0^\infty e^{-ct} \left| \pi_x(\widehat{X}_t^{x+\epsilon}, \theta) - \pi_x(\widehat{X}_t^x, \theta) \right| dt \right] \\
& \quad + \widehat{\mathbb{E}} \left[\int_0^\infty \left| e^{\int_0^t b_x(\widehat{X}_s^{x+\epsilon}, Y_s^i) ds} - e^{\int_0^t b_x(\widehat{X}_s^x, Y_s^i) ds} \right| \cdot \left| \pi_x(\widehat{X}_t^x, \theta) \right| dt \right].
\end{aligned}$$

Following the same arguments as in Theorem 38 from Chapter V.7 in [46], we can prove that $x \mapsto \widehat{X}_t^x$ is continuous $\widehat{\mathbb{P}} \otimes dt$ -a.s., so that

$$\text{(A.7)} \quad \left| e^{\int_0^t b_x(\widehat{X}_s^{x+\epsilon}, Y_s^i) ds} - e^{\int_0^t b_x(\widehat{X}_s^x, Y_s^i) ds} \right| \rightarrow 0, \quad \epsilon \downarrow 0 \quad \widehat{\mathbb{P}} \otimes dt\text{-a.s.}$$

Furthermore,

$$\text{(A.8)} \quad \left| e^{\int_0^t b_x(\widehat{X}_s^{x+\epsilon}, Y_s^i) ds} - e^{\int_0^t b_x(\widehat{X}_s^x, Y_s^i) ds} \right| \leq 2e^{-ct}.$$

Hence, combining (A.7), (A.8) with Condition (5) in Assumption 2.2 we obtain that the second expectation on the right-hand side of (A.6) vanishes as $\epsilon \downarrow 0$. On the other hand, one can prove a version of comparison principle for hybrid-SDEs¹, i.e. $\widehat{X}_t^x \leq \widehat{X}_t^{x+\epsilon}$, $\widehat{\mathbb{P}} \otimes dt$ -a.s. and using that $\pi_x(x, \theta) \geq 0$ for any $(x, \theta) \in \mathcal{I} \times \mathbb{R}_+$ and $x \mapsto \pi_x(x, \theta)$ is decreasing for any $\theta \in \mathbb{R}_+$, we obtain

$$\text{(A.9)} \quad \left| \pi_x(\widehat{X}_t^{x+\epsilon}, \theta) - \pi_x(\widehat{X}_t^x, \theta) \right| \leq 2\pi_x(\widehat{X}_t^x, \theta), \quad \widehat{\mathbb{P}} \otimes dt\text{-a.s.}$$

Therefore, using again Condition (5) in Assumption 2.2, we can let $\epsilon \downarrow 0$ and we observe that also the first expectation on the right-hand side of (A.5) converges to zero, due to the dominated convergence theorem.

¹Writing $[0, \infty) = \bigcup_{n=1}^\infty [\tau_{n-1}, \tau_n)$, where τ_n is a \mathbb{F} -stopping when Y makes a jump, thus applying Proposition 5.2.18 from [36] to each $[\tau_{n-1}, \tau_n)$, $n \in \mathbb{N}$ and using the flow property of $(\widehat{X}_t)_{t \geq 0}$ we obtain the result.

Now we focus on the two other terms on the right-hand side of (A.4). By the differentiability of the map $x \mapsto \widehat{X}^x$ (cf. Theorem 39 Chapter V.7 of [46]) and by the mean value theorem, for $\tilde{x}^\epsilon \in (x, x+\epsilon)$, we have that,

$$(A.10) \quad e^{\int_0^t b_x(\widehat{X}_s^{x+\epsilon}, Y_s^i) ds} - e^{\int_0^t b_x(\widehat{X}_s^x, Y_s^i) ds} = \epsilon e^{\int_0^t b_x(\widehat{X}_s^{\tilde{x}^\epsilon}, Y_s^i) ds} \left(\int_0^t b_{xx}(\widehat{X}_s^{\tilde{x}^\epsilon}, Y_s^i) \partial_x \widehat{X}_s^x \Big|_{x=\tilde{x}^\epsilon} ds \right), \quad \widehat{\mathbb{P}} \otimes dt\text{-a.s.}$$

This, combined it with Assumption 2.1-(3) gives us, for fixed $\eta \in \mathcal{T}$,

$$(A.11) \quad \begin{aligned} B^\epsilon(\eta) &\leq \epsilon \widehat{\mathbb{E}} \left[\left[e^{\int_0^\eta b_x(\widehat{X}_s^{\tilde{x}^\epsilon}, Y_s^i) ds} \left(\int_0^\eta b_{xx}(\widehat{X}_s^{\tilde{x}^\epsilon}, Y_s^i) \partial_x \widehat{X}_s^x \Big|_{x=\tilde{x}^\epsilon} ds \right) \right] \right] \\ &\leq \epsilon \widehat{\mathbb{E}} \left[e^{-c\eta} \int_0^\eta |b_{xx}(\widehat{X}_s^{\tilde{x}^\epsilon}, Y_s^i)| |\partial_x \widehat{X}_s^x \Big|_{x=\tilde{x}^\epsilon} ds \right] \\ &\leq \epsilon \widehat{\mathbb{E}} \left[\int_0^\infty e^{-cs} |b_{xx}(\widehat{X}_s^{\tilde{x}^\epsilon}, Y_s^i)| |\partial_x \widehat{X}_s^x \Big|_{x=\tilde{x}^\epsilon} ds \right], \end{aligned}$$

hence sending $\epsilon \downarrow 0$ in (A.11) by Assumption 2.2-(6) invoking dominated convergence theorem we can exchange limit with expectation and we obtain (cf. (A.4))

$$\lim_{\epsilon \downarrow 0} (k_1 B^\epsilon(\tau^*(\theta)) + k_2 B^\epsilon(\sigma_\epsilon^*(\theta))) = 0.$$

