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## Convergence of infinitesimal generators and stability of convex monotone semigroups

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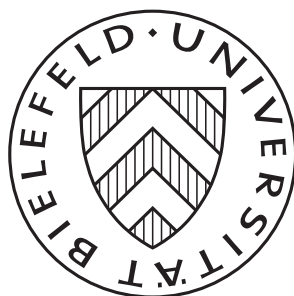


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# Convergence of Infinitesimal Generators and Stability of Convex Monotone Semigroups

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Jonas Blessing, Michael Kupper, and Max Nendel



# CONVERGENCE OF INFINITESIMAL GENERATORS AND STABILITY OF CONVEX MONOTONE SEMIGROUPS

JONAS BLESSING<sup>\*,1</sup>, MICHAEL KUPPER<sup>\*,2</sup>, AND MAX NENDEL<sup>\*\*,3</sup>

**ABSTRACT.** Based on the convergence of their infinitesimal generators in the mixed topology, we provide a stability result for strongly continuous convex monotone semigroups on spaces of continuous functions. In contrast to previous results, we do not rely on the theory of viscosity solutions but use a recent comparison principle which uniquely determines the semigroup via its  $\Gamma$ -generator defined on the Lipschitz set and therefore resembles the classical analogue from the linear case. The framework also allows for discretizations both in time and space and covers a variety of applications. This includes Euler schemes and Yosida-type approximations for upper envelopes of families of linear semigroups, stability results and finite-difference schemes for convex HJB equations, Freidlin–Wentzell-type results and Markov chain approximations for a class of stochastic optimal control problems and continuous-time Markov processes with uncertain transition probabilities.

*Key words:* convex monotone semigroup, infinitesimal generator, convergence of semigroups, Euler formula, optimal control, finite-difference scheme, Markov chain approximation, large deviations

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## 1. INTRODUCTION

In this article, we consider sequences  $(S_n)_{n \in \mathbb{N}}$  of convex monotone semigroups on spaces of continuous functions and give explicit conditions under which convergence of their infinitesimal generators  $(A_n)_{n \in \mathbb{N}}$  guarantees convergence of the semigroups  $(S_n)_{n \in \mathbb{N}}$ . In the linear case, such results are classical, cf. Kurtz [66] and Trotter [81], and can be applied, for instance, to obtain convergence results for Markov processes, see, e.g., Ethier and Kurtz [35], Kurtz [55, 56] and Kurtz [68]. In addition, based on the Crandall–Liggett theorem, c.f. Crandall and Liggett [26], the results can be extended to nonlinear semigroups which are generated by  $m$ -accretive or dissipative operators, see Brézis and Pazy [14] and Kurtz [67]. While this approach closely resembles the theory of linear semigroups, the definition of the nonlinear resolvent typically requires the existence of a unique classical solution of a corresponding fully nonlinear elliptic PDE. As pointed out in Evans [36] and Feng and Kurtz [40], the necessary regularity of classical solutions is, in general, delicate. This observation was, among others, one of the motivations for the introduction of viscosity solutions, cf. Crandall et al. [25] and Crandall and Lions [27]. In contrast to classical solutions, the latter have the stability property that, under mild conditions, limits of viscosity solutions are again viscosity solutions. A prominent example in this context is the vanishing viscosity method, where smooth solutions of semilinear second order equations converge uniformly to the unique viscosity solution of a fully nonlinear first order equation. Another remarkable stability property of viscosity solutions is that every monotone approximation scheme, which is consistent and stable, converges to the exact solution provided that the latter is unique, cf. Barles and Souganidis [3]. Based on this result, it is possible to derive explicit convergence rates for several numerical schemes for HJB equations, see, e.g., Barles and Jakobsen [1, 2], Briani et al. [16], Caffarelli and Souganidis [18], Jakobsen et al. [53] and Krylov [60–62]. Returning to the convergence of nonlinear semigroups, we would like to discuss the results from Feng and Kurtz [40] which are motivated by the large deviations principle for Markov processes. The approach in [40] combines results on dissipative operators with key arguments from viscosity theory and has been applied and extended in Feng [38], Feng et al. [39], Kraaij [57, 58] and Popovic [78]. Here, the resolvent equation  $(\text{id} - \lambda A)u = f$  for the limit operator  $Au := \lim_{n \rightarrow \infty} A_n u$  is solved in the viscosity sense and needs to satisfy a comparison principle which does not hold a priori but has to be verified on a case-by-case basis. Since the existence of a solution is always guaranteed, the limit semigroup can then be constructed via the Euler formula  $S(t)f := \lim_{n \rightarrow \infty} R(t/n)^n f$ , where  $R(\lambda)f$  denotes the unique viscosity solution of the resolvent equation. By construction, the semigroup  $(S(t))_{t \geq 0}$  is generated by the operator  $A$  and one can show that  $S(t)f = \lim_{n \rightarrow \infty} S_n(t)f$ .

In contrast to the previously mentioned results, the present approach neither relies on the existence of nonlinear resolvents nor on the theory of viscosity solutions. Instead, the key arguments are based on a recent comparison principle for strongly continuous convex monotone semigroups, cf. Blessing et al. [10] and Blessing and Kupper [11] which uniquely determines semigroups via their  $\Gamma$ -generators defined on their Lipschitz sets. Furthermore, due to an approximation procedure, it is typically sufficient to determine the generators for smooth functions. These results allow us to proceed in the following way. First, the limit semigroup is defined as  $S(t)f := \lim_{l \rightarrow \infty} S_{n_l}(t)f$  for all  $(t, f)$  in a countable dense set, where the convergence for a subsequence  $(n_l)_{l \in \mathbb{N}}$  is guaranteed by a relative compactness argument. Since we only require convergence w.r.t. the mixed topology rather than convergence w.r.t. the supremum norm, the latter can be

verified by means of Arz la-Ascoli's theorem. After an extension to arbitrary  $(t, f)$ , we then show that the generator of  $(S(t))_{t \geq 0}$  is given by  $Af = \lim_{n \rightarrow \infty} A_n f$  for smooth functions  $f$ . At this point we would like to emphasize that the semigroups  $(S_n)_{n \in \mathbb{N}}$  satisfy the comparison principle from [10, 11] which transfers to  $(S(t))_{t \geq 0}$ . In particular, the limit semigroup does not depend on the choice of the convergent subsequence  $(n_l)_{l \in \mathbb{N}}$  and therefore satisfies  $S(t)f = \lim_{n \rightarrow \infty} S_n(t)f$ . This stability result for convex monotone semigroups is stated in Theorem 2.3 while Theorem 2.7 also allows for discretizations in time and space. The latter covers a variety of approximation schemes of the form  $S(t)f = \lim_{n \rightarrow \infty} I_n^{k_n} f$ , where  $(I_n)_{n \in \mathbb{N}}$  is a family of one-step operators describing the dynamics on a discrete time scale of size  $h_n > 0$  with  $h_n \rightarrow 0$  and  $k_n h_n \rightarrow t$ , see Theorem 2.9. This extension of the classical Chernoff approximation, cf. Chernoff [22, 23], includes finite-difference methods for HJB equations, cf. Barles and Jakobsen [2], Bonnans and Zidani [13] and Krylov [61, 62], and Markov chain approximations for stochastic control problems, cf. Dupuis and Kushner [33] and Fleming and Soner [41]. Moreover, Theorem 2.9 allows to construct nonlinear semigroups without relying on the existence of nonlinear resolvents, e.g., in the context of upper envelopes for families of linear semigroups, cf. Denk et al. [30], Nendel and R ckner [71] and Nisio [73]. Finally, we remark that our notion of a strongly continuous convex monotone semigroup coincides with the one in Goldys et al. [49], where it is shown that the function  $u(t) := S(t)f$  is a viscosity solution of the abstract Cauchy problem  $\partial_t u = Au$  with initial condition  $u(0) = f$ . Hence, our approach is consistent with the theory of viscosity solutions and thus Chernoff-type approximations can be seen as monotone schemes. We refer to Fleming and Soner [41, Chapter II.3] for a broad discussion on the relation between semigroups and viscosity solutions and to Yong and Zhou [83, Chapter 4] for an illustration of the interplay between the dynamic programming principle and viscosity solutions in a stochastic optimal control setting. In addition, convex monotone semigroups appear in the context of stochastic processes under model uncertainty, cf. Coquet et al. [24], Criens and Niemann [28], Fadina et al. [37], Hu and Peng [51], Krak et al. [59], K hn [63], Neufeld and Nutz [72] and Peng [77]. In many situations, these semigroups admit a stochastic representation via BSDEs and 2BSDEs, cf. Cheridito et al. [21], El Karoui et al. [34], Kazi-Tani et al. [54] and Soner et al. [80]. For stability and approximation results for BSDEs, we refer to Briand et al. [15], Hu and Peng [52], Geiss et al. [47] and Papapantoleon et al. [74].

The abstract results are illustrated in a variety of applications which we briefly discuss at this point. In Subsection 3.1, we derive an implicit Euler formula and a Yosida approximation for upper envelopes of families of linear semigroups. Here, we would like to emphasize that difficulties in defining the resolvent of the supremum of linear operators are avoided by considering the supremum over linear resolvents rather than the nonlinear resolvent. We refer to Budde and Farkas [17], Cerrai [19], K hnemund [64] and Pazy [75] for the corresponding results on linear semigroups in different settings. In Subsection 3.2 and Subsection 3.3, we study finite-difference methods, cf. Barles and Jakobsen [2], Bonnans and Zidani [13], Dupuis and Kushner [33] and Krylov [61, 62], as well as the stability of convex HJB equations, cf. Barron and Jensen [4, 5], Crandall and Lions [27], Frankowska [42] and Kraaij [58]. In Subsection 3.4, we consider explicit Euler schemes for Lipschitz ODEs with an additive noise term, where the distribution of the noise can be uncertain. Depending on the scaling of the noise this can either be seen as a robustness result which states that, regardless of possible numerical errors, the Euler scheme still converges to the solution of the ODE or as an Euler-Maruyama scheme for SDEs driven by G-Brownian motions, cf. Geng et al. [48, Section 3], Hu et

al. [50] and Peng [76, 77]. The analysis of randomized Euler schemes is continued in Subsection 3.5, where we focus on asymptotic convergence rates by means of a large deviations approach, cf. Dupuis and Ellis [32], Dembo and Zeitouni [29], Feng and Kurtz [40] and Varadhan [82] leading to Freidlin–Wentzell-type results, cf. Feng and Kurtz [40] and Freidlin and Wentzell [43]. Finally, in Subsection 3.6 and Subsection 3.7, we provide Markov chain approximations for a class of stochastic control problems, cf. Dupuis and Kushner [33], Fleming and Soner [41] and Krylov [60], and continuous-time Markov processes with uncertain transition probabilities, cf. Bartl et al. [8] and Fuhrmann et al. [45].

## 2. SETUP AND MAIN RESULTS

**2.1. The mixed topology and basic notations.** Let  $\kappa: \mathbb{R}^d \rightarrow (0, \infty)$  be a bounded continuous function with

$$c_\kappa := \sup_{x \in \mathbb{R}^d} \sup_{|y| \leq 1} \frac{\kappa(x)}{\kappa(x-y)} < \infty, \quad (2.1)$$

and denote by  $C_\kappa(\mathbb{R}^d)$  the space of all continuous functions  $f: \mathbb{R}^d \rightarrow \mathbb{R}$  satisfying

$$\|f\|_\kappa = \sup_{x \in \mathbb{R}^d} |f(x)|\kappa(x) < \infty.$$

We endow  $C_\kappa(\mathbb{R}^d)$  with the mixed topology between  $\|\cdot\|_\kappa$  and the topology of uniform convergence on compact sets, i.e., the strongest locally convex topology on  $C_\kappa(\mathbb{R}^d)$  that coincides on  $\|\cdot\|_\kappa$ -bounded sets with the topology of uniform convergence on compact subsets. It is well-known, see [49, Proposition A.4], that a sequence  $(f_n)_{n \in \mathbb{N}} \subset C_\kappa(\mathbb{R}^d)$  converges to  $f \in C_\kappa(\mathbb{R}^d)$  w.r.t. the mixed topology if and only if

$$\sup_{n \in \mathbb{N}} \|f_n\|_\kappa < \infty \quad \text{and} \quad \lim_{n \rightarrow \infty} \|f - f_n\|_{\infty, K} = 0$$

for all compact subsets  $K \Subset \mathbb{R}^d$ , where  $\|f\|_{\infty, K} := \sup_{x \in K} |f(x)|$ . Similarly, for a family of functions  $(f_s)_{s \geq 0} \subset C_\kappa(\mathbb{R}^d)$  and  $t \geq 0$ , it holds  $f_t = \lim_{s \rightarrow t} f_s$  if and only if  $\sup_{|s-t| \leq \delta_0} \|f_s\|_\kappa < \infty$  for some  $\delta_0 > 0$  and, for every  $\varepsilon > 0$  and  $K \Subset \mathbb{R}^d$ , there exists  $\delta > 0$  with  $\|f_s - f_t\|_{\infty, K} < \varepsilon$  for all  $|s - t| \leq \delta$ . Subsequently, if not stated otherwise, all limits in  $C_\kappa(\mathbb{R}^d)$  are understood w.r.t. the mixed topology and compact subsets are denoted by  $K \Subset \mathbb{R}^d$ . We point out that the mixed topology is, in general, not metrizable but generated by the uncountable family of seminorms

$$p_{(a_n), (K_n)}(f) := \sup_{n \in \mathbb{N}} \sup_{x \in K_n} a_n |f(x)|\kappa(x),$$

where  $(a_n)_{n \in \mathbb{N}} \subset (0, \infty)$  is a sequence with  $a_n \rightarrow 0$  and  $(K_n)_{n \in \mathbb{N}}$  is a sequence of compact sets  $K_n \Subset \mathbb{R}^d$ . Moreover, the mixed topology has no neighbourhood of zero which is  $\|\cdot\|_\kappa$ -bounded or, equivalently, bounded in the mixed topology. Nevertheless, for monotone operators  $S: C_\kappa(\mathbb{R}^d) \rightarrow C_\kappa(\mathbb{R}^d)$ , sequential continuity is equivalent to continuity and continuity is equivalent to continuity on  $\|\cdot\|_\kappa$ -bounded subsets, see [70]. We further remark that the mixed topology coincides with the Mackey topology of the dual pair  $(C_\kappa(\mathbb{R}^d), \mathcal{M}_\kappa(\mathbb{R}^d))$ , where  $\mathcal{M}_\kappa(\mathbb{R}^d)$  denotes the space of all countably additive signed Borel measures  $\mu$  with

$$\int_{\mathbb{R}^d} \frac{1}{\kappa(x)} |\mu|(dx) < \infty,$$

where  $|\mu|$  denotes the total variation measure of  $\mu$ , see, e.g., [49]. In particular, the mixed topology belongs to the class of strict topologies, c.f. [44, 65, 79]. Finally, relative

compactness w.r.t. the mixed topology can be characterized by means of Aréla Ascoli's theorem and, due to the Stone–Weierstraß theorem, there exists a countable set of smooth functions with compact support which is dense in  $C_\kappa(\mathbb{R}^d)$ .

For functions  $f, g: \mathbb{R}^d \rightarrow \mathbb{R}$ , we write  $f \leq g$  if and only if  $f(x) \leq g(x)$  for all  $x \in \mathbb{R}^d$ . In addition, let  $f^+(x) := \max\{f(x), 0\}$  be the positive part and  $(\tau_x f)(y) := f(x + y)$  be the shift of a function  $f: \mathbb{R}^d \rightarrow \mathbb{R}$  for all  $x, y \in \mathbb{R}^d$ . Closed balls with radius  $r \geq 0$  around zero are denoted by

$$B_{C_\kappa(\mathbb{R}^d)}(r) := \{f \in C_\kappa(\mathbb{R}^d) : \|f\|_\kappa \leq r\} \quad \text{and} \quad B_{\mathbb{R}^d}(r) := \{x \in \mathbb{R}^d : |x| \leq r\},$$

where  $|\cdot|$  denotes the Euclidean distance. The space  $\text{Lip}_b(\mathbb{R}^d)$  consists of all bounded Lipschitz continuous functions  $f: \mathbb{R}^d \rightarrow \mathbb{R}$  and, for every  $r \geq 0$ , the set of all  $r$ -Lipschitz functions  $f: \mathbb{R}^d \rightarrow \mathbb{R}$  with  $\sup_{x \in \mathbb{R}^d} |f(x)| \leq r$  is denoted by  $\text{Lip}_b(\mathbb{R}^d, r)$ . Moreover, we define  $C_b^\infty(\mathbb{R}^d)$  as the space of all bounded infinitely differentiable functions  $f: \mathbb{R}^d \rightarrow \mathbb{R}$  such that the partial derivatives of all orders are bounded. Let  $\mathbb{R}_+ := \{x \in \mathbb{R} : x \geq 0\}$ .

## 2.2. Stability of convex monotone semigroups.

**Definition 2.1.** A strongly continuous convex monotone semigroup is a family  $(S(t))_{t \geq 0}$  of operators  $S(t): C_\kappa(\mathbb{R}^d) \rightarrow C_\kappa(\mathbb{R}^d)$  satisfying the following conditions:

- (i)  $S(0) = \text{id}_{C_\kappa(\mathbb{R}^d)}$  and  $S(s+t)f = S(s)S(t)f$  for all  $s, t \geq 0$  and  $f \in C_\kappa(\mathbb{R}^d)$ .
- (ii)  $S(t)f \leq S(t)g$  for all  $t \geq 0$  and  $f, g \in C_\kappa(\mathbb{R}^d)$  with  $f \leq g$ .
- (iii)  $S(t)(\lambda f + (1 - \lambda)g) \leq \lambda S(t)f + (1 - \lambda)S(t)g$  for all  $t \geq 0$ ,  $f, g \in C_\kappa(\mathbb{R}^d)$  and  $\lambda \in [0, 1]$ .
- (iv) The mapping  $\mathbb{R}_+ \rightarrow C_\kappa(\mathbb{R}^d)$ ,  $t \mapsto S(t)f$  is continuous for all  $f \in C_\kappa(\mathbb{R}^d)$ .

Furthermore, the generator of the semigroup is defined by

$$A: D(A) \rightarrow C_\kappa(\mathbb{R}^d), \quad f \mapsto \lim_{h \downarrow 0} \frac{S(h)f - f}{h},$$

where the domain  $D(A)$  consists of all  $f \in C_\kappa(\mathbb{R}^d)$  such that the previous limit exists. The Lipschitz set  $\mathcal{L}^S$  consists of all  $f \in C_\kappa(\mathbb{R}^d)$  such that there exist  $c \geq 0$  and  $t_0 > 0$  with

$$\|S(t)f - f\|_\kappa \leq ct \quad \text{for all } t \in [0, t_0],$$

while the upper Lipschitz set  $\mathcal{L}_+^S$  consists of all  $f \in C_\kappa(\mathbb{R}^d)$  such that there exist  $c \geq 0$  and  $t_0 > 0$  with

$$\|(S(t)f - f)^+\|_\kappa \leq ct \quad \text{for all } t \in [0, t_0].$$

We say that a set  $\mathcal{C} \subset C_\kappa(\mathbb{R}^d)$  is invariant if  $S(t): \mathcal{C} \rightarrow \mathcal{C}$  for all  $t \geq 0$ .

By definition, we have  $D(A) \subset \mathcal{L}^S \subset \mathcal{L}_+^S$ . Furthermore, in contrast to the theory of strongly continuous linear semigroups, the domain  $D(A)$  is, in general, not invariant under the nonlinear semigroup, see [31, Example 5.2]. This observation motivated the study of invariant Lipschitz sets [12] and the upper  $\Gamma$ -generator to obtain a comparison principle for convex monotone semigroups [10, 11]. The invariance of the sets  $\mathcal{L}^S$  and  $\mathcal{L}_+^S$  and the comparison principle are stated in Theorem 2.3 and Theorem 2.4 below.

**Assumption 2.2.** Let  $(S_n)_{n \in \mathbb{N}}$  be a sequence of strongly continuous convex monotone semigroups  $(S_n(t))_{t \geq 0}$  on  $C_\kappa(\mathbb{R}^d)$  with  $S_n(t)0 = 0$  for all  $n \in \mathbb{N}$  and  $t \geq 0$ . We denote by  $A_n$  the generator of  $S_n$  and impose the following conditions:

- (i) For every  $r, T \geq 0$ , there exists  $c \geq 0$  with

$$\|S_n(t)f - S_n(t)g\|_\kappa \leq c\|f - g\|_\kappa$$

for all  $n \in \mathbb{N}$ ,  $t \in [0, T]$  and  $f, g \in B_{C_\kappa(\mathbb{R}^d)}(r)$ .



- (ii) For every  $\varepsilon > 0$ ,  $r, T \geq 0$  and  $K \in \mathbb{R}^d$ , there exist  $c \geq 0$  and  $K' \in \mathbb{R}^d$  with

$$\|S_n(t)f - S_n(t)g\|_{\infty, K} \leq c\|f - g\|_{\infty, K'} + \varepsilon$$

for all  $n \in \mathbb{N}$ ,  $t \in [0, T]$  and  $f, g \in B_{C_\kappa(\mathbb{R}^d)}(r)$ .

- (iii) For every  $f \in \text{Lip}_b(\mathbb{R}^d)$  and  $\varepsilon > 0$ , there exist  $\delta, t_0 > 0$  and  $(f_n)_{n \in \mathbb{N}} \subset C_\kappa(\mathbb{R}^d)$  with  $f_n \rightarrow f$  and

$$\|S_n(t)(\tau_x f_n) - \tau_x S_n(t)f_n\|_\kappa \leq \varepsilon t$$

for all  $n \in \mathbb{N}$ ,  $t \in [0, t_0]$  and  $x \in B_{\mathbb{R}^d}(\delta)$ .

- (iv) For every  $f \in \text{Lip}_b(\mathbb{R}^d)$  and  $T \geq 0$ , there exists  $r \geq 0$  with  $S_n(t)f \in \text{Lip}_b(\mathbb{R}^d, r)$  for all  $n \in \mathbb{N}$  and  $t \in [0, T]$ .
- (v) For every  $f \in C_b^\infty(\mathbb{R}^d)$ , there exists a sequence  $(f_n)_{n \in \mathbb{N}}$  with  $f_n \in D(A_n)$  for all  $n \in \mathbb{N}$  and  $f_n \rightarrow f$  such that the sequence  $(A_n f_n)_{n \in \mathbb{N}}$  converges in  $C_\kappa(\mathbb{R}^d)$ .

The following theorem is a consequence of the more general result that will be proved in the next subsection.

**Theorem 2.3.** *Let  $(S_n)_{n \in \mathbb{N}}$  be a sequence satisfying Assumption 2.2. Then, there exists a strongly continuous convex monotone semigroup  $(S(t))_{t \geq 0}$  on  $C_\kappa(\mathbb{R}^d)$  with*

$$S(t)f = \lim_{n \rightarrow \infty} S_n(t_n)f_n \quad \text{for all } (f, t) \in C_\kappa(\mathbb{R}^d) \times \mathbb{R}_+,$$

where  $(t_n)_{n \in \mathbb{N}} \subset \mathbb{R}_+$  and  $(f_n)_{n \in \mathbb{N}} \subset C_\kappa(\mathbb{R}^d)$  are arbitrary sequences with  $t_n \rightarrow t$  and  $f_n \rightarrow f$ . Furthermore, the semigroup  $(S(t))_{t \geq 0}$  has the following properties:

- (i) Let  $f \in C_\kappa(\mathbb{R}^d)$  and  $(f_n)_{n \in \mathbb{N}}$  be a sequence with  $f_n \in D(A_n)$  for all  $n \in \mathbb{N}$  and  $f_n \rightarrow f$  such that the sequence  $(A_n f_n)_{n \in \mathbb{N}}$  converges in  $C_\kappa(\mathbb{R}^d)$ . Then, it holds  $f \in D(A)$  and  $Af = \lim_{n \rightarrow \infty} A_n f_n$ . In particular, this is valid for all  $f \in C_b^\infty(\mathbb{R}^d)$ .
- (ii) For every  $r, T \geq 0$ , there exists  $c \geq 0$  with

$$\|S(t)f - S(t)g\|_\kappa \leq c\|f - g\|_\kappa \quad \text{for all } t \in [0, T] \text{ and } f, g \in B_{C_\kappa(\mathbb{R}^d)}(r).$$

Furthermore, it holds  $S(t)0 = 0$  for all  $t \geq 0$ .

- (iii) For every  $\varepsilon > 0$ ,  $r, T \geq 0$  and  $K \in \mathbb{R}^d$ , there exist  $c \geq 0$  and  $K' \in \mathbb{R}^d$  with

$$\|S(t)f - S(t)g\|_{\infty, K} \leq c\|f - g\|_{\infty, K'} + \varepsilon$$

for all  $t \in [0, T]$  and  $f, g \in B_{C_\kappa(\mathbb{R}^d)}(r)$ .

- (iv) For every  $f \in \text{Lip}_b(\mathbb{R}^d)$  and  $\varepsilon > 0$ , there exist  $\delta, t_0 > 0$  with

$$\|S(t)(\tau_x f) - \tau_x S(t)f\|_\kappa \leq \varepsilon t$$

for all  $t \in [0, t_0]$  and  $x \in B_{\mathbb{R}^d}(\delta)$ .

- (v) The sets  $\mathcal{L}^S$ ,  $\mathcal{L}_+^S$  and  $\text{Lip}_b(\mathbb{R}^d)$  are invariant.

Due to the comparison principle in [10, 11], the semigroup  $(S(t))_{t \geq 0}$  from the previous theorem is uniquely determined by its infinitesimal generator. For later reference, we state the results from [10, 11] in the following form.

**Theorem 2.4.** *Let  $(S_1(t))_{t \geq 0}$  and  $(S_2(t))_{t \geq 0}$  be two strongly continuous convex monotone semigroups on  $C_\kappa(\mathbb{R}^d)$  with  $S_i(t)0 = 0$  and generators  $A_1$  and  $A_2$ , respectively, such that the following conditions are satisfied for  $i = 1, 2$ :*

- (i) It holds  $C_b^\infty(\mathbb{R}^d) \subset D(A_i)$  and  $A_1 f = A_2 f$  for all  $f \in C_b^\infty(\mathbb{R}^d)$ .
- (ii) For every  $r, T \geq 0$ , there exists  $c \geq 0$  with

$$\|S_i(t)f - S_i(t)g\|_\kappa \leq c\|f - g\|_\kappa \quad \text{for all } t \in [0, T] \text{ and } f, g \in B_{C_\kappa(\mathbb{R}^d)}(r).$$



(iii) For every  $\varepsilon > 0$ ,  $r, T \geq 0$  and  $K \subseteq \mathbb{R}^d$ , there exist  $c \geq 0$  and  $K' \subseteq \mathbb{R}^d$  with

$$\|S_i(t)f - S_i(t)g\|_{\infty, K} \leq c\|f - g\|_{\infty, K'} + \varepsilon$$

for all  $t \in [0, T]$  and  $f, g \in B_{C_\kappa(\mathbb{R}^d)}(r)$ .

(iv) For every  $f \in \text{Lip}_b(\mathbb{R}^d)$  and  $\varepsilon > 0$ , there exist  $\delta, t_0 > 0$  with

$$\|S_i(t)(\tau_x f) - \tau_x S_i(t)f\|_\kappa \leq \varepsilon t$$

for all  $t \in [0, t_0]$  and  $x \in B_{\mathbb{R}^d}(\delta)$ .

(v) It holds  $S_i(t): \text{Lip}_b(\mathbb{R}^d) \rightarrow \text{Lip}_b(\mathbb{R}^d)$  for all  $t \geq 0$ .

Then, it holds  $S_1(t)f = S_2(t)f$  for all  $t \geq 0$  and  $f \in C_\kappa(\mathbb{R}^d)$ .

*Proof.* It follows from [11, Theorem 2.5] that

$$\mathcal{L}_+^{S_1} \cap \text{Lip}_b(\mathbb{R}^d) = \mathcal{L}_+^{S_2} \cap \text{Lip}_b(\mathbb{R}^d) \quad (2.2)$$

and therefore [10, Corollary 2.11] implies

$$S_1(t)f = S_2(t)f \quad \text{for all } (f, t) \in C_b^\infty(\mathbb{R}^d) \times \mathbb{R}_+. \quad (2.3)$$

Indeed, the semigroups  $(S_1(t))_{t \geq 0}$  and  $(S_2(t))_{t \geq 0}$  clearly satisfy [10, Assumption 2.4] while equation (2.2), condition (ii) and condition (v) imply  $\mathcal{L}^{S_1} \cap \text{Lip}_b(\mathbb{R}^d) \subset \mathcal{L}_+^{S_2}$  and

$$S_1(t): \mathcal{L}^{S_1} \cap \text{Lip}_b(\mathbb{R}^d) \rightarrow \mathcal{L}^{S_1} \cap \text{Lip}_b(\mathbb{R}^d) \quad \text{for all } t \geq 0.$$

Moreover, due to condition (i) and (iv), we can apply [10, Lemma 3.6] to conclude that  $(S_1(t))_{t \geq 0}$  and  $(S_2(t))_{t \geq 0}$  have the same upper  $\Gamma$ -generator on  $\mathcal{L}^{S_1} \cap \text{Lip}_b(\mathbb{R}^d)$ . Hence,

$$S_1(t)f \leq S_2(t)f \quad \text{for all } t \geq 0 \text{ and } f \in C_b^\infty(\mathbb{R}^d)$$

and reversing the roles of  $(S_1(t))_{t \geq 0}$  and  $(S_2(t))_{t \geq 0}$  yields that equation (2.3) is valid. Since  $C_b^\infty(\mathbb{R}^d) \subset C_\kappa(\mathbb{R}^d)$  is dense, it follows from condition (iii) that

$$S_1(t)f = S_2(t)f \quad \text{for all } (f, t) \in C_\kappa(\mathbb{R}^d) \times \mathbb{R}_+. \quad \square$$

**2.3. Discretization in time and space.** Let  $(\mathcal{T}_n)_{n \in \mathbb{N}}$  and  $(X_n)_{n \in \mathbb{N}}$  be two sequences of subsets  $\mathcal{T}_n \subset \mathbb{R}_+$  and  $X_n \subset \mathbb{R}^d$  such that, for every  $t \geq 0$  and  $x \in \mathbb{R}^d$ , there exist  $t_n \in \mathcal{T}_n$  and  $x_n \in X_n$  with  $t_n \rightarrow t$  and  $x_n \rightarrow x$ , respectively. We assume that  $0 \in \mathcal{T}_n$  and  $s \pm t \in \mathcal{T}_n$  for all  $n \in \mathbb{N}$  and  $s, t \in \mathcal{T}_n$  with  $s \geq t$ . If  $h_n := \inf(\mathcal{T}_n \setminus \{0\}) > 0$ , it follows that  $\mathcal{T}_n = \{kh_n : k \in \mathbb{N}_0\}$  is an equidistant grid. Otherwise, the set  $\mathcal{T}_n$  is already dense. In addition, the sets  $X_n$  are supposed to be closed and satisfy  $x + y \in X_n$  for all  $n \in \mathbb{N}$  and  $x, y \in X_n$ . The space  $C_\kappa(X_n)$  consists of all continuous functions  $f: X_n \rightarrow \mathbb{R}$  with

$$\|f\|_{\kappa, X_n} := \sup_{x \in X_n} |f(x)|\kappa(x) < \infty.$$

Let  $B_{C_\kappa(X_n)}(r) := \{f \in C_\kappa(X_n) : \|f\|_{\kappa, X_n} \leq r\}$  for all  $r \geq 0$ . Moreover, for a sequence  $(f_n)_{n \in \mathbb{N}}$  with  $f_n \in C_\kappa(X_n)$  and  $f \in C_\kappa(\mathbb{R}^d)$ , we define  $f := \lim_{n \rightarrow \infty} f_n$  if and only if

$$\sup_{n \in \mathbb{N}} \|f_n\|_{\kappa, X_n} < \infty \quad \text{and} \quad \lim_{n \rightarrow \infty} \|f - f_n\|_{\infty, K_n} = 0$$

for all  $K \subseteq \mathbb{R}^d$  with  $K \cap X_n \neq \emptyset$  for all  $n \in \mathbb{N}$ , where  $K_n := K \cap X_n$ . The continuity of  $f$  guarantees that the previous limit is uniquely determined. Indeed, assume by contradiction that there exists another function  $g \in C_\kappa(\mathbb{R}^d)$  with  $g = \lim_{n \rightarrow \infty} f_n$  and  $f \neq g$ . Then, there exists  $\varepsilon > 0$  with  $A := \{x \in \mathbb{R}^d : |f(x) - g(x)| > \varepsilon\} \neq \emptyset$ . Fix  $x \in A$  and choose a sequence  $(x_n)_{n \in \mathbb{N}}$  with  $x_n \in X_n$  and  $x_n \rightarrow x$ . Since  $A$  is open, there exists  $n_0 \in \mathbb{N}$  with  $x_n \in A$  for all  $n \geq n_0$ . For  $K := \{x_n : n \in \mathbb{N}\} \cup \{x\}$ , we obtain

$$\varepsilon \leq \|f - g\|_{\infty, K_n} \leq \|f - f_n\|_{\infty, K_n} + \|f_n - g\|_{\infty, K_n} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

In addition, we also fix a family  $\{S_n(t) : n \in \mathbb{N}, t \in \mathcal{T}_n\}$  of operators

$$S_n(t) : C_\kappa(X_n) \rightarrow C_\kappa(X_n),$$

and define  $S_n(t)f := S_n(t)(f|_{X_n})$  for all  $n \in \mathbb{N}$ ,  $t \in \mathcal{T}_n$  and  $f \in C_\kappa(\mathbb{R}^d)$ , i.e., we consider  $C_\kappa(\mathbb{R}^d)$  as a subset of  $C_\kappa(X_n)$  using the possibility to restrict functions from  $\mathbb{R}^d$  to  $X_n$ . Similar to Definition 2.1, a family  $(S_n(t))_{t \in \mathcal{T}_n}$  is called convex monotone semigroup on  $C_\kappa(X_n)$  if the operators  $S_n(t) : C_\kappa(X_n) \rightarrow C_\kappa(X_n)$  are convex and monotone for all  $n \in \mathbb{N}$  and  $t \in \mathcal{T}_n$ , where the space  $C_\kappa(X_n)$  is endowed with the pointwise order, and satisfy  $S_n(0)f = f$  and  $S_n(s+t)f = S_n(s)S_n(t)f$  for all  $n \in \mathbb{N}$ ,  $s, t \in \mathcal{T}_n$  and  $f \in C_\kappa(X_n)$ .