Step 2: Let $(i, \theta) \in \mathbb{Y} \times \mathbb{R}_+$. By Theorem 3.1, we know that $v(\cdot, i; \theta) = k_1$, $x \in (\underline{x}, \alpha_i(\theta))$ and $v(\cdot, i; \theta) = k_2$, $x \in (\beta_i(\theta), \bar{x})$ for any $(i, \theta) \in \mathbb{Y} \times \mathbb{R}_+$. Hence, it is sufficient to show that $v(\cdot, i; \theta) \in C^2(\mathcal{C}^\theta)$, for any $(i, \theta) \in \mathbb{Y} \times \mathbb{R}_+$. We then define the following boundary value problem:

$$(A.12) \quad \begin{cases} \mathcal{L}_{\widehat{X}} w(x, i; \theta) + (b_x(x, i) - q_{ii}) w(x, i; \theta) = f(x, i; \theta), & (x, i) \in \mathcal{C}^\theta \\ w(\alpha_i(\theta), i; \theta) = v(\alpha_i(\theta), i; \theta), & w(\beta_i(\theta), i; \theta) = v(\beta_i(\theta), i; \theta), \end{cases}$$

where $\mathcal{L}_{\widehat{X}} w(x, i; \theta) := \frac{1}{2} \sigma^2(x, i) w_{xx}(x, i; \theta) + (b(x, i) + \sigma \sigma_x(x, i)) w_x(x, i; \theta)$ and $f(x, i; \theta) := -\sum_{j \neq i} q_{ij} v(x, j; \theta) - \pi_x(x, \theta)$. Using Condition (4) in Assumption 2.1 we conclude that (A.12) has a unique classical solution $w(\cdot, i; \theta) \in C^2(\alpha_i(\theta), \beta_i(\theta))$ (see Theorem 6.13 in [28]). We then define the function $\bar{w} : (\alpha_i(\theta), \beta_i(\theta)) \times \mathbb{Y} \times \mathbb{R}_+ \mapsto \mathbb{R}$ by:

$$\bar{w}(x, j; \theta) := \begin{cases} w(x, i; \theta), & j = i \\ v(x, j; \theta), & j \neq i, \end{cases}$$

and the stopping time $\tau_Y := \inf\{t \geq 0 : Y_t^i \neq i\}$. Hence, setting $\eta(\theta) := \tau_Y \wedge \tau^*(\theta) \wedge \sigma^*(\theta)$, from Dynkin's formula we have:

$$(A.13) \quad \bar{w}(x, i; \theta) = w(x, i; \theta) = \widehat{\mathbb{E}}_{(x, i)} \left[v(\widehat{X}_{\eta(\theta)}, Y_{\eta(\theta)}; \theta) e^{\int_0^{\eta(\theta)} b_x(\widehat{X}_s, Y_s) ds} + \int_0^{\eta(\theta)} e^{\int_0^t b_x(\widehat{X}_s, Y_s) ds} \pi_x(\widehat{X}_t, \theta) dt \right]$$

because $\bar{w}(\widehat{X}_{\eta(\theta)}, Y_{\eta(\theta)}; \theta) = v(\widehat{X}_{\eta(\theta)}, Y_{\eta(\theta)}; \theta)$, and, on $s < \eta(\theta)$,

$$\begin{aligned} &\frac{1}{2} \sigma^2(\widehat{X}_s, Y_s) \bar{w}_{xx}(\widehat{X}_s, Y_s; \theta) + (b(\widehat{X}_s, Y_s) + \sigma \sigma_x(\widehat{X}_s, Y_s)) \bar{w}_x(\widehat{X}_s, Y_s; \theta) + (b_x(\widehat{X}_s, Y_s) - q_{ii}) \bar{w}(\widehat{X}_s, Y_s; \theta) = \\ &\frac{1}{2} \sigma^2(\widehat{X}_s, Y_s) w_{xx}(\widehat{X}_s, Y_s; \theta) + (b(\widehat{X}_s, Y_s) + \sigma \sigma_x(\widehat{X}_s, Y_s)) w_x(\widehat{X}_s, Y_s; \theta) + (b_x(\widehat{X}_s, Y_s) - q_{ii}) w(\widehat{X}_s, Y_s; \theta) \\ &= f(\widehat{X}_s, Y_s; \theta), \quad \widehat{\mathbb{P}}_{(x, i)}\text{-a.s.} \end{aligned}$$

However, $\eta(\theta) \leq \tau^*(\theta) \wedge \sigma^*(\theta)$, $\widehat{\mathbb{P}}_{(x, i)}$ -a.s., which, by the martingale property (3) in Proposition A.1, gives

$$v(x, i; \theta) = \widehat{\mathbb{E}}_{(x, i)} \left[v(\widehat{X}_{\eta(\theta)}, Y_{\eta(\theta)}; \theta) e^{\int_0^{\eta(\theta)} b_x(\widehat{X}_s, Y_s) ds} + \int_0^{\eta(\theta)} e^{\int_0^t b_x(\widehat{X}_s, Y_s) ds} \pi_x(\widehat{X}_t, \theta) dt \right] = w(x, i; \theta).$$

Since the latter holds for every $x \in (\alpha_i(\theta), \beta_i(\theta))$ and for arbitrary $i \in \mathbb{Y}$, we obtain $v \equiv w$, which finally yields that $v(\cdot, i; \theta) \in C^2((\alpha_i(\theta), \beta_i(\theta)))$.