If  $\inf(T_n \setminus \{0\}) > 0$ , the generator of  $(S_n(t))_{t \in \mathcal{T}_n}$  can not be defined as the limit

$$A_n f = \lim_{h \downarrow 0} \frac{S_n(h)f - f}{h}.$$

This observation motivates the following rather technical definition and corresponding condition (v) in Assumption 2.6. However, if  $\mathcal{T}_n = \mathbb{R}_+$  for all  $n \in \mathbb{N}$ , it is sufficient to show that, for every  $f \in C_b^\infty(\mathbb{R}^d)$ , there exist  $f_n \in D(A_n)$  with  $f_n \rightarrow f$  such that the sequence  $(A_n f_n)_{n \in \mathbb{N}}$  converges in  $C_\kappa(\mathbb{R}^d)$ . Moreover, for Chernoff-type approximations, we only have to verify that the approximation scheme satisfies a reasonable consistency condition. For details, we refer to Theorem 2.7 and Theorem 2.9 below.

**Definition 2.5.** The asymptotic Lipschitz set  $\mathcal{L}_0$  consists of all  $f \in C_\kappa(\mathbb{R}^d)$  such that there exist  $c \geq 0$  and sequences  $(t_n)_{n \in \mathbb{N}}$  and  $(f_n)_{n \in \mathbb{N}}$  with  $t_n \in \mathcal{T}_n \setminus \{0\}$ ,  $f_n \in C_\kappa(X_n)$ ,  $f_n \rightarrow f$  and

$$\|S_n(t)f_n - f_n\|_{\kappa, X_n} \leq ct \quad \text{for all } n \in \mathbb{N} \text{ and } t \in [0, t_n] \cap \mathcal{T}_n. \quad (2.4)$$

Furthermore, the asymptotic domain  $\mathcal{D}_0$  consists of all  $f \in C_\kappa(\mathbb{R}^d)$  such that there exist  $g \in C_\kappa(\mathbb{R}^d)$  and a sequence  $(f_n)_{n \in \mathbb{N}}$  with  $f_n \in C_\kappa(X_n)$  and  $f_n \rightarrow f$  which satisfy the following conditions:

- (i) There exist  $c \geq 0$  and  $(t_n)_{n \in \mathbb{N}}$  with  $t_n \in \mathcal{T}_n \setminus \{0\}$  such that inequality (2.4) is valid.
- (ii) For every  $K \in \mathbb{R}^d$ , there exists  $(h_n)_{n \in \mathbb{N}}$  with  $h_n \in (0, t_n] \cap \mathcal{T}_n$ ,  $h_n \rightarrow 0$  and

$$\lim_{n \rightarrow \infty} \left\| \frac{S_n(h_n)f_n - f_n}{h_n} - g \right\|_{\infty, K_n} = 0. \quad (2.5)$$

For every  $n \in \mathbb{N}$  and  $r \geq 0$ , we denote by  $\text{Lip}_b(X_n, r)$  the set of all  $r$ -Lipschitz functions  $f : X_n \rightarrow \mathbb{R}$  with  $\|f\|_{\infty, X_n} \leq r$ .

**Assumption 2.6.** Suppose that  $(S_n)_{n \in \mathbb{N}}$  is a sequence of convex monotone semigroups  $(S_n(t))_{t \in \mathcal{T}_n}$  on  $C_\kappa(X_n)$  with  $S_n(t)0 = 0$  for all  $n \in \mathbb{N}$  and  $t \in \mathcal{T}_n$ . In addition, the following conditions are satisfied:

- (i) For every  $r, T \geq 0$ , there exists  $c \geq 0$  with

$$\|S_n(t)f - S_n(t)g\|_{\kappa, X_n} \leq c\|f - g\|_{\kappa, X_n}$$

for all  $n \in \mathbb{N}$ ,  $t \in [0, T] \cap \mathcal{T}_n$  and  $f, g \in B_{C_\kappa(X_n)}(r)$ .

- (ii) For every  $\varepsilon > 0$ ,  $r, T \geq 0$  and  $K \in \mathbb{R}^d$ , there exist  $c \geq 0$  and  $K' \in \mathbb{R}^d$  with

$$\|S_n(t)f - S_n(t)g\|_{\infty, K_n} \leq c\|f - g\|_{\infty, K'_n} + \varepsilon$$

for all  $n \in \mathbb{N}$ ,  $t \in [0, T] \cap \mathcal{T}_n$  and  $f, g \in B_{C_\kappa(X_n)}(r)$ .

- (iii) For every  $f \in \text{Lip}_b(\mathbb{R}^d)$  and  $\varepsilon > 0$ , there exist  $\delta, t_0 > 0$  and a sequence  $(f_n)_{n \in \mathbb{N}}$  with  $f_n \in C_\kappa(X_n)$  for all  $n \in \mathbb{N}$ ,  $f_n \rightarrow f$  and

$$\|S_n(t)(\tau_x f_n) - \tau_x S_n(t)f_n\|_{\kappa, X_n} \leq \varepsilon t$$

- for all  $n \in \mathbb{N}$ ,  $t \in [0, t_0] \cap \mathcal{T}_n$ ,  $x \in B_{X_n}(\delta)$  and  $y \in X_n$ .
- (iv) For every  $f \in \text{Lip}_b(\mathbb{R}^d)$  and  $T \geq 0$ , there exists  $r \geq 0$  with  $S_n(t)f \in \text{Lip}_b(X_n, r)$  for all  $n \in \mathbb{N}$  and  $t \in [0, T] \cap \mathcal{T}_n$ .
- (v) It holds  $C_b^\infty(\mathbb{R}^d) \subset \mathcal{D}_0$ .

The following theorem is the main result of this article.

**Theorem 2.7.** *Let  $(S_n)_{n \in \mathbb{N}}$  be a sequence satisfying Assumption 2.6. Then, there exists a strongly continuous convex monotone semigroup  $(S(t))_{t \geq 0}$  on  $C_\kappa(\mathbb{R}^d)$  with*

$$S(t)f = \lim_{n \rightarrow \infty} S_n(t_n)f_n \quad \text{for all } (f, t) \in C_\kappa(\mathbb{R}^d) \times \mathbb{R}_+, \quad (2.6)$$

where  $(t_n)_{n \in \mathbb{N}}$  and  $(f_n)_{n \in \mathbb{N}}$  are arbitrary sequences with  $t_n \in \mathcal{T}_n$  and  $f_n \in C_\kappa(X_n)$  for all  $n \in \mathbb{N}$ ,  $t_n \rightarrow t$  and  $f_n \rightarrow f$ . Furthermore, the semigroup  $(S(t))_{t \geq 0}$  has the following properties:

- (i) It holds  $\mathcal{L}_0 \subset \mathcal{L}^S$  and  $\mathcal{D}_0 \subset D(A)$ . Moreover, for every  $f \in \mathcal{D}_0$  and  $K \in \mathbb{R}^d$ ,

$$\lim_{n \rightarrow \infty} \left\| \frac{S_n(h_n)f_n - f_n}{h_n} - Af \right\|_{\infty, K_n} = 0, \quad (2.7)$$

where  $(h_n)_{n \in \mathbb{N}}$  and  $(f_n)_{n \in \mathbb{N}}$  are arbitrary sequences satisfying the conditions (i) and (ii) in Definition 2.5. In particular, (2.7) is valid for all  $f \in C_b^\infty(\mathbb{R}^d)$ .

- (ii) For every  $r, T \geq 0$ , there exists  $c \geq 0$  with

$$\|S(t)f - S(t)g\|_\kappa \leq c\|f - g\|_\kappa \quad \text{for all } t \in [0, T] \text{ and } f, g \in B_{C_\kappa(\mathbb{R}^d)}(r).$$

Furthermore, it holds  $S(t)0 = 0$  for all  $t \geq 0$ .

- (iii) For every  $\varepsilon > 0$ ,  $r, T \geq 0$  and  $K \in \mathbb{R}^d$ , there exist  $c \geq 0$  and  $K' \in \mathbb{R}^d$  with

$$\|S(t)f - S(t)g\|_{\infty, K} \leq c\|f - g\|_{\infty, K'} + \varepsilon$$

for all  $t \in [0, T]$  and  $f, g \in B_{C_\kappa(\mathbb{R}^d)}(r)$ .

- (iv) For every  $f \in \text{Lip}_b(\mathbb{R}^d)$  and  $\varepsilon > 0$ , there exist  $\delta, t_0 > 0$  with

$$\|S(t)(\tau_x f) - \tau_x S(t)f\|_\kappa \leq \varepsilon t$$

for all  $t \in [0, t_0]$  and  $x \in B_{\mathbb{R}^d}(\delta)$ .

- (v) The sets  $\mathcal{L}^S$ ,  $\mathcal{L}_+^S$  and  $\text{Lip}_b(\mathbb{R}^d)$  are invariant.

*Proof.* First, we construct the family  $(S(t))_{t \geq 0}$  of operators  $S(t): C_\kappa(\mathbb{R}^d) \rightarrow C_\kappa(\mathbb{R}^d)$ . Let  $\mathcal{D} \subset C_b^\infty(\mathbb{R}^d)$  and  $\mathcal{T} \subset \mathbb{R}_+$  be countable dense sets with  $0 \in \mathcal{T}$ . Moreover, for every  $t \in \mathcal{T}$ , let  $(t_n)_{n \in \mathbb{N}}$  be a sequence with  $t_n \in \mathcal{T}_n$  and  $t_n \rightarrow t$ . For  $t = 0$ , we choose  $t_n := 0$  for all  $n \in \mathbb{N}$ . Since Assumption 2.6(i) and (iv) imply that the sequence  $(S_n(t_n)f)_{n \in \mathbb{N}}$  is bounded and uniformly equicontinuous, we can use Lemma A.1 and a diagonalization argument to choose a subsequence  $(n_l)_{l \in \mathbb{N}} \subset \mathbb{N}$  such that the limit

$$S(t)f := \lim_{l \rightarrow \infty} S_{n_l}(t_{n_l})f \in C_\kappa(\mathbb{R}^d) \quad (2.8)$$

exists for all  $(f, t) \in \mathcal{D} \times \mathcal{T}$ . In the order to extend the family  $(S(t))_{t \in \mathcal{T}}$  to arbitrary points in time, we fix  $f \in \mathcal{D}$  and  $T \geq 0$ . Since  $f \in \mathcal{L}_0$ , there exist  $c_1 \geq 0$  as well as sequences  $(h_n)_{n \in \mathbb{N}}$  and  $(f_n)_{n \in \mathbb{N}}$  with  $h_n \in \mathcal{T}_n \setminus \{0\}$  and  $f_n \in C_\kappa(X_n)$  for all  $n \in \mathbb{N}$ ,  $f_n \rightarrow f$  and

$$\|S_n(h)f_n - f_n\|_{\kappa, X_n} \leq c_1 h \quad \text{for all } n \in \mathbb{N} \text{ and } h \in [0, h_n] \cap \mathcal{T}_n.$$

Let  $s, t \in [0, T] \cap \mathcal{T}$  with  $s \leq t$ . We choose  $s_n, t_n \in \mathcal{T}_n$  with  $s_n \rightarrow s$  and  $t_n \rightarrow t$  and define  $k_n := \max\{k \in \mathbb{N}_0 : kh_n \leq t_n - s_n\}$  for all  $n \in \mathbb{N}$ . By Assumption 2.6(i), there exists  $c_2 \geq 0$  with

$$\begin{aligned} \|S_n(s_n)f_n - S_n(t_n)f_n\|_{\kappa, X_n} &= \|S_n(s_n)S_n(t_n - s_n)f_n - S_n(s_n)f_n\|_{\kappa, X_n} \\ &\leq c_2 \|S_n(t_n - s_n)f_n - f_n\|_{\kappa, X_n} \\ &\leq c_2 \|S_n(k_n h_n)S_n(t_n - s_n - k_n h_n)f_n - S_n(k_n h_n)f_n\|_{\kappa, X_n} \\ &\quad + c_2 \sum_{i=0}^{k_n-1} \|S_n(ih_n)S_n(h_n)f_n - S_n(ih_n)f_n\|_{\kappa, X_n} \\ &\leq c_1 c_2^2 (t_n - s_n - k_n h_n) + c_2^2 k_n \|S_n(h_n)f_n - f_n\|_{\kappa, X_n} \\ &\leq c(t_n - s_n) \end{aligned}$$

for all  $n \in \mathbb{N}$  and  $c := 2c_1 c_2^2$ . Equation (2.8) yields

$$\|S(s)f - S(t)f\|_{\kappa} \leq c|s - t| \quad \text{for all } s, t \in [0, T] \cap \mathcal{T}. \quad (2.9)$$

Now, let  $t \in [0, T]$  be arbitrary and choose a sequence  $(\tilde{t}_n)_{n \in \mathbb{N}} \subset [0, T] \cap \mathcal{T}$  with  $\tilde{t}_n \rightarrow t$ . Inequality (2.9) guarantees that the limit

$$S(t)f := \lim_{n \rightarrow \infty} S(\tilde{t}_n)f \in C_{\kappa}(\mathbb{R}^d) \quad (2.10)$$

exists and does not depend on the choice of the approximating sequence  $(\tilde{t}_n)_{n \in \mathbb{N}}$ . In order to extend  $(S(t))_{t \geq 0}$  to the whole space  $C_{\kappa}(\mathbb{R}^d)$ , we fix  $t \geq 0$  and  $f \in C_{\kappa}(\mathbb{R}^d)$ . By Assumption 2.6(ii) and the previous construction, for every  $\varepsilon > 0$ ,  $r \geq 0$  and  $K \Subset \mathbb{R}^d$ , there exist  $c \geq 0$  and  $K' \Subset \mathbb{R}^d$  with

$$\|S(t)g_1 - S(t)g_2\|_{\infty, K} \leq c\|g_1 - g_2\|_{\infty, K'} + \varepsilon \quad (2.11)$$

for all  $g_1, g_2 \in B_{C_{\kappa}(\mathbb{R}^d)}(r) \cap \mathcal{D}$ . Let  $(f_n)_{n \in \mathbb{N}} \subset \mathcal{D}$  be a sequence with  $f_n \rightarrow f$ . Due to Assumption 2.6(i) and equation (2.10), the sequence  $(S(t)f_n)_{n \in \mathbb{N}}$  is bounded. Hence, by inequality (2.11), the limit

$$S(t)f := \lim_{n \rightarrow \infty} S(t)f_n \in C_{\kappa}(\mathbb{R}^d)$$

exists and does not depend on the choice of the approximating sequence  $(f_n)_{n \in \mathbb{N}}$ . The family  $(S(t))_{t \geq 0}$  consists of convex monotone operators  $S(t): C_{\kappa}(\mathbb{R}^d) \rightarrow C_{\kappa}(\mathbb{R}^d)$  which have the desired properties (ii) and (iii) such that  $S(t): \text{Lip}_b(\mathbb{R}^d) \rightarrow \text{Lip}_b(\mathbb{R}^d)$  for all  $t \geq 0$ . Indeed, these properties are an immediate consequence Assumption 2.6 and the previous construction.

Second, we show that the family  $(S(t))_{t \geq 0}$  is strongly continuous and satisfies

$$S(t)f = \lim_{l \rightarrow \infty} S_{n_l}(t_{n_l})f \quad \text{for all } (f, t) \in C_{\kappa}(\mathbb{R}^d) \times \mathcal{T}, \quad (2.12)$$

where  $(t_{n_l})_{l \in \mathbb{N}}$  is the same sequence as in equation (2.8). To do so, let  $f \in C_{\kappa}(\mathbb{R}^d)$ ,  $t \geq 0$ ,  $\varepsilon > 0$  and  $K \Subset \mathbb{R}^d$ . Furthermore, let  $(f_n)_{n \in \mathbb{N}} \subset \mathcal{D}$  be a sequence with  $f_n \rightarrow f$  and  $r := \sup_{n \in \mathbb{N}} \|f_n\|_{\kappa}$ . Due to property (iii), there exist  $c \geq 0$  and  $K' \subset \mathbb{R}^d$  with

$$\|S(s)g_1 - S(s)g_2\|_{\infty, K} \leq c\|g_1 - g_2\|_{\infty, K'} + \frac{\varepsilon}{6}$$

for all  $s \in [0, t+1]$  and  $g_1, g_2 \in B_{C_{\kappa}(\mathbb{R}^d)}(r)$ . Subsequently, we fix  $n \in \mathbb{N}$  with

$$6c\|f - f_n\|_{\infty, K'} \leq \varepsilon.$$

Due to inequality (2.9) and  $\inf_{x \in K} \kappa(x) > 0$ , there exists  $\delta \in (0, 1]$  with

$$\begin{aligned} \|S(s)f - S(t)f\|_{\infty, K} &\leq \|S(s)f - S(s)f_n\|_{\infty, K} + \|S(s)f_n - S(t)f_n\|_{\infty, K} \\ &\quad + \|S(t)f_n - S(t)f\|_{\infty, K} \\ &\leq 2c\|f - f_n\|_{\infty, K'} + \|S(s)f_n - S(t)f_n\|_{\infty, K} < \varepsilon \end{aligned}$$

for all  $s \geq 0$  with  $|s - t| < \delta$  which shows that the mapping  $\mathbb{R}_+ \rightarrow C_\kappa(\mathbb{R}^d)$ ,  $t \mapsto S(t)f$  is continuous. In order to verify equation (2.12), we fix  $f \in C_\kappa(\mathbb{R}^d)$ ,  $t \in \mathcal{T}$ ,  $\varepsilon > 0$  and  $K \Subset \mathbb{R}^d$ . Let  $(f_n)_{n \in \mathbb{N}} \subset \mathcal{D}$  be a sequence with  $f_n \rightarrow f$  and define  $r := \sup_{n \in \mathbb{N}} \|f_n\|_\kappa$ . Due to Assumption 2.6(ii) and property (iii), there exist  $c \geq 0$  and  $K' \Subset \mathbb{R}^d$  with

$$\begin{aligned} \|S_n(t_n)g_1 - S_n(t_n)g_2\|_{\infty, K_n} &\leq c\|g_1 - g_2\|_{\infty, K'} + \frac{\varepsilon}{6}, \\ \|S(t)g_1 - S(t)g_2\|_{\infty, K} &\leq c\|g_1 - g_2\|_{\infty, K'} + \frac{\varepsilon}{6} \end{aligned} \quad (2.13)$$

for all  $n \in \mathbb{N}$  and  $g_1, g_2 \in B_{C_\kappa(\mathbb{R}^d)}(r)$ . Subsequently, we fix  $k \in \mathbb{N}$  with

$$6c\|f - f_k\|_{\infty, K'} \leq \varepsilon.$$

Equation (2.8) yields  $l_0 \in \mathbb{N}$  with

$$\|S(t)f_k - S_{n_l}(t_{n_l})f_k\|_{\infty, K_{n_l}} \leq \frac{\varepsilon}{3} \quad \text{for all } l \geq l_0. \quad (2.14)$$

We combine inequality (2.13) and inequality (2.14) to obtain

$$\begin{aligned} \|S(t)f - S_{n_l}(t_{n_l})f\|_{\infty, K_{n_l}} &\leq \|S(t)f - S(t)f_k\|_{\infty, K} + \|S(t)f_k - S_{n_l}(t_{n_l})f_k\|_{\infty, K_{n_l}} \\ &\quad + \|S_{n_l}(t_{n_l})f_k - S_{n_l}(t_{n_l})f\|_{\infty, K_{n_l}} \\ &\leq 2c\|f - f_k\|_{\infty, K'} + \|S(t)f_k - S_{n_l}(t_{n_l})f_k\|_{\infty, K_{n_l}} + \frac{\varepsilon}{3} < \varepsilon \end{aligned}$$

for all  $l \geq l_0$  and therefore  $S(t)f = \lim_{l \rightarrow \infty} S_{n_l}(t_{n_l})f$ .

Third, for every  $t \geq 0$  and  $f \in C_\kappa(\mathbb{R}^d)$ , we show that

$$S(t)f = \lim_{l \rightarrow \infty} S_{n_l}(t_{n_l})f_{n_l}, \quad (2.15)$$

where  $(t_n)_{n \in \mathbb{N}}$  and  $(f_n)_{n \in \mathbb{N}}$  are arbitrary sequences with  $t_n \in \mathcal{T}_n$  and  $f_n \in C_\kappa(X_n)$  for all  $n \in \mathbb{N}$ ,  $t_n \rightarrow t$  and  $f_n \rightarrow f$ . We define  $T := \sup_{n \in \mathbb{N}} t_n$  and  $r := \sup_{n \in \mathbb{N}} c_\kappa \|f_n\|_{\kappa, X_n}$ , where  $c_\kappa$  is defined by equation (2.1). Let  $\varepsilon > 0$  and  $K \Subset \mathbb{R}^d$ . Due to Assumption 2.6(ii), there exist  $c \geq 0$  and  $K' \Subset \mathbb{R}^d$  with

$$\begin{aligned} \|S_n(s)g_1 - S_n(s)g_2\|_{\infty, K_n} &\leq c\|g_1 - g_2\|_{\infty, K'_n} + \frac{\varepsilon}{7}, \\ \|S(t)g_1 - S(t)g_2\|_{\infty, K} &\leq c\|g_1 - g_2\|_{\infty, K'} + \frac{\varepsilon}{7} \end{aligned} \quad (2.16)$$

for all  $n \in \mathbb{N}$ ,  $s \in [0, T] \cap \mathcal{T}_n$  and  $g_1, g_2 \in B_{C_\kappa(X_n)}(r)$ . Moreover, there exist  $n_0 \in \mathbb{N}$  and  $g \in B_{C_\kappa(\mathbb{R}^d)}(r) \cap C_b^\infty(\mathbb{R}^d)$  with

$$\max\{\|f - g\|_{\infty, K'}, \|f_n - g\|_{\infty, K'_n}\} \leq \frac{\varepsilon}{7c} \quad \text{for all } n \geq n_0. \quad (2.17)$$

Indeed, let  $\eta: \mathbb{R}^d \rightarrow \mathbb{R}$  be an infinitely differentiable function with  $\eta \geq 0$ ,  $\text{supp}(\eta) \subset B_{\mathbb{R}^d}(1)$  and  $\int_{\mathbb{R}^d} \eta(x) dx = 1$ . For every  $n \in \mathbb{N}$  and  $x \in \mathbb{R}^d$ , we define

$$g_n(x) := \int_{\mathbb{R}^d} ((f(x - y) \wedge n) \vee (-n)) \eta_n(y) dy,$$

where  $\eta_n(y) := n^d \eta(ny)$ . It holds  $g_n \in C_b^\infty(\mathbb{R}^d)$ ,  $\|f - g_n\|_{\infty, K'} \rightarrow 0$  and equation (2.1) implies  $\|g_n\|_\kappa \leq c_\kappa \|f\|_\kappa \leq r$  for all  $n \in \mathbb{N}$ . In combination with  $\|f - f_n\|_{\infty, K'_n} \rightarrow 0$  this proves equation (2.17). It follows from the proof of equation (2.9) and  $\inf_{x \in K'} \kappa(x) > 0$  that there exists  $\delta > 0$  with

$$\|S_n(s_1)g - S_n(s_2)g\|_{\infty, K_n} \leq \frac{\varepsilon}{7} \quad (2.18)$$

for all  $n \in \mathbb{N}$  and  $s_1, s_2 \in [0, T] \cap \mathcal{T}_n$  with  $|s_1 - s_2| < \delta$ . Furthermore, since  $(S(t))_{t \geq 0}$  is strongly continuous, we can fix  $s \in [0, T] \cap \mathcal{T}$  with  $|s - t| < \delta$  and

$$\|S(s)g - S(t)g\|_{\infty, K} \leq \frac{\varepsilon}{7}. \quad (2.19)$$

By equation (2.12), there exist a sequence  $(s_n)_{n \in \mathbb{N}}$  with  $s_n \in [0, T] \cap \mathcal{T}_n$  and  $l_0 \in \mathbb{N}$  with  $n_{l_0} \geq n_0$ ,  $|s - t_{n_l}| < \delta$  and

$$\|S_{n_l}(s_{n_l})g - S(s)g\|_{\infty, K_{n_l}} \leq \frac{\varepsilon}{7} \quad \text{for all } l \geq l_0. \quad (2.20)$$

We combine the inequalities (2.16)-(2.20) to obtain

$$\begin{aligned} & \|S_{n_l}(t_{n_l})f_{n_l} - S(t)f\|_{\infty, K_{n_l}} \\ & \leq \|S_{n_l}(t_{n_l})f_{n_l} - S_{n_l}(t_{n_l})g\|_{\infty, K_{n_l}} + \|S_{n_l}(t_{n_l})g - S_{n_l}(s_{n_l})g\|_{\infty, K_{n_l}} \\ & \quad + \|S_{n_l}(s_{n_l})g - S(s)g\|_{\infty, K_{n_l}} + \|S(s)g - S(t)g\|_{\infty, K} + \|S(t)g - S(t)f\|_{\infty, K} \\ & \leq c\|f_{n_l} - g\|_{\infty, K'_{n_l}} + c\|f - g\|_{\infty, K} + \frac{5\varepsilon}{7} \leq \varepsilon \end{aligned}$$

for all  $l \geq l_0$ . This implies  $S_{n_l}(t_{n_l})f_{n_l} \rightarrow S(t)f$ .

Fourth, we show that  $(S(t))_{t \geq 0}$  is a semigroup. Clearly, it holds  $S(0) = \text{id}_{C_\kappa(\mathbb{R}^d)}$ . In order to show the semigroup property, let  $s, t \geq 0$  and  $f \in C_\kappa(\mathbb{R}^d)$ . Choose  $s_n, t_n \in \mathcal{T}_n$  with  $s_n \rightarrow s$  and  $t_n \rightarrow t$ . For every  $l \in \mathbb{N}$ ,

$$\begin{aligned} & S(s+t)f - S(s)S(t)f \\ & = (S(s+t)f - S_{n_l}(s_{n_l} + t_{n_l})f) + (S_{n_l}(s_{n_l})S_{n_l}(t_{n_l})f - S_{n_l}(s_{n_l})S(t)f) \\ & \quad + (S_{n_l}(s_{n_l})S(t)f - S(s)S(t)f). \end{aligned}$$

Equation (2.15) implies that the first and third term on the right-hand side convergence to zero as  $l \rightarrow \infty$ . Moreover, for every  $\varepsilon > 0$  and  $K \Subset \mathbb{R}^d$ , it follows from Assumption 2.6(i) and (ii) that there exist  $c \geq 0$  and  $K' \Subset \mathbb{R}^d$  with

$$\|S_{n_l}(s_{n_l})S_{n_l}(t_{n_l})f - S_{n_l}(s_{n_l})S(t)f\|_{\infty, K_{n_l}} \leq c\|S_{n_l}(t_{n_l})f - S(t)f\|_{\infty, K'_{n_l}} + \varepsilon$$

for all  $l \in \mathbb{N}$ . Hence, we can use equation (2.15) to obtain

$$\lim_{l \rightarrow \infty} (S_{n_l}(s_{n_l})S_{n_l}(t_{n_l})f - S_{n_l}(s_{n_l})S(t)f) = 0$$

and therefore  $S(s+t)f = S(s)S(t)f$ . In particular, for every  $t \geq 0$  and  $f \in \mathcal{L}^S$ , there exist  $c \geq 0$  and  $h_0 > 0$  with

$$\|S(h)S(t)f - S(t)f\|_\kappa \leq c\|S(h)f - f\|_\kappa \leq ch \quad \text{for all } h \in (0, h_0]$$

which shows that  $S(t): \mathcal{L}^S \rightarrow \mathcal{L}^S$  for all  $t \geq 0$ . Furthermore, the invariance of  $\mathcal{L}_+^S$  is guaranteed by Lemma B.2.