Step 3: Thanks to the previous steps, it is now sufficient to prove the continuity of $v_x(\cdot, i; \theta)$ only across the free boundaries $\alpha_i(\theta)$ and $\beta_i(\theta)$, for any $(i, \theta) \in \mathbb{Y} \times \mathbb{R}_+$. We prove this only for $\alpha_i(\theta)$, being the proof across $\beta_i(\theta)$ similar. Notice that from (A.4) we have

$$(A.14) \quad \left| \frac{v(x + \epsilon, i; \theta) - v(x, i; \theta)}{\epsilon} \right| \leq \frac{1}{\epsilon} A^\epsilon + \frac{k_1}{\epsilon} B^\epsilon(\tau^*(\theta)) + \frac{k_2}{\epsilon} B^\epsilon(\sigma_\epsilon^*(\theta)).$$

We recall that the map $x \mapsto \widehat{X}^x$ is differentiable and by regularity of profit the function (cf. Assumption 2.2), for $x_1^\epsilon \in (x, x + \epsilon)$ we have that,

$$(A.15) \quad \left| \pi_x(\widehat{X}_t^{x+\epsilon}, \theta) - \pi_x(\widehat{X}_t^x, \theta) \right| \leq \epsilon \left| \pi_{xx}(\widehat{X}_t^{x_1^\epsilon}, \theta) \right| \partial_x \widehat{X}_t^x \Big|_{x=x_1^\epsilon}, \quad \widehat{\mathbb{P}} \otimes dt\text{-a.s.}$$

Using (A.15) in the first expectation on the right-hand side of (A.6), and applying the mean value theorem in the second expectation (cf. (A.10)), we can find $x_2^\epsilon \in (x, x + \epsilon)$ such that

$$(A.16) \quad \begin{aligned} \frac{1}{\epsilon} A_\epsilon &\leq \widehat{\mathbb{E}} \left[\int_0^{\tau^*(\theta) \wedge \sigma_\epsilon^*(\theta)} e^{-ct} \left| \pi_{xx}(\widehat{X}_t^{x_1^\epsilon}, \theta) \right| \partial_x \widehat{X}_s^x \Big|_{x=x_1^\epsilon} ds \right] \\ &+ \widehat{\mathbb{E}} \left[\int_0^{\tau^*(\theta) \wedge \sigma_\epsilon^*(\theta)} \left(e^{\int_0^t b_x(\widehat{X}_s^{x_2^\epsilon}, Y_s^i) ds} \cdot \left| \pi_x(\widehat{X}_t^x, \theta) \right| \cdot \left| \int_0^t b_{xx}(\widehat{X}_s^{x_2^\epsilon}, Y_s^i) \partial_x \widehat{X}_s^x \Big|_{x=x_2^\epsilon} ds \right| \right) dt \right]. \end{aligned}$$

On the other hand, by (A.11), for any $\eta \in \mathcal{T}$, we have

$$(A.17) \quad \frac{1}{\epsilon} B^\epsilon(\eta) \leq \widehat{\mathbb{E}} \left[\int_0^\infty e^{-cs} |b_{xx}(\widehat{X}_s^{\tilde{x}^\epsilon}, Y_s^i)| \partial_x \widehat{X}_s^x \Big|_{x=\tilde{x}^\epsilon} ds \right],$$

for some $x_\epsilon \in (x, x + \epsilon)$. Hence, sending $\epsilon \rightarrow 0$, we can interchange limits and expectation thanks to Conditions (5) and (6) in Assumption 2.2, and find from (A.14) that (cf. (A.16) and (A.17))

$$(A.18) \quad \begin{aligned} 0 \leq |v_x(x, i; \theta)| &\leq \widehat{\mathbb{E}} \left[\int_0^{\tau^*(\theta) \wedge \sigma^*(\theta)} e^{-ct} \left| \pi_{xx}(\widehat{X}_t^x, \theta) \right| \partial_x \widehat{X}_t^x ds \right] \\ &+ \widehat{\mathbb{E}} \left[\int_0^{\tau^*(\theta) \wedge \sigma^*(\theta)} e^{-ct} \cdot \left| \pi_x(\widehat{X}_t^x, \theta) \right| \cdot \left(\int_0^t |b_{xx}(\widehat{X}_s^x, Y_s^i)| \partial_x \widehat{X}_s^x ds \right) dt \right] \\ &+ k_1 \widehat{\mathbb{E}} \left[e^{\int_0^{\tau^*(\theta)} b_x(\widehat{X}_s^x, Y_s^i) ds} \left(\int_0^{\tau^*(\theta)} |b_{xx}(\widehat{X}_s^x, Y_s^i)| \partial_x \widehat{X}_s^x \Big|_x ds \right) \right] \\ &+ k_2 \widehat{\mathbb{E}} \left[e^{\int_0^{\sigma^*(\theta)} b_x(\widehat{X}_s^x, Y_s^i) ds} \left(\int_0^{\sigma^*(\theta)} |b_{xx}(\widehat{X}_s^x, Y_s^i)| \partial_x \widehat{X}_s^x ds \right) \right], \end{aligned}$$

where we have also used that $\sigma_\epsilon^*(\theta) \rightarrow \sigma^*(\theta)$ $\widehat{\mathbb{P}}_{(x,i)} \otimes d\theta$ -a.s. Finally, sending $x \downarrow \alpha_i(\theta)$, $\tau^*(\theta) \rightarrow 0$, $\widehat{\mathbb{P}}_{(x,i)} \otimes d\theta$ -a.s. so that $v(\cdot, i; \theta)$ is C^1 across $\alpha_i(\theta)$. \square

Finally, arguing as in the proofs of Steps 1 and 3 in the proof of Proposition 3.3, one can evaluate upper and lower bounds for $\frac{1}{\epsilon}(v(x + \epsilon, i; \theta) - v(x, i; \theta))$, which, after taking limits as $\epsilon \downarrow 0$ yield the following probabilistic representation of v_x .

Lemma A.1. *Under the Assumptions 2.2-(5) and (6)-(2.1), the representation*

$$\begin{aligned}
 v_x(x, i; \theta) = & \widehat{\mathbb{E}} \left[\int_0^{\tau^*(\theta) \wedge \sigma^*(\theta)} e^{\int_0^t b_x(\widehat{X}_s^x, Y_s^i) ds} \left(\pi_{xx}(\widehat{X}_t^x, \theta) \partial_x \widehat{X}_t^x \right. \right. \\
 & \left. \left. + \left(\int_0^t b_{xx}(\widehat{X}_s^x, Y_s^i) \partial_x \widehat{X}_s^x ds \right) \pi_x(\widehat{X}_t^x, \theta) \right) dt \right] \\
 & + k_1 \widehat{\mathbb{E}} \left[e^{\int_0^{\tau^*(\theta)} b_x(\widehat{X}_s^x, Y_s^i) ds} \left(\int_0^{\tau^*(\theta)} b_{xx}(\widehat{X}_s^x, Y_s^i) \partial_x \widehat{X}_s^x ds \right) \mathbf{1}_{\{\tau^*(\theta) < \sigma^*(\theta)\}} \right] \\
 & + k_2 \widehat{\mathbb{E}} \left[e^{\int_0^{\sigma^*(\theta)} b_x(\widehat{X}_s^x, Y_s^i) ds} \left(\int_0^{\sigma^*(\theta)} b_{xx}(\widehat{X}_s^x, Y_s^i) \partial_x \widehat{X}_s^x ds \right) \mathbf{1}_{\{\sigma^*(\theta) < \tau^*(\theta)\}} \right]
 \end{aligned}$$

holds for any $(x, i, \theta) \in \mathcal{O} \times \mathbb{R}_+$, where $\tau^*(\theta) = \tau^*(x, i; \theta)$ and $\sigma^*(\theta) = \sigma^*(x, i; \theta)$ are as in Theorem 3.1.

APPENDIX B. SKOROKHOD REFLECTION PROBLEM WITH REGIME-SWITCHING AND PROOF OF THEOREM 3.2

In this section we focus on the existence and uniqueness of a solution to $\mathbf{SP}(\alpha, \beta, x, i)$ as defined in Definition 3.1. Then we show a comparison result and finally we prove Theorem 3.2.

Recall I as in (3.16) and denote by $\Gamma : \mathcal{D}[0, \infty) \rightarrow \mathcal{D}[0, \infty)$ the following operator

$$\begin{aligned}
 \Gamma(X)_t := & I(X)_t - \max \left\{ (x - \beta_i(\theta))^+ \wedge \inf_{s \leq t} (I(X)_s - \alpha_{Y_s}(\theta)), \right. \\
 \text{(B.1)} \quad & \left. \sup_{s \leq t} \left((I(X)_s - \beta_{Y_s}(\theta)) \wedge \inf_{u \in [s, t]} (I(X)_u - \alpha_{Y_u}(\theta)) \right) \right\} =: I(X)_t - \Xi(X)_t.
 \end{aligned}$$

We first prove that the operator has a fixed point.

Proposition B.1. *There exists unique solution to Skorokhod problem $\mathbf{SP}(\alpha, \beta, x, i)$.*

Proof. Let $T > 0$ and $X, X' \in \mathcal{D}[0, \infty)$, then using the inequalities

$$\begin{aligned}
 \max \{ \alpha_1, \beta_1 \} - \max \{ \alpha_2, \beta_2 \} & \leq \max \{ \alpha_1 - \alpha_2, \beta_1 - \beta_2 \} \\
 \left| \sup_{s \leq t} |f(s)| - \sup_{s \leq t} |g(s)| \right| & \leq \sup_{s \leq t} |f(s) - g(s)| \\
 \left| \inf_{s \leq t} |f(s)| - \inf_{s \leq t} |g(s)| \right| & \leq \sup_{s \leq t} |f(s) - g(s)|,
 \end{aligned}$$

we obtain that

$$\text{(B.2)} \quad \sup_{s \leq t} |\Gamma(X)_s - \Gamma(X')_s|^2 \leq 4 \sup_{s \leq t} |I(X)_s - I(X')_s|^2, \quad t \in [0, T].$$

We now derive an estimate for $\sup_{s \leq t} |I(X)_s - I(X')_s|^2$. In particular, we have that

$$\begin{aligned}
 \mathbb{E} \left[\sup_{s \leq t} |I(X)_s - I(X')_s|^2 \right] & \leq 2 \mathbb{E} \left[\sup_{s \leq t} \left| \int_0^s (b(X_s, Y_s) - b(X'_s, Y_s)) ds \right|^2 \right. \\
 & \left. + \sup_{s \leq t} \left| \int_0^s (\sigma(X_s, Y_s) - \sigma(X'_s, Y_s)) dW_s \right|^2 \right].
 \end{aligned}$$