Fifth, we show that, for every  $f \in \text{Lip}_b(\mathbb{R}^d)$  and  $\varepsilon > 0$ , there exist  $\delta, t_0 > 0$  with

$$\|S(t)(\tau_x f) - \tau_x S(t)f\|_\kappa \leq \varepsilon t$$

for all  $t \in [0, t_0]$  and  $x \in B_{\mathbb{R}^d}(\delta)$ . Let  $f \in \text{Lip}_b(\mathbb{R}^d)$  and  $\varepsilon > 0$ . By Assumption 2.6(iii), there exist  $\delta, t_0 > 0$  and a sequence  $(f_n)_{n \in \mathbb{N}}$  with  $f_n \in C_\kappa(X_n)$ ,  $f_n \rightarrow f$  and

$$\|S_n(t)(\tau_x f_n) - \tau_x S_n(t)f_n\|_{\kappa, X_n} \leq \varepsilon t \quad (2.21)$$

for all  $n \in \mathbb{N}$ ,  $t \in [0, t_0] \cap \mathcal{T}_n$ ,  $x \in B_{X_n}(\delta)$  and  $y \in X_n$ . Let  $t \in [0, t_0]$ ,  $x \in B_{\mathbb{R}^d}(\delta)$  and  $y \in \mathbb{R}^d$ . Choose  $t_n \in [0, t_0] \cap \mathcal{T}_n$  and  $x_n, y_n \in X_n$  with  $t_n \rightarrow t$ ,  $x_n \rightarrow x$  and  $y_n \rightarrow y$ . For

every  $l \in \mathbb{N}$ ,

$$\begin{aligned} & |S(t)(\tau_x f) - \tau_x S(t)f|(y) \\ & \leq |(S(t)(\tau_x f))(y) - (S(t)(\tau_x f))(y_{n_l})| + |S(t)(\tau_x f) - S_{n_l}(t_{n_l})(\tau_{x_{n_l}} f)|(y_{n_l}) \\ & \quad + |S_{n_l}(t_{n_l})(\tau_{x_{n_l}} f) - \tau_{x_{n_l}} S_{n_l}(t_{n_l})f|(y_{n_l}) + |\tau_{x_{n_l}} S_{n_l}(t_{n_l})f - \tau_{x_{n_l}} S(t)f|(y_{n_l}) \\ & \quad + |(\tau_{x_{n_l}} S(t)f)(y_{n_l}) - (\tau_x S(t)f)(y)|. \end{aligned}$$

The first and the last term on the right-hand side converge to zero because  $S(t)(\tau_x f)$  and  $S(t)f$  are continuous functions while equation (2.15) guarantees that the second and fourth term vanish as well. Moreover, due to inequality (2.21), the third term can be bounded by  $\frac{\varepsilon t_{n_l}}{\kappa(y_{n_l})}$  which converges to  $\frac{\varepsilon t}{\kappa(y)}$ . We obtain

$$|(S(t)(\tau_x f))(y) - (\tau_x S(t)f)(y)|\kappa(y) \leq \varepsilon t.$$

Sixth, we show that  $\mathcal{L}_0 \subset \mathcal{L}^S$  and  $\mathcal{D}_0 \subset D(A)$ . In addition, for every  $f \in \mathcal{D}_0$  and  $K \Subset \mathbb{R}^d$ , we prove that

$$\lim_{n \rightarrow \infty} \left\| \frac{S_n(h_n)f_n - f_n}{h_n} - Af \right\|_{\infty, K_n} = 0, \quad (2.22)$$

where  $(h_n)_{n \in \mathbb{N}}$  and  $(f_n)_{n \in \mathbb{N}}$  are arbitrary sequences satisfying the conditions (i) and (ii) in Definition 2.5. Inequality (2.4), Assumption 2.6(i) and equation (2.15) guarantee that  $\mathcal{L}_0 \subset \mathcal{L}^S$ . In the sequel, we fix  $f \in \mathcal{D}_0$  and  $K \Subset \mathbb{R}^d$ . Let  $(f_n)_{n \in \mathbb{N}}$  and  $(t_n)_{n \in \mathbb{N}}$  be sequences as in the definition of the asymptotic domain. Furthermore, we also fix  $\varepsilon > 0$  and define  $T := \sup_{n \in \mathbb{N}} t_n$ . Assumption 2.6(i) and the fact that  $\kappa > 0$  is continuous yield  $\lambda_0 \in (0, 1]$  with

$$\sup_{n \in \mathbb{N}} \sup_{t \in [0, T] \cap \mathcal{T}_n} \lambda_0 \|S_n(t)f_n\|_{\kappa, X_n} \leq \inf_{x \in K} \frac{\kappa(x)\varepsilon}{3}. \quad (2.23)$$

We fix  $\lambda \in (0, \lambda_0]$  with  $\lambda T \leq \lambda_0$  and use inequality (2.4) to define

$$r := \sup_{n \in \mathbb{N}} \sup_{h \in (0, t_n] \cap \mathcal{T}_n} \left\| \frac{S_n(h)f_n - f_n}{\lambda h} + f_n \right\|_{\kappa, X_n} < \infty. \quad (2.24)$$

Assumption 2.6(ii) guarantees the existence of  $c \geq 0$  and  $K' \Subset \mathbb{R}^d$  with

$$\|S_n(t)g_1 - S_n(t)g_2\|_{\infty, K_n} \leq c\|g_1 - g_2\|_{\infty, K'_n} + \frac{\varepsilon}{3} \quad (2.25)$$

for all  $n \in \mathbb{N}$ ,  $t \in [0, T] \cap \mathcal{T}_n$  and  $g_1, g_2 \in B_{C_\kappa(X_n)}(r)$ . Choose  $g \in C_\kappa(\mathbb{R}^d)$  as in the definition of the asymptotic domain and a sequence  $(h_n)_{n \in \mathbb{N}}$  with  $h_n \in (0, t_n] \cap \mathcal{T}_n$  and  $h_n \rightarrow 0$  satisfying equation (2.5) with  $K'$ . In particular, there exists  $n_0 \in \mathbb{N}$  with

$$\left\| \frac{S_n(h_n)f_n - f_n}{h_n} - g \right\|_{\infty, K'_n} \leq \frac{\varepsilon}{3c} \quad \text{for all } n \geq n_0. \quad (2.26)$$

In the sequel, all functions are evaluated at a fixed point  $x \in K$ . For every  $n \geq n_0$  and  $k \in \mathbb{N}$  with  $kh_n \leq T$ , we use the semigroup property of  $(S_n(t))_{t \in \mathcal{T}_n}$ , Lemma B.1 and



inequality (2.23) to estimate

$$\begin{aligned}
S_n(kh_n)f_n - f_n &= \sum_{i=1}^k (S_n((i-1)h_n)S_n(h_n)f_n - S_n((i-1)h_n)f_n) \\
&\leq \lambda h_n \sum_{i=1}^k \left( S_n((i-1)h_n) \left( \frac{S_n(h_n)f_n - f_n}{\lambda h_n} + f_n \right) - S_n((i-1)h_n)f_n \right) \\
&\leq \lambda h_n \sum_{i=1}^k S_n((i-1)h_n) \left( \frac{S_n(h_n)f_n - f_n}{\lambda h_n} + f_n \right) + \frac{kh_n\varepsilon}{3}.
\end{aligned}$$

Furthermore, it follows from equation (2.24) and inequality (2.25) that

$$\begin{aligned}
&\lambda h_n \sum_{i=1}^k S_n((i-1)h_n) \left( \frac{S_n(h_n)f_n - f_n}{\lambda h_n} + f_n \right) + \frac{kh_n\varepsilon}{3} \\
&= \lambda h_n \sum_{i=1}^k S_n((i-1)h_n) \left( \frac{1}{\lambda}g + f_n \right) + \frac{kh_n\varepsilon}{3} \\
&\quad + \lambda h_n \sum_{i=1}^k \left( S_n((i-1)h_n) \left( \frac{S_n(h_n)f_n - f_n}{\lambda h_n} + f_n \right) - S_n((i-1)h_n) \left( \frac{1}{\lambda}g + f_n \right) \right) \\
&\leq \lambda h_n \sum_{i=1}^k S_n((i-1)h_n) \left( \frac{1}{\lambda}g + f_n \right) + ck h_n \left\| \frac{S_n(h_n)f_n - f_n}{h_n} - g \right\|_{\infty, K'_n} + \frac{2kh_n\varepsilon}{3}.
\end{aligned}$$

Combining the previous estimates with inequality (2.26) yields

$$S_n(kh_n)f_n - f_n \leq \lambda h_n \sum_{i=1}^k S_n((i-1)h_n) \left( \frac{1}{\lambda}g + f_n \right) + kh_n\varepsilon. \quad (2.27)$$

Fix  $t \in [0, T]$  and define  $k_n := \max\{k \in \mathbb{N}_0 : kh_n \leq t\}$  for all  $n \in \mathbb{N}$ . In addition, we define  $i_n^s := \max\{i \in \mathbb{N} : (i-1)h_n \leq s\}$  for all  $s \in [0, t]$  and  $n \in \mathbb{N}$ . Since  $k_n h_n \rightarrow t$  and  $i_n^s \rightarrow s$  for all  $s \in [0, t]$ , we can use equation (2.15), inequality (2.27), Assumption 2.6(i) and the dominated convergence theorem to conclude

$$\begin{aligned}
S(t)f - f &= \lim_{l \rightarrow \infty} (S_{n_l}(k_{n_l}h_{n_l})f_{n_l} - f_{n_l}) \\
&\leq \lim_{l \rightarrow \infty} \lambda \int_0^t \sum_{i=1}^{k_{n_l}} S_{n_l}((i-1)h_{n_l}) \left( \frac{1}{\lambda}g + f_{n_l} \right) \mathbb{1}_{[(i-1)h_{n_l}, ih_{n_l})}(s) \, ds + \varepsilon t \\
&= \lambda \int_0^t \lim_{l \rightarrow \infty} S_{n_l}((i_{n_l}^s - 1)h_{n_l}) \left( \frac{1}{\lambda}g + f_{n_l} \right) \, ds + \varepsilon t \\
&= \lambda \int_0^t S(s) \left( \frac{1}{\lambda}g + f \right) \, ds + \varepsilon t.
\end{aligned} \quad (2.28)$$

The previous estimate holds uniformly for all  $\lambda \in (0, \lambda_0]$  with  $\lambda T \leq \lambda_0$  and  $t \in [0, T]$ , where  $\lambda_0$  depends on  $\varepsilon > 0$  and  $K \Subset \mathbb{R}^d$ . Concerning the lower bound, we observe that

the semigroup property and Lemma B.1 imply

$$\begin{aligned}
& S_n(kh_n)f_n - f_n \\
&= - \sum_{i=1}^k (S_n((i-1)h_n)f_n - S_n((i-1)h_n)S_n(h_n)f_n) \\
&\geq -\lambda h_n \sum_{i=1}^k \left( S_n((i-1)h_n) \left( S_n(h_n)f_n - \frac{S_n(h_n)f_n - f_n}{\lambda h_n} \right) - S_n(ih_n)f_n \right)
\end{aligned}$$

for all  $k, n \in \mathbb{N}$  and  $\lambda > 0$  with  $\lambda h_n \leq 1$ . Hence, similar to the previous arguments, one can show that there exists  $\lambda_1 \in (0, 1]$  with

$$S(t)f - f \geq -\lambda \int_0^t S(s) \left( -\frac{1}{\lambda}g + f \right) ds - \varepsilon t \quad (2.29)$$

for all  $\lambda \in (0, \lambda_1]$  with  $\lambda T \leq \lambda_1$  and  $t \in [0, T]$ , where  $\lambda_1$  also depends on  $\varepsilon > 0$  and  $K \Subset \mathbb{R}^d$ . Since inequality (2.28) and inequality (2.29) hold uniformly for all  $x \in K$ , it follows from the strong continuity of  $(S(t))_{t \geq 0}$  that

$$\lim_{h \downarrow 0} \left\| \frac{S(h)f - f}{h} - g \right\|_{\infty, K} = 0.$$

Note that the function  $g \in C_\kappa(\mathbb{R}^d)$  does neither depend on the choice of  $K' \Subset \mathbb{R}^d$  nor on  $\varepsilon > 0$ . Since  $K \Subset \mathbb{R}^d$  was arbitrary, we obtain  $f \in D(A)$  with  $Af = g$ .

Sixth, we verify equation (2.6) by showing that the semigroup  $(S(t))_{t \geq 0}$  does not depend on the choice of the convergent subsequence in equation (2.8). For every subsequence  $(\tilde{n}_k)_{k \in \mathbb{N}} \subset \mathbb{N}$ , there exist a further subsequence  $(\tilde{n}_{k_l})_{l \in \mathbb{N}}$  and a strongly continuous convex monotone semigroup  $(\tilde{S}(t))_{t \geq 0}$  which has the properties (i)-(v) and satisfies

$$\tilde{S}(t)f = \lim_{l \rightarrow \infty} S_{\tilde{n}_{k_l}}(t_{\tilde{n}_{k_l}})f_{\tilde{n}_{k_l}} \quad \text{for all } (f, t) \in C_\kappa(\mathbb{R}^d) \times \mathbb{R}_+,$$

where  $(t_n)_{n \in \mathbb{N}}$  and  $(f_n)_{n \in \mathbb{N}}$  are arbitrary sequences with  $t_n \in \mathcal{T}_n$  and  $f_n \in C_\kappa(X_n)$  for all  $n \in \mathbb{N}$ ,  $t_n \rightarrow t$  and  $f_n \rightarrow f$ . Hence, we can apply Theorem 2.4 to obtain

$$S(t)f = \tilde{S}(t)f \quad \text{for all } (f, t) \in C_\kappa(\mathbb{R}^d) \times \mathbb{R}_+.$$

Since every subsequence has a further subsequence which converges to a limit that is independent of the choice of the subsequence, we conclude

$$S(t)f = \lim_{n \rightarrow \infty} S_n(t_n)f_n \quad \text{for all } (f, t) \in C_\kappa(\mathbb{R}^d) \times \mathbb{R}_+,$$

where  $(t_n)_{n \in \mathbb{N}}$  and  $(f_n)_{n \in \mathbb{N}}$  are arbitrary sequences with  $t_n \in \mathcal{T}_n$  and  $f_n \in C_\kappa(X_n)$  for all  $n \in \mathbb{N}$ ,  $t_n \rightarrow t$  and  $f_n \rightarrow f$ .  $\square$

Theorem 2.3 follows immediately from Theorem 2.7 if we verify that  $C_b^\infty(\mathbb{R}^d) \subset \mathcal{D}_0$ .

*Proof of Theorem 2.3.* Let  $f \in C_\kappa(\mathbb{R}^d)$  and  $(f_n)_{n \in \mathbb{N}}$  be a sequence with  $f_n \in D(A_n)$  for all  $n \in \mathbb{N}$  and  $f_n \rightarrow f$  such that the sequence  $(A_n f_n)_{n \in \mathbb{N}}$  converges in  $C_\kappa(\mathbb{R}^d)$ . For every  $n \in \mathbb{N}$  and  $t \geq 0$ , it follows from [11, Lemma 2.3] that

$$S_n(t)f_n - f_n \leq \int_0^t (S_n(s)(f_n + A_n f_n) - S_n(s)f_n) ds. \quad (2.30)$$

In the sequel, we argue similarly to obtain a lower bound. Due to  $f_n \in D(A_n) \subset \mathcal{L}^{S_n}$  and Assumption 2.2(i), the mapping  $[0, \infty) \rightarrow \mathbb{R}$ ,  $t \mapsto (S_n(t)f_n)(x)$  is Lipschitz continuous

and therefore satisfies

$$(S_n(t)f_n - f_n)(x) = \int_0^t \frac{d}{ds}(S_n(s)f_n)(x) ds \quad \text{for all } t \geq 0 \text{ and } x \in \mathbb{R}^d.$$

Moreover, for every  $n \in \mathbb{N}$ ,  $t \geq 0$  and  $h \in (0, 1]$ , we can use Lemma B.1 to estimate

$$\begin{aligned} & \frac{S_n(t)f_n - S_n(t)S_n(h)f_n}{h} - S_n(t)(f_n - A_nf_n) - S_n(t)f_n \\ & \leq S_n(t) \left( \frac{f_n - S_n(h)f_n}{h} + S_n(h)f_n \right) - S_n(t)S_n(h)f_n + S_n(t)f_n - S_n(t)(f_n - A_nf_n) \\ & \leq \frac{1}{2}S_n(t) \left( 2 \left( S_n(h)f_n - f_n + A_nf_n - \frac{S_n(h)f_n - f_n}{h} \right) + f_n - A_nf_n \right) \\ & \quad - \frac{1}{2}S_n(t)(f_n - A_nf_n) - S_n(t)S_n(h)f_n + S_n(t)f_n. \end{aligned}$$

The strong continuity of  $(S_n(t))_{t \geq 0}$ , Assumption 2.2(ii) and  $f_n \in D(A_n)$  imply that the right-hand side converges to zero as  $h \rightarrow 0$ . Hence,

$$S_n(t)f_n - f_n \geq - \int_0^t (S_n(s)(f_n - A_nf_n) - S_n(s)f_n) ds \quad (2.31)$$

for all  $n \in \mathbb{N}$  and  $t \geq 0$ . Since  $(A_nf_n)_{n \in \mathbb{N}}$  is bounded, it follows from inequality (2.30), inequality (2.31) and Assumption 2.2(i) that there exists  $c \geq 0$  with

$$\|S_n(t)f_n - f_n\|_\kappa \leq ct \quad \text{for all } n \in \mathbb{N} \text{ and } t \in [0, 1].$$

This shows that  $f \in \mathcal{L}_0$ . Furthermore, the limit  $g := \lim_{n \rightarrow \infty} A_nf_n \in C_\kappa(\mathbb{R}^d)$  exists by assumption. Hence, for every  $n \in \mathbb{N}$  and  $K \Subset \mathbb{R}^d$ , we can choose  $h_n \in (0, 1]$  with

$$\begin{aligned} \left\| \frac{S_n(h_n)f_n - f_n}{h_n} - g \right\|_{\infty, K} & \leq \left\| \frac{S_n(h_n)f_n - f_n}{h_n} - A_nf_n \right\|_{\infty, K} + \|A_nf_n - g\|_{\infty, K} \\ & \leq \|A_nf_n - g\|_{\infty, K} + \frac{1}{n} \rightarrow 0. \end{aligned}$$

This shows that  $f \in \mathcal{D}_0$ . In particular, Assumption 2.2(v) implies  $C_b^\infty(\mathbb{R}^d) \subset \mathcal{D}_0$ . Now, the claim follows from Theorem 2.7.  $\square$

**2.4. Chernoff-type approximations.** Subsequently, we focus on the case, where  $\mathcal{T}_n$  is an equidistant grid. For this purpose, let  $(h_n)_{n \in \mathbb{N}} \subset (0, \infty)$  be a sequence with  $h_n \rightarrow 0$  and define  $\mathcal{T}_n := \{kh_n : k \in \mathbb{N}_0\}$  for all  $n \in \mathbb{N}$ . Furthermore, let  $(X_n)_{n \in \mathbb{N}}$  be a sequence of closed sets  $X_n \subset \mathbb{R}^d$  with  $x + y \in X_n$  for all  $x, y \in X_n$  such that, for every  $x \in \mathbb{R}^d$ , there exist  $x_n \in X_n$  with  $x_n \rightarrow x$ . Then, every semigroup  $(S(t))_{t \in \mathcal{T}_n}$  is of the form

$$S_n(kh_n) = I_n^k f = \underbrace{(I_n \circ \dots \circ I_n)}_{k \text{ times}} f \quad \text{with} \quad I_n f := S(h_n)f,$$

where  $I_n^0 f := f$ . Hence, the corresponding limit semigroup  $(S(t))_{t \geq 0}$  from Theorem 2.7 is obtained by iterating a sequence  $(I_n)_{n \in \mathbb{N}}$  of one-step operators. In many applications, the latter have an explicit representation from which one can derive properties of the one-step operators that are transferred to the limit semigroup. In the case  $X_n = \mathbb{R}^d$ , this procedure has been used to construct and approximate nonlinear semigroups, see, e.g., [10–12] and the references therein. Now, due to the possibility of discretizing the space via a sequence of subsets  $X_n \subset \mathbb{R}^d$ , we can show that, under reasonable stability and consistency conditions, every approximation scheme consisting of convex monotone

operators converges to a convex monotone semigroup. Moreover, the latter is uniquely determined by its infinitesimal generator which satisfies

$$Af = \lim_{n \rightarrow \infty} \frac{I_n f - f}{h_n} \quad \text{for all } f \in C_b^\infty(\mathbb{R}^d).$$

In the sequel, we consider a sequence  $(I_n)_{n \in \mathbb{N}}$  of operators  $I_n: C_\kappa(X_n) \rightarrow C_\kappa(X_n)$  and impose the following assumption. Note that, apart from condition (ii), all the required properties are preserved during the iteration. Moreover, in Subsection 2.5, we discuss sufficient conditions on the one-step operators  $I_n$  which guarantee that the iterated operators  $I_n^k$  are uniformly equicontinuous on bounded sets w.r.t. the mixed topology.

**Assumption 2.8.** The operators  $(I_n)_{n \in \mathbb{N}}$  are convex and monotone with  $I_n 0 = 0$  for all  $n \in \mathbb{N}$ . Furthermore, they satisfy the following conditions:

(i) There exists  $\omega \geq 0$  with

$$\|I_n f - I_n g\|_{\kappa, X_n} \leq e^{\omega h_n} \|f - g\|_{\kappa, X_n} \quad \text{for all } n \in \mathbb{N} \text{ and } f, g \in C_\kappa(X_n).$$

(ii) For every  $\varepsilon > 0$ ,  $r, T \geq 0$  and  $K \Subset \mathbb{R}^d$ , there exist  $c \geq 0$  and  $K' \Subset \mathbb{R}^d$  with

$$\|I_n^k f - I_n^k g\|_{\infty, K_n} \leq c \|f - g\|_{\infty, K'_n} + \varepsilon$$

for all  $k, n \in \mathbb{N}$  with  $kh_n \leq T$  and  $f, g \in B_{C_\kappa(X_n)}(r)$ .

(iii) There exist  $\delta_0 \in (0, 1]$  and  $L \geq 0$  with

$$\|I_n(\tau_x f) - \tau_x I_n f\|_{\kappa, X_n} \leq L r h_n |x|$$

for all  $n \in \mathbb{N}$ ,  $x \in B_{X_n}(\delta_0)$ ,  $r \geq 0$  and  $f \in \text{Lip}_b(X_n, r)$ .

(iv) It holds  $I_n: \text{Lip}_b(X_n, r) \rightarrow \text{Lip}_b(X_n, e^{\omega h_n} r)$  for all  $r \geq 0$  and  $n \in \mathbb{N}$ .

(v) For every  $f \in C_b^\infty(\mathbb{R}^d)$ , there exists  $g \in C_\kappa(\mathbb{R}^d)$  with  $g = \lim_{n \rightarrow \infty} \frac{I_n f - f}{h_n}$ .

We recall from Subsection 2.3 that the convergence in condition (v) means that

$$\sup_{n \in \mathbb{N}} \left\| \frac{I_n f - f}{h_n} \right\|_{\kappa, X_n} < \infty \quad \text{and} \quad \lim_{n \rightarrow \infty} \left\| \frac{I_n f - f}{h_n} - g \right\|_{\infty, K_n} = 0 \quad \text{for all } K \Subset \mathbb{R}^d.$$

We obtain the following result that will be frequently used in the rest of this article.

**Theorem 2.9.** *Let  $(I_n)_{n \in \mathbb{N}}$  be a sequence satisfying Assumption 2.8. Then, there exists a strongly continuous convex monotone semigroup  $(S(t))_{t \geq 0}$  on  $C_\kappa(\mathbb{R}^d)$  with  $S(t)0 = 0$  such that*

$$S(t)f = \lim_{n \rightarrow \infty} I_n^{k_n^t} f_n \quad \text{for all } (f, t) \in C_\kappa(\mathbb{R}^d) \times \mathbb{R}_+,$$

where  $(k_n^t)_{n \in \mathbb{N}} \subset \mathbb{N}$  and  $(f_n)_{n \in \mathbb{N}}$  with  $f_n \in C_\kappa(X_n)$  are arbitrary sequences satisfying  $k_n^t h_n \rightarrow t$  and  $f_n \rightarrow f$ . Moreover, the following statements are valid:

(i) It holds  $C_b^\infty(\mathbb{R}^d) \subset D(A)$  and

$$Af = \lim_{n \rightarrow \infty} \frac{I_n f - f}{h_n} \quad \text{for all } f \in C_b^\infty(\mathbb{R}^d).^1$$

(ii) It holds  $\|S(t)f - S(t)g\|_\kappa \leq e^{\omega t} \|f - g\|_\kappa$  for all  $t \geq 0$  and  $f, g \in C_\kappa(\mathbb{R}^d)$ .

(iii) For every  $\varepsilon > 0$ ,  $r, T \geq 0$  and  $K \Subset \mathbb{R}^d$ , there exist  $c \geq 0$  and  $K' \Subset \mathbb{R}^d$  with

$$\|S(t)f - S(t)g\|_{\infty, K} \leq c \|f - g\|_{\infty, K'} + \varepsilon$$

for all  $t \in [0, T]$  and  $f, g \in B_{C_\kappa(\mathbb{R}^d)}(r)$ .

<sup>1</sup>The statement is valid for all  $f \in C_\kappa(\mathbb{R}^d)$  such that the limit  $g := \lim_{n \rightarrow \infty} \frac{I_n f - f}{h_n} \in C_\kappa$  exists.

(iv) For every  $r, t \geq 0$ ,  $f \in \text{Lip}_b(r)$  and  $x \in B_{\mathbb{R}^d}(\delta_0)$ ,

$$\|S(t)(\tau_x f) - \tau_x S(t)f\|_\kappa \leq Lrt e^{\omega t} |x|.$$

Furthermore, it holds  $S(t): \text{Lip}_b(r) \rightarrow \text{Lip}_b(e^{\omega t} r)$  for all  $r, t \geq 0$ .

*Proof.* We define  $S_n(t) := I_n^k f$  for all  $n \in \mathbb{N}$ ,  $t := kh_n \in \mathcal{T}_n$  and  $f \in C_\kappa(X_n)$ . In order to apply Theorem 2.7, we have to verify Assumption 2.6. The sequence  $(S_n)_{n \in \mathbb{N}}$  consists of convex monotone semigroups on  $C_\kappa(X_n)$  with  $S_n(t)0 = 0$  and Assumption 2.6(ii) is valid due to Assumption 2.8(ii). For every  $k, n \in \mathbb{N}$  and  $f, g \in C_\kappa(X_n)$ , it follows by induction from Assumption 2.8(i) that

$$\|I_n^k f - I_n^k g\|_{\kappa, X_n} \leq e^{\omega kh_n} \|f - g\|_{\kappa, X_n} \quad (2.32)$$

showing that Assumption 2.6(i) is satisfied. Next, we show by induction that

$$\|I_n^k(\tau_x f) - \tau_x I_n^k f\|_{\kappa, X_n} \leq Lrk h_n e^{\omega kh_n} |x|$$

for all  $k, n \in \mathbb{N}$ ,  $x \in B_{X_n}(\delta_0)$ ,  $r \geq 0$  and  $f \in \text{Lip}_b(X_n, r)$ . For  $k = 1$ , the claim holds due to Assumption 2.8(iii). Moreover, for the induction step, we use inequality (2.32) and Assumption 2.8(iv) to obtain

$$\begin{aligned} & \|I_n^{k+1}(\tau_x f) - \tau_x I_n^{k+1} f\|_{\kappa, X_n} \\ & \leq \|I_n^k I_n(\tau_x f) - I_n^k(\tau_x I_n f)\|_{\kappa, X_n} + \|I_n^k(\tau_x I_n f) - \tau_x I_n^k I_n f\|_{\kappa, X_n} \\ & \leq e^{\omega kh_n} \|I_n(\tau_x f) - \tau_x I_n f\|_{\kappa, X_n} + L e^{\omega h_n} r k h_n e^{\omega kh_n} |x| \\ & \leq e^{\omega kh_n} L r h_n |x| + L r k h_n e^{\omega(k+1)k_n} |x| \leq L r (k+1) h_n e^{\omega(k+1)h_n} |x|. \end{aligned}$$

Hence, Assumption 2.6(iii) is satisfied and Assumption 2.6(iv) and (v) are an immediate consequence of Assumption 2.8(iv) and (v). Now, Theorem 2.7 yields the claim.  $\square$

**2.5. Discussion of Assumption 2.6.** In the sequel, we discuss sufficient conditions that guarantee the required uniform equicontinuity on bounded sets w.r.t. the mixed topology. For families of convex monotone operators, this condition can be equivalently characterized by continuity from above, see Appendix C. Throughout this section, let  $(\mathcal{T}_n)_{n \in \mathbb{N}}$  and  $(X_n)_{n \in \mathbb{N}}$  be sequences of subsets  $\mathcal{T}_n \subset \mathbb{R}_+$  and  $X_n \subset \mathbb{R}^d$ , respectively, such that  $X_n$  is closed for all  $n \in \mathbb{N}$ . Moreover, let  $\{S_n(t): n \in \mathbb{N}, t \in \mathcal{T}_n\}$  be a family of convex monotone operators  $S_n(t): C_\kappa(X_n) \rightarrow F_\kappa(X_n)$  with  $S_n(t)0 = 0$  and

$$\sup_{n \in \mathbb{N}} \sup_{t \in [0, T] \cap \mathcal{T}_n} \sup_{f \in B_{C_\kappa(X_n)}(r)} \|S_n(t)f\|_{\kappa, X_n} < \infty \quad \text{for all } r, T \geq 0. \quad (2.33)$$

Here, the space  $F_\kappa(X_n)$  consists of all functions  $f: \mathbb{R}^d \rightarrow \mathbb{R}$  with  $\|f\|_\kappa < \infty$ .

**Lemma 2.10.** *The following two statements are equivalent:*

(i) For every  $T \geq 0$ ,  $K \Subset \mathbb{R}^d$  and  $(f_k)_{k \in \mathbb{N}} \subset C_\kappa(\mathbb{R}^d)$  with  $f_k \downarrow 0$ ,

$$\sup_{n \in \mathbb{N}} \sup_{t \in [0, T] \cap \mathcal{T}_n} \sup_{x \in K_n} (S_n(t)f_k)(x) \downarrow 0 \quad \text{as } k \rightarrow \infty.$$

(ii) For every  $\varepsilon > 0$ ,  $r, T \geq 0$  and  $K \Subset \mathbb{R}^d$ , there exist  $c \geq 0$  and  $K' \Subset \mathbb{R}^d$  with

$$\|S_n(t)f - S_n(t)g\|_{\infty, K_n} \leq c \|f - g\|_{\infty, K'_n} + \varepsilon$$

for all  $n \in \mathbb{N}$ ,  $t \in [0, T] \cap \mathcal{T}_n$  and  $f, g \in B_{C_\kappa(X_n)}(r)$ .

*Proof.* Apply Corollary C.2 with  $I := \mathbb{N} \times [0, T]$ ,  $X_{(n,t)} := X_n$  and  $\Phi_{(n,t)} := S_n(t)$ .  $\square$

The next lemma is, despite its simple nature, very useful in applications to Markovian transition semigroups under model uncertainty, since it typically reduces to bounding certain moments.

**Lemma 2.11.** *Let  $\tilde{\kappa}: \mathbb{R}^d \rightarrow (0, \infty)$  be another bounded continuous function such that, for every  $\varepsilon > 0$ , there exists  $K \Subset X$  with*

$$\sup_{x \in K^c} \frac{\tilde{\kappa}(x)}{\kappa(x)} \leq \varepsilon.$$

*Moreover, for every  $r, T \geq 0$ , there exists  $c \geq 0$  with*

$$\|S_n(t)f\|_{\tilde{\kappa}, X_n} \leq c\|f\|_{\tilde{\kappa}, X_n}$$

*for all  $n \in \mathbb{N}$ ,  $t \in [0, T] \cap \mathcal{T}_n$  and  $f \in B_{C_\kappa(X_n)}(r)$ . Then, for every  $\varepsilon > 0$ ,  $r, T \geq 0$  and  $K \Subset \mathbb{R}^d$ , there exist  $c \geq 0$  and  $K' \Subset \mathbb{R}^d$  with*

$$\|S_n(t)f - S_n(t)g\|_{\infty, K_n} \leq c\|f - g\|_{\infty, K'_n} + \varepsilon$$

*for all  $n \in \mathbb{N}$ ,  $t \in [0, T] \cap \mathcal{T}_n$  and  $f, g \in B_{C_\kappa(X_n)}(r)$ .*

*Proof.* Apply Corollary C.3 with  $I := \mathbb{N} \times [0, T]$ ,  $X_{(n,t)} := X_n$  and  $\Phi_{(n,t)} := S_n(t)$ .  $\square$

The next lemma characterizes continuity from above by means of suitable cut-off functions. Remarkably, in many applications, the family  $(\zeta_x)_{x \in K}$  can be constructed by scaling and shifting a single smooth functions with compact support, see the proof of Corollary 2.14 below. Furthermore, this criterion allows us to guarantee continuity from above by verifying sufficient conditions for the generators rather than the semigroups, see Corollary 2.15 and Corollary 2.16 below.

**Lemma 2.12.** *Assume that, for every  $\varepsilon > 0$ ,  $r, T \geq 0$  and  $K \Subset \mathbb{R}^d$ , there exist a family  $(\zeta_x)_{x \in K}$  of continuous functions  $\zeta_x: \mathbb{R}^d \rightarrow \mathbb{R}$  and  $\tilde{K} \Subset \mathbb{R}^d$  with*

- (i)  $0 \leq \zeta_x \leq 1$  for all  $x \in K$ ,
- (ii)  $\sup_{y \in \tilde{K}^c} \zeta_x(y) \leq \varepsilon$  for all  $x \in K$ ,
- (iii)  $(S_n(t)(\frac{r}{\kappa}(1 - \zeta_x)))(x) \leq \varepsilon$  for all  $n \in \mathbb{N}$ ,  $t \in [0, T] \cap \mathcal{T}_n$  and  $x \in K_n$ .

*Then, for every  $\varepsilon > 0$ ,  $r, T \geq 0$  and  $K \Subset \mathbb{R}^d$ , there exist  $c \geq 0$  and  $K' \Subset \mathbb{R}^d$  with*

$$\|S_n(t)f - S_n(t)g\|_{\infty, K_n} \leq c\|f - g\|_{\infty, K'_n} + \varepsilon$$

*for all  $n \in \mathbb{N}$ ,  $t \in [0, T] \cap \mathcal{T}_n$  and  $f, g \in B_{C_\kappa(X_n)}(r)$ .*

*Proof.* Apply Corollary C.4 with  $I := \mathbb{N} \times [0, T]$ ,  $X_{(n,t)} := X_n$  and  $\Phi_{(n,t)} := S_n(t)$ .  $\square$

Subsequently, we derive several corollaries to Lemma 2.12. For the next two results, we assume that  $s \pm t \in \mathcal{T}_n$  for all  $n \in \mathbb{N}$  and  $s, t \in \mathcal{T}_n$  with  $s \geq t$ . Let  $(S_n)_{n \in \mathbb{N}}$  be a sequence of convex monotone semigroups  $(S_n(t))_{t \in \mathcal{T}_n}$  on  $C_\kappa(X_n)$  with  $S_n(t)0 = 0$ . Moreover, for every  $r, T \geq 0$ , there exists  $c \geq 0$  with

$$\|S_n(t)f - S_n(t)g\|_\kappa \leq c\|f - g\|_\kappa \tag{2.34}$$

for all  $n \in \mathbb{N}$ ,  $t \in [0, T] \cap \mathcal{T}_n$  and  $f, g \in B_{C_\kappa(X_n)}(r)$ .

**Corollary 2.13.** *Assume that, for every  $\varepsilon > 0$ ,  $r \geq 0$  and  $K \Subset \mathbb{R}^d$ , there exist a family  $(\zeta_x)_{x \in K}$  of continuous functions  $\zeta_x: \mathbb{R}^d \rightarrow \mathbb{R}$  and  $\tilde{K} \Subset \mathbb{R}^d$  such that*

- (i)  $0 \leq \zeta_x \leq 1$  and  $\zeta_x(x) = 1$  for all  $x \in K$ ,
- (ii)  $\sup_{y \in \tilde{K}^c} \zeta_x(y) \leq \varepsilon$  for all  $x \in K$ ,

(iii) there exists  $(t_n)_{n \in \mathbb{N}}$  with  $t_n \in \mathcal{T}_n \setminus \{0\}$  and

$$\left\| (S_n(t) \left( \frac{r}{\kappa} (1 - \zeta_x) \right) - \frac{r}{\kappa} (1 - \zeta_x)) \right\|_{\kappa, X_n} \leq \varepsilon t$$

for all  $n \in \mathbb{N}$ ,  $t \in [0, t_n] \cap \mathcal{T}_n$  and  $x \in K_n$ .

Then, for every  $\varepsilon > 0$ ,  $r, T \geq 0$  and  $K \Subset \mathbb{R}^d$ , there exist  $c \geq 0$  and  $K' \Subset \mathbb{R}^d$  with

$$\|S_n(t)f - S_n(t)g\|_{\infty, K_n} \leq c\|f - g\|_{\infty, K'_n} + \varepsilon$$

for all  $n \in \mathbb{N}$ ,  $t \in [0, T] \cap \mathcal{T}_n$  and  $f, g \in B_{C_\kappa(X_n)}(r)$ .

*Proof.* Let  $\varepsilon > 0$ ,  $r, T \geq 0$ ,  $K \Subset \mathbb{R}^d$  and  $(\zeta_x)_{x \in K}$  be a family of continuous functions  $\zeta_x: \mathbb{R}^d \rightarrow \mathbb{R}$  satisfying the conditions (i)-(iii) with  $\varepsilon/c$ . Let  $n \in \mathbb{N}$  and  $t \in [0, T] \cap \mathcal{T}_n$ . Since  $\mathcal{T}_n$  is closed under addition and subtraction, there exist  $t_1, t_2 \in [0, t_0] \cap \mathcal{T}_n$  and  $k \in \mathbb{N}$  with  $t = kt_1 + t_2$ . Let  $x \in K$  and  $f := \frac{r}{\kappa} (1 - \zeta_x)$ . We use the semigroup property, inequality (2.34) and condition (iii) to obtain

$$\begin{aligned} \|S_n(t)f - f\|_{\kappa, X_n} &\leq \|S_n(kt_1)S_n(t_2)f - S_n(kt_1)f\|_{\kappa, X_n} \\ &\quad + \sum_{i=0}^{k-1} \|S_n(it_1)S_n(t_1)f - S_n(it_1)f\|_{\kappa, X_n} \\ &\leq c\|S_n(t_2)f - f\|_{\kappa, X_n} + ck\|S_n(t_1)f - f\|_{\kappa, X_n} \leq \varepsilon t. \end{aligned}$$

Now, the claim follows from  $f(x) = 0$ ,  $\inf_{x \in K} \kappa(x) > 0$  and Lemma 2.12.  $\square$

The next result is particularly useful when the generators are differential operators. Furthermore, the condition imposed on  $\kappa$  is clearly satisfied if  $\kappa \equiv 1$ . For every infinitely differentiable function  $f: \mathbb{R}^d \rightarrow \mathbb{R}$  and  $i \in \mathbb{N}$ , let

$$\|D^i f\|_\kappa := \sum_{|\alpha|=i} \|D^\alpha f\|_\kappa \in [0, \infty],$$

where  $D^\alpha f := \partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d} f$  and  $|\alpha| := \sum_{j=1}^d \alpha_j$  for all  $\alpha := (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d$ . The space  $C_c^\infty(\mathbb{R}^d)$  consists of all infinitely differentiable functions  $f: \mathbb{R}^d \rightarrow \mathbb{R}$  with compact support.

**Corollary 2.14.** *Suppose that the function  $1/\kappa$  is infinitely differentiable and, for every  $\varepsilon > 0$ , there exists  $K \Subset \mathbb{R}^d$  with*

$$\sup_{x \in K^c} |(D^\alpha \frac{1}{\kappa})\kappa|(x) \leq \varepsilon \quad \text{for all } 1 \leq |\alpha| \leq N.$$

Furthermore, there exist a sequence  $(h_n)_{n \in \mathbb{N}}$  with  $h_n \in \mathcal{T}_n \setminus \{0\}$ ,  $N \in \mathbb{N}$  and a non-decreasing function  $\rho: \mathbb{R}_+ \rightarrow \mathbb{R}_+$  with  $\rho(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$  such that

$$\left\| \frac{S_n(h)f - f}{h} \right\|_{\kappa, X_n} \leq \rho \left( \sum_{i=1}^N \|D^i f\|_\kappa \right) \quad (2.35)$$

for all  $n \in \mathbb{N}$ ,  $h \in (0, h_n] \cap \mathcal{T}_n$  and  $f := \frac{r}{\kappa} (1 - \zeta)$  with  $r \geq 0$  and  $\zeta \in C_c^\infty(\mathbb{R}^d)$ . Then, for every  $\varepsilon > 0$ ,  $r, T \geq 0$  and  $K \Subset \mathbb{R}^d$ , there exist  $c \geq 0$  and  $K' \Subset \mathbb{R}^d$  with

$$\|S_n(t)f - S_n(t)g\|_{\infty, K_n} \leq c\|f - g\|_{\infty, K'_n} + \varepsilon$$

for all  $n \in \mathbb{N}$ ,  $t \in [0, T] \cap \mathcal{T}_n$  and  $f, g \in B_{C_\kappa(X_n)}(r)$ .



*Proof.* Let  $\varepsilon > 0$ ,  $r \geq 0$  and  $K \Subset \mathbb{R}^d$ . Moreover, let  $\xi: \mathbb{R}^d \rightarrow \mathbb{R}$  be infinitely differentiable with compact support,  $0 \leq \xi \leq 1$  and  $\xi(x) = 1$  for all  $x \in B_{\mathbb{R}^d}(1)$ . Choose  $\delta_0 > 0$  with  $\delta_0 r \leq \varepsilon/2$  and

$$\delta_0 r \sum_{\beta < \alpha} \binom{\alpha}{\beta} \|D^{\beta \frac{1}{\kappa}}\|_{\kappa} \|D^{\alpha-\beta} \xi\|_{\infty} \leq \frac{\varepsilon}{2} \quad \text{for all } 1 \leq |\alpha| \leq N. \quad (2.36)$$

Furthermore, there exists  $\tilde{K} \Subset \mathbb{R}^d$  with  $\sup_{x \in \tilde{K}^c} |(D^{\alpha \frac{1}{\kappa}} \kappa)(x)| \leq \delta_0$  for all  $1 \leq |\alpha| \leq N$ . Let  $\delta \in (0, \delta_0]$  with  $\delta|y - x| \leq 1$  for all  $(x, y) \in K \times \tilde{K}$  and define  $\zeta_x(y) := \xi(\delta(y - x))$  for all  $(x, y) \in K \times \mathbb{R}^d$ . We verify the conditions (i)-(iii) from Corollary 2.13.

- (i) Clearly, it holds  $0 \leq \zeta_x \leq 1$  and  $\zeta_x(x) = \xi(0) = 1$  for all  $x \in K$ .
- (ii) Choose  $R \geq 0$  with  $\text{supp}(\xi) \subset B_{\mathbb{R}^d}(R)$  and  $K \subset B_{\mathbb{R}^d}(R)$ . For every  $x \in K$  and  $y \in B_{\mathbb{R}^d}(R + R/\delta)^c$ , it holds  $\delta(y - x) \geq R$  and therefore  $\zeta_x(y) = \xi(\delta(y - x)) = 0$ .
- (iii) For every  $x \in K$  and  $f := \frac{r}{\kappa}(1 - \zeta_x)$ , we use the product formula, inequality (2.36) and the fact that  $(1 - \zeta_x)(y) = 0$  for all  $(x, y) \in K \times \tilde{K}$  to obtain

$$\begin{aligned} |D^{\alpha} f|_{\kappa} &\leq \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} |D^{\beta}(\frac{r}{\kappa}) D^{\alpha-\beta}(1 - \zeta_x)|_{\kappa} \\ &= r \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} |(D^{\beta \frac{1}{\kappa}} \kappa) \delta^{|\alpha-\beta|} D^{\alpha-\beta} \xi| \\ &\leq \delta r \sum_{\beta < \alpha} \binom{\alpha}{\beta} \|D^{\beta \frac{1}{\kappa}}\|_{\kappa} \|D^{\alpha-\beta} \xi\|_{\infty} + r |(D^{\alpha \frac{1}{\kappa}} \kappa)(1 - \zeta_x)| \\ &\leq \sup_{y \in \tilde{K}^c} r |(D^{\alpha \frac{1}{\kappa}} \kappa)(y)| + \frac{\varepsilon}{2} \leq \varepsilon. \end{aligned}$$

Combining the previous estimate with inequality (2.35) yields

$$\left\| \frac{S_n(h)f - f}{h} \right\|_{\kappa} \leq \rho(c\varepsilon) \quad \text{for all } n \in \mathbb{N} \text{ and } h \in (0, h_n] \cap \mathcal{T}_n,$$

where  $c := \sum_{i=1}^N \#\{\alpha \in \mathbb{N}_0^d: |\alpha| = i\}$ . Furthermore, it holds  $\rho(c\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . Now, the claim follows from Corollary 2.13.  $\square$

For discrete time  $\mathcal{T}_n := \{kh_n: k \in \mathbb{N}_0\}$  with  $h_n > 0$ , the semigroups are given by  $S_n(kh_n) := I_n^k = I_n \circ \dots \circ I_n$  with  $I_n := S_n(h_n)$  and the verification of condition (iii) in Corollary 2.13 and Corollary 2.14 only requires knowledge of the one-step operators  $I_n$  rather than the entire semigroups  $S_n$ . Furthermore, in many applications, the operators  $I_n$  have an explicit representation that is very useful in the verification of condition (iii). However, in the continuous-time case, explicit representations for the semigroups  $S_n$  are not available or rather complicated. Here, it is important to replace condition (iii) in the previous results by a corresponding condition on the infinitesimal generators. For the following two corollaries, we assume that  $\mathcal{T}_n := \mathbb{R}_+$  for all  $n \in \mathbb{N}$ . Let  $(S_n)_{n \in \mathbb{N}}$  be a family of strongly continuous convex monotone semigroups  $(S_n(t))_{t \geq 0}$  on  $C_{\kappa}(\mathbb{R}^d)$  with  $S_n(t)0 = 0$  satisfying inequality (2.34). Here, it is crucial to require norm convergence in equation (2.37) rather than convergence w.r.t. the mixed topology.

**Corollary 2.15.** *Assume that, for every  $\varepsilon > 0$ ,  $r \geq 0$  and  $K \Subset \mathbb{R}^d$ , there exist a family  $(\zeta_x)_{x \in K}$  of continuous functions  $\zeta_x: \mathbb{R}^d \rightarrow \mathbb{R}$  and  $\tilde{K} \Subset \mathbb{R}^d$  such that*

- (i)  $0 \leq \zeta_x \leq 1$  and  $\zeta_x(x) = 1$  for all  $x \in K$ ,
- (ii)  $\sup_{y \in \tilde{K}^c} \zeta_x(y) \leq \varepsilon$  for all  $x \in K$ ,

(iii) the function  $f := \frac{r}{\kappa}(1 - \zeta_x)$  satisfies  $f \in D(A_n)$ ,  $\|A_n f\|_\kappa \leq \varepsilon$  and

$$\lim_{h \downarrow 0} \left\| \frac{S_n(h)f - f}{h} - A_n f \right\|_{\kappa, X_n} = 0 \quad \text{for all } n \in \mathbb{N} \text{ and } x \in K. \quad (2.37)$$

Then, for every  $\varepsilon > 0$ ,  $r, T \geq 0$  and  $K \Subset \mathbb{R}^d$ , there exist  $c \geq 0$  and  $K' \Subset \mathbb{R}^d$  with

$$\|S_n(t)f - S_n(t)g\|_{\infty, K_n} \leq c\|f - g\|_{\infty, K'_n} + \varepsilon$$

for all  $n \in \mathbb{N}$ ,  $t \in [0, T]$  and  $f, g \in B_{C_\kappa(X_n)}(r)$ .

*Proof.* Let  $\varepsilon > 0$ ,  $r, T \geq 0$  and  $K \Subset \mathbb{R}^d$ . We choose  $c \geq 0$  such that inequality (2.34) is satisfied with  $r + \varepsilon$  and a family  $(\zeta_x)_{x \in K}$  of continuous functions  $\zeta_x: \mathbb{R}^d \rightarrow \mathbb{R}$  satisfying the conditions (i)-(iii) with  $\varepsilon/c$ . Let  $x \in K$  and  $f := \frac{r}{\kappa}(1 - \zeta_x)$ . For every  $n \in \mathbb{N}$  and  $t \in [0, T]$ , similar to inequality (2.30) and inequality (2.31), one can use inequality (2.34) and equation (2.37) to show that

$$\begin{aligned} (S_n(t)f - f)\kappa &\leq \int_0^t (S_n(s)(f + A_n f) - S_n(s)f)\kappa \, ds, \\ (S_n(t)f - f)\kappa &\geq - \int_0^t (S_n(s)(f - A_n f) - S_n(s)f)\kappa \, ds. \end{aligned}$$

Hence, inequality (2.34) and condition (iii) imply

$$-\varepsilon t \leq -c\|A_n f\|_\kappa \leq \|S_n(t)f - f\|_\kappa \leq c\|A_n f\|_\kappa t \leq \varepsilon t.$$

Now, the claim follows from  $f(x) = 0$ ,  $\inf_{x \in K} \kappa(x) > 0$  and Lemma 2.12.  $\square$

**Corollary 2.16.** Suppose that the function  $1/\kappa$  is infinitely differentiable and, for every  $\varepsilon > 0$ , there exists  $K \Subset \mathbb{R}^d$  with

$$\sup_{x \in K^c} |(D^\alpha \frac{1}{\kappa})\kappa|(x) \leq \varepsilon \quad \text{for all } 1 \leq |\alpha| \leq N.$$

For every  $r \geq 0$  and  $\zeta \in C_c^\infty(\mathbb{R}^d)$ , the function  $f := \frac{r}{\kappa}(1 - \zeta)$  satisfies  $f \in D(A_n)$  and

$$\lim_{h \downarrow 0} \left\| \frac{S_n(h)f - f}{h} - A_n f \right\|_{\kappa, X_n} = 0 \quad \text{for all } n \in \mathbb{N}.$$

Furthermore, there exist  $N \in \mathbb{N}$  and a non-decreasing function  $\rho: \mathbb{R}_+ \rightarrow \mathbb{R}_+$  with  $\rho(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$  such that

$$\|A_n f\|_{\kappa, X_n} \leq \rho \left( \sum_{i=1}^N \|D^i f\|_\kappa \right)$$

for all  $n \in \mathbb{N}$  and  $f := \frac{r}{\kappa}(1 - \zeta)$  with  $r \geq 0$  and  $\zeta \in C_c^\infty(\mathbb{R}^d)$ . Then, for every  $\varepsilon > 0$ ,  $r, T \geq 0$  and  $K \Subset \mathbb{R}^d$ , there exist  $c \geq 0$  and  $K' \Subset \mathbb{R}^d$  with

$$\|S_n(t)f - S_n(t)g\|_{\infty, K_n} \leq c\|f - g\|_{\infty, K'_n} + \varepsilon$$

for all  $n \in \mathbb{N}$ ,  $t \in [0, T] \cap \mathcal{T}_n$  and  $f, g \in B_{C_\kappa(X_n)}(r)$ .

*Proof.* Using Corollary 2.15, one can proceed similar to the proof of Corollary 2.14.  $\square$

**Remark 2.17.** In many applications, it is easy to show that  $S_n(t): C_\kappa(X_n) \rightarrow F_\kappa(X_n)$  but verifying the continuity of  $S_n(t)f$  is more complicated. However, due to the proofs of the corollaries 2.13–2.16 it is sufficient to require that

$$S_n(t)(\frac{r}{\kappa}(1 - \zeta_x)) \in C_\kappa(X_n) \quad \text{and} \quad S_n(s+t)(\frac{r}{\kappa}(1 - \zeta_x)) = S_n(s)S_n(t)(\frac{r}{\kappa}(1 - \zeta_x)).$$

Furthermore, it is often straightforward to show that  $S_n(t): \text{Lip}_b(X_n) \rightarrow \text{Lip}_b(X_n)$ . Hence, for every  $f \in C_\kappa(\mathbb{R}^d)$ , we can choose a sequence  $(f_k)_{k \in \mathbb{N}} \subset \text{Lip}_b(\mathbb{R}^d)$  with  $f_k \rightarrow f$  to obtain  $S_n(t)f = \lim_{k \rightarrow \infty} S_n(t)f_k \in C_\kappa(X_n)$  for all  $n \in \mathbb{N}$  and  $t \in \mathcal{T}_n$ . In particular, in the case  $\kappa \equiv 1$ , one can usually choose  $\frac{r}{\kappa}(1 - \zeta_x) \in \text{Lip}_b(\mathbb{R}^d)$ .

### 3. EXAMPLES

**3.1. Euler formula and Yosida approximation for upper envelopes.** In this subsection, we derive an Euler formula and a Yosida-type approximation for upper envelopes of families of linear semigroups. The construction uses the supremum of linear resolvents rather than the resolvent of the supremum of linear operators as it is common both in the classical theory of m-accretive operators and the theory of viscosity solutions. We refer to [17] and [64] for an Euler formula for bi-continuous linear semigroups as well as to [19] for a Hille–Yosida theorem for weakly continuous linear semigroups on the space of bounded uniformly continuous functions and to [75] for the classical Yosida approximation. To that end, we consider a non-empty family  $(A^\theta)_{\theta \in \Theta}$  of linear operators  $A^\theta: D(A^\theta) \rightarrow C_\kappa(\mathbb{R}^d)$  with domain  $D(A^\theta) \subset C_\kappa(\mathbb{R}^d)$  and resolvent  $\varrho(A^\theta) \subset \mathbb{C}$ . In the sequel, we provide certain nonlinear versions of the classical Euler formula and the Yosida approximation in order to construct a strongly continuous convex monotone semigroup  $(S(t))_{t \geq 0}$  on  $C_\kappa(\mathbb{R}^d)$  whose generator is given by

$$Af = \sup_{\theta \in \Theta} A^\theta f \quad \text{for all } f \in C_b^\infty(\mathbb{R}^d).$$

**Assumption 3.1.** Suppose that the conditions (i)–(iv) or the conditions (i), (ii'), (iii) and (iv) from the following list are satisfied:

- (i) There exists  $\omega \in \mathbb{R}$  with  $(\omega, \infty) \subset \bigcap_{\theta \in \Theta} \varrho(A^\theta)$  such that, for every  $\lambda \in (\omega, \infty)$  and  $\theta \in \Theta$ ,

$$\begin{aligned} \|(\lambda - \omega)(\lambda - A^\theta)^{-1}f\|_\infty &\leq \|f\|_\infty \quad \text{for all } f \in \text{Lip}_b(\mathbb{R}^d), \\ \|(\lambda - \omega)(\lambda - A^\theta)^{-1}f\|_\kappa &\leq \|f\|_\kappa \quad \text{for all } f \in C_\kappa(\mathbb{R}^d). \end{aligned}$$

In addition, the operators  $(\lambda - A^\theta)^{-1}: C_\kappa(\mathbb{R}^d) \rightarrow C_\kappa(\mathbb{R}^d)$  are monotone for all  $\lambda \in (\omega, \infty)$  and  $\theta \in \Theta$ .

- (ii) There exists a bounded continuous function  $\tilde{\kappa}: \mathbb{R}^d \rightarrow (0, \infty)$  with

$$\lim_{|x| \rightarrow \infty} \frac{\tilde{\kappa}(x)}{\kappa(x)} = 0 \quad \text{and} \quad \|(\lambda - \omega)(\lambda - A^\theta)^{-1}f\|_{\tilde{\kappa}} \leq \|f\|_{\tilde{\kappa}}$$

for all  $\lambda \in (\omega, \infty)$  and  $f \in C_\kappa(\mathbb{R}^d)$ .

- (ii') For every  $\varepsilon > 0$ ,  $r \geq 0$  and  $K \Subset \mathbb{R}^d$ , there exist a family  $(\zeta_x)_{x \in K}$  of continuous functions  $\zeta_x: \mathbb{R}^d \rightarrow \mathbb{R}$  and  $\tilde{K} \Subset \mathbb{R}^d$  such that

- (a)  $0 \leq \zeta_x \leq 1$  and  $\varphi_x(x) = 1$  for all  $x \in K$ ,
- (b)  $\sup_{y \in \tilde{K}^c} \zeta_x(y) \leq \varepsilon$  for all  $x \in K$ ,
- (c)  $\frac{r}{\kappa}(1 - \zeta_x) \in \bigcap_{\theta \in \Theta} D(A^\theta)$  and  $\sup_{\theta \in \Theta} \|A^\theta(\frac{r}{\kappa}(1 - \zeta_x))\|_\kappa \leq \varepsilon$  for all  $x \in K$ .

- (iii) There exists  $L \geq 0$  with

$$(\lambda - \omega)^2 \|\tau_x(\lambda - A^\theta)^{-1}f - (\lambda - A^\theta)^{-1}(\tau_x f)\|_\infty \leq Lr|x|$$

for all  $r \geq 0$ ,  $f \in \text{Lip}_b(\mathbb{R}^d, r)$ ,  $x \in \mathbb{R}^d$ ,  $\lambda \in (\omega, \infty)$  and  $\theta \in \Theta$ .

- (iv) It holds  $C_b^\infty(\mathbb{R}^d) \subset \bigcap_{\theta \in \Theta} D(A^\theta)$ . Moreover, for every  $f \in C_b^\infty(\mathbb{R}^d)$  and  $K \Subset \mathbb{R}^d$ ,

$$\sup_{\theta \in \Theta} \|A^\theta f\|_\kappa < \infty \quad \text{and} \quad \lim_{\lambda \rightarrow \infty} \|(\lambda - \omega)(\lambda - A^\theta)^{-1}A^\theta f - A^\theta f\|_{\infty, K} = 0.$$

The previous conditions are, for instance, satisfied for suitably bounded families of generators of Lévy processes, Ornstein–Uhlenbeck processes and geometric Brownian motions. In these cases, it is straightforward to verify the respective conditions for the transition semigroups and use the Laplace transform in order to transfer them to the resolvents. For a detailed illustration how the conditions can be verified for a large class of transition semigroups, we refer to [71, Section 6.3 and Section 7]. At this point we would like to emphasise that the nonlinear semigroups in [71] are not constructed via the resolvents but as so-called Nisio semigroups. Furthermore, due to Lemma 2.11 and Corollary 2.13, the conditions (ii) and (ii') both guarantee that Assumption 2.8(ii) is valid. While condition (ii) is satisfied for Ornstein–Uhlenbeck processes and geometric Brownian motions, it does not cover Lévy processes with non-integrable jumps. On the other hand, condition (ii') applies to Lévy processes with possibly non-integrable jumps but neither to Ornstein–Uhlenbeck processes nor geometric Brownian motions. In the sequel, let  $(\lambda_n)_{n \in \mathbb{N}} \subset (\omega, \infty)$  be a sequence with  $\lambda_n \rightarrow \infty$  and

$$h_n := \frac{1}{\lambda_n - \omega} \quad \text{for all } n \in \mathbb{N}.$$

For every  $n \in \mathbb{N}$  and  $f \in C_\kappa(\mathbb{R}^d)$ , we define

$$I_n f := \sup_{\theta \in \Theta} \lambda_n (\lambda_n - A^\theta)^{-1} f.$$

Moreover, let  $X_n := \mathbb{R}^d$  and  $\mathcal{T}_n := \{kh_n : k \in \mathbb{N}_0\}$  for all  $n \in \mathbb{N}$ . For every  $n \in \mathbb{N}$  and  $f \in C_\kappa(\mathbb{R}^d)$ , the resolvent identity implies

$$I_n f = f + \sup_{\theta \in \Theta} A^\theta (\lambda_n - A^\theta)^{-1} f.$$

Moreover, for every  $f \in \bigcap_{\theta \in \Theta} D(A^\theta)$ , it follows that

$$I_n f = f + \sup_{\theta \in \Theta} (\lambda_n - A^\theta)^{-1} A^\theta f. \quad (3.1)$$

**Theorem 3.2.** *Suppose that Assumption 3.1 is satisfied. Then, there exists a strongly continuous convex monotone semigroup  $(S(t))_{t \geq 0}$  on  $C_\kappa(\mathbb{R}^d)$  with  $S(t)0 = 0$  given by*

$$S(t)f = \lim_{n \rightarrow \infty} I_n^{k_n^t} f \quad \text{for all } (f, t) \in C_\kappa(\mathbb{R}^d) \times \mathbb{R}_+,$$

where  $(k_n^t)_{n \in \mathbb{N}} \subset \mathbb{N}$  is an arbitrary sequence satisfying  $k_n^t h_n \rightarrow t$ , such that

$$C_b^\infty(\mathbb{R}^d) \subset D(A) \quad \text{and} \quad Af = \sup_{\theta \in \Theta} A^\theta f \quad \text{for all } f \in C_b^\infty(\mathbb{R}^d).$$

Moreover, the conditions of Theorem 2.4 are satisfied.

*Proof.* In order to apply Theorem 2.9, we have to verify Assumption 2.8. First, we show that  $I_n : \text{Lip}_b(\mathbb{R}^d, r) \rightarrow \text{Lip}_b(\mathbb{R}^d, e^{\beta h_n} r)$  for all  $n \in \mathbb{N}$  and  $r \geq 0$ , where

$$\beta := \omega + \frac{L\lambda_0}{\lambda_0 - \omega} \quad \text{and} \quad \lambda_0 := \min_{n \in \mathbb{N}} \lambda_n > \omega.$$

For every  $n \in \mathbb{N}$  and  $f, g \in \text{Lip}_b(\mathbb{R}^d)$ , it follows from Assumption 3.1(i) that

$$\|I_n f - I_n g\|_\infty \leq \frac{\lambda_n}{\lambda_n - \omega} \|f - g\|_\infty = (1 + \omega h_n) \|f - g\|_\infty \leq e^{\omega h_n} \|f - g\|_\infty.$$

Moreover, it holds  $I_n 0 = 0$  and Assumption 3.1(iii) implies

$$\|I_n(\tau_x f) - \tau_x I_n f\|_\infty \leq \frac{L\lambda_n}{\lambda_n - \omega} r h_n |x| \leq \frac{L\lambda_0}{\lambda_0 - \omega} r h_n |x| \quad (3.2)$$

for all  $n \in \mathbb{N}$ ,  $r \geq 0$ ,  $f \in \text{Lip}_b(\mathbb{R}^d, r)$  and  $x \in \mathbb{R}^d$ . Hence, for every  $x, y \in \mathbb{R}^d$ ,

$$\begin{aligned} |(I_n f)(x + y) - (I_n f)(y)| &\leq |\tau_x I_n f - I_n(\tau_x f)|(y) + |I_n(\tau_x f) - I_n f|(y) \\ &\leq \frac{\lambda_0}{\lambda_0 - \omega} L r h_n |x| + (1 + \omega h_n) r |x| \\ &= \left(1 + \left(\omega + \frac{L \lambda_0}{\lambda_0 - \omega}\right) h_n\right) r |x| \leq e^{\beta h_n} r |x|. \end{aligned}$$

We obtain that  $I_n f \in \text{Lip}_b(\mathbb{R}^d, e^{\beta h_n} r)$  and therefore Assumption 2.8(iv) is satisfied. Furthermore, inequality (3.2) and the fact that  $\|\kappa\|_\infty < \infty$  yield Assumption 2.8(iii).