By an application of Hölder's inequality in the drift term and an application of Burkholder-Davis-Gundy's inequality to the local martingale $M_t := \int_0^t (\sigma(X_s, Y_s) - \sigma(X'_s, Y_s)) dW_s$, we find that

$$\begin{aligned} \mathbb{E} \left[\sup_{s \leq t} |I(X)_s - I(X')_s|^2 \right] &\leq \mathbb{E} \left[C(T+1) \int_0^t |b(X_s, Y_s) - b(X'_s, Y_s)|^2 ds \right] \\ &\leq CK^2(1+T) \mathbb{E} \left[\int_0^t \sup_{u \leq s} |X_u - X'_u|^2 ds \right], \end{aligned}$$

where the last inequality is due to the Lipschitz property of b and σ . Finally, for a constant $C_1 = C_1(K, T) > 0$ it holds

$$(B.3) \quad \mathbb{E} \left[\sup_{s \leq t} |\Gamma(X)_s - \Gamma(X')_s|^2 \right] \leq C_1 \mathbb{E} \left[\int_0^t \sup_{u \leq s} |X_u - X'_u|^2 ds \right], \quad t \in [0, T].$$

Let $\{X^{(n)}\}_{n \in \mathbb{N}}$ be a \mathbb{F} -adapted process such that

$$\begin{cases} X_t^{(0)} := x, & t \in [0, T] \\ X_t^{(n)} := \Gamma(X^{(n-1)})_t, & t \in [0, T]. \end{cases}$$

Then, by (B.3) we have that

$$(B.4) \quad \mathbb{E} \left[\sup_{s \leq t} |\Gamma(X^{(n)})_s - \Gamma(X^{(n-1)})_s|^2 \right] \leq C_1 \mathbb{E} \left[\int_0^t \sup_{u \leq s} |X_u^{(n-1)} - X_u^{(n-2)}|^2 ds \right],$$

which yields

$$(B.5) \quad \mathbb{E} \left[\sup_{s \leq t} |\Gamma(X^{(n)})_s - \Gamma(X^{(n-1)})_s|^2 \right] \leq \frac{(Rt)^n}{n!},$$

where $\mathbb{E}[\sup_{s \leq t} |X_s^{(1)} - x|^2] \leq Rt$. Hence, by using (B.5) and Markov inequality, we obtain that

$$\int_0^t b(X_s^{(n)}, Y_s) ds + \int_0^t \sigma(X_s^{(n)}, Y_s) dW_s \rightarrow \int_0^t b(X_s, Y_s) ds + \int_0^t \sigma(X_s, Y_s) dW_s,$$

uniformly in compact sets of $[0, T]$ as $n \uparrow \infty$. Defining $X := \lim_{n \rightarrow \infty} \Gamma(X^{(n)})$, and taking $\xi = \xi^+ - \xi^- = \Xi(X)$, we write $X^\xi = I(X^\xi) + \xi$ so that the pair (X^ξ, ξ) satisfies the properties of Skorokhod problem in Definition 3.1 for every $t \in [0, T]$. To extend the solution to $[0, \infty)$, we observe that (B.3) does not depend from the initial conditions. Hence, the result holds on each interval with length T and, by writing $[0, \infty) := \bigcup_{n=1}^{\infty} [(n-1)T, nT)$, the proof is complete. \square

The next Proposition establishes a comparison principle for the optimally controlled state with respect to the mean-field parameter.

Proposition B.2. *Let $x_1, x_2 \in \mathcal{I}$ and $\theta_1, \theta_2 \in \mathbb{R}_+$ such that $x_1 \leq x_2$ and $\theta_1 \leq \theta_2$ then it is true that:*

$$(B.6) \quad X_t^{x_1, \xi^*(\theta_1)} \leq X_t^{x_2, \xi^*(\theta_2)}, \quad \mathbb{P} \otimes dt\text{-a.s.}$$

Proof. Let $\{X^{(n,x,\theta)}\}_{n \in \mathbb{N}}$ to be a sequence of \mathbb{F} -adapted processes such that

$$(B.7) \quad dX_t^{(n,x,\theta)} = (b(X_t^{(n,x,\theta)}, Y_t) + g_n^{(\theta)}(X_t^{(n,x,\theta)}, Y_t)) dt + \sigma(X_t^{(n,x,\theta)}, Y_t) dW_t,$$

where,

$$g_n^{(\theta)}(x, i) := \begin{cases} 0, & x \in [\alpha_i(\theta), \beta_i(\theta)] \\ -n(x - \alpha_i(\theta)), & x < \alpha_i(\theta) \\ -n(x - \beta_i(\theta)), & x > \beta_i(\theta). \end{cases}$$

Let $\{\tau_k\}_{k \in \mathbb{N}}$ be the sequence of random times where the jumps of Y occurred. Thanks to Proposition 5.2.18 in [36] and Lemma 4.1, by an induction argument we find

$$X_t^{(n, x_1, \theta_1)} \leq X_t^{(n, x_2, \theta_2)}, \quad \mathbb{P}\text{-a.s. for any } t \in [\tau_k, \tau_{k+1}), \quad \text{for any } k \in \mathbb{N}_0.$$

Then, using Theorem 1.4.1 in [45] (with estimates as in Proposition 2.3 in [51]), we obtain, $\mathbb{P} \otimes dt$ -a.s.,

$$X_t^{x_1, \xi^*(\theta_1)} = \lim_{n \uparrow \infty} X_t^{(n, x_1, \theta_1)} \leq \lim_{n \uparrow \infty} X_t^{(n, x_2, \theta_2)} = X_t^{x_2, \xi^*(\theta_2)},$$

thus completing the proof. \square

We are now in the position to provide the proof of Theorem 3.2

Proof of Theorem 3.2. By Proposition B.1, there exists a unique solution to $\mathbf{SP}(\alpha(\theta), \beta(\theta), x, i)$, denoted by $(X^{\xi^*(\theta)}, \xi^*(\theta))$. This satisfies the properties collected in Definition 3.1. We prove that $\xi^*(\theta) \in \mathcal{A}_e$ and verify that the optimal policy has the form as in (3.18). By Property (1) in Definition 3.1, we have that

$$X_t^{\xi^*(\theta)} \in [\alpha_{Y_t}(\theta), \beta_{Y_t}(\theta)], \quad \forall t \geq 0 \quad \mathbb{P}_{(x, i)}\text{-a.s.},$$

so that

$$(B.8) \quad \limsup_{T \uparrow \infty} \frac{1}{T} \mathbb{E}_{(x, i)} [|X_T^{\xi^*(\theta)}|] = 0.$$