Second, we show that  $I_n : C_\kappa(\mathbb{R}^d) \rightarrow C_\kappa(\mathbb{R}^d)$  for all  $n \in \mathbb{N}$ . Assumption 3.1(i) yields

$$\|I_n f - I_n g\|_\kappa \leq \frac{\lambda_n}{\lambda_n - \omega} \|f - g\|_\kappa = (1 + \omega h_n) \|f - g\|_\kappa \leq e^{\omega h_n} \|f - g\|_\kappa$$

and thus Assumption 2.8(i) is satisfied. Moreover, the operators  $I_n : C_\kappa(\mathbb{R}^d) \rightarrow F_\kappa(\mathbb{R}^d)$  are convex and monotone with  $I_n 0 = 0$ . Let  $f \in C_\kappa(\mathbb{R}^d)$  and  $(f_k)_{k \in \mathbb{N}} \subset \text{Lip}_b(\mathbb{R}^d)$  be a sequence with  $f_k \rightarrow f$ . In case that Assumption 3.1(ii) is valid, Corollary C.3 implies  $I_n f = \lim_{k \rightarrow \infty} I_n f_k \in C_\kappa(\mathbb{R}^d)$  and thus  $I_n : C_\kappa(\mathbb{R}^d) \rightarrow C_\kappa(\mathbb{R}^d)$ . In addition, it follows by induction that  $\|I_n^k f\|_{\tilde{\kappa}} \leq e^{\omega k h_n} \|f\|_{\tilde{\kappa}}$  for all  $k, n \in \mathbb{N}$  and  $f \in C_\kappa(\mathbb{R}^d)$ . Hence, we can apply Lemma 2.11 to obtain that Assumption 2.8(ii) is satisfied. If Assumption 3.1(ii') is valid, we can use Corollary 2.13 and Remark 2.17 instead. Indeed, equation (3.1) and Assumption 3.1(i) imply

$$\begin{aligned} \|I_n\left(\frac{r}{\kappa}(1 - \zeta_x)\right) - \frac{r}{\kappa}(1 - \zeta_x)\|_\kappa &\leq \sup_{\theta \in \Theta} \|(\lambda_n - A^\theta)^{-1} A^\theta \left(\frac{r}{\kappa}(1 - \zeta_x)\right)\|_\kappa \\ &\leq \sup_{\theta \in \Theta} \|A^\theta \left(\frac{r}{\kappa}(1 - \zeta_x)\right)\|_\kappa h_n \leq \varepsilon h_n. \end{aligned}$$

Third, we verify Assumption 2.8(v). Let  $f \in C_b^\infty(\mathbb{R}^d)$  and define

$$A_n f := \frac{I_n f - f}{h_n} \quad \text{for all } n \in \mathbb{N}.$$

For every  $n \in \mathbb{N}$ , it follows from equation (3.1) that

$$A_n f = \sup_{\theta \in \Theta} ((\lambda_n - \omega) \lambda_n (\lambda_n - A^\theta)^{-1} f) - (\lambda_n - \omega) = \sup_{\theta \in \Theta} (\lambda_n - \omega) (\lambda_n - A^\theta)^{-1} A^\theta f.$$

Hence, for every  $n \in \mathbb{N}$  and  $K \in \mathbb{R}^d$ , Assumption 3.1(i) yields

$$\begin{aligned} \|A_n f\|_\kappa &\leq \sup_{\theta \in \Theta} \|(\lambda_n - \omega) (\lambda_n - A^\theta)^{-1} A^\theta f\|_\kappa \leq \sup_{\theta \in \Theta} \|A^\theta f\|_\kappa, \\ \|A_n f - A f\|_{\infty, K} &\leq \sup_{\theta \in \Theta} \|(\lambda_n - \omega) (\lambda_n - A^\theta)^{-1} A^\theta f - A^\theta f\|_{\infty, K}. \end{aligned}$$

and Assumption 3.1(iv) guarantees that Assumption 2.8(v) is satisfied. Now, the claim follows from Theorem 2.9, where property (iv) holds with the constants  $L$  and  $\omega + L$  rather than  $\frac{L \lambda_0}{\lambda_0 - \omega}$  and  $\beta$  because of the fact that  $\lambda_n \rightarrow \infty$ .  $\square$

For the Yosida approximation, we additionally have to require norm convergence of the generators in condition (iv).

**Assumption 3.3.** Suppose that the conditions (i)–(iii) or the conditions (i), (ii') and (iii) from Assumption 3.1 are satisfied. Furthermore, the following statement is valid:

(iv') It holds  $C_b^\infty(\mathbb{R}^d) \subset \bigcap_{\theta \in \Theta} D(A^\theta)$ . Moreover, for every  $f \in C_b^\infty(\mathbb{R}^d)$ ,

$$\sup_{\theta \in \Theta} \|A^\theta f\|_\kappa < \infty \quad \text{and} \quad \lim_{\lambda \rightarrow \infty} \sup_{\theta \in \Theta} \|(\lambda - \omega)(\lambda - A^\theta)^{-1} A^\theta f - A^\theta f\|_\kappa = 0.$$

For every  $n \in \mathbb{N}$  and  $\theta \in \Theta$ , we define

$$A_n^\theta := (\lambda_n - \omega)A^\theta(\lambda_n - A^\theta)^{-1}.$$

Since  $A_n^\theta: C_\kappa(\mathbb{R}^d) \rightarrow C_\kappa(\mathbb{R}^d)$  is a bounded linear operator, we can define

$$J_n f := \sup_{\theta \in \Theta} e^{h_n A_n^\theta} f \quad \text{for all } n \in \mathbb{N} \text{ and } f \in C_\kappa(\mathbb{R}^d),$$

where the operator exponential is given as the power series

$$e^{h_n A_n^\theta} := \sum_{k=1}^{\infty} \frac{(h_n A_n^\theta)^k}{k!}.$$

Since the resolvent identity

$$h_n A_n^\theta f = \lambda_n (\lambda_n - A^\theta)^{-1} f - f \tag{3.3}$$

holds for all  $n \in \mathbb{N}$ ,  $f \in C_\kappa(\mathbb{R}^d)$  and  $\theta \in \Theta$ , we obtain

$$J_n f = e^{-1} \sup_{\theta \in \Theta} e^{\lambda_n (\lambda_n - A^\theta)^{-1}} f \quad \text{for all } n \in \mathbb{N} \text{ and } f \in C_\kappa(\mathbb{R}^d). \tag{3.4}$$

**Theorem 3.4.** *Suppose that Assumption 3.3 is satisfied. Then, there exists a strongly continuous convex monotone semigroup  $(T(t))_{t \geq 0}$  on  $C_\kappa(\mathbb{R}^d)$  with  $T(t)0 = 0$  given by*

$$T(t)f = \lim_{n \rightarrow \infty} J_n^{k_n^t} f \quad \text{for all } (f, t) \in C_\kappa(\mathbb{R}^d) \times \mathbb{R}_+, \tag{3.5}$$

where  $(k_n^t)_{n \in \mathbb{N}} \subset \mathbb{N}$  is an arbitrary sequence satisfying  $k_n^t h_n \rightarrow t$ , such that

$$C_b^\infty(\mathbb{R}^d) \subset D(B) \quad \text{and} \quad Bf = \sup_{\theta \in \Theta} A^\theta f \quad \text{for all } f \in C_b^\infty(\mathbb{R}^d).^2$$

Moreover, the conditions of Theorem 2.4 are satisfied and therefore

$$S(t)f = T(t)f \quad \text{for all } t \geq 0 \text{ and } f \in C_\kappa(\mathbb{R}^d).$$

*Proof.* First, for every  $n \in \mathbb{N}$ ,  $r \geq 0$ ,  $f \in \text{Lip}_b(\mathbb{R}^d, r)$  and  $x \in \mathbb{R}^d$ , we show that

$$\|\tau_x J_n f - J_n(\tau_x f)\|_\infty \leq \frac{L\lambda_0 e^{\beta h_0}}{\lambda_0 - \omega} r h_n |x|, \tag{3.6}$$

where

$$\beta := \omega + \frac{L\lambda_0}{\lambda_0 - \omega} \quad \text{and} \quad \lambda_0 := \min_{n \in \mathbb{N}} \lambda_n > \omega.$$

Subsequently, we write  $B_{n,\theta} := \lambda_n (\lambda_n - A^\theta)^{-1}$  for all  $n \in \mathbb{N}$  and  $\theta \in \Theta$ . For every  $n \in \mathbb{N}$ ,  $r \geq 0$ ,  $f \in \text{Lip}_b(\mathbb{R}^d, r)$ ,  $x \in \mathbb{R}^d$  and  $\theta \in \Theta$ , Assumption 3.1(iii) implies

$$\|\tau_x B_{n,\theta} f - B_{n,\theta}(\tau_x f)\|_\infty \leq \frac{L\lambda_0}{\lambda_0 - \omega} r h_n |x|. \tag{3.7}$$

Furthermore, by induction, it follows from Assumption 3.1(i) that

$$\|B_{n,\theta} f - B_{n,\theta} g\|_\infty \leq (1 + \omega h_n)^k \|f - g\|_\infty. \tag{3.8}$$

for all  $k, n \in \mathbb{N}$ ,  $f, g \in \text{Lip}_b(\mathbb{R}^d)$  and  $\theta \in \Theta$ . Hence, similar to the proof of Theorem 3.2, one can show that

$$B_{n,\theta}: \text{Lip}_b(\mathbb{R}^d, r) \rightarrow \text{Lip}_b(\mathbb{R}^d, (1 + \beta h_n)^k r). \tag{3.9}$$

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<sup>2</sup>Here, we denote by  $B$  the generator of  $(T(t))_{t \geq 0}$ .

for all  $n \in \mathbb{N}$ ,  $r \geq 0$  and  $\theta \in \Theta$ . Combining the inequalities (3.7)–(3.9) yields

$$\begin{aligned}
\|\tau_x B_{n,\theta}^k f - B_{n,\theta}^k(\tau_x f)\|_\infty &\leq \sum_{l=1}^k \|B_{n,\theta}^{k-l}(\tau_x B_{n,\theta}^l f) - B_{n,\theta}^{k-l} B_{n,\theta}(\tau_x B_{n,\theta}^{l-1} f)\|_\infty \\
&\leq \sum_{l=1}^k (1 + \omega h_n)^{k-l} \|\tau_x B_{n,\theta} B_{n,\theta}^{l-1} f - B_{n,\theta}(\tau_x B_{n,\theta}^{l-1} f)\|_\infty \\
&\leq \sum_{l=1}^k (1 + \omega h_n)^{k-l} \frac{L\lambda_0}{\lambda_0 - \omega} (1 + \beta h_n)^{l-1} r h_n |x| \\
&\leq k(1 + \beta h_n)^{k-1} \frac{L\lambda_0}{\lambda_0 - \omega} r h_n |x|
\end{aligned}$$

for all  $n \in \mathbb{N}$ ,  $r \geq 0$ ,  $f \in \text{Lip}_b(\mathbb{R}^d, r)$ ,  $x \in \mathbb{R}^d$  and  $\theta \in \Theta$ . Equation (3.4) implies

$$\begin{aligned}
\|\tau_x J_n f - J_n(\tau_x f)\|_\infty &\leq e^{-1} \sup_{\theta \in \Theta} \|\tau_x e^{B_{n,\theta}} - e^{B_{n,\theta}}(\tau_x f)\|_\infty \\
&\leq e^{-1} \sup_{\theta \in \Theta} \sum_{k=0}^{\infty} \frac{\|\tau_x B_{n,\theta}^k f - B_{n,\theta}^k(\tau_x f)\|_\infty}{k!} \\
&\leq e^{-1} \sum_{k=1}^{\infty} \frac{(1 + \beta h_n)^{k-1}}{(k-1)!} \frac{L\lambda_0}{\lambda_0 - \omega} r h_n |x| \\
&= \frac{L\lambda_0}{\lambda_0 - \omega} e^{\omega h_n} r h_n |x| = \frac{L\lambda_0 e^{\omega h_0}}{\lambda_0 - \omega} r h_n |x|
\end{aligned}$$

for all  $n \in \mathbb{N}$ ,  $r \geq 0$ ,  $f \in \text{Lip}_b(\mathbb{R}^d, r)$  and  $x \in \mathbb{R}^d$ .

Second, we verify Assumption 2.8. We have already shown that Assumption 2.8(iii) is satisfied. Moreover, it follows from equation (3.4) and equation (3.9) that

$$J_n : \text{Lip}_b(\mathbb{R}^d, r) \rightarrow \text{Lip}_b(\mathbb{R}^d, e^{\beta h_n} r) \quad \text{for all } n \in \mathbb{N} \text{ and } r \geq 0.$$

Similarly, one can derive from Assumption 3.1(i) that

$$\|J_n f - J_n g\|_\kappa \leq e^{\omega h_n} \|f - g\|_\kappa \quad \text{for all } n \in \mathbb{N} \text{ and } f, g \in C_\kappa(\mathbb{R}^d).$$

Moreover, the operators  $I_n : C_\kappa(\mathbb{R}^d) \rightarrow F_\kappa(\mathbb{R}^d)$  are convex and monotone with  $I_n 0 = 0$ . Let  $f \in C_\kappa(\mathbb{R}^d)$  and  $(f_k)_{k \in \mathbb{N}} \subset \text{Lip}_b(\mathbb{R}^d)$  be a sequence with  $f_k \rightarrow f$ . In case that Assumption 3.1(ii) is valid, Corollary C.3 implies  $I_n f = \lim_{k \rightarrow \infty} I_n f_k \in C_\kappa(\mathbb{R}^d)$  and thus  $I_n : C_\kappa(\mathbb{R}^d) \rightarrow C_\kappa(\mathbb{R}^d)$ . In addition, it follows by induction that  $\|I_n^k f\|_{\tilde{\kappa}} \leq e^{\omega k h_n} \|f\|_{\tilde{\kappa}}$  for all  $k, n \in \mathbb{N}$  and  $f \in C_\kappa(\mathbb{R}^d)$ . Hence, we can apply Lemma 2.11 to obtain that Assumption 2.8(ii) is satisfied. If Assumption 3.1(ii') is valid, we can use Corollary 2.13 and Remark 2.17 instead. Indeed, Assumption 3.1(i) implies

$$\begin{aligned}
\|J_n(\frac{r}{\kappa}(1 - \zeta_x)) - \frac{r}{\kappa}(1 - \zeta_x)\|_\kappa &\leq \sup_{\theta \in \Theta} \int_0^1 \|e^{s h_n A_n^\theta} h_n A_n^\theta(\frac{r}{\kappa}(1 - \zeta_x))\|_\kappa ds \\
&\leq \sup_{\theta \in \Theta} e^{\omega h_n} h_n \|A^\theta(\frac{r}{\kappa}(1 - \zeta_x))\|_\kappa \leq e^{\omega h_0} \varepsilon h_n.
\end{aligned}$$



It remains to verify Assumption 2.8(v). For every  $n \in \mathbb{N}$ ,  $f \in C_b^\infty(\mathbb{R}^d)$  and  $\theta \in \Theta$ , we use Assumption 3.1(i) and equation (3.3) to estimate

$$\begin{aligned} \left\| \frac{e^{h_n A_n^\theta} f - f}{h_n} - A_n^\theta f \right\|_\kappa &\leq \int_0^1 \|e^{sh_n A_n^\theta} A_n^\theta f - A_n^\theta f\|_\kappa \\ &\leq \int_0^1 \|e^{sh_n A_n^\theta} A^\theta f - A^\theta f\|_\kappa \\ &\leq e^{\omega h_n} h_n \|A_n^\theta A^\theta f\|_\kappa \\ &= e^{\omega h_n} \|\lambda_n(\lambda_n - A^\theta)^{-1} A^\theta f - A^\theta f\|_\kappa \\ &\leq e^{\omega h_n} \|(\lambda - \omega)(\lambda_n - A^\theta)^{-1} A^\theta f - A^\theta f\|_\kappa + \omega h_n \|A^\theta f\|_\kappa. \end{aligned}$$

Assumption 3.3(iv') and equation (3.4) imply

$$\lim_{n \rightarrow \infty} \left\| \frac{J_n f - f}{h_n} - \sup_{\theta \in \Theta} A^\theta f \right\|_\kappa = 0.$$

Now, the claim follows from Theorem 2.9 and Theorem 2.4.  $\square$

**3.2. Finite-difference schemes for convex HJB equations.** In this subsection, we provide finite-difference approximations for convex HJB equations. For simplicity, we focus on the one-dimensional case and constant coefficients and refer to [2, 13, 33, 61, 62] for the multi-dimensional case with Lipschitz coefficients. We fix  $\kappa \equiv 1$  and denote by  $C_b(\mathbb{R})$  the space of all bounded continuous functions  $f: \mathbb{R} \rightarrow \mathbb{R}$ . Let  $(\delta_n)_{n \in \mathbb{N}}$ ,  $(h_n)_{n \in \mathbb{N}}$  and  $(\sigma_n)_{n \in \mathbb{N}}$  be sequences in  $(0, \infty)$  with  $\delta_n, h_n \rightarrow 0$  and  $\sigma_n \rightarrow \infty$ . Furthermore, let  $\varphi: \mathbb{R}_+ \rightarrow [0, \infty]$  be a function. We show that the finite-difference scheme

$$(I_n f)(x) := f(x) + h_n \sup_{\sigma \in [0, \sigma_n]} \left( \frac{\sigma^2}{2} \frac{f(x + \delta_n) - 2f(x) + f(x - \delta_n)}{\delta_n^2} - \varphi(\sigma) \right),$$

which is defined for all  $n \in \mathbb{N}$ ,  $f \in C_b(\mathbb{R})$  and  $x \in \mathbb{R}$ , converges to a strongly continuous convex monotone semigroup  $(S(t))_{t \geq 0}$  on  $C_b(\mathbb{R})$  whose generator is given by

$$Af = \sup_{\sigma \geq 0} \left( \frac{1}{2} \sigma^2 f'' - \varphi(\sigma) \right) \quad \text{for all } f \in C_b^\infty(\mathbb{R}).$$

**Assumption 3.5.** Suppose that the following conditions are satisfied:

- (i) For every  $n \in \mathbb{N}$ , there exists  $\sigma \in [0, \sigma_n]$  with  $\varphi(\sigma) = 0$ .
- (ii) It holds  $\lim_{\sigma \rightarrow \infty} \varphi(\sigma)/\sigma^2 = \infty$ .
- (iii) It holds  $\frac{\sigma_n^2 h_n}{\delta_n^2} \leq 1$  for all  $n \in \mathbb{N}$ .

Condition (iii) is classical and guarantees that the finite-difference scheme is stable w.r.t. the supremum norm on the grid. Due to Theorem 2.4, the semigroup  $(S(t))_{t \geq 0}$  appearing in the following theorem is unique.

**Theorem 3.6.** *Suppose that Assumption 3.5 is satisfied. Then, there exists a strongly continuous convex monotone semigroup  $(S(t))_{t \geq 0}$  on  $C_b(\mathbb{R})$  with  $S(t)0 = 0$  given by*

$$S(t)f = \lim_{n \rightarrow \infty} I_n^{k_n} f \quad \text{for all } (f, t) \in C_b(\mathbb{R}) \times \mathbb{R}_+,$$

where  $(k_n^t)_{n \in \mathbb{N}} \subset \mathbb{N}$  is an arbitrary sequence satisfying  $k_n^t h_n \rightarrow t$ , such that

$$C_b^\infty(\mathbb{R}^d) \subset D(A) \quad \text{and} \quad Af = \sup_{\sigma \geq 0} \left( \frac{1}{2} \sigma^2 f'' - \varphi(\sigma) \right) \quad \text{for all } f \in C_b^\infty(\mathbb{R}^d).$$

Moreover, the conditions of Theorem 2.4 are satisfied.

*Proof.* We verify Assumption 2.8. For every  $n \in \mathbb{N}$ ,  $f \in C_b(\mathbb{R})$  and  $x \in \mathbb{R}$ ,

$$(I_n f)(x) = \sup_{\sigma \in [0, \sigma_n]} \left( \left( 1 - \frac{\sigma^2 h_n}{\delta_n^2} \right) f(x) + \frac{\sigma^2 h_n}{2\delta_n^2} (f(x + 2\delta_n) + f(x - 2\delta_n)) - \varphi(\sigma) \right).$$

Hence, Assumption 3.5(iii) yields  $I_n: \text{Lip}_b(\mathbb{R}^d, r) \rightarrow \text{Lip}_b(\mathbb{R}^d, r)$  for all  $n \in \mathbb{N}$  and  $r \geq 0$  while the identity  $I_n(\tau_x f) = \tau_x I_n f$  holds by definition for all  $f \in C_b(\mathbb{R})$ . Moreover, the convex monotone operators  $I_n: C_b(\mathbb{R}) \rightarrow F_b(\mathbb{R})$  satisfy  $I_n 0 = 0$  and

$$\|I_n f - I_n g\|_\infty \leq \|f - g\|_\infty \quad \text{for all } n \in \mathbb{N} \text{ and } f, g \in C_b(\mathbb{R}).$$

Now, let  $f \in C_b^\infty(\mathbb{R})$ . It follows from Taylor's formula that

$$\frac{\sigma^2}{2} \frac{f(x + \delta_n) - 2f(x) + f(x - \delta_n)}{\delta_n^2} = \frac{\sigma^2}{2\delta_n^2} \int_0^{\delta_n} (\delta_n - s)(f''(x + s) + f''(x - s)) ds$$

for all  $n \in \mathbb{N}$ ,  $\sigma \geq 0$  and  $x \in \mathbb{R}$ . Hence, by Assumption 3.5(ii), there exists  $r \geq 0$  with

$$\left( \frac{I_n f - f}{h_n} \right)(x) = \sup_{\sigma \in [0, r]} \left( \frac{\sigma^2}{2\delta_n^2} \int_0^{\delta_n} (\delta_n - s)(f''(x + s) + f''(x - s)) ds - \varphi(\sigma) \right)$$

for all  $n \in \mathbb{N}$ . Moreover, by increasing  $r \geq 0$ , we can assume that

$$\sup_{\sigma \geq 0} \left( \frac{1}{2} \sigma^2 f'' - \varphi(\sigma) \right) = \sup_{\sigma \in [0, r]} \left( \frac{1}{2} \sigma^2 f'' - \varphi(\sigma) \right).$$

We obtain

$$\frac{I_n f - f}{h_n} \rightarrow \sup_{\sigma \geq 0} \left( \frac{1}{2} \sigma^2 f'' - \varphi(\sigma) \right) \quad \text{as } n \rightarrow \infty.$$

Moreover, for every  $n \in \mathbb{N}$ ,

$$\left\| \frac{I_n f - f}{h_n} \right\|_\infty \leq \sup_{\sigma \geq 0} (\sigma \|f''\|_\infty - \varphi(\sigma)).$$

and therefore we obtain from Assumption 3.5(ii), Corollary 2.14 and Remark 2.17 that  $I_n: C_b(\mathbb{R}) \rightarrow C_b(\mathbb{R})$  for all  $n \in \mathbb{N}$  and that Assumption 2.8(ii) is satisfied.  $\square$

**3.3. Vanishing viscosity and stability of convex HJB equations.** In this subsection, we provide a stability result for convex HJB equations which covers, as a special case, the vanishing viscosity method. We fix  $\kappa \equiv 1$  and denote by  $C_b(\mathbb{R}^d)$  the space of all bounded continuous functions  $f: \mathbb{R}^d \rightarrow \mathbb{R}$ . Let  $(\varphi_n)_{n \in \mathbb{N}}$  be a sequence of functions  $\varphi_n: \mathbb{S}_+^d \times \mathbb{R}^d \rightarrow [0, \infty]$ , where  $\mathbb{S}_+^d$  consists of all symmetric positive semi-definite  $d \times d$ -matrices, and define

$$H_n(a, b) := \sup_{(a', b') \in \mathbb{S}_+^d \times \mathbb{R}^d} \left( \frac{1}{2} \text{tr}(aa') + b^T b' - \varphi_n(a', b') \right) \quad \text{for all } (a, b) \in \mathbb{R}^{d \times d} \times \mathbb{R}^d.$$

Here, we denote by  $\text{tr}(aa')$  the matrix trace and by  $b^T b'$  the euclidean inner product. Under the assumption that  $H_n \rightarrow H$ , we will show that a sequence of semigroups  $(S_n(t))_{t \geq 0}$ , whose generators are given by

$$(A_n f)(x) = H_n(D^2 f(x), Df(x)) \quad \text{for all } f \in C_b^\infty(\mathbb{R}^d),$$

converges to a semigroup  $(S(t))_{t \geq 0}$  with generator  $Af = H(D^2 f, Df)$ .

**Assumption 3.7.** Suppose that the following conditions are satisfied:

- (i) For every  $n \in \mathbb{N}$ , there exists  $(a, b) \in \mathbb{S}_+^d \times \mathbb{R}^d$  with  $\varphi_n(a, b) = 0$ .

(ii) The penalization functions grow superlinearly, i.e.,

$$\lim_{|a|+|b| \rightarrow \infty} \inf_{n \in \mathbb{N}} \frac{\varphi_n(a, b)}{|a| + |b|} = \infty.$$

(iii) There exists a function  $H: \mathbb{R}^{d \times d} \times \mathbb{R}^d \rightarrow \mathbb{R}$  with

$$\lim_{n \rightarrow \infty} \sup_{(a, b) \in K} |H_n(a, b) - H(a, b)| = 0 \quad \text{for all } K \Subset \mathbb{S}_+^d \times \mathbb{R}^d.$$

In particular, condition (ii) guarantees that  $(H_n)_{n \in \mathbb{N}}$  is a sequence of real-valued functions. Note that all the semigroups appearing in the following theorem are unique due to Theorem 2.4.

**Theorem 3.8.** *Suppose that Assumption 3.7 is satisfied. Then, there exists a sequence  $(S_n)_{n \in \mathbb{N}}$  of strongly continuous convex monotone semigroups  $(S_n(t))_{t \geq 0}$  on  $C_b(\mathbb{R}^d)$  with*

$$C_b^\infty(\mathbb{R}^d) \subset D(A_n) \quad \text{and} \quad A_n f = H_n(D^2 f, Df)$$

*for all  $n \in \mathbb{N}$  and  $f \in C_b^\infty(\mathbb{R}^d)$  satisfying Assumption 2.6. Hence, there exists another strongly continuous convex monotone semigroup  $(S(t))_{t \geq 0}$  on  $C_b(\mathbb{R}^d)$  with*

$$S(t)f = \lim_{n \rightarrow \infty} S_n(t)f \quad \text{for all } (f, t) \in C_b(\mathbb{R}^d) \times \mathbb{R}_+$$

*satisfying  $C_b^\infty(\mathbb{R}^d) \subset D(A)$  and  $Af = H(D^2 f, Df)$  for all  $f \in C_b^\infty(\mathbb{R}^d)$ .*

*Proof.* First, we construct the semigroups  $(S_n(t))_{t \geq 0}$  by applying Theorem 2.9. To do so, for every  $n \in \mathbb{N}$ ,  $t \geq 0$ ,  $f \in C_b(\mathbb{R}^d)$  and  $x \in \mathbb{R}^d$ , we define

$$(I_n(t)f)(x) := \sup_{(a, b) \in \mathbb{S}_+^d \times \mathbb{R}^d} (\mathbb{E}[f(x + \sqrt{a}W_t + bt)] - \varphi_n(a, b)t),$$

where  $\sqrt{a} \in \mathbb{S}_+^d$  denotes the unique matrix with  $\sqrt{a}\sqrt{a} = a$ . It is straightforward to show that the operators  $I_n(t): C_b(\mathbb{R}^d) \rightarrow C_b(\mathbb{R}^d)$  are well-defined and satisfy

- $I_n(t)$  is convex and monotone with  $I_n(t)0 = 0$  for all  $n \in \mathbb{N}$  and  $t \geq 0$ ,
- $\|I_n(t)f - I_n(t)g\|_\infty \leq \|f - g\|_\infty$  for all  $n \in \mathbb{N}$ ,  $t \geq 0$  and  $f, g \in C_b(\mathbb{R}^d)$ ,
- $I_n(t)(\tau_x f) = \tau_x I_n(t)f$  for all  $n \in \mathbb{N}$ ,  $t \geq 0$ ,  $f \in C_b(\mathbb{R}^d)$  and  $x \in \mathbb{R}^d$ ,
- $I_n(t): \text{Lip}_b(\mathbb{R}^d, r) \rightarrow \text{Lip}_b(\mathbb{R}^d, r)$  for all  $n \in \mathbb{N}$  and  $r, t \geq 0$ .

Let  $n \in \mathbb{N}$  and  $f \in C_b^\infty(\mathbb{R}^d)$ . Due to Assumption 3.7(ii), there exists  $r \geq 0$  with

$$\begin{aligned} H_n(D^2 f(x), Df(x)) &= \sup_{|a|+|b| \leq r} \left( \frac{1}{2} \text{tr}(aD^2 f(x)) + b^T Df(x) - \varphi_n(a, b) \right), \\ (I_n(t)f)(x) &= \sup_{|a|+|b| \leq r} (\mathbb{E}[f(x + \sqrt{a}W_t + bt)] - \varphi_n(a, b)t) \end{aligned}$$

for all  $t \geq 0$  and  $x \in \mathbb{R}^d$ . Hence, for every  $h > 0$ , Itô's formula implies

$$\frac{I_n(h)f - f}{h} = \sup_{|a|+|b| \leq r} \left( \mathbb{E} \left[ \frac{1}{h} \int_0^h \left( \frac{1}{2} \text{tr}(aD^2 f + b^T Df) \right) (\cdot + \sqrt{a}W_s + bs) ds \right] - \varphi_n(a, b) \right)$$

and the right-hand side converges to  $H_n(D^2 f, Df)$  as  $h \rightarrow 0$ . The previous equation also yields the estimate

$$\left\| \frac{I_n(h)f - f}{h} \right\|_\infty \leq \sup_{(a, b) \in \mathbb{S}_+^d \times \mathbb{R}^d} \left( \frac{1}{2} |a| \cdot \|D^2 f\|_\infty + |b| \cdot \|Df\|_\infty - \varphi_n(a, b) \right).$$

In the sequel, we fix  $n \in \mathbb{N}$  and choose a sequence  $(h_k)_{k \in \mathbb{N}} \subset (0, \infty)$  with  $h_k \rightarrow 0$ . Define  $I_{n,k}f := I_n(h_k)f$  for all  $f \in C_b(\mathbb{R}^d)$ . Due to the previous arguments and Corollary 2.14,

the sequence  $(I_{n,k})_{k \in \mathbb{N}}$  satisfies Assumption 2.8. Hence, by Theorem 2.9, there exists a strongly continuous convex monotone semigroup  $(S_n(t))_{t \geq 0}$  on  $C_b(\mathbb{R}^d)$  with  $S_n(t)0 = 0$  such that

$$S_n(t)f = \lim_{k \rightarrow \infty} (I_{n,k})^{m_k^t} f \quad \text{for all } (f, t) \in C_b(\mathbb{R}^d) \times \mathbb{R}_+,$$

where  $(m_k^t)_{k \in \mathbb{N}} \subset \mathbb{N}$  is an arbitrary sequence satisfying  $m_k^t h_k \rightarrow t$  as  $k \rightarrow \infty$ . Moreover, the following statements are valid:

- (i)  $C_b^\infty(\mathbb{R}^d) \subset D(A_n)$  and  $A_n f = H_n(D^2 f, Df)$  for all  $f \in C_b^\infty(\mathbb{R}^d)$ .
- (ii)  $\|S_n(t)f - S_n(t)g\|_\infty \leq \|f - g\|_\infty$  for all  $n \in \mathbb{N}$ ,  $t \geq 0$  and  $f, g \in C_b(\mathbb{R}^d)$ .
- (iii) For every  $\varepsilon > 0$ ,  $r, T \geq 0$  and  $K \in \mathbb{R}^d$ , there exist  $c \geq 0$  and  $K' \in \mathbb{R}^d$  with

$$\|S_n(t)f - S_n(t)g\|_{\infty, K} \leq c\|f - g\|_{\infty, K'} + \varepsilon$$

for all  $t \in [0, T]$  and  $f, g \in B_{C_b(\mathbb{R}^d)}(r)$ .