It remains to show that $\mathbb{E}[|\xi_T^*(\theta)|] < \infty$ for any $T < \infty$. To that end, let $T < \infty$ and $\psi : \mathcal{O} \rightarrow \mathbb{R}$ be any classical solution of

$$(B.9) \quad \begin{cases} \mathcal{L}_{(X, Y)} \psi(x, i) = 0, & x \in (\alpha_i(\theta), \beta_i(\theta)), \quad i \in \mathbb{Y} \\ \psi_x(\alpha_i(\theta), i) = 1, & \psi_x(\beta_i(\theta), i) = -1. \end{cases}$$

Applying Itô-Meyer's formula to $\{\psi(X_t^{\xi^*(\theta)}, Y_t)\}_{t \geq 0}$ on $[0, T]$ we obtain

$$\begin{aligned} \mathbb{E}_{(x, i)} [\psi(X_T^{\xi^*(\theta)}, Y_T)] &= \psi(x, i) + \mathbb{E}_{(x, i)} \left[\int_0^T \mathcal{L}_{(X, Y)} \psi(X_t^{\xi^*(\theta)}, Y_t) dt \right] \\ &\quad + \mathbb{E}_{(x, i)} \left[\int_0^T \psi_x(X_t^{\xi^*(\theta)}, Y_t) d\xi_t^*(\theta) \right], \end{aligned}$$

and by using the fact that ψ is a solution to (B.9) we conclude that

$$\mathbb{E}_{(x, i)} [\psi(X_T^{\xi^*(\theta)}, Y_T)] = \psi(x, i) + \mathbb{E}_{(x, i)} [|\xi_T^*(\theta)|].$$

Since $X^{\xi^*(\theta)} \in [\alpha_{Y_t}(\theta), \beta_{Y_t}(\theta)]$, for any $t \geq 0$ $\mathbb{P}_{(x, i)}$ -a.s., we obtain that

$$\mathbb{E}_{(x, i)} [\psi(X_T^{\xi^*(\theta)}, Y_T)] \leq \max_{i \in \mathbb{Y}} \max_{x \in [\alpha_i(\theta), \beta_i(\theta)]} |\psi(x, i)| < \infty,$$

and therefore

$$(B.10) \quad \mathbb{E}_{(x, i)} [|\xi_T^*(\theta)|] \leq \mathbb{E}_{(x, i)} [\psi(X_T^{\xi^*(\theta)}, Y_T)] + \psi(x, i) < \infty, \quad \text{for any } T < \infty.$$

Combining (B.8) and (B.10) we conclude that $\xi^* \in \mathcal{A}_e$. Finally, defining

$$\widehat{\xi}_t^+(\theta) := \sup_{0 \leq s \leq t} \left(\alpha_{Y_s}(\theta) - I(X^{\widehat{\xi}(\theta)})_s + \widehat{\xi}_s^-(\theta) \right)^+ \quad \text{and} \quad \widehat{\xi}_t^-(\theta) := \sup_{0 \leq s \leq t} \left(I(X^{\widehat{\xi}(\theta)})_s + \widehat{\xi}_s^+(\theta) - \beta_{Y_s}(\theta) \right)^+$$

it is easy to see that $\widehat{\xi}(\theta) := \widehat{\xi}^+(\theta) - \widehat{\xi}^-(\theta)$ satisfies the properties in Definition 3.1. Since, however, the solution to $\mathbf{SP}(\alpha(\theta), \beta(\theta), x, i)$ is unique, it follows that $\xi^{*,+} \equiv \widehat{\xi}^+$ and $\xi^{*,-} \equiv \widehat{\xi}^-$. \square

APPENDIX C. PROOF OF (1) IN THEOREM 4.1: THE FOKKER-PLANCK COUPLED SYSTEM

We follow closely the proof of Theorem 1 in [23]. Let μ^θ be the cumulative distribution function of the stationary distribution ν^θ (cf. Proposition 4.2). To simplify the notation, we drop the dependence on θ ; that is, we set $\mu := \mu^\theta$ as well as $\alpha_i := \alpha_i(\theta)$, $\beta_i := \beta_i(\theta)$ for any $i \in \mathbb{Y}$. We divide the rest of the proof in three steps.

Step 1. We first derive a variational equation for the function μ . To this end, introduce the generator $\mathcal{L}_{(X,Y)}^{\alpha,\beta}$ of the process (X^ξ, Y) , where X^ξ is reflected at the boundaries $(\alpha, \beta) := \{(\alpha_i, \beta_i)\}_{i \in \mathbb{Y}}$. Namely, for any $\phi := (\phi(\cdot, i))_{i \in \mathbb{Y}} \in C^2(\mathcal{I}; \mathbb{R}^d)$ define

$$\mathcal{L}_{(X,Y)}^{\alpha,\beta} \phi(x, i) := \frac{1}{2} \sigma^2(x, i) \phi_{xx}(x, i) + b(x, i) \phi_x(x, i) + \sum_{j \in \mathbb{Y}} q_{ij} \widehat{\phi}(x, j), \quad (x, i) \in \mathcal{O},$$

where $\widehat{\phi}(x, i) := \phi(\alpha_i^- \vee x \wedge \beta_i, i)$. From Chapter IV in [24], we have that μ solves the equation

$$(C.1) \quad \sum_{i \in \mathbb{Y}} \int_{\alpha_i^-}^{\beta_i} \mathcal{L}_{(X,Y)}^{\alpha,\beta} \phi(x, i) d\mu(x, i) = 0, \quad \text{for any } \phi := (\phi(\cdot, i))_{i \in \mathbb{Y}} \in C^2(\mathcal{I}; \mathbb{R}^d).$$

Hence, by Theorem 5.3.4 in [6] we obtain that μ is absolutely continuous with respect to the Lebesgue measure with density function μ_x .

Next, split the interval $[\min_{i \in \mathbb{Y}} \alpha_i, \max_{i \in \mathbb{Y}} \beta_i]$ in the disjoint sub-intervals

$$I_k = (l_{k-1}, l_k), \quad k = 1, \dots, K,$$

with $l_0 = \min_{i \in \mathbb{Y}} \{\alpha_i\}$, $l_K = \max_{i \in \mathbb{Y}} \{\beta_i\}$ and

$$l_{k+1} = \min_{\substack{i \in \mathbb{Y}: \\ \alpha_i > l_k}} \{\alpha_i\} \wedge \min_{\substack{i \in \mathbb{Y}: \\ \beta_i > l_k}} \{\beta_i\}.$$

Introduce the set of test functions

$$(C.2) \quad \mathcal{V} := \{ \phi \in C^2(\mathcal{I}; \mathbb{R}^d) : \phi_x(x, i) = \phi_{xx}(x, i) = 0, \quad x \in \mathcal{I} \setminus (\alpha_i(\theta), \beta_i(\theta)), \quad \text{for any } i \in \mathbb{Y} \}.$$

Since $\mu(\alpha_i^-, i) = 0$, for $\phi \in \mathcal{V}$ we observe that

$$\int_{\alpha_i^-}^{\beta_i} \widehat{\phi}(x, j) d\mu(x, i) = \mu(\beta_i, i) \widehat{\phi}(\beta_j, j) - \int_{\alpha_j}^{\beta_j} \mu(\alpha_i^- \vee x \wedge \beta_i, i) \phi_x(x, j) dx,$$

so that, exchanging the indexes in the sum, we find

$$(C.3) \quad \begin{aligned} \sum_{i \in \mathbb{Y}} \sum_{j \in \mathbb{Y}} q_{ij} \int_{\alpha_i^-}^{\beta_i} \widehat{\phi}(x, j) d\mu(x, i) &= \sum_{i \in \mathbb{Y}} \sum_{j \in \mathbb{Y}} q_{ij} \mu(\beta_i, i) \widehat{\phi}(\beta_j, j) \\ &\quad - \sum_{i \in \mathbb{Y}} \sum_{j \in \mathbb{Y}} q_{ji} \int_{\alpha_i}^{\beta_i} \mu(\alpha_j^- \vee x \wedge \beta_j, j) \phi_x(x, i) dx. \end{aligned}$$

By substituting (C.3) in (C.1) and rearranging the terms, for any $\phi \in \mathcal{V}$, we obtain

$$(C.4) \quad \sum_{i \in \mathbb{Y}} (I_w^i(\mu, \phi) + \Sigma^i(\mu, \phi)) = 0,$$

where

$$\begin{aligned} I_w^i(\mu, \phi) &:= \int_{\alpha_i}^{\beta_i} \frac{1}{2} \sigma^2(x, i) \mu_x(x, i) \phi_{xx}(x, i) dx \\ &\quad + \int_{\alpha_i}^{\beta_i} \left(b(x, i) \mu_x(x, i) - \sum_{j \in \mathbb{Y}} q_{ji} \mu(\alpha_j^- \vee x \wedge \beta_j, j) \right) \phi_x(x, i) dx, \\ \Sigma^i(\mu, \phi) &:= \sum_{j \in \mathbb{Y}} q_{ji} \mu(\beta_j, j) \widehat{\phi}(\beta_i, i). \end{aligned}$$

Step 2. The aim of this step is to gradually use the variational equation (C.4) to obtain the system (4.10) and the proper regularity of μ . We argue as follows:

- (1) First, for any $i \in \mathbb{Y}$ and k such that $I_k \subset (\alpha_i, \beta_i)$, choose $\phi(\cdot, j) = 0$ for $j \neq i$ and $\phi(\cdot, i)$ such that $\text{supp } \phi(\cdot, i) \subset I_k$. For such a choice of ϕ , we have $\Sigma^i(\mu, \phi) = 0$, so that from (C.4) we find $I_w^i(\mu, \phi) = 0$. Thus, setting $\psi := \phi_x(\cdot, i)$, we deduce that the function $\mu(\cdot, i)$ is a solution to the variational equation

$$\begin{aligned} &\int_{I_k} \left(\frac{1}{2} \sigma^2(x, i) \mu_x(x, i) \psi_x(x, i) dx \right. \\ &\quad \left. + \left(b(x, i) \mu_x(x, i) - \sum_{j \in \mathbb{Y}} q_{ji} \mu(\alpha_j^- \vee x \wedge \beta_j, j) \right) \psi(x, i) \right) dx = 0, \end{aligned}$$

for any $\psi \in C_c^2(I_k)$. Therefore, by the interior regularity for elliptic equations (see Theorem 8.10 in [28]), we obtain that $\mu(\cdot, i) \in C^2(I_k; \mathbb{R})$. Since the interval I_k is arbitrary, we obtain

$$(C.5) \quad \mu(\cdot, i) \in C^2\left((\alpha_i, \beta_i) \setminus \bigcup_{j \in \mathbb{Y}} \{\alpha_j, \beta_j\}\right),$$

for any $i \in \mathbb{Y}$.