- (iv)  $S_n(t)(\tau_x f) = \tau_x S_n(t)f$  for all  $n \in \mathbb{N}$ ,  $t \geq 0$ ,  $f \in C_b(\mathbb{R}^d)$  and  $x \in \mathbb{R}^d$ .
- (v)  $S_n(t): \text{Lip}_b(\mathbb{R}^d, r) \rightarrow \text{Lip}_b(\mathbb{R}^d, r)$  for all  $n \in \mathbb{N}$  and  $r, t \geq 0$ .

In particular, due to Theorem 2.4, the semigroup  $(S_n(t))_{t \geq 0}$  does not depend on the choice of the sequence  $(h_k)_{k \in \mathbb{N}}$ .

Second, we note that the conditions (i), (iii) and (iv) of Assumption 2.2 are clearly satisfied and that Assumption 3.7(iii) yields condition (v). Moreover, Assumption 3.7(ii) and Corollary 2.16 guarantee that Assumption 2.2(ii) is also valid. Hence, the claim about the semigroup  $(S(t))_{t \geq 0}$  follows from Theorem 2.3. Again, due to Theorem 2.4, the latter is unique.  $\square$

**3.4. Randomized Euler schemes for Lipschitz ODEs.** Let  $\psi: \mathbb{R}^d \rightarrow \mathbb{R}^d$  be a bounded Lipschitz continuous function. The aim of this subsection is to show that the unique solution of the parameter dependent ordinary differential equation

$$\begin{cases} \partial_t u(t, x) = \psi(u(t, x)), & t \geq 0, \\ u(0, x) = x, & x \in \mathbb{R}^d, \end{cases} \quad (3.10)$$

can be approximated by a randomized Euler scheme under model uncertainty. For that purpose, let  $(h_n)_{n \in \mathbb{N}} \subset (0, \infty)$  be a sequence with  $h_n \rightarrow 0$  and let  $(\xi_n)_{n \in \mathbb{N}}$  be a sequence of i.i.d. random variables on a sublinear expectation space  $(\Omega, \mathcal{H}, \mathcal{E})$  satisfying  $\mathcal{E}[|\xi_1|^3] < \infty$  and  $\mathcal{E}[a\xi_1] = 0$  for all  $a \in \mathbb{R}^d$ . If  $\mathcal{E}[\cdot] = \mathbb{E}[\cdot]$  is a linear expectation, then there exists a unique probability measure  $\mu$  on  $\mathcal{B}(\mathbb{R}^d)$  with

$$\mathbb{E}[f(\xi_n)] = \int_{\mathbb{R}^d} f(\xi_n) d\mu \quad \text{for all } n \in \mathbb{N} \text{ and } f \in C_b(\mathbb{R}^d).$$

Furthermore, if the distribution  $\mu$  of the random variables  $(\xi_n)_{n \in \mathbb{N}}$  is uncertain, a worst case approach consists in taking the supremum over a set of measures, i.e.,

$$\mathcal{E}[f(\xi_n)] := \sup_{\mu \in M} \int_{\mathbb{R}^d} f(\xi_n) d\mu.$$

For more details, we refer to [77] and [11, Appendix B]. For every  $n \in \mathbb{N}$ ,  $\delta > 0$ ,  $t \geq 0$  and  $x \in \mathbb{R}^d$ , we recursively define a randomized explicit Euler scheme under model uncertainty by

$$X_0^{n, \delta, x} := x \quad \text{and} \quad X_{(k+1)h_n}^{n, \delta, x} := X_{kh_n}^{n, \delta, x} + h_n \psi(X_{kh_n}^{n, \delta, x}) + \delta \sqrt{h_n} \xi_{k+1}$$

and show that  $X_{k_n^t h_n}^{n, \delta_n, x} \rightarrow u(t, x)$  weakly for all  $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$ , where  $(\delta_n)_{n \in \mathbb{N}} \subset (0, \infty)$  and  $(k_n^t)_{n \in \mathbb{N}} \subset \mathbb{N}$  are arbitrary sequences with  $\delta_n \rightarrow 0$  and  $k_n^t h_n \rightarrow t$ . To be precise, we show that

$$\mathcal{E} \left[ f \left( X_{k_n^t h_n}^{n, \delta_n, x} \right) \right] \rightarrow f(u(t, x)) \quad \text{for all } f \in C_b(\mathbb{R}^d).$$

This means that, regardless of possible numerical errors, the Euler scheme still converges to the solution of (3.10). Furthermore, in the case that  $\delta_n \rightarrow \delta > 0$ , the Euler scheme converges to the solution of a stochastic differential equation which might be driven by a G-Brownian motion. We define

$$(I_{n, \delta} f)(x) := \mathcal{E} \left[ f \left( x + h_n \psi(x) + \delta \sqrt{h_n} \xi_1 \right) \right]$$

for all  $n \in \mathbb{N}$ ,  $\delta > 0$ ,  $f \in C_b(\mathbb{R}^d)$  and  $x \in \mathbb{R}^d$ . Let  $L$  be the Lipschitz constant of  $\psi$ .

**Theorem 3.9.** *Let  $(\delta_n)_{n \in \mathbb{N}} \subset (0, \infty)$  with  $\delta_n \rightarrow \delta \in \mathbb{R}_+$ . Then, there exists a strongly continuous convex monotone semigroup  $(S_\delta(t))_{t \geq 0}$  on  $C_b(\mathbb{R}^d)$  with  $S(t)0 = 0$  such that*

$$S_\delta(t)f = \lim_{n \rightarrow \infty} (I_{n, \delta_n})^{k_n^t} f_n = \lim_{n \rightarrow \infty} \mathcal{E} \left[ f \left( X_{k_n^t h_n}^{n, \delta_n, \cdot} \right) \right] \quad \text{for all } (f, t) \in C_b(\mathbb{R}^d) \times \mathbb{R}_+,$$

where  $(k_n^t)_{n \in \mathbb{N}} \subset \mathbb{N}$  is an arbitrary sequence with  $k_n^t h_n \rightarrow t$ . Moreover, the following statements are valid:

(i) It holds  $C_b^\infty(\mathbb{R}^d) \subset D(A_\delta)$  and

$$(A_\delta f)(x) = \frac{1}{2} \delta^2 \mathcal{E} [\xi_1^T D^2 f(x) \xi_1] + Df(x)^T \psi(x) \quad \text{for all } f \in C_b^\infty(\mathbb{R}^d).$$

(ii) It holds  $\|S_\delta(t)f - S_\delta(t)g\|_\infty \leq \|f - g\|_\infty$  for all  $t \geq 0$  and  $f, g \in C_b(\mathbb{R}^d)$ .

(iii) For every  $\varepsilon > 0$ ,  $r, T \geq 0$  and  $K \Subset \mathbb{R}^d$ , there exist  $c \geq 0$  and  $K' \Subset \mathbb{R}^d$  with

$$\|S_\delta(t)f - S_\delta(t)g\|_{\infty, K} \leq c\|f - g\|_{\infty, K'} + \varepsilon$$

for all  $t \in [0, T]$  and  $f, g \in B_{C_b(\mathbb{R}^d)}(r)$ .

(iv) For every  $r, t \geq 0$ ,  $f \in \text{Lip}_b(r)$  and  $x \in \mathbb{R}^d$ ,

$$\|S_\delta(t)(\tau_x f) - \tau_x S_\delta(t)f\|_\kappa \leq L e^{Lt} r t |x|.$$

Furthermore, it holds  $S_\delta(t) : \text{Lip}_b(r) \rightarrow \text{Lip}_b(e^{Lt}r)$  for all  $r, t \geq 0$ .

In particular, we obtain  $(S_\delta(t)f)(x) = \mathcal{E}[f(X_t^{\delta, x})]$  for all  $t \geq 0$ ,  $f \in C_b(\mathbb{R}^d)$  and  $x \in \mathbb{R}^d$ , where  $X_t^{\delta, x}$  denotes the unique solution of the parameter dependent ordinary (stochastic) differential equation

$$\begin{cases} dX_t^{\delta, x} = \psi(X_t^{\delta, x})dt + \delta dW_t, & t \geq 0, \\ X_0^{\delta, x} = x, & x \in \mathbb{R}^d, \end{cases} \quad (3.11)$$

which is driven by a G-Brownian motion  $(W_t)_{t \geq 0}$  with  $\mathcal{E}[W_1 W_1^T] = \mathcal{E}[\xi_1 \xi_1^T]$ .

*Proof.* In order to apply Theorem 2.9, we have to verify Assumption 2.8. It is straightforward to show that the operators  $I_{n, \delta_n} : C_b(\mathbb{R}^d) \rightarrow C_b(\mathbb{R}^d)$  are well-defined and satisfy

- $I_{n, \delta_n}$  is convex and monotone with  $I_{n, \delta_n}(t)0 = 0$  for all  $n \in \mathbb{N}$ ,
- $\|I_{n, \delta_n} f - I_{n, \delta_n} g\|_\infty \leq \|f - g\|_\infty$  for all  $n \in \mathbb{N}$  and  $f, g \in C_b(\mathbb{R}^d)$ ,
- $\|\tau_x I_{n, \delta_n} f - I_{n, \delta_n}(\tau_x f)\|_\infty \leq L r h_n |x|$  for all  $n \in \mathbb{N}$ ,  $f \in C_b(\mathbb{R}^d)$  and  $x \in \mathbb{R}^d$ ,
- $I_{n, \delta_n} : \text{Lip}_b(\mathbb{R}^d, r) \rightarrow \text{Lip}_b(\mathbb{R}^d, (1 + L h_n)r)$  for all  $n \in \mathbb{N}$  and  $r \geq 0$ .

Moreover, for every  $n \in \mathbb{N}$ ,  $f \in C_b^\infty(\mathbb{R}^d)$  and  $x \in \mathbb{R}^d$ , Taylor's formula yields

$$\begin{aligned} & f(x + h_n \psi(x) + \delta_n \sqrt{h_n} \xi_1) \\ &= f(x) + Df(x)^T (h_n \psi(x) + \delta_n \sqrt{h_n} \xi_1) \\ & \quad + \frac{1}{2} (h_n \psi(x) + \delta_n \sqrt{h_n} \xi_1)^T D^2 f(x) (h_n \psi(x) + \sqrt{h_n} \delta_n \xi_1) + (R_n(f, x, \xi_1), \end{aligned}$$

where the reminder term can be estimated by

$$|R_n(f, x, \xi_1)| \leq \frac{|h_n \psi(x) + \delta_n \sqrt{h_n} \xi_1|^3}{6} \|D^3 f\|_\infty.$$

Since  $\mathcal{E}[a\xi_1] = 0$  for all  $a \in \mathbb{R}^d$ , we obtain

$$\begin{aligned} \frac{(I_{n,\delta_n} f - f)(x)}{h_n} &= \mathcal{E} \left[ \frac{f(x + h_n \psi(x) + \delta_n \sqrt{h_n} \xi_1) - f(x)}{h_n} \right] \\ &= Df(x)^T \psi(x) + \frac{\delta_n^2}{2} \mathcal{E} [\xi^T D^2 f(x) \xi] + (\tilde{R}_n(f, x, \xi_1), \end{aligned}$$

where the reminder term can be estimated by

$$|\tilde{R}_n(f, x, \xi_1)| \leq \frac{1}{2} h_n |\psi(x)^T D^2 f(x) \psi(x)| + \frac{\mathcal{E}[|R_n(f, x, \xi_1)|]}{h_n}.$$

Since  $\mathcal{E}[|\xi_1|^3] < \infty$  and  $\|\psi\|_\infty < \infty$ , we obtain

$$\frac{I_{n,\delta_n} f - f}{h_n} \rightarrow Df(\cdot)^T \psi + \frac{\delta^2}{2} \mathcal{E} [\xi^T D^2 f(\cdot) \xi] \quad \text{for all } f \in C_b^\infty(\mathbb{R}^d).$$

Moreover, the previous estimates and Corollary 2.14 guarantee that Assumption 2.8(ii) is satisfied and, since the random variables  $(\xi_n)_{n \in \mathbb{N}}$  are iid, we obtain

$$(I_{n,\delta_n})^k f_n = \mathcal{E} \left[ f \left( X_{kh_n}^{n,\delta_n,x} \right) \right]$$

for all  $k, n \in \mathbb{N}$ ,  $f \in C_b(\mathbb{R}^d)$  and  $x \in \mathbb{R}^d$ . Finally, by relying on the strong Markov property for stochastic differential equations driven by G-Brownian motions [50], one can show that  $(T_\delta(t)f)(x) := \mathcal{E}[f(X_t^{\delta,x})]$  defines a strongly continuous monotone semigroup on  $C_b(\mathbb{R}^d)$  satisfying the conditions of Theorem 2.4. Now, the claim follows from Theorem 2.9 and Theorem 2.4.  $\square$

**Corollary 3.10.** *Let  $\delta \geq 0$  and  $(\delta_n)_{n \in \mathbb{N}} \subset \mathbb{R}_+$  be a sequence with  $\delta_n \rightarrow \delta$ . Then,*

$$S_\delta(t)f = \lim_{n \rightarrow \infty} S_{\delta_n}(t)f \quad \text{for all } t \geq 0 \text{ and } f \in C_b(\mathbb{R}^d).$$

*Proof.* Theorem 3.9 and Corollary 2.16 guarantee that Assumption 2.2 is satisfied. Indeed, it follows from the proof of Theorem 3.9 that

$$\lim_{n \rightarrow \infty} \left\| \frac{I_{n,\delta} f - f}{h_n} - A_\delta f \right\|_\infty = 0 \quad \text{for all } f \in C_b^\infty(\mathbb{R}^d)$$

and therefore [12, Theorem 4.3] implies

$$\lim_{h \downarrow 0} \left\| \frac{S_\delta(h)f - f}{h} - A_\delta f \right\|_\infty = 0 \quad \text{for all } f \in C_b^\infty(\mathbb{R}^d).$$

Hence, the claim follows from Theorem 2.3.  $\square$

**3.5. Large deviations for randomized Euler schemes.** We continue our analysis of randomized Euler schemes from the previous subsection and focus on the convergence rate of the scheme by means of a large deviations approach, see [29, 32, 40, 82]. While the choice of the parameters  $\delta_n$  and  $h_n$  is still arbitrary, the choice of the additional parameter  $\alpha_n$  depends on  $\delta_n$  and  $h_n$ . Moreover, we obtain a large deviations result for stochastic differential equations in the spirit of Freidlin–Wentzell [43]. Let  $(h_n)_{n \in \mathbb{N}}$ ,  $(\delta_n)_{n \in \mathbb{N}}$  and  $(\alpha_n)_{n \in \mathbb{N}}$  be sequences in  $(0, \infty)$ . Moreover, let  $(\xi_n)_{n \in \mathbb{N}}$  be a sequence of i.i.d. random variables on a sublinear expectation space  $(\Omega, \mathcal{H}, \mathcal{E})$  satisfying  $\mathcal{E}[|\xi_1|^3] < \infty$  and  $\mathcal{E}[a\xi_1] = 0$  for all  $a \in \mathbb{R}^d$ . For every  $n \in \mathbb{N}$  and  $f \in C_b(\mathbb{R}^d)$ , we define

$$J_n f := \frac{1}{\alpha_n} \log (I_n e^{\alpha_n f}),$$

where  $I_n: C_b(\mathbb{R}^d) \rightarrow C_b(\mathbb{R}^d)$  is given by

$$(I_n f)(x) := \mathcal{E} \left[ f \left( x + h_n \psi(x) + \delta_n \sqrt{h_n} \xi_1 \right) \right].$$

In addition, for every  $n \in \mathbb{N}$ ,  $\delta > 0$ ,  $t \geq 0$  and  $x \in \mathbb{R}^d$ , we recursively define

$$X_0^{n, \delta, x} := x \quad \text{and} \quad X_{(k+1)h_n}^{n, \delta, x} := X_{kh_n}^{n, \delta, x} + h_n \psi \left( X_{kh_n}^{n, \delta, x} \right) + \delta \sqrt{h_n} \xi_{k+1}.$$

**Theorem 3.11.** *Assume that there exists  $\delta, \gamma \geq 0$  with  $\delta_n \rightarrow \delta$  and  $\alpha_n \delta_n^2 \rightarrow \gamma$ . Furthermore, let  $h_n \rightarrow 0$  and  $\alpha_n h_n \rightarrow 0$ . Then, there exists a strongly continuous convex monotone semigroup  $(T_{\gamma, \delta}(t))_{t \geq 0}$  on  $C_b(\mathbb{R}^d)$  with  $T_{\gamma, \delta}(t)0 = 0$  such that*

$$T_{\gamma, \delta}(t)f = \lim_{n \rightarrow \infty} J_n^{k_n^t} f = \lim_{n \rightarrow \infty} \frac{1}{\alpha_n} \log \left( \mathcal{E} \left[ \exp \left( \alpha_n f \left( X_{k_n^t h_n}^{n, \delta_n, \cdot} \right) \right) \right] \right)$$

for all  $t \geq 0$ ,  $f \in C_b(\mathbb{R}^d)$  and sequences  $(k_n^t)_{n \in \mathbb{N}} \subset \mathbb{N}$  with  $k_n^t \rightarrow t$ . Moreover, the following statements are valid:

(i) It holds  $C_b^\infty(\mathbb{R}^d) \subset D(B_{\gamma, \delta})$  and

$$(B_{\gamma, \delta} f)(x) = \frac{1}{2} \mathcal{E} [\delta^2 \xi_1^T D^2 f(x) \xi_1 + \gamma |Df(x)^T \xi_1|^2] + Df(x)^T \psi(x)$$

for all  $f \in C_b^\infty(\mathbb{R}^d)$  and  $x \in \mathbb{R}^d$ .

(ii) It holds  $\|T_{\gamma, \delta}(t)f - T_{\gamma, \delta}(t)g\|_\infty \leq \|f - g\|_\infty$  for all  $t \geq 0$  and  $f, g \in C_b(\mathbb{R}^d)$ .

(iii) For every  $\varepsilon > 0$ ,  $r, T \geq 0$  and  $K \Subset \mathbb{R}^d$ , there exist  $c \geq 0$  and  $K' \Subset \mathbb{R}^d$  with

$$\|T_{\gamma, \delta}(t)f - T_{\gamma, \delta}(t)g\|_{\infty, K} \leq c\|f - g\|_{\infty, K'} + \varepsilon$$

for all  $t \in [0, T]$  and  $f, g \in B_{C_b(\mathbb{R}^d)}(r)$ .

(iv) For every  $r, t \geq 0$ ,  $f \in \text{Lip}_b(r)$  and  $x \in \mathbb{R}^d$ ,

$$\|T_{\gamma, \delta}(t)(\tau_x f) - \tau_x T_{\gamma, \delta}(t)f\|_\kappa \leq L e^{Lt} r t |x|.$$

Furthermore, it holds  $T_{\gamma, \delta}(t): \text{Lip}_b(r) \rightarrow \text{Lip}_b(e^{Lt}r)$  for all  $r, t \geq 0$ .

*Proof.* By relying on the estimates in the proof of Theorem 3.9, it is straightforward to show that the operators  $J_n: C_b(\mathbb{R}^d) \rightarrow C_b(\mathbb{R}^d)$  are well-defined and satisfy

- $J_n$  is convex and monotone with  $J_n(t)0 = 0$  for all  $n \in \mathbb{N}$ ,
- $\|J_n f - J_n g\|_\infty \leq \|f - g\|_\infty$  for all  $n \in \mathbb{N}$  and  $f, g \in C_b(\mathbb{R}^d)$ ,
- $\|\tau_x J_n f - J_n(\tau_x f)\|_\infty \leq L r h_n |x|$  for all  $n \in \mathbb{N}$ ,  $f \in C_b(\mathbb{R}^d)$  and  $x \in \mathbb{R}^d$ ,
- $J_n: \text{Lip}_b(\mathbb{R}^d, r) \rightarrow \text{Lip}_b(\mathbb{R}^d, (1 + L h_n)r)$  for all  $n \in \mathbb{N}$  and  $r \geq 0$ .



Indeed, one can show that  $J_n(f + c) = J_nf + c$  and  $J_n(\lambda f + (1 - \lambda)g) \leq \max\{J_nf, J_ng\}$  for all  $n \in \mathbb{N}$ ,  $f, g \in C_b(\mathbb{R}^d)$ ,  $c \in \mathbb{R}$  and  $\lambda \in [0, 1]$ . Hence,

$$\begin{aligned} & (J_n(\lambda f + (1 - \lambda)g))(x) - \lambda(J_nf)(x) - (1 - \lambda)(J_ng)(x) \\ &= \left( J_n(\lambda(f - (J_nf)(x)) + (1 - \lambda)(g - (J_ng)(x))) \right)(x) \\ &\leq \max \left\{ (J_n(f - (J_nf)(x)))(x), (J_n(g - (J_ng)(x)))(x) \right\} = 0 \end{aligned}$$

for all  $n \in \mathbb{N}$ ,  $f, g \in C_b(\mathbb{R}^d)$  and  $\lambda \in [0, 1]$  which shows that  $J_n$  is convex. Moreover,

$$J_nf \leq J_n(g + \|f - g\|_\infty) = J_ng + \|f - g\|_\infty$$

and reversing the roles of  $f$  and  $g$  yields  $\|J_nf - J_ng\|_\infty \leq \|f - g\|_\infty$ . The remaining properties can be obtained similarly.

For every  $n \in \mathbb{N}$ ,  $f \in C_b^\infty(\mathbb{R}^d)$  and  $x \in \mathbb{R}^d$ , Taylor's formula yields

$$\begin{aligned} & \exp \left( \alpha_n (f(x + h_n \psi(x) + \delta_n \sqrt{h_n} \xi) - f(x)) \right) \\ &= 1 + \alpha_n Df(x)^T (h_n \psi(x) + \delta_n \sqrt{h_n} \xi_1) + \frac{\alpha_n^2}{2} |Df(x)^T (h_n \psi(x) + \delta_n \sqrt{h_n} \xi_1)|^2 \\ &\quad + \frac{\alpha_n}{2} (h_n \psi(x) + \delta_n \sqrt{h_n} \xi_1^T D^2 f(x) (h_n \psi(x) + \delta_n \sqrt{h_n} \xi_1) + R_n(f, x, \xi_1), \end{aligned}$$

where the reminder term can be estimated by

$$|R_n(f, x, \xi_1)| \leq \frac{\alpha_n |h_n \psi(x) + \delta_n \sqrt{h_n} \xi_1|^3}{6} \rho \left( \sum_{i=1}^3 \|D^i f\|_\infty \right)$$

for a suitable function  $\rho: \mathbb{R}_+ \rightarrow \mathbb{R}_+$  which satisfies  $\rho(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . We use the fact that  $\mathcal{E}[a\xi_1] = 0$  for all  $a \in \mathbb{R}^d$  to conclude

$$\begin{aligned} & \mathcal{E} \left[ \exp \left( \alpha_n (f(x + h_n \psi(x) + \delta_n \sqrt{h_n} \xi_1) - f(x)) \right) \right] \\ &= 1 + \alpha_n h_n Df(x)^T \psi(x) + \frac{\alpha_n^2 h_n^2}{2} |Df(x)^T \psi(x)|^2 + \frac{\alpha_n h_n^2}{2} \psi(x)^T D^2 f(x) \psi(x) \\ &\quad + \frac{\alpha_n h_n}{2} \mathcal{E} [\alpha_n \delta_n^2 |Df(x)^T \psi(x)| + \delta_n^2 \xi_1^T D^2 f(x) \xi_1] + \tilde{R}_n(f, x, \xi_1), \end{aligned}$$

where the reminder term can be estimated by  $|\tilde{R}_n(f, x, \xi_1)| \leq \mathcal{E}[|R_n(f, x, \xi_1)|]$ . Moreover, by assumption, it holds  $h_n \rightarrow 0$  and  $\alpha_n h_n \rightarrow 0$ . Hence, Taylor's formula implies

$$\begin{aligned} \frac{J_nf - f}{h_n} &= \frac{1}{\alpha_n h_n} \log \left( \mathcal{E} \left[ \exp \left( \alpha_n (f(x + h_n \psi(x) + \delta_n \sqrt{h_n} \xi_1) - f(x)) \right) \right] \right) \\ &= Df(x)^T \psi(x) + \frac{\alpha_n h_n}{2} |Df(x)^T \psi(x)|^2 + \frac{h_n}{2} \psi(x)^T D^2 f(x) \psi(x) \\ &\quad + \frac{1}{2} \mathcal{E} [\alpha_n \delta_n |Df(x)^T \xi_1|^2 + \delta_n^2 \xi_1^T D^2 f(x) \xi_1] + \hat{R}_n(f, x, \xi_1), \end{aligned}$$

where the reminder term can be estimated by

$$|\hat{R}_n(f, x, \xi_1)| \leq \hat{\rho} \left( \sum_{i=1}^3 \|D^i f\|_\infty \right) r_n$$

for a suitable function  $\hat{\rho}: \mathbb{R}_+ \rightarrow \mathbb{R}_+$  satisfying  $\hat{\rho}(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$  and a sequence  $(r_n)_{n \in \mathbb{N}} \subset \mathbb{R}_+$  with  $r_n \rightarrow 0$ . Since  $\delta_n \rightarrow \delta$ ,  $\alpha_n \delta_n^2 \rightarrow \gamma$ ,  $h_n \rightarrow 0$  and  $\alpha_n h_n \rightarrow 0$ , we obtain

$$\lim_{n \rightarrow \infty} \left\| \frac{J_nf - f}{h_n} - \frac{1}{2} \mathcal{E} [\delta^2 \xi_1^T D^2 f(x) \xi_1 + \gamma |Df(x)^T \xi_1|^2] - Df(x)^T \psi(x) \right\|_\infty = 0$$

for all  $f \in C_b^\infty(\mathbb{R}^d)$ . In addition, the previous estimates and Corollary 2.14 guarantee that Assumption 2.8(ii) is satisfied and, since the random variables  $(\xi_n)_{n \in \mathbb{N}}$  are iid, we obtain

$$(J_n^k f)(x) = \frac{1}{\alpha_n} \log \left( \mathcal{E} \left[ \exp \left( \alpha_n f \left( X_{k_n^t h_n}^{n, \delta_n, x} \right) \right) \right] \right)$$

for all  $k, n \in \mathbb{N}$ ,  $f \in C_b(\mathbb{R}^d)$  and  $x \in \mathbb{R}^d$ . Now, the claim follows from Theorem 2.9.  $\square$

If the limit is a Hamilton–Jacobi semigroup, we obtain the following Laplace principle.

**Corollary 3.12.** *Let  $(\delta_n)_{n \in \mathbb{N}} \subset (0, \infty)$  be a sequence with  $\delta_n \rightarrow 0$  and  $\delta_n^2 n \rightarrow \infty$ . Then,*

$$\delta_n^2 \log \mathcal{E} \left[ \exp \left( \frac{1}{\delta_n^2} f \left( \frac{\delta_n}{\sqrt{n}} \sum_{i=1}^n \xi_i \right) \right) \right] \rightarrow \sup_{y \in \mathbb{R}^d} (f(y) - \varphi(y))$$

for all  $f \in C_b(\mathbb{R}^d)$ , where the rate function  $\varphi: \mathbb{R}^d \rightarrow [0, \infty]$  is given by

$$\varphi(y) := \sup_{z \in \mathbb{R}^d} (y^T z - \frac{1}{2} \mathcal{E}[|z^T \xi_1|^2]).$$

*Proof.* Applying Theorem 3.11 with  $h_n := 1/n$ ,  $\alpha_n := 1/\delta_n^2$ , and  $\psi \equiv 0$  yields

$$T_{1,0}(t)f = \lim_{n \rightarrow \infty} \delta_n^2 \log \mathcal{E} \left[ \exp \left( \frac{1}{\delta_n^2} f \left( \cdot + \frac{\delta_n}{\sqrt{n}} \sum_{i=1}^n \xi_i \right) \right) \right] \quad \text{for all } f \in C_b(\mathbb{R}^d),$$

$$B_{1,0}f = \frac{1}{2} \mathcal{E}[|Df(x)^T \xi_1|^2] \quad \text{for all } f \in C_b^\infty(\mathbb{R}^d).$$

In addition, similar to the proof of [11, Theorem 3.4], one can show that  $(T_{1,0}(t))_{t \geq 0}$  can be represented by the Hopf–Lax formula

$$(T_{1,0}(t)f)(x) = \sup_{x \in \mathbb{R}^d} (f(x + ty) - \varphi(y)t) \quad \text{for all } t \geq 0 \text{ and } f \in C_b(\mathbb{R}^d). \quad \square$$

Moreover, we obtain a Laplace principle corresponding to the Freidlin–Wentzell large deviations principle [43] which characterizes the convergence rate of the solution  $X_t^{\delta, x}$  of the SDE (3.11), driven by a standard Brownian motion, to the solution  $X_t^{0, x} = u(t, x)$  of the ODE (3.10). In the sublinear case, where the solution  $X_t^{\delta, x}$  of the SDE (3.11) is driven by a G-Brownian motion, we refer to [20, 46] for corresponding large deviations principles.