- (2) Next, for $i \in \mathbb{Y}$ we use the enumeration $l_k = \alpha_i < l_{k+1} < \dots < l_{m-1} < l_m = \beta_i$ and define $\Delta f(l_j, i) = f(l_j^+, i) - f(l_j^-, i)$ for $k < j < m$, $\Delta f(l_k, i) = -f(l_k^-, i)$ and $\Delta f(l_m, i) = f(l_m^+, i)$. Since $\mu(\cdot, i) \in C^2(I_\ell)$ for $\ell = k+1, \dots, m$, dividing the domain of integration into the I_ℓ 's and applying integration by parts, for $\phi \in \mathcal{V}$ we obtain

$$(C.6) \quad I_w^i(\mu, \phi) = I_s^i(\mu, \phi) + \Psi_\sigma^i(\mu, \phi),$$

where

$$\begin{aligned} I_s^i(\mu, \phi) &:= - \int_{\alpha_i}^{\beta_i} \left(\frac{1}{2} \sigma^2(x, i) \mu_{xx}(x, i) - (b(x, i) - \sigma \sigma_x(x, i)) \mu_x(x, i) \right. \\ &\quad \left. + \sum_{j \in \mathbb{Y}} q_{ji} \mu(\alpha_j^- \vee x \wedge \beta_j, j) \right) \phi_x(x, i) dx, \\ \Psi_\sigma^i(\mu, \phi) &:= - \sum_{j=k}^m \frac{1}{2} \sigma^2(l_j, i) \Delta \mu_x(l_j, i) \phi_x(l_j, i). \end{aligned}$$

In particular, for any $i \in \mathbb{Y}$ and ℓ such that $I_\ell \subset (\alpha_i, \beta_i)$, by choosing $\phi(\cdot, j) = 0$ for $j \neq i$ and $\phi(\cdot, i)$ such that $\text{supp } \phi(\cdot, i) \subset I_\ell$, μ solves the equation $I_s^i(\mu, \phi) = 0$. Hence, since the i, I_ℓ are arbitrary, we conclude that

$$\frac{1}{2} \sigma^2(x, i) \mu_{xx}(x, i) - (b(x, i) - \sigma \sigma_x(x, i)) \mu_x(x, i) + \sum_{j \in \mathbb{Y}} q_{ji} \mu(\alpha_j^- \vee x \wedge \beta_j, j) = 0,$$

in (α_i, β_i) , for any $i \in \mathbb{Y}$; that is, μ solves the equation (4.10) pointwise.

- (3) Furthermore, using the latter equation, we deduce that $I_s^i(\mu, \phi) = 0$ for any $\phi \in \mathcal{V}$. Therefore, by choosing $\phi \in \mathcal{V}$ such that $\phi_x(l_j, i) = \phi_{xx}(l_j, i) = 0$ for any $j, i \in \mathbb{Y}$, from (C.4) and (C.6) we obtain $\Sigma^i(\mu, \phi) = 0$. Since the values of $\phi(\beta_i, i)$ are arbitrary, this in turn implies that

$$\sum_{j \in \mathbb{Y}} q_{ji} \mu(\beta_j, j) = 0, \quad i \in \mathbb{Y}.$$

Such a system of equations corresponds to the eigenvector problem with zero eigenvalue of the matrix \mathbb{Q} and its solution is given by the stationary distribution of Y . Hence, we have $\mu(\beta_i, i) = p(i)$ for any $i \in \mathbb{Y}$, proving that the boundary conditions of (4.10) are met.

- (4) Finally, using $I_s^i(\mu, \phi) = \Sigma^i(\mu, \phi) = 0$ in (C.4), we have

$$\sum_{i \in \mathbb{Y}} \sum_{\substack{j=k: \\ l_k=\alpha_i, \\ l_m=\beta_i}}^m \frac{1}{2} \sigma^2(l_j, i) \Delta \mu_x(l_j, i) \phi_x(l_j, i) = 0, \quad \text{for any } \phi \in \mathcal{V},$$

from which we deduce (thanks also to (C.5)) that $\mu \in C^1(\mathcal{I}; \mathbb{R}^d)$.

Concluding, $\mu(\cdot, i) \in C^1([\alpha_i, \beta_i]) \cap C^2((\alpha_i, \beta_i) \setminus \bigcup_{j \in \mathbb{Y}} \{\alpha_j, \beta_j\})$ is a classical solution to the boundary value problem in (4.10).

Step 3. Now we prove that (4.10) admits a unique solution. Assume that (4.10) has two solutions, denoted by $\mu_1 = (\mu_1(\cdot, i))_{i \in \mathbb{Y}}$ and $\mu_2 = (\mu_2(\cdot, i))_{i \in \mathbb{Y}}$, then the difference $\tilde{\mu} := \mu_2 - \mu_1 = (\mu_2(\cdot, i) - \mu_1(\cdot, i))_{i \in \mathbb{Y}}$ solves (4.10) with $\tilde{\mu}(x, i) = 0$ for $x \in \{\alpha_i, \beta_i\}$. Therefore, by Theorem 1 in [48], we obtain that

$$(C.7) \quad 0 \leq \sup_{(\alpha_i, \beta_i)} |\tilde{\mu}(x, i)| \leq C \max_{\{\alpha_i, \beta_i\}} |\tilde{\mu}(x, i)| = 0,$$

which in turn implies that, for any fixed $i \in \mathbb{Y}$, $\mu_2(x, i) = \mu_1(x, i)$, $x \in [\alpha_i, \beta_i]$. Hence, by Proposition 4.2 the unique stationary distribution ν admits a cumulative function μ which coincides with the unique solution to (4.10).

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