**Corollary 3.13.** *Let  $(\delta_n)_{n \in \mathbb{N}} \subset (0, \infty)$  be a sequence with  $\delta_n \rightarrow 0$ . Then,*

$$\delta_n^2 \log \left( \mathcal{E} \left[ \exp \left( \frac{1}{\delta_n^2} f \left( X_t^{\delta_n, x} \right) \right) \right] \right) \rightarrow \sup_{y \in \mathbb{R}^d} (f(y) - \varphi(y))$$

for all  $f \in C_b(\mathbb{R}^d)$ , where the rate function  $\varphi: \mathbb{R}^d \rightarrow [0, \infty]$  is given by

$$\varphi(y) := \sup_{z \in \mathbb{R}^d} (y^T z - \frac{1}{2} \mathcal{E}[|z^T \xi_1|^2]).$$

*Proof.* For every  $n \in \mathbb{N}$ , we denote by  $(S_{\delta_n}(t))_{t \geq 0}$  the corresponding semigroup from Theorem 3.9. Moreover, for a fixed  $n \in \mathbb{N}$ , we choose  $\alpha_k := 1/\delta_n^2$  for all  $k \in \mathbb{N}$  and choose a sequence  $(h_k)_{k \in \mathbb{N}} \subset (0, \infty)$  with  $h_k \rightarrow \infty$ . Define

$$(I_{k, \delta_n} f)(x) := \mathcal{E} \left[ f \left( x + h_n \psi(x) + \delta_n \sqrt{h_n} \xi_1 \right) \right] \quad \text{and} \quad J_{k, \delta_n} := \frac{1}{\alpha_k} \log (I_{k, \delta_n} e^{\alpha_k f})$$

for all  $k, n \in \mathbb{N}$ ,  $f \in C_b(\mathbb{R}^d)$  and  $x \in \mathbb{R}^d$ . Theorem 3.9 and Theorem 3.11 imply

$$\begin{aligned} \delta_n^2 \log \left( \mathcal{E} \left[ \exp \left( \frac{1}{\delta_n^2} f \left( X_t^{\delta, \cdot} \right) \right) \right] \right) &= \delta_n^2 \log \left( S_{\delta_n}(t) e^{f/\delta_n^2} \right) \\ &= \lim_{k \rightarrow \infty} \delta_n^2 \log \left( (I_{k, \delta_n}^{m_k^t} e^{f/\delta_n^2}) \right) \\ &= \lim_{k \rightarrow \infty} (J_{k, \delta_n})^{m_k^t} f = T_{1, \delta_n}(t) f \end{aligned}$$

for all  $t \geq 0$ ,  $f \in C_b(\mathbb{R}^d)$  and sequences  $(m_k^t)_{k \in \mathbb{N}} \subset \mathbb{N}$  with  $m_k^t h_k \rightarrow t$ . Furthermore, Theorem 3.11 and Corollary 2.16 guarantee that Assumption 2.2 is satisfied. Indeed, it follows from the proof of Theorem 3.11 and [12, Theorem 4.3] that

$$\lim_{h \downarrow 0} \left\| \frac{T_{1,0}(h)f - f}{h} - B_{1,0}f \right\|_\infty = 0 \quad \text{for all } f \in C_b^\infty(\mathbb{R}^d).$$

Hence, the claim follows from Theorem 2.3 and the representation of  $(T_{1,0}(t))_{t \geq 0}$  by the Hopf–Lax formula from Corollary 3.12.  $\square$

**3.6. Discretization of stochastic optimal control problems.** In this subsection, we fix a Borel measurable function  $\varphi: \mathbb{S}_+^d \times \mathbb{R}^d \rightarrow [0, \infty]$  and consider the value function of a dynamic stochastic control problem with finite time horizon given by

$$(T(t)f)(x) := \sup_{(a,b) \in \mathcal{A}} \left( \mathbb{E} \left[ f \left( x + \int_0^t \sqrt{a_s} dW_s + \int_0^t b_s ds \right) \right] - \mathbb{E} \left[ \int_0^t \varphi(a_s, b_s) ds \right] \right),$$

where  $(W_t)_{t \geq 0}$  is a  $d$ -dimensional standard Brownian motion on a complete filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$  satisfying the usual conditions and  $\mathcal{A}$  consists of all predictable processes  $(a, b): \Omega \times \mathbb{R}_+ \rightarrow \mathbb{S}_+^d \times \mathbb{R}^d$  such that

$$\mathbb{E} \left[ \int_0^t |a_s| + |b_s| ds \right] < \infty \quad \text{for all } t \geq 0.$$

The aim of this subsection is to approximate  $T(t)f$  by iterating a sequence of discrete static control problems given by

$$(I_n f)(x) := \sup_{(a,b) \in \mathcal{A}_n} \left( \mathbb{E} [f(x + \sqrt{a h_n} \xi_n + b h_n)] - \varphi_n(a, b) h_n \right),$$

where  $\mathcal{A}_n \subset \mathbb{S}_+^d \times \mathbb{R}^d$ ,  $x \in X_n \subset \mathbb{R}^d$  and  $\varphi_n: \mathcal{A}_n \rightarrow [0, \infty]$ . To be precise, we show that

$$T(t)f = \lim_{n \rightarrow \infty} I_n^{k_n^t} f \quad \text{for all } (f, t) \in C_b(\mathbb{R}^d) \times \mathbb{R}_+,$$

where  $h_n \rightarrow 0$  and  $k_n^t h_n \rightarrow t$ .

Subsequently, we formalize the definition of the operators  $(I_n)_{n \in \mathbb{N}}$  and impose sufficient conditions to guarantee the convergence. Let  $(h_n)_{n \in \mathbb{N}} \subset (0, \infty)$  be a sequence with  $h_n \rightarrow 0$  and define  $\mathcal{T}_n := \{k h_n : k \in \mathbb{N}_0\}$  for all  $n \in \mathbb{N}$ . Moreover, let  $(X_n)_{n \in \mathbb{N}}$  be a sequence of closed sets  $X_n \subset \mathbb{R}^d$  with  $x + y \in X_n$  for all  $x, y \in X_n$ , such that, for every  $x \in \mathbb{R}^d$ , there exists  $x_n \in X_n$  with  $x_n \rightarrow x$ . We choose random variables  $\xi_n: \Omega \rightarrow \mathbb{R}^d$  with  $\mathbb{E}[\xi_n] = 0$ ,  $\mathbb{E}[\xi_n \xi_n^T] = \mathbf{1} \in \mathbb{R}^{d \times d}$  and  $\sup_{n \in \mathbb{N}} \mathbb{E}[|\xi_n|^3] < \infty$  and subsets  $\mathcal{A}_n \subset \mathbb{S}_+^d \times \mathbb{R}^d$ . Suppose that

$$x + \sqrt{a h_n} \xi_n(\omega) + b h_n \in X_n$$

for all  $n \in \mathbb{N}$ ,  $x, y \in X_n$ ,  $(a, b) \in \mathcal{A}_n$  and  $\omega \in \Omega$ . Finally, let  $(\varphi_n)_{n \in \mathbb{N}}$  be a sequence of functions  $\varphi_n: \mathcal{A}_n \rightarrow [0, \infty]$  and define

$$\begin{aligned}\varphi_n^*: \mathbb{R}^{d \times d} \times \mathbb{R}^d &\rightarrow \mathbb{R}, (a, b) \mapsto \sup_{(a', b') \in \mathcal{A}_n} \left( \frac{1}{2} \operatorname{tr}(aa') + bb' - \varphi_n(a', b') \right), \\ \varphi^*: \mathbb{R}^{d \times d} \times \mathbb{R}^d &\rightarrow \mathbb{R}, (a, b) \mapsto \sup_{(a', b') \in \mathbb{S}_+^d \times \mathbb{R}^d} \left( \frac{1}{2} \operatorname{tr}(aa') + bb' - \varphi(a', b') \right).\end{aligned}$$

**Assumption 3.14.** Suppose that the following conditions are satisfied:

- (i) There exists  $(a^*, b^*) \in \mathbb{S}_+^d \times \mathbb{R}^d$  with  $\varphi(a^*, b^*) = 0$ .
- (ii) It holds  $\lim_{|a|+|b| \rightarrow \infty} \frac{\varphi(a, b)}{|a|+|b|} = \infty$ .
- (iii) For every  $n \in \mathbb{N}$ , there exists  $(a_n^*, b_n^*) \in \mathcal{A}_n$  with  $\varphi_n(a_n^*, b_n^*) = 0$ . Furthermore, it holds  $\sup_{n \in \mathbb{N}} (|a_n^*| + |b_n^*|) < \infty$ .
- (iv) It holds  $\lim_{|a|+|b| \rightarrow \infty} \inf_{n \in \mathbb{N}} \frac{\varphi_n(a, b)}{|a|+|b|} = \infty$ .
- (v) It holds  $\lim_{n \rightarrow \infty} \sup_{|a|+|b| \leq r} |\varphi_n^*(a, b) - \varphi^*(a, b)| = 0$  for all  $r \geq 0$ .

**Theorem 3.15.** Suppose that Assumption 3.14 is satisfied. Then, there exists a strongly continuous convex monotone semigroup  $(S(t))_{t \geq 0}$  on  $C_b(\mathbb{R}^d)$  with  $S(t)0 = 0$  given by

$$S(t)f = \lim_{n \rightarrow \infty} I_n^{k_n^t} f \quad \text{for all } (f, t) \in C_b(\mathbb{R}^d) \times \mathbb{R}_+,$$

where  $(k_n^t)_{n \in \mathbb{N}} \subset \mathbb{N}$  is an arbitrary sequence with  $k_n^t h_n \rightarrow t$ , such that

$$C_b^\infty(\mathbb{R}^d) \subset D(A) \quad \text{and} \quad Af = \sup_{(a, b) \in \mathbb{S}_+^d \times \mathbb{R}^d} \left( \frac{1}{2} \operatorname{tr}(aD^2f(x)) + b^T Df(x) - \varphi(a, b) \right)$$

for all  $f \in C_b^\infty(\mathbb{R}^d)$ . Moreover, the conditions of Theorem 2.4 are satisfied and therefore

$$S(t)f = T(t)f \quad \text{for all } t \geq 0 \text{ and } f \in C_\kappa(\mathbb{R}^d).$$

*Proof.* In order to apply Theorem 2.9, we have to verify Assumption 2.8. It is straightforward to show that the operators  $I_n: C_b(X_n) \rightarrow C_b(X_n)$  are well-defined and satisfy

- $I_n$  is convex and monotone with  $I_n 0 = 0$  for all  $n \in \mathbb{N}$ ,
- $\|I_n f - I_n g\|_{\infty, X_n} \leq \|f - g\|_{\infty, X_n}$  for all  $n \in \mathbb{N}$  and  $f, g \in C_b(X_n)$ ,
- $\tau_x I_n f = I_n(\tau_x f)$  for all  $n \in \mathbb{N}$ ,  $f \in C_b(X_n)$  and  $x \in X_n$ ,
- $I_n: \operatorname{Lip}_b(X_n, r) \rightarrow \operatorname{Lip}_b(X_n, r)$  for all  $n \in \mathbb{N}$  and  $r \geq 0$ .

Now, let  $f \in C_b^\infty(\mathbb{R}^d)$ . For every  $n \in \mathbb{N}$  and  $x \in X_n$ ,

$$\begin{aligned}& f(x + \sqrt{ah_n}\xi_n + bh_n) - f(x) \\&= \int_0^1 Df(x + t(\sqrt{ah_n}\xi_n + bh_n))^T (\sqrt{ah_n}\xi_n + bh_n) dt \\&= \int_0^1 Df(x + t(\sqrt{ah_n}\xi_n + bh_n))bh_n dt + \int_0^1 Df(x + tbh_n)^T \sqrt{ah_n}\xi_n dt \\&\quad + \int_0^1 \int_0^1 \sqrt{ah_n}\xi_n^T D^2 f(x + st\sqrt{ah_n}\xi_n + tbh_n) \sqrt{ah_n}\xi_n t ds dt.\end{aligned}$$

Since  $\mathbb{E}[\xi_n] = 0$  and  $\mathbb{E}[|\xi_n|^2] = d$ , we obtain

$$(I_n^{a, b} f - f)(x) - \varphi_n(a, b)h_n \leq \left( \frac{1}{2} d|a| \cdot \|D^2 f\|_\infty + |b| \cdot \|Df\|_\infty - \varphi_n(a, b) \right) h_n$$

for all  $n \in \mathbb{N}$ ,  $(a, b) \in \mathcal{A}_n$  and  $x \in X_n$ , where  $(I_n^{a,b}f)(x) := \mathbb{E}[f(x + \sqrt{ah_n}\xi_n + bh_n)]$ . Hence, due to Assumption 3.14(iv), there exists  $r \geq 0$  with

$$I_nf = \sup_{(a,b) \in \mathcal{A}_n^r} (I_n^{a,b}f - \varphi_n(a, b)h_n) \quad \text{for all } n \in \mathbb{N}, \quad (3.12)$$

where  $\mathcal{A}_n^r := \{(a, b) \in \mathcal{A}_n : |a| + |b| \leq r\}$ . By increasing  $r \geq 0$ , we can assume that

$$\varphi_n^*(D^2f(x), Df(x)) = \sup_{(a,b) \in \mathcal{A}_n^r} \left( \frac{1}{2} \text{tr}(aD^2f(x)) + b^T Df(x) - \varphi_n(a, b) \right) \quad (3.13)$$

for all  $n \in \mathbb{N}$  and  $x \in \mathbb{R}^d$ . For every  $n \in \mathbb{N}$ ,  $(a, b) \in \mathcal{A}_n^r$  and  $x \in X_n$ , Taylor's formula yields

$$\begin{aligned} f(x + \sqrt{ah_n}\xi_n + bh_n) &= f(x) + (\sqrt{ah_n}\xi_n + bh_n)^T Df(x) \\ &\quad + \frac{1}{2} (\sqrt{ah_n}\xi_n + bh_n)^T D^2f(x) (\sqrt{ah_n}\xi_n + bh_n) + R_n(f, x, \xi_n), \end{aligned}$$

where the remainder term can be estimated by

$$|R_n(f, x, \xi_n)| \leq \frac{|\sqrt{ah_n}\xi_n + bh_n|^3}{6} \|D^3f\|_\infty.$$

It follows from  $\mathbb{E}[\xi_n] = 0$  and  $\mathbb{E}[\xi_n \xi_n^T] = \mathbf{1}$  that

$$\begin{aligned} \frac{I_n^{a,b}f - f}{h_n} &= \mathbb{E} \left[ \frac{f(x + \sqrt{ah_n}\xi_n + bh_n) - f(x)}{h_n} \right] \\ &= b^T Df(x) + \frac{1}{2} \text{tr}(aD^2f(x)) + \tilde{R}_n(f, x, \xi_n), \end{aligned}$$

where the remainder term can be estimated by

$$|\tilde{R}_n(f, x, \xi_n)| \leq \frac{1}{2} |b|^2 h_n |D^2f(x)| + \mathbb{E}[|R_n(f, x, \xi_n)|].$$

Hence, we can use equation (3.12), equation (3.13), the condition  $\sup_{n \in \mathbb{N}} \mathbb{E}[|\xi_n|^3] < \infty$  and Assumption 3.14(v) to obtain

$$\lim_{n \rightarrow \infty} \left\| \frac{I_nf - f}{h_n} - \varphi^*(D^2f, Df) \right\|_\infty = 0 \quad \text{for all } f \in C_b^\infty(\mathbb{R}^d).$$

Moreover, the previous estimates and Corollary 2.14 guarantee that Assumption 2.8(ii) is satisfied while the results in [10, Subsection 5.1] yield that the semigroup  $(T(t))_{t \geq 0}$  satisfies the assumptions of Theorem 2.4. Now, claim follows from Theorem 2.9 and Theorem 2.4.  $\square$

**3.7. Markov chain approximations for perturbed transition semigroups.** Let  $(\psi_t)_{t \geq 0}$  be a family of functions  $\psi_t : \mathbb{R}^d \rightarrow \mathbb{R}^d$  and  $(\mu_t)_{t \geq 0} \subset \mathcal{P}_p(\mathbb{R}^d)$ , where  $\mathcal{P}_p(\mathbb{R}^d)$  consists of all probability measures  $\mu : \mathcal{B}(\mathbb{R}^d) \rightarrow [0, 1]$  with  $\int_{\mathbb{R}^d} |x|^p \mu(dx) < \infty$ . In [8, 45], the authors studied Wasserstein perturbations of the linear transition semigroup

$$(R(t)f)(x) := \int_{\mathbb{R}^d} f(\psi_t(x) + y) \mu_t(dy)$$

which can be constructed as the monotone limit

$$(T(t)f)(x) = \inf_{n \in \mathbb{N}} J(2^{-n}t)^{2^n} f = \lim_{n \rightarrow \infty} J(2^{-n}t)^{2^n} f,$$

where the one step operators are defined by

$$(J(t)f)(x) := \sup_{\nu \in \mathcal{P}_p(\mathbb{R}^d)} \left( \int_{\mathbb{R}^d} (f(\psi_t(x) + y)) \nu(dy) - t\varphi \left( \frac{\mathcal{W}_p(\mu_t, \nu)}{t} \right) \right)$$

for all  $t \geq 0$ ,  $f \in C_b(\mathbb{R}^d)$  and  $x \in \mathbb{R}^d$ . Here, we denote by

$$\mathcal{W}_p(\mu, \nu) := \left( \inf_{\pi \in \Pi(\mu, \nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^p \pi(dx, dy) \right)^{\frac{1}{p}} \quad \text{for all } \mu, \nu \in \mathcal{P}_p(\mathbb{R}^d)$$

the  $p$ -Wasserstein distance, where the set  $\Pi(\mu, \nu)$  consists of all probability measures on  $\mathcal{B}(\mathbb{R}^d \times \mathbb{R}^d)$  with first marginal  $\mu$  and second marginal  $\nu$ . Furthermore,

$$t\varphi(c/t) := \begin{cases} 0, & c = t = 0, \\ \infty, & c \neq 0, t = 0, \end{cases}$$

for a non-decreasing function  $\varphi: \mathbb{R}_+ \rightarrow [0, \infty]$ . In this context, the family  $(R(t))_{t \geq 0}$  is called the reference model and  $(T(t))_{t \geq 0}$  is referred to as the perturbed transition semigroup. For related distributionally robust optimization problems and the description of nonparametric uncertainty using Wasserstein distances, we refer to [7, 9, 69].

The aim of this subsection is to provide a Markov chain approximation for the perturbed transition semigroup. To that end, let  $(h_n)_{n \in \mathbb{N}} \subset (0, \infty)$  be a sequence with  $h_n \rightarrow 0$  and define  $\mathcal{T}_n := \{kh_n: k \in \mathbb{N}_0\}$  for all  $n \in \mathbb{N}$ . Moreover, let  $(X_n)_{n \in \mathbb{N}}$  be a sequence of closed sets  $X_n \subset \mathbb{R}^d$  with  $x + y \in X_n$  for all  $x, y \in X_n$ , such that, for every  $x \in \mathbb{R}^d$ , there exists  $x_n \in X_n$  with  $x_n \rightarrow x$ . For every  $n \in \mathbb{N}$ , the set  $\mathcal{P}_p(X_n)$  consists of all probability measures  $\mu: \mathcal{B}(X_n) \rightarrow [0, 1]$  with  $\int_{X_n} |x|^p \mu(dx) < \infty$  and is endowed with the  $p$ -Wasserstein distance

$$\mathcal{W}_p^{X_n}(\mu, \nu) := \left( \inf_{\pi \in \Pi(\mu, \nu)} \int_{X_n \times X_n} |x - y|^p \pi(dx, dy) \right)^{\frac{1}{p}}.$$

Let  $(\psi_n)_{n \in \mathbb{N}}$  be a sequence of functions  $\psi_n: X_n \rightarrow X_n$  and  $(\mu_n)_{n \in \mathbb{N}}$  be a sequence of measures  $\mu_n \in \mathcal{P}_p(X_n)$ . For every  $n \in \mathbb{N}$ ,  $f \in C_b(X_n)$  and  $x \in X_n$ , we define

$$(I_n f)(x) := \sup_{\nu \in \mathcal{P}_p(X_n)} \left( \int_{X_n} f(\psi_n(x) + y) \nu(dy) - \varphi \left( \frac{\mathcal{W}_p^{X_n}(\mu_n, \nu)}{h_n} \right) h_n \right).$$

Furthermore, we define

$$A_0: D(A_0) \rightarrow C_b(\mathbb{R}^d), \quad f \mapsto \lim_{n \rightarrow \infty} \frac{1}{h_n} \left( \int_{X_n} f(\psi_n(\cdot) + y) \mu_n(dy) - f \right),$$

where  $D(A_0)$  consists of all functions  $f \in C_b(\mathbb{R}^d)$  such that the previous limit exists.

**Assumption 3.16.** Suppose that the following conditions are satisfied:

- (i) It holds  $\varphi(0) = 0$  and  $\lim_{c \rightarrow \infty} \frac{\varphi(c)}{c} = \infty$ .
- (ii) It holds  $\lim_{n \rightarrow \infty} \int_{X_n} |y| \mu_n(dy) = 0$ .
- (iii) There exists  $L \geq 0$  such that, for every  $n \in \mathbb{N}$  and  $x, y \in X_n$ ,

$$|\psi_n(x) - \psi_n(y)| \leq e^{Lh_n} |x - y| \quad \text{and} \quad |\psi_n(x + y) - x - \psi_n(y)| \leq Lh_n |x|.$$

Moreover, we assume that  $\sup_{x \in K_n} |\psi_n(x) - x| \rightarrow 0$  for all  $K \in \mathbb{R}^d$ .

- (iv) For every  $r \geq 0$ , there exists a sequence  $(\delta_n)_{n \in \mathbb{N}} \subset (0, 1]$  with  $\delta_n \downarrow 0$  such that, for every  $\lambda \in B_{\mathbb{R}^d}(r)$ , there exists a sequence  $(\lambda_n)_{n \in \mathbb{N}}$  with  $\lambda_n \in X_n$  and

$$\left| \frac{\lambda_n}{h_n} - \lambda \right| \leq \delta_n \quad \text{and} \quad \varphi \left( \frac{|\lambda_n|}{h_n} \right) - \varphi(|\lambda|) \leq \delta_n \quad \text{for all } n \in \mathbb{N}.$$

- (v) It holds  $C_b^\infty(\mathbb{R}^d) \subset D(A_0)$ . Moreover, there exist  $N \in \mathbb{N}$  and a non-decreasing function  $\rho: \mathbb{R}_+ \rightarrow \mathbb{R}_+$  with

$$\left\| \int_{X_n} f(\psi_n(\cdot) + y) \mu_n(dy) - f \right\|_\infty \leq \rho \left( \sum_{i=1}^N \|D^i f\|_\infty \right) h_n$$

for all  $n \in \mathbb{N}$  and  $f \in C_b^\infty(\mathbb{R}^d)$ .

Assumption 3.16(i) guarantees that

$$\varphi^*(a) := \sup_{b \geq 0} (ab - \varphi(b)) < \infty \quad \text{for all } a \geq 0.$$

In particular, the real-valued convex function  $\varphi^*: \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is continuous. Moreover, the following conditions from [10, Section 5.3] guarantee that the semigroup  $(T(t))_{t \geq 0}$  satisfies the assumptions of Theorem 2.4.

**Assumption 3.17.** Suppose that the following conditions are satisfied:

- (i) It holds  $\lim_{t \downarrow 0} \int_{\mathbb{R}^d} |y|^p d\mu_t(y) = 0$ .
- (ii) There exist  $r > 0$  and  $c \geq 0$  with  $\mu_t(B_{\mathbb{R}^d}(r)^c) \leq ct$  for all  $t \in [0, 1]$ .
- (iii) It holds  $\psi_t(0) = 0$  for all  $t \geq 0$  and there exists  $c \geq 0$  with

$$|\psi_t(x) - \psi_t(y) - (x - y)| \leq ct|x - y|$$

for all  $x, y \in \mathbb{R}^d$  and  $t \in [0, 1]$ .

- (iv) It holds  $A_0 f = \lim_{h \downarrow 0} \frac{R(h)f - f}{h}$  for all  $f \in C_b^\infty(\mathbb{R}^d)$ .

**Theorem 3.18.** Suppose that Assumption 3.16 is satisfied. Then, there exists a strongly continuous convex monotone semigroup  $(S(t))_{t \geq 0}$  on  $C_b(\mathbb{R}^d)$  with  $S(t)0 = 0$  given by

$$S(t)f = \lim_{n \rightarrow \infty} I_n^{k_n^t} f_n \quad \text{for all } (f, t) \in C_b(\mathbb{R}^d) \times \mathbb{R}_+,$$

where  $(k_n^t)_{n \in \mathbb{N}} \subset \mathbb{N}$  is arbitrary sequence with  $k_n^t h_n \rightarrow t$ , such that

$$C_b^\infty(\mathbb{R}^d) \subset D(A) \quad \text{and} \quad Af = A_0 f + \varphi^*(|Df|) \quad \text{for all } f \in C_b^\infty(\mathbb{R}^d).$$

Moreover, the conditions of Theorem 2.4 are satisfied and therefore

$$S(t)f = T(t)f \quad \text{for all } t \geq 0 \text{ and } f \in C_\kappa(\mathbb{R}^d).$$

*Proof.* In order to apply Theorem 2.9, we have to verify Assumption 2.8. For every  $n \in \mathbb{N}$ ,  $r \geq 0$ ,  $f \in \text{Lip}_b(X_n, r)$  and  $x, y \in X_n$ , it follows from Assumption 3.16(iv) that

$$\begin{aligned} |(I_n f)(x) - (I_n f)(y)| &\leq r |\psi_n(x) - \psi_n(y)| \leq e^{Lh_n} r |x - y|, \\ |\tau_x I_n f - I_n(\tau_x f)|(y) &\leq r |\psi_n(x + y) - x - \psi_n(y)| \leq Lr h_n |x|. \end{aligned}$$

The operators  $I_n: C_b(X_n) \rightarrow F_b(X_n)$  are convex and monotone with  $I_n 0 = 0$  such that

$$\|I_n f - I_n g\|_{\infty, X_n} \leq \|f - g\|_{\infty, X_n} \quad \text{for all } n \in \mathbb{N} \text{ and } f, g \in C_b(X_n).$$

Moreover, for every  $r \geq 0$  and  $\varepsilon > 0$ , there exists  $n_0 \in \mathbb{N}$  with

$$(I_n f)(x) = \sup_{\mathcal{W}_p^{X_n}(\mu_n, \nu) \leq \varepsilon} (I_n^\nu f)(x) \tag{3.14}$$

for all  $n \geq n_0$ ,  $f \in B_{C_b(X_n)}(r)$  and  $x \in X_n$ , where

$$(I_n^\nu f)(x) := \int_{X_n} f(\psi_n(x) + y) \nu(dy) - \varphi \left( \frac{\mathcal{W}_p^{X_n}(\mu_n, \nu)}{h_n} \right) h_n.$$



Indeed, by Assumption 3.16(ii), there exists  $n_0 \in \mathbb{N}$  with  $\varphi(\varepsilon/h_n)h_n \geq 2r$  and thus

$$I_n^\nu f \leq -r \leq I_n^{\mu_n} f \quad \text{for all } n \geq n_0 \text{ and } \mathcal{W}_p(\mu_n, \nu) \geq \varepsilon.$$

Now, let  $f \in C_b^\infty(\mathbb{R}^d)$ . For every  $n \in \mathbb{N}$ ,  $x \in X_n$  and  $\nu \in \mathcal{P}_p(X_n)$ ,

$$\begin{aligned} & \frac{1}{h_n} \left( \int_{X_n} f(\psi_n(x) + y) \nu(dy) - f(x) - \varphi \left( \frac{\mathcal{W}_p^{X_n}(\mu_n, \nu)}{h_n} \right) h_n \right) \\ &= \frac{1}{h_n} \left( \int_{X_n} f(\psi_n(x) + y) \nu(dy) - \int_{X_n} f(\psi_n(x) + y) \mu_n(dy) \right) - \varphi \left( \frac{\mathcal{W}_p^{X_n}(\mu_n, \nu)}{h_n} \right) \\ & \quad + \frac{1}{h_n} \left( \int_{X_n} f(\psi_n(x) + y) \mu_n(dy) - f(x) \right). \end{aligned} \quad (3.15)$$

It follows from Assumption 3.16(v), the Kantorowitsch transport duality, the fact that  $f$  is  $\|Df\|_\infty$ -Lipschitz and Hölder's inequality that

$$\begin{aligned} & \frac{1}{h_n} \left( \int_{X_n} f(\psi_n(x) + y) \nu(dy) - f(x) - \varphi \left( \frac{\mathcal{W}_p^{X_n}(\mu_n, \nu)}{h_n} \right) h_n \right) \\ & \leq \frac{\mathcal{W}_p^{X_n}(\mu_n, \nu) \|Df\|_\infty}{h_n} - \varphi \left( \frac{\mathcal{W}_p^{X_n}(\mu_n, \nu)}{h_n} \right) + \rho \left( \sum_{i=1}^N \|D^i f\|_\infty \right) \\ & \leq \varphi^*(\|Df\|_\infty) + \rho \left( \sum_{i=1}^N \|D^i f\|_\infty \right) \end{aligned}$$

for all  $n \in \mathbb{N}$ ,  $x \in X_n$  and  $\nu \in \mathcal{P}_p(X_n)$  and therefore

$$\left\| \frac{I_n f - f}{h_n} \right\|_\infty \leq \varphi^*(\|Df\|_\infty) + \rho \left( \sum_{i=1}^N \|D^i f\|_\infty \right) \quad \text{for all } n \in \mathbb{N}. \quad (3.16)$$

Next, we show that, for every  $K \Subset \mathbb{R}^d$  and  $\varepsilon > 0$ , there exists  $n_0 \in \mathbb{N}$  with

$$\left( \frac{I_n f - f}{h_n} \right)(x) \leq A_0 f(x) + \varphi^*(|Df(x)| + \varepsilon) + \varepsilon \quad (3.17)$$

for all  $n \geq n_0$  and  $x \in K_n$ . Since  $Df$  is continuous, there exists  $\delta \in (0, \varepsilon/4]$  with

$$|Df(x) - Df(x+y)| < \frac{\varepsilon}{4} \quad \text{for all } x \in K \text{ and } y \in B_{\mathbb{R}^d}(4\delta). \quad (3.18)$$

Furthermore, by Assumption 3.16(i) and equation (3.14), we can choose  $n_0 \in \mathbb{N}$  with

$$\mu_n(B_{X_n}(\delta)^c) < \frac{\varepsilon}{4} \quad \text{and} \quad \frac{1}{\delta} \int_{X_n} |y| \mu_n(dy) < \frac{\varepsilon}{4} \quad \text{for all } n \geq n_0 \quad (3.19)$$

such that

$$(I_n f)(x) = \sup_{\mathcal{W}_p^{X_n}(\mu_n, \nu) \leq \delta^2} (I_n^\nu f)(x) \quad \text{for all } n \geq n_0 \text{ and } x \in X_n. \quad (3.20)$$

Let  $n \geq n_0$ ,  $x \in K_n$  and  $\nu \in \mathcal{P}_p(X_n)$  with  $\mathcal{W}_p^{X_n}(\mu_n, \nu) \leq \delta^2$ . For an optimal coupling  $\pi \in \Pi(\nu, \mu_n)$  and  $q := \frac{p}{p-1}$ , Hölder's inequality implies

$$\begin{aligned}
& \frac{1}{h_n} \left( \int_{X_n} f(\psi_n(x) + y) \nu(dy) - \int_{X_n} f(\psi_n(x) + z) \mu_n(dz) \right) - \varphi \left( \frac{\mathcal{W}_p^{X_n}(\mu_n, \nu)}{h_n} \right) \\
&= \frac{1}{h_n} \int_{X_n \times X_n} f(\psi_n(x) + y) - f(\psi_n(x) + z) \pi(dy, dz) - \varphi \left( \frac{\mathcal{W}_p^{X_n}(\mu_n, \nu)}{h_n} \right) \\
&= \frac{1}{h_n} \int_0^1 \int_{X_n \times X_n} Df(\psi_n(x) + y + t(y - z))(y - z) \pi(dy, dz) dt - \varphi \left( \frac{\mathcal{W}_p^{X_n}(\mu_n, \nu)}{h_n} \right) \\
&\leq \frac{a_n^\nu(x)}{h_n} \left( \int_{X_n \times X_n} |y - z|^p \pi(dy, dz) \right)^{\frac{1}{p}} - \varphi \left( \frac{\mathcal{W}_p^{X_n}(\mu_n, \nu)}{h_n} \right) \\
&= a_n^\nu(x) \frac{\mathcal{W}_p^{X_n}(\mu_n, \nu)}{h_n} - \varphi_n \left( \frac{\mathcal{W}_p^{X_n}(\mu_n, \nu)}{h_n} \right) \leq \varphi^*(a_n^\nu(x)), \tag{3.21}
\end{aligned}$$

where

$$a_n^\nu(x) := \int_0^1 \left( \int_{X_n \times X_n} |Df(\psi_n(x) + y + t(y - z))|^q \pi(dy, dz) \right)^{\frac{1}{q}} dt.$$

It remains to show that

$$a_n^\nu(x) \leq |Df(x)| + \varepsilon \quad \text{for all } n \geq n_0. \tag{3.22}$$

For every  $n \geq 0$  and  $t \in [0, 1]$ , Minkowski's inequality and inequality (3.18) yield

$$\begin{aligned}
& \left( \int_{X_n \times X_n} |Df(\psi_n(x) + y + t(y - z))|^q \pi(dy, dz) \right)^{\frac{1}{q}} - |Df(x)| \\
&\leq \left( \int_{X_n \times X_n} |Df(\psi_n(x) + y + t(y - z)) - Df(x)|^q \pi(dy, dz) \right)^{\frac{1}{q}} \\
&\leq \left( \int_{B_{X_n \times X_n}(\delta)} |Df(\psi_n(x) + y + t(y - z)) - Df(x)|^q \pi(dy, dz) \right)^{\frac{1}{q}} \\
&\quad + 2\|Df\|_\infty \pi(B_{X_n \times X_n}(\delta)^c) \\
&\leq 2\|Df\|_\infty \pi(B_{X_n \times X_n}(\delta)^c) + \varepsilon/4.
\end{aligned}$$

Moreover, it follows from inequality (3.19) and Hölder's inequality that

$$\begin{aligned}
\pi(B_{X_n \times X_n}(\delta)^c) &\leq \pi(B_{X_n}(\delta)^c \times X_n) + \pi(X_n \times B_{X_n}(\delta)^c) \\
&= \nu(B_{X_n}(\delta)^c) + \mu_n(B_{X_n}(\delta)^c) \\
&\leq \frac{1}{\delta} \int_{X_n} |y| \nu(dy) + \frac{\varepsilon}{4} \\
&\leq \frac{1}{\delta} \mathcal{W}_p^{X_n}(\mu_n, \nu) + \frac{1}{\delta} \int_{X_n} |y| \mu_n(dy) + \frac{\varepsilon}{4} \leq \delta + \frac{\varepsilon}{2} \leq \frac{3\varepsilon}{4}.
\end{aligned}$$

Moreover, due to Assumption 3.16(v), there exists  $n_1 \geq n_0$  with

$$\left\| \frac{1}{h_n} \left( \int_{X_n} f(\psi_n(\cdot) + y) \mu_n(dy) - f \right) - A_0 f \right\|_{\infty, K_n} \leq \varepsilon \quad \text{for all } n \geq n_1. \tag{3.23}$$

Combing the equations (3.15) and (3.20) with the inequalities (3.21)–(3.23) yields that inequality (3.17) is valid. In order to obtain a similar lower bound, we show that, for every  $K \subseteq \mathbb{R}^d$  and  $\varepsilon > 0$ , there exists  $n_0 \in \mathbb{N}$  with

$$\left( \frac{I_n f - f}{h_n} \right) (x) \geq A_0 f(x) - \varphi^*(|Df(x)|) - 2\varepsilon \quad (3.24)$$

for all  $n \geq n_0$  and  $x \in K_n$ . By Assumption 3.16(i), there exists  $r \geq 0$  with

$$\varphi^*(|Df(x)|) = \sup_{\lambda \in B_{\mathbb{R}^d}(r)} (\lambda Df(x) - \varphi(|\lambda|)) \quad \text{for all } x \in \mathbb{R}^d. \quad (3.25)$$

Moreover, by Assumption 3.16(iv), there exists a sequence  $(\delta_n)_{n \in \mathbb{N}} \subset (0, 1]$  with  $\delta_n \downarrow 0$  such that, for every  $\lambda \in B_{\mathbb{R}^d}(r)$ , there exists a sequence  $(\lambda_n)_{n \in \mathbb{N}}$  with  $\lambda_n \in X_n$  and

$$\left| \frac{\lambda_n}{h_n} - \lambda \right| \leq \delta_n \quad \text{and} \quad \varphi \left( \frac{|\lambda_n|}{h_n} \right) - \varphi(|\lambda|) \leq \delta_n \quad \text{for all } n \in \mathbb{N}. \quad (3.26)$$

In the sequel, we fix  $\lambda \in B_{\mathbb{R}^d}(r)$  and choose such a sequence  $(\lambda_n)_{n \in \mathbb{N}}$ . Since the measures  $\nu_n := \mu_n * \delta_{\lambda_n} \in \mathcal{P}_p(X_n)$  satisfy  $\mathcal{W}_p^{X_n}(\mu_n, \nu_n) = |\lambda_n|$ , we obtain

$$\begin{aligned} & \frac{1}{h_n} \left( \int_{X_n} f(\psi_n(x) + y) \nu_n(dy) - \int_{X_n} f(\psi_n(x) + y) \mu_n(dy) \right) - \varphi \left( \frac{\mathcal{W}_p^{X_n}(\mu_n, \nu_n)}{h_n} \right) \\ &= \frac{1}{h_n} \int_{X_n} f(\psi_n(x) + y + \lambda_n) - f(\psi_n(x) + y) \mu_n(dy) - \varphi \left( \frac{|\lambda_n|}{h_n} \right) \\ &= \frac{\lambda_n}{h_n} \int_0^1 \int_{X_n} Df(\psi_n(x) + y + t\lambda_n) \mu_n(dy) dt - \varphi \left( \frac{|\lambda_n|}{h_n} \right) \end{aligned} \quad (3.27)$$

for all  $n \in \mathbb{N}$  and  $x \in X_n$ . Moreover, there exists  $\delta > 0$  with

$$|Df(x) - Df(x + y)| \leq \frac{\varepsilon}{4(1+r)} \quad \text{for all } x \in K \text{ and } y \in B_{\mathbb{R}^d}(3\delta). \quad (3.28)$$

Since inequality (3.26) guarantees  $|\lambda_n| \leq (1+r)h_n \rightarrow 0$  and due Assumption 3.16(ii) and (iii), we can choose  $n_0 \in \mathbb{N}$  such that  $|\lambda_n| \leq \delta$ ,  $\sup_{x \in K_n} |\psi_n(x) - x| \leq \delta$ ,

$$2(1+r)\|Df\|_{\infty} \mu_n(B_{X_n}(\delta)^c) \leq \frac{\varepsilon}{4} \quad \text{and} \quad (1 + \|Df\|_{\infty})\delta_n \leq \frac{\varepsilon}{4} \quad (3.29)$$

for all  $n \geq n_0$ . Inequality (3.26), inequality (3.28) and inequality (3.21) imply

$$\begin{aligned} & \frac{\lambda_n}{h_n} \int_0^1 \int_{X_n} Df(\psi_n(x) + y + t\lambda_n) \mu_n(dy) dt - \varphi \left( \frac{|\lambda_n|}{h_n} \right) - (\lambda Df(x) - \varphi(|\lambda|)) \\ & \geq -\frac{|\lambda_n|}{h_n} \int_0^1 \int_{X_n} |Df(\psi_n(x) + y + t\lambda_n) - Df(x)| \mu_n(dy) dt - \left| \frac{\lambda_n}{h_n} - \lambda \right| \cdot |Df(x)| \\ & \quad + \varphi(|\lambda|) - \varphi \left( \frac{|\lambda_n|}{h_n} \right) \\ & \geq -(1+r) \int_0^1 \int_{B_{X_n}(\delta)} |Df(\psi_n(x) + y + t\lambda_n) - Df(x)| \mu_n(dy) dt \\ & \quad - 2(1+r)\|Df\|_{\infty} \mu_n(B_{X_n}(\delta)^c) - \frac{\varepsilon}{2} \geq -\varepsilon \end{aligned} \quad (3.30)$$

for all  $n \geq n_0$  and  $x \in K_n$ . Combining equation (3.15), inequality (3.23), equation (3.25) and inequality (3.30) yields that inequality (3.24) is valid. Hence, from inequality (3.17), inequality (3.24) and the continuity of  $\varphi^*$ , we obtain

$$\lim_{n \rightarrow \infty} \left\| \frac{I_n f - f}{h_n} - A_0 f - \varphi^*(|Df|) \right\|_{\infty, K_n} = 0.$$

In addition, inequality (3.16), Corollary 2.13 and Remark 2.17 guarantee that the operators  $I_n: C_b(X_n) \rightarrow C_b(X_n)$  are well-defined and satisfy Assumption 2.8(ii) while the results in [10, Subsection 5.3] yield that  $(T(t))_{t \geq 0}$  satisfies the assumptions of Theorem 2.4. Now, claim follows from Theorem 2.9 and Theorem 2.4.  $\square$

#### APPENDIX A. A VERSION OF ARZÉLA ASCOLI'S THEOREM

Let  $(X_n)_{n \in \mathbb{N}}$  be a sequence of subsets  $X_n \subset \mathbb{R}^d$  such that, for every  $x \in \mathbb{R}^d$ , there exists a sequence  $(x_n)_{n \in \mathbb{N}}$  with  $x_n \in X_n$  and  $x_n \rightarrow x$ . A sequence  $(f_n)_{n \in \mathbb{N}}$  of functions  $f_n: X_n \rightarrow \mathbb{R}$  is called bounded if  $\sup_{n \in \mathbb{N}} \|f_n\|_{\kappa, X_n} < \infty$ . Furthermore, the sequence is called uniformly equicontinuous if, for every  $\varepsilon > 0$ , there exists  $\delta > 0$  with

$$|f_n(x) - f_n(y)| < \varepsilon \quad \text{for all } n \in \mathbb{N} \text{ and } x, y \in X_n \text{ and } |x - y| < \delta.$$

**Lemma A.1.** *Let  $(f_n)_{n \in \mathbb{N}}$  be a bounded uniformly equicontinuous sequence of functions  $f_n: X_n \rightarrow \mathbb{R}$ . Then, there exist  $f \in C_\kappa(\mathbb{R}^d)$  and a subsequence  $(n_l)_{l \in \mathbb{N}} \subset \mathbb{N}$  with*

$$\lim_{l \rightarrow \infty} \|f - f_{n_l}\|_{\infty, K_{n_l}} = 0$$

for all  $K \Subset \mathbb{R}^d$  with  $K \cap X_{n_l} \neq \emptyset$  for all  $l \in \mathbb{N}$ . Furthermore, it holds

$$f(x) = \lim_{l \rightarrow \infty} f_{n_l}(x_{n_l})$$

for all  $x \in \mathbb{R}^d$  and sequences  $(x_n)_{n \in \mathbb{N}}$  with  $x_n \in X_n$  and  $x_n \rightarrow x$ .

*Proof.* Let  $D \subset \mathbb{R}^d$  be countable and dense. Moreover, for every  $x \in D$ , let  $(x_n)_{n \in \mathbb{N}}$  be a sequence with  $x_n \in X_n$  and  $x_n \rightarrow x$ . We use the Bolzano–Weierstrass theorem and a diagonalization argument to choose a subsequence  $(n_l)_{l \in \mathbb{N}} \subset \mathbb{N}$  such that the limit

$$f(x) := \lim_{l \rightarrow \infty} f_{n_l}(x_{n_l}) \in \mathbb{R}$$

exists for all  $x \in D$ . This defines a uniformly continuous function  $f: D \rightarrow \mathbb{R}$  satisfying  $\sup_{x \in D} |f(x)|\kappa(x) < \infty$  which can be uniquely extended to a uniformly continuous function  $f \in C_\kappa(\mathbb{R}^d)$ . Moreover, for every  $x \in \mathbb{R}^d$  and sequence  $(x_n)_{n \in \mathbb{N}}$  with  $x_n \in X_n$  and  $x_n \rightarrow x$ , it follows from the uniform equicontinuity that  $f_{n_l}(x_{n_l}) \rightarrow f(x)$ . Let  $K \Subset X$  with  $K \cap X_{n_l} \neq \emptyset$  for all  $l \in \mathbb{N}$ . We assume, by contradiction, that there exist  $\varepsilon > 0$ , a subsequence  $(n_{l,1})_{l \in \mathbb{N}}$  of  $(n_l)_{l \in \mathbb{N}}$  and  $x_{n_{l,1}} \in K_{n_{l,1}}$  with

$$|f_{n_{l,1}}(x_{n_{l,1}}) - f(x_{n_{l,1}})| \geq \varepsilon \quad \text{for all } l \in \mathbb{N}.$$

Since  $K$  is compact, we can choose a further subsequence  $(n_{l,2})_{l \in \mathbb{N}}$  of  $(n_{l,1})_{l \in \mathbb{N}}$  and  $x \in \mathbb{R}^d$  with  $x_{n_{l,2}} \rightarrow x$ . It holds  $f(x_{n_{l,2}}) \rightarrow f(x)$  and  $f_{n_{l,2}}(x_{n_{l,2}}) \rightarrow f(x)$  which leads to a contradiction. We obtain  $\lim_{l \rightarrow \infty} \|f - f_{n_l}\|_{\infty, K_{n_l}} = 0$ .  $\square$

#### APPENDIX B. A BASIC CONVEXITY ESTIMATE

**Lemma B.1.** *Let  $V$  be a vector space and  $\phi: V \rightarrow \mathbb{R}$  be a convex functional. Then,*

$$\phi(v) - \phi(w) \leq \lambda \left( \phi \left( \frac{v-w}{\lambda} + w \right) - \phi(w) \right) \quad \text{for all } v, w \in V \text{ and } \lambda \in (0, 1].$$

*Proof.* We use the convexity to estimate

$$\begin{aligned}\phi(v) - \phi(w) &= \phi\left(\lambda\left(\frac{v-w}{\lambda} + w\right) + (1-\lambda)w\right) - \phi(w) \\ &\leq \lambda\phi\left(\frac{v-w}{\lambda} + w\right) + (1-\lambda)\phi(w) - \phi(w) \\ &= \lambda\left(\phi\left(\frac{v-w}{\lambda} + w\right) - \phi(w)\right)\end{aligned}$$

for all  $v, w \in V$  and  $\lambda \in (0, 1]$ .  $\square$

**Lemma B.2.** *Let  $(S(t))_{t \geq 0}$  be a convex monotone semigroup on  $C_\kappa(\mathbb{R}^d)$  such that, for every  $r, T \geq 0$ , there exists  $c \geq 0$  with  $\|S(t)f - S(t)g\|_\kappa \leq c\|f - g\|_\kappa$  for all  $t \in [0, T]$  and  $f, g \in B_{C_\kappa(\mathbb{R}^d)}(r)$ . Then, it holds  $S(t): \mathcal{L}_+^S \rightarrow \mathcal{L}_+^S$  for all  $t \geq 0$ .*

*Proof.* Let  $f \in \mathcal{L}_+^S$  and  $t \geq 0$ . By definition, there exists  $h_0 > 0$  with

$$r := \sup_{h \in (0, h_0]} \left\| \frac{(S(h)f - f)^+}{h} \right\|_\kappa + \|f\|_\kappa < \infty$$

and the semigroup property yields  $S(h)S(t)f = S(t)S(h)f$  for all  $h \geq 0$ . By Lemma B.1, the monotonicity and the uniform Lipschitz continuity, there exists  $c \geq 0$  with

$$\begin{aligned}\left(\frac{S(h)S(t)f - S(t)f}{h}\right)_\kappa &\leq \left(S(t)\left(f + \frac{(S(h)f - f)^+}{h}\right) - S(t)f\right)_\kappa \\ &\leq c \left\| \frac{(S(h)f - f)^+}{h} \right\|_\kappa \leq cr \quad \text{for all } h \in (0, h_0].\end{aligned} \quad \square$$

## APPENDIX C. CONTINUITY FROM ABOVE

Let  $\text{ca}_\kappa^+(\mathbb{R}^d)$  be the set of all Borel measures  $\mu: \mathcal{B}(\mathbb{R}^d) \rightarrow [0, \infty]$  with  $\int_{\mathbb{R}^d} \frac{1}{\kappa} d\mu < \infty$ . Moreover, the convex conjugate of a functional  $\phi: C_\kappa(\mathbb{R}^d) \rightarrow \mathbb{R}$  is defined by

$$\phi^*: \text{ca}_\kappa^+(\mathbb{R}^d) \rightarrow [0, \infty], \quad \mu \mapsto \sup_{f \in C_\kappa(\mathbb{R}^d)} (\mu f - \phi(f)), \quad \text{where} \quad \mu f := \int_{\mathbb{R}^d} f d\mu.$$

In the sequel, let  $(X_i)_{i \in I}$  be a family of closed sets  $X_i \subset \mathbb{R}^d$  and  $(\phi_i)_{i \in I}$  be a family of convex monotone functionals  $\phi_i: C_\kappa(X_i) \rightarrow \mathbb{R}$  with  $\phi_i(0) = 0$  and

$$\sup_{i \in I} \sup_{f \in B_{C_\kappa(X_i)}(r)} |\phi_i(f)| < \infty \quad \text{for all } r \geq 0. \quad (\text{C.1})$$

For every  $i \in I$  and  $f \in C_\kappa(\mathbb{R}^d)$ , we define  $\phi_i(f) := \phi_i(f|_{X_i})$ , i.e., we consider  $C_\kappa(\mathbb{R}^d)$  as a subset of  $C_\kappa(X_i)$  using the possibility to restrict functions from  $\mathbb{R}^d$  to  $X_i$ . Moreover, let  $K_i := K \cap X_i$  for all  $K \in \mathbb{R}^d$  and  $i \in I$ .

**Lemma C.1.** *The following two statements are equivalent:*

- (i) *It holds  $\sup_{i \in I} \phi_i(f_n) \downarrow 0$  for all sequences  $(f_n)_{n \in \mathbb{N}} \subset C_\kappa(\mathbb{R}^d)$  with  $f_n \downarrow 0$ .*
- (ii) *For every  $\varepsilon > 0$  and  $r \geq 0$ , there exist  $c \geq 0$  and  $K \in \mathbb{R}^d$  with*

$$\sup_{i \in I} |\phi_i(f) - \phi_i(g)| \leq c\|f - g\|_{\infty, K_i} + \varepsilon$$

*for all  $i \in I$  and  $f, g \in B_{C_\kappa(X_i)}(r)$ .*

*Proof.* First, we assume that condition (i) is satisfied. Let  $\varepsilon > 0$  and  $r \geq 0$ . Since the functional  $\phi_i: C_\kappa(\mathbb{R}^d) \rightarrow \mathbb{R}$  is continuous from above, we can apply [10, Theorem C.1] to obtain

$$\phi_i(f) = \max_{\mu \in M_i} (\mu f - \phi_i^*(\mu)) \quad \text{for all } i \in I \text{ and } f \in B_{C_\kappa(\mathbb{R}^d)}(r), \quad (\text{C.2})$$

where  $M_i := \{\mu \in \text{ca}_\kappa^+(\mathbb{R}^d): \phi_i^*(\mu) \leq \phi_i(\frac{2r}{\kappa}) - 2\phi_i(-\frac{r}{\kappa})\}$ . Furthermore, it holds

$$\mu(X_i^c) = 0 \quad \text{for all } \mu \in M_i \text{ and } i \in I. \quad (\text{C.3})$$

Indeed, by contradiction, we suppose that there exists  $\mu \in M_i$  with  $\mu(X_i^c) > 0$ . Then, due to Ulam's theorem, there exists  $K \subseteq X_i^c$  with  $\mu(K) > 0$ . Moreover, by Urysohn's lemma, there exists a continuous function  $f: \mathbb{R}^d \rightarrow [0, 1]$  with  $f(x) = 0$  for all  $x \in X_i$  and  $f(x) = 1$  for all  $x \in K$ . We use  $\phi_i(\lambda f) = \phi_i(\lambda f|_{X_i}) = \phi_i(0) = 0$  to conclude

$$\phi_i^*(\mu) \geq \sup_{\lambda \geq 0} (\mu(\lambda f) - \phi_i(\lambda f)) \geq \sup_{\lambda \geq 0} \lambda \mu(K) = \infty.$$

This contradicts the fact that  $\mu \in M_i$  guarantees  $\phi_i^*(\mu) < \infty$ . Next, we show that

$$\phi_i(f) = \max_{\mu \in M_i} (\mu f - \phi_i^*(\mu)) \quad \text{for all } i \in I \text{ and } f \in B_{C_\kappa(X_i)}(r), \quad (\text{C.4})$$

where  $\mu f = \int_{X_i} f d\mu$  is well-defined due to equation (C.3). Let  $i \in I$  and  $f \in B_{C_\kappa(X_i)}(r)$ . Since  $X_i \subset \mathbb{R}^d$  is closed and  $f\kappa \in C_b(X_i)$ , by Tietze's extension theorem, there exists a function  $\tilde{g} \in C_b(\mathbb{R}^d)$  with  $(f\kappa)(x) = \tilde{g}(x)$  for all  $x \in X_i$  and  $\|f\kappa\|_{\infty, X_i} = \|\tilde{g}\|_\infty$ . Hence, the function  $g := \frac{1}{\kappa}\tilde{g} \in C_\kappa(\mathbb{R}^d)$  satisfies  $f(x) = g(x)$  for all  $x \in X_i$  and  $\|f\|_{\kappa, X_i} = \|g\|_\kappa$ . It follows from equation (C.2) and equation (C.3) that

$$\phi_i(f) = \phi_i(g) = \max_{\mu \in M_i} (\mu g - \phi_i^*(\mu)) = \max_{\mu \in M_i} (\mu f - \phi_i^*(\mu)).$$

Condition (i) guarantees that the convex monotone functional

$$\phi: C_\kappa(\mathbb{R}^d) \rightarrow \mathbb{R}, \quad f \mapsto \sup_{i \in I} \phi_i(f)$$

is continuous from above with  $\phi(0) = 0$ . In addition, it follows from condition (C.1) and [6, Theorem 2.2] that

$$M := \left\{ \mu \in \text{ca}_\kappa^+(\mathbb{R}^d): \phi^*(\mu) \leq \sup_{i \in I} \left( \phi_i\left(\frac{2r}{\kappa}\right) - 2\phi_i\left(-\frac{r}{\kappa}\right) \right) \right\}$$

is  $\sigma(\text{ca}_\kappa^+(\mathbb{R}^d), C_\kappa(\mathbb{R}^d))$ -relatively compact and thus Prokhorov's theorem guarantees the existence of  $K \subseteq \mathbb{R}^d$  with  $\sup_{\mu \in M} \int_{K^c} \frac{1}{\kappa} d\mu \leq \frac{\varepsilon}{2r}$ . For every  $i \in I$  and  $f, g \in B_{C_\kappa(X_i)}(r)$ , we use equation (C.4) and  $M_i \subset M$  to obtain

$$\begin{aligned} |\phi_i(f) - \phi_i(g)| &\leq \sup_{\mu \in M_i} |\mu f - \mu g| \\ &\leq \sup_{\mu \in M_i} \left( \int_K |f - g| d\mu + \int_{K^c} |f - g| d\mu \right) \\ &\leq \sup_{\mu \in M_i} \left( \mu(K) \|f - g\|_{\infty, K_i} + \int_{K^c} \frac{2r}{\kappa} d\mu \right) \\ &\leq c \|f - g\|_{\infty, K_i} + \varepsilon, \end{aligned}$$

where  $c := \sup_{\mu \in M} \mu(K) \leq \phi(1) + \sup_{\mu \in M} \phi^*(\mu) < \infty$ .

Second, we assume that condition (ii) is satisfied. Let  $(f_n)_{n \in \mathbb{N}} \subset C_\kappa(\mathbb{R}^d)$  be a sequence with  $f_n \downarrow 0$  and define  $r := \|f_1\|_\kappa$ . For every  $\varepsilon > 0$ , we can choose  $c \geq 0$  and  $K \Subset \mathbb{R}^d$  with

$$\sup_{i \in I} \phi_i(f_n) \leq c \|f_n\|_{\infty, K} + \frac{\varepsilon}{2} \quad \text{for all } n \in \mathbb{N}.$$

Hence, by Dini's theorem, there exists  $n_0 \in \mathbb{N}$  with  $\sup_{i \in I} \phi_i(f_n) \leq \varepsilon$  for all  $n \geq n_0$ .  $\square$

Now, let  $(\Phi_i)_{i \in I}$  be a family of convex monotone operators  $\Phi_i: C_\kappa(X_i) \rightarrow F_\kappa(X_i)$  with  $\Phi_i(0) = 0$  and

$$\sup_{i \in I} \sup_{f \in B_{C_\kappa(X_i)}(r)} \|\Phi_i(f)\|_\kappa < \infty \quad \text{for all } r \geq 0. \quad (\text{C.5})$$

Here, the space  $F_\kappa(X_i)$  consists of all functions  $f: X_i \rightarrow \mathbb{R}$  with  $\|f\|_\kappa < \infty$ .

**Corollary C.2.** *The following two statements are equivalent:*

(i) *For every  $K \Subset \mathbb{R}^d$  and  $(f_n)_{n \in \mathbb{N}} \subset C_\kappa(\mathbb{R}^d)$  with  $f_n \downarrow 0$ ,*

$$\sup_{i \in I} \sup_{x \in K_i} (\Phi_i f_n)(x) \downarrow 0 \quad \text{as } n \rightarrow \infty.$$

(ii) *For every  $\varepsilon > 0$ ,  $r \geq 0$  and  $K \Subset \mathbb{R}^d$ , there exist  $c \geq 0$  and  $K' \Subset \mathbb{R}^d$  with*

$$\|\Phi_i f - \Phi_i g\|_{\infty, K_i} \leq c \|f - g\|_{\infty, K'_i} + \varepsilon \quad \text{for all } f, g \in B_{C_\kappa(X_i)}(r).$$

*Proof.* This follows from Lemma C.1 with  $\tilde{I} := \{(i, x): i \in I, x \in K_i\}$ ,  $\tilde{X}_{(i, x)} := X_i$  and  $\tilde{\phi}_{(i, x)}(f) := (\Phi_i f)(x)$  for all  $(i, x) \in \tilde{I}$  and  $f \in C_\kappa(\mathbb{R}^d)$ .  $\square$

**Corollary C.3.** *Let  $\tilde{\kappa}: \mathbb{R}^d \rightarrow (0, \infty)$  be another bounded continuous function such that, for every  $\varepsilon > 0$ , there exists  $K \Subset X$  with*

$$\sup_{x \in K^c} \frac{\tilde{\kappa}(x)}{\kappa(x)} \leq \varepsilon.$$

*Moreover, for every  $r \geq 0$ , there exists  $c \geq 0$  with*

$$\|\Phi_i f\|_{\tilde{\kappa}, X_i} \leq c \|f\|_{\tilde{\kappa}, X_i} \quad \text{for all } i \in I \text{ and } f \in B_{C_\kappa(X_i)}(r).$$

*Then, for every  $\varepsilon > 0$ ,  $r \geq 0$  and  $K \Subset \mathbb{R}^d$ , there exist  $c \geq 0$  and  $K' \Subset \mathbb{R}^d$  with*

$$\|\Phi_i f - \Phi_i g\|_{\infty, K_i} \leq c \|f - g\|_{\infty, K'_i} + \varepsilon \quad \text{for all } i \in I \text{ and } f, g \in B_{C_\kappa(X_i)}(r).$$

*Proof.* It suffices to verify condition (i) from Corollary C.2. To do so, let  $K \Subset \mathbb{R}^d$  and  $(f_n)_{n \in \mathbb{N}} \subset C_\kappa(\mathbb{R}^d)$  be a sequence with  $f_n \downarrow 0$ . Let  $\varepsilon > 0$  and choose  $K \Subset \mathbb{R}^d$  with

$$\sup_{x \in K^c} \frac{\tilde{\kappa}(x)}{\kappa(x)} \leq \varepsilon.$$

By Dini's theorem, there exists  $n_0 \in \mathbb{N}$  with  $f_n(x) \leq \varepsilon$  for all  $n \geq n_0$  and  $x \in K$ . Hence,

$$\|f_n\|_{\tilde{\kappa}} = \sup_{x \in K} |f_n(x)| \tilde{\kappa}(x) + \sup_{x \in K^c} |f_n(x)| \kappa(x) \frac{\tilde{\kappa}(x)}{\kappa(x)} \leq \left( \sup_{x \in K} \tilde{\kappa}(x) + \|f_n\|_\kappa \right) \varepsilon.$$

We obtain  $\|f_n\|_{\tilde{\kappa}} \rightarrow 0$  as  $n \rightarrow \infty$  and thus

$$\sup_{i \in I} \|\Phi_i f_n\|_{\tilde{\kappa}} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

For every  $K \Subset \mathbb{R}^d$ , we use the fact that  $\inf_{x \in K} \kappa(x) > 0$  to conclude

$$\sup_{i \in I} \sup_{x \in K_i} (\Phi_i f_n)(x) \downarrow 0 \quad \text{as } n \rightarrow \infty.$$

Now, the claim follows from Corollary C.2.  $\square$



**Corollary C.4.** Assume that, for every  $\varepsilon > 0$ ,  $r \geq 0$  and  $K \Subset \mathbb{R}^d$ , there exist a family  $(\zeta_x)_{x \in K}$  of continuous functions  $\zeta_x: \mathbb{R}^d \rightarrow \mathbb{R}$  and  $\tilde{K} \Subset \mathbb{R}^d$  with

- (i)  $0 \leq \zeta_x \leq 1$  for all  $x \in K$ ,
- (ii)  $\sup_{y \in \tilde{K}^c} \zeta_x(y) \leq \varepsilon$  for all  $x \in K$ ,
- (iii)  $(\Phi_i(\frac{r}{\kappa}(1 - \zeta_x)))(x) \leq \varepsilon$  for all  $i \in I$  and  $x \in K_i$ .

Then, for every  $\varepsilon > 0$ ,  $r \geq 0$  and  $K \Subset \mathbb{R}^d$ , there exist  $c \geq 0$  and  $K' \Subset \mathbb{R}^d$  with

$$\|\Phi_i f - \Phi_i g\|_{\infty, K_i} \leq c \|f - g\|_{\infty, K'_i} + \varepsilon \quad \text{for all } i \in I \text{ and } f, g \in B_{C_\kappa(X_i)}(r).$$

*Proof.* It suffices to verify condition (i) from Corollary C.2. To do so, let  $K \Subset \mathbb{R}^d$  and  $(f_n)_{n \in \mathbb{N}} \subset C_\kappa(\mathbb{R}^d)$  be a sequence with  $f_n \downarrow 0$ . Let  $\varepsilon \in (0, 1]$  and  $r := 2\|f_1\|_\kappa + 1$ . It follows from condition (C.5) and  $\inf_{x \in K} \kappa(x) > 0$  that

$$c := \sup_{i \in I} \sup_{x \in K_i} (\Phi_i(\frac{r}{\kappa}))(x) + 1 < \infty.$$

By assumption, there exist a family  $(\zeta_x)_{x \in K}$  of continuous functions  $\zeta_x: \mathbb{R}^d \rightarrow \mathbb{R}$  and  $\tilde{K} \Subset \mathbb{R}^d$  satisfying the conditions (i)-(iii) with  $\varepsilon/c$ . For every  $i \in I$  and  $x \in K_i$ , the convexity and monotonicity of  $\Phi_i$ , condition (i) and condition (iii) imply

$$\begin{aligned} (\Phi_i f_n)(x) &\leq \frac{1}{2}(\Phi_i(2f_n \zeta_x))(x) + \frac{1}{2}(\Phi_i(2f_n(1 - \zeta_x)))(x) \\ &\leq \frac{1}{2}(\Phi_i(2f_n \zeta_x))(x) + \frac{\varepsilon}{2}. \end{aligned}$$

By condition (i), condition (ii) and Dini's theorem, there exists  $n_0 \in \mathbb{N}$  with

$$2f_n \zeta_x \leq \frac{r\varepsilon}{c\kappa} \quad \text{for all } n \geq n_0 \text{ and } x \in K.$$

Hence, for every  $i \in I$ ,  $x \in K_i$  and  $n \geq n_0$ , we use the fact that  $\Phi_i$  is convex and monotone with  $\Phi_i(0) = 0$  to obtain

$$(\Phi_i(2f_n \zeta_x))(x) \leq (\Phi_i(\frac{r\varepsilon}{c\kappa}))(x) \leq \frac{\varepsilon}{c}(\Phi_i(\frac{r}{\kappa}))(x) \leq \varepsilon. \quad \square$$

